

COMPUTATION OF STABILIZING PI CONTROLLERS FOR INTERVAL SYSTEMS

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Abstract: In this paper, a simple method for the computation of the parameters of a PI controller which stabilize control system with fixed parameters is given. The proposed method is based on plotting the stability boundary locus in the (k_p, k_i) -plane and then computing stabilizing PI controllers. The method presented does not require sweeping over the parameters and also does not need a linear programming to solve a set of inequalities. Thus, it has several important advantages over existing results. The proposed method is also used to compute all the parameters of a PI controller which stabilize a control system with and interval plant family. Examples are given to show the benefit of the method presented.

Keywords: Stabilization, PI control, Kharitonov theorem, Interval systems.

1. Introduction

There has been a great amount of research work on the tuning of PI, PID and lag/lead controllers since these types of controllers have been widely used in industries for several decades (see [1-5] and references therein). However, many important results have been recently reported on computation of all stabilizing P, PI and PID controllers after the publication of work by Ho *et al.* [6-9]. A new and complete analytical solution which is based on the generalized version of the Hermite-Biehler theorem has been provided in [6] for computation of all stabilizing constant gain controllers for a given plant. A linear programming characterization of all stabilizing PI and PID controllers for a given plant has been obtained in [7,9]. This characterization besides being computationally efficient has revealed important structural properties of PI and PID controllers. For example, it was shown that for a fixed proportional gain, the set of stabilizing integral

and derivative gains lie in a convex set. This method is very important since it can cope with systems that are open loop stable or unstable, minimum or nonminimum phase. However, the computation time for this approach increases in an exponential manner with the order of the system being considered. It also needs sweeping over proportional gain to find all stabilizing PI and PID controllers which is a disadvantage of the method. An alternative fast approach to this problem based on the use of the Nyquist plot has been given in [10-11]. An extension of the method given in [7] to the lag/lead controller structure has been given in [12]. A parameter space approach using singular frequency concept has been given in [13] for design of robust PID controllers. More direct graphical approaches to this problem based on frequency response plots have been given in [14-15]. However, the frequency gridding has become the major problem for this approach. On the other hand, compensator design in classical control engineering is based on a plant with fixed parameters. In the real world, however, most practical system models are not known exactly, meaning that the system contains uncertainties. So, in designing a control system for both stability and performance robustness, it is always required to take uncertainties into account. This requirement has attracted the attention of many researchers over the years to find solutions for the problems of robust stability analysis and controller synthesis for uncertain systems especially control systems with parametric uncertainty. Much recent work on systems with uncertain parameters has been based on Kharitonov's result [16] on the stability of interval polynomials. Kharitonov showed that for the interval polynomial

$$P(s) = a_0 + a_1s + a_2s^2 + a_3s^3 + \dots + a_ns^n \quad (1)$$

where $a_i \in [\underline{a_i}, \overline{a_i}]$, $i = 1, 2, \dots, n$, the stability of the set could be found by applying the Routh criterion to the following four polynomials

$$\begin{aligned}
p_1(s) &= \underline{a_0} + \underline{a_1}s + \overline{a_2}s^2 + \overline{a_3}s^3 + \dots \\
p_2(s) &= \overline{a_0} + \underline{a_1}s + \overline{a_2}s^2 + \underline{a_3}s^3 + \dots \\
p_3(s) &= \underline{a_0} + \overline{a_1}s + \underline{a_2}s^2 + \overline{a_3}s^3 + \dots \\
p_4(s) &= \overline{a_0} + \underline{a_1}s + \underline{a_2}s^2 + \underline{a_3}s^3 + \dots
\end{aligned} \quad (2)$$

In this paper, a new approach is given for computation of stabilizing PI controllers in the parameter plane, (k_p, k_i) -plane. The result of [11] is used to avoid the problem of frequency gridding. Thus, a very fast way of calculating the stabilizing values of PI controllers for a given SISO control system is given. The proposed method is then used for computation of PI controllers for the stabilization of interval systems.

The paper is organized as follows: The proposed method is described in section 2. In section 3, the computation of PI controllers for interval plant stabilization is given. Concluding remarks are given in section 4.

2. Stabilization Using a PI Controller

Consider the single-input single-output (SISO) control system of Figure 1 where

$$G(s) = \frac{N(s)}{D(s)} \quad (3)$$

is the plant to be controlled and $C(s)$ is a PI controller of the form

$$C(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s} \quad (4)$$

The problem is to compute the parameters of the PI controller of Eq. (4) which stabilize the system of Figure 1.

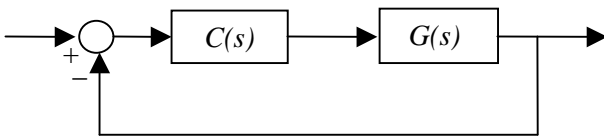


Figure 1: A SISO control system

Decomposing the numerator and the denominator polynomials of Eq. (3) into their even and odd parts, and substituting $s = j\omega$, gives

$$G(j\omega) = \frac{N_e(-\omega^2) + j\omega N_o(-\omega^2)}{D_e(-\omega^2) + j\omega D_o(-\omega^2)} \quad (5)$$

The closed loop characteristic polynomial of the system can be written as

$$\begin{aligned}
\Delta(s) &= [k_i N_e(-\omega^2) - k_p \omega^2 N_o(-\omega^2) - \omega^2 D_o(-\omega^2)] \\
&+ j[k_p \omega N_e(-\omega^2) + k_i \omega N_o(-\omega^2) + \omega D_e(-\omega^2)] = 0
\end{aligned} \quad (6)$$

Then, equating the real and imaginary parts of $\Delta(s)$ to zero, one obtains

$$k_p(-\omega^2 N_o(-\omega^2)) + k_i(N_e(-\omega^2)) = \omega^2 D_o(-\omega^2) \quad (7)$$

and

$$k_p(N_e(-\omega^2)) + k_i(N_o(-\omega^2)) = -D_e(-\omega^2) \quad (8)$$

Let

$$\begin{aligned}
Q(\omega) &= -\omega^2 N_o(-\omega^2) \\
R(\omega) &= N_e(-\omega^2) \\
S(\omega) &= N_o(-\omega^2) \\
U(\omega) &= N_o(-\omega^2) \\
X(\omega) &= \omega^2 D_o(-\omega^2) \\
Y(\omega) &= -D_e(-\omega^2)
\end{aligned} \quad (9)$$

Then, Eq. (7) and Eq. (8) can be written as

$$\begin{aligned}
k_p Q(\omega) + k_i R(\omega) &= X(\omega) \\
k_p S(\omega) + k_i U(\omega) &= Y(\omega)
\end{aligned} \quad (10)$$

From this equation

$$k_p = \frac{X(\omega)U(\omega) - Y(\omega)R(\omega)}{Q(\omega)U(\omega) - R(\omega)S(\omega)} \quad (11)$$

and

$$k_i = \frac{Y(\omega)Q(\omega) - X(\omega)S(\omega)}{Q(\omega)U(\omega) - R(\omega)S(\omega)} \quad (12)$$

Solving these two equations simultaneously, the stability boundary locus, $l(k_p, k_i, \omega)$, in (k_p, k_i) -plane can be obtained. The stability boundary locus divides the parameter plane $((k_p, k_i)$ -plane) into stable and unstable regions. Choosing a test point within each region, the stable region which contains the values of stabilizing k_p and k_i parameters can be determined.

Example 1: Consider the control system of Figure 1 with transfer function

$$G(s) = \frac{N(s)}{D(s)} = \frac{s^3 + 4s^2 - s + 1}{s^5 + 2s^4 + 32s^3 + 14s^2 - 4s + 50} \quad (13)$$

which has 2 right-half plane poles and 2 right half-plane zeros. From Eq. (11) and Eq. (12)

$$k_p = \frac{-\omega^8 + 23\omega^6 + 94\omega^4 - 210\omega^2 + 50}{-\omega^6 - 18\omega^4 + 7\omega^2 - 1} \quad (14)$$

and

$$k_i = \frac{2\omega^8 - 117\omega^6 - 20\omega^4 - 46\omega^2}{-\omega^6 - 18\omega^4 + 7\omega^2 - 1} \quad (15)$$

The aim is to compute all the stabilizing values of k_p and k_i which make the characteristic polynomial of Eq. (6) Hurwitz stable. For a range of frequency, the stability boundary locus can be easily computed.

For example, for $\omega \in [0.45, 7.8]$, $l(k_p, k_i, \omega)$ is shown in Figure 2. From this figure it can be seen that there are a few regions namely R1, R2, R3, R4 and R5 in which one needs to choose a test point in order to find the stability region. For example, choosing a test point within region R5 such as $k_p = 13$ and $k_i = 25$, it can be calculated that the characteristic polynomial has two right half plane complex roots, therefore, the system is unstable for these values of parameters. Thus, the region R5 is not a stability region. It has been computed that the only stabilizing region is the region denoted by R1. For example, for $k_p = 5$ and $k_i = 20$ within region R1, the characteristic polynomial is

$$\Delta(s) = s^6 + 2s^5 + 37s^4 + 54s^3 + 71s^2 + 35s + 20$$

which is a stable polynomial. The all stabilizing values of k_p and k_i are shown in Figure 3.

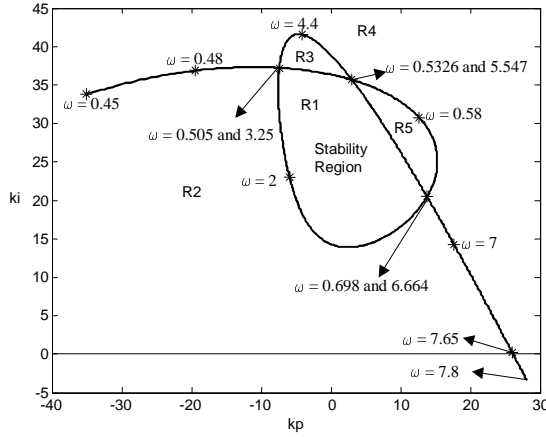


Figure 2: Stability boundary locus

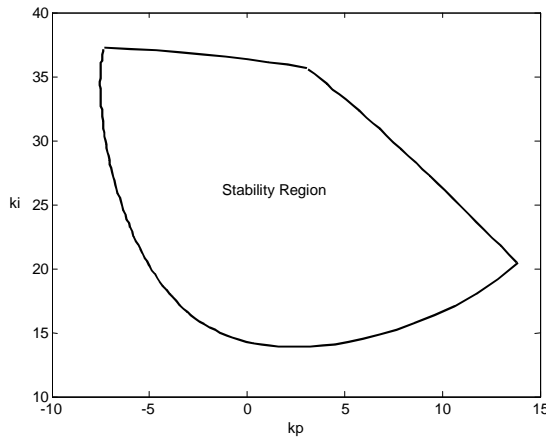


Figure 3: All stabilizing PI controllers

This example show that the method is very fast and effective, however, frequency gridding becomes important. An efficient approach to avoid frequency gridding can be obtained by using the Nyquist plot based approach of [11]. In this case, it is only necessary to find real values of ω that satisfy

$$\text{Im}[G(s)] = 0 \quad (16)$$

where $s = j\omega$. Thus, the frequency axis can be divided into finite number of intervals and then by testing each interval the stability region can be computed. For example, consider a second order system

$$G(s) = \frac{s-1}{s^2 + 0.8s - 0.2} \quad (17)$$

Form Eq. (11) and Eq. (12)

$$k_p = \frac{1.8\omega^2 + 0.2}{-\omega^2 - 1} \quad (18)$$

and

$$k_i = \frac{-\omega^4 + 0.6\omega^2}{-\omega^2 - 1} \quad (19)$$

The stability boundary locus for 100 frequency points within $\omega \in [0, 10]$ is shown in Figure 4. From this figure it is not possible to determine the region of stability. However, for $G(s)$ of Eq. (17), the real frequency values which satisfy Eq. (14) is 0.77 rad/sec. Thus the frequency axis can be divided into two intervals such as $\omega \in (0, 0.77)$ and $\omega \in (0.77, \infty)$. For 100 points within $\omega \in [0, 0.9]$, $l(k_p, k_i, \omega)$ is shown in Figure 5 where it can be seen that there are stabilizing values of k_p and k_i when $\omega \in (0, 0.77)$ as shown in Figure 5. For $G(s)$ of Eq. (13), $\text{Im}[G(j\omega)] = 0$ for $\omega = 7.65$. Thus, one needs to plot stability boundary locus for ω changing between 0 and 7.65. Then, stabilizing region can be computed as shown in example 1.

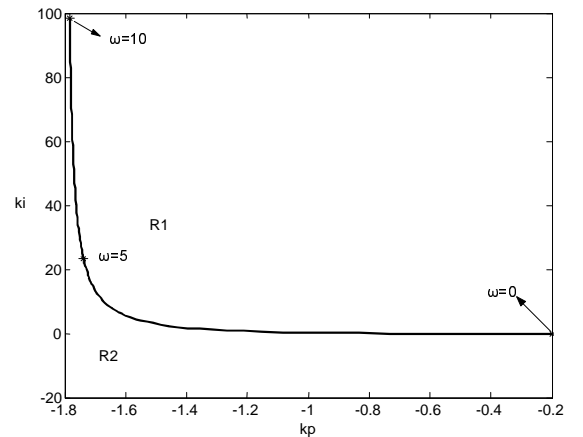


Figure 4: Stability boundary locus

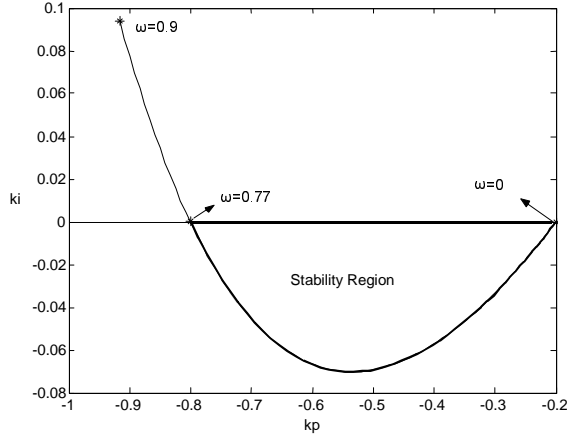


Figure 5: Stability boundary locus

3. Interval Plant Stabilization

There are some important results in the literature about stabilization of interval systems. For example, in [17], it was shown that a constant gain controller stabilizes an interval plant family if and only if it stabilizes a set of eight of the extreme plants. In [18], it was shown that a first order controller stabilizes an interval plant if it stabilizes the set of extreme plants. The best results regarding this subject were given in [19-20] where it was proved that a first order controller stabilizes an interval plant if and only if it simultaneously stabilizes the sixteen Kharitonov plants family. In [21], the generalized version of the Hermite-Biehler theorem has been used for the stabilization of interval systems. In this section, instead of using Routh tables, which were used in [19] in order to characterize all the parameters of a first order controller which stabilize an interval plant, the stability boundary locus is used to find all the values of the parameters of a PI controller for which the given interval plant is Hurwitz stable.

Consider a unity feedback system with a PI controller of Eq. (4) and an interval plant

$$G(s) = \frac{N(s)}{D(s)} = \frac{q_m s^m + q_{m-1} s^{m-1} + \dots + q_0}{p_n s^n + p_{n-1} s^{n-1} + \dots + p_0} \quad (20)$$

where $q_i \in [\underline{q}_i, \overline{q}_i]$, $i=0,1,2,\dots,m$ and $p_i \in [\underline{p}_i, \overline{p}_i]$, $j=0,1,2,\dots,n$. Let the Kharitonov polynomials associated with $N(s)$ and $D(s)$ be respectively:

$$\begin{aligned} N_1(s) &= \underline{q}_0 + \underline{q}_1 s + \underline{q}_2 s^2 + \underline{q}_3 s^3 + \dots \\ N_2(s) &= \underline{q}_0 + \underline{q}_1 s + \underline{q}_2 s^2 + \underline{q}_3 s^3 + \dots \\ N_3(s) &= \underline{q}_0 + \underline{q}_1 s + \underline{q}_2 s^2 + \underline{q}_3 s^3 + \dots \\ N_4(s) &= \underline{q}_0 + \underline{q}_1 s + \underline{q}_2 s^2 + \underline{q}_3 s^3 + \dots \end{aligned} \quad (21)$$

and

$$\begin{aligned} D_1(s) &= \underline{p}_0 + \underline{p}_1 s + \underline{p}_2 s^2 + \underline{p}_3 s^3 + \dots \\ D_2(s) &= \underline{p}_0 + \underline{p}_1 s + \underline{p}_2 s^2 + \underline{p}_3 s^3 + \dots \\ D_3(s) &= \underline{p}_0 + \underline{p}_1 s + \underline{p}_2 s^2 + \underline{p}_3 s^3 + \dots \\ D_4(s) &= \underline{p}_0 + \underline{p}_1 s + \underline{p}_2 s^2 + \underline{p}_3 s^3 + \dots \end{aligned} \quad (22)$$

By taking all combinations of the $N_i(s)$ and $D_j(s)$ for $i, j=1,2,3,4$, the following sixteen Kharitonov plants family can be obtained

$$G_K(s) = G_{ij}(s) = \frac{N_i(s)}{D_j(s)} \quad (23)$$

where $i, j=1,2,3,4$.

Define the set $S(C(s)G(s))$ which contains all the values of the parameters of the controller $C(s)$ which stabilize $G(s)$, then the set of all the stabilizing values of parameters of a PI controller which stabilize the interval plant of Eq. (20) can be written as

$$\begin{aligned} S(C(s)G(s)) &= S(C(s)G_K(s)) = \\ &= S(C(s)G_{11}(s)) \cap S(C(s)G_{12}(s)) \dots S(C(s)G_{44}(s)) \end{aligned} \quad (24)$$

where $G_K(s)$ represents the sixteen Kharitonov plant family which is given in Eq. (23).

Example 2: Consider the control system of Figure 1 with an interval transfer function

$$G(s) = \frac{K}{s^4 + a_2 s^3 + a_1 s^2 + a_0 s} \quad (25)$$

where $K \in [10,30]$, $a_2 \in [85,95]$, $a_1 \in [1900,2000]$ and $a_0 \in [3450,3750]$. The objective is to calculate all the parameters of a PI controller which stabilize $G(s)$. Consider the first Kharitonov plant ($i=1$ and $j=1$) which is

$$G_{11}(s) = \frac{10}{s^4 + 95s^3 + 2000s^2 + 3450s} \quad (26)$$

Since $\text{Im}[G_{11}(j\omega)] = 0$ is only satisfied for $\omega = 6.0263$ rad/sec, it is necessary to obtain stability boundary locus for $\omega \in (0, 6.0263)$. Then, from Eqs. (11) and (12)

$$k_p = -0.1\omega^4 + 200\omega^2 \quad (27)$$

and

$$k_i = -9.5\omega^4 + 345\omega^2 \quad (28)$$

All stabilizing values of k_p and k_i are shown in Figure 6. Figure 7 shows the stability regions of the eight Kharitonov plants (the interval plant of Eq. (25) has eight Kharitonov plants since there are only two Kharitonov polynomials for the numerator) where the intersection of these regions, which can be obtained from the stability region of $G_{21}(s)$ and

$G_{23}(s)$ as shown in Figure 7, is the stability region which is shown in Figure 8.

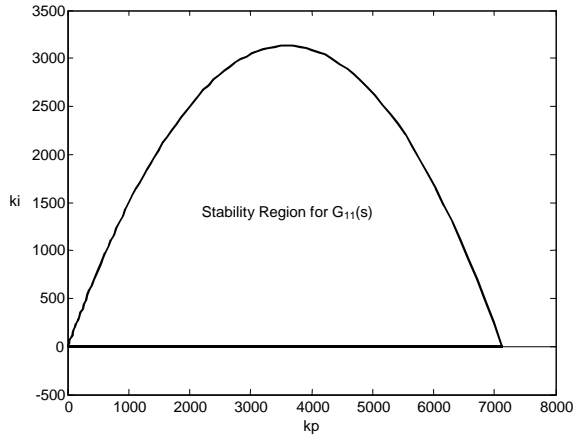


Figure 6: Stability region for $G_{11}(s)$

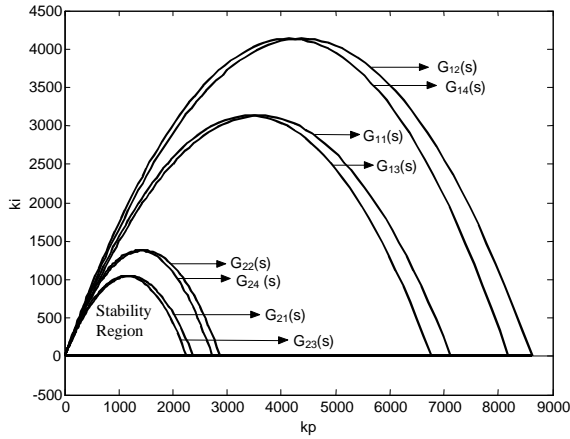


Figure 7: Stability region for eight Kharitonov plants

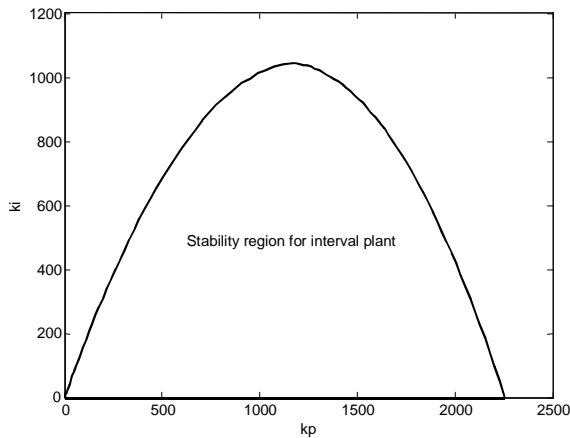


Figure 8: Stability region for the interval plants of Eq. (25)

4. Conclusions

In this paper, a new approach has been presented for the computation of the boundaries of the limiting values of PI controllers parameters that guarantee stability. The approach is based on the stability boundary locus which can be easily obtained by equating the real and the imaginary parts of the characteristic equation to zero. Computation of PI compensator parameters for interval plant stabilization has also been studied. The method presented does not require sweeping over the parameters. Also, it does not need linear programming to solve a set of inequalities. Therefore, the method has advantages over existing results. Given examples clearly show the value of the method presented.

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