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A SMALL GAIN THEOREM FOR LOCALLY INPUT TO OUTPUT STABLE INTERCONNECTED SYSTEMS

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Abstract. The results of the present paper extend the nonlinear small-gain theorem to the case of "local input-to-output practical stability". The criterion which must be satisfied by the gain functions can give results for a wide class of systems, since it can be applied to gain functions that cannot be overbounded by any linear function in a neighborhood of zero.

Key Words: nonlinear systems, small-gain theorem, input-to-output practical stability (IOpS)

1. INTRODUCTION

Input-to-state stability (ISS), first introduced by Sontag in [4], combines the idea of *bounded input bounded state stability (BIBS)* together with decay of states under zero input and provides a way of expressing the dependence of the state of nonlinear systems on the magnitude of the input.

The notion of nonlinear gains, introduced by ISS, was the basis for the derivation in [2] and [6] of two nonlinear small-gain theorems for feedback interconnected systems. These small gain theorems referred to ISS and some extensions of ISS, like *input-to-output practical stability* (*IOpS*) ([2]), and have been used for the derivation of stabilization and robust control results ([1],[2],[6]). The sufficient condition involved in these small gain theorems requires that an appropriate composition of the gain functions of the two subsystems must be smaller than the identity function.

In [7], a small-gain theorem for locally ISS systems, that involved a more relaxed condition, was established. More specifically, the aforementioned composition of the gain functions was now required to be smaller than the identity function plus a constant, thus resulting to the local input-to-state practical stability for the interconnection. Hence, the property of asymptotic stability under zero input is lost. The results of [7] are extended in the present paper for the case of local input-to-output practical stability. Thus the small-gain theorem, introduced here, can be applied to systems interconnected through their output vectors. The advantage of the proposed condition, like the corresponding of [7], is that it can give results for a wider class of systems, since it can be satisfied even by gain functions that cannot be overbounded by any linear function in a neighborhood of zero.

1.1 Notations and Facts

- $|x|$ denotes the usual Euclidean norm of the vector $x \in \Re^n$.
- For any measurable function $u:\Re^+ \to \Re^n$, $\|u\|_{t_0}$ denotes the ess-sup $\{|u(t)|, t \ge t_0\}$ and $\|u\|$ denotes the ess-sup $\{|u(t)|, t \ge 0\}$.
- For any measurable function $u:\Re^+ \to \Re^n$, and for any time $0 \le t_1$, u^{t_1} denotes the truncation
$$u^{t_1} = \begin{cases} u(t), & t \in [0, t_1] \\ 0, & \text{otherwise} \end{cases}$$
- Let $f:\Re^+ \to \Re^+, g:\Re^+ \to \Re^+$. Then, $(f \circ g)(s) := f(g(s)), \forall s \ge 0$.
- A function $a:\Re^+ \to \Re^+$ is of class-K, if it is continuous (C^0), vanishing at zero and strictly increasing. By K_∞ we denote the subclass of K

consisting of all functions $a \in K$ with $a(s) \to +\infty$ as $s \to +\infty$.

- A function $\beta: \Re^+ \times \Re^+ \to \Re^+$ is of class- KL, if for each fixed $t \in \Re^+$ the function $\beta(\cdot,t)$ is of class- K and for each fixed $s \in \Re^+$ the function $\beta(s,\cdot)$ is decreasing and $\lim_{t\to\infty} \beta(s,t) = 0$.
- *Weak triangular inequality* [2]: For any function γ of class- K, any $\rho > 0$, $a \geq 0$ and $b \geq 0$ we have:

$$\gamma(a+b) \leq \gamma\big((1+\rho)(a)\big) + \gamma\big((1+\rho^{-1})b\big) \qquad (1)$$

2. STABILITY CONCEPTS

In the sequel, the concept of (ε,δ) *-input-to-output practical stability* ((ε,δ) *-IOpS*) is introduced. This property constitutes a local version of the *input-to-output practical stability* proposed in [2].

Definition 2.1. The system

$$\dot{x} = f(x,u), \; y = h(x,u) \qquad (2)$$

with $x \in \Re^n$, $u \in \Re^m$ and $y \in \Re^k$, is said to be (ε,δ) *-input-to-output practically stable,* if there exist a class- KL function β_y, a class- K function γ_y, called the *gain*, a nonnegative constant d and strictly positive real numbers δ, ε such that, for each initial state $x(0)$, with $|x(0)| \leq \delta$, and for each measurable input function $u(\cdot)$, with $\|u\| \leq \varepsilon$, the state x and the output y of the system exist for each $t \geq 0$ and the output satisfies the condition

$$|y(t)| \leq \beta_y(|x(0)|,t) + \gamma_y(\|u\|) + d, \; \forall t \geq 0 \qquad (3)$$

When $y = x$ in (3), the system is called (ε,δ) *-input-to-state practically stable* ((ε,δ) -(ISpS)). Whenever (3) holds with $d = 0$, input-to-output practical stability coincides with the concept of input-to-output stability.

3. NONLINEAR SMALL-GAIN THEOREM

Consider, now, the interconnected systems

$$\dot{x}_1 = f_1(x_1,y_2,u_1), \; y_1 = h_1(x_1,y_2,u_1) \qquad (4a)$$

$$\dot{x}_2 = f_2(x_2,y_1,u_2), \; y_2 = h_2(x_2,y_1,u_2) \qquad (4b)$$

where $x_i \in \Re^{n_i}$, $u_i \in \Re^{m_i}$, $y_i \in \Re^{k_i}$ and f_i, h_i, $i = 1,2$ are locally Lipschitz in their arguments, with $f_i(0,0,0) = 0$ and $h_i(0,0,0) = 0$. In addition, there exists a unique locally Lipschitz function h, such that the output $\bar{y} = (y_1, y_2)$ of the interconnection (4) can be expressed in the form $\bar{y} = h(x_1,x_2,u_1,u_2)$. The following small-gain theorem is introduced:

Theorem 3.1 Suppose that the systems (4a) and (4b), with inputs (u_1,y_2) and (u_2,y_1), respectively, are (ε_i,δ_i) -ISS and (ε_i,δ_i) -IOS, $i = 1,2$ in the following sense: there exist strictly positive constants $\varepsilon_{x_i}, \hat{\varepsilon}_{x_i}, \delta_{x_i}, \varepsilon_{y_i}, \hat{\varepsilon}_{y_i}, \delta_{y_i}$, $i = 1,2$ such that for all $t \geq 0$

$$\begin{aligned}&|x_1(t)| \leq \beta_{x1}(|x_1(0)|,t) + \gamma_{x1}^y(\|y_2\|) + \gamma_{x1}^u(\|u_1\|),\\ &\text{for } |x_1(0)| \leq \delta_{x1}, \; \|u_1\| \leq \varepsilon_{x1}, \; \|y_2\| \leq \hat{\varepsilon}_{x1}\end{aligned} \qquad (5a)$$

$$\begin{aligned}&|y_1(t)| \leq \beta_{y1}(|x_1(0)|,t) + \gamma_{y1}^y(\|y_2\|) + \gamma_{y1}^u(\|u_1\|),\\ &\text{for } |x_1(0)| \leq \delta_{y1}, \; \|u_1\| \leq \varepsilon_{y1}, \; \|y_2\| \leq \hat{\varepsilon}_{y1}\end{aligned} \qquad (5b)$$

$$\begin{aligned}&|x_2(t)| \leq \beta_{x2}(|x_2(0)|,t) + \gamma_{x2}^y(\|y_1\|) + \gamma_{x2}^u(\|u_2\|),\\ &\text{for } |x_2(0)| \leq \delta_{x2}, \; \|u_2\| \leq \varepsilon_{x2}, \; \|y_1\| \leq \hat{\varepsilon}_{x2}\end{aligned} \qquad (5c)$$

$$\begin{aligned}&|y_2(t)| \leq \beta_{y2}(|x_2(0)|,t) + \gamma_{y2}^y(\|y_1\|) + \gamma_{y2}^u(\|u_2\|),\\ &\text{for } |x_2(0)| \leq \delta_{y2}, \; \|u_2\| \leq \varepsilon_{y2}, \; \|y_1\| \leq \hat{\varepsilon}_{y2}\end{aligned} \qquad (5d)$$

Then, if there exist strictly positive constants λ, τ and ω and nonnegative constants d_1 and d_2, such that the conditions

$$\left.\begin{aligned}(1+\lambda)\gamma_{y1}^y \circ (1+\lambda)\gamma_{y2}^y(s) \leq s + d_1\\ (1+\lambda)\gamma_{y2}^y \circ (1+\lambda)\gamma_{y1}^y(s) \leq s + d_2\end{aligned}\right\}, \quad \forall\, 0 \leq s \leq \omega \qquad (6)$$

$$\frac{d_1 + d_2}{\lambda} < \bar{\delta}_1 := \min\{\omega, \hat{\varepsilon}_{x1}, \hat{\varepsilon}_{x2}, \hat{\varepsilon}_{y1}, \hat{\varepsilon}_{y2}\} \qquad (7)$$

and

$$\begin{aligned}&\gamma_{x1}^y\left((1+\tau)\tfrac{d_2}{\lambda}\right) + \gamma_{x2}^y\left((1+\tau)\tfrac{d_1}{\lambda}\right) < \bar{\delta}_2 := \\ &\min\{\delta_{x1}, \delta_{x2}, \delta_{y1}, \delta_{y2}\}\end{aligned} \qquad (8)$$

are satisfied, then there can be found δ, ε such that the system (4) with state $\bar{x} = (x_1, x_2)$, output $\bar{y} = (y_1, y_2)$ and input $\bar{u} = (u_1, u_2)$ is (ε,δ) -ISpS and (ε,δ) -IOpS. More specifically, let

$$\begin{aligned}\phi_{y1}(s) &:= (1+\lambda^{-1})\left[\beta_{y1}(s,0) + \gamma_{y1}^y\left((1+\lambda^{-1})^2\beta_{y2}(s,0)\right)\right]\\ \phi_{y2}(s) &:= (1+\lambda^{-1})\left[\beta_{y2}(s,0) + \gamma_{y2}^y\left((1+\lambda^{-1})^2\beta_{y1}(s,0)\right)\right]\\ \phi_{\bar{y}}(s) &:= \phi_{y1}(s) + \phi_{y2}(s)\end{aligned} \qquad (9)$$

$$\begin{aligned}\phi_{x1}(s) &:= \beta_{x1}(s,0) + \gamma_{x1}^y \circ 2(1+\tau^{-1})\phi_{y2}(s),\\ \phi_{x2}(s) &:= \beta_{x2}(s,0) + \gamma_{x2}^y \circ 2(1+\tau^{-1})\phi_{y1}(s),\\ \phi_{\bar{x}}(s) &:= \phi_{x1}(s) + \phi_{x2}(s)\end{aligned} \qquad (10)$$

$$r_{y1}(s) := (1+\lambda^{-1})\left[\gamma_{y1}^{u}(s) + \gamma_{y1}^{y} \circ (1+\lambda)(1+\lambda^{-1})\gamma_{y2}^{u}(s)\right]$$
$$r_{y2}(s) := (1+\lambda^{-1})\left[\gamma_{y2}^{u}(s) + \gamma_{y2}^{y} \circ (1+\lambda)(1+\lambda^{-1})\gamma_{y1}^{u}(s)\right]$$
$$r_{\bar{y}}(s) := r_{y1}(s) + r_{y2}(s) \tag{11}$$

$$r_{x1}(s) := \gamma_{x1}^{u}(s) + \gamma_{x1}^{y} \circ 2(1+\tau^{-1}) r_{y2}(s),$$
$$r_{x2}(s) := \gamma_{x2}^{u}(s) + \gamma_{x2}^{y} \circ 2(1+\tau^{-1}) r_{y1}(s), \tag{12}$$
$$r_{\bar{x}}(s) := r_{x1}(s) + r_{x2}(s)$$

$$l_1 := \frac{d_1}{\lambda},\ l_2 := \frac{d_2}{\lambda},\ l := l_1 + l_2 \tag{13a}$$

$$n_1 := \gamma_{x1}^{y}\left((1+\tau) l_2\right), n_2 := \gamma_{x2}^{y}\left((1+\tau) l_1\right),\ n := n_1 + n_2 \tag{13b}$$

Then, for any pair of strictly positive constants (ε, δ) satisfying

$$\varepsilon \le \min\left\{\varepsilon_{x1}, \varepsilon_{x2}, \varepsilon_{y1}, \varepsilon_{y2}\right\} \tag{14a}$$

$$\phi_{\bar{y}}(\delta) + r_{\bar{y}}(\varepsilon) + l < \bar{\delta}_1 \tag{14b}$$

$$\phi_{\bar{x}}(\delta) + r_{\bar{x}}(\varepsilon) + n < \bar{\delta}_2 \tag{14c}$$

and for each function a of class-K_∞, there exist functions $\beta_{\bar{y}}, \beta_{\bar{x}}$ of class-KL, a function $\gamma_{\bar{x}}$ of class-K and a positive constant $\hat{n}$, such that for each initial state $\bar{x}(0)$, with $|\bar{x}(0)| \le \delta$, and each measurable input $\bar{u}$, with $\|\bar{u}\| \le \varepsilon$, the state $\bar{x}$ and the output $\bar{y}$ of the system (4) exist for each $t \ge 0$ and satisfy the conditions

$$|\bar{x}(t)| \le \beta_{\bar{x}}\left(|\bar{x}(0)|, t\right) + \gamma_{\bar{x}}(\|\bar{u}\|) + \hat{n},\ \forall t \ge 0 \tag{15a}$$

$$|\bar{y}(t)| \le \beta_{\bar{y}}\left(|\bar{x}(0)|, t\right) + \gamma_{\bar{y}}(\|\bar{u}\|) + l,\ \forall t \ge 0 \tag{15b}$$

with $\gamma_{\bar{y}}(s) := r_{\bar{y}}(s) + a(s)$.

Proof: The proof follows with a few modifications of some basic steps that have appeared in [2], [3], [5], [6] and [7].

Setting $\|y_2\| = \|u_1\| = \|y_1\| = \|u_2\| = 0$ in (5), it follows, with the use of (9) and (10), that for all δ for which (14b,c) are satisfied, the condition

$$\delta < \bar{\delta}_2 \tag{16}$$

as well as the implication

$$|\bar{x}(0)| \le \delta \Rightarrow |\bar{y}(0)| < \bar{\delta}_1 \tag{17}$$

are also satisfied. Since the functions $f_i, h_i, i = 1{,}2$ in (4) are supposed to be locally Lipschitz, (16) and (17) imply that for each initial condition $\bar{x}(0)$, with $|\bar{x}(0)| \le \delta$, and any measurable function $\bar{u}$, with $\|\bar{u}\| \le \varepsilon$, with (ε, δ) satisfying (14), there exists a strictly positive number T, such that the interval $[0, T)$ is the maximum interval inside which the state and the output of the system (4) are uniquely defined and satisfy the conditions (see also [2], [6])

$$|\bar{x}(t)| < \bar{\delta}_2,\ |\bar{y}(t)| < \bar{\delta}_1,\ \ \forall t \in [0, T) \tag{18}$$

Let $\gamma_{xi} := \gamma_{xi}^{y}$, $\gamma_{yi} := \gamma_{yi}^{y}$, $\sigma_{xi} := \gamma_{xi}^{u}\left(\|u_i^T\|\right)$, $\sigma_{yi} := \gamma_{yi}^{u}\left(\|u_i^T\|\right)$, $i = 1{,}2$. Then, from (5) and the causality of the system (4), it is implied that

$$\left.\begin{aligned} |y_1(t)| &\le \beta_{y1}\left(|x_1(0)|, t\right) + \gamma_{y1}\left(\|y_2^T\|\right) + \sigma_{y1}, \\ |y_2(t)| &\le \beta_{y2}\left(|x_2(0)|, t\right) + \gamma_{y2}\left(\|y_1^T\|\right) + \sigma_{y2} \end{aligned}\right\},\ \forall t \in [0, T) \tag{19}$$

From (19), it follows with the application of (1) and (6) that

$$\begin{aligned} \|y_1^T\| \le\ & \beta_{y1}(|\bar{x}(0)|, 0) + \\ & \gamma_{y1}\left(\left(1+\lambda^{-1}\right)^2 \beta_{y2}(|\bar{x}(0)|, 0)\right) + \\ & \gamma_{y1}\left(\left(1+\lambda^{-1}\right)(1+\lambda)\sigma_{y2}\right) + \\ & + (1+\lambda)^{-1}\|y_1^T\| + (1+\lambda)^{-1} d_1 + \sigma_{y1} \end{aligned} \tag{20}$$

Taking into account (9), (11), (13a) and the definition of σ_{yi}, it is directly implied by (20) that

$$\|y_1^T\| \le \phi_{y1}\left(|\bar{x}(0)|\right) + r_{y1}\left(\|\bar{u}^T\|\right) + l_1 \tag{21}$$

Working in a similar way, the following inequality is deduced

$$\|y_2^T\| \le \phi_{y2}\left(|\bar{x}(0)|\right) + r_{y2}\left(\|\bar{u}^T\|\right) + l_2 \tag{22}$$

Combining (21) and (22) and taking into account (9), (11), (13a) and (14b), it is concluded that, for all (ε, δ) satisfying (14), for all initial conditions $|\bar{x}(0)| \le \delta$ and input functions $\|\bar{u}\| \le \varepsilon$, the following holds:

$$\begin{aligned} \|\bar{y}^T\| &\le \phi_{\bar{y}}\left(|\bar{x}(0)|\right) + r_{\bar{y}}(\|\bar{u}\|) + l \le \\ & \phi_{\bar{y}}(\delta) + r_{\bar{y}}(\varepsilon) + l < \bar{\delta}_1 \end{aligned} \tag{23}$$

Using now (5a), (21) and (22), it is implied that

$$\begin{aligned} |x_1(t)| \le\ & \beta_{x1}\left(|\bar{x}(0)|, t\right) + \\ & \gamma_{x1}\left(\phi_{y_2}(|\bar{x}(0)|) + r_{y_2}\left(\|\bar{u}^T\|\right) + l_2\right) + \sigma_{x1} \end{aligned} \tag{24}$$

Applying (1) it is proved that for all $\tau > 0$

$$\begin{aligned} |x_1(t)| \le\ & \beta_{x1}(|\bar{x}(0)|, 0) + \\ & \gamma_{x1}\left(2(1+\tau^{-1})\phi_{y_2}(|\bar{x}(0)|)\right) + \\ & \gamma_{x1}\left(2(1+\tau^{-1}) r_{y_2}\left(\|\bar{u}^T\|\right)\right) + \\ & \gamma_{x1}\left((1+\tau) l_2\right) + \sigma_{x1},\ \forall t \in [0, T) \end{aligned} \tag{25}$$

Working similarly for x_2, choosing τ in such a way that (8) is satisfied and using the definitions of (10), (12), (13b) and the condition (14c), it follows that

$$\begin{aligned}\left\|\bar{x}^T\right\| &\le \phi_{\bar{x}}\left(\left|\bar{x}(0)\right|\right)+r_{\bar{x}}(\|\bar{u}\|)+n \le \\ &\phi_{\bar{x}}(\delta)+r_{\bar{x}}(\varepsilon)+n<\bar{\delta}_2\end{aligned} \tag{26}$$

Conditions (23) and (26) imply that $\left|\bar{x}(T)\right|<\bar{\delta}_2$ and $\left|\bar{y}(T)\right|<\bar{\delta}_1$. Hence, there exists $T_1>0$, such that the state and the output of the system (4) exist, are uniquely defined and, in addition, $\left|\bar{x}(t)\right|<\bar{\delta}_2$ and $\left|\bar{y}(t)\right|<\bar{\delta}_1$ for all t, such that $T\le t<T+T_1$. Then, it follows by contradiction that condition (18) is satisfied with T being infinite ([2], [6]).

The proof continues by showing first that the system (4) has the following property ([6]):

Property I: For all δ,ε that satisfy (14) and for all strictly positive constants η and m, if the conditions

$$\left|\bar{x}(0)\right|\le\delta,\ \|\bar{u}\|\le\varepsilon \tag{27a}$$

and

$$\max\left\{\phi_{\bar{y}}(\left|\bar{x}(0)\right|)+r_{\bar{y}}(\|\bar{u}\|),\phi_{\bar{x}}(\left|\bar{x}(0)\right|)+r_{\bar{x}}(\|\bar{u}\|)\right\}\le m \tag{27b}$$

are satisfied, then, there exists $\tilde{T}>0$, that depends only on η and m, such that

$$\left|\bar{y}(t)\right|\le\eta+r_{\bar{y}}(\|\bar{u}\|)+l,\ \forall t\ge\tilde{T} \tag{28}$$

Proof of Property I: From (5b), (26) and (27b) it follows that for all $t_0\ge 0$, $t_1\ge 0$

$$\begin{aligned}\left|y_2(t)\right| &\le \beta_{y2}(m+n,t_1)+\gamma_{y2}\left(\left\|y_1\right\|_{t_0}\right)+ \\ &\sigma_{y2},\ \forall t\ge t_1+t_0\end{aligned} \tag{29}$$

$$\begin{aligned}\left|y_1(t)\right| &\le \beta_{y1}(m+n,t_1)+\gamma_{y1}\left(\left\|y_2\right\|_{t_1+t_0}\right)+ \\ &\sigma_{y1},\ \forall t\ge 2t_1+t_0\end{aligned} \tag{30}$$

Combining (29) with (30), it is implied with the use of (1), (6), (11) and (13a) that

$$\begin{aligned}\left|y_1(t)\right| &\le \beta_{y1}(m+n,t_1)+ \\ &\gamma_{y1}\left((1+\lambda^{-1})^2\beta_{y2}(m+n,t_1)\right)+ \\ &(1+\lambda)^{-1}\left\|y_1\right\|_{t_0}+(1+\lambda^{-1})^{-1}r_{y1}(\|\bar{u}\|)+ \\ &(1+\lambda^{-1})^{-1}l_1,\quad \forall t\ge 2t_1+t_0\end{aligned} \tag{31}$$

Choosing t_1 in such a way that

$$\beta_{y1}(m+n,t_1)+\gamma_{y1}\left((1+\lambda^{-1})^2\beta_{y2}(m+n,t_1)\right)\le \frac{\eta}{4(1+\lambda^{-1})}$$

and using (9), (21) and (27b), condition (31) implies that

$$\left|y_1(t)\right|\le\frac{\eta}{4}+(1+\lambda)^{-1}m+r_{y1}(\|\bar{u}\|)+l_1,\forall t\ge 2t_1+t_0$$

In a similar way, it can be inductively shown that for all $k\in\mathbb{N}^*$

$$\begin{aligned}\left|y_1(t)\right| &\le\frac{\eta}{4}+(1+\lambda)^{-k}m+r_{y1}(\|\bar{u}\|)+l_1, \\ &\forall t\ge 2kt_1+t_0\end{aligned} \tag{32}$$

Choosing $k_1\in\mathbb{N}^*$ in such a way that

$$(1+\lambda)^{-k_1}m\le\frac{\eta}{4}$$

condition (32) implies that, for $\tilde{T}_1=2t_1k_1+t_0$, the following condition holds

$$\left|y_1(t)\right|\le\frac{\eta}{2}+r_{y1}(\|\bar{u}\|)+l_1,\ \forall t\ge\tilde{T}_1 \tag{33}$$

In a similar way, it can be shown that there exists $\tilde{T}_2>0$ such that

$$\left|y_2(t)\right|\le\frac{\eta}{2}+r_{y2}(\|\bar{u}\|)+l_2,\ \forall t\ge\tilde{T}_2 \tag{34}$$

From (33) and (34) it follows that (28) is satisfied for $\tilde{T}=\max\{\tilde{T}_1,\tilde{T}_2\}$. Hence, the proof of Property I has been completed.

Using Property I and combining results of [3], [5] and [6], condition (15b) can be proved. More specifically, consider any function a of class-K_∞ and a function $\hat{a}$ of class-K_∞, such that $\phi_{\bar{y}}(\hat{a}(s))\le a(s),\forall s\ge 0$. Then (15b) is satisfied with $\beta_{\bar{y}}(\cdot,\cdot)$ determined by ([3], [6])

$$\beta_{\bar{y}}(s,t):=\phi_{\bar{y}}(s)^{1/2}\left(\int_s^{s+1}\bar{\psi}(\rho,\hat{a}^{-1}(\rho),t)\,d\rho+\frac{s}{(s+1)(t+1)}\right)^{1/2}\quad \forall s\ge 0,\ t\ge 0$$

where

$$\bar{\psi}(s,b,t):=\min\left\{\phi_{\bar{y}}(s),\ \inf\left\{\psi_m(t):m>\max\{\phi_{\bar{y}}(s)+r_{\bar{y}}(b),\phi_{\bar{x}}(s)+r_{\bar{x}}(b)\}\right\}\right\}$$

and

$$\psi_m(t):=\begin{cases}T_m^{-1}(t), & t>0\\ +\infty, & t=0\end{cases},\quad m>0$$

The family of mappings $\{T_m\}_{m>0}:(\Re^+)^*\to(\Re^+)^*$ is defined by the relation ([5], [6])

$$T_m(\eta):=\frac{2}{\eta}\int_{\eta/2}^{\eta}\hat{T}_m(s)\,ds+\frac{m}{\eta}$$

where

$$\hat{T}_m(\eta):=\inf\{\tilde{T}>0:(14),(27)\Rightarrow\ (28)\}$$

Since (15b) is satisfied, condition (15a) is proved with the following steps ([4]). Using (5a), it follows that for all initial conditions $\left|\bar{x}(0)\right| \le \delta$, input functions $\|\bar{u}\| \le \varepsilon$ and for all $t \ge 0$, it holds that

$$\left|x_1(t)\right| \le \beta_{x1}\left(\left|x_1(t/2)\right|, t/2\right) + \gamma_{x1}^{y}\left(\left\|y_2^t\right\|_{t/2}\right) + \gamma_{x1}^{u}\left(\|u_1\|\right) \tag{35}$$

Moreover, condition (15b) implies that

$$\left\|y_2^t\right\|_{t/2} \le \left\|\bar{y}^t\right\|_{t/2} \le \beta_{\bar{y}}\left(\left|\bar{x}(0)\right|, t/2\right) + \gamma_{\bar{y}}\left(\|\bar{u}\|\right) + l \tag{36}$$

From (35) and (36), it follows with the use of (1) that

$$\begin{aligned}\left|x_1(t)\right| \le\ & \beta_{x1}\left(\left|x_1(t/2)\right|, t/2\right) + \\ & \gamma_{x1}^{y}\left(2(1+\tau^{-1})\beta_{\bar{y}}\left(\left|\bar{x}(0)\right|, t/2\right)\right) + \\ & \gamma_{x1}^{y}\left(2(1+\tau^{-1})\gamma_{\bar{y}}\left(\|\bar{u}\|\right)\right) + \\ & \gamma_{x1}^{y}\left((1+\tau)l\right) + \gamma_{x1}^{u}\left(\|\bar{u}\|\right), \quad \forall t \ge 0\end{aligned} \tag{37}$$

Using again (1), (5a) and (15b), it is shown that

$$\begin{aligned}\left|x_1(t/2)\right| \le\ & \beta_{x1}\left(\left|\bar{x}(0)\right|, t/2\right) + \\ & \gamma_{x1}^{y}\left(2(1+\tau^{-1})\beta_{\bar{y}}\left(\left|\bar{x}(0)\right|, 0\right)\right) + \\ & \gamma_{x1}^{y}\left(2(1+\tau^{-1})\gamma_{\bar{y}}\left(\|\bar{u}\|\right)\right) + \\ & \gamma_{x1}^{y}\left((1+\tau)l\right) + \gamma_{x1}^{u}\left(\|\bar{u}\|\right), \quad \forall t \ge 0\end{aligned} \tag{38}$$

Substituting (37) in (38), it follows that for all $\left|\bar{x}(0)\right| \le \delta$ and $\|\bar{u}\| \le \varepsilon$, the following holds

$$\begin{aligned}\left|x_1(t)\right| \le\ & \beta_{x1}\Big(2(1+\tau^{-1})\Big(\beta_{x1}\left(\left|\bar{x}(0)\right|, t/2\right) + \\ & \gamma_{x1}^{y}\left(2(1+\tau^{-1})\beta_{\bar{y}}\left(\left|\bar{x}(0)\right|, 0\right)\right)\Big), t/2\Big) + \\ & \gamma_{x1}^{y}\left(2(1+\tau^{-1})\beta_{\bar{y}}\left(\left|\bar{x}(0)\right|, t/2\right)\right) + \\ & \beta_{x1}\left(2(1+\tau^{-1})\left(\gamma_{x1}^{y}\left(2(1+\tau^{-1})\gamma_{\bar{y}}\left(\|\bar{u}\|\right)\right) + \gamma_{x1}^{u}\left(\|\bar{u}\|\right)\right), 0\right) + \\ & \gamma_{x1}^{y}\left(2(1+\tau^{-1})\gamma_{\bar{y}}\left(\|\bar{u}\|\right)\right) + \gamma_{x1}^{u}\left(\|\bar{u}\|\right) + \\ & \beta_{x1}\left((1+\tau)\gamma_{x1}^{y}\left((1+\tau)l\right), 0\right) + \gamma_{x1}^{y}\left((1+\tau)l\right), \ \forall t \ge 0\end{aligned}$$

Following a similar procedure for the state variable x_2 it is proved that (15a) is satisfied for

$$\begin{aligned}\beta_{\bar{x}}(s,t) :=\ & \\ & \beta_{x1}\left(2(1+\tau^{-1})\left(\beta_{x1}\left(s, \tfrac{t}{2}\right) + \gamma_{x1}^{y}\left(2(1+\tau^{-1})\beta_{\bar{y}}(s,0)\right)\right), \tfrac{t}{2}\right) + \\ & \gamma_{x1}^{y}\left(2(1+\tau^{-1})\beta_{\bar{y}}\left(s, \tfrac{t}{2}\right)\right) + \\ & \beta_{x2}\left(2(1+\tau^{-1})\left(\beta_{x2}\left(s, \tfrac{t}{2}\right) + \gamma_{x2}^{y}\left(2(1+\tau^{-1})\beta_{\bar{y}}(s,0)\right)\right), \tfrac{t}{2}\right) + \\ & \gamma_{x2}^{y}\left(2(1+\tau^{-1})\beta_{\bar{y}}\left(s, \tfrac{t}{2}\right)\right)\end{aligned}$$

$$\begin{aligned}\gamma_{\bar{x}}(s) :=\ & \gamma_{x1}^{y}\left(2(1+\tau^{-1})\gamma_{\bar{y}}(s)\right) + \gamma_{x1}^{u}(s) + \\ & \beta_{x1}\left(2(1+\tau^{-1})\left(\gamma_{x1}^{y}\left(2(1+\tau^{-1})\gamma_{\bar{y}}(s)\right) + \gamma_{x1}^{u}(s)\right), 0\right) + \\ & \gamma_{x2}^{y}\left(2(1+\tau^{-1})\gamma_{\bar{y}}(s)\right) + \gamma_{x2}^{u}(s) \\ & \beta_{x2}\left(2(1+\tau^{-1})\left(\gamma_{x2}^{y}\left(2(1+\tau^{-1})\gamma_{\bar{y}}(s)\right) + \gamma_{x2}^{u}(s)\right), 0\right) +\end{aligned}$$

and

$$\begin{aligned}\hat{n} :=\ & \beta_{x1}\left((1+\tau)\gamma_{x1}^{y}\left((1+\tau)l\right), 0\right) + \gamma_{x1}^{y}\left((1+\tau)l\right) + \\ & \beta_{x2}\left((1+\tau)\gamma_{x2}^{y}\left((1+\tau)l\right), 0\right) + \gamma_{x2}^{y}\left((1+\tau)l\right)\end{aligned}$$

■

Remark 1: The sufficient condition involved in the small-gain theorems established in [2] and [6], for the case of local input-to-output stability, has the form

$$\left.\begin{aligned}(1+\lambda)\gamma_{y1}^{y} \circ (1+\lambda)\gamma_{y2}^{y}(s) \le s \\ (1+\lambda)\gamma_{y2}^{y} \circ (1+\lambda)\gamma_{y1}^{y}(s) \le s\end{aligned}\right\}, \ \forall\, 0 \le s \le \omega$$

Obviously, this condition coincides with (5), when $d_1 = d_2 = 0$. The form of the sufficient condition (5) was first introduced in [7] for the case of local ISpS. The presence of the constant d at the right hand side of (5) is responsible for loosing the asymptotic stability property of the interconnection.

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