

ZERO STRUCTURE ASSIGNMENT OF MATRIX PENCILS BY ADDITIVE STRUCTURED TRANSFORMATIONS

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Abstract. Matrix Pencil Models are natural descriptions of linear networks and systems. Changing the values of elements of networks, that is redesigning them implies changes in the zero structure of the associated pencil by structured additive transformations. The paper examines the problem of zero assignment of regular matrix pencils by a special type of structured additive transformations. For a certain family of network redesign problems the additive perturbations may be described as diagonal perturbations and such modifications are considered here. This problem has certain common features with the pole assignment of linear systems by structured static compensators and thus the new powerful methodology of global linearisation [1,2] can be used. For regular pencils with infinite zeros families of structured degenerate additive transformations are defined and parameterized and this lead to the derivation of conditions for zero structure assignment, as well as methodology for computing such solutions. Finally the case of regular pencils with no infinite zeros is considered and conditions of zero assignment are developed. The results here may provide the means for studying problems of linear network redesign by modification of the nondynamic elements.

Key Words: Linear Systems, Systems Redesign, Matrix Pencils, Frequency Assignment, Algebraic methods and Global Linearisation.

1. INTRODUCTION

The problem which we consider here is equivalent to a zero assignment of the regular matrix pencil $sF+G+H$, where $sF+G$ may express the internal dynamics matrix of a system and $H=U\Lambda V$ may represent a static structural change where U , V are known graph incidence matrices (they may express a topology modification) and Λ is a diagonal matrix of continuous design parameters. In reality, the three matrices U , V , Λ are design parameters [15]. Here we shall assume that the incidence matrices U , V are fixed and thus only the diagonal matrix Λ is free for the assignment of zeros $sF+G+U\Lambda V$. A large family of such problems can be reduced to the case of diagonal additive perturbations and this is the problem considered here in some detail. The paper

deals with both the study of solvability conditions, as well as the derivation of solutions, whenever such solutions exist [15].

The general properties of the frequency assignment map are considered first and the notion of degenerate transformations, i.e. those making the pencil $sF+G+H$ singular is defined. For the case of pencils with infinite zeros, a parameterization of the set of degenerate transformations H is given according to the nature of the resulting singularity of the pencil. The significance of degenerate solutions is emphasized by establishing the property that if the differential of the frequency assignment map at a degenerate point H_0 is onto, then this implies assignability of zero structure of the pencil by some appropriate H . The explicit form of the differential at a degenerate point is computed and it is shown that

for a generic pencil there exist degenerate points H_0 such that the corresponding differential is onto. Using as the starting point such degenerate solutions, it is shown that transformations H , which are nondegenerate, may be constructed to assign the zeros of $sF+G+H$ in the neighborhood of any arbitrary symmetric set of complex numbers. The proposed methodology is a Quasi-Newton type numerical approach and its convergence properties are examined. Finally, the case of pencils with no infinite zeros is considered and conditions for the complex zero assignment are derived in terms of invariants associated with the pencil. The results are stated without proof and their proof may be found in the full report [12].

2. ZERO ASSIGNMENT OF MATRIX PENCILS BY DIAGONAL ADDITIVE PERTURBATIONS: PROBLEM FORMULATIONS AND BACKGROUND RESULTS

The mathematical formulation of the problem can be stated as follows:

Problem formulation: Given a square pencil $sA + B$ such that $A, B \in R^{n \times n}$, $\text{rank} A = n_1 < n$ the problem to be examined here is to investigate the solvability of the equation:

$$\det(sA + B + \Lambda) = \varphi(s) \quad (2.1)$$

with respect to $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ when $\varphi(s)$ is a given polynomial of n_1 degree.

Notation: Q_{mn} is the set of lexicographically ordered sequences of m integers from n set of integers and D_n is any sequence of n integers from $(1, 2, \dots, n)$ with possible repetition and any order.

Definition (1): A sequence $\omega = (i_1, i_2, \dots, i_n) \in Q_{n, 2n}$ characterises a minor α_ω of $C_n[I_n, \Lambda]$.

On such sequences we define the following:

(a) The operation π on $\omega \in Q_{n, 2n}$ as:

$$\Delta \pi(\omega) = (\pi(i_1), \pi(i_2), \dots, \pi(i_n)) = (j_1, \dots, j_n)$$

where:

$$\pi(i_k) = \begin{cases} i_k & \text{if } i_k \leq n \\ i_k - n & \text{if } i_k > n \end{cases}$$

(b) A sequence $\omega = (i_1, i_2, \dots, i_n) \in Q_{n, 2n}$ is called degenerate, if $\pi(\omega) = (j_1, j_2, \dots, j_n)$ has at least two equal elements (ie $j_l = j_k$) and it is nondegenerate, if $\pi(\omega) = (j_1, j_2, \dots, j_n)$ has distinct elements.

(c) For a sequence $\omega \in Q_{n, 2n}$, which is nondegenerate we define as the sign of ω ,

$\Delta \text{sgn}(\omega) = \sigma(\omega) = \text{sign}(j_1, j_2, \dots, j_n)$ and as the trace of ω the subset of the elements of $\pi(\omega) = (j_1, j_2, \dots, j_n)$ which correspond to $i_k > n$

and thus is the set $\langle \omega \rangle = (\hat{i}_{k_1}, \hat{i}_{k_2}, \dots, \hat{i}_{k_\mu})$ $i \leq n$.

Proposition (1): Let $[I_n, \Lambda] \in \mathfrak{R}^{n \times 2n}$ and denote

$$C_n[I_n, \Lambda] = [\dots, a_\omega, \dots] \in \mathfrak{R}^{1 \times \binom{2n}{n}}, \omega \in Q_{n, 2n}$$

Then a_ω are defined as follows:

- $a_\omega = 0$, if ω is degenerate
- $a_\omega \neq 0$, if ω is nondegenerate

Furthermore, if ω is nondegenerate, $\sigma(\omega)$ is the sign

of ω and $\langle \omega \rangle = \{\hat{i}_{k_1}, \hat{i}_{k_2}, \dots, \hat{i}_{k_\mu}\}$ is the trace of ω ,

then $a_\omega = \sigma(\omega) \lambda_{\hat{i}_{k_1}} \lambda_{\hat{i}_{k_2}} \dots \lambda_{\hat{i}_{k_\mu}}$.

The set of $Q_{n, 2n}$ sequences may thus be divided into two disjoint sets, the set $Q_{n, 2n}^d$ of degenerate sequences and the set $Q_{n, 2n}^{nd}$ of nondegenerate sequences. Both subsets of sequences are assumed to be lexicographically ordered. Consider now the characteristic redesigned polynomial:

$$\Delta \Phi(s) = \det(sA + B + \Lambda) = \Phi(s, B, \Lambda)$$

By the Binet-Cauchy theorem we have that:

$$\begin{aligned} \det[sA + B + \Lambda] &= \det([I_n, \Lambda] \cdot [sA^t + B^t, I_n]^t) = \\ &= C_n([I_n, \Lambda]) \cdot C_n([sA^t + B^t, I_n]^t) = \varphi(s). \end{aligned} \quad (2.2)$$

Definition (2): Let $Q_{n, 2n}^D$ $Q_{n, 2n}^{ND}$ be the ordered subjects of degenerate and nondegenerate of $Q_{n, 2n}$ associated with the $[I_n, I_n]$ structure. We shall denote by $\tilde{C}_n([I_n, \Lambda])$ the subvector of $C_n([I_n, \Lambda])$ obtained by omitting all zero coordinates corresponding to $Q_{n, 2n}^d$ sequences (indices) and similarly by $\tilde{C}_n([sA^t + B^t, I_n])$ the reduced dimension subvector of $C_n([sA^t + B^t, I_n])$ derived by deleting the $Q_{n, 2n}^d$ set of coordinates. The subvectors $\tilde{C}_n([I_n, \Lambda])$, $\tilde{C}_n([sA^t + B^t, I_n])$ will be referred to as $[I_n, I_n]$ -structured projections. Note that

$$\begin{aligned} C_n[I_n, \Lambda] C_n \left(\begin{bmatrix} sA + B \\ I_n \end{bmatrix} \right) &= \\ \tilde{C}_n([I_n, \Lambda]) \tilde{C}_n \left(\begin{bmatrix} sA + B \\ I_n \end{bmatrix} \right) &= \varphi(s) \end{aligned} \quad (2.3)$$

and given that:

$$\begin{aligned}\tilde{C}_n([I_n, \Lambda]) &= [\dots, a_\omega, \dots] \omega \in Q_{n,2n}^{nD} \\ &= \begin{bmatrix} \dots, \sigma(\omega) \lambda_{i_{k_1}} & \dots \lambda_{i_{k_\mu}} & \dots \end{bmatrix} \\ &= \begin{bmatrix} \dots, \lambda_{i_{k_1}} & \dots \lambda_{i_{k_\mu}} & \dots \end{bmatrix} \text{diag}\{\dots, \sigma(\omega), \dots\} \\ &= \hat{C}_n([I_n, \Lambda]) D(\sigma(\omega))\end{aligned}\quad (2.4)$$

then

$$\begin{aligned}\varphi(s) &= \tilde{C}_n([I_n, \Lambda]) D(\sigma(\omega)) \tilde{C}_n \left(\begin{bmatrix} sA+B \\ I_n \end{bmatrix} \right) \\ &= \tilde{C}_n([I_n, \Lambda]) \tilde{C}_n \left(\begin{bmatrix} sA+B \\ I_n \end{bmatrix} \right)\end{aligned}\quad (2.5)$$

The vectors

$$\begin{aligned}\hat{C}_n([I_n, \Lambda]) &\overset{\Delta}{=} \tilde{C}_n([I_n, \Lambda]) D\{\sigma(\omega)\} \in \mathfrak{R}^{1 \times 2^n} \quad (2.6) \\ \hat{C}_n \left(\begin{bmatrix} sA+B \\ I_n \end{bmatrix} \right) &\overset{\Delta}{=} D\{\sigma(\omega)\} \tilde{C}_n \left(\begin{bmatrix} sA+B \\ I_n \end{bmatrix} \right) = \hat{P}(s) \in \mathfrak{R}^2[s]\end{aligned}$$

will be referred to as normalised $[I_n, I_n]$ -structured projections of $C_n([I_n, \Lambda])$, $C_n \left(\begin{bmatrix} sA+B \\ I_n \end{bmatrix} \right)^t$ respectively. In particular,

$\hat{C}_n \left(\begin{bmatrix} sA+B \\ I_n \end{bmatrix} \right)^t = \hat{P}(s)$, will be called the $[I_n, I_n]$ -Grassmann representative of the system.

Proposition (2) The normalised $[I_n, I_n]$ -structured projection of $\hat{C}_n([I_n, \Lambda])$ may be expressed as:

$$\tilde{C}_n([I_n, \Lambda]) = (1, \lambda_1) \otimes (1, \lambda_2) \otimes \dots \otimes (1, \lambda_n) \quad (2.7)$$

where \otimes denotes the standard tensor product.

The above result follows by inspection of the expression of $\hat{C}_n([I_n, \Lambda])$. The characteristic polynomial is expressed as in (2.5) and it is generated by the $[I_n, I_n]$ -Grassmann representative of the system ie

$$\hat{P}(s) = \hat{C}_n \left(\begin{bmatrix} sA+B \\ I_n \end{bmatrix} \right)$$

Remark (1): For any $sA+B$, $C_n \left(\begin{bmatrix} sA+B \\ I_n \end{bmatrix} \right)$ is a polynomial vector; however, $\hat{P}(s)$ is not necessarily coprime.

Definition (3): The greatest common divisor of the entries of $\hat{P}(s)$ will be denoted by $\varphi_{A,B}(s)$ and this will be referred to as the $[I_n, I_n]$ -fixed polynomial of the system. A system for which $\varphi_{A,B}(s) = 1$ will be called $[I_n, I_n]$ -irreducible; otherwise, it will be called $[I_n, I_n]$ -reducible.

The following result can be readily established

Theorem (1): The fixed zeros of the redesigned polynomial $\Phi(A, B, \Lambda)$ for all possible Λ are only the roots of $\Phi_{A,B}(s)$ polynomial.

We can now easily establish [1] that:

$$\det[sA+B+\Lambda] = (1, \lambda_1) \otimes (1, \lambda_2) \otimes \dots \otimes (1, \lambda_n) \hat{P}(s)$$

By equating the coefficients of the powers of s we get $(1, \lambda_1) \otimes (1, \lambda_2) \dots \otimes (1, \lambda_n) \cdot P = \varphi$, where φ is the coefficient vector of $\varphi(s)$ and P is called the **Plucker matrix** for the problem.

Example (1): Let a system matrix of a circuit be:

$$sA+B = \begin{bmatrix} s+5 & s-1 & s \\ 2s & s & s+3 \\ 1 & 2 & -1 \end{bmatrix}$$

In this case the $C_3([I_3, A_3])$ matrix is:

$$C_3[I_3, \Lambda_3] \equiv C_3 \begin{bmatrix} 1 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \lambda_3 \end{bmatrix}$$

And can be calculated to be:

$$(1, 0, 0, \lambda_3, 0, -\lambda_2, 0, 0, 0, \lambda_2 \lambda_3, \lambda_1, 0, 0, 0, -\lambda_1 \lambda_3, 0, \lambda_1 \lambda_2, 0, 0, \lambda_1 \lambda_2 \lambda_3)$$

The $[sA+B, I_n]^t$ matrix is expressed as:

$$\begin{bmatrix} sA+B \\ I_3 \end{bmatrix} \equiv \begin{bmatrix} s+5 & s-1 & s \\ 2s & s & s+3 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The nonzero elements of $C_3([I_3, \Lambda_3])$ are $(1, \lambda_3, -\lambda_2, \lambda_2 \lambda_3, \lambda_1, -\lambda_1 \lambda_3, \lambda_1 \lambda_2, \lambda_1 \lambda_2 \lambda_3)$ and the corresponding elements of $C_3([sA+B, I_3]^1)$ are $(3s^2-21s-33, -s^2+7s, 2s+5, s+5, -3s-6, -s, -1, 1)$. Therefore:

$$\det(sA + B + \Lambda) = 1(3s^2 - 21s - 33) + \lambda_3(-s^2 + 7s) + \lambda_2(2s + 5) + \lambda_2 \lambda_3(s + 5) + \lambda_1(-3s - 6) + \lambda_1 \lambda_3(-s) + \lambda_1 \lambda_2(-1) + \lambda_1 \lambda_2 \lambda_3$$

The problem described involves the solution of a set of nonlinear algebraic equations. When the number of solutions is finite, this number is combinatorially large (one can prove that the degree is $n!$) and this makes the problem difficult to be investigated via the standard Groebner basis tools especially when n is large. To construct a solution of the problem we will follow the methodology in [10] by studying the local properties of degenerate solutions.

The *Frequency Assignment Map* associated with the problem is the map assigning Λ to the coefficient vector φ ie. $F: \mathbb{R}^n \rightarrow \mathbb{R}^n: F(\Lambda) = \varphi$

A diagonal matrix Λ_0 is *degenerate* iff: $F(\Lambda_0)=0$, or equivalently $\det(sA + B + \Lambda_0) = 0$.

In other words, Λ_0 is degenerate if the pencil $sA+B+\Lambda_0$ becomes singular. The following theorem shows the great importance of degenerate matrices

Theorem (2): If there exists a degenerate matrix Λ_0 such that the differential DF_{Λ_0} is onto then any set of n sequences can be assigned via some diagonal perturbation.

For a generic $n \times n$ pencil when n is small the set of all degenerate matrices may be constructed via Groebner Basis.

Example (2): Consider the Pencil:

$$sA + B = \begin{bmatrix} -3s & 2+4s & -1-s \\ -3+4s & 5+s & -1-2s \\ -4+s & 6+5s & -1-3s \end{bmatrix}$$

then the set of equations defining all the degenerate matrices $\text{diag}\{x, y, z\}$ is given by:

$$\begin{aligned} x - 4y - xy + 6z + 5xz + xyz &= 0 \\ -5 + x - 3xy - 11z + xz - 3yz &= 0 \\ -2 + 7x + 10y - 19z &= 0 \end{aligned}$$

A Groebner Basis for the above set of equations is:

$$480 + 5312x + 16433x^2 + 21474x^3 + 15452x^4 + 5726x^5 + 147x^6 = 0$$

$$\begin{aligned} &1579680 - 10392988x - 18923271x^2 - 12885549x^3 - \\ &3302425x^4 - 81879x^5 + 2714400y = 0 \\ &2122560 - 12293068x - 18923271x^2 - 12885549x^3 - \\ &3302425x^4 - 81879x^5 + 5157360z = 0 \end{aligned}$$

which gives 3! solutions: 4 real and 2 complex.

One can calculate the number of degenerate matrices for a generic pencil as follows:

Theorem (3): For a generic $n \times n$ Pencil $sA+B$ such that $\text{rank}(A)=n-1$ the number of degenerate diagonal matrices is finite and equal to $n!$.

3. CLASSIFICATION OF DEGENERATE COMPENSATORS.

We may classify the degenerate matrices Λ of a Pencil $sA+B$ according to the sizes of row or column minimal indices of $sA+B-\Lambda$.

Definition: A degenerate matrix Λ of a Pencil $sA+B$ is of degree k if the polynomial module that spans the right Kernel of $sA+B-\Lambda$ has Forney dynamical order k .

Theorem (4): For a generic $n \times n$ Pencil $sA+B$ such that $\text{rank}(A)=n-1$ the number of degenerate diagonal matrices of degree d , B_d , $(0 \leq d \leq n-1)$ is finite and equal to:

$$B_d = \begin{cases} \binom{n}{d+1} A_{d+1} & \text{if } d > 0 \\ 1 & \text{if } d = 0 \end{cases}$$

where A_{d+1} is the number of permutations of $d+1$ objects with no fixed points.

Although the construction of degenerate matrices looks as though it has the same complexity of the problem we have started, there is certain degenerate matrix that can be easily constructed via linear equations. These are the degenerate diagonal matrices of degree 0 and $n-1$.

Proposition (3): Let v^t, w vectors such that:

$v^t A = 0, A w = 0$, then the diagonal matrices

$$\Lambda_0 = -\text{diag}\left\{ \frac{v^t b_1}{v_1}, \dots, \frac{v^t b_n}{v_n} \right\},$$

$$\Lambda_{n-1} = -\text{diag}\left\{ \frac{b_1^t w}{w_1}, \dots, \frac{b_n^t w}{w_n} \right\}$$

where b_i , (b_i^t) are the columns (rows) of B and v_i (w_i) are the coordinates of v (w), of the degenerate solution.

Another classification of the degenerate matrices are into infinite and finite. Infinite are those solutions that are taken as limits of sequences Λ_n whose one or more elements tend to infinity. The degenerate

matrices constructed in Proposition 4 are finite iff $v_i \neq 0$. If V is the basis matrix of the left kernel of A , the next theorem characterizes V so that there exists at least one finite degenerate matrix.

Theorem (5): If $V = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$ is a basis matrix of the left kernel of A then there exists a $v \in V$ such that the corresponding degenerate matrix produced by v is finite iff $v_i \neq 0$.

Note that if the above defined V has not the desired properties, there exists a $k \times n$ submatrix of A , say A' , such that $\text{rank}(A) = \text{rank}(A')$.

4. GENERICITY RESULTS AND CONSTRUCTION OF SOLUTIONS.

The differential of the frequency assignment map F related to our problem, plays a very important role in the determination of the onto properties of the map and therefore in the solvability of the problem. This can be calculated in many ways and for a general square rank deficient polynomial matrix $A(s)$ it can be proved that:

Lemma (1): $\det(A(s) + xB(s)) = x \cdot \text{trace}(A \text{adj}(A(s))B(s)) + O(x^2)$

this shows that if $\text{adj}(sA + B - \Lambda_0) = g(s) \cdot v^t(s)$ then DF_{Λ_0} can be represented by the coefficient matrix of the polynomial vector $(g_1(s)v_1(s), \dots, g_n(s)v_n(s))$. Next we will prove that

Proposition (4): For a generic Pencil the degenerate diagonal matrix Λ_0 of the zero assignment map of the problem, satisfies $\text{rank} DF_{\Lambda_0} = n$.

Next we will show that a Quasi-Newton type of numerical method starting from a regular degenerate matrix can produce diagonal matrices which assign the desired frequencies and it is within a r distance from the degenerate matrix.

Theorem(6): Let $M = n \|DF_{\Lambda_0}^{-1}\| \|T\|, a = \|DF_{\Lambda_0}^{-1}\| \varphi$ and ϵ, r are such that:

$$\epsilon \leq \frac{r}{a} < \frac{(M^{-1} + \|\Lambda_0\|^{n-1}) \frac{1}{n-1} - \|\Lambda_0\|}{a} = \epsilon_0$$

then sequence Λ_k produced by the iteration:

$$\Lambda_{k+1} = \Lambda_k - DF_{\Lambda_0}^{-1} (F(\Lambda_k) - \epsilon \phi)$$

converges to a Λ that satisfies:

$$F(\Lambda) = \epsilon \phi, \quad \|\Lambda - \Lambda_0\| \leq r$$

The above suggest the following methodology for the solution of the problem:

Computational Procedure:

1. Construct the degenerate matrix Λ_0 as above
2. Use the iteration:

$$\Lambda_{k+1} = \Lambda_k - DF_{\Lambda_0}^{-1} (F(\Lambda_k) - \epsilon \phi)$$

with ϵ as in the Theorem 6 and starting from Λ_0 , until convergence is achieved.

Example (3): Consider a network whose system matrix $T(s)$ is defined by:

$$T(s) = \begin{bmatrix} G_1 + G_2 + sC & -G_2 \\ -G_2 & G_2 + G_3 + 1/[(sL + (1/G_4))] \end{bmatrix}$$

Letting $X_3 = 1/[(sL + (1/G_4))]X_2$ $T(s)$ is transformed to:

$$T_1(s) = \begin{bmatrix} G_1 + G_2 + sC & -G_2 & 0 \\ -G_2 & G_2 + G_3 & 1 \\ 0 & 1 & -sL - (1/G) \end{bmatrix}$$

when the values are: $C=1, L=1, G_1=4, G_2=1, G_3=0, G_4=\infty$ the system matrix becomes:

$$T_1(s) = \begin{bmatrix} s+5 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -s \end{bmatrix}$$

assuming that we would like to change the natural frequencies of the above system by tuning the values of G_2, G_3, G_4 , we get the following perturbation:

$$\begin{bmatrix} G_2 & -G_2 & 0 \\ -G_2 & G_2 + G_3 & 0 \\ 0 & 0 & G_4 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} G_2 & 0 & 0 \\ 0 & G_3 & 0 \\ 0 & 0 & G_4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U \Lambda U^t$$

Which is equivalent to applying a diagonal perturbation $\Lambda = \text{diag}(G_2, G_3, G_4)$ to the system

$$U^{-1} T_1(s) (U^T)^{-1} = \begin{bmatrix} s+5 & s+4 & 0 \\ s+4 & s+4 & 1 \\ 0 & 1 & -s \end{bmatrix}$$

The equations defining the degenerate perturbations are:

$$f_2(G_2, G_3, G_4) = -1 - G_2 - G_3 = 0$$

$$f_1(G_2, G_3, G_4) = -5 - 4G_2 - 5G_3 - G_2G_3 + G_4 + G_2G_4 + G_3G_4 = 0$$

$$f_0(G_2, G_3, G_4) = -5 - G_2 + 4G_4 + 4G_2G_4 + 5G_3G_4 + G_2G_3G_4 = 0$$

and the finite solutions of these equations are given by:

$$a) G_2 = -2, G_3 = 1, G_4 = -3$$

$$b) G_2 = 0, G_3 = -1, G_4 = -5$$

both of them are full (or regular), so both can be used as starting points for a numerical Quasi-Newton

method to place the characteristic polynomial at any given second order one, $p(s)$:

$$x_{n+1} = x_n - (Jf)_{x_0}^{-1} (f - ep)$$

where $x=(G_2, G_3, G_4)^t$, $p=[1, 8, 15]^t$, $f=[f_2, f_1, f_0]^t$ and $x_0=(-2, 1, -3)^t$. Starting with $e=0.5$ the method converges after about 60 iterations to $x_{60}=(-2, 5507, 1, 050697, -2, 74137)^t$. Taking now this as a starting point we repeat the method for $e=1.2$ and so on. The following table displays the various solutions we obtain through this algorithm the last column being the Euclidean distance of the solution from the degenerate one:

Iterations	E	G2	G3	G4	Dist. from degenerate perturbation
0	0	-2	1	-3	0
60	0,5	-2,55	1,050	-2,741	0,610
50	1,2	-3,325	1,125	-2,652	1,375
85	2,5	-4,706	1,206	-2,611	2,741
135	5	-7,278	1,278	-2,594	5,301
250	10	-12,33	1,333	-2,588	10,34
80	18	-20,36	1,365	-2,586	18,37

5. NECESSARY AND SUFFICIENT CONDITION FOR ARBITRARY ASSIGNMENT $n=n_i$

The onto properties of a polynomial map such as F can be examined in terms of its differential. The rank of the differential of a complex algebraic map although it is a local invariant may determine its global properties [6].

Proposition (5): If F is an algebraic map between two complex varieties X, Y such that $\dim X \leq \dim Y$ then: there exists x in X : $\text{rank} DF_x = \dim Y$ iff F is (almost) onto.

This shows that the invariant that characterizes the onto property of the map F is the n -th exterior product of its differential DF_x and in the case we examine, this invariant is the determinant of the Jacobian of F , ie $\det(J(F)_x)$. Due to the $F(x)=f(x).P$ where $(x)=[1, x_1, x_2, \dots, x_1 x_2 \dots x_n]$ the Jacobian of the pole placement map can be calculated in terms of the Jacobian of f and the Plucker matrix P ie

$$\det(J(F)_x) = C_n(J(f)).C_n(P)$$

To calculate therefore $\det(J(F)_x)$ comes down to calculating $C_n(J(f))$. The calculation of $J(f)$ is easily implied by the following result:

Proposition (6): The partial derivative of f with respect to x_i , is given by:
 $\otimes(1, x_1) \otimes \dots \otimes (1, x_{i-1}) \otimes (0, 1) \otimes (1, x_{i+1}) \otimes \dots \otimes (1, x_n)$

Select n entries of the vector $f(x)$ say $a=[a_1, a_2, \dots, a_n]$, and call the Jacobian of the function a , $J(a)$; then this is a square $n \times n$ matrix whose determinant is one of the coordinates of the vector $C_n(J(f))$, conversely, all the coordinates $C_n(J(f))$ are of the form $\det(J(a))$ for some a . Next is a key theorem on the description of the compound $C_n(J(f))$.

Proposition (7): The Jacobian $J(a)$ is given by:
 $J(a) = \text{diag}(x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}) I(a) \text{diag}(a_1, a_2, \dots, a_n)$
 where the ij entry of $I(a)$ is 1 if a_j contains x_i and 0 otherwise. Therefore the determinant of $J(a)$ is equal to: $\det(J(a)) = I(a) a_1 a_2 \dots a_n / x_1 x_2 \dots x_n$.

Now every selection of p monomials $a=[a_1, a_2, \dots, a_n]$ correspond to a minor M_a of P . For a given monomial m consider the sum $P_m = \sum \det(I(a)) M_a$ where the sum is taken when $a_1 a_2 \dots a_n / x_1 x_2 \dots x_n = m$ and $\det(I(a)) \neq 0$. Then the collection of all P_m constitutes a system invariant characterizing the onto properties of the pole placement map. In fact, this leads to:

Theorem (7): The complex pole placement map is onto if there exists m such that $P_m \neq 0$.

6. CONCLUSIONS

The problem of zero assignment of matrix pencils by additive structured transformations has been considered and necessary and sufficient conditions were given, the problem is central in the study of redesign of composite systems having a certain topology by changing the values of individual elements. The case of structured models is a subject of further work.

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