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IMPROVING PERFORMANCE OF SINGLE INPUT STABLE LTI SYSTEMS WITH POSITIVE CONTROLS

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Abstract. An easy applicable regulator design for stable Linear Time Invariant (LTI) systems with positive controls is presented. The synthesis algorithm is derived on the basis of Lyapunov stability theory and quadratic performance measures. The synthesis algorithm is given as a convex optimization routine, a set of Linear Matrix Inequality (LMI) conditions together with the maximization of some variables. Performance improvement with respect to the open loop system will be attained, if the loop is closed with the a priori imposed static state feedback control law, $u = \max(0, Fx)$, of which the feedback matrix F has to be determined. Examples are given that illustrate the applicability of the derived synthesis algorithms. More specifically, the examples show that costs, i.e. output energy can be reduced and significant disturbance reduction can be achieved.

Key Words. Piecewise Linear Models, Positive Controls, Stabilization, Performance Analysis.

1. INTRODUCTION

This paper focusses on the regulator problem for single input LTI stable systems with a positivity constraint on the control variable. In particular the control corresponding to the equilibrium position belongs to the boundary of the control set. This makes the control problem more difficult since it is not possible to design controllers that never reach the bound imposed by the constraint. This in contrast to constrained controller designs where the equilibrium point is an interior point of the constrained set.

Control system design that can explicitly deal with a positivity constraint on the input is of great interest. Not in the last place because positivity constraints often arise quite naturally in industrial practice. One could think of mechanical systems subjected to a unilateral force, regulation of chemical processes with one-way valves, population control [6], control of product chains for sustainable production (recycling rates), economic stabilization policy (investments and taxes), to mention a few.

In this paper simple control laws will be derived that fulfil pre-imposed performance requirements, and assure stability of the closed loop system via a quadratic Lyapunov function, for the a priori chosen static state feedback control law $u = \max(0, Fx)$. More specific, output energy can be reduced and significant disturbance reduction can be achieved compared to the open-loop system for a suitably chosen F. Examples illustrate the theoretically derived results.

2. PROBLEM STATEMENT

Consider the following single-input LTI system

$$\Sigma : \begin{cases} \dot{x} = Ax + bu + B_w w \\ z = Cx + du + D_w w \end{cases} \tag{1}$$

interconnected with a given positive feedback controller, with unknown F

$$C_\Sigma : u = \max(0, Fx) \tag{2}$$

where the state $x \in X \subseteq \mathbb{R}^n$ is assumed to be available for feedback, the input $u \in \mathbb{R}_+$, the to be controlled variable $z \in \mathbb{R}^p$, the disturbance or exogeneous input $w \in \mathbb{R}^m$ that can not be manupulated, and A,b,B_w,C,d,D_w,F matrices of appropriate dimensions. $X \supseteq 0$, is assumed to be a convex set such that $x = 0$ is an equilibrium point of the closed loop system Σ_c, i.e. Σ interconnected with C_Σ. The a priori imposed control law was suggested in [4], where stabilizability of open loop unstable LTI systems is studied. The closed loop system system is well-posed, i.e. the existence of a unique non-sliding solution is assured globally for the closed loop system [5]. The setup of the control configuration is shown in Fig. 1.

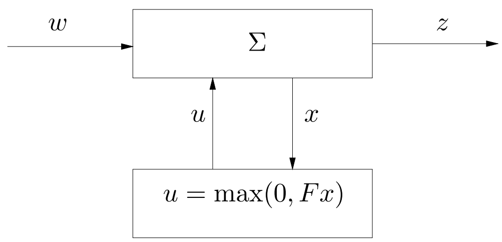

Fig.1. The general control system setup with the a priori imposed positive control law.

The objective is to stabilize the origin and to improve performance of the closed-loop positive control system in comparison to the autonomous system, i.e. (1) with $u = 0$. Performance will be specified on the channel $w \to z$, or performance requirements will be imposed on the controlled variables z.

The closed-loop system, Σ_c can be rewritten as

$$\Sigma_c : \begin{cases} \dot{x} = Ax + B_w w \\ z = Cx + D_w w \end{cases} \text{if} \quad Fx \leq 0 \qquad \begin{cases} \dot{x} = (A + bF)x + B_w w \\ z = (C + dF)x + D_w w \end{cases} \text{if} \quad Fx \geq 0 \tag{3}$$

This leads to the following observations. First of all we deal with a switching system were the switching region, the halfspaces $X_{\gtreqless} = \{x \mid Fx \gtreqless 0\}$, $\gtreqless \in \{\leq, \geq\}$ depend on F, the static state feedback gain, to be determined by the synthesis algorithm. This in contrast with the usual problem formulation from the field of hybrid systems, where it is assumed that the switch regions are fixed [8]. A second difficulty of this control system is that the control $u = 0$, corresponding to the equilibrium position $x = 0$, corresponds to the boundary of the control set $U = \{u \mid u = \max(0, Fx) \geq 0\}$. Therefore in contrast to constrained controller designs where the equilibrium point is an interior point of the constrained set U, i.e. $U = \{u \mid \|u(x)\| \leq 1\}$, we expect that in a neighbourhood of $u = 0$ the constraint $U = \{u \mid u \geq 0\}$ is important [9].

A desirable property of the closed-loop system is, that controller design based on quadratic performance measures on the variables w and z, can be automated and outperforms the open loop system. This tractable objective, together with the fact that the a priori imposed control structure leaves the closed-loop well posed, makes the particular choice for the positive control structure clear.

3. CONTROLLER SYNTHESIS

As a first result, a sufficient condition for positive stabilizability, will be derived. From a control point of view, these conditions may not seem to be very interesting since the system Σ is already assumed to be stable (the control law $u = 0$ stabilizes the system), but they illustrate the transformations that are involved in ariving at LMIs for the quadratic performance problems that are of interest. Furthermore it is a well known fact that switching between stable systems, (3) may produce an unstable system [7]. The positive stabilizability conditions parametrize the family of positive controllers (2) that quadratically stabilize the system (3).

After the derivation of quadratic stabilizability conditions, controller synthesis algorithms will be derived based on performance measures that involve the variables w and z. An algorithm will be derived that computes a controller that minimizes the 'energy' of the controlled variable z. Also an algorithm will be proposed that computes a controller, that minimizes (a popular measure of) the influence of the exogeneous input w on the controlled variable z.

As a first step the state-space X will be constrained by $X_a = \{x \mid |Fx| \leq a\}$ where $a \in \mathbb{R}_+$ will be chosen sufficiently large, i.e. such that X_a covers at least the region of interest. Then the halfspaces $X_{\gtreqless}$ change into the slabs $X_{\leq} = \{x \mid -a \leq Fx \leq 0\}$ and $X_{\geq} = \{x \mid 0 \leq Fx \leq a\}$. Furthermore to arrive at the synthesis inequalities it is neccesary to replace these connected slabs by slabs that do not contain the origin, and therefore are not overlapping but their distance is small, $X_{\leq} = \{x \mid -a \leq Fx \leq -\varepsilon\}$ and $X_{\geq} = \{x \mid \varepsilon \leq Fx \leq a\}$ where $\varepsilon > 0$ can be chosen arbitrarily small. These regions can then also in a non conservative way described by ellipsoids, i.e. $X_{\leq} = \{x \mid -a \leq Fx \leq -\varepsilon\} \Leftrightarrow \{x \mid \|\xi Fx + \tau\|_2 \leq 1\}$ and $X_{\geq} = \{x \mid \varepsilon \leq Fx \leq a\} \Leftrightarrow \{x \mid \|\xi Fx - \tau\|_2 \leq 1\}$ with $\xi = 2/(-\varepsilon + a)$ and $\tau = -(\varepsilon + a)/(\varepsilon - a)$. The closed-loop system to study then becomes

$$\Sigma_c : \begin{cases} \dot{x} = Ax + B_w w \\ z = Cx + D_w w \end{cases} \text{if} \quad \|\xi Fx + \tau\|_2 \leq 1 \qquad \begin{cases} \dot{x} = (A + bF)x + B_w w \\ z = (C + dF)x + D_w w \end{cases} \text{if} \quad \|\xi Fx - \tau\|_2 \leq 1 \tag{4}$$

3.1. Quadratic Stabilizability

To analyse quadratic stabilizability, we consider the state equation of (4) without exogeneous input w i.e.,

$$\begin{array}{ll} \dot{x} = Ax & \text{if} \quad \|\xi Fx + \tau\|_2 \leq 1 \\ \dot{x} = (A + bF)x & \text{if} \quad \|\xi Fx - \tau\|_2 \leq 1 \end{array} \tag{5}$$

and investigate stabilizability. This means we investigate whether all trajectories converge to $X_\varepsilon = \{x \mid |Fx| \leq \varepsilon\}$ and finally since we deal with a stable system the control input can always be chosen such that all trajectories converge to zero as $t \to \infty$. This means asymptotic stability of the closed loop system. A sufficient condition for asymptotic stability of the positive control system is therefore a quadratic stabilizability condition, i.e. the existence of a quadratic function

$$V(x) = x^T Px, \quad P = P^T > 0 \tag{6}$$

that decreases along every nonzero trajectory of the closed loop system (5). Since $\frac{d}{dt}V(x) = x^T\{A^TP + PA\}x$ if $\|\xi Fx + \tau\|_2 \leq 1$ and $\frac{d}{dt}V(x) = x^T\{(A+bF)^TP + P(A+bF)\}x$ if $\|\xi Fx - \tau\|_2 \leq 1$ a sufficient condition for stabilizability can be obtained by introducing the $\mathcal{S}$-method, see e.g. [2]. The condition for stability thus requires the existence of a positive definite matrix $P > 0$, parameters $\lambda_1 \leq 0, \lambda_2 \leq 0$, and a feedback matrix F such that

$$\begin{bmatrix} A^TP + PA + \lambda_1\xi^2F^TF & \lambda_1\xi\tau F^T \\ \lambda_1\xi\tau F & -\lambda_1(1-\tau^2) \end{bmatrix} < 0 \tag{7}$$

and

$$\begin{bmatrix} A_{cl}^TP + PA_{cl} + \lambda_2\xi^2F^TF & -\lambda_2\xi\tau F^T \\ -\lambda_2\xi\tau F & -\lambda_2(1-\tau^2) \end{bmatrix} < 0 \tag{8}$$

with $A_{cl} = A + bF$. These conditions are not convex in the free variables, but after applying standard LMI results, i.e. taking the Schur complement of (7,8) followed by a congruence transformation with $Q = P^{-1}$, again followed by a change of variables $FQ = Y$, and once more the Schur complement as in [3], the following equivalent requirements to (7,8) are obtained, namely there should exist a positive definite matrix $Q > 0$, parameters $\mu_1 \leq 0, \mu_2 \leq 0$, and a matrix Y such that

$$\begin{bmatrix} QA^T + AQ & \xi Y^T \\ \xi Y & -\mu_1(1-\tau^2) \end{bmatrix} < 0 \tag{9}$$

and

$$\begin{bmatrix} QA^T + AQ + Y^Tb^T + bY & \xi Y^T \\ \xi Y & -\mu_2(1-\tau^2) \end{bmatrix} < 0 \tag{10}$$

Hence we arrived at stability conditions in terms of LMIs. If a solution to (7,8) exists, then a stabilizing controller can be computed as $u = \max(0, YQ^{-1}x)$ and $V(x) = x^TQ^{-1}x$ is a Lyapunov function for the system. Stability is assured for the regions $\{x \mid V(x) \leq c\} \subseteq X_a$ where $c > 0$ a constant, since the regions $\{x \mid V(x) \leq c\}$ are (controlled) invariant sets for the closed loop system.

For the transformation from (7,8) to (9,10) to be possible it is neccesary that $\tau^2 > 1$. This is achieved by constraining the state-space and choosing slabs that do not contain the origin. Furthermore, we succeeded in arriving at LMI conditions by converting the feedback matrix F in a non conservative way from the switch region into the LMI formulation. Because a quadratic Lyapunov function candidate is used only closed-loop stability of open-loop stable systems can be proved. This can be seen from (9). For this LMI to be negative definite the $(1,1)$ block has to be negative which means that A has to be Hurwitz. Furthermore it can be seen that the derivative condition on the candidate Lyapunov function for the regime where $u = 0$ (9), depends on the control vatiable Y. This is a desired property because the control variable can be chosen in such a way that it influences the closed loop dynamics of the system positively.

Introducing more sophisticated candidate Lyapunov functions, e.g. piecewise quadratic functions, will by our present knowledge not lead to LMI conditions. However more sophisticated candidate Lyapunov functions are needed to prove positive stabilizability for systems that are open-loop unstable. Another possibility is of course to use a different feedback strategy. We think that both approaches could be fruitful in proving positive stabilizability for more general system descriptions but at the cost of more computationally expensive algorithms. The transformation from (7,8) to the finally obtained LMIs (9,10) in the new variables makes it possible to synthesize several controllers based on quadratic performance analysis.

3.2. Minimizing the Output Energy

Given (4) with $B_w = D_w = 0$. A positive controller $u = \max(0, YQ^{-1}x)$, that reduces the output energy $L(x_0, x, u) = \int_{t=0}^{\infty} z(t)^T z(t)dt$, with initial condition $x_0 = x(0)$ contained in a symmetric set around $x_0 = 0$ with respect to the system without controls can be obtained as a solution of the minimization of trace(Z), with Z a slack variable, subject to a positive definite matrix $Q > 0$, parameters $\mu_1 \leq 0, \mu_2 \leq 0$, and a matrix Y such that

$$\begin{bmatrix} QA^T + AQ & \xi Y^T & QC^T \\ \xi Y & -\mu_1(1-\tau^2) & 0 \\ CQ & 0 & -I \end{bmatrix} < 0, \tag{11}$$

$$\left[\begin{array}{ccc} M & \xi Y^T & QC^T + Y^T d^T \\ \xi Y & -\mu_2(1-\tau^2) & 0 \\ CQ + dY & 0 & -I \end{array}\right] < 0 \tag{12}$$

and

$$\left[\begin{array}{cc} Z & I \\ I & Q \end{array}\right] > 0 \tag{13}$$

with $M = QA^T + AQ + Y^T b^T + bY$. Furthermore $V_u(x) = x^T Q^{-1} x$ is a Lyapunov function for the system and $V_u(x_0)$ is the smallest upperbound for $L(x_0, x, u)$ that can be obtained using quadratic Lyapunov functions.

Also a lowerbound for the cost $L(x_0, x, u)$ can be computed by maximizing trace P subject to $P > 0, \lambda_1 \leq 0, \lambda_2 \leq 0$ and

$$\left[\begin{array}{cc} -L + \lambda_1 \xi^2 F^T F & \lambda_1 \xi \tau F^T \\ \lambda_1 \xi \tau F & -\lambda_1(1-\tau^2) \end{array}\right] < 0 \tag{14}$$

and

$$\left[\begin{array}{cc} -L_{cl} + \lambda_2 \xi^2 F^T F & -\lambda_2 \xi \tau F^T \\ -\lambda_2 \xi \tau F & -\lambda_2(1-\tau^2) \end{array}\right] < 0 \tag{15}$$

with $L = A^T P + PA + C^T C$, $L_{cl} = A_{cl}{}^T P + PA_{cl} + C_{cl}^T C_{cl}$, $A_{cl} = A + bF$, $C_{cl} = C + dF$ and with $F = YQ^{-1}$ obtained from the synthesis problem above. Furthermore $V_l(x_0) = x_0^T P x_0$ is the largest lowerbound for the output energy $L(x_0, x, \max(0, YQ^{-1}x))$ that can be obtained using quadratic Lyapunov functions and the suggested feedback. So $V_l(x_0) \leq L(x_0, x, \max(0, YQ^{-1}x)) \leq V_u(x_0)$. Furthermore the cost $L(x_0, x, 0)$ associated with the autonomous system, $V_a(x_0) = x_0^T P x_0$ can be computed exactly with the solution P of $A^T P + PA + C^T C = 0$. The minimal cost $L(x_0, x, K_{LQR}(P)x)$ which is associated with a linear quadratic regulator (LQR), $V_{LQR}(x_0) = x_0^T P x_0$ can be computed with the solution P of the matrix Ricatti equation $A^T P + PA + C^T C - (PB + C^T d)(d^T d)^{-1}(B^T P + d^T C) = 0$. This means that $V_{LQR}(x_0) \leq L(x_0, x, \max(0, YQ^{-1}x)) \leq V_a(x_0)$ also holds. Of course, since we restrict ourselves to positive controls, serious performance degradation can occur for certain initial conditions compared to the optimal LQR design. However we can do at least as good as the autonomous system. Furthermore there are always initial conditions for which we can do better then the autonomous system. By comparing the bounds $V_{LQR}(x_0), V_l(x_0), V_u(x_0), V_a(x_0)$ with each other, following the analysis presented in [1], it is possible to relate performance accuracy, e.g. minimal or maximal costs $L(x_0, x, \max(0, YQ^{-1}x))$, to regions in the state-space without doing simulations.

A slightly modified version of this performance problem for which the cross term $2x^T C^T du$ is omitted in the costfunction is given in the Appendix.

3.3. Disturbance Reduction

The next objective is to compute a feedback that minimizes the influence of the disturbance w on the output z as much as possible. The intention is to improve performance of the closed-loop positive control system with respect to the system without controls. Formally a positive controller, (2), will be computed that reduces γ subject to $\sup_{\|w\|_2 \neq 0} \frac{\|z\|_2}{\|w\|_2} < \gamma$ and (4) with respect to the system without controls. Here $\|y\|_2^2 = \int_0^\infty y(t)^T y(t) dt$, w is assumed to be Lebesgue integrable. The gain $\sup_{\|w\|_2 \neq 0} \frac{\|z\|_2}{\|w\|_2}$ is known as the L_2 induced gain of the system and is in the case of a LTI system equal to the H_∞ norm of its transfer function. The solution to this problem can be obtained by the minimization of γ subject to $Q > 0, \mu_1 \leq 0, \mu_2 \leq 0$, and a matrix Y such that

$$\left[\begin{array}{cccc} QA^T + AQ & * & * & * \\ \xi Y & -\mu_1(1-\tau^2) & * & * \\ CQ & 0 & -I & * \\ B_w^T + D_w^T CQ & 0 & 0 & N \end{array}\right] < 0 \tag{16}$$

and

$$\left[\begin{array}{cccc} M & * & * & * \\ \xi Y & -\mu_2(1-\tau^2) & * & * \\ CQ + dY & 0 & -I & * \\ O & 0 & 0 & N \end{array}\right] < 0 \tag{17}$$

with $M = QA^T + AQ + Y^T b^T + bY$, $N = -I\gamma^2 + D_w^T D_w$ and $O = B_w^T + D_w^T CQ + D_w^T dY$. The $*$ elements follow from the symmetry of the matrices.

4. EXAMPLES

Consider a LTI mechanical oscillator (1) with $A = \left[\begin{array}{cc} 0 & 1 \\ -1 & -0.4 \end{array}\right]$ and $b = [0\ 1]^T$ and suppose that we have to stabilize it applying a force in only one direction, i.e. $u \geq 0$. We suggest the positive controller (2) with feedback matrix F unknown. Furthermore we want to impose performance requirements on the variables w and z. The exogeneous inputs w and the controlled variables z will be specified later as a result of the specific control problem formulation.

The slabs $X_{\leq} = \{x \mid -a \leq Fx \leq -\varepsilon\}$ and $X_{\geq} = \{x \mid \varepsilon \leq Fx \leq a\}$ are parametrized with $a = 100$ and $\varepsilon = 0.0001$. The examples were evaluated in MATLAB with the developed positive control algorithms. Typically the function $[F, Q] = H2\_pos(A, b, C^T C, d^T d, 2C^T d)$ computes the controller parameters that minimizes the output energy. The function $[F, Q, \gamma] =$

$H\inf\_pos(A, b, B_wC, d, D_w)$ computes the controller parameters that maximizes disturbance reduction.

4.1. Minimizing the Output Energy

The objective in this example is to compute a positive controller that minimizes the energy $L(x_0, x, u)$ as defined in the Appendix. with $C^TC = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ and $d^Td = 1$. As a solution to the quadratic performance problem, the minimization of the output energy we obtain: $Q = \begin{bmatrix} 0.0385 & -0.0077 \\ -0.0077 & 0.0415 \end{bmatrix}$ and $F = YQ^{-1} = [-0.2543\ -1.2714]$. In Fig. 2 and Fig. 3 simulation results are shown for the initial condition $x_0 = [0.7\ -0.7]^T$. They illustrate the performance improvement of the positive control system with respect to the autonomous sytem.More specifically, in Fig. 2 the open-loop system response versus the closed-loop system response is depicted. The costs associated with these trajectories are $L(x_0, x, u = 0) = 21$ for the open-loop system, and $L(x_0, x, u = \max(0, Fx)) = 5.64$ for the closed-loop positive control system. Also the corresponding positive control is shown. The costs associated with an optimal LQR design without unilateral control constraint equals $L(x_0, x, u = K_{LQR}x)$ $= 5.47$. In Fig. 3 the open-loop versus closed-loop trajectory in the phase space is shown for the same initial condition. Also the computed switch line $Fx = 0$, is shown in this picture. The region above this line corresponds to $u = 0$. Also, the elliptical level curves from the quadratic Lyapunov (energy) function, i.e., the controlled invariant sets are shown. Energy decreases along every nonzero trajectory of the closed loop system.

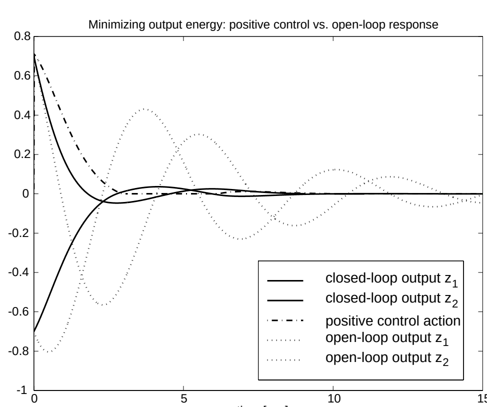

Fig.2. Open-loop system response versus closed-loop system response.

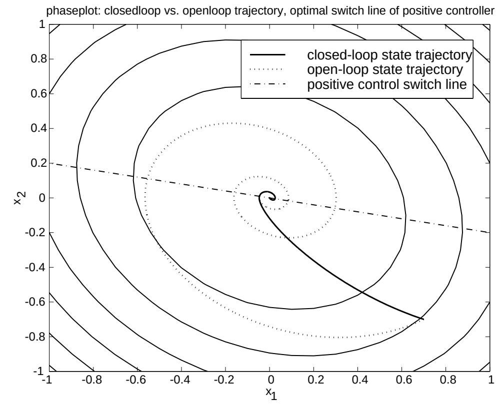

Fig.3. Open-loop versus closed-loop trajectory in the phase space.

4.2. Disturbance Reduction

In this example the objective is to minimize the influence of w on z, i.e. the L_2 induced gain. Therefore we consider again the mechanical oscillator but with disturbance input matrix $B_w = [0\ 1]^T$, and output function $C = [1\ 0]$, $d = 0$, and $D_w = 0$. For the simulation examples the disturbance input is chosen as $w = \sin(w_0t)$. Here the frequency of the disturbance input is chosen $w_0 = 1$ which is equal to the eigen-frequency of the open-loop oscillator system. As a solution to this problem, the minimization of the influence of w on z, we obtain: $Q = \begin{bmatrix} 0.4002 & -0.0802 \\ -0.0802 & 0.4003 \end{bmatrix}$ and $F = YQ^{-1} = [-0.1950\ -0.9734]$. Fig. 4and Fig. 5 illustrate performance improvement of the positive control system with respect to the autonomous sytem. The L_2 induced gain of the open-loop system, which in this case equals the H_∞ norm, can also be obtained from Fig. 4 as the maximum amplitude of the response.

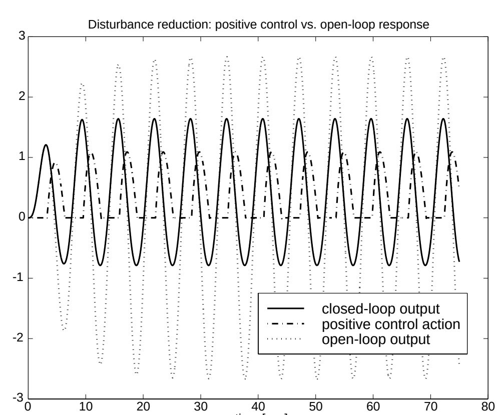

Fig.4. Simulation example showing significant disturbance reduction of the positive control system compared to the autonomous system.

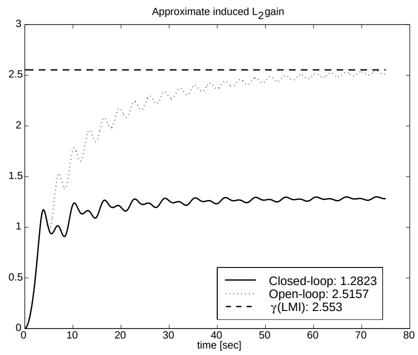

Fig.5. Approximate L_2 induced gain obtained from the simulation example

5. CONCLUSIONS

State feedback synthesis algorithms are derived for single input LTI systems with positive controls. These algorithms are based on Lyapunov stability theory and performance considerations. The examples show that the closed loop system outperforms the open-loop system.

Extensions are possible in several directions, in general performance improvement can be obtained for performance criteria that are based on quadratic functions of the exogenous inputs and controlled variables (outputs). Also the synthesis inequalities for dynamic state feedback, and observers can be derived. This to deal with situations for which the state isn't accessible.

6. REFERENCES

[1] G.Z. Angelis, M.J.G. van de Molengraft, R.J.P. van der Linden, and J.J. Kok. Controller design for polytopic models based on robust performance. In *Proceedings of the ECC, Karlsruhe*, 1999.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.

[3] A. Hassibi and S. Boyd. Quadratic stabilization and control of piecewise-linear systems. In *Proceedings of the American Control Conference*, pages 3659–3664, June 1998.

[4] W.P.M.H. Heemels, S.J.L. van Eijndhoven, and A.A. Stoorvogel. Linear quadratic regulator problem with positive controls. *International Journal of Control*, 1999.

[5] J.-I. Imura and A.J. van der Schaft. Characterization of well-posedness of piecewise linear systems. *Memorandum No. 1475, Faculty of Mathematical Sciences, University of Twente, The Netherlands*, December 1998.

[6] S. Muratori, S. Rinaldi, and B. Trinchera. Performance evaluation of positive regulators for population control. *Modelling, Identification and Control*, 10(3):125–134, 1989.

[7] S. Petterson and B. Lennartson. Stability and robustness for hybrid systems. *Technical Report, Control Engineering Lab, Chalmers University of Technology, Gothenburg, Sweden*, 1996.

[8] S. Petterson and B. Lennartson. An LMI approach for stability analysis of nonlinear systems. In *Proceedings of ECC'97, Bruxelles, Belgium*, July 1-4 1997.

[9] G.V. Smirnov. Stabilization by constrained controls. *SIAM J. Control and Optimization*, 34(5):1616–1649, September 1996.

APPENDIX: Minimizing the output energy without cross-term

Given (4) with $B_w = D_w = 0$. A positive controller $u = \max(0, YQ^{-1}x)$ that reduces the output energy $L(x_0, x, u) = \int_{t=0}^{\infty}(z^T z - 2x^T C^T du)dt$, with initial condition $x_0 = x(0)$ contained in a symmetric set around $x_0 = 0$ with respect to the system without controls can be obtained as the solution of the minimization of trace(Z) subject to a positive definite matrix $Q > 0$, parameters $\mu_1 \leq 0, \mu_2 \leq 0$, and

$$\begin{bmatrix} QA^T + AQ & \xi Y^T & QC^T \\ \xi Y & -\mu_1(1-\tau^2) & 0 \\ CQ & 0 & -I \end{bmatrix} < 0, \quad (18)$$

$$\begin{bmatrix} M & \xi Y^T & QC^T & Y^T d^T \\ \xi Y & -\mu_2(1-\tau^2) & 0 & 0 \\ CQ & 0 & -I & 0 \\ dY & 0 & 0 & -I \end{bmatrix} < 0 \quad (19)$$

and

$$\begin{bmatrix} Z & I \\ I & Q \end{bmatrix} > 0 \quad (20)$$

with $M = QA^T + AQ + Y^T b^T + bY$. Furthermore $V(x) = x^T Q^{-1} x$ is a Lyapunov function for the system and $x_0^T Q^{-1} x_0$ is the smallest upperbound provable via quadratic functions for the energy $L(x_0, x, u)$. Also a lowerbound for the cost $L(x_0, x, u)$ can be computed by maximizing trace P subject to $P > 0, \lambda_1 \leq 0, \lambda_2 \leq 0$ and

$$\begin{bmatrix} -L + \lambda_1 \xi^2 F^T F & \lambda_1 \xi \tau F^T \\ \lambda_1 \xi \tau F & -\lambda_1(1-\tau^2) \end{bmatrix} < 0 \quad (21)$$

and

$$\begin{bmatrix} -L_{cl} + \lambda_2 \xi^2 F^T F & -\lambda_2 \xi \tau F^T \\ -\lambda_2 \xi \tau F & -\lambda_2(1-\tau^2) \end{bmatrix} < 0 \quad (22)$$

with $L = A^T P + PA + C^T C$, $L_{cl} = {A_{cl}}^T P + PA_{cl} + C_{cl}^T C_{cl} - C^T dF - F^T d^T C$ $A_{cl} = A + bF$ and $C_{cl} = C + dF$ and with $F = YQ^{-1}$ obtained from the synthesis problem above. Furthermore $V_l(x_0) = x_0^T P x_0$ is the largest lowerbound provable via quadratic functions for the output energy $L(x_0, x, \max(0, YQ^{-1}x))$.