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OUTPUT-REFERENCE TRACKING PROBLEM FOR DISCRETE-TIME SYSTEMS WITH INPUT SATURATIONS AND CONSTANT REFERENCES

S. TARBOURIECH, I. QUEINNEC, C. PITTET

L.A.A.S - C.N.R.S,7, Avenue du Colonel Roche, 31077 Toulouse cedex 4, France.
E-mail: tarbour@laas.fr - Fax: +33 (0) 561 33 69 69

Abstract. Nonlinear state feedback controllers are exhibited for locally stabilizing linear discrete-time systems with both saturating actuators and additive disturbances when the output must track a certain reference level. The objective is then to bring the steady-state error due to disturbances to zero by using a saturated controller and a dead-zone function. Thus, we want to determine both a stabilizing controller and a region of the state space over which the stability of the resulting closed-loop system is ensured, when the controls are allowed to saturate.

1 Introduction

For technological, economical or safety reasons, the energy delivered by the actuators to the system is bounded. This can be described by a bound on the control input amplitude. If this limitation is omitted in the control design, some unexpected phenonena can affect the closed-loop system: a degradation of performances, the occurence of parasitic equilibrium points or limit cycles, and in the worst cases the closed-loop system can become unstable. One main improvement in linear theory then consists in taking into account actuators limitations. In the last years, there has been a renewed interest in the problem of local [8], global [14] or semi-global [11] stabilization of systems subject to input saturations.

In addition to the problem of stability induced by actuators saturations, the control must meet some performance requirements, among of them the reference tracking in presence of disturbances. It is well-known that for linear systems the disturbances can be eliminated by using an integral-error feedback, but when the control saturates, a windup phenomenon can occur. Several approaches based on the addition of a dead-zone nonlinearity have been proposed in the continuous-time framework to overcome the windup problem [6], [9], [10]. Unfortunately, they do not provide any systematic and single step methodology to compute both a stabilizing and saturating controller and a related domain of stability for the closed-loop system. Either one need extensive simulations [10], or a two-steps strategy is developed, firstly by designing a linear controller for the nonsaturating plant and then by modifying the controller to take the saturation into account [6], [9], [19].

The objective of this paper is to propose a systematic method to compute a saturating controller and a domain of local stability for discrete-time systems, while addressing the problem of reference traking with additive disturbances on the state. No assumption on the open-loop stability is made. In this sense, the addressed problem is a problem of local stabilization. Solutions, when some assumptions on the open-loop stability are considered, are proposed in a global and semi-global context by [1] and [13]. Here, the objective is expressed in terms of output tracking of a given reference level by bringing the steady-state error to zero. The saturating controller containing nonlinear actions and the local ellipsoidal domain of stability are computed by using some relaxation schemes and a Linear Matrix Inequalities (LMIs) formulation. Moreover, an ellipsoidal domain of admissible disturbances and reference is computed, for which the stability is ensured. This paper is the same vein as the ones developed in [17], [18] for continuous-time systems.

Notation. The transpose of a vector $y(t)$ is denoted by $y(t)'$. The index e indicates that the variable is considered at the equilibrium. Matrix I_n denotes the identity matrix in $\Re^{n\times n}$. For any matrix M, $M_{(i)}$ (resp. $M_{(i,j)}$) denotes its ith row vector (resp. its component of the ith row and jth column). Given any vector $\beta \in \Re^g$, the matrix $\Delta(\beta)$ denotes the diagonal matrix with components $\beta_{(i)}$, for $i = 1, ..., g$. For any matrix $M \in \Re^{n\times n}$, $M > 0$ (resp. $M \geq 0$) means that M is positive definite (resp. semi-definite). For two vectors x, $y \in \Re^n$, the notation $x \geq y$ is component-wise, that is, means that $x_{(i)} \geq y_{(i)}$, $i = 1, ..., n$. $co\{\}$ denotes the convex hull. For any matrix $N \in \Re^{m\times n}$, with $m \leq n$ and $rank(N) = m$, $N^{\#} = N^T(NN^T)^{-1}$ denotes the right pseudo-inverse of N, that is, $NN^{\#} = I_m$. Finally, in the sequel the saturation function $sat_{w_0}(w)$ is generically described by its components: $sat_{w_0}(w_{(i)}) = sign(w_{(i)})\min(|w_{(i)}|, w_{0(i)})$, $i = 1, ..., m$.

2 Problem formulation

Consider the discrete-time system subject to actuator saturation described by

$$\left\{\begin{array}{l} x(k+1) = \tilde{A}x(k) + \tilde{B}sat_{u_0}(u(k)) + \tilde{d} \\ y(k) = \tilde{C}x(k) + \tilde{w} \\ \epsilon(k) = y(k) - r \end{array}\right. \tag{1}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^m$ is the control input, $y \in \Re^l$ is the output vector, $\tilde{d} \in \Re^n$ and $\tilde{w} \in \Re^l$ are vectors of disturbances, $r \in \Re^l$ is the desired reference to follow and $\epsilon \in \Re^l$ is the tracking error. The matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are constant real matrices of appropriate dimensions. Furthermore we assume the following.
A1. $rank(\tilde{B}) = m$ and $rank(\tilde{C}) = l$.
A2. $m \geq l$ [20].
A3. Pairs $(\tilde{A}, \tilde{B})$ and $(\tilde{C}, \tilde{A})$ are controllable and observable. Furthermore, $rank(\begin{bmatrix} I_n - \tilde{A} & -\tilde{B} \\ -\tilde{C} & 0 \end{bmatrix}) = n + l$.

The control objective is to achieve asymptotic output-reference tracking in spite of both actuator saturation and external disturbances. In other words, our aim consists in computing suitable nonlinear state feedback controllers in order to bring the output $y(k)$ at the reference r: that is, in order to obtain $\epsilon(k) \to 0$ as $k \to +\infty$.

Thus introduce an additional state variable $q(k) \in \Re^l$ with an anti-windup term [2]:

$$q(k+1) = q(k) + \epsilon(k) + \Delta(h)(sat_{v_0}(v(k)) - v(k))$$

with $v \in \Re^l$ an additionnal control input and $\Delta(h) \in \Re^{l\times l}$ a diagonal positive matrix. $\Delta(h)$ is the anti-windup gain matrix. As in [10] for the continuous-time case, we introduce the error coordinates representation using the new state vector

$$z = \begin{bmatrix} \epsilon' & x_2' & q' \end{bmatrix}' \in \Re^{n+l} \tag{2}$$

where $x_2 \in \Re^{n-l}$ is defined by $x_2 = M_1 x$, $M_1 \in \Re^{(n-l)\times n}$, M_1 being chosen such that $M_2 = \begin{bmatrix} \tilde{C} \\ M_1 \end{bmatrix} \in \Re^{n\times n}$ is nonsingular. Then the initial system (1) can be written as

$$\begin{array}{l} z(k+1) = Az(k) + B_1 sat_{u_0}(u(k)) \\ \quad +B_2(sat_{v_0}(v(k)) - v(k)) + B_3 d \end{array} \tag{3}$$

with $A = \begin{bmatrix} M_2\tilde{A}M_2^{-1} & 0 \\ E' & I_l \end{bmatrix} = \begin{bmatrix} \bar{A} & 0 \\ E' & I_l \end{bmatrix} \in \Re^{(n+l)\times(n+l)}$, $E = \begin{bmatrix} I_l \\ 0 \end{bmatrix} \in \Re^{n\times l}$, $B_1 = \begin{bmatrix} M_2\tilde{B} \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} \in \Re^{(n+l)\times m}$, $B_2 = \begin{bmatrix} 0 \\ \Delta(h) \end{bmatrix} \in \Re^{(n+l)\times l}$, $B_3 = \begin{bmatrix} M_2 & (I_n - \bar{A})E & -(I_n - \bar{A})E \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{B}_3 \\ 0 \end{bmatrix} \in \Re^{(n+l)\times(n+2l)}$ and $d = \begin{bmatrix} \tilde{d}' & \tilde{w}' & r' \end{bmatrix}' \in \Re^{n+2l}$. Our control objective can be formulated into the following problem.

Problem 1 *Compute two matrices $F_1 \in \Re^{m\times(n+l)}$ and $F_2 \in \Re^{l\times(n+l)}$, a set of initial conditions $\mathcal{X}_0$ and a set $\mathcal{D}_0$ of admissible disturbances d such that $u(k) = F_1 z(k)$ and $v(k) = F_2 z(k)$ locally asymptotically stabilize system (3) for any initial condition in $\mathcal{X}_0$ and any disturbance in $\mathcal{D}_0$, that is, asymptotically stabilize the closed-loop system:*

$$\begin{array}{l} z(k+1) = Az(k) + B_1 sat_{u_0}(F_1 z(k)) \\ \quad +B_2(sat_{v_0}(F_2 z(k)) - F_2 z(k)) + B_3 d \end{array} \tag{4}$$

In the disturbance free case (i.e. $d = 0$, that is, $\tilde{d} = \tilde{w} = r = 0$) stabilizing feedback gains F_1 and F_2 being given, the resulting nonlinear closed-loop system (4) possesses a basin of attraction of the equilibrium point $z_e = 0$ [12], [15]. When $d \neq 0$ one can define equilibrium points for the closed-loop system : $z(k+1) = z(k) = z_e \neq 0$. Thus, the closed-loop system (4) exhibits local behaviors around these equilibrium points whose study may be very hard (if not impossible). An interesting way to overcome this difficulty is therefore to determine a set of admissible initial conditions $\mathcal{X}_0$ from which the stability of system (4) with respect to the wished equilibrium points is guaranteed. Since our control objective consists in particular in satisfying the tracking condition $\epsilon_e = 0$, or equivalently $y(k) \to r$ as $k \to \infty$, the interesting equilibrium points can be defined as follows:

$$z(k+1) = z(k) = z_e = \begin{bmatrix} 0 & x_{2e}' & q_e' \end{bmatrix}' \tag{5}$$

The resolution of Problem 1 consists in being able to characterize the sets $\mathcal{X}_0$ and $\mathcal{D}_0$ such that the following properties hold with respect to system (4):
P1. When $d = 0$ (disturbance free case), $z(k)$ asymptotically converges to the origin for any $z(0) \in \mathcal{X}_0$.
P2. When $d \neq 0$, $z(k)$ converges to z_e, as defined in (5), for any $z(0) \in \mathcal{X}_0$ and $d \in \mathcal{D}_0$.

Remark 1 *When saturations do not occur, that is, when $z(k) \in S(F_1, u_0) \cap S(F_2, v_0)$ defined by*

$$S(F_1, u_0) = \{z \in \Re^{n+l}; -u_0 \le F_1 z \le u_0\} \quad (6)$$
$$S(F_2, v_0) = \{z \in \Re^{n+l}; -v_0 \le F_2 z \le v_0\} \quad (7)$$

the closed-loop system (4) admits the linear model

$$z(k+1) = (A + B_1 F_1) z(k) + B_3 d \qquad (8)$$

Note that we cannot conclude, without additional conditions, that any trajectory initiated in $S(F_1, u_0) \cap S(F_2, v_0)$ is a trajectory of system (8), that is, remains confined in $S(F_1, u_0) \cap S(F_2, v_0)$.

Remark 2 *When the open-loop system is stable, the global stabilization of system (4) can be studied. In this case, the set $\mathcal{X}_0$ in Problem 1 is equal to $\Re^{n+l}$ [5], [7]. Throughout the paper, no assumption on the open-loop stability is done (it can be unstable). In this sense, the problem to be solved is a problem of local stabilization.*

Remark 3 *In the current literature on anti-windup, it is often assumed that the dead-zone nonlinearity directly acts on u [2]. Here we have relaxed this constraint by introducing an additionnal control input v. This brings an additionnal degree of freedom to the control that may be used to improve the performance of the closed-loop system (in terms of convergence rate for example). However, this cannot be directly specified in the synthesis problem, but can be a posteriori analyzed.*

3 Preliminaries

3.1 Existence conditions of the equilibrium set

We set different conditions to obtain for system (4) an equilibrium point z_e in the form (5).

Lemma 1 *Suppose that there exists an equilibrium point z_e for system (4). Then it is defined as in (5) provided that $sat_{v_0}(F_2 z_e) = F_2 z_e$, that is, provided that $z_e \in S(F_2, v_0)$.*

Proof. Consider system (4) at the equilibrium z_e. Then one gets: $z_e = A z_e + B_1 sat_{u_0}(F_1 z_e) + B_2(sat_{v_0}(F_2 z_e) - F_2 z_e) + B_3 d_e$. From (3), this equality can be decomposed as:

$$0 = -(I_n - \bar{A}) \begin{bmatrix} \epsilon_e \\ x_{2e} \end{bmatrix} + \bar{B}_1 sat_{u_0}(F_1 z_e) + \bar{B}_3 d_e$$
$$0 = E' \begin{bmatrix} \epsilon_e \\ x_{2e} \end{bmatrix} + \Delta(h)(sat_{v_0}(F_2 z_e) - F_2 z_e)$$

Hence, one obtains $\epsilon_e = 0$ provided that $sat_{v_0}(F_2 z_e) - F_2 z_e = 0$ (since by definition, $\Delta(h)$ is a positive definite diagonal matrix), which is equivalent to $z_e \in S(F_2, v_0)$. □

The saturation term $sat_{u_0}(F_1 z_e) \in \Re^m$ allows to describe 3^m regions of saturation in $\Re^{n+l}$ [8]. Hence, according to this description for one value d_e, one can consider 3^m points $\begin{bmatrix} 0 & x_{2e}' & q_e' \end{bmatrix}'$. Each of these points could be locally studied in terms of both existence and stability. However, in order to simplify our study, we restrict our attention to the case where $sat_{u_0}(F_1 z_e) = F_1 z_e$.

The two following lemmas, for which the proofs mimic the ones described [17], present the conditions on disturbance d to verify that the desired equilibrium point is in the region of linearity.

Lemma 2 *If $A + B_1 F_1$ is asymptotically stable and d satisfies*

$$-u_0 \le F_1(I_{n+l} - (A + B_1 F_1))^{-1} B_3 d \le u_0 \qquad (9)$$
$$-v_0 \le F_2(I_{n+l} - (A + B_1 F_1))^{-1} B_3 d \le v_0 \qquad (10)$$

then the following properties hold:
1. *the equilibrium point z_e is given by :*

$$z_e = (I_{n+l} - (A + B_1 F_1))^{-1} B_3 d_e \qquad (11)$$

$\forall$ d_e satisfying (9) and (10).
2. *$z_e \in \mathcal{S}(F_1, u_0) \cap \mathcal{S}(F_2, v_0)$, and is the unique equilibrium point for system (4).*

Lemma 3 *The equilibrium point z_e as defined in (5) and (11) belongs to $S(F_1, u_0)$ if $d \in S(\mathcal{F}, u_0)$ which is the polyhedral set defined by:*

$$S(\mathcal{F}, u_0) = \{d \in \Re^{n+2l}; -u_0 \le \mathcal{F} d \le u_0\}$$
$$\text{with } \mathcal{F} = \begin{bmatrix} 0 & I_m \end{bmatrix} \begin{bmatrix} I_n - \bar{A} & -\bar{B}_1 \\ -E' & 0 \end{bmatrix}^{\#} B_3 \qquad (12)$$

Remark 4 *When $m = l$, it suffices to replace $\begin{bmatrix} I_n - \bar{A} & -\bar{B}_1 \\ -E' & 0 \end{bmatrix}^{\#}$ by $\begin{bmatrix} I_n - \bar{A} & -\bar{B}_1 \\ -E' & 0 \end{bmatrix}^{-1}$ in (12).*

Remark 5 *Lemmas 1, 2 and 3 point out that the output-tracking objective cannot be carried out for any disturbances d, that is, for any disturbance $\tilde{d}$ and $\tilde{w}$ and any reference input r. The only possibility to solve our output-tracking objective for any reference r is to verify $(I_n - \bar{A})E = 0$. Such a condition requires some structural properties between matrices $\tilde{A}$ and $\tilde{C}$.*

3.2 Polytopic model

Define the scalars $\alpha_{1(i)}(z(k))$, $i = 1, ..., m$ and $\alpha_{2(g)}(z(k))$, $g = 1, ..., l$, as [16]:

$$0 < \alpha_{1(i)}(z(k)) = \frac{sat_{u_0}(F_{1(i)} z(k))}{F_{1(i)} z(k))} \le 1$$
$$0 < \alpha_{2(g)}(z(k)) = \frac{sat_{v_0}(F_{2(g)} z(k))}{F_{2(g)} z(k))} \le 1$$

Thus the following lemma can be stated.

Lemma 4 *Consider any compact set $\Omega_0 \subset \Re^{n+l}$, then for $z(k) \in \Omega_0$ the following properties hold:*
1. *vectors $\alpha_1(z(k))$ and $\alpha_2(z(k))$ admit lower bounds:*

$$\begin{array}{l}\alpha_{1min(i)} = \min\{\alpha_{1(i)}(z); z \in \Omega_0\}, i = 1, ..., m \\ \alpha_{2min(g)} = \min\{\alpha_{2(g)}(z); z \in \Omega_0\}, g = 1, ..., l\end{array}$$

2. *we can define vertex matrices $\mathcal{A}_j$, $j = 1, ..., 2^m$ and $\mathcal{B}_q$, $q = 1, ..., 2^l$:*

$$\mathcal{A}_j = A + B_1\Delta(\gamma_j)F_1 \quad (13)$$

$$\mathcal{B}_q = B_2(\Delta(\bar{\gamma}_q) - I_l)F_2 \quad (14)$$

where $\Delta(\gamma_j)$ (resp. $\Delta(\bar{\gamma}_q)$) is a diagonal matrix for which $\Delta(\gamma_j)_{(i,i)} = \gamma_{j(i)}$ (resp. $\Delta(\bar{\gamma}_q)_{(g,g)} = \bar{\gamma}_{q(g)}$) take the values 1 or $\alpha_{1min(i)}$, $i = 1, ..., m$, (resp. 1 or $\alpha_{2min(g)}$, $g = 1, ..., l$).
3. *$z(k+1)$ can be determined by the following polytopic system:*

$$z(k+1) = \sum_{j=1}^{2^m} \lambda_j \mathcal{A}_j z(k) + \sum_{q=1}^{2^l} \bar{\lambda}_q \mathcal{B}_q z(k) + B_3 d \quad (15)$$

with $\sum_{j=1}^{2^m} \lambda_j = 1$, $\lambda_j \geq 0$, $\sum_{q=1}^{2^l} \bar{\lambda}_q = 1$, $\bar{\lambda}_q \geq 0$.

By definition, the set $S(F_1, u_0^{\alpha_1}) \cap S(F_2, {v_0}^{\alpha_2})$

$$\begin{array}{l} S(F_1, u_0^{\alpha_1}) = \{z \in \Re^{n+l}; -u_0^{\alpha_1} \leq F_1 z \leq u_0^{\alpha_1}\} \\ S(F_2, {v_0}^{\alpha_2}) = \{z \in \Re^{n+l}; -{v_0}^{\alpha_2} \leq F_2 z \leq {v_0}^{\alpha_2}\}\end{array}$$

with $u_{0(i)}^{\alpha_1} = \frac{u_{0(i)}}{\alpha_{1min(i)}}$, $i = 1, ..., m$ (resp. ${v_{0(g)}}^{\alpha_2} = \frac{v_{0(g)}}{\alpha_{2min(g)}}$, $g = 1, ..., l$), contains Ω_0 and corresponds to the maximal set in which model (15) represents system (4).

Remark 6 *In order to solve Problem 1, we need to determine matrices F_1, F_2, set Ω_0 and vectors α_{1min}, α_{2min}.*

4 Main results

We choose ellipsoidal sets $\mathcal{X}_0$ and $\mathcal{D}_0$ derived from symmetric positive definite matrices P and S and from positive scalars ξ and σ, as follows:

$$\mathcal{X}_0 = \mathcal{E}(P, \xi) = \left\{z \in \Re^{n+l}; z'Pz \leq \xi^{-1}\right\} \quad (16)$$

$$\mathcal{D}_0 = \mathcal{E}(S, \sigma) = \left\{d \in \Re^{n+2l}; d'Sd \leq \sigma^{-1}\right\} \quad (17)$$

Let us define $\mathcal{H}(j,q) = P(A + B_1\Delta(\gamma_j)F_1 + B_2(\Delta(\bar{\gamma}_q) - I_l)Z(I_{n+l} - A - B_1F_1))$ and state the following proposition.

Proposition 1 *If there exist matrices F_1, Z, $P = P' > 0$, $S = S' > 0$, vectors α_{1min} and α_{2min}, positive scalars $\xi > 0$, $\sigma > 0$, $\mu > 0$ and $\omega > 0$ satisfying* [1]*:*

$$\begin{bmatrix} -\mu P & \star & \star & \star \\ 0 & -\omega S & \star & \star \\ \mathcal{H}(j,q) & PB_3 & -P & \star \\ 0 & 0 & 0 & \mu\sigma + \omega\xi - \sigma \end{bmatrix} \leq 0 \quad (18)$$
$$\forall j = 1, ..., 2^m,\ q = 1, ..., 2^l$$

$$0 < \alpha_{1min(i)} \leq 1,\ \forall i = 1, ..., m \quad (19)$$

$$0 < \alpha_{2min(g)} \leq 1,\ \forall g = 1, ..., l \quad (20)$$

$$\begin{bmatrix} P & \star \\ \alpha_{1min(i)}F_{1(i)} & \xi u_{0(i)}^2 \end{bmatrix} \geq 0,\ \forall i = 1, ..., m \quad (21)$$

$$\begin{bmatrix} P & \star \\ \alpha_{2min(g)}Z_{(g)}(I_{n+l} - A - B_1F_1) & \xi v_{0(g)}^2 \end{bmatrix} \geq 0$$
$$\forall g = 1, ..., l \quad (22)$$

$$\begin{bmatrix} S & \star \\ \mathcal{F}_{(i)} & \sigma u_{0(i)}^2 \end{bmatrix} \geq 0,\ \forall i = 1, ..., m \quad (23)$$

$$\begin{bmatrix} S & \star \\ Z_{(g)}B_3 & \sigma v_{0(g)}^2 \end{bmatrix} \geq 0,\ \forall g = 1, ..., l \quad (24)$$

with $\mathcal{F}$ defined in (12), then the gains F_1 and $F_2 = Z(I_{n+l} - (A + B_1F_1))$, the set of admissible initial conditions $\mathcal{E}(P, \xi)$ and the set of admissible disturbances $\mathcal{E}(S, \sigma)$ solve Problem 1.

Proof. Relations (21)-(22) mean that $\mathcal{E}(P, \xi) \subseteq S(F_1, u_0^{\alpha_1}) \cap S(F_2, v_0^{\alpha_2})$. If there exist matrices F_1, Z and P, vectors α_{1min} and α_{2min} satisfying relations (19) and (20), and inclusion relations (21)-(22) then the polytopic model (15) can represent the saturated system (4). Moreover, by considering the quadratic function $\mathcal{V}(z) = z'Pz$ and by computing $\mathcal{V}(z(k+1))$ along the trajectories of system (15) it follows $\mathcal{V}(z(k+1)) = [z(k)'\mathcal{M}(j,q)' + d'B_3']P[\mathcal{M}(j,q)z(k) + B_3d]$ with from (13)-(14) $\mathcal{M}(j,q) = \sum_{j=1}^{2^m}\lambda_j\mathcal{A}_j + \sum_{q=1}^{2^l}\bar{\lambda}_q\mathcal{B}_q$.
First, we have to prove that $\mathcal{V}(z(k+1)) \leq \xi^{-1}$, $\forall z(k)$ and $\forall d$ satisfying $z(k)'Pz(k) \leq \xi^{-1}$ and $d'Sd \leq \sigma^{-1}$, respectively. By using the $\mathcal{S}$-procedure [3] it is possible to show that it is equivalent to seek $\mu > 0$ and $\omega > 0$ such that

$$\begin{array}{c}\mathcal{V}(z(k+1)) - \frac{1}{\xi} - \mu\left(z(k)'Pz(k) - \frac{1}{\xi}\right) \\ -\omega\left(d'Sd - \frac{1}{\sigma}\right) \leq 0\end{array} \quad (25)$$

[1] In (18), $\star$ is the substitute for blocks ensuring matrix symmetry.

Thanks to convexity properties, if relation (18) is verified then inequality (25) holds. Furthermore, the satisfaction of relation (18) implies the asymptotic stability of matrix $A + B_1 F_1$. Moreover, if relations (23)-(24) are verified then the inclusion relation $\mathcal{E}(S,\sigma) \subseteq (S(\mathcal{F},u_0) \cap S(ZB_3,v_0))$, or, equivalently, $\mathcal{E}(S,\sigma) \subseteq (S(\mathcal{F},u_0) \cap S(F_2(I_{n+l} - (A + B_1F_1))^{-1}B_3, v_0))$ is satisfied and therefore z_e is defined as in (11) and belongs to the set $S(F_1,u_0) \cap S(F_2,v_0)$. Hence, provided that there exist matrices F_1, Z, P and S, vectors α_{1min} and α_{2min}, positive scalars ξ, σ, μ and ω satisfying relations (18), (19), (20), (21), (22), (23) and (24), Problem 1 is solved. □

Remark 7 *A special case of the anti-windup term is given by $\Delta(h) = 0$. Then the control reduces to a simple integrating action and reads $q(k+1) = q(k) + \epsilon(k)$. Therefore the closed-loop system (4) becomes:*

$$z(k+1) = Az(k) + B_1 sat_{u_0}(F_1 z(k)) + B_3 d \quad (26)$$

This case is also equivalent to set $F_2 = 0$. Thus, Proposition 1 applies by considering relation (18) in which we set $Z = 0$ and relations (19), (21) and (23).

Remark 8 *The direct application of Proposition 1 for solving Problem 1 is difficult due to some nonlinearities in the variables in relations (18), (21) and (22). The problem formulation and the use of Semi-Definite Programming do not allow to solve the complete synthesis problem (as defined by Problem 1) in one single step (it is not possible to compute F_1 and F_2 (or Z) together). This is however possible for continuous-time systems [17]. At this time, we can emphasize that our problem formulation entails different sources of conservatism such that the representation of the saturated system by a polytopic model, the search of a unique Lyapunov matrix P shared by each vertex of the matrix polytope or still the use of the $\mathcal{S}$-procedure. Moreover, some global optimization methods for solving the nonlinear problem introduced in Proposition 1 could be used [4], but their worst-case complexities and the required computational effort may make the solution untractable. Even if we do not attain the global optimum, we will be satisfied with an (approximative) feasible solution, that is a suboptimal solution.*

Hence, relations of Proposition 1 become linear as soon as some variables are fixed. In this sense some relaxation schemes can be considered as described below:

- *Relax 1.* Fix F_1, Z (or equivalently F_2), P, S, ξ, σ and search α_{1min}, α_{2min}, ω, μ which solve relations (18)-(22).
- *Relax 2.* Fix F_1, Z (or equivalently F_2), α_{1min}, α_{2min}, ω, μ and search P, S, ξ, σ which solve relations (18), (21)-(24).
- *Relax 3.* Fix P, Z, α_{1min}, α_{2min}, ω, μ and search F_1, S, ξ, σ which solve relations (18), (21)-(24).
- *Relax 4.* Fix P, F_1, α_{1min}, α_{2min}, ω, μ and search Z, S, ξ, σ which solve relations (18), (21)-(24).

Furthermore, Proposition 1 provides a sufficient condition in order to derive the gains F_1 and F_2 and the sets $\mathcal{E}(P,\xi)$ and $\mathcal{E}(S,\sigma)$. It is then interesting to orient the solutions in order to obtain ellipsoids $\mathcal{E}(P,\xi)$ and $\mathcal{E}(S,\sigma)$ as large as possible. First recall that the size of $\mathcal{E}(P,\xi)$ (resp. $\mathcal{E}(S,\sigma)$) is related both to P (resp. S) and ξ (resp. σ). In that sense, we can consider optimization problems described as follows:

- *Optim 1.* $\min \sum_{i=1}^{m} (\alpha_{1min(i)} + \alpha_{2min(i)})$.
- *Optim 2.* $\min log(det(\xi\ P)) + log(det(\sigma\ S))$ corresponds to the maximization of the two volumes of ellipsoids $\mathcal{E}(P,\xi)$ and $\mathcal{E}(S,\sigma)$.
- *Optim 3.* $\min \xi + \sigma + trace(P) + trace(S)$ may be considered as an approximation of Optim 2 which allows to use LMI toolbox of Matlab.
- *Optim 4.* $\min \xi + \sigma$.

We can therefore combine these optimization problems with the previous relaxations schemes.

Remark 9 *The optimization criterion Optim 1, associated with Relax 1, allows to decrease α_{1min} and α_{2min}, which corresponds, according to the definition of α_{1min} and α_{2min}, to increase the tolerance to the saturation of the closed-loop system.*

Remark 10 *Note that the optimization of both $\mathcal{E}(P,\xi)$ and $\mathcal{E}(S,\sigma)$ leads to some trade-off between these two sets, since one increases while the other one decreases. Depending on the control objective, the optimization criterion may be weighted to emphasize the initial admissible states set $\mathcal{E}(P,\xi)$ or the set $\mathcal{E}(S,\sigma)$ involving both output tracking references and admissible disturbances.*

Remark 11 *Two maximal admissible tracking references may be derived for $\mathcal{E}(P,\xi)$ and $\mathcal{E}(S,\sigma)$. The first one corresponds to the maximal initial value $\begin{bmatrix} -r_1' & 0 & 0 \end{bmatrix}'$ belonging to $\mathcal{E}(P,\xi)$, i.e., the maximal distance between the initial output y at the origin and the reference signal r. The second one corresponds to the maximal reference, when no disturbance occurs, $\begin{bmatrix} 0 & 0 & r_2' \end{bmatrix}'$ belonging to $\mathcal{E}(S,\sigma)$. Finally, $r_{max} = \min(r_1, r_2)$. Note however that these maximal values r_1 and r_2 do not correspond to the maximal value r which may be attained by exploring the whole surface of $\mathcal{E}(P,\xi)$ and $\mathcal{E}(S,\sigma)$, respectively.*

Remark 12 *Note that the initialisation step (choice of admissible solutions P, F_1, F_2, choice of μ, criterion) and the optimization criteria of Relax 1, 2, 3 and 4 have a strong influence on the solution, both in terms of size of sets $\mathcal{E}(P,\xi)$ and $\mathcal{E}(S,\sigma)$ and in terms of closed-loop spectrum. They have to be chosen according to the control objective.*

5 Conclusion

The problem of local stabilization and reference tracking of linear discrete-time systems subject to input saturation and disturbances has been addressed. By using some relaxation schemes and an LMI formulation we have proposed an solution allowing the simultaneous computation of a saturating controller, a local domain of stability and a domain of admissible reference and disturbances. By using iterative procedures induced by relaxation schemes and different optimization criteria, we can emphasize either the stability or the performance, in terms of sizes of either the stability domain or the reference and disturbances domains.

We have only considered constant disturbances and references. Thus, in the future an interesting way could be to consider time-varying disturbances and references, with a knowledge or an estimation of their dynamics, and eventually with a tolerance concerning the tracking performance (admissible delay or tracking error). Another way could be the synthesis of output feedback type controller instead of state feedback one. In this case, the first question is: Does one has to compute a dynamic output controller for the initial system (1) or for the augmented one (3) ?

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