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# ROBUST STABILITY OF POLYNOMIALS WITH POLYNOMIC STRUCTURE OF COEFFICIENTS*

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**Abstract.** This paper deals with robust stability analysis of polynomials where uncertain coefficients are polynomic functions of the second order of interval parameters. The method consists in determination of a convex hull overbounding the value set of such a family and using Zero Exclusion Theorem. An arbitrary stability region can be chosen. Presented method states only sufficient condition of robust stability due to nonconvexity of the value set. Both computational and graphical way of using this method are possible. The computational efficiency of presented method and more general method based on Sign-decomposition is compared.

**Keywords.** robust stability, polynomials, polynomic uncertainty, value set

## 1. INTRODUCTION

Robust stability of linear systems with parametric uncertainty has been strongly developing in past two decades since the well-known Kharitonov theorem [5] have been published. This theorem solves the robust stability for interval polynomials. The next significant step has been achieved by solving of robust stability for polynomials with parametric uncertainty of affine structure by Edge Theorem [1]. In the last few years polynomials with multilinear and polynomic structure of coefficients are of great interest. The robust stability analysis of those polynomials appears to be too complicated for statement of a simple solution.

Nowadays there are only few approaches which treat this problem generally. One of those is the technique using Bernstein Expansion [4]. This method states a necessary and sufficient condition for Hurwitz stability of polynomial with polynomic parameter dependency by checking the Hurwitz determinant for positivity using the Bernstein iterative algorithm.

An interesting method for the analysis of robust stability of polynomic interval polynomials was introduced in [3] for continuous case and in [2] for discrete case. Both methods state a necessary and sufficient condition on the coefficient space using the Modified Routh and the Modified Jury table respectively. Positivity of elements of both tables is tested by Sign-decomposition.

The traditional tool for robust stability analysis consists in using Zero Exclusion Principle. This principle makes it possible to replace multidimensional problem in the coefficient space by two-dimensional analysis of the value set in the complex plane. Surprisingly simple result on robust stability of multilinear interval polynomials based on this principle is given by the Mapping Theorem [7]. This theorem states that the value set of such a family of polynomials evaluated in an arbitrary point in the complex plane is contained in the convex hull of the value set of vertices polynomials. This theorem gives only sufficient condition for robust stability but an arbitrary stability region can be chosen. However the Mapping Theorem cannot be used for polynomic parameter dependency.

A method using Zero Exclusion Principle is described in [8]. This method combines Bernstein algorithm and the analysis of the value set to state a necessary and sufficient condition for Hurwitz robust stability of a polynomial with polynomic structure of its coefficients.

An efficient method analyzing robust stability of polynomials with uncertain coefficients being polynomic functions of the second order of interval parameters is presented in this paper. A sufficient condition is derived by overbounding the (nonconvex) value set by a convex hull for an arbitrary point in the complex plane lying on the boundary of chosen stability region and by determination whether zero is excluded or included. This test can be done either in computational or in graphical way. Profitting from appropriate properties of presented procedure the former is recommended especially for high number of parameters. This method can be used in principle for polynomials where the coefficients are polynomic functions of higher order than two but in such cases numerical algorithms for determining the zero points of multivariate polynomic functions have to be used and whole procedure becomes very complicated.

The main advantage of presented method consists in its high computational efficiency which is shown by comparison with the method derived in [2].

Even if the presented method can be used in practice for stability analysis of polynomials with coefficients being polynomic functions of the second order only, it has great importance for example for stability analysis of Takagi-Sugeno fuzzy systems.

## 2. PRELIMINARIES

First of all it is necessary to introduce the concept of multilinear and polynomic interval polynomials.

Let us consider a family of polynomials whose coefficients are polynomic functions of $\mathbf{q} \in Q \subset \Re^l$ in the form:

$$P(s,\mathbf{q}) = c_n(\mathbf{q})s^n + \cdots + c_1(\mathbf{q})s + c_0(\mathbf{q}) \tag{1}$$

where $s$ is not necessarily the Laplace operator and

$$\begin{aligned} &\mathbf{q} = [q_1 \quad q_2 \quad \cdots \quad q_l]^T \\ &q_i \in [q_i^-, q_i^+] \quad i = 1,\ldots,l \end{aligned} \tag{2}$$

is an interval (vector) parameter.
Let us suppose that each coefficient $c_k(\mathbf{q})$ can be written as

$$\begin{aligned} c_k(\mathbf{q}) = {} & \beta_k + \alpha_{k1}q_1 + \alpha_{k2}q_2 + \cdots + \alpha_{kl}q_l + \\ & + \left(\gamma_{k11}^1 q_1 + \cdots + \gamma_{kl1}^1 q_l\right) q_1 + \\ & + \left(\gamma_{k12}^1 q_1 + \cdots + \gamma_{kl2}^1 q_l\right) q_1^2 + \cdots \\ & + \left(\gamma_{k1m}^1 q_1 + \cdots + \gamma_{klm}^1 q_l\right) q_1^m + \cdots \\ & + \left(\gamma_{k1m}^l q_1 + \cdots + \gamma_{klm}^l q_l\right) q_l^m \\ k = {} & 0,\ldots,n \end{aligned} \tag{3}$$

Then the polynomial $P(s,\mathbf{q})$ is referred to as a polynomic interval polynomial. For $m$=1 for all $k$=0,...,$n$ ($c_j(\mathbf{q})$ are polynomic functions of the second order) the polynomial $P(s,\mathbf{q})$ is said to be a polynomic interval polynomial of the second order (of degree $n$). If $c_k(\mathbf{q})$ for all $k$=0,...,$n$ are multilinear functions (i.e. if for all $i$=1,...,$l$ $c_k(\mathbf{q})$ are linear functions of $q_i$ when the $q_r$, $r \neq i$ are held constant), the polynomial $P(s,\mathbf{q})$ is said to be a multilinear interval polynomial.

In the rest of this paper if $\mathbf{A} \in \Re^{l,l}$ is a ($l \times l$) matrix then $A_{ij}$ denotes the element of $\mathbf{A}$ lying on the position ($i,j$), if $\mathbf{b} \in \Re^l$ is a vector then $b_i$ denotes the element of $\mathbf{b}$ lying on the $i$-th position.

## 3. ALGORITHM

Let

$$\begin{aligned} &P(s,\mathbf{q}) = c_n(\mathbf{q})s^n + \cdots + c_1(\mathbf{q})s + c_0(\mathbf{q}) \\ &\mathbf{q} \in Q \subset \Re^l, \ \mathbf{q} = [q_1 \quad q_2 \quad \cdots \quad q_l]^T \\ &Q = [q_1^-, q_1^+] \times \cdots \times [q_l^-, q_l^+] \\ &q_i \in [q_i^-, q_i^+], \ q_i^- < q_i^+ \quad i = 1,\ldots,l \end{aligned} \tag{4}$$

be a polynomic interval polynomial of the second order. To avoid dropping in degree, $c_n(\mathbf{q}) \neq 0$ for all $\mathbf{q} \in Q$ is assumed. Then each coefficient $c_k(\mathbf{q})$ can be expressed as

$$\begin{aligned} &c_k(\mathbf{q}) = \mathbf{q}^T \mathbf{C}^{(k)} \mathbf{q} + \left(\mathbf{d}^{(k)}\right)^T \mathbf{q} + v^{(k)} \\ &\mathbf{C}^{(k)} \in \Re^{l,l}, \ \mathbf{d}^{(k)} \in \Re^l, \ v^{(k)} \in \Re, \ k = 0,\ldots,n \end{aligned} \tag{5}$$

Presented method deals with the value set of $P(s,\mathbf{q})$ evaluated at some complex point $s = s_0 = |s_0| e^{j\psi_0}$. The image $P(s_0,\mathbf{q})$ can be expressed as

$$\begin{aligned} P(s_0,\mathbf{q}) &= c_n(\mathbf{q})s_0^n + \cdots + c_1(\mathbf{q})s_0 + c_0(\mathbf{q}) \\ &= c_{\mathrm{Re}}^{s_0}(\mathbf{q}) + j.c_{\mathrm{Im}}^{s_0}(\mathbf{q}) \end{aligned} \tag{6}$$

where $c_{\mathrm{Re}}^{s_0}(\mathbf{q}), c_{\mathrm{Im}}^{s_0}(\mathbf{q})$ are polynomic functions of the second order and are given by

$$c_{\mathrm{Re}}^{s_0}(\mathbf{q}) = \sum_{k=0}^{n} c_k(\mathbf{q}) |s_0|^k \cos(k\psi_0)$$
$$c_{\mathrm{Im}}^{s_0}(\mathbf{q}) = \sum_{k=0}^{n} c_k(\mathbf{q}) |s_0|^k \sin(k\psi_0) \tag{7}$$

Our task is to determine the minimum and maximum distances $h_{\min}^{s_0}(\varphi), h_{\max}^{s_0}(\varphi)$ of the point [0,*j*0] from the set $P(s_0,\mathbf{q})$ in the complex plane in some direction $\varphi$, $\varphi\in[0,\pi]$ respectively (see Fig.1).

**Remark 1**: It has to be noted that the distance is measured from the point [0,*j*0] in the direction $\varphi$, $\varphi\in[0,\pi]$. It means that the distance can be negative (in such a case the distance is measured from the point [0,*j*0] in the direction $\pi+\varphi$).

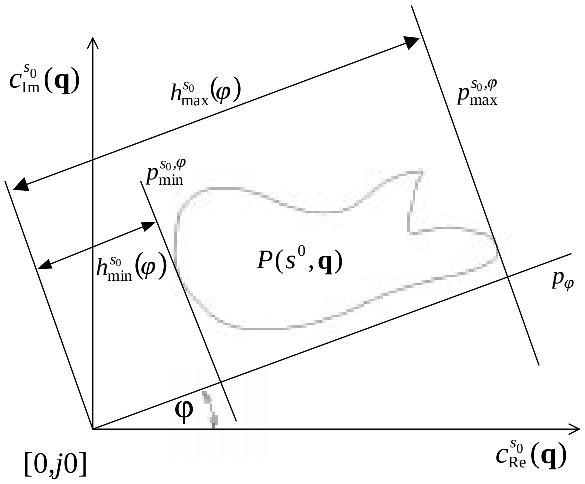


Fig. 1 Minimum and maximum distance of $P(s_0,\mathbf{q})$ in a direction $\varphi$

It can be easily shown that finding the minimum and maximum distances is equivalent to finding the minimum and maximum value of the function $c_\varphi^{s_0}(\mathbf{q})$, $\mathbf{q}\in Q$

$$\begin{aligned} c_\varphi^{s_0}(\mathbf{q}) &= c_{\mathrm{Re}}^{s_0}(\mathbf{q})\cos(\varphi) + c_{\mathrm{Im}}^{s_0}(\mathbf{q})\sin(\varphi) \\ &= \left[c_{\mathrm{Re}}^{s_0}(\mathbf{q}), c_{\mathrm{Im}}^{s_0}(\mathbf{q})\right].[\cos(\varphi),\sin(\varphi)]^T \\ &= \left[c_{\mathrm{Re}}^{s_0}(\mathbf{q}), c_{\mathrm{Im}}^{s_0}(\mathbf{q})\right].\mathbf{g}(\varphi) \end{aligned} \tag{8}$$

From (9) follows that $c_\varphi^{s_0}(\mathbf{q})$ is a polynomic function of the second order of $\mathbf{q}$. It means that $c_\varphi^{s_0}(\mathbf{q})$ is bounded and $h_{\min}^{s_0}(\varphi), h_{\max}^{s_0}(\varphi)$ are finite.

The problem of finding extreme values of $c_\varphi^{s_0}(\mathbf{q})$ on a box $Q$ is a task of mathematical programming. General formulation of a task of mathematical programming is as follows.

Let us consider the problem of minimization of a function $f_0(\mathbf{x})$, where the constraints are given in the form of inequalities

$$\min\left\{f_0(\mathbf{x}) \middle| f_j(\mathbf{x}) \le b_j,\ j = 1,\ldots,m\right\} \tag{9}$$

Necessary conditions of extreme values can be determined by the following theorem.

**Definition 1**: Let a point ${}^{\mathbf{0}}\mathbf{x}$ satisfy all constraints of (9). Let $J({}^{\mathbf{0}}\mathbf{x})$ be the set of indices, for which the corresponding constraints are active (e.g. inequality changes to equality):

$$J\left({}^{\mathbf{0}}\mathbf{x}\right) = \left\{j \middle| f_j\left({}^{\mathbf{0}}\mathbf{x}\right) = b_j\right\} \tag{10}$$

The point ${}^{\mathbf{0}}\mathbf{x}$ is said to be a regular point of the set $X$ given by constraints in (10), if the gradients $\nabla f_j({}^{\mathbf{0}}\mathbf{x})$ are linearly independent $\forall j\in J({}^{\mathbf{0}}\mathbf{x})$.

**Theorem 1** [6] Let ${}^*\mathbf{x}$ be a regular point of a set $X$ and a function $f_0(\mathbf{x})$ has in some neighbourhood of ${}^*\mathbf{x}$ continuous first partial derivatives. If the function $f_0(\mathbf{x})$ has in the point ${}^*\mathbf{x}$ the local minimum on $X$, then there exists a (Lagrange) vector ${}^*\boldsymbol{\lambda}\in\Re^m$ that

$$\begin{aligned} \nabla f_0\left({}^*\mathbf{x}\right) + \sum_{r=1}^{m} {}^*\lambda_r \nabla f_r\left({}^*\mathbf{x}\right) &= 0 \\ {}^*\lambda_j\left(f_j\left({}^*\mathbf{x}\right) - b_j\right) &= 0 \\ \lambda_j^* &\ge 0 \end{aligned} \tag{11}$$

hold $\forall j=1,\ldots,m$.

**Remark 2**: For maximization of a function $f_0(\mathbf{x})$ the last inequality of (11) is replaced by ${}^*\lambda_j \le 0$.

To apply Theorem 1 for solving our problem it is necessary to check whether the preconditions of this theorem are satisfied. As $c_\varphi^{s_0}(\mathbf{q})$ is a polynomic function of the second order, its first partial derivatives are continuous $\forall\mathbf{q}\in Q$ and the second assumption is satisfied. In our case

$$\begin{aligned} &f_0(\mathbf{q}) = c_\varphi^{s_0}(\mathbf{q}) \\ &f_j(\mathbf{q}) = (-1)^{j+1} q_i,\quad i = 1,\ldots,l \quad \mathrm{j} = 1,\ldots,2l \\ &\qquad\qquad j = 2i\text{-}1, 2i \\ &b_j = -q_i^- \text{ for } j \text{ even} \\ &b_j = q_i^+ \text{ for } j \text{ odd} \end{aligned} \tag{12}$$

Then

$$\nabla f_j(\mathbf{q}) = (-1)^{j+1}\mathbf{e}^{(i)} \quad \forall \mathbf{q} \in Q,\ j = 1,\dots,2l$$
$$i = \frac{j+1}{2} \text{ for } j \text{ odd},\ i = \frac{j}{2} \text{ for } j \text{ even} \tag{13}$$

where $\mathbf{e}^{(i)}=[0,...,0,1,0,...,0]^T$ with 1 on the $i$-th position. Because for some $\mathbf{q}\in Q$ only even or only odd constraints (or none of them) can be active $(q_i^- < q_i^+)\ \forall i = 1,\dots,l$, $\nabla f_j(\mathbf{q})$ are linearly independent $\forall \mathbf{q}\in Q$, $j\in J(\mathbf{q})$. It means that all points $\mathbf{q}\in Q$ are regular ones.

According to Theorem 1 it is necessary to determine the gradient $\nabla c_\varphi^{s_0}(\mathbf{q})$. From (8)

$$\nabla c_\varphi^{s_0}(\mathbf{q}) = \left[\nabla c_{\mathrm{Re}}^{s_0}(\mathbf{q}), \nabla c_{\mathrm{Im}}^{s_0}(\mathbf{q})\right].g(\varphi) \tag{14}$$

The components of $\nabla c_k(\mathbf{q})$

$$\nabla c_k(\mathbf{q}) = \left[\frac{\partial c_k(\mathbf{q})}{\partial q_1},\dots,\frac{\partial c_k(\mathbf{q})}{\partial q_l}\right]^T \tag{15}$$

follows from (5)

$$\frac{\partial c_k(\mathbf{q})}{\partial q_i} = 2C_{ii}^{(k)}q_i + \sum_{\substack{r=1\\ r\neq i}}^{l}\left(C_{ir}^{(k)} + C_{ri}^{(k)}\right)q_r$$
$$k = 0,\dots,n \quad i = 1,\dots,l \tag{16}$$

From (7)

$$\nabla c_{\mathrm{Re}}^{s_0}(\mathbf{q}) = \sum_{k=0}^{n}\nabla c_k(\mathbf{q})|s_0|^k\cos(k\psi_0)$$
$$\nabla c_{\mathrm{Im}}^{s_0}(\mathbf{q}) = \sum_{k=0}^{n}\nabla c_k(\mathbf{q})|s_0|^k\sin(k\psi_0) \tag{17}$$

After substituting (12), (13), (14), (15), (16) and (17) to (11) the following system of equations and inequalities is obtained:

$$\left[\begin{array}{ccc|ccccc} W_{11} & \cdots & W_{1l} & 1 & -1 & 0 & \cdots & \cdots \\ \vdots & \ddots & \vdots & 0 & 0 & 1 & -1 & \cdots \\ & \ddots & & \vdots & \ddots & & \ddots & \ddots \\ W_{l1} & \cdots & W_{ll} & 0 & \cdots & 0 & 1 & -1 \end{array}\right] . \begin{bmatrix} q_1 \\ \vdots \\ q_l \\ \lambda_1 \\ \vdots \\ \lambda_{2l} \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ \vdots \\ w_l \end{bmatrix} \tag{i}$$

$$\begin{aligned} \lambda_1(q_1 - q_1^+) &= 0 \\ \lambda_2(-q_1 - q_1^-) &= 0 \\ \lambda_3(q_2 - q_2^+) &= 0 \\ \lambda_4(-q_2 - q_2^-) &= 0 \\ &\vdots \\ \lambda_{2l-1}(q_l - q_l^+) &= 0 \\ \lambda_{2l}(-q_l - q_l^-) &= 0 \end{aligned} \tag{ii}$$

(18)

$$\lambda_1,\dots,\lambda_{2l} \geq 0 \text{ for minimization}$$
$$\lambda_1,\dots,\lambda_{2l} \leq 0 \text{ for maximization}$$

where

$$W_{uv} = \left[\sum_{k=0}^{n}\left(C_{uv}^{(k)} + C_{vu}^{(k)}\right).|s_0|^k\cos(k\psi_0)\right].\cos(\varphi) + \left[\sum_{k=0}^{n}\left(C_{uv}^{(k)} + C_{vu}^{(k)}\right).|s_0|^k\sin(k\psi_0)\right].\sin(\varphi)$$
$$w_u = \left[\sum_{k=0}^{n}d_u^{(k)}.|s_0|^k\cos(k\psi_0)\right].\cos(\varphi) + \left[\sum_{k=0}^{n}d_u^{(k)}.|s_0|^k\sin(k\psi_0)\right].\sin(\varphi)$$
$$u,v = 1,\dots,l$$

The important fact is that the equation (18-i) is linear. The computational procedure of solving (18) runs as follows. At first all solutions of (18-ii) (nonlinear) are determined. This corresponds to determining of all the parts of the box $Q$ – the interior and all the parts of the boundary of $Q$ (all manifolds with the dimension $i$, $i$=0,...,$l$-1 containing only points on the boundary of $Q$). Each solution of (18-ii) corresponds to $2l$ linear equations (from (18-ii) it follows that at least one of $\lambda_{2i-1}$, $\lambda_{2i}$, $\forall i$=1,...,$l$ has to equal zero; if $\lambda_{2i-1}$=0 then either $\lambda_{2i}$=0 or $q_i=-q_i^-$, if $\lambda_{2i}$=0 then either $\lambda_{2i-1}$=0 or $q_i=q_i^+$ $\forall i$=1,...,$l$). These $2l$ equations together with $l$ equations of (18-i) form $3l$ linearly independent linear equations for $3l$ unknown variables. It means that there exists a unique solution $({}^*\boldsymbol{\lambda},{}^*\mathbf{q})$ (for each solution of (18-ii)) of (18). As the number of manifolds with the dimension $i$, $i$=0,...,$l$-1 containing only points on the boundary of $Q$ is $2^i\binom{l}{i}$, the total number $n_s$ of all solutions of (18) is given by

$$n_s = \sum_{i=0}^{l}2^i\binom{l}{i} = 3^l \tag{19}$$

In the next step it is checked whether ${}^*\lambda_j^{(t)} \geq 0$ $\left({}^*\lambda_j^{(t)} \leq 0\right)$ $\forall j = 1,\dots,2l$ and ${}^*\mathbf{q}^{(t)} \in Q$ for each $t = 1,\dots,n_s$. Denote by $T_{\min}$ ($T_{\max}$) the set of $t$ for which these conditions are satisfied.

$$T_{\min} = \left\{t: {}^*\mathbf{q}^{(t)} \in Q, {}^*\lambda_j^{(t)} \geq 0\ \forall j = 1,...,2l\right\}$$
$$T_{\max} = \left\{t: {}^*\mathbf{q}^{(t)} \in Q, {}^*\lambda_j^{(t)} \leq 0\ \forall j = 1,...,2l\right\} \tag{20}$$

Then

$$
\begin{aligned}
h_{\min}^{s_0}(\varphi) &= \min_{t \in T_{\min}} \left[ c_\varphi^{s_0}\left( {}^*\mathbf{q}^{(t)} \right) \right] \\
h_{\max}^{s_0}(\varphi) &= \max_{t \in T_{\max}} \left[ c_\varphi^{s_0}\left( {}^*\mathbf{q}^{(t)} \right) \right]
\end{aligned}
\tag{21}
$$

The minimum and maximum distances indicate that the set $P(s_0,\mathbf{q})$ lies in the complex plane in the space between the lines $p_{\min}^{s_0,\varphi}$ and $p_{\max}^{s_0,\varphi}$:

$$
\begin{aligned}
p_{\min}^{s_0,\varphi} &: \; c_{\mathrm{Im}}^{s_0}(\mathbf{q}) = -\frac{1}{\tan(\varphi)} c_{\mathrm{Re}}^{s_0}(\mathbf{q}) + \frac{h_{\min}^{s_0}(\varphi)}{\sin(\varphi)} \\
p_{\max}^{s_0,\varphi} &: \; c_{\mathrm{Im}}^{s_0}(\mathbf{q}) = -\frac{1}{\tan(\varphi)} c_{\mathrm{Re}}^{s_0}(\mathbf{q}) + \frac{h_{\max}^{s_0}(\varphi)}{\sin(\varphi)}
\end{aligned}
\tag{22}
$$

To determine a convex hull overbounding the set $P(s_0,\mathbf{q})$ $\mathbf{q} \in Q$, the procedure described above is performed for a set of $\varphi_r \in \Phi$,

$$
\Phi = \left\{ \begin{matrix} \varphi_r : 0 \le \varphi_1 \le \cdots \le \varphi_{R-1} \le \varphi_R \le \pi, \\ r = 1, \ldots, R \end{matrix} \right\}
\tag{23}
$$

(it means that the system (18) is solved for a set of $\varphi$). The higher the number $R$ is, the "more tight" convex hull is obtained.

In the next step the set $V_\Phi(s_0)$ of intersections of the following lines is determined:

$$
\begin{aligned}
V_\Phi(s_0) &= \left\{ S_m^{s_0} : m = 1, \ldots, 2R \right\} \\
V_r^{s_0} &= \mathrm{insec}\left( p_{\min}^{s_0,\varphi_r}, p_{\min}^{s_0,\varphi_{r+1}} \right) \\
V_R^{s_0} &= \mathrm{insec}\left( p_{\min}^{s_0,\varphi_R}, p_{\max}^{s_0,\varphi_1} \right) \\
V_{r+R}^{s_0} &= \mathrm{insec}\left( p_{\max}^{s_0,\varphi_r}, p_{\max}^{s_0,\varphi_{r+1}} \right) \\
V_{2R}^{s_0} &= \mathrm{insec}\left( p_{\max}^{s_0,\varphi_R}, p_{\min}^{s_0,\varphi_1} \right) \\
r &= 1, \ldots, R-1
\end{aligned}
\tag{24}
$$

where $\mathrm{insec}(p_x,p_y)$ denotes the intersection of the lines $p_x$ and $p_y$. (see Fig. 2)

The coordinates of intersections are given by

$$
\mathrm{insec}\left( p_{term}^{s_0,\varphi_m}, p_{term}^{s_0,\varphi_{m+1}} \right) =
= \left[ \begin{matrix} \dfrac{h_{term}^{s_0}(\varphi_2)\sin(\varphi_1) - h_{term}^{s_0}(\varphi_1)\sin(\varphi_2)}{\sin(\varphi_1 - \varphi_2)}, \\ \dfrac{h_{term}^{s_0}(\varphi_2)\cos(\varphi_1) - h_{term}^{s_0}(\varphi_1)\cos(\varphi_2)}{\sin(\varphi_1 - \varphi_2)} \end{matrix} \right]
\tag{25}
$$

Now the key theorem of this paper can be stated.

**Theorem 2**: Denote by Conv $A$ the convex hull of a set $A$. Then

$$
P(s_0,\mathbf{q}) \subseteq Conv\ V_\Phi(s_0)
\tag{26}
$$

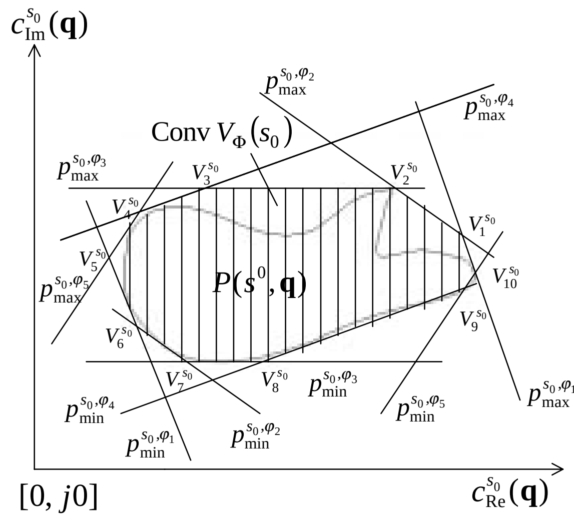


Fig.2 Convex hull $V_\Phi(s_0)$ for $R$=5

Using Theorem 2 the Zero Exclusion Principle gives a necessary condition for stability of a family of polynomials (1).

**Theorem 3 (Zero Exclusion Principle)**: The family of polynomials (1) of constant degree containing at least one stable polynomial is robustly stable with respect to $S$ if

$$
0 \notin Conv\ V_\Phi(s_0) \text{ for all } s_0 \in \partial S
\tag{27}
$$

where $\partial S$ denotes the boundary of $S$.

The zero exclusion test can be performed in both graphical and computational way. The latter is recommended as described below because of saving a lot of time.

**Theorem 4 (Zero inclusion test)**: $0 \in \mathrm{Conv}\ V_\Phi(s_0)$ if and only if

$$
h_{\min}^{s_0}(\varphi) \le 0,\ h_{\max}^{s_0}(\varphi) \ge 0 \text{ for all } \varphi \in \Phi
\tag{28}
$$

Theorem 4 makes it possible to decide about zero exclusion or inclusion without computing the set of intersections $V_\Phi(s_0)$.

To demonstrate the efficiency the computational times (on Pentium II 400MHz/64MB) of presented method for testing Schur stability and the method based on Sign-decomposition [2] are compared in Table 1.

The values in the upper and lower rows correspond to the presented method and the method based on Sign-decomposition respectively. The values the for presented algorithm have been found out for $R$=6 and a set of 30 points $s_0$ regularly distributed around the upper unit semicircle in the complex plane. It is necessary to note that the computational times for on

Table 1 Comparison of computational times (in seconds)

| n \| l | 2 | 3 | 5 | 7 | 10 |
|---|---|---|---|---|---|
| **2** | 2 | 3 | 22 | 130 | 500 |
| | 0,1 | 10 | 600 | $10^3$ | $10^5$ |
| **5** | 2,5 | 7 | 35 | 180 | 800 |
| | 0,2 | 55 | 1000 | $10^4$ | - |
| **10** | 6 | 15 | 80 | 350 | 1800 |
| | 0,5 | 260 | $10^4$ | - | - |
| **20** | 15 | 35 | 180 | 650 | 3500 |
| | 2 | 1200 | - | - | - |

Sign-decomposition based method vary in a big interval (in contrast to the presented algorithm). In Table 1 the highest values are shown.

## 4. EXAMPLE

Let a family of discrete polynomials be given by

$$P(z,\mathbf{q}) = c_2(\mathbf{q})z^2 + c_1(\mathbf{q})z + c_0(\mathbf{q})$$

where

$$\mathbf{q} = [q_1 \quad q_2]^T, \; q_i \in [0,1]$$

and

$$\begin{aligned} c_2(\mathbf{q}) &= 1 \\ c_1(\mathbf{q}) &= 0.2 \cdot q_2 + 0.1 \cdot q_1 \cdot q_2 - 0.5 \cdot q_2^2 \\ c_0(\mathbf{q}) &= -0.3 \cdot q_1 + 0.2 \cdot q_1^2 + q_1 \cdot q_2 - 0.5 \cdot q_2^2 \end{aligned}$$

The question is whether this family of polynomials is Schur stable.

In this case the stability region $S$ is the unit circle, therefore the boundary $\partial S = e^{j\omega}$, $\omega \in [0,2\pi]$. The Zero Exclusion Principle will be tested graphically. Due to symmetricity it is sufficient to plot the value set only for $s_0 = e^{j\omega}$, $\omega \in [0,\pi]$. The corresponding plot of the convex hulls of value sets is shown on Fig.3 ($R$=6). As $0 \notin V_\Phi(s_0)$ for all $s_0 \in \partial S$, the polynomial $P(z,\mathbf{q})$ is robustly Schur stable.

## 5. CONCLUSION

An algorithm for checking robust stability of a family of polynomials with coefficient being polynomic functions of the second order of an interval parameter has been presented. A sufficient condition has been derived by determination of a convex hull of the value set. For checking whether zero is excluded or included both computational and graphical way have been used. The former is recommended because of saving a lot of computational work. The main advantage of this algorithm compared to general methods for an arbitrary order of polynomic function

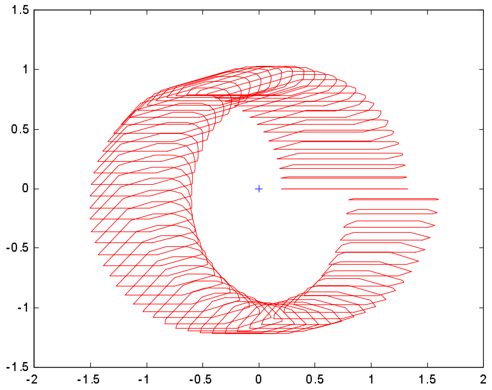


Fig. 3 Plot of the convex hulls of the value sets

consists in the high efficiency which is demonstrated by comparison with the method based on Sign-decomposition. Moreover an arbitrary stability region can be chosen. The presented method has been illustrated on an easy example.

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