DESIGN OF LOW-ORDER CONTROLLERS FOR LINEAR DISCRETE-TIME SYSTEMS WITH NONRANDOM DISTURBANCES

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Abstract

We consider LTI SISO systems with nonrandom disturbances. The problem is to synthesize low-order controllers for optimal disturbance attenuation in such systems. The main idea is to fix a desired closed-loop characteristic polynomial, then a performance index is a convex function of controller coefficients. The case of l_1 and l_∞-bounded disturbances is under consideration. New algorithms for solving the arising linear programming problems are proposed. The example demonstrates the advantages of the new techniques.

Introduction

In the recent literature, a lot of attention is paid for optimal feedback design of controllers in the presence of bounded nonrandom perturbations (see e.g. [1-6] and references therein). However, standard approaches such as l_1-optimization [3,4] result in high-order optimal controllers, which are not well suited for practical applications. Recently new techniques for design of low-order controllers have been proposed [7, 8]; they deal with the new performance index which guarantees uniform boundedness of the output in time domain. In the present paper we provide the new approach to optimal design, which allows to synthesize fixed-order controllers for standard performance indices such as l_1 and l_∞ norms. The main idea is that we fix the closed-loop characteristic polynomial, as it has been proposed by Ya. Tsypkin [1,2] (see also [6]); then a performance index is a convex function of controller parameters.

1. Problem statement

We consider a LTI SISO discrete-time control system described by a difference equation

$$Q(q)y(n) = qP(q)u(n) + S(q)w(n), \qquad (1)$$

where n are time instances, $y(n)$ is an input, $u(n)$ is a control, while $w(n)$ is external (unmeasured) disturbance. Variable q denotes the shift operator, i.e. $q^m u(n) = u(n-m)$. The polynomials $Q(q)$, $P(q)$ and $S(q)$ are assumed to be known and coprime, with $Q(0) = 1$.

The goal is to synthesize a feedback $u(n)$ to minimize a guaranteed norm of the output. The feedback is seeked in the form

$$R(q)u(n) = -T(q)y(n), \qquad (2)$$

where $R(q)$ and $T(q)$ are polynomials to be found.

From equations (1) and (2) we get

$$y(n) = W(q)w(n),$$
$$W(q) = \frac{F(q)}{G(q)} = \frac{S(q)R(q)}{Q(q)R(q) + qP(q)T(q)}, \qquad (3)$$

If we fix the denominator of the transfer function $W(q)$ as $G(q)$:

$$G(q) = Q(q)R(q) + qP(q)T(q), \qquad (4)$$

then we choose the desired dynamical properties of the closed-loop system. For instance, dead-beat control (finite impulse response) corresponds to $G(q) = 1$. In general, equation (4) provides the constraints for the controller coefficients (2).

Now let us specify the class of external disturbances. We assume them to be nonrandom and bounded in some norm. If we suppose it to be uniformly bounded for all time instances, it means that

$$||w(n)||_\infty = \max_n |w(n)| \le 1. \qquad (5)$$

Then it is easy to show that (provided $W(q)$ is stable) the guaranteed steady-state value of the output is bounded by the quantity

$$\max_{w \in \mathcal{B}} ||y(n)||_\infty = ||W(q)||_1,$$

where the unit ball $\mathcal{B}$ is given by (5). We say that a controller is l^1-optimal, if it minimizes $||W(q)||_1$.

Similarly, if the input is bounded in l_1-norm

$$||w(n)||_1 = \sum_n |w(n)| \le 1, \qquad (6)$$

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then the guaranteed steady-state output is bounded by $||W(q)||_\infty$. Minimizing this performance index we get l^∞-optimal controller. The last model arises when we deal with outliers as external disturbances.

2. Optimal design

While the polynomial $G(q)$ is fixed, equation (4) is a Diophantine equaton with respect to polynomials $R(q)$ and $T(q)$. Denote $R^0(q)$ and $T^0(q)$ the minimal order solution of (4), then a general solution is given by the formula

$$\begin{array}{rcl} R(q) & = & R^0(q) - qP(q)X(q), \\ T(q) & = & T^0(q) + Q(q)X(q), \end{array} \qquad (7)$$

where $X(q)$ is an arbitrary polynomial. The order of the controller is defined by the order of $X(q)$:

$$\begin{array}{rcl} \deg R(q) & = & \deg P(q) + \deg X(q) + 1, \\ \deg T(q) & = & \deg Q(q) + \deg X(q). \end{array} \qquad (8)$$

In contrast with the standard l_1-optimization [3–5], we can choose the order of $X(q)$, and, hence, the order of the controller.

A polynomial $X(q)$ provides the parametrization of closed-loop transfer functions:

$$W(q) = V(q) - U(q)X(q), \qquad (9)$$

$$V(q) = \frac{S(q)R^0(q)}{G(q)}, U(q) = \frac{qP(q)X(q)S(q)}{G(q)}.$$

Thus the optimal controller can be found by solving of one of the following optimization problems, see (5–6):

$$\begin{array}{rcl} J_1 = ||V(q) - U(q)X(q)||_1 & \to & \min\limits_X \\ J_\infty = ||V(q) - U(q)X(q)||_\infty & \to & \min\limits_X . \end{array} \qquad (10)$$

The norms $||\ldots||_1$ и $||\ldots||_\infty$ are defined as

$$\begin{array}{rcl} ||W(q)||_1 & = & |h_0| + |h_1| + \ldots, \\ ||W(q)||_\infty & = & \max\limits_n |h_n|,\ n = 0, 1, \ldots, \end{array} \qquad (11)$$

where h_n are found from the series $W(q) = h_0 + h_1 q + \ldots + h_n q^n + \ldots$. The transfer function $W(q)$ is stable, thus the series converge. Notice that $||W(q)||_\infty$ is not H^∞ norm, but the l_∞-norm of the sequence h_n (11).

In general, optimization of J_1 and J_∞ can be reduced to a linear programming problem, but for deadbeat control ($G(q) = 1$), the special structure of optimization problems can be exploited to construct effective iterative methods, see Sections 4, 5. If $G(q) \neq 1$, the series for $h(n)$ can be truncated, thus the same methods can be applied for approximate solution.

For the particular case $G(q) = 1$ and l_1-optimization the similar approach (with no special methods for linear programming problems) has been proposed in [6].

3. l^1-optimization

For deadbeat control ($G(q) = 1$), the transfer functions $V(q)$ and $U(q)$ in (10) degenerate to polynomials:

$$\begin{array}{rcl} V(q) & = & v_0 + v_1 q + \ldots + v_s, \\ U(q) & = & 0 + u_1 q + \ldots + u_l, \\ X(q) & = & x_0 + x_1 q + \ldots + x_m. \end{array} \qquad (12)$$

The orders s, l, m are fixed; we suppose $s \leq l + m$. Rewrite the optimization problem in vector-matrix form:

$$J_1 = |v_0| + ||v - Ux||_1 \longrightarrow \min_x, \qquad (13)$$

where $v \in R^{l+m}$, $x \in R^{m+1}$, and U is $(l+m) \times (m+1)$ matrix of the special form:

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_s \\ 0 \\ \vdots \\ 0 \end{pmatrix}, U = \begin{pmatrix} u_1 & 0 & \cdots & 0 \\ \vdots & u_1 & \cdots & 0 \\ u_l & \vdots & \ddots & \vdots \\ 0 & u_l & & u_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_l \end{pmatrix} \qquad (14)$$

The iterative method for solving (13) constructs the sequences $\lambda^k \in R^1$, $\phi^k \in R^{l+m}$ ($\phi^0 = 0$), $\sigma^k \in R^{l+m}$ and $x^k \in R^{m+1}$, $(k = 0; 1; \cdots)$, as follows:

$$\begin{array}{rcl} x^k & = & \arg\min_x \sum_{i=1}^{l+m} \sigma_i^k \left(v_i - U_i x\right)^2, \\ \sigma_i^k & = & (1 - |\phi_i^k|)^2, \\ \phi_i^{k+1} & = & \phi_i^k - \frac{\sigma_i^k}{\sqrt{\lambda^k}} \left(v_i - U_i x^k\right), \\ \lambda^k & = & \sum_{i=1}^{l+m} \sigma_i^k \left(v_i - U_i x^k\right)^2, \end{array} \qquad (15)$$

here U_i is the i-th row of U.

4. l^∞-optimization

For $G(q) = 1$ we can rewrite the problem of J_∞-optimization (10) in similar form:

$$J_\infty = \max\{|v_0|, ||v - Ux||_\infty\} \longrightarrow \min_x, \qquad (16)$$

where v, x, U are defined in (14). The iterative method generates sequences $\lambda^k \in R^1$, $\gamma^k \in R^1$, $\sigma^k \in R^{l+m}$, $\phi^k \in R^{l+m}$ and $x^k \in R^{m+1}$, $(k = 0; 1; \cdots)$. The initial approximation is

$$\begin{array}{rcl} x^0 & = & \arg\min_x \sum_{i=1}^{l+m} \left(v_i - U_i x\right)^2, \\ \phi^1 & = & \frac{\gamma^1}{\sqrt{\lambda^1}} \left(v_i - U_i x^0\right), \\ \gamma^1 & = & \max_i \left|v_i - U_i x^0\right|, \\ \lambda^1 & = & \sum_{i=1}^{l+m} \left(v_i - U_i x^0\right)^2, \end{array} \qquad (17)$$

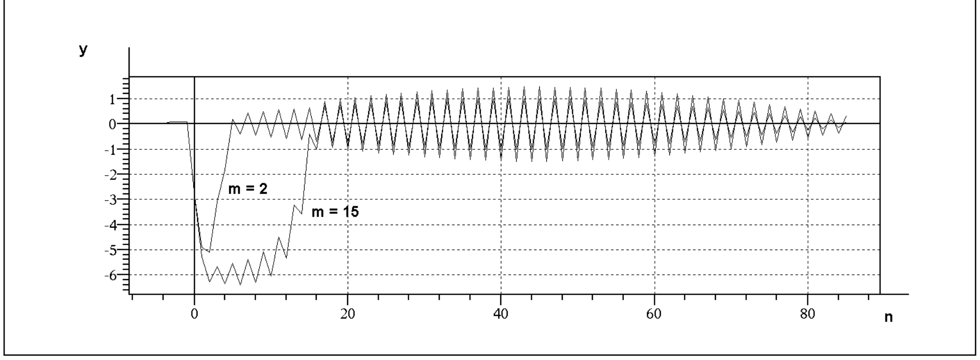

Fig. 1.

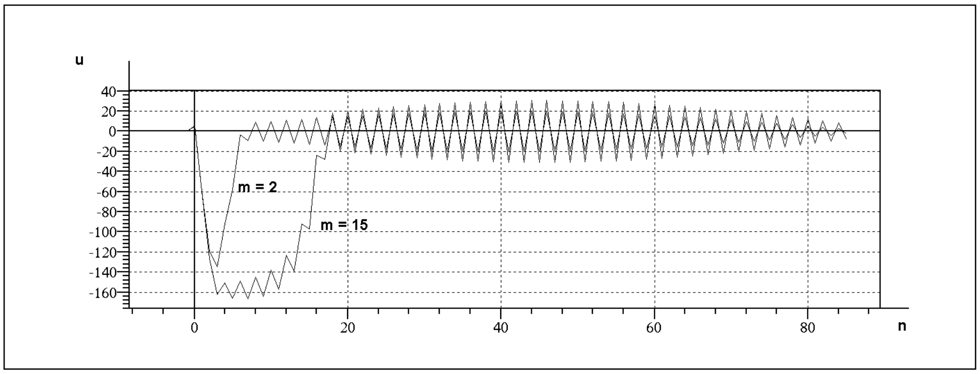

Fig. 2.

Next iterations are defined by the formulae

$$x^k = \arg\min_x \sum_{i=1}^{l+m} (\sigma_i^k)^{-1} \left(v_i - \phi_i^k - U_i x\right)^2, \qquad (18)$$
$$\sigma_i^k = (\gamma^k - |\phi_i^k|)^2,$$

$$\lambda^k = \sum_{i=1}^{l+m} (\sigma_i^k)^{-1} \left(v_i - \phi_i^k - U_i x^k\right)^2, \qquad (19)$$

if $\lambda^k > 1$ then

$$\phi_i^{k+1} = \phi_i^k + \frac{1}{\sqrt{\lambda^k}} \left(v_i - \phi_i^k - U_i x^k\right), \qquad (20)$$

and $\gamma^{k+1} = \gamma^k$, while for $\lambda^k \leq 1$

$$\phi^{k+1} = \frac{\gamma^{k+1}}{\gamma^1}\phi^1,\ \gamma^{k+1} = \max_i \left|v_i - U_i x^k\right|. \qquad (21)$$

5. Example

Consider the example, borrowed from [7], [8]:

$$(1 - 2.7q + 23.5q^2 + 4.6q^3)y(n) =$$
$$u(n-1) + (1 - 2.5q + 1.501q^2)w(n).$$

We require the closed-loop system to be FIR, that is $G(q) = 1$. Minimal order solution of (4) is obvious and equal to

$$R^0(q) = 1,\ T^0 = q^{-1}(1 - Q(q)) = 2.7 - 23.5q - 4.6q^2.$$

The results of l_1-optimization as functions of m — the order of $X(q)$ (7) are presented in the table.

m	X = 0	0	1	3	7	15
J_1^*	5.00	3.80	3.42	3.15	3.03	3.01

Notice that for $m = 15$ ($\deg R = 16$, $\deg T = 18$) the controller is l_1-optimal [8]. The comparison of the results confirms that low-order controller $m = 2$ ($\deg R = 3$, $\deg T = 5$) provides less than 10% loss of the cost function if compared with much more complicated optimal controller of 18th order.

Fig.1 shows time response of the system with low-order controller $m = 2$ and l_1-optimal high-order controller $m = 15$ under disturbance $w(n) \simeq (-1)^n$ and nonzero initial conditions $y(n) = 0.1, n < 0$. We conclude that time responses are very close, while the effect of nonzero initial conditions is more articulated for the high-order controller. The controls $u(n)$ are shown at Fig.2; we observe that high-order control requires higher control efforts.

Next we compare impulse responses $h_1(n)$, $h_\infty(n)$ for control systems, optimizing cost functions J_1 and J_∞ (10) respectively. For $m = 2$ and $G(q) = 1$ the number of impulse responses is 6; their values are given in the table.

n	0	1	2	3	4	5
h_1	1.00	-1.62	0.00	0.00	0.00	0.63
h_∞	1.00	-0.91	-0.91	-0.91	0.83	0.91

In accordance with the theory, l^∞-optimization provides smaller values of maximal amplitudes of $h(n)$, while l_1-optimal system guarantees less "weight" of impulse response. Next table summarizes the optimal values of J_1, J_∞.

	J_1	J_∞
h_1	3.247	1.622
h_∞	5.470	1.000

6. Conclusions

This method has been implemented for various problem formulations, based on available a priori information about disturbances and about desired poles of the system. The software tools allow to design low-order controllers under wide variety of system specifications. Numerous examples demonstrate the effectiveness of the approach.

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