

# NONLINEAR CONTROLLERS FOR NONLINEAR PASSIVE SYSTEMS WITH ARBITRARY INPUT NONLINEARITIES

WASSIM M. HADDAD<sup>‡</sup> and VIJAYSEKHAR CHELLABOINA<sup>†</sup>

<sup>‡</sup>School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150

<sup>†</sup>Mechanical and Aerospace Engineering, University of Missouri, Columbia, MO 65211

**Abstract.** A nonlinear dynamic compensator framework for nonlinear passive systems with arbitrary input nonlinearities is proposed. For this class of systems we prove global closed-loop stability by modifying the dynamic compensator to include a suitable input nonlinearity. The proof of this result is based on dissipativity theory and shows that the nonlinear controller modification counteracts the effects of the input nonlinearity by recovering the passivity of the plant and the compensator.

**Key Words.** Nonlinear control, passive systems, actuator nonlinearities, dissipativity theory

## 1. Introduction

In [1], the authors present a novel nonlinear control design framework for positive real linear systems with arbitrary input nonlinearities. Specifically, for positive real linear plants controlled by positive real controllers, a nonlinear controller modification is proposed that effectively counteracts the effects of *arbitrary* input actuator nonlinearities. The main contribution of [1] is that the feedback interconnection results are not based on absolute stability criteria [2] and are valid for input nonlinearities that are not necessarily either sector-bounded or odd or monotonic.

In this paper we generalize the results of [1] to nonlinear passive continuous-time and discrete-time systems controlled by nonlinear passive compensators. Our main result guarantees global asymptotic closed-loop stability for nonlinear passive systems with arbitrary input nonlinearities so long as the nonlinear dynamic compensator is modified to include a suitable input nonlinearity. The only restriction on the input nonlinearity is that it be memoryless and that either its characteristics be known or its output be measurable. The proof of this result is based on dissipativity theory [3–7] and shows that the nonlinear controller modification counteracts the effects of the input nonlinearity by recovering the passivity of the plant and compensator with respect to a modified set of inputs and outputs. Finally, in the case where the plant and compensator are linear, our continuous-time results specialize to the results obtained in [1].

## 2. Mathematical Preliminaries

In this section we establish definitions, notation, and a key result used in the paper. Let  $\mathbb{R}$  denote the real numbers, let  $\mathbb{R}^n$  denote the set of  $n \times 1$  real column vectors, let  $\mathbb{R}^{m \times n}$  denote the set of  $m \times n$  real matrices, and let  $\mathcal{N}$  denote the set of nonnegative integers.

Furthermore, we write  $\|\cdot\|$  for the Euclidean vector norm and  $V'(x)$  for the Fréchet derivative of  $V$  at  $x$ . Finally, let  $C^0$  denote the set of continuous functions and  $C^r$  denote the set of functions with  $r$  continuous derivatives.

In this paper we consider nonlinear dynamical systems  $\mathcal{G}$  of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \quad (1) \\ y(t) &= h(x(t)) + J(x(t))u(t), \quad (2) \end{aligned}$$

where  $x \in \mathbb{R}^n$ ,  $u, y \in \mathbb{R}^m$ ,  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $G : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ ,  $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , and  $J : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ . We assume that  $f(\cdot)$ ,  $G(\cdot)$ ,  $h(\cdot)$ , and  $J(\cdot)$  are smooth (at least  $C^1$  mappings) and  $f(\cdot)$  has at least one equilibrium so that, without loss of generality,  $f(0) = 0$  and  $h(0) = 0$ . Furthermore, for the nonlinear dynamical system  $\mathcal{G}$  we assume that the required properties for the existence and uniqueness of solutions are satisfied, that is,  $u(\cdot)$  satisfies sufficient regularity conditions such that the system (1) has a unique solution forward in time. For the dynamical system  $\mathcal{G}$  given by (1) and (2) we assume that for all input-output pairs  $u, y \in \mathbb{R}^m$ ,  $\int_{t_1}^{t_2} |u^T(s)y(s)|ds < \infty$ ,  $t_1, t_2 \geq 0$ .

**Definition 2.1** [7]. A nonlinear dynamical system  $\mathcal{G}$  of the form (1), (2) is exponentially passive (resp., passive) if there exists a  $C^0$  nonnegative-definite function  $V_s : \mathbb{R}^n \rightarrow \mathbb{R}$  called a *storage function* and a scalar  $\varepsilon > 0$  (resp.,  $\varepsilon = 0$ ) such that the *dissipation inequality*

$$e^{\varepsilon t} V_s(x(t)) \leq e^{\varepsilon t_0} V_s(x(t_0)) + \int_{t_0}^t 2e^{\varepsilon s} u^T(s)y(s)ds, \quad (3)$$

is satisfied for all  $t \geq t_0 \geq 0$ , where  $x(t)$ ,  $t \geq 0$ , is the solution of (1) with  $u \in \mathbb{R}^m$ .

**Remark 2.1.** If  $V_s(\cdot)$  is  $C^1$  continuous then an equivalent statement for exponential passivity of  $\mathcal{G}$  is

$$\dot{V}_s(x(t)) + \varepsilon V_s(x(t)) \leq 2u^T(t)y(t), \quad t \geq 0, \quad (4)$$

where  $\dot{V}_s(\cdot)$  denotes the total derivative of  $V_s(x)$  along the state trajectories  $x(t)$ ,  $t \geq 0$ , of (1).

---

This research was supported in part by the National Science Foundation under Grant ECS-9496249 and the Air Force Office of Scientific Research under Grant F49620-96-1-0125.

**Definition 2.2** [6]. A dynamical system  $\mathcal{G}$  is *zero-state observable* if for all  $x \in \mathbb{R}^n$ ,  $u(t) \equiv 0$ ,  $y(t) \equiv 0$  implies  $x(t) \equiv 0$ .

Next, we consider feedback interconnections of passive and exponentially passive dynamical systems. Specifically, we consider the nonlinear dynamical system  $\mathcal{G}$  given by (1), (2) with the nonlinear feedback system  $\mathcal{G}_c$  given by

$$\begin{aligned}\dot{x}_c(t) &= f_c(x_c(t)) + G_c(x_c(t))u_c(t), \quad x_c(0) = x_{c0}, \quad (5) \\ y_c(t) &= h_c(x_c(t)) + J_c(x_c(t))u_c(t), \quad (6)\end{aligned}$$

where  $t \geq 0$ ,  $x_c \in \mathbb{R}^{n_c}$ ,  $u_c \in \mathbb{R}^{m_c}$ ,  $y_c \in \mathbb{R}^{l_c}$ ,  $f_c : \mathbb{R}^{n_c} \rightarrow \mathbb{R}^{n_c}$  and satisfies  $f_c(0) = 0$ ,  $G_c : \mathbb{R}^{n_c} \rightarrow \mathbb{R}^{n_c \times m}$ ,  $h_c : \mathbb{R}^{n_c} \rightarrow \mathbb{R}^m$  and satisfies  $h_c(0) = 0$ ,  $J_c : \mathbb{R}^{n_c} \rightarrow \mathbb{R}^{m \times m}$ ,  $m_c = l_c = m$ . Here, we assume that the negative feedback interconnection of  $\mathcal{G}$  and  $\mathcal{G}_c$  is well posed; that is, with  $u_c = y$  and  $y_c = -u$ ,  $\det[I_m + J_c(x_c)J(x)] \neq 0$  for all  $x_c$  and  $x$ . The following result gives a sufficient condition for global asymptotic stability of the negative feedback interconnection of  $\mathcal{G}$  and  $\mathcal{G}_c$ .

**Theorem 2.1** [7]. Consider the closed-loop system consisting of the nonlinear dynamical systems  $\mathcal{G}$  given by (1), (2) and  $\mathcal{G}_c$  given by (5), (6) and assume  $\mathcal{G}$  and  $\mathcal{G}_c$  are zero-state observable. If  $\mathcal{G}$  is passive with a  $C^1$  radially unbounded, positive-definite storage function,  $\mathcal{G}_c$  is exponentially passive with a  $C^1$  radially unbounded, positive-definite storage function, and  $\text{rank}[G_c(0)] = m$ , then the negative feedback interconnection of  $\mathcal{G}$  and  $\mathcal{G}_c$  is globally asymptotically stable.

### 3. Nonlinear Controllers for Systems with Arbitrary Input Nonlinearities

In this section we present a framework to construct nonlinear controllers for nonlinear passive systems with *arbitrary* input nonlinearities. Specifically, we suppose that the nonlinear plant (1), (2) is found to possess an input nonlinearity so that (1), (2) is not valid. Rather, in place of (1), (2), a more accurate model is

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + \delta G(x(t))\sigma(u(t)), \quad x(0) = x_0, \quad (7) \\ y(t) &= h(x(t)) + \delta J(x(t))\sigma(u(t)), \quad (8)\end{aligned}$$

where  $t \geq 0$ ,  $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^m$  denotes an input nonlinearity and  $\delta > 0$  is an unknown scalar. We assume that for all  $i = 1, \dots, m$ , if  $u_i = 0$  then  $\sigma_i(u) = 0$ , where  $u_i$  and  $\sigma_i(u)$  denote the  $i$ th components of  $u$  and  $\sigma(u)$ , respectively. Furthermore, we assume that with input  $v = \delta\sigma(u)$  and output  $y$ ,  $\mathcal{G}$  is passive with a  $C^1$  radially unbounded, positive-definite storage function. Note that if  $\sigma(u) = u$  and  $\delta = 1$ , that is,  $u = v$ , it follows that  $\mathcal{G}$  is passive with input  $u$  and output  $y$ .

To illustrate the allowable input nonlinearities, consider first the special case  $\sigma(u) = [\sigma_1(u_1) \dots \sigma_m(u_m)]^T$  of decoupled nonlinearities. In this case, the  $i$ th component  $\sigma_i(u_i)$  of  $\sigma(\cdot)$  depends only upon the  $i$ th component  $u_i$  of  $u$ . Now  $\sigma_i(\cdot)$  can represent an arbitrary scalar nonlinearity that vanishes at the origin. For example, the saturation nonlinearity  $\sigma_1(u_1) = \text{sat}(u_1)$  is allowable as well as deadzone, quantization, and relay nonlinearities. Note that in the case where  $\sigma_i(\cdot)$  represents a saturation nonlinearity, the unknown scalar  $\delta > 0$  allows for the consideration of saturation nonlinearities with unknown amplitude and slope. Similar remarks hold for the

other types of nonlinearities cited above. Also note that different types of nonlinearities are permissible. For example,  $\sigma(u) = [\text{sat}(u_1) \text{sgn}(u_2)]^T$  is allowed, where  $\text{sgn}(0) = 0$ . More generally,  $\sigma(u)$  may also denote a nonlinearity whose coordinates are not necessarily decoupled. For example, the radial saturation nonlinearity

$$\sigma(u) = \begin{cases} u, & \|u\|_2 \leq 1, \\ 1, & \|u\|_2 > 1, \end{cases}$$

where  $\|u\|_2$  denotes the Euclidean norm of  $u$ , can also be considered.

Note that if  $\sigma(u) = u$  and  $\delta = 1$ , that is,  $u = v$ , it follows from Theorem 2.1 that the negative feedback interconnection of  $\mathcal{G}$  and  $\mathcal{G}_c$  is globally asymptotically stable, where  $\mathcal{G}_c$  is given by (5), (6) and is such that  $\text{rank}[G_c(0)] = m$ , and  $\mathcal{G}_c$  is exponentially passive with input  $u_c$ , output  $y_c$ , and a  $C^1$  radially unbounded, positive-definite storage function. However, in the presence of the input nonlinearity  $\sigma(\cdot)$  Theorem 2.1 is no longer valid and hence closed-loop stability and performance may be effected. Next, we modify the controller (5), (6) to account for the input nonlinearity  $\sigma(\cdot)$  in order to guarantee closed-loop stability.

To counteract the effect of the input nonlinearity  $\sigma(u)$  in (7), (8) we modify the controller (5), (6) by replacing the compensator dynamics (5) and control inputs (6) by

$$\begin{aligned}\dot{x}_c(t) &= f_c(x_c(t)) + G_c(x_c(t))\beta(u(t))y(t), \quad (9) \\ u(t) &= -[h_c(x_c(t)) + J_c(x_c(t))\beta(u(t))y(t)], \quad (10)\end{aligned}$$

where  $x_c(0) = x_{c0}$ ,  $t \geq 0$ ,  $\beta : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times m}$  is given by  $\beta(u) = \text{diag}(\beta_1(u), \beta_2(u), \dots, \beta_m(u))$ , where for  $i = 1, \dots, m$ ,

$$\beta_i(u) = \begin{cases} \sigma_i(u)/u_i, & u_i \neq 0, \\ \text{arbitrary}, & u_i = 0. \end{cases} \quad (11)$$

Since  $\sigma_i(u) = 0$  if  $u_i = 0$  it follows that  $\beta_i(u)u_i = \sigma_i(u)$ , for all  $i = 1, \dots, m$ , and  $u \in \mathbb{R}^m$ . Consequently, it follows that

$$\beta(u)u = \sigma(u), \quad u \in \mathbb{R}^m. \quad (12)$$

Finally, we assume that  $\mathcal{G}_c$ , with input  $v_c = \beta(u)y$  and output  $y_c = -u$ , is exponentially passive with a  $C^1$  radially unbounded, positive-definite storage function. Note that if  $\beta(u) = I_m$ , that is,  $v_c = y$ , it follows that  $\mathcal{G}_c$  is exponentially passive with input  $u_c = y$  and output  $y_c = -u$ .

The form of the controller input nonlinearity  $\beta(u)$  in (9) and (10) is quite simple, requiring only knowledge of  $\sigma(u)$  and division by  $u_i$ . For the case  $m = 1$  and several common nonlinearities, the required controller nonlinearity  $\beta(u)$  is illustrated in Table 1. Note that in the case of nonlinearities such as the relay nonlinearity  $\sigma(u) = \text{sgn}(u)$  leads to unbounded  $\beta(u)$  for  $u$  near zero. Hence in this case it may be desirable to artificially implement a deadzone so that  $\beta(u)$  is bounded. Finally, although all of the input nonlinearities shown in Table 1 are sector-bounded and odd monotonic, our results are valid for nonlinearities that are not necessarily either sector-bounded or odd or monotonic.

The nonlinear dynamic compensator (9), (10) can be implemented in two ways. If  $\sigma(u)$  is known, then  $\beta(u)$  can be constructed from (11) by evaluating  $\sigma(u)$  in real time for each value of  $u$ . If, however, the model  $\sigma(u)$  is not available but  $\sigma(u(t))$  can be measured during the closed-loop operation, then  $\beta(u(t))$  can be formed from  $u(t)$  and  $\sigma(u(t))$  by implementing (11) with  $u = u(t)$ .

This scheme is illustrated in Fig. 1. If, however, neither a model of  $\sigma(u)$  nor a measurement of  $\sigma(u(t))$  is available, then  $\beta(u(t))$  cannot be formed and our approach does not apply. Hence, in the sequel, we assume that either an accurate model of  $\sigma(u)$  is available or that the signal  $\sigma(u(t))$  is available for feedback. Finally, note that in the case where  $J_c(x_c) \neq 0$  the controller output equation contains an algebraic constraint on  $u$ . For each choice of  $J_c(x_c)$  and  $\beta(u)$  this equation must be examined for solvability in terms of  $u$ .

Next, we present the main result of this paper which shows that in spite of the input nonlinearity  $\sigma(u)$  in (7), (8), closed-loop stability is guaranteed if the modified nonlinear controller (9), (10) is implemented in place of (5), (6).

**Theorem 3.1.** Consider the closed-loop system consisting of the nonlinear plant  $\mathcal{G}$  given by (7), (8) and the nonlinear dynamic compensator  $\mathcal{G}_c$  given by (9), (10), where the input nonlinearity  $\sigma(\cdot)$  is such that  $\sigma_i(u) = 0$  if  $u_i = 0$  and  $\beta_i(u) \neq 0$ ,  $u \in \mathbb{R}^m$ ,  $i = 1, \dots, m$ . Assume that  $\mathcal{G}$  is zero-state observable,  $\text{rank}[G_c(0)] = m$ , and  $\mathcal{G}$  is passive with input  $v = \delta\sigma(u)$ , output  $y$ , and a  $C^1$  radially unbounded, positive-definite storage function  $V_s(\cdot)$ . Furthermore, assume that  $\mathcal{G}_c$  is exponentially passive with input  $v_c = \beta(u)y$ , output  $y_c = -u$ , and a  $C^1$  radially unbounded, positive-definite storage function  $V_{sc}(\cdot)$ . Then the closed-loop system (7)–(10) is globally asymptotically stable.

**Proof.** Consider the Lyapunov function candidate  $V(x, x_c) = (1/\delta)V_s(x) + V_{sc}(x_c)$ . Now, the corresponding Lyapunov derivative is given by

$$\begin{aligned}\dot{V}(x, x_c) &= (1/\delta)\dot{V}_s(x) + \dot{V}_{sc}(x_c) \\ &= (1/\delta)V'_s(x)[f(x) + \delta G(x)\sigma(u)] \\ &\quad + V'_{sc}(x_c)[f_c(x_c) + G_c(x_c)\beta(u)y] \\ &= (1/\delta)V'_s(x)[f(x) + G(x)v] \\ &\quad + V'_{sc}(x_c)[f_c(x_c) + G_c(x_c)v_c].\end{aligned}$$

Since  $\mathcal{G}$  is passive with input  $v$  and output  $y$  and  $\mathcal{G}_c$  is exponentially passive with input  $v_c$  and output  $y_c$  it follows that there exists  $\varepsilon > 0$  such that

$$\begin{aligned}\dot{V}(x, x_c) &= (1/\delta)V'_s(x)[f(x) + G(x)v] \\ &\quad + V'_{sc}(x_c)[f_c(x_c) + G_c(x_c)v_c] \\ &\leq (2/\delta)y^T v + 2v_c^T y_c - \varepsilon V_{sc}(x_c) \\ &= 2y^T \sigma(u) - 2y^T \beta(u)u - \varepsilon V_{sc}(x_c) \\ &= -\varepsilon V_{sc}(x_c) \\ &\leq 0,\end{aligned}$$

which implies that the negative feedback interconnection of  $\mathcal{G}$  and  $\mathcal{G}_c$  is Lyapunov stable. To show asymptotic stability let  $\mathcal{R} \triangleq \{(x, x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c} : \dot{V}(x, x_c) = 0\}$  and, since  $V_{sc}(x_c)$  is positive definite, note that  $\dot{V}(x, x_c) = 0$  only if  $x_c = 0$ . Now, since  $\text{rank}[G_c(0)] = m$  and  $\beta_i(u) \neq 0$ ,  $u \in \mathbb{R}^m$ ,  $i = 1, \dots, m$ , it follows that on every invariant set  $\mathcal{M}$  contained in  $\mathcal{R}$ ,  $v_c(t) = y(t) \equiv 0$  and hence  $u(t) \equiv 0$  so that  $\dot{x}(t) = f(x(t))$ . Now, since  $\mathcal{G}$  is zero-state observable it follows that  $\mathcal{M} = \{(0, 0)\}$  is the largest invariant set contained in  $\mathcal{R}$ . Hence, it follows from LaSalle's invariance principle [8] that  $(x(t), x_c(t)) \rightarrow \mathcal{M} = \{(0, 0)\}$  as  $t \rightarrow \infty$ . Now, global asymptotic stability of the closed-loop system follows from the fact that  $V_s(\cdot)$  and  $V_{sc}(\cdot)$  are, by assumption, radially unbounded.  $\square$

**Remark 3.1.** It is important to note that if in Theorem 3.1 the nonlinear plant  $\mathcal{G}$  is exponentially passive, then global asymptotic stability of the negative feedback interconnection of  $\mathcal{G}$  and  $\mathcal{G}_c$  is guaranteed without the assumptions of zero-state observability of  $\mathcal{G}$ ,  $\beta_i(u) \neq 0$ ,  $u \in \mathbb{R}^m$  (or, equivalently,  $\sigma_i(u) \neq 0$ ,  $u \neq 0$ ), and  $\text{rank}[G_c(0)] = m$ . Alternatively, global asymptotic stability of the negative feedback interconnection of  $\mathcal{G}$  and  $\mathcal{G}_c$  is also guaranteed if the nonlinear plant  $\mathcal{G}$  is input strict passive [6] (resp., output strict passive [6]) and the nonlinear compensator  $\mathcal{G}_c$  is input strict passive (resp., output strict passive). A similar remark holds for Theorem 4.1 below.

## 4. Extensions to Discrete-Time Nonlinear Systems

In this section we extend the results of Section 3 to discrete-time systems. Specifically, we consider discrete-time nonlinear dynamical systems  $\mathcal{G}$  of the form

$$\begin{aligned}x(k+1) &= f(x(k)) + G(x(k))u(k), \quad x(0) = x_0, \quad (13) \\ y(k) &= h(x(k)) + J(x(k))u(k), \quad (14)\end{aligned}$$

where  $k \in \mathcal{N}$ ,  $x \in \mathbb{R}^n$ ,  $u, y \in \mathbb{R}^m$ ,  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $G : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ ,  $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , and  $J : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ . Here, we assume that  $f(\cdot)$ ,  $G(\cdot)$ ,  $h(\cdot)$ , and  $J(\cdot)$  are  $C^0$  mappings and  $f(\cdot)$  has at least one equilibrium so that, without loss of generality,  $f(0) = 0$  and  $h(0) = 0$ . The following definition gives the discrete-time counterpart of exponential passivity.

**Definition 4.1.** A discrete-time nonlinear dynamical system  $\mathcal{G}$  of the form (13), (14) is geometrically passive (resp., passive) if there exists a  $C^0$  nonnegative-definite function  $V_s : \mathbb{R}^n \rightarrow \mathbb{R}$  called a *storage function* and a scalar  $\rho \in (0, 1)$  (resp.,  $\rho = 1$ ) such that the *dissipation inequality*

$$\rho^{-k_1} V_s(x(k_1)) \leq \rho^{-k_0} V_s(x(k_0)) + \sum_{k=k_0}^{k_1-1} 2\rho^{-k-1} u^T(k) y(k), \quad (15)$$

is satisfied for all  $k_1 > k_0 \geq 0$ , where  $x(k)$ ,  $k \in \mathcal{N}$ , is the solution of (13) with  $u \in \mathbb{R}^m$ .

As in the continuous-time case we assume that the nonlinear plant (13), (14) possesses an input nonlinearity so that (13), (14) is more accurately given by

$$x(k+1) = f(x(k)) + \delta G(x(k))\sigma(u(k)), \quad (16)$$

$$y(k) = h(x(k)) + \delta J(x(k))\sigma(u(k)), \quad (17)$$

where  $x(0) = x_0$ ,  $k \in \mathcal{N}$  and  $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^m$  denotes an input nonlinearity satisfying the assumptions given in Section 3 and  $\delta > 0$  is an unknown scalar. Furthermore, we assume that with input  $v = \delta\sigma(u)$  and output  $y$ ,  $\mathcal{G}$  is passive with a  $C^0$  radially unbounded, positive-definite storage function.

To counteract the effect of the input nonlinearity  $\sigma(u)$  in (16), (17) we propose a discrete-time nonlinear compensator given by

$$x_c(k+1) = f_c(x_c(k)) + G_c(x_c(k))\beta(u(k))y(k), \quad (18)$$

$$u(k) = -[h_c(x_c(k)) + J_c(x_c(k))\beta(u(k))y(k)], \quad (19)$$

where  $k \in \mathcal{N}$ ,  $x_c(0) = x_{c0}$ , and  $\beta : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times m}$  is given by  $\beta(u) = \text{diag}(\beta_1(u), \beta_2(u), \dots, \beta_m(u))$ , and for  $i = 1, \dots, m$ ,  $\beta_i(\cdot)$  is given by (11). Finally, we assume that  $\mathcal{G}_c$ , with input  $v_c = \beta(u)y$  and output  $y_c = -u$ , is geometrically passive with a  $C^0$  radially unbounded, positive-definite storage function.

**Theorem 4.1.** Consider the closed-loop system consisting of the nonlinear plant  $\mathcal{G}$  given by (16), (17) and the nonlinear dynamic compensator  $\mathcal{G}_c$  given by (18), (19), where the input nonlinearity  $\sigma(\cdot)$  is such that  $\sigma_i(u) = 0$  if  $u_i = 0$  and  $\beta_i(u) \neq 0$ ,  $u \in \mathbb{R}^m$ ,  $i = 1, \dots, m$ . Assume that  $\mathcal{G}$  is zero-state observable,  $\text{rank}[G_c(0)] = m$ , and  $\mathcal{G}$  is passive with input  $v = \delta\sigma(u)$ , output  $y$ , and a  $C^0$  radially unbounded, positive-definite storage function  $V_s(\cdot)$ . Furthermore, assume that  $\mathcal{G}_c$  is geometrically passive with input  $v_c = \beta(u)y$ , output  $y_c = -u$ , and a  $C^0$  radially unbounded, positive-definite storage function  $V_{sc}(\cdot)$ . Then the closed-loop system (16)–(19) is globally asymptotically stable.

**Proof.** Consider the Lyapunov function candidate  $V(x, x_c) = (1/\delta)V_s(x) + V_{sc}(x_c)$ . Now, the corresponding Lyapunov difference is given by

$$\begin{aligned}\Delta V(x, x_c) &= (1/\delta)\Delta V_s(x) + \Delta V_{sc}(x_c) \\ &= (1/\delta)[V_s(f(x) + \delta G(x)\sigma(u)) - V_s(x)] \\ &\quad + V_{sc}(f_c(x_c) + G_c(x_c)\beta(u)y) - V_{sc}(x_c) \\ &= (1/\delta)[V_s(f(x) + G(x)v) - V_s(x)] \\ &\quad + V_{sc}(f_c(x_c) + G_c(x_c)v_c) - V_{sc}(x_c).\end{aligned}$$

Since  $\mathcal{G}$  is passive with input  $v$  and output  $y$  and  $\mathcal{G}_c$  is geometrically passive with input  $v_c$  and output  $y_c$  it follows that there exists  $\rho \in (0, 1)$  such that

$$\begin{aligned}\Delta V(x, x_c) &= (1/\delta)[V_s(f(x) + G(x)v) - V_s(x)] \\ &\quad + V_{sc}(f_c(x_c) + G_c(x_c)v_c) - V_{sc}(x_c) \\ &\leq (2/\delta)y^T v + 2v_c^T y_c - (1 - \rho)V_{sc}(x_c) \\ &= 2y^T \sigma(u) - 2y^T \beta(u)u - (1 - \rho)V_{sc}(x_c) \\ &= -(1 - \rho)V_{sc}(x_c) \\ &\leq 0,\end{aligned}$$

which implies that the negative feedback interconnection of  $\mathcal{G}$  and  $\mathcal{G}_c$  is Lyapunov stable. Now, the proof of global asymptotic stability of the closed-loop system is identical to that of the continuous-time and hence is omitted.  $\square$

## 5. Illustrative Numerical Example

Consider the controlled nonlinear oscillator given by the undamped Duffing's equation

$$\ddot{x}(t) + (2 + x^2(t))x(t) = 2u(t), \quad x(0) = x_0, \quad (20)$$

$$y(t) = \dot{x}(t), \quad (21)$$

where  $t \geq 0$ ,  $x(t), u(t), y(t) \in \mathbb{R}$ , and input nonlinearity  $\sigma(u) = \sin(u)$ . Defining  $x_1 \triangleq x$  and  $x_2 \triangleq \dot{x}$ , (20), (21) can be written in the state space form (1), (2) with  $x = [x_1 \ x_2]^T$ ,  $f(x) = [x_2 - (2 + x_1^2)x_1]^T$ ,  $G(x) = [0 \ 2]^T$ ,  $h(x) = x_2$ , and  $J(x) = 0$ . Now, with input  $v = \sigma(u) = \sin(u)$  and output  $y$ , the nonlinear dynamical system given by (20), (21) can be shown to be passive [7] with storage function  $V_s(x) = 2x_1^2 + \frac{1}{2}x_1^4 + x_2^2$ . Furthermore, note that (20), (21) is zero-state observable.

In order to stabilize the dynamical system (20), (21) we consider a compensator emulating a nonlinear damped oscillator given by

$$\ddot{x}_c(t) + \eta(x_c(t), \dot{x}_c(t))(\dot{x}_c(t) + x_c(t)) = y(t), \quad (22)$$

$$u(t) = -x_c(t) - \dot{x}_c(t), \quad (23)$$

where  $t \geq 0$ ,  $x_c(0) = x_{c0}$ ,  $x_c(t) \in \mathbb{R}$ , and  $\eta(x_c, \dot{x}_c) \triangleq 2 + (x_c + \dot{x}_c)^2$ . Defining  $x_{c1} \triangleq x_c$  and  $x_{c2} \triangleq \dot{x}_c$ , (22),

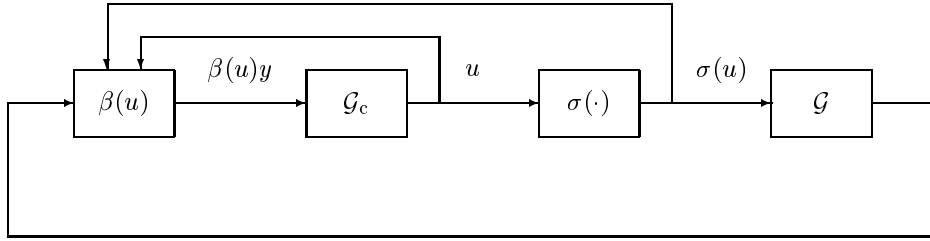
(23) can be written in the state space form (5), (6) with  $x_c = [x_{c1} \ x_{c2}]^T$ ,  $f_c(x_c) = [x_{c2} - \eta(x_{c1}, x_{c2})(x_{c1} + x_{c2})]^T$ ,  $G_c(x_c) = [0 \ 1]^T$ ,  $h_c(x_c) = x_{c1} + x_{c2}$ , and  $J_c(x_c) = 0$ . With input  $u_c = y$ , output  $y_c = -u$ , and  $V_{sc}(x_c) = x_{c1}^2 + (x_{c1} + x_{c2})^2$  the nonlinear compensator (22), (23) can be shown to be exponentially passive [7]. With the sinusoidal nonlinearity  $\sigma(u) = \sin(u)$  the compensator (22), (23) leads to a limit cycle instability for the initial condition  $x_1(0) = 2$ ,  $x_2(0) = 0$ . Alternatively, the modified nonlinear compensator (9), (10) guarantees global closed-loop asymptotic stability. The comparison of the time responses for position and velocity for both controlled systems is given in Figs. 2 and 3, respectively. Finally, Figs. 4 and 5 compare the control efforts of the unmodified and modified compensators, respectively.

## 6. Conclusion

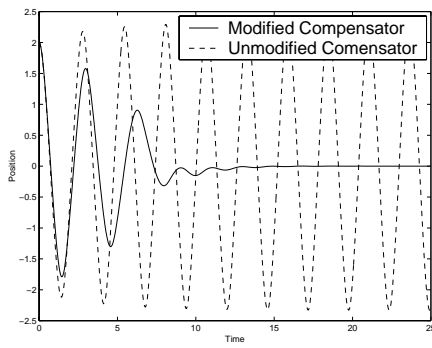
A novel nonlinear control approach based on dissipativity theory was developed for addressing the problem of input nonlinearities in nonlinear passive plants. The approach assumes that the nonlinear plant in the absence of input nonlinearities is passive and the nonlinear controller is exponentially passive, while the class of input nonlinearities that can be addressed is quite general. Global closed-loop stability in the face of arbitrary input nonlinearities is guaranteed by modifying the input to the nonlinear compensator to counteract the effects of the input nonlinearity. This modification results in recovering the passivity of the plant and the compensator.

## References

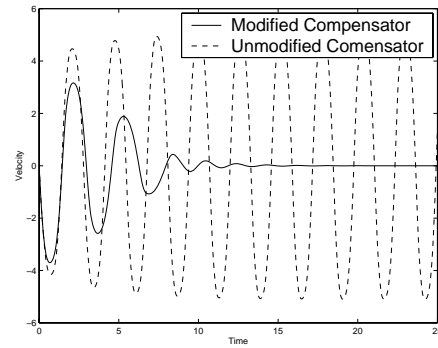
- [1] D. S. Bernstein and W. M. Haddad, "Nonlinear controllers for positive real systems with arbitrary input nonlinearities," *IEEE Trans. Autom. Contr.*, vol. AC-39, pp. 1513–1517, 1994.
- [2] K. S. Narendra and J. H. Taylor, *Frequency Domain Criteria for Absolute Stability*. New York: Academic Press, 1973.
- [3] J. C. Willems, "Dissipative dynamical systems part I: General theory," *Arch. Rational Mech. Anal.*, vol. 45, pp. 321–351, 1972.
- [4] P. J. Moylan, "Implications of passivity in a class of nonlinear systems," *IEEE Trans. Autom. Contr.*, vol. 19, pp. 373–381, 1974.
- [5] D. J. Hill and P. J. Moylan, "The stability of nonlinear dissipative systems," *IEEE Trans. Autom. Contr.*, vol. 21, pp. 708–711, 1976.
- [6] D. J. Hill and P. J. Moylan, "Stability results for nonlinear feedback systems," *Automatica*, vol. 13, pp. 377–382, 1977.
- [7] V. Chellaboina and W. M. Haddad, "Exponentially dissipative systems: A nonlinear extension of strict positive realness," in *Proc. Amer. Contr. Conf.*, (Chicago, IL), 2000.
- [8] J. P. LaSalle, "Some extensions of Liapunov's second method," *IRE Trans. Circ. Thy.*, vol. CT-7, pp. 520–527, 1960.



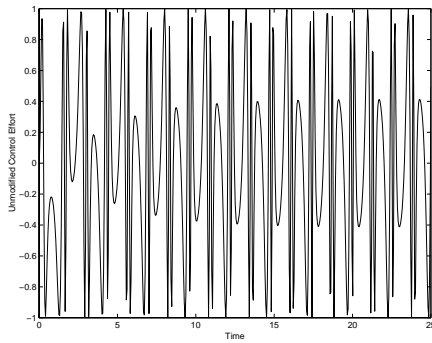
**Fig. 1:** Feedback scheme for constructing  $\beta(u)$



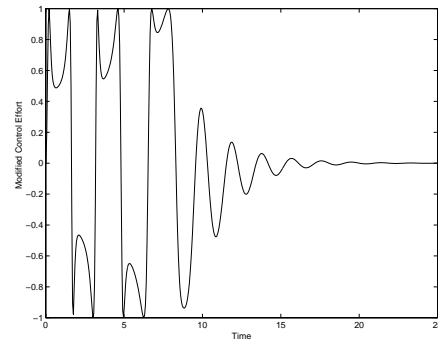
**Fig. 2:** Position versus time



**Fig. 3:** Velocity versus time

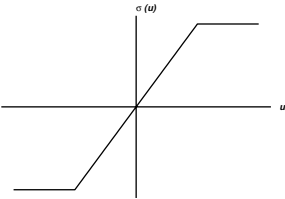
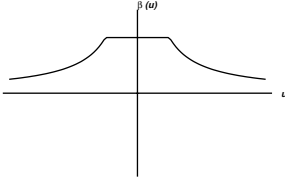
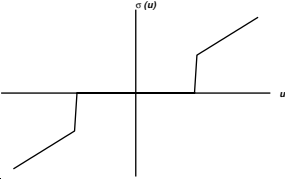
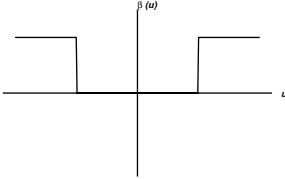
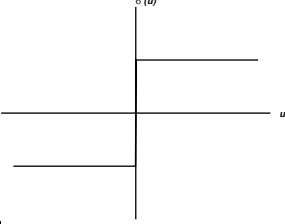
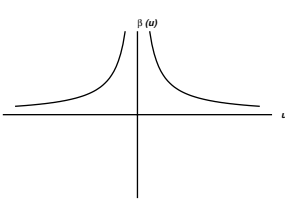
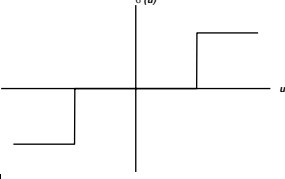
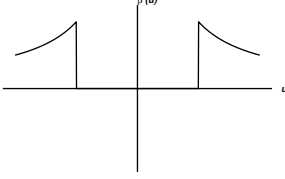


**Fig. 4:** Control effort versus time



**Fig. 5:** Control effort versus time

**Table 1:** Representative input nonlinearities

	Input Nonlinearity $\sigma(u)$	Controller Nonlinearity $\beta(u) = \sigma(u)/u$
Saturation		
Deadzone		
Relay		
Relay with Deadzone		
Quantization	