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STABILIZATION OF PROGRAM MOTION OF NONLINEAR OBJECT WITH INCOMPLETE INFORMATION ABOUT THE STATE VECTOR

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ABSTRACT

The method of nonlocal stabilization of the program motion is developed under condition of incomplete information about vectors of state when some coordinates are unobservable.

1. INTRODUCTION

The problem of control law design is a key one in control theory. The methods that have been developed in automatic control theory are associated with bringing the controlled system to a given constant state under the fundamental assumption that perturbations are small. Since the actions on a system can have finite values, nonlocal stabilization problems arise. These problems should be considered in their original nonlinear statements without using any simplifications, in conditions of controls boundedness. The control goal can be either stabilization of a given state or stabilization of a given program motion. These problems are considered in the paper in conditions of incomplete information about the state vector when certain coordinates are unobservable.

2. STATEMENT OF THE PROBLEM

The deviations x of controlled system from program motion are described by ODE

$$\dot{x} = f(x,u,t), \quad x = \|x_i\|_{i=1}^n, \quad x \in G\{\|x\| \leq h\},$$

$$u = \|u_\nu(t)\|_{\nu=1}^m, \quad u \in U \tag{2.1}$$

Functions $f(x,u,t)$ are continuous in common. Controls $u(t)$ are integrable functions on a finite interval and take their values in a bounded convex domain U. The control goal is formalized by the identity

$$x(t) \equiv 0 \tag{2.2}$$

Subvector $x^1 = \|x_s\|_{s=1}^p, \quad 0 < p \leq n$ contains observable variables. The varying domain of unobservable variables $x^2 = \|x_k\|_{k=p+1}^n$ is denoted by $G(x^1)$.

The problem is to find a depending on observable coordinates control $u(x^1,t)$ for which the zero solution (2.2) of corresponding closed system

$$\dot{x} = f(x, u(x^1,t), t) \tag{2.3}$$

is asymtotically stable in the large, i.e. in a finite region $\tilde{G} \subset G$ [1]. Admissible functions $u(x^1,t)$ are chosen from the set D of multivalued upper semicontinuous (with respect to inclusion) functions $u(x^1,t)$ [1,2]. Remind the corresponding definition [3]. A (generally multivalued) function $u(x,t)$ is referred to as upper semicontinuous one with respect to inclusion at a point $\{x,t\}$ if the deviation $d(U(x^1,t^1),U(x,t))$, $(d(A,B) = \sup_{a\in A}\rho(a,B))$ of the values set $U(x^1,t^1)$ of the function at the point $\{x^1,t^1\}$ from the set $U(x,t)$ at the point $\{x,t\}$ tends to zero as $\{x^1,t^1\} \to \{x,t\}$. It is well known [3] that a bounded function $r(\xi)$ is upper semicontinuous one with respect to inclusion on a closed set M if and only if its graph (i.e. the set $\{(y,z) : y \in M, \quad z = r(y)\}$ is the closed set. The functions $u(x,p,t)$ which satisfy the extremum conditions $(p, f(x,u(x,p,t),t) = \min_{u\in U}(p, f(x,u,t))$ are upper semicontinuous ones with respect to inclusion. Thus, the value set $U(x^1,t)$ of function $u(x^1,t)$ is closed and bounded.

A solution $x(t)$ of (2.3) when $u(x^1,t) \in D$ is an absolute continuous solution of the corresponding differential inclusion [2,3]

$$\dot{x} \in F(x,t), \quad F(x,t) = \{y : y = \sum_{s=1}^{n+1} \alpha_s f(x,u^s),$$

$$u^s \in \mathrm{co}U(x^1,t), \quad \alpha \in A = \{\alpha_s \geq 0, \ \sum_{s=1}^{n+1} \alpha_s = 1\}\} \tag{2.4}$$

where u^s is chosen independently from the convex hall of the upper semicontinuous set $U(x^1,t)$ of values of a function $u(x^1,t) \in D$ at the point $\{x,t\}$. Asymptotic stability of the zero solution of (2.3) is understood as that for inclusion (2.4). Therefore the derivative of the Lyapunov function $v(x,t) \in \mathbf{C}^1\{G,[0,\infty)\}$ by virtue of the inclusion (2.4) is expressed by the relation [4]

$$(\dot{v})_{(2.3)} \triangleq (\dot{v})_{(2.4)} = \frac{\partial v}{\partial t} + \max_{z \in F(x,t)} \left(\frac{\partial v}{\partial x}, z \right) \tag{2.5}$$

The Lyapunov functions method is the universal one in stability theory because the classical Lyapunov theorems are inverted in most cases. It could be believed that requirements of the Lyapunov function existance for the differential inclusion (2.4) is not natural since invertion of the Lyapunov theorems is obtained for differential equations. However, the inclusion in question is generated by the differential equation (2.3), and therefore requirement mentioned above is not an additional assumption. The inverse theorems for the selector-linear inclusions are obtained in [4].

To establish stability of the zero solution of (2.3) (or (2.4)) is a quite complicated problem because we have no efficient method of determining whether some control $u(x^1,t)$ is stabilizing or not. Thus the efficient solution of stabilization problem should include two functions: a stabilizing admissible control $u(x^1,t)$ and a Lyapunov function $v(x,t)$ that proves asymptotic stability of the solution $x=0$ of (2.3). Taking this reason into account we introduce the following definitions:

Definition 2.1. Admissible control $u(x^1,t)$ and Lyapunov function $v(x,t)$ are referred to as stabilizing pair (S-pair) for the system (2.1) in domain G if $u(x^1,t)$ is stabilizing control and $v(x,t)$ is a Lyapunov function satisfying criterion of asymptotic stability of $x=0$ of the closed system (2.3) (or inclusion (2.4) if $u(x^1,t) \in D$).

Definition 2.2. We say that system (2.1) is s-stabilizable in G if in this domain there exists an S-pair

$$S = \{v(x,t), \ u(x^1,t)\} \tag{2.6}$$

The stabilizability is understood in this paper as s-stabilizability. The only difference between the two concepts of stabilizability is thaht the s-stabilizability requires existance of the Lyapunov function $v(x,t)$ that ensures stability in addition to existance of stabilizing control.

Definition 2.3. Let us say that the stationary system

$$\dot{x} = f(x,u) \tag{2.7}$$

is stationary stabilizable in G if in this domain there exists a stationary S-pair

$$S = \{v(x), \ u(x^1)\} \tag{2.8}$$

Admissible controls can depend on x if all coordinates x_i are observable, i.e. $x^1 = x$. It is possible to introduce similar definitions of exponential stability, strong stability, etc.

In accordance with the Lyapunpv method the derivative (2.5) must be negative definite, $\dot{v}(x,t) < 0$, where the inequality $\psi(x,t) < 0$ is understood here and further in the Lyapunov sense, i.e. there exists a function $\psi_0(x)$ such that $\psi(x,t) \leq \psi_0(x) < 0$ for all $x \neq 0, \quad x \in G, \quad t \geq t_0$.

Since components x^1, x^2 of the state vector x are different from the observability point of view, we will use the following simple criterion of negative definiteness.

Lemma 2.1. A function $w(x)$ is negative definite in G if and only if the functions

$$w_1(x^1) = \max_{x^2 \in G(x^1)} w(x) < 0, \qquad x^1 \neq 0 \tag{2.9}$$

$$w_2(x^2) = [w(x)]_{x^1=0} < 0, \qquad x^2 \neq 0 \tag{2.10}$$

are negative definite.

Necessity of the lemma directly follows from negative definiteness of $w(x)$. Prove sufficiency by reductio ad absurdum. Let functions (2.9), (2.10) be negative definite while $w(x)$ be not. Then there exists a point $x_0 = (x_0^1, x_0^2) \neq 0$ such that $w(x_0) \geq 0$. So we have

$$0 \leq w(x_0) \leq w_1(x_0^1) \leq 0$$

that implies $w(x_0) = w_1(x_0^1) = 0$ and $x_0^1 = 0$. The last means $x_0^2 \neq 0$ Then

$$0 = w(x_0) = w(x_0^1, x_0^2) = w(0, x_0^2) = w_2(x_0^2) < 0.$$

This contradiction proves the lemma.

3. CRITERIA OF STABILIZABILITY

Taking into account the lemma 2.1 consider the first-order partial differential inequality

$$L\{v(x,t)\} \triangleq$$

$$\min_{u \in U} \max_{x^2 \in G(x^1)} \left[\frac{\partial v(x,t)}{\partial t} + \left(\frac{\partial v(x,t)}{\partial x}, f(x,u,t) \right) \right] < 0$$

$$x^1 \neq 0 \tag{3.1}$$

In the case of complete information about the state vector inequality (3.1) takes simpler form

$$L_0\{v(x,t)\} \triangleq \frac{\partial v(x,t)}{\partial t} + \min_{u \in U} \left(\frac{\partial v(x,t)}{\partial x}, f(x,u,t) \right) < 0$$

$$x \neq 0 \tag{3.2}$$

For the sake of simplicity consider in detail only the stationary stabilization problem.

THEOREM 3.1 The system (2.7) is stationary stabilizable in the class D of controls $u(x^1) \in U$ if and only if there exists in G a positive definite solution $v(x) \in \mathbf{C}^1(G)$ of the partial differential inequalities

$$L_s\{v(x)\} = \min_{u\in U} \max_{x^2\in G(x^1)} \left(\frac{\partial v(x)}{\partial x}, f(x,u) \right) < 0,$$

$$x^1 \neq 0 \tag{3.3}$$

$$L_1\{v(x)\} =$$

$$\max \left[\left(\frac{\partial v(x)}{\partial x}, f(x,u) \right) \mid x^1 = 0, u \in U^* \subset U \right] < 0,$$

$$x^2 \neq 0 \tag{3.4}$$

where

$$U^* = U(0),$$

$$U(x^1) = Arg \min_{u\in U} \max_{x^2\in G(x^1)} \left(\frac{\partial v(x)}{\partial x}, f(x,u) \right). \tag{3.4a}$$

If the function $\tilde{v}(x)$ is such a solution of (3.3), (3.4), the functions

$$\tilde{v}(x); \tilde{u}(x^1) \in \tilde{U}(x^1) =$$

$$\operatorname{Arg} \min_{u\in U} \max_{x^2\in G(x^1)} \left(\frac{\partial \tilde{v}(x)}{\partial x}, f(x,u) \right) \tag{3.5}$$

yield a stationary S-pair of (2.7).

PROOF. Necessity. Let $\{\bar{v}(x); \bar{u}(x^1), \bar{u}(x^1) \in \bar{U}(x^1)\}$ be a stationary S-pair of (2.7). It means that $\bar{v}(x) > 0, x \neq 0, x \in G$ and its derivative by virtue of the system (2.7) with $u = \bar{u}(x^1) \in \bar{U}(x^1)$ has the form

$$(\dot{\bar{v}})_{(2.7)} = \max_{z\in F(x)} \left(\frac{\partial \bar{v}(x)}{\partial x}, z \right) =$$

$$\max_{u^s\in \mathrm{co}\bar{U}(x^1)} \max_{\alpha\in A} \sum_{s=1}^{n+1} \alpha_s \left(\frac{\partial \bar{v}(x)}{\partial x}, f(x,u^s) \right) =$$

$$= \max_{u\in \mathrm{co}\bar{U}(x^1)} \left(\frac{\partial \bar{v}(x)}{\partial x}, f(x,u) \right) = -w(x) < 0,$$

$$x \neq 0, \quad x \in G \tag{3.6}$$

The inequality (3.3) follows immediately from (3.6) since

$$L_s\{\bar{v}(x)\} = \min_{u\in U} \max_{x^2\in G(x^1)} \left(\frac{\partial \bar{v}(x)}{\partial x}, f(x,u) \right)$$

$$\leq \max_{u\in \mathrm{co}\bar{U}(x^1)} \max_{x^2\in G(x^1)} \left(\frac{\partial \bar{v}(x)}{\partial x}, f(x,u) \right) =$$

$$= \max_{x^2\in G(x^1)} \dot{\bar{v}}(x) = \max_{x^2\in G(x^1)} [-w(x)] < 0, \quad x^1 \neq 0 \tag{3.7}$$

In the similar way choosing $U^* = \bar{U}(0)$ we obtain

$$L_1\{\bar{v}(x)\} = \left[\max_{u\in \mathrm{co}\bar{U}(x^1)} \left(\frac{\partial \bar{v}(x)}{\partial x}, f(x,u) \right) \right]_{x^1=0}$$

$$= [-w(x)]_{x^1=0} < 0, \quad x^2 \neq 0 \tag{3.8}$$

Sufficiency. Let function $\tilde{v}(x)$ be a positive definite solution of the inequalities (3.3), (3.4) and control $\tilde{u}(x^1)$ be chosen in accordance with (3.5). Now we show that the functions (3.5) form an S-pair of (2.7) in domain G. According to (3.6) the derivative of $\tilde{v}(x) > 0$ by virtue of corresponding differential inclusion

$$\dot{\tilde{v}}(x) = \max_{\alpha\in A} \max_{u^\nu\in \mathrm{co}\tilde{U}(x^1)} \sum_{\nu=1}^{n+1} \alpha_\nu \left(\frac{\partial \tilde{v}(x)}{\partial x}, f(x,u^\nu) \right)$$

$$\leq \max_{\alpha\in A} \max_{u^\nu\in \mathrm{co}\tilde{U}(x^1)} \max_{x^2\in G(x^1)} \sum_{\nu=1}^{n+1} \alpha_\nu \left(\frac{\partial \tilde{v}(x)}{\partial x}, f(x,u^\nu) \right)$$

$$\leq \max_{\alpha\in A} \max_{u^\nu\in \mathrm{co}\tilde{U}(x^1)} \sum_{\nu=1}^{n+1} \alpha_\nu \max_{x^2\in G(x^1)} \left(\frac{\partial \tilde{v}(x)}{\partial x}, f(x,u^\nu) \right)$$

$$= \max_{\alpha\in A} \max_{u^s\in \mathrm{co}\tilde{U}(x^1)} \sum_{\nu=1}^{n+1} \alpha_\nu L_s\{\tilde{v}(x)\} = L_s\{\tilde{v}(x)\} < 0,$$

$$x^1 \neq 0 \tag{3.9}$$

is negative definite in respect to x^1 because $\psi(x^1, u^s) = \min[\psi(x^1, u) \mid u \in \mathrm{co}U]$ where

$$\psi(x^1, u) = \max_{x^2\in G(x^1)} \left(\frac{\partial \tilde{v}(x)}{\partial x}, f(x,u) \right).$$

The inequality (3.4) leads us to the conclusion (taking into account (3.6))

$$[\dot{\tilde{v}}(x)]_{x^1=0} = \left[\max_{u\in \mathrm{co}\tilde{U}(x^1)} \left(\frac{\partial \tilde{v}(x)}{\partial x}, f(x,u) \right) \right]_{x^1=0}$$

$$= L_1\{\tilde{v}(x)\} < 0, \quad x^2 \neq 0 \tag{3.10}$$

It means that $\dot{\tilde{v}}(x) < 0$, i.e. $\{\tilde{v}(x), \tilde{u}(x^1)\}$ is S-pair in G because $\tilde{v}(x) > 0$.

Note that distinctions between necessary and sufficient conditions is only the equality $U^* = \tilde{U}(0)$. As it follows from Theorem 3.1, the final stage of solving the control synthesis problem (not associated with solution of partial differential inequalities (3.3), (3.4)) is reduced according to (3.4) to a rather difficult minimax problem. This circumstance reflects essentially all features of the nonlocal stabilization problem in the conditions of incomplete information about the state vector. It will be noted that inequality (3.3) generates the required control $\tilde{u}(x^1)$ which ensures asymptotic stability in respect to observable coordinates x^1 and simple stability in respect to unobservable coordinates x^2. The inequality (3.5) ensures satisfying limit relation $x^2(t) \to 0, t \to \infty$ if (3.4) is satisfied.

THEOREM 3.2. The system (2.1) is stabilizable in the class D of controls $u(x^1,t) \in U$ that depends on observable variables x^1 if and only if there exists a positive definite in G solution $v(x,t)$ with the infinite small upper limit of the partial differential inequality (3.1) and that of type (3.4). If $\tilde{v}(x,t)$ is such a solution, functions of the type (3.5) make an S-pair of (2.1).

The proof of the Theorem 3.2 can be obtained with the proper transformations the same way as that of the Theorem 3.1.

If the state vector x is observable the stabilizability criterion becomes simpler.

THEOREM 3.3. System (2.1) is stabilizable when the state vector x is observable (i.e. $x^1 = x$) if and only if there exists a positive definite in G solution $v(x,t)$ with the infinite small upper limit of the partial differential inequality

$$L_c\{v(x,t)\} = \frac{\partial v(x,t)}{\partial t} + \min_{u \in U}\left(\frac{\partial v(x,t)}{\partial x}, f(x,u,t)\right) < 0,$$

$$x \neq 0, x \in G \tag{3.11}$$

If $\tilde{v}(x,t)$ is such a solution functions of the type (3.5) make an S-pair of (2.1).

All criteria stated above remain valid with proper transformations for discrete control systems described by the difference equations

$$x(s+1) = f(x(s), u(s), s)$$

if S-pairs are taken in the form

$$S_d = \{v(x,s); u(x^1,s)\}$$

and the functional inequalities

$$L_d\{v(x,s)\} =$$

$$= \min_{u \in U} \max_{x^2 \in G(x^1)} [v(f(x(s),u(s),s)) - v(x,s)]$$

are considered instead of the partial differential inequalities.

4. CONSTRUCTIVE METHOD OF STABILIZING PAIRS DESIGN

The stabilizability criteria stated above have the Lyapunov type. Such criteria give only principal solution to the problem of stabilizability. To find the required control solution of nonlinear inequalities in partial derivatives should be obtained. No general methods to solve such inequalities are known. It turns out that this problem can be solved with computer [1,5]. As a result, a composed method of solution of these inequalities is presented in this section. This method is based, firstly, on solution of inequalities in a sufficiently small neighborhood of the coordinate origin $x = 0$, and, secondly, on solution of inequalities in a region $\Gamma = G \setminus B$ where B is small enough ball $\|x\| < \epsilon$. For simplicity, we consider only a stationary system (2.7).

For sufficiently small x and u system (2.7) may be presented in the form

$$\dot{x} = Ax + Bu + \ldots \tag{4.1}$$

A stabilizing pair of linear system

$$\dot{x} = Ax + Bu \tag{4.2}$$

can be chosen in the form

$$S = \{v = x'Lx, \quad u = Cx^1\}, \qquad \|x\| \leq \epsilon \tag{4.3}$$

if number $\epsilon > 0$ is small enough.

Denote by δ the radius of a ball $\|x\| \leq \delta$ in the invariant domain

$$x'Lx \leq \lambda, \quad \lambda = \max_{\|x\| \leq \epsilon} x'Lx, \quad \lambda > 0 \tag{4.4}$$

of the system $\dot{x} = f(x, Cx^1)$. In order to solve the control design problem we have now to construct an admissible control $u(x^1)$ in the domain

$$\Gamma = \{x : \quad 0 < \gamma\delta \leq \|x\| \leq h\}, \qquad 0 < \gamma < 1 \tag{4.5}$$

that ensures motion of (2.7) from Γ to the ball $\|x\| \leq \delta$ in finite time. Thereafter the control is switched from $u(x^1)$ to the linear control Cx^1 that assures that the motion of the system (2.7) remains forever in the ball $\|x\| \leq \delta$ and also tends asymptotically to $x = 0$. This means that we have to solve the stabilization problem not for the point $x = 0$ but for the ball $\|x\| \leq \delta$ taking into account mentioned above possibility of swithing the control from $u(x^1)$ to Cx^1.

To solve the last problem the inequalities (3.3), (3.4) must be solved in Γ (4.5).

If $\tilde{v}(x)$ is a solution of these inequalities in the domain $G = \|x\| \leq h$, it will satisfy conditions

$$\tilde{v}(x) \geq \Delta > 0, \qquad \dot{\tilde{v}}(x) \leq -\Delta, \qquad x \in \Gamma, \tag{4.6}$$

where $\dot{\tilde{v}}(x)$ is derivative calculated by virtue of the closed loop system (2.7) with control $\tilde{u}(x^1)$ that is determined in accordance with (3.5).

Applying the Weierstrass theorem one can find from (4.6) that there exists in Γ a solution of (3.3), (3.4) in the class of polynomials of finite degree

$$v(x) = \sum_{s=1}^{N} \beta_s \phi_s(x), \tag{4.7}$$

where $\phi_s(x)$ denote some ordered monomials $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$, $\sum_{i=1}^{N} m_i \leq m_N$. Of course, we can use any other full system of functions (trigonometric functions for instance) instead of monomials as basis of series (4.7).

Thus the problem of solution of (3.3), (3.4) in Γ is reduced to that for inequalities

$$g_1(\beta, x^1) =$$

$$\min_{u\in U}\max_{x^2\in G(x^1)\cap\Gamma}\sum_{s=1}^{N}\beta_s\left(\frac{\partial\phi_s(x)}{\partial x}, f(x,u)\right) < 0,$$

$$x^1 \neq 0$$

$$g_2(\beta, x^2) = \max_{x^1=0, u\in U^*}\sum_{s=1}^{N}\beta_s\left(\frac{\partial\phi_s(x)}{\partial x}, f(x,u)\right) < 0,$$

$$x^2 \neq 0 \tag{4.8}$$

where U^* has the form (3.4a). Inequalities (4.8) do not contain unknown functions. This feature allowes to solve these inequalities with the aid of computer, in particular, by using methods of mathematical programming. If $\tilde{\beta}$ represents such a solution, we will have an S-pair in Γ:

$$S = \left\{\sum_{s=1}^{N}\tilde{\beta}_s\phi_s(x), \quad \tilde{u}(\tilde{\beta}, x^1)\right\}. \tag{4.9}$$

Functions (4.9) taken together with functions (4.3) give solution of stabilization problem.

CONCLUSION

The developed method allowes to solve a nonlocal stabilization problem in condition of incomplete information about state vector. The same approach can be applied to nonstationary case. The only difference is consideration of finite time intervals on which corresponded inequalities must be solved in the case of exponential stabilization.

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