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ACTIVE CONTROL OF COMBUSTION INSTABILITIES VIA HYBRID RESETTING CONTROLLERS

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Abstract. In this paper we propose an energy flow resetting control architecture as a means for achieving enhanced energy dissipation in combustion systems. The overall framework is based on a hybrid controller architecture wherein the closed-loop dynamical system is characterized by impulsive differential equations. The proposed framework is used to design high-performance hybrid controllers for suppressing thermoacoustic oscillations in combustion chambers by efficiently dissipating modal system energy.

Key Words. Hybrid resetting controllers, thermoacoustic instabilities, combustion control, energy flow, impulsive dynamical systems

1. INTRODUCTION

Engineering applications involving steam and gas turbines and jet and ramjet engines for power generation and propulsion technology involve combustion processes. Due to the inherent coupling between several intricate physical phenomena in these processes involving acoustics, thermodynamics, fluid mechanics, and chemical kinetics, the dynamic behavior of combustion systems is characterized by highly complex nonlinear models (see [1] and the references therein). The unstable dynamic coupling between heat release in combustion processes generated by reacting mixtures releasing chemical energy and unsteady motions in the combustor develop acoustic pressure and velocity oscillations which can severely impact operating conditions and system performance [1]. These pressure oscillations, known as thermoacoustic instabilities, often lead to high vibration levels causing mechanical failures, high levels of acoustic noise, high burn rates, and even component melting.

In [2] a novel class of controllers, called resetting virtual absorbers, were proposed for vibration control of lossless structural systems. The key idea of resetting control is to achieve enhanced energy dissipation between interconnected systems. Specifically, if a dissipative or lossless plant is at a high energy level, and a dissipative feedback controller at a low energy level is attached to it, then energy will generally tend to flow from the plant into the controller, decreasing the plant energy and increasing the controller energy [3]. Of course, emulated energy, and not physical energy, is accumulated by the controller. Conversely, if the attached controller is at a high energy level and the plant is at a low energy level, then energy can flow from the controller to the plant, since a controller can generate real, physical energy to effect the required energy flow. Hence, if and when the controller states coincide with a high emulated energy level, then we can *reset* these states to remove the emulated energy so that the emulated energy is not returned to the plant. Since active energy flow resetting control for interconnected systems gives rise to discontinuous closed-loop motions, impulsive differential equations [4–6] provide the mathematical foundation for analyzing hybrid resetting controllers. Motivated by the results of [2], the authors in [5,6] develop a general framework for feedback systems possessing discontinuous motions by addressing stability, dissipativity, feedback interconnections, and optimality of nonlinear impulsive dynamical systems. The results in [5, 6] provide a general analysis and synthesis framework for hybrid feedback control systems in that they apply to nonlinear dynamical systems with abstract energy notions for which a physical system energy interpretation is not necessary.

Utilizing the time-averaged combustion model developed in [1] for capturing thermoacoustic instabilities, in this paper we develop active energy flow resetting controllers to mitigate combustion induced pressure instabilities in combustion systems. The hybrid resetting controller can be viewed as a specialized technique for severing the coupling between the acoustics and unsteady combustion to effectively enhance the removal of energy in the combustor. In particular, significant modal energy dissipation is achieved via the hybrid resetting controller to suppress thermoacoustic oscillations. The proposed framework is used to design two kinds of hybrid resetting controllers; namely, time-dependent and input/state-dependent resetting controllers. The overall framework demonstrates that hybrid resetting controllers provide an extremely efficient mechanism for dissipating energy in combustion processes.

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2. HYBRID RESETTING CONTROLLERS

In this section we apply the theory of impulsive differential equations [4] to the analysis and design of hybrid resetting controllers. We consider a continuous-time nonlinear plant of the form

$$\dot{x}_{\rm p}(t) = f_{\rm p}(x_{\rm p}(t)) + G_{\rm p}(x_{\rm p}(t))u(t), \quad x(0) = x_0, \quad (1)$$

$$y(t) = h_{\rm p}(x_{\rm p}(t)), \quad (2)$$

where $t \geq 0$, $x_{\rm p}(t) \in \mathbb{R}^{n_{\rm p}}$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^l$, $f_{\rm p} : \mathbb{R}^{n_{\rm p}} \to \mathbb{R}^{n_{\rm p}}$ is Lipschitz continuous and satisfies $f_{\rm p}(0) = 0$, $G_{\rm p} : \mathbb{R}^{n_{\rm p}} \to \mathbb{R}^{n_{\rm p} \times m}$, and $h_{\rm p} : \mathbb{R}^{n_{\rm p}} \to \mathbb{R}^l$ and satisfies $h_{\rm p}(0) = 0$. Furthermore, we consider a hybrid resetting controller of the form

$$\dot{x}_{\rm c}(t) = f_{\rm cc}(x_{\rm c}(t)) + G_{\rm cc}(x_{\rm c}(t))y(t), \quad x_{\rm c}(0) = x_{\rm c0}, \quad (t, x_{\rm c}(t), y(t)) \notin \mathcal{S}_{\rm c}, \quad (3)$$

$$\Delta x_{\rm c}(t) = f_{\rm dc}(x_{\rm c}(t)) + G_{\rm dc}(x_{\rm c}(t))y(t), \quad (t, x_{\rm c}(t), y(t)) \in \mathcal{S}_{\rm c}, \quad (4)$$

$$u(t) = h_{\rm cc}(x_{\rm c}(t)), \quad (5)$$

where $t \geq 0$, $x_{\rm c}(t) \in \mathbb{R}^{n_{\rm c}}$, $f_{\rm cc} : \mathbb{R}^{n_{\rm c}} \to \mathbb{R}^{n_{\rm c}}$ is Lipschitz continuous and satisfies $f_{\rm cc}(0) = 0$, $G_{\rm cc} : \mathbb{R}^{n_{\rm c}} \to \mathbb{R}^{n_{\rm c} \times l}$, $f_{\rm dc} : \mathbb{R}^{n_{\rm c}} \to \mathbb{R}^{n_{\rm c}}$ is continuous and satisfies $f_{\rm dc}(0) = 0$, $G_{\rm dc} : \mathbb{R}^{n_{\rm c}} \to \mathbb{R}^{n_{\rm c} \times l}$, $h_{\rm cc} : \mathbb{R}^{n_{\rm c}} \to \mathbb{R}^m$ and satisfies $h_{\rm cc}(0) = 0$, and $\mathcal{S}_{\rm c} \subset [0, \infty) \times \mathbb{R}^{n_{\rm c}} \times \mathbb{R}^l$ is the *resetting set*. Here, we assume that $u(\cdot)$ is restricted to the class of *admissible* inputs consisting of measurable functions $u(t) \in \mathcal{U}$ for all $t \geq 0$, where the constraint set $\mathcal{U}$ is given with $0 \in \mathcal{U}$. We refer to the differential equation (3) as the *continuous-time controller dynamics*, and we refer to the difference equation (4) as the *resetting control law*. Hence, a hybrid resetting dynamic controller consists of three elements; namely, a continuous-time differential equation, which governs the controller states between resetting events; a difference equation, which governs the way the controller states are instantaneously changed when a resetting event occurs; and a criterion for determining when the controller states are to be reset.

The closed-loop system (1)–(5) is given by

$$\dot{x}(t) = f_{\rm c}(x(t)), \quad (t, x(t)) \notin \mathcal{S}, \quad (6)$$

$$\Delta x(t) = f_{\rm d}(x(t)), \quad (t, x(t)) \in \mathcal{S}, \quad (7)$$

where

$$f_{\rm c}(x) \triangleq \begin{bmatrix} f_{\rm p}(x_{\rm p}) + G_{\rm p}(x_{\rm p})h_{\rm cc}(x_{\rm c}) \\ f_{\rm cc}(x_{\rm c}) + G_{\rm cc}(x_{\rm c})h_{\rm p}(x_{\rm p}) \end{bmatrix}, \quad (8)$$

$$f_{\rm d}(x) \triangleq \begin{bmatrix} 0 \\ f_{\rm dc}(x_{\rm c}) + G_{\rm dc}(x_{\rm c})h_{\rm p}(x_{\rm p}) \end{bmatrix}, \quad (9)$$

$$x \triangleq \begin{bmatrix} x_{\rm p} \\ x_{\rm c} \end{bmatrix} \in \mathbb{R}^n, \quad n = n_{\rm p} + n_{\rm c}, \quad (10)$$

and $\mathcal{S} \triangleq \{(t, x) : (t, x_{\rm c}, h_{\rm p}(x_{\rm p})) \in \mathcal{S}_{\rm c}\}$. Note that the closed-loop system (6), (7) has the form of an impulsive differential equation [4–6]. However, while the closed-loop state vector consists of the plant states and controller states, it is clear from (9) that only those states associated with the controller are reset.

Next, we present a key stability result from [5,6] on feedback interconnections of dissipative impulsive dynamical systems used later in the paper. For this result, we view (1), (2) and (3)–(5) as separate interconnected subsystems with inputs u and $u_{\rm c} \triangleq y$ and outputs y and $y_{\rm c} \triangleq -u$, respectively. Now, a *time-dependent resetting controller* with input $u_{\rm c} = y$ and output $y_{\rm c} = -u$ can be written as (3)–(5) with $\mathcal{S}_{\rm c}$ defined as $\mathcal{S}_{\rm c} \triangleq \mathcal{T} \times \mathbb{R}^{n_{\rm c}} \times \mathcal{U}$, where $\mathcal{T} \triangleq \{t_1, t_2, \ldots\}$ and $0 < t_1 < t_2 < \ldots$ are prescribed resetting times. Alternatively, a *state-dependent resetting controller* with input $u_{\rm c} = y$ and output $y_{\rm c} = -u$ can be written as (3)–(5) with $\mathcal{S}_{\rm c}$ defined as $\mathcal{S}_{\rm c} \triangleq [0, \infty) \times \mathcal{Z}_{\rm c}$, where $\mathcal{Z}_{\rm c} \triangleq \mathcal{Z}_{{\rm c}x_{\rm c}} \times \mathcal{U}$ and $\mathcal{Z}_{{\rm c}x_{\rm c}} \subset \mathbb{R}^{n_{\rm c}}$. More generally, an *input/state-dependent resetting controller* with input $u_{\rm c} = y$ and output $y_{\rm c} = -u$ can be written as (3)–(5) with $\mathcal{S}_{\rm c}$ defined as $\mathcal{S}_{\rm c} \triangleq [0, \infty) \times \mathcal{Z}_{{\rm c}x_{\rm c}} \times \mathcal{Z}_{{\rm c}u_{\rm c}}$, where $\mathcal{Z}_{{\rm c}u_{\rm c}} \subset \mathcal{U}$. For the following definition let $k \in \mathcal{N}_{[t, \hat{t})} \triangleq \{k : t \leq t_k < \hat{t}\}$, where t_k denotes the resetting times of the hybrid controller (3)–(5).

Definition 2.1. A hybrid resetting controller $\mathcal{G}_{\rm c}$ of the form (3)–(5) with $u_{\rm c} \triangleq y$ and $y_{\rm c} \triangleq -u$ is *exponentially passive* if there exists a continuous nonnegative-definite function $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, called a *storage function*, and a scalar $\varepsilon > 0$ such that the *dissipation inequality*

$$e^{\varepsilon T} V_{\rm s}(x_{\rm c}(T)) \leq e^{\varepsilon t_0} V_{\rm s}(x_{\rm c}(t_0)) + \int_{t_0}^{T} 2e^{\varepsilon t} u_{\rm c}^{\rm T}(t) y_{\rm c}(t) {\rm d}t + \sum_{k \in \mathcal{N}_{[t_0, T)}} 2e^{\varepsilon t_k} u_{\rm c}^{\rm T}(t_k) y_{\rm c}(t_k), \quad (11)$$

is satisfied for all t_0, $T \geq 0$, and where $x_{\rm c}(t)$, $t \geq t_0$, is the solution to (3)–(5) with $u_{\rm c} \in \mathbb{R}^l$ and $x_{\rm c}(t_0) = x_{\rm c0}$. A hybrid resetting controller $\mathcal{G}_{\rm c}$ of the form (3)–(5) is *passive* if the dissipation inequality (11) is satisfied with $\varepsilon = 0$.

For the statement of the following theorem recall the standard definitions of passivity and zero-state observability of a nonlinear dynamical system given by (1), (2). Furthermore, we assume all storage functions associated with the plant and the controller are positive definite. See [5] for further details.

Theorem 2.1 [5]. Consider the closed-loop system consisting of the nonlinear dynamical system $\mathcal{G}$ given by (1), (2) and the nonlinear hybrid resetting controller $\mathcal{G}_{\rm c}$ given by (3)–(5). If $\mathcal{S}_{\rm c} = \mathcal{T} \times \mathbb{R}^{n_{\rm c}} \times \mathcal{U}$ or $\mathcal{S}_{\rm c} = [0, \infty) \times \mathcal{Z}_{{\rm c}x_{\rm c}} \times \mathcal{Z}_{{\rm c}u_{\rm c}}$ and $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are passive, then the closed-loop system (6), (7) is Lyapunov stable. If, alternatively, $\mathcal{S}_{\rm c} = \mathcal{T} \times \mathbb{R}^{n_{\rm c}} \times \mathcal{U}$ or $\mathcal{S}_{\rm c} = [0, \infty) \times \mathcal{Z}_{{\rm c}x_{\rm c}} \times \mathcal{Z}_{{\rm c}u_{\rm c}}$ and $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are exponentially passive, then the closed-loop system (1), (2) is asymptotically stable. Finally, if $\mathcal{S}_{\rm c} = [0, \infty) \times \mathcal{Z}_{{\rm c}x_{\rm c}} \times \mathcal{Z}_{{\rm c}u_{\rm c}}$, $\mathcal{G}$ is passive and zero-state observable, $\mathcal{G}_{\rm c}$ is exponentially passive, ${\rm rank}[G_{\rm cc}(0)] = l$, and Assumption 3.1 of [5] holds, then the closed-loop system (6), (7) is asymptotically stable.

3. STATE SPACE MODELING OF COMBUSTION PROCESSES

In order to develop a state space model for combustion processes that capture the coupling between unsteady combustion and acoustics, we consider the mass, momentum, and energy conservation equations for a two phase mixture in a combustor. Specifically, the conservation equations are given by [1]

$$\frac{\partial \rho}{\partial t} + v_{\rm g} \cdot \nabla p = \mathcal{W}, \quad (12)$$

$$\rho \frac{\partial v_{\rm g}}{\partial t} + \rho v_{\rm g} \cdot \nabla v_{\rm g} + \nabla p = \mathcal{F}, \quad (13)$$

$$\frac{\partial p}{\partial t} + \gamma p \nabla \cdot v_{\rm g} + v_{\rm g} \cdot \nabla p = \mathcal{P}, \quad (14)$$

where ρ is the local density of the mixture, $v_{\rm g}$ is the local velocity of the gas phase, p is the local pressure, γ is the mixture ratio of specific heats, $\mathcal{W}$ represents the mass conversion rate of condensed phases to gases per unit volume, $\mathcal{F}$ is the force interaction between the gas and condensed phases, and $\mathcal{P}$ is the sum of heat release associated with chemical reactions and energy transfer between the gas-liquid phase. In this formulation we assume that droplets are dispersed in the gas, which implies that, if p_l and p_g are the local pressures of the liquid and gas phase, respectively, $p_\mathrm{l} << p_\mathrm{g}$, $p \cong p_\mathrm{g}$ and hence ([1]) $p = \rho \bar{R} T_\mathrm{g}$, where $\bar{R}$ is the gas constant for the mixture and T_g is the temperature of the gas.

The framework for analyzing combustion instabilities is based on the conservation equations (12), (13), and (14) for total mass, momentum, and energy, with the energy equation written with the pressure as the dependent variable. Writing all dependent variables as sums of mean $\bar{(\cdot)}$ and fluctuating $(\cdot)'$ parts given by

$$p(r_1,r_2,r_3,t) \ = \ \bar{p} + p'(r_1,r_2,r_3,t), \tag{15}$$
$$\rho(r_1,r_2,r_3,t) \ = \ \bar{\rho}(r_1,r_2,r_3) + \rho'(r_1,r_2,r_3,t), \tag{16}$$
$$v_\mathrm{g}(r_1,r_2,r_3,t) \ = \ \bar{v}_\mathrm{g}(r_1,r_2,r_3) + v'_\mathrm{g}(r_1,r_2,r_3,t), \tag{17}$$
$$T_\mathrm{g}(r_1,r_2,r_3,t) \ = \ \bar{T}_\mathrm{g}(r_1,r_2,r_3) + T'_\mathrm{g}(r_1,r_2,r_3,t), \tag{18}$$

where (r_1, r_2, r_3) represent generalized coordinates, and assuming that the average values $\bar{p}, \bar{\rho}, \bar{v}_\mathrm{g}, \bar{T}_\mathrm{g}$ do not vary with time and the average pressure $\bar{p}$ is uniform inside the combustion chamber, a second-order approximation of (12), (13), (14), and $p = \rho \bar{R} T_\mathrm{g}$ yields

$$\nabla^2 p' - \frac{1}{\bar{a}^2}\frac{\partial^2 p'}{\partial t^2} = \varphi, \qquad \hat{n}\cdot\nabla p' = -\vartheta, \tag{19}$$

where $\bar{a} \triangleq \sqrt{\gamma \frac{\bar{p}}{\bar{\rho}}}$ is the local average sound velocity inside the combustor, $\hat{n}$ is the outward normal vector of the combustor chamber surface, and φ and ϑ are nonlinear terms containing all physical processes of acoustic motions, mean flow, and combustion under conditions with no external forcing [1].

To control combustion instabilities appropriate external forces are needed to influence the unsteady mass, momentum, and energy in the combustion chamber. Hence, control forces are included in the conservation equations by modifying the nonhomogeneous terms of (12), (13), and (14) to include control input terms of the form $\mathcal{W}_\mathrm{c}$, $\mathcal{F}_\mathrm{c}$, and $\mathcal{P}_\mathrm{c}$, respectively. The specific forms of $\mathcal{W}_\mathrm{c}$, $\mathcal{F}_\mathrm{c}$, and $\mathcal{P}_\mathrm{c}$ depend on the type of control actuation used. In this case, (19) becomes

$$\nabla^2 p'(r_1,r_2,r_3,t) - \frac{1}{\bar{a}^2}\frac{\partial^2 p'(r_1,r_2,r_3,t)}{\partial t^2} = \varphi(r_1,r_2,r_3,t) + \varphi_\mathrm{c}(r_1,r_2,r_3,t), \tag{20}$$

$$\hat{n}\cdot\nabla p'(r_1,r_2,r_3,t) = -\vartheta(r_1,r_2,r_3,t) - \vartheta_\mathrm{c}(r_1,r_2,r_3,t), \tag{21}$$

where

$$\varphi_\mathrm{c} \triangleq \nabla\cdot\mathcal{F}'_\mathrm{c} - \frac{1}{\bar{a}^2}\frac{\partial \mathcal{P}'_\mathrm{c}}{\partial t}, \quad \vartheta_\mathrm{c} \triangleq -\mathcal{F}'_\mathrm{c}\cdot\hat{n}, \tag{22}$$

represent external inputs due to control actuation. Since the input terms in (20), (21) are treated as small perturbations to the acoustic field, the solution for the unsteady pressure field $p'(r_1,r_2,r_3,t)$ can be approximated by

$$p'(r_1,r_2,r_3,t) = \bar{p}\sum_{i,j,k=0}^{\infty} \eta_{ijk}(t)\psi_{ijk}(r_1,r_2,r_3), \tag{23}$$

where ψ_{ijk} are the normal modes of the system forming a complete set of orthogonal basis functions satisfying

$$0 \ = \ \nabla^2\psi_{ijk}(r_1,r_2,r_3) + k_{ijk}^2\psi_{ijk}(r_1,r_2,r_3), \tag{24}$$
$$0 \ = \ \hat{n}\cdot\nabla\psi_{ijk}(r_1,r_2,r_3), \quad i,j,k = 0,1,2,\ldots, \tag{25}$$

where k_{ijk}, $i,j,k = 0,1,2,\ldots$, are the wave numbers defined by $k_{ijk} \triangleq \frac{\omega_{ijk}}{\bar{a}}$, and ω_{ijk}, $i,j,k = 0,1,2,\ldots$, are the natural frequencies. Now, using a Galerkin decomposition it follows from (20), (23), and (24) that

$$\ddot{\eta}_{ijk} + \omega_{ijk}^2\eta_{ijk} = F_{ijk} + u_{ijk}, \tag{26}$$

where

$$F_{ijk} \triangleq -\frac{\bar{a}^2}{\bar{p}E_{ijk}^2}\left(\int_{\mathcal{V}_\mathrm{m}} \varphi\psi_{ijk}\mathrm{d}\mathcal{V} + \oint_{\mathcal{S}_\mathrm{m}} \vartheta\psi_{ijk}\mathrm{d}\mathcal{S}\right), \tag{27}$$
$$u_{ijk} \triangleq -\frac{\bar{a}^2}{\bar{p}E_{ijk}^2}\left(\int_{\mathcal{V}_\mathrm{m}} \varphi_\mathrm{c}\psi_{ijk}\mathrm{d}\mathcal{V} + \oint_{\mathcal{S}_\mathrm{m}} \vartheta_\mathrm{c}\psi_{ijk}\mathrm{d}\mathcal{S}\right), \tag{28}$$
$$E_{ijk}^2 \triangleq \int_{\mathcal{V}_\mathrm{m}} \psi_{ijk}^2\mathrm{d}\mathcal{V}, \tag{29}$$

where $\mathcal{V}_\mathrm{m}$ is any arbitrary material volume within the continuum, $\mathcal{S}_\mathrm{m}$ is the surface that encloses $\mathcal{V}_\mathrm{m}$, and $\mathrm{d}\mathcal{V}$ and $\mathrm{d}\mathcal{S}$ are the infinitesimal volume and surface elements, respectively.

Finally, using a one-dimensional combustor model whose geometry is such that the longitudinal modes are decoupled from the transverse modes, it follows that the index i is the only index in the triple i,j,k that applies. Furthermore, we substitute x for the generalized coordinates (r_1,r_2,r_3) so that $\mathcal{V} = \int_0^L \mathcal{A}_\mathrm{c}(x)\mathrm{d}x$, where $\mathcal{A}_\mathrm{c}(x)$ represents the cross sectional area of the combustor and L is the combustor length. In this case, (26) becomes

$$\ddot{\eta}_i(t) + \omega_i^2\eta_i(t) + \sum_{p=1}^{\infty}(d_{ip}\dot{\eta}_p(t) + e_{ip}\eta_p(t)) + \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}(a_{ipq}\dot{\eta}_p(t)\dot{\eta}_q(t) + b_{ipq}\eta_p(t)\eta_q(t)) = u_i(t), \tag{30}$$

where the constants d_{ip}, e_{ip}, a_{ipq}, and b_{ipq} depend on the unperturbed mode shapes and natural frequencies of the combustor [1] and the control input to the i^{th} mode is given by

$$u_i(t) = \frac{\bar{a}^2}{\bar{p}E_i^2}\sum_{a=1}^{m}\hat{u}_a(t)\psi_i(x). \tag{31}$$

Since in typical combustion devices pressure amplitude and phase fluctuations vary slowly with time, we apply the method of time-averaging to approximate the second-order nonlinear dynamical system given by (30) by a corresponding system of two first-order differential equations. In this case, the pressure amplitudes $\eta_i(t)$, $i = 0,1,2,\ldots$, can be written in the form

$$\eta_i(t) \ = \ A_i(t)\sin(\omega_i t) + B_i(t)\cos(\omega_i t), \tag{32}$$

where $A_i(t)$ and $B_i(t)$ are assumed to be slowly time-varying functions. Applying the method of averaging to (30) over an interval τ with $u_i(t) \equiv 0$ yields

$$\dot{A}_i(t) \ = \ \frac{1}{\omega_i\tau}\int_t^{t+\tau} F_i\cos(\omega_i\sigma)\mathrm{d}\sigma, \tag{33}$$
$$\dot{B}_i(t) \ = \ -\frac{1}{\omega_i\tau}\int_t^{t+\tau} F_i\sin(\omega_i\sigma)\mathrm{d}\sigma. \tag{34}$$

For longitudinal vibrational modes with τ taken to be the period of the fundamental mode, that is, $\tau = 2\pi/\omega_1$, it follows that $\omega_i = i\omega_1$ and

$$\dot{A}_i \ = \ \alpha_i A_i + \theta_i B_i + \frac{i\beta}{2}\sum_{p=1}^{i-1}(A_pA_{i-p} - B_pB_{i-p}) - i\beta\sum_{p=1}^{\infty}(A_{i+p}A_p + B_{i+p}B_p), \tag{35}$$
$$\dot{B}_i \ = \ \alpha_i B_i - \theta_i A_i + \frac{i\beta}{2}\sum_{p=1}^{i-1}(A_pB_{i-p} + B_pA_{i-p})$$

$$+i\beta\sum_{p=1}(A_{i+p}B_p - B_{i+p}A_p), \quad (36)$$

where $\alpha_i = -\frac{1}{2}d_{ii}$ represents a growth/decay constant, $\theta_i = -\frac{1}{2}\frac{e_{ii}}{\omega_i}$ represents a frequency shift constant, and $\beta = ((\gamma+1)/8\gamma)\omega_1$. For details of this formulation see [1].

4. HYBRID RESETTING CONTROLLERS FOR COMBUSTION CONTROL

In this section we apply the concepts of active energy flow resetting control to the control of thermoacoustic instabilities in combustion processes. To design hybrid resetting controllers, we associate with the plant a positive-definite, continuously differentiable function $V_{\rm p}(x_{\rm p})$, satisfying $V_{\rm p}(0) = 0$, which we will refer to as the *plant energy*. Furthermore, we associate with the controller a positive-definite, continuously differentiable function $V_{\rm c}(x_{\rm c})$, satisfying $V_{\rm c}(0) = 0$, which we refer to as the *emulated controller energy*. Finally, we associate with the closed-loop system the function $V(x) \triangleq V_{\rm p}(x_{\rm p}) + V_{\rm c}(x_{\rm c})$, which we call the *total energy*.

4.1. Time-Dependent Resetting Controllers

To design hybrid resetting controllers for combustion systems we concentrate on a two-mode, nonlinear time-averaged combustion model with nonlinearities present due to the second-order gas dynamics. Furthermore, we assume that actuation is provided by load speakers while we measure pressure fluctuations via pressure-type microphones. Now, using (35) and (36), a two-mode, time-averaged combustion plant model is given by

$$\dot{x}_{\rm p1}(t) = \alpha_1 x_{\rm p1}(t) + \theta_1 x_{\rm p2}(t) - \beta(x_{\rm p1}(t)x_{\rm p3}(t) + x_{\rm p2}(t)x_{\rm p4}(t)) + u_{\rm s1}(t), \quad x_{\rm p1}(0) = x_{\rm p10}, \quad t \geq 0, \quad (37)$$

$$\dot{x}_{\rm p2}(t) = -\theta_1 x_{\rm p1}(t) + \alpha_1 x_{\rm p2}(t) + \beta(x_{\rm p2}(t)x_{\rm p3}(t) - x_{\rm p1}(t)x_{\rm p4}(t)) + u_{\rm s2}(t), \quad x_{\rm p2}(0) = x_{\rm p20}, \quad (38)$$

$$\dot{x}_{\rm p3}(t) = \alpha_2 x_{\rm p3}(t) + \theta_2 x_{\rm p4}(t) + \beta(x_{\rm p1}^2(t) - x_{\rm p2}^2(t)) + u_{\rm s3}(t), \quad x_{\rm p3}(0) = x_{\rm p30}, \quad (39)$$

$$\dot{x}_{\rm p4}(t) = -\theta_2 x_{\rm p3}(t) + \alpha_2 x_{\rm p4}(t) + 2\beta x_{\rm p1}(t)x_{\rm p2}(t) + u_{\rm s4}(t), \quad x_{\rm p4}(0) = x_{\rm p40}, \quad (40)$$

where $x_{\rm p1} = A_1$, $x_{\rm p2} = B_1$, $x_{\rm p3} = A_2$, $x_{\rm p4} = B_2$, $\alpha_1, \alpha_2, \beta, \theta_1, \theta_2 \in \mathbb{R}$, and $u_{{\rm s}i}$, $i = 1, \ldots, 4$, are added control input signals. For the data parameters $\alpha_1 = 5$, $\alpha_2 = -55$, $\theta_1 = 4$, $\theta_2 = 32$, $\gamma = 1.4$, $\omega_1 = 1\frac{\rm rad}{\rm sec}$, and $x_{\rm p0} = [1\ 1\ 1\ 1]^{\rm T}$, the open-loop ($u_{{\rm s}i}(t) \equiv 0, i = 1, \ldots, 4$) dynamics (37)–(40) result in a limit cycle instability.

To design a stabilizing time-dependent resetting controller for (37)–(40) we first design a control law $u_{\rm s} = -K_{\rm s}x_{\rm p} + u$, where $K_{\rm s} \triangleq {\rm diag}[k_{\rm s1}, k_{\rm s2}, k_{\rm s3}, k_{\rm s4}]$, $x_{\rm p} \triangleq [x_{\rm p1}, x_{\rm p2}, x_{\rm p3}, x_{\rm p4}]^{\rm T}$, $u_{\rm s} \triangleq [u_{\rm s1}, u_{\rm s2}, u_{\rm s3}, u_{\rm s4}]^{\rm T}$, and $u \triangleq [u_1, u_2, u_3, u_4]^{\rm T}$, that renders the dynamical system (37)–(40) passive. In this case, (37)–(40) are given by (1) with

$$f_{\rm p}(x_{\rm p}) = \begin{bmatrix} \alpha_1 x_{\rm p1} + \theta_1 x_{\rm p2} - \beta(x_{\rm p1}x_{\rm p3} + x_{\rm p2}x_{\rm p4}) - k_{\rm s1}x_{\rm p1} \\ -\theta_1 x_{\rm p1} + \alpha_1 x_{\rm p2} + \beta(x_{\rm p2}x_{\rm p3} - x_{\rm p1}x_{\rm p4}) - k_{\rm s2}x_{\rm p2} \\ \alpha_2 x_{\rm p3} + \theta_2 x_{\rm p4} + \beta(x_{\rm p1}^2 - x_{\rm p2}^2) - k_{\rm s3}x_{\rm p3} \\ -\theta_2 x_{\rm p3} + \alpha_2 x_{\rm p4} + 2\beta x_{\rm p1}x_{\rm p2} - k_{\rm s4}x_{\rm p4} \end{bmatrix}, \quad (41)$$

$$G_{\rm p}(x_{\rm p}) = I_4. \quad (42)$$

Now, with $y = x_{\rm p}$, $k_{\rm s1} = k_{\rm s2} = \alpha_1$, and $k_{\rm s3} = k_{\rm s4} = 0$, it follows that (1), (2), with $f_{\rm p}(x_{\rm p})$ and $G_{\rm p}(x_{\rm p})$ given by (41), (42), and $h_{\rm p}(x_{\rm p}) = x_{\rm p}$, is passive with input u, output y, and plant energy, or storage function, $V_{\rm p}(x_{\rm p}) = x_{\rm p1}^2 + x_{\rm p2}^2 + x_{\rm p3}^2 + x_{\rm p4}^2$. Hence, $V_{\rm p}'(x_{\rm p})f_{\rm p}(x_{\rm p}) \leq 0$, $x_{\rm p} \in \mathbb{R}^4$. Furthermore, (1), (2), with $f_{\rm p}(x_{\rm p})$ and $G_{\rm p}(x_{\rm p})$ given by (41), (42), and $h_{\rm p}(x_{\rm p}) = x_{\rm p}$, is zero-state observable.

To improve the performance of the above controller and enhance energy dissipation, we use the flexibility in u to design a hybrid resetting controller. Specifically, consider the hybrid controller emulating the plant structure given by (3)–(5), with $\mathcal{S}_{\rm c} = \mathcal{T} \times \mathbb{R}^{n_{\rm c}} \times \mathbb{R}^l$, and

$$f_{\rm cc}(x_{\rm c}) = \begin{bmatrix} \alpha_1 x_{\rm c1} + \theta_1 x_{\rm c2} - \beta(x_{\rm c1}x_{\rm c3} + x_{\rm c2}x_{\rm c4}) - k_{\rm c1}x_{\rm c1} \\ -\theta_1 x_{\rm c1} + \alpha_1 x_{\rm c2} + \beta(x_{\rm c2}x_{\rm c3} - x_{\rm c1}x_{\rm c4}) - k_{\rm c2}x_{\rm c2} \\ \alpha_2 x_{\rm c3} + \theta_2 x_{\rm c4} + \beta(x_{\rm c1}^2 - x_{\rm c2}^2) - k_{\rm c3}x_{\rm c3} \\ -\theta_2 x_{\rm c3} + \alpha_2 x_{\rm c4} + 2\beta x_{\rm c1}x_{\rm c2} - k_{\rm c4}x_{\rm c4} \end{bmatrix}, \quad (43)$$

$$f_{\rm dc}(x_{\rm c}) = \begin{bmatrix} -x_{\rm c1} \\ -x_{\rm c2} \\ -x_{\rm c3} \\ -x_{\rm c4} \end{bmatrix}, \quad G_{\rm cc}(x_{\rm c}) = I_4, \quad G_{\rm dc}(x_{\rm c}) = 0, \quad (44)$$

$$h_{\rm cc}(x_{\rm c}) = -[\, x_{\rm c1},\ x_{\rm c2},\ x_{\rm c3},\ x_{\rm c4}]^{\rm T}, \quad (45)$$

where $k_{\rm c1} > \alpha_1$, $k_{\rm c2} > \alpha_1$, $k_{\rm c3} > \alpha_2$, and $k_{\rm c4} > \alpha_2$. It can be easily shown using Corollary 5.1 and Remark 5.3 of [5] that the hybrid resetting controller (3)–(5), with dynamics given by (43)–(45), resetting set $\mathcal{S}_{\rm c} = \mathcal{T} \times \mathbb{R}^{n_{\rm c}} \times \mathbb{R}^l$, input y, and output $-u$, is exponentially passive with emulated controller energy, or storage function, $V_{\rm c}(x_{\rm c}) = x_{\rm c1}^2 + x_{\rm c2}^2 + x_{\rm c3}^2 + x_{\rm c4}^2$. Hence, $V_{\rm c}'(x_{\rm c})f_{\rm cc}(x_{\rm c}) \leq -\varepsilon V_{\rm c}(x_{\rm c})$, $x_{\rm c} \in \mathbb{R}^4$, where $\varepsilon = \min\{\alpha_1 - k_{\rm c1}, \alpha_1 - k_{\rm c2}, \alpha_2 - k_{\rm c3}, \alpha_2 - k_{\rm c4}\}$. Furthermore, note that rank $[G_{\rm cc}(0)] = 4$. Hence, stability of the closed-loop system (1)–(5) is guaranteed by Theorem 2.1. Finally, we note that the total energy of the closed-loop system (1)–(5) is given by

$$V(x) = x_{\rm p1}^2 + x_{\rm p2}^2 + x_{\rm p3}^2 + x_{\rm p4}^2 + x_{\rm c1}^2 + x_{\rm c2}^2 + x_{\rm c3}^2 + x_{\rm c4}^2. \quad (46)$$

The effect of the resetting law (4) with $f_{\rm dc}(x_{\rm c})$ and $G_{\rm dc}(x_{\rm c})$ given by (44), is to cause all controller states to be instantaneously reset to zero; that is, the resetting law (4) implies $V_{\rm c}(x_{\rm c} + \Delta x_{\rm c}) = 0$. The closed-loop resetting law is thus given by

$$\Delta x = f_{\rm d}(x) = [\ 0\ 0\ 0\ 0\ -x_{\rm c1}\ -x_{\rm c2}\ -x_{\rm c3}\ -x_{\rm c4}]^{\rm T}. \quad (47)$$

Note that since

$$x + \Delta x = \begin{bmatrix} x_{\rm p1} & x_{\rm p2} & x_{\rm p3} & x_{\rm p4} & 0 & 0 & 0 & 0 \end{bmatrix}^{\rm T}, \quad (48)$$

it follows that

$$V(x + \Delta x) = V_{\rm p}(x_{\rm p}), \quad (49)$$

and

$$V(x + \Delta x) - V(x) = -V_{\rm c}(x_{\rm c}) \leq 0. \quad (50)$$

Now, from (50) it follows that the resetting law (4) causes the total energy to instantaneously decrease by an amount equal to the accumulated controller energy.

To illustrate the dynamic behaviour of the closed-loop system, let $\alpha_1 = 5$, $\alpha_2 = -55$, $k_{\rm s1} = \alpha_1$, $k_{\rm s2} = \alpha_1$, $k_{\rm s3} = 0$, $k_{\rm s4} = 0$, $k_{\rm c1} = \alpha_1 + 0.1$, $k_{\rm c2} = \alpha_1 + 0.1$, $k_{\rm c3} = 0$, $k_{\rm c4} = 0$, and $\mathcal{T} = \{2, 4, 6, \ldots\}$, so that the controller resets periodically with a period of 2 seconds. The response of the controlled system (1), (2) with the hybrid resetting controller (3)–(5) and initial condition $x_0 = [1\ 1\ 1\ 1\ 0\ 0\ 0\ 0]^{\rm T}$ is shown in Figure 1. Note that the control force versus time is discontinuous at the resetting times. A comparison of the plant energy, emulated control energy, and total energy is given in Figure 2.

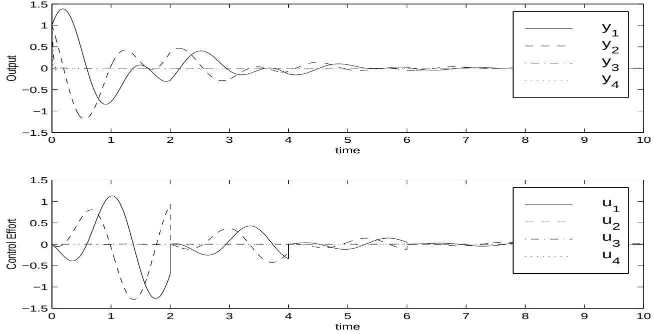

Figure 1: Time-dependent resetting controller: Output and control effort versus time

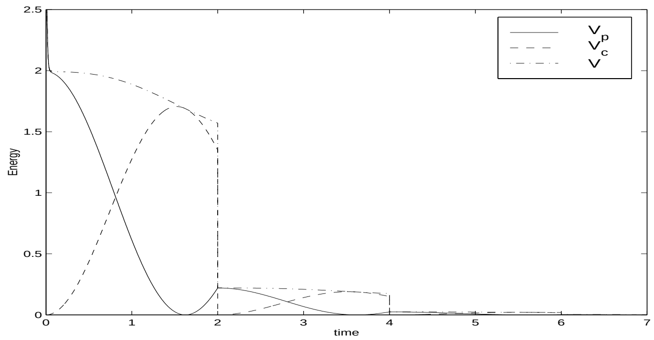

Figure 2: Time-dependent resetting controller: Plant, controller, and total energy

In this example the resetting times were chosen arbitrarily. However, with the same choice of controller parameters we can choose a resetting time to achieve finite-time stabilization. Specifically, in this case the resetting time would correspond to the time at which all of the energy of the plant is drawn to the controller. This resetting time can be obtained from the energy history of the closed-loop system without resetting. In particular, the time instant when the plant and controller interchange energies such that plant energy is at zero will correspond to the resetting time that achieves finite-time stabilization. For this example, finite-time stability is achieved by choosing the resetting instant at $t = 1.6223$ sec.

4.2. Input/State-Dependent Resetting Controllers

In this subsection we describe the mathematical setting and design of an input/state-dependent resetting controller. We consider the plant and hybrid resetting controller as described in Subsection 4.1 with $\mathcal{S}_\mathrm{c} = [0, \infty) \times \mathcal{Z}_{\mathrm{c}x_\mathrm{c}} \times \mathcal{Z}_{\mathrm{c}y}$, where

$$\mathcal{Z}_{\mathrm{c}x_\mathrm{c}} \times \mathcal{Z}_{\mathrm{c}y} \triangleq \{(x_\mathrm{c}, y) : f_{\mathrm{dc}}(x_\mathrm{c}) \neq 0 \quad \text{and} \quad V_\mathrm{c}'(x_\mathrm{c})[f_{\mathrm{cc}}(x_\mathrm{c}) + G_{\mathrm{cc}}(x_\mathrm{c})y] \leq 0\}. \quad (51)$$

(Note that since $u_\mathrm{c} = y$, $\mathcal{Z}_{\mathrm{c}u_\mathrm{c}} = \mathcal{Z}_{\mathrm{c}y}$.) The resetting set (51) is thus defined to be the set of all controller state and input points that represent non-increasing controller energy while the controller is in the loop, except for state points that satisfy $f_{\mathrm{dc}}(x_\mathrm{c}) = 0$. As mentioned in Remark 2.7 of [5], the states x_c that satisfy $f_{\mathrm{dc}}(x_\mathrm{c}) = 0$ are states that do not change under the action of the resetting law, and thus we need to exclude these states from the resetting set to ensure that the Assumption A2 of [5] is not violated. Furthermore, since the hybrid resetting controller given in Subsection 4.1 is exponentially passive for $\mathcal{S}_\mathrm{c} = [0, \infty) \times \mathbb{R}^{n_\mathrm{c}} \times \mathbb{R}^l$, it follows that the hybrid resetting controller is exponentially passive for $\mathcal{S}_\mathrm{c} = [0, \infty) \times \mathcal{Z}_{\mathrm{c}x_\mathrm{c}} \times \mathcal{Z}_{\mathrm{c}y}$. Hence, asymptotic stability of the closed-loop system (1)–(5) is guaranteed by Theorem 2.1.

To illustrate the dynamic behaviour of the closed-loop system we again choose $\alpha_1 = 5$, $\alpha_2 = -55$, $k_{\mathrm{s}1} = \alpha_1$, $k_{\mathrm{s}2} = \alpha_1$, $k_{\mathrm{s}3} = 0$, $k_{\mathrm{s}4} = 0$, $k_{\mathrm{c}1} = \alpha_1 + 0.1$, $k_{\mathrm{c}2} = \alpha_1 + 0.1$, $k_{\mathrm{c}3} = 0$, and $k_{\mathrm{c}4} = 0$, with initial condition $x_0 = [1\ 1\ 1\ 1\ 0\ 0\ 0\ 0]^\mathrm{T}$. The response of the controlled system (1), (2), with dynamics (41), (42), and $h_\mathrm{p}(x_\mathrm{p}) = x_\mathrm{p}$, and the input/state-dependent resetting controller given by (3)–(5) with dynamics (43)–(45) and resetting set (51) is given in Figure 3. The total energy, plant energy, and emulated controller energy versus time are shown in Figure 4. Note that the proposed input/state-dependent resetting controller achieves finite-time stabilization.

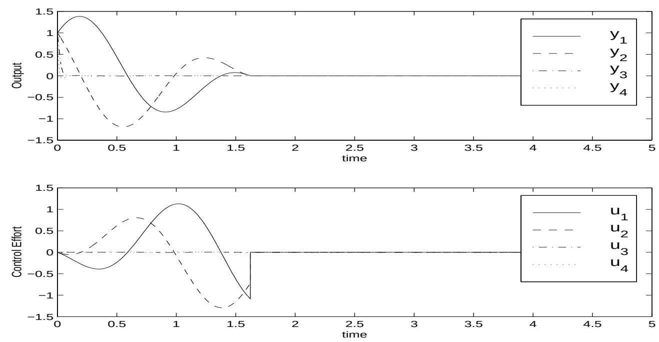

Figure 3: Input/state-dependent resetting controller: Output and control effort versus time

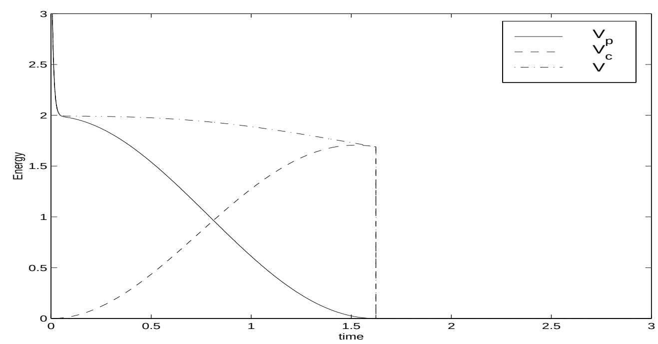

Figure 4: Input/state-dependent resetting controller: Plant, controller, and total energy

References

[1] F. E. C. Culick, “Combustion Instabilities in Liquid-Fueled Propulsion Systems - An Overview,” *AGARD Conference Proceedings*, no. 450, 1988.

[2] R. T. Bupp, D. S. Bernstein, V. Chellaboina, and W. M. Haddad, “Resseting Virtual Absorbers for Vibration Control,” *J. Vibr. Contr.*, vol. 6, pp. 61–83, 2000.

[3] Y. Kishimoto, D. S. Bernstein, and S. R. Hall, “Energy Flow Control of Interconnected Structures: I. Modal Subsystems,” *Contr. -Thy. and Adv. Tech.*, vol. 10, no. 4, pp. 1563–1590, 1995.

[4] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*. Singapore, World Scientific, 1989.

[5] W. M. Haddad, V. Chellabonia, and N. A. Kablar, “Nonlinear Impulsive Dynamical Systems Part I: Stability and Dissipativity,” *Proc. IEEE Conf. Dec. Contr.*, pp. 4404–4422, Phoenix, AZ, December 1999.

[6] W. M. Haddad, V. Chellabonia, and N. A. Kablar, “Nonlinear Impulsive Dynamical Systems Part II: Feedback Interconnections and Optimality,” *Proc. IEEE Conf. Dec. Contr.*, pp. 5225–5234, Phoenix, AZ, December 1999.