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DEGENERATE HOPF BIFURCATION TO MULTIPLE REGENERATIVE CHATTER

M. S. FOFANA

Manufacturing Engineering, Worcester Polytechnic Institute
Worcester, MA 01609-2280, USA, msfofana@wpi.edu

Abstract. This paper deals with the application of infinite-dimensional delay dynamical systems in the study of multiple regenerative chatter machining. Bifurcation equations describing the dynamics when linearized stability is lost have been derived. Two cases are considered. First, when Hopf's conditions hold, and second when one of them is violated.

Key Words. Regenerative chatter, degenerate bifurcation, delay differential equations.

1. INTRODUCTION

The classical theorem of Hopf Bifurcation for the study of periodic solutions of ordinary differential equations at an equilibrium point has become the standard mathematical tool for reducing higher dimensional nonlinear dynamical systems to lower dimensional systems, while at the same time preserving the emanating dynamics. This is not surprising, since the reduction and the emanating dynamics can be established according to prescribed degenerate assumptions, known as the conditions or the hypotheses of Hopf's theorem. The conditions state that among a known number of eigenvalues associated with the linearized stability of a dynamical system at a particular equilibrium point, there is a pair of complex-conjugate eigenvalues crossing the imaginary axis with nonzero velocity, and while the remaining eigenvalues lie in the left hand side of the complex plane. The description of the dynamics are often achieved by the determination of the zeros of a single scalar bifurcation equation of the form $g(a, \mu) = 0$. The variables a and μ denote, respectively, the amplitude response and the varying bifurcation parameter of the dynamical system under investigation. Sketches of the solutions of $g(a, \mu) = 0$, presented in the form of bifurcation diagrams in the $a - \mu$ plane, qualitatively display the boundaries of stable and unstable dynamics. When anyone of Hopf's conditions is violated, the system will experience degenerate Hopf bifurcation that typically entails distinctive wealth of dynamics [2]. For instance, the existence of additional eigenvalues on the imaginary axis can give rise to degenerate dynamics of dimension > 2. This degeneracy is the simplest form to violate Hopf's conditions, and there is substantial amount of literature dealing with this degeneracy.

In this paper degenerate Hopf bifurcation as applied to orthogonal chip removal process subject to multiple regenerative chatter is considered. Chatter is an instability machining dynamics that violently occurs when the tool is cutting a surface $x(t)$ from a workpiece at time t that is already modulated from the pre-machined surface profile $x(t - h)$ at time $t - h$, where h is the time delay between successive tool cuts. Owing to this instability dynamics - which is the coupled dynamics of the interactions between the cutting process, tool, workpiece and machine-tool structure, irregularities or regenerative chatter marks on both the surface finish and on the tool are produced, and transitively fluctuating cutting forces in a nonlinear delay manner. Equations governing the chatter machining models are typically delay differential equations of the retarded type with a distinct parameter μ representing some prescribed cutting conditions. The analysis will begin by first considering the linearized stability of transcendental characteristic equations for steady state machining as pairs of complex conjugate eigenvalues cross the imaginary axis. Sufficient conditions ensuring stable and unstable machining under Hopf and degenerate bifurcations will be established. Second, the corresponding dynamics when stable machining is lost will be examined by the construction of a generalized center manifold in an infinite-dimensional space for a fixed and multiple time delays. The integral averaging method will be employed to obtain the required bifurcation equations of the form $g(a, \mu) = 0$.

2. HOPF BIFURCATION

They are called retarded delay differential equations (RDDEs), if the delay term appears in the restoring force of the described system. Using the standard notation according to [5], the system considered is described by RDDEs of the form

$$\begin{aligned} \dot{x}(t) &= f(x_t(\theta), \mu, \varepsilon), \quad x(t) \subseteq \Re^n, \\ x_t(\theta) &\subseteq C := C([-h, 0], \Re^n), \end{aligned} \tag{1}$$

with an initial continuous function $\phi(\theta)$ defined on the interval $[-h, 0]$. By the fundamental definition $x_t(\theta) = x(t+\theta)$, $-h \leq \theta \leq 0$, it is required that $\phi(\theta) = x_t(\theta)$, or equivalently, $\phi(\theta) = x(t+\theta)$, $-h \leq \theta \leq 0$, $t \geq 0$. The element $x_t(\theta)$ is the past history of the future variable $x(t+\theta)$, and a solution trajectory of $x_t(\theta)$ will always coincide with that of $x(t+\theta)$ at time $t = 0$. $f = f(x_t(\theta), \mu) : \Re x C \to \Re^n$ is a nonlinear functional mapping, and it is continuously differentiable with respect to the arguments $x_t(\theta)$, μ. ε is a scaling parameter that takes values between 0 and 1. For equation (1), there is an equilibrium point which is assumed without loss of generality to be the trivial solution $x_t(\theta) = 0$. Furthermore, at this equilibrium, the function f satisfies $f(0, \mu, 0) \equiv \partial f(0, \mu, 0)/\partial x_t(\theta) \equiv 0$, and as the bifurcation parameter μ varies near μ_c, the trivial solution loses stability and undergoes a degenerate bifurcation. In this way, the Fréchet derivative $\dot{f}(0, \mu, 0)$ of f evaluated at the equilibrium point is assumed to have finite pairs of eigenvalues that cross the imaginary axis in the complex plane. Thus, the variational delay differential equation is obtained,

$$\dot{x}(t) = L(\phi(\theta), \mu)\phi(\theta) + \varepsilon \Delta f(\phi(\theta), \mu, \varepsilon), \tag{2}$$

where the linear functional mapping $L = L(x_t(\theta), \mu) : C x \Re \to \Re^n$ is given by $L(\phi(\theta), \mu) \equiv \dot{f}_\phi(0, \mu, 0) = \int_{-h}^{0} [d\eta(\theta, \mu)]\phi(\theta)$ and $\Delta f = \Delta f(x_t(\theta), \mu, \varepsilon)$ is strictly nonlinear. The element $\eta(\theta, \mu) : [-h, 0] \to \Re^n$ is a function of bounded variation in $[-h, 0]$ which is described by $\eta(\theta, \mu) = \{-L((-h), \mu), \, when \; \theta = -h, \, 0, \, when \; -h < \theta < 0$ and $L((0), \mu) \; when \; \theta = 0$. The linearized part of equation (2) has the transcendental characteristic equation $\Delta(\lambda, \mu) := \det\{\lambda I - \int_{-h}^{0} [d\eta(\theta, \mu)] e^{\lambda\theta}\} = 0$, where I is the identity matrix. The eigenvalues of $\Delta(\lambda, \mu) = 0$, which may be real, or occur in complex conjugate pairs, are infinite in number, and vary continuously and uniquely with the bifurcation parameter μ. To this regard, we claim that $\Delta(\lambda, \mu) = 0$ has the two pair of roots $\pm i\omega_1$, $\pm i\omega_2$ and all other eigenvalues of $\Delta(\lambda, \mu) = 0$ with $\mu = \mu_c$ have a negative real parts. Then, it has been shown that there exits the direct sum decomposition of $C = P \oplus Q$ by all the eigenvalues of $\Delta(\lambda, \mu) = 0$, where the subspace $P(\lambda, \mu) \subseteq C$ is the generalized eigenspace corresponding to the roots, and $Q = Q(\lambda, \mu) \subseteq C$ is the infinite-dimensional, complementary subspace associated with the remaining eigenvalues of $\Delta(\lambda, \mu) = 0$. Tangent to the space P, is a parabolic smooth curve that represents a local generalized centre manifold $M_\mu = M_\mu(\lambda, A(\theta, \mu))$ in C, where $A(\theta, \mu)$ is the infinitesimal generator of the semigroup $J(t, \mu)$, $t, \mu \geq 0$ of bounded linear operator, and it is defined by $D(A(\theta, \mu)) = \{\phi(\theta) \subseteq C, \frac{d\phi(\theta)}{d\theta} \subseteq C, \frac{d\phi}{d\theta}(0) = L(\phi(\theta), \mu)\phi(\theta)\}$, $A(\theta, \mu)\phi(\theta) = \frac{d\phi(\theta)}{d\theta}$. The semigroup $J(t, \mu)$ maps the space C into itself, namely $J(t, \mu) : C x \Re \longrightarrow C$, or equivalently, the it maps the past history $x_t(\theta)$ onto the future by the relation $x_t(\phi(\theta), \mu) = J(t, \mu)\phi(\theta)$ with $J(0, \mu) = I$. On the centre manifold, it is known that long term behaviour of the variation of constants-integral equation $x_t(\phi(\theta), \mu) = J(t, \mu)\phi(\theta) + \varepsilon \int_{0}^{t} J((t-\zeta), \mu) X_0(\theta) \Delta f(\phi(\theta), \mu, \varepsilon) d\zeta$ associated with $P \subseteq C$, and which is a solution to the nonlinear delay equation (2), can well be approximated by n-dimensional ordinary differential equations (ODEs). The dimension of n corresponds to the number of eigenvalues of $\Delta(\lambda, \mu) = 0$ associated to P. In this case, they are four, and thus $n = 4$ in C. The function $X_0(\theta)$ is a matrix defined as $X_0(\theta) = [0, 0]^T$, $-h \leq \theta < 0$, $X_0(0) = [0, I]^T$, $\theta = 0$. Since the integral solution $x_t(\phi(\theta), \mu)$ is in C, then there is the unique representation $x_t(\phi(\theta), \mu) = x_t^P(\phi(\theta), \mu) + x_t^Q(\phi(\theta), \mu)$, where $x_t^P(\phi(\theta), \mu)$, $x_t^Q(\phi(\theta), \mu)$ are the projections of this equation onto P, Q, respectively. Similar representations for $\phi(\theta)$, $X_0(\theta) \subseteq C$ are given as follows: $\phi(\theta) = \phi^P(\theta) + \phi^Q(\theta)$ and $X_0(\theta) =$

$X_0^P(\theta) + X_0^Q(\theta)$. Efforts to construct the centre manifold proceed by considering further the linear delay equation $\dot{u}(\hat{t}) = \hat{L}(u_{\hat{t}}(s), \mu)u_{\hat{t}}(s)$, $u(\hat{t})\epsilon\Re^n$, $u_{\hat{t}}(s), \psi(s)\epsilon\hat{C}$, $u_{\hat{t}}(s) = \psi(s)$, $0 \leq s \leq h$ in the Banach space $\hat{C} := \hat{C}([0, h], \Re^n)$. This linear delay equation in $\hat{C}$ is formally adjoint to $\dot{x}(t) = L(\phi(\theta), \mu)\phi(\theta)$ with respect to the bilinear relation $(\psi_j(s), \phi_k(\theta)) = (\psi_j(0), \phi_k(0)) - \int_{-h}^{0}\int_0^{\theta} \psi_j(\zeta - s)[d\eta(\theta, \mu)]\phi_k(\zeta)d\zeta$, $j,\ k = 1, 2, 3, 4, ..n$, where $\phi_k(\theta) \subseteq C$ are the values of the initial function $\phi(\theta) \subseteq C$, while $\psi_j(s) \subseteq \hat{C}$ are the values of the prescribed initial continuous function $\psi(s)$ in $\hat{C}$. $\hat{L} = \hat{L}(\psi(s), \mu)\psi(s) : \hat{C}x\Re^n \rightarrow \Re^n$ is the transpose of L, and for the bounded variational function $\hat{\eta}(s, \mu) : [0, h] \rightarrow \Re^n$, $\hat{L}$ is described as $\hat{L}(\psi(s), \mu) = -\int\limits_{-h}^{0} [d\hat{\eta}(s, \mu)]\psi(s)$. The the infinitesimal generator $\hat{A}(s, \mu)$ in C is given by $D(\hat{A}(s, \mu)) = \{\psi(s) \subseteq \hat{C},\ \frac{d\psi(s)}{ds} \subseteq \hat{C},\ \frac{d\psi}{ds}(0) = \hat{L}(\psi(s), \mu)\psi(s)\}$, $\hat{A}(s, \mu)\psi(s) = \frac{d\psi(s)}{ds}$. It is known that the corresponding infinite eigenvalues of $\dot{u}(\hat{t}) = \hat{L}(\psi(s), \mu)\psi(s)$ are exactly the same as those of $\Delta(\lambda, \mu) = 0$ associated with $\dot{x}(t) = L(\phi(\theta), \mu)\phi(\theta)$. Incidently, there is the four-dimensional local center manifold $\hat{M}_\mu = \hat{M}_\mu(\lambda, \hat{A}(s, \mu))$ in $\hat{C}$ of the generalized eigenspace $\hat{P} = \hat{P}(\lambda, \mu) \subseteq \hat{C}$ spanned by the solutions of $\dot{u}(\hat{t}) = \hat{L}(\psi(s), \mu)\psi(s)$ corresponding to $\pm i\omega_1$, $\pm i\omega_2$. Indeed, solutions of $\dot{x}(t) = L(\phi(\theta), \mu)\phi(\theta) \subseteq C$ and $\dot{u}(\hat{t}) = \hat{L}(\psi(s), \mu)\psi(s) \subseteq \hat{C}$ corresponding to the eigenvalues $\pm i\omega_1$, $\pm i\omega_2$ constitute the required initial functions $\phi(\theta) = \Phi(\theta)b \subseteq C$, $\psi(s) = \Psi(s)\hat{b} \subseteq \hat{C}$, where the values $\Phi(\theta) = [\phi_1(\theta),\ \phi_2(\theta), \ldots\ \ \phi_k(\theta)]$, $\phi_k(\theta) = \chi_k(\theta, \mu)e^{\lambda\theta}$ and $\Psi(s) = [\psi_1(s),\ \psi_2(s),\ .\ \ .\ \psi_j(s)]$, $\psi_j(s) = \hat{\chi}_j(s, \mu)e^{-\lambda s}$ are the bases for $P \subseteq C$, $\hat{P} \subseteq \hat{C}$, respectively, and $b, \hat{b}$ are some constant vectors. The adjoint nature of $\dot{x}(t) = L(\phi(\theta), \mu)\phi(\theta) \subseteq C$ and $\dot{u}(\hat{t}) = \hat{L}(\psi(s), \mu)\psi(s) \subseteq \hat{C}$ yields the identity $(\Psi(s), \Phi(\theta))B \equiv \hat{B}(\Psi(s), \Phi(\theta))$, where $(\Psi(s), \Phi(\theta))$ is an inner product matrix and its elements are the bilinear relation $(\psi_j(s), \phi_k(\theta))$. The matrices $B \subseteq C$ and $\hat{B} \subseteq \hat{C}$ are the real values of $\Delta(\lambda, \mu) = 0$ at criticality. By the identity $A\Phi(\theta) = \Phi(\theta)B$ which comes from the definition of the infinitesimal generator $A(\theta, \mu) \subseteq C$, the matrix B can be obtained. A simple computation will shown that $\Phi(\theta) = \Phi(0)e^{B\theta}, -h \leq \theta \leq 0$, and thus $J(t, \mu)\Phi(\theta) = \Phi(0)e^{B(\theta+t)}$, $t \geq 0$ is a solution operator of the linear part of equation (2). The matrix B is equivalent to $\hat{B} \subseteq \hat{C}$, if and only if the resulting inner product matrix after integration $(\Psi, \Phi)_{nsg} = I$ is the identity matrix. Typically, $(\Psi(s), \Phi(\theta)) \neq I$, and for such a situation, the basis $\Psi(s)$ for $\hat{P} \subseteq \hat{C}$ can be normalized to a new set of functions $\bar{\Psi}(s) \subseteq \hat{C}$. This set is obtained by evaluating $\bar{\Psi}(s) = (\Psi, \Phi)^{-1}_{nsg}\Psi(s) = [\bar{\psi}_1(s),\ \bar{\psi}_2(s), ..\ \bar{\psi}_j(s)]$, and the substitution of the new elements $(\bar{\psi}_j(s), \phi_k(\theta))$, $j,\ k = 1, 2, 3, 4, ..n$ into the bilinear equation will yield $(\bar{\Psi}, \Phi) = I$. With the decomposition of $C = P \oplus Q$ and the characterization of $\Phi(\theta) \subseteq C$, $\Psi(s) \subseteq \hat{C}$ such that $(\Psi(s), \Phi(\theta)) = I$, then the elements on the generalized centre manifold M_μ, tangent to $P \subseteq C$, are described by $M_\mu \equiv P \equiv \{\phi(\theta)\ \epsilon\ C,\ \phi(\theta) = \phi^P(\theta) + \phi^Q(\theta) \mid \phi^P(\theta) = \Phi(\theta)b,\ b := (\Psi(s), \phi^P(\theta))\}$, while the corresponding elements on the complementary space are given by $Q = \{(\phi(\theta)\ \epsilon\ C,\ \phi^Q(\theta) = \phi(\theta) - \phi^P(\theta) \mid (\Psi(s), \phi^Q(\theta)) = 0\}$. Similarly the projection of $X_0(\theta) \subseteq C$ onto P and Q are given by $X_0^P(\theta) = \Phi(\theta)\Psi(s)$ and $X_0^Q(\theta) = X_0(\theta) - \Phi(\theta)\Psi(s)$. Indeed, the solution to equation (2) on the centre manifold is given by $M_\mu = \{x_t(\theta)\ \epsilon\ C \mid x_t^P(\theta) = \Phi(\theta)z(t) + x_t^Q(\theta)$, where $z(t)\ \epsilon\ \Re^4$ and $z(t) = (\Psi(s), \phi(\theta))\}$. $x_t^Q(\theta)$ is the variation of constant integral solution on a local centre manifold tangent to Q. By the exponential estimates of the integral solutions $x_t^P(\theta)$ on $P \subseteq C$ and $x_t^Q(\theta)$ on $Q \subseteq C$, it is known that the integral solution $x_t^Q(\theta)$ on the centre manifold tangent to Q is bounded for all values of $t \geq 0$, while $x_t^P(\theta)$ is locally equivalent to the ODEs corresponding to the integral solution $x_t^P(\theta)$ on $M_\mu \subseteq C$ tangent to P. Thus, on this centre manifold, the equivalent ODEs $\dot{z}(t) = Bz(t) + \varepsilon\Psi(0)\Delta f(\Phi(\theta)z(t), \mu, \varepsilon)$, $-h \leq t < \infty$ can be obtained by differentiating $x_t^P(\theta) = \Phi(\theta)z(t)$, and using the characterizations on P and the solution operator $J(t, \mu)\Phi(\theta) = \Phi(0)e^{B(\theta+t)}$. Illustrations of these ideas in the context of single and multiple regenerative chatter in orthogonal machining are presented in the next two sections.

3. APPLICATION

A machine-tool and workpiece in a chatter situation is often modeled as relative vibratory motion between the tool and workpiece. In this way, single regenerative chatter can be examined. They were the authors [5, 6], who suggested that the feed modulations by the tool and workpiece displacements may vary due to the fluctuating chatter marks on the tool and the workpiece. They showed that there are two or more traces of the cutting tool left on the earlier surface profile being machined, and as a result of this, multiple regenerative chatter can be transparent as the cutting condition continued to vary. To quantify their experimental observations, single delay terms instead of multiple ones were adopted. Conditions for stable and unstable machining were derived for small time delay. In this section, it is assumed that Hopf's condition are not violated. That is, the bifurcation parameter μ varies in such way that the characteristic equation $\Delta(\lambda,\mu)=0$ of the linearized stability has the two roots $\upsilon(\mu)\pm i\omega(\mu)$ satisfying at criticality the conditions $\upsilon(\mu_c)=0$, $\omega(\mu_c)\neq 0$, $\Re e\{d\Delta(\lambda,\mu)/d\mu\}\neq 0$, and all the remaining infinite number are negative real parts. The multiple regenerative chatter based on a single degree of freedom turning model is described by the nonlinear multiple delay differential equations

$$
\begin{aligned}
\dot{x}_1(t) &= x_2 \qquad (3)\\
\dot{x}_2(t) &= -2\delta_0\omega_0\Omega^{-1}x_2-\omega_0^2\Omega^{-2}x_1+\\
&\quad \omega_0^2\Omega^{-2}\mu_1\{x_1-x_1(t-h_1)\}-\\
&\quad \omega_0^2\Omega^{-2}\mu_2\{x_1-x_1(t-h_2)\}-\\
&\quad \varepsilon\omega_0^2\Omega^{-2}\{\sum_{j=2}^{q}\sigma_j\{x_1-x_1(t-h_2)\}^j,
\end{aligned}
$$

where $\mu_1,\ \mu_2$ the cutting force coefficients due to the feed modulations caused by the difference between current regenerative workpiece and tool displacements at times t, $t-h_1$, $t-h_2$, respectively, and $h_1\leq h\leq h_2$, $h_1<h_2$. The physical quantities $\delta_0=c/2\sqrt{mk}$, $\omega_0=\sqrt{k/m}$, m is the mass of the tool, c is the damping coefficient, k, the linear spring constant, and σ_j, $j=2,3,....q$ are the coefficients of the cutting force nonlinearity. Putting $\mu_1=0$ and $q=3$, equations (3) become the variant form of the delay equations according to the authors [4,7,8]. The linear delay equations

$$
\begin{aligned}
\dot{x}_1(t) &= x_2 \qquad (4)\\
\dot{x}_2(t) &= -2\delta_0\omega_0\Omega^{-1}x_2-\omega_0^2\Omega^{-2}x_1+\\
&\quad \omega_0^2\Omega^{-2}\mu_1\{x_1-x_1(t-h_1)\}-\\
&\quad \omega_0^2\Omega^{-2}\mu_2\{x_1-x_1(t-h_2)\},
\end{aligned}
$$

in $C([-h,0],\Re^2)$ and their adjoint form

$$
\begin{aligned}
\dot{u}_1(\hat{t}) &= \omega_0^2\Omega^{-2}(1-\mu_1+\mu_2)u_2+ \qquad (5)\\
&\quad \omega_0^2\Omega^{-2}(\mu_1-\mu_2)u_2(\hat{t}+h)\\
\dot{u}_2(\hat{t}) &= -u_1+2\delta_0\omega_0\Omega^{-1}u_2, \quad h_1\leq h\leq h_2.
\end{aligned}
$$

in $\hat{C}([0,h],\Re^2)$ with respect to the bilinear pairing

$$
\begin{aligned}
(\psi_j(s),\phi_k(\theta)) &= (\psi_j(0),\phi_k(0))- \qquad (6)\\
&\quad \mu_1\omega_0^2\Omega^{-2}\int_{-h_1}^{0}\psi_j(\zeta+h_1)\times\\
&\quad \phi_k(\zeta)d\zeta+\mu_2\omega_0^2\Omega^{-2}\times\\
&\quad \int_{-h_2}^{0}\psi_j(\zeta+h_2)\phi_k(\zeta)d\zeta,\\
j,\ k &= 1,2,\quad \phi_k(\theta)\subseteq C,\ \psi_j(s)\subseteq\hat{C},
\end{aligned}
$$

in $C([-h,0],\Re^2)\times\hat{C}([0,h],\Re^2)$ have eigenvalues that satisfy the characteristic equation

$$
\begin{aligned}
\Delta(\lambda,\mu) &: = \lambda^2+2\delta_0\omega_0\Omega^{-1}\lambda+ \qquad (7)\\
&\quad \omega_0^2\Omega^{-2}-\omega_0^2\Omega^{-2}\mu_1\times\\
&\quad (1-e^{-\lambda h_1})+\omega_0^2\Omega^{-2}\mu_2\times\\
(1-e^{-\lambda h_2}) &= 0.
\end{aligned}
$$

At Hopf bifurcation $\lambda_{1,2}=\pm i\omega$ are solutions to the characteristic equation $\Delta(\lambda,\mu_2)=0$. Thus, the substitution of $\lambda_1=i\omega$ into $\Delta(\lambda,\mu_2)=0$, setting the real and imaginary parts to zero, and solving for Ω, μ_2, will lead to the following expressions $\mu_2=-(\omega_0\sin\omega h_2)^{-1}(2\delta_0\omega\Omega-\mu_1\omega_0\sin\omega h_1)$, $\Omega=\beta_{11}\omega^{-1}+\left\{\beta_{11}^2+\omega_0^2(1-\mu_1\beta_{22})\right\}^{1/2}$, in which the notations $\beta_{11}=(\sin\omega h_2)^{-1}\delta_0\omega_0(1-\cos\omega h_2)$ and $\beta_{22}=(\sin\omega h_2)^{-1}(1-\cos\omega h_2)\sin\omega h_1+(1-\cos\omega h_1)$. For some typical values of the model parameters appearing in these equations, the characterization of stable and unstable machining can be displayed in the parameter plane (Ω,μ_2). When $\mu_1=0$, these equations become the well known expressions (see [8]) for the boundaries of stable and unstable machining with single regenerative chatter. The impact of the multiple regenerative effect (*i.e.*, $\mu_1\neq 0$) on these boundaries can be positive (increased stable machining region), or negative (increased unstable machining region), depending upon in particular, the magnitude of the numerical values of the cutting force coefficient μ_1 and the time delay h_1. Corresponding to the simple roots $\pm i\omega$ of equation (7) are the bases $\Phi(\theta)\subseteq C([-h,0],\Re^2)$, $\Psi(s)\subseteq\hat{C}([0,h],\Re^2)$ for the generalized eigenspaces $P\subseteq C$ and $\hat{P}\subseteq\hat{C}$, where $\Phi(\theta)=[\phi_1(\theta),\ \phi_2(\theta)]$, $\phi_1(\theta)=[\cos\omega\theta,\ \sin\omega\theta]^T$, $\phi_2(\theta)=[-\sin\omega\theta,\ \cos\omega\theta]^T$ and $\Psi(s)=[\psi_1(s),\ \psi_2(s)]$, $\psi_1(s)=[\cos\omega s,\ -\sin\omega s]^T$, $\psi_2(s)=[\sin\omega s,\ \cos\omega s]^T$. The substitution of the elements $(\psi_j(s),\ \phi_k(\theta))$, $j,k=1,2$ of $(\Psi(s),\Phi(\theta))$, into equation (6) yields the nonsingular, scalar matrix $(\Psi,\Phi)_{nsg}=[[\psi_{11},\psi_{21}]^T,[-\psi_{12},\psi_{22}]^T]$, where $\psi_{11}\equiv\psi_{22}\equiv 1-\omega_0^2\Omega^{-2}(\mu_1h_1\cos\omega h_1-\mu_2h_2\cos\omega h_2)$, $\psi_{12}=-\omega_0^2\Omega^{-2}(\mu_1h_1\sin\omega h_1-$

$\mu_2 h_2 \sin\omega h_2)$, and $\psi_{21} = -\psi_{12}$. Thus $\Psi(s) \subseteq \hat{C}$ is normalized to the new basis $\bar{\Psi}(s) = [\ [\bar{\psi}_{11}(s), \bar{\psi}_{21}(s)]^T,\ [\bar{\psi}_{12}(s), \bar{\psi}_{22}(s)]^T\ ]$, where $\bar{\psi}_{11}(s) = (\psi_{11}^2 + \psi_{12}^2)^{-1}(\psi_{22}\cos\omega s + \psi_{12}\sin\omega s)$, $\bar{\psi}_{12}(s) = (\psi_{11}^2 + \psi_{12}^2)^{-1}(\psi_{22}\sin\omega s - \psi_{12}\cos\omega s)$, $\bar{\psi}_{21} = -(\psi_{11}^2 + \psi_{12}^2)^{-1}(\psi_{21}\cos\omega s + \psi_{11}\sin\omega s)$ and $\bar{\psi}_{22}(s) = -(\psi_{11}^2 + \psi_{12}^2)^{-1}(\psi_{21}\sin\omega s - \psi_{11}\cos\omega s)$ are obtained by evaluating the relation $\bar{\Psi}(s) = (\Psi, \Phi)^{-1}_{nsg}\Psi(s)$. Again the substitution of $(\bar{\psi}_j(s), \phi_k(\theta))$, $j,k = 1,2$ of $(\bar{\Psi}(s), \Phi(\theta))$ into equation (6) will lead to the $2x2$ identity matrix $(\bar{\Psi}, \Phi)_{id} = (\psi_{11}^2+\psi_{12}^2)^{-1}\{[[(\psi_{11}^2+\psi_{12}^2), 0]^T, [0, (\psi_{11}^2 + \psi_{12}^2)]^T]\} = I$. Therefore, the transformation $x_t^P(\theta) = \Phi(\theta)z(t)$, $z(t)\epsilon\Re^2$, $z(t) = (\bar{\Psi}(s), \phi^P(\theta))$ will give rise to $x_1 = z_1$, $x_2 = z_2$, when $\theta = 0$, and $x_1(t-h) = z_1\cos\omega h + z_2\sin\omega h$, $x_2(t-h) = -z_1\sin\omega h + z_2\cos\omega h$, *when* $\theta = -h$, where $h_1 \le h \le h_2$. Moreover $B = [[0,\omega]^T, [-\omega, 0]^T]$, and in the generalized centre manifold $M_\mu \subseteq C([-h,0],\Re^2)$, $z(t)$ satisfies the ODEs

$$\begin{aligned}\dot{z}_1(t) &= -\omega z_2 - \varepsilon\bar{\psi}_{12}(0)\omega_0^2\Omega^{-2}\{z_1 - \\ &\qquad (z_1\cos\omega h_2 + z_2\sin\omega h_2)\}\left\{\sum_{j=2}^{3}\sigma_j\{z_1 - \right. \\ &\qquad (z_1\cos\omega h_2 + z_2\sin\omega h_2)\}^{j-1} + \tilde{\mu}_2,\}\end{aligned} \tag{8a}$$

$$\begin{aligned}\dot{z}_2(t) &= \omega z_1 - \varepsilon\bar{\psi}_{22}(0)\omega_0^2\Omega^{-2}\{z_1 - \\ &\qquad (z_1\cos\omega h_2 + z_2\sin\omega h_2)\}\left\{\sum_{j=2}^{3}\sigma_j\{z_1 - \right. \\ &\qquad \left.(z_1\cos\omega h_2 + z_2\sin\omega h_2)\}^{j-1} + \tilde{\mu}_2\right\},\end{aligned} \tag{8b}$$

where $\bar{\psi}_{12}(0) = -(\psi_{11}^2 + \psi_{12}^2)^{-1}\psi_{12}$ and $\bar{\psi}_{22}(0) = (\psi_{11}^2 + \psi_{12}^2)^{-1}\psi_{11}$. To examine the bifurcations and their stability property, equations (8) are converted into amplitude $a = a(a, \varphi, \mu_2)$ and phase $\varphi = \varphi(a, \varphi, \mu_2)$ relations by means of the polar coordinate transformation $z_1 = a\sin\Theta$, $z_2 = -a\cos\Theta$, $\Theta = \omega t + \varphi$. This will yield periodic equations satisfied by a and φ. Then, applying the integral averaging method according to [1] on these equations, and for the sake of presentation simplicity, the resulting averaged equations for the amplitude are given by $\dot{a}(t) = -\varepsilon\omega_0^2\Omega^{-2}a\{\sigma_3\gamma_{113}a^2 + \tilde{\mu}_2\gamma_{111}\}$, where $\gamma_{113} = \frac{3}{4}(\cos\omega h_2 - 1)\{\bar{\psi}_{22}(0)\sin\omega h_2 + \bar{\psi}_{12}(0)(\cos\omega h_2 - 1)\}$ and $\gamma_{111} = -\frac{1}{2}\left\{\bar{\psi}_{22}(0)\sin\omega h_2 + \bar{\psi}_{12}(0)(\cos\omega h_2 - 1)\right\}$. Putting $\dot{a} = 0$ the required bifurcation equations can be derived. Then, the super- and subcritical bifurcations when stable machining is lost can be determined qualitatively by the signs of the coefficients in these equations. Indeed, if $\sigma_3\gamma_{113} < 0$ and $\tilde{\mu}_2\gamma_{111} > 0$, then the behaviour of the chatter response is subcritical bifurcation. For this situation, a branch of unstable periodic solutions will bifurcate from the bifurcation point $\mu_2 = \mu_{2c}$ for values of μ_2 larger than μ_{2c}. On the other hand, when $\sigma_3\gamma_{113} > 0$ and $\tilde{\mu}_2\gamma_{111} > 0$, the behaviour is supercritical bifurcation, and thus a branch of stable periodic solutions bifurcates from $\mu_2 = \mu_{2c}$ for values of μ_2 larger than μ_{2c}.

4. DEGENERATE APPLICATION

Next the degenerate bifurcation when two pairs of simple roots cross the imaginary axis in the complex plane as the bifurcation parameter μ_2 is varied near its critical value μ_{2c} is considered. The turning operation is modeled as two degrees of freedom model, and the governing delay differential equations in $C := C([-h,0],\Re^4)$, are of the form

$$\begin{aligned}&\ddot{x}_1(t) + 2\delta_0\omega_0\Omega^{-1}\dot{x}_1 + \omega_0^2\Omega^{-2}x_1 - \\ &\omega_0^2\Omega^{-2}\mu_1\{x_1 - x_1(t-h_1)\} + \\ &\omega_0^2\Omega^{-2}\mu_2\{x_1 - x_1(t-h_2)\} = \varepsilon\omega_0^2\Omega^{-2}\Delta f_1,\end{aligned} \tag{9a}$$

$$\begin{aligned}&\ddot{x}_2(t) + 2\delta_0\omega_0\Omega^{-1}\dot{x}_2 + \omega_0^2\Omega^{-2}x_2 - \\ &\omega_0^2\Omega^{-2}\mu_1\{x_2 - x_2(t-h_1)\} + \\ &\omega_0^2\Omega^{-2}\mu_2\{x_2 - x_2(t-h_2)\} = \varepsilon\omega_0^2\Omega^{-2}\Delta f_2,\end{aligned} \tag{9b}$$

where x_1, x_2 represent both the tool and workpiece displacements and Δf_1, Δf_2 denote the nonlinear perturbations. Putting $\varepsilon = 0$ in these equations, the linearized delay equations are obtained. With proper characterization of the parameters in these equations, the corresponding characteristic equation has two simple roots $\pm i\omega_1$, $\pm i\omega_2$, and all others have negative real parts. Thus, the elements of the basis $\Phi(\theta) = [\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta)]$, $-h \le \theta \le 0$ for the generalized eigenspace $P \subseteq C$ are $\phi_1(\theta) = [\cos\omega_1\theta, \sin\omega_1\theta, 0, 0]^T$, $\phi_2(\theta) = [-\sin\omega_1\theta, \cos\omega_1\theta, 0, 0]^T$, $\phi_3(\theta) = [0, 0, \cos\omega_2\theta, \sin\omega_2\theta]^T$, $\phi_4(\theta) = [0, 0, -\sin\omega_2\theta, \cos\omega_2\theta]^T$, while those elements of $\Psi(s) = [\psi_1(s), \psi_2(s), \psi_3(s), \psi_4(s)]$, $0 \le s \le h$ for the basis $\hat{P} \subseteq \hat{C} := \hat{C}([0,h],\Re^4)$ are $\psi_1(s) = [\cos\omega_1 s, \sin\omega_1 s, 0, 0]^T$, $\psi_2(s) = [-\sin\omega_1 s, \cos\omega_1 s, 0, 0]^T$, $\psi_3(s) = [0, 0, \cos\omega_2 s, \sin\omega_2 s]^T$, $\psi_4(s) = [0, 0, -\sin\omega_2 s, \cos\omega_2 s]^T$. The elements $(\psi_j(s), \phi_k(\theta))$, $j,k = 1,2,3,4$ of the 4×4 inner product matrix $(\Psi(s), \Phi(\theta))$ are computed similarly as in Section 2. In particular, with $\omega = \omega_1$ and $\omega = \omega_2$, the following scalar values for the nonsingular matrix $(\Psi, \Phi)_{nsg} = [[\psi_{11}, \psi_{21}, 0, 0]^T, [-\psi_{12}, \psi_{22}, 0, 0]^T, [0, 0, \psi_{33}, \psi_{43}]^T, [0, 0, -\psi_{34}, \psi_{44}]^T]$, where $\psi_{11} \equiv \psi_{22} \equiv 1 - \omega_0^2\Omega^{-2}(\mu_1 h_1 \cos\omega_1 h_1 - \mu_2 h_2 \cos\omega_1 h_2)$, $\psi_{12} = -\omega_0^2\Omega^{-2}(\mu_1 h_1 \sin\omega_1 h_1 - \mu_2 h_2 \sin\omega_1 h_2)$, $\psi_{21} = -\psi_{12}$, and $\psi_{33} \equiv \psi_{44} \equiv 1 - \omega_0^2\Omega^{-2}(\mu_1 h_1 \cos\omega_2 h_1 - \mu_2 h_2 \cos\omega_2 h_2)$, $\psi_{34} = -\omega_0^2\Omega^{-2}(\mu_1 h_1 \sin\omega_2 h_1 - \mu_2 h_2 \sin\omega_2 h_2)$, $\psi_{43} = -\psi_{34}$. $\Psi(s) \subseteq \hat{C}$ is normalized to the new basis $\bar{\Psi}(s) = [[\bar{\psi}_{11}(s), \bar{\psi}_{21}(s), 0, 0]^T, [\bar{\psi}_{12}(s), \bar{\psi}_{22}(s), 0, 0]^T, [0, 0, \bar{\psi}_{33}(s), \bar{\psi}_{43}(s)]^T, [0, 0, \bar{\psi}_{34}(s), \bar{\psi}_{44}(s)]^T]$, where $\bar{\psi}_{11}(s) = (\psi_{11}^2 + \psi_{12}^2)^{-1}(\psi_{22}\cos\omega_1 s + \psi_{12}\sin\omega_1 s)$, $\bar{\psi}_{12}(s) = (\psi_{11}^2 + \psi_{12}^2)^{-1}(\psi_{22}\sin\omega_1 s - \psi_{12}\cos\omega_1 s)$, $\bar{\psi}_{21} = -(\psi_{11}^2 + \psi_{12}^2)^{-1}(\psi_{21}\cos\omega_1 s + \psi_{11}\sin\omega_1 s)$ and $\bar{\psi}_{22}(s) = -(\psi_{11}^2 + \psi_{12}^2)^{-1}(\psi_{21}\sin\omega_1 s - \psi_{11}\cos\omega_1 s)$. By replacing ω_1 with ω_2 in these same relations, the remaining elements $\bar{\psi}_{33}(s)$, $\bar{\psi}_{34}(s)$, $\bar{\psi}_{43}(s)$ and $\bar{\psi}_{44}(s)$ can be obtained. Thus, it can be shown that $(\bar{\Psi}(s), \Phi(\theta)) = I$ is the 4×4 identity matrix. Similarly as before, $x_t^P(\theta) = \Phi(\theta)z(t)$, $z(t)\epsilon\Re^4$, $z(t) = (\bar{\Psi}(s), \phi^P(\theta))$ will yield the ODEs on the centre manifold $M_\mu \subseteq C([-h, 0], \Re^4)$, namely

$$\begin{aligned}\dot{z}_1(t) &= -\omega_1 z_2 - \varepsilon\bar{\psi}_{12}(0)\omega_0^2\Omega^{-2}\{\tilde{\mu}_2\{ \\ &\quad z_1 - (z_1\cos\omega_1 h_2 + z_2\sin\omega_1 h_2)\} + \\ &\quad \Delta f(\Phi(\theta)z(t))\},\end{aligned} \tag{10a}$$

$$\begin{aligned}\dot{z}_2(t) &= \omega_1 z_1 - \varepsilon\bar{\psi}_{22}(0)\omega_0^2\Omega^{-2}\{\tilde{\mu}_2\{ \\ &\quad z_1 - (z_1\cos\omega_1 h_2 + z_2\sin\omega_1 h_2)\} + \\ &\quad \Delta f(\Phi(\theta)z(t))\},\end{aligned} \tag{10b}$$

$$\begin{aligned}\dot{z}_3(t) &= -\omega_2 z_4 - \varepsilon\bar{\psi}_{34}(0)\omega_0^2\Omega^{-2}\{\tilde{\mu}_2\{ \\ &\quad z_3 - (z_3\cos\omega_2 h_2 + z_4\sin\omega_2 h_2)\} + \\ &\quad \Delta f(\Phi(\theta)z(t))\},\end{aligned} \tag{10c}$$

$$\begin{aligned}\dot{z}_4(t) &= \omega_2 z_3 - \varepsilon\bar{\psi}_{44}(0)\omega_0^2\Omega^{-2}\{\tilde{\mu}_2\{ \\ &\quad z_3 - (z_3\cos\omega_2 h_2 + z_4\sin\omega_2 h_2)\} + \\ &\quad \Delta f(\Phi(\theta)z(t))\}.\end{aligned} \tag{10d}$$

Again, transforming these equations to amplitude and phase relations using the polar coordinates $z_j = a_j\sin\Theta_j$, $z_j = -a_j\cos\Theta_j$, $\Theta_j = \omega_j t + \varphi_j$, $j = 1, 2, 3, 4$, and then applying the integral averaging method to the resulting equations lead to the averaged equations. Also from the averaged equations, the bifurcation equations describing the dynamics of multiple regenerative chatter on the centre manifold can be derived. Depending upon the nature of the nonlinearity in equations (10), machining can experience qualitative changes ranging from super- and/or subcritical stability to complex dynamics.

In this paper, bifurcation to multiple regenerative chatter has been investigated. Using the infinite-dimensional delay dynamical systems, the equations governing multiple regenerative chatter machining are reduced to finite dimensional ODEs on generalized centre manifolds. Both Hopf and degenerate bifurcations are considered for fixed time delays.

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