Proceedings of the 8th IEEE Mediterranean Conference
on Control and Automation (MED 2000)
Rio, Patras, GREECE 17-19 July, 2000

IMPROVED POLYNOMIAL MATRIX DETERMINANT COMPUTATION

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Abstract. An early result on the Smith-MacMillan form of a rational matrix is used for evaluating the degree of the determinant of a polynomial matrix using numerically reliable techniques. This allows for accurate determinant zeroing and determinant interpolation, thus improving existing numerical methods for polynomial matrix determinant computation.

Key Words. Polynomial Matrix, Determinant Computation, Numerical Methods.

1 Motivation

As pointed out in [5], one of the main stumbling block when designing algorithms for numerical computation of the determinant

$$d(s) = d_0 + d_1 s + \cdots + d_\delta s^\delta$$

of a given non-singular $n \times n$ polynomial matrix

$$A(s) = A_0 + A_1 s + \cdots + A_\alpha s^\alpha$$

is the correct evaluation of the degree δ.

Due to unavoidable numerical roundoff errors, it is sometimes difficult to decide which leading coefficients are relevant in the computed determinant

$$\hat{d}(s) = \hat{d}_0 + \hat{d}_1 s + \cdots + \hat{d}_{\hat{\delta}} s^{\hat{\delta}}$$

where $\hat{\delta} \geq \delta$. Typically, some threshold ϵ is selected as a function of n, α and the machine precision. Then, the smallest index i such that $|\hat{d}_i| < \epsilon$ is determined and coefficients $\hat{d}_i, \hat{d}_{i+1}, \ldots, \hat{d}_{\hat{\delta}}$ are artificially set to zero. Such an operation, referred to as zeroing, is a popular technique that proves necessary for preventing unduly degree swelling when performing successive operations on polynomial matrices.

The same kind of problem arises when deriving $\hat{d}(s)$ via interpolation [5] or fast Fourier transform [2], the later being the most powerful and reliable numerical method to date for performing polynomial matrix determinant computation, as illustrated by intensive computational experiments [2]. An estimate is always required on the number of sample points needed for performing interpolation or fast Fourier transform. The sum of column or row degrees of $A(s)$ is usually chosen as a rough upper bound on $\delta + 1$, the correct number of points.

In this note, we solve the determinant degree evaluation problem while keeping with our main impetus, which is the development of reliable numerical algorithms for dealing with polynomial matrices [1] and their implementation in a user-friendly MATLAB package called the Polynomial Toolbox [4].

In this regard, the aim of this note is to point out that the value of d can be evaluated in a numerically stable way, thus allowing for proper determinant zeroing and determinant interpolation. The result that proves instrumental to this computation is not new. It was published some twenty years ago by Van Dooren and co-workers [6]. It was derived when studying the Smith-MacMillan of a rational matrix from its Laurent expansion at its poles and zeros. Our main contribution is in showing that this result can actually be used for improving polynomial matrix determinant computation.

2 Main Result

Theorem 1 *Let*

$$T_i = \begin{bmatrix} A_\alpha & \cdots & A_{-i+1} & A_{-i} \\ & \ddots & & \vdots \\ & & A_\alpha & A_{\alpha-1} \\ 0 & & & A_\alpha \end{bmatrix}$$

denote a Toeplitz matrix built from matrix coefficients of a non-singular $n \times n$ polynomial matrix $A(s)$, where it is assumed that $T_{i-\alpha} = 0$ and $A_i = 0$ when $i < 0$. Let

$$r_i = \operatorname{rank} T_i - \operatorname{rank} T_{i-1} \tag{1}$$

and

$$k = \min\{i : r_i = n, i \geq 0\}.$$

Then the degree δ of the determinant of $A(s)$ is given by

$$\delta = \operatorname{rank} T_k - n(k+1) \tag{2}$$

Proof: It is well-known that any non-singular rational matrix features the same number of poles and zeros, finite or infinite, counting multiplicities [7]. Moreover, degree δ is equal to the number of finite zeros of $A(s)$. Since a polynomial matrix has no finite poles [7], it follows that

$$\delta = p_\infty - z_\infty \tag{3}$$

where p_∞ and z_∞ denote algebraic multiplicities of the pole and zero of $A(s)$ at infinity, respectively. Using the Smith-MacMillan form of $A(s)$ at infinity – see Corollary 3.7 and Remark 4 in Section IV in [6] – one can show that

$$p_\infty = \sum_{i=-\alpha}^{-1} r_i \tag{4}$$

and

$$z_\infty = \sum_{i=0}^{k} (n - r_i). \tag{5}$$

Equation (2) readily follows from relations (1), (3), (4) and (5). ■

It must be underlined that a fast recursive algorithm is described in [6] for performing the successive rank computations and determining index k. The algorithm takes advantage of the special Toeplitz structure of matrices T_i and hinges upon the singular value decomposition, a numerically stable operation. Therefore Theorem 1 provides a numerically reliable method for evaluating the degree of the determinant of a polynomial matrix using coefficients of the matrix only.

3 Illustration

Let

$$A(s) = \begin{bmatrix} 1+2s^2 & 2s^2+s^3 & s^2 \\ -3+s-2s^2 & -1+s-s^2-s^3 & -1+s-s^2 \\ 2 & 2+s & 1 \end{bmatrix}.$$

Build Toeplitz matrices

$$T_{-3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T_{-2} = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 2 & 2 & 1 \\ 0 & -1 & 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\cdots$$

Rank offsets (1) are collected in Table 1. Note that $r_k = n = 3$ for $k = 4$.

i	-3	-2	-1	0	1	2	3	**4**	5	6
r_i	1	1	2	2	2	2	2	**3**	3	3

Table 1: Rank offsets r_i.

Recalling equation (2), the degree of the determinant of $A(s)$ is given by

$$\delta = \operatorname{rank} T_4 - 15 = 0.$$

Matrix $A(s)$ is therefore unimodular and its determinant $d(s)$ can readily be computed by interpolation at only one point, i.e.

$$d(s) = \det A(0) = \det \begin{bmatrix} 1 & 0 & 0 \\ -3 & -1 & -1 \\ 2 & 2 & 1 \end{bmatrix} = 1.$$

4 Conclusion

As pointed out by a reviewer, there is a number of important issues concerning the design of efficient algorithms dealing with Toeplitz matrices, see e.g. the recent textbook [3] and references therein. A detailed study of these algorithms and their application to the problem at hand remain however out of the scope of the present note, but could be an interesting topic for future research.

5 Acknowledgment

This work was supported by the Barrande Project No. 97/005-97/026, by the Grant Agency of the Czech

Republic under contract No. 102/99/1368 and by the Ministry of Education of the Czech Republic under contract No. VS97/034.

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