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CONVOLUTION PROFILES FOR NONCAUSAL INVERSION OF MULTIVARIABLE DISCRETE-TIME SYSTEMS

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Abstract. Noncausal inversion of discrete-time linear multivariable systems is analyzed in the geometric approach framework with the aim of computing convolution profiles that ensure perfect tracking with infinite preaction and infinite postaction time. It is shown how this computation is related to the concepts of multivariable output relative degree and invariant zeros of the plant. Then, a computational setting for the convolution profiles is derived by using the standard geometric approach tools.

Key Words. Geometric approach, noncausal inversion, discrete-time systems, convolution profiles.

1. INTRODUCTION

The problem of deriving a right or left inverse for a multivariable dynamical system has been widely studied in the past. Structural conditions for multivariable system invertibility were almost contemporarily derived by Dorato [5], Sain and Massey [12], and Silverman [13]. Equivalent structural conditions, expressed in geometric terms, were stated by Basile and Marro in [1], where the maximum subspace of perfect output controllability with respect to the generic i-th derivative in the output space was also derived as an application of the conditioned invariant algorithm, thus paving the way for a formal definition of the multivariable relative degree in geometric terms. It is well-known that the right inversion is related to the perfect tracking problem. Davison and Scherzinger [3] and Qiu and Davison [11] showed that perfect tracking is not possible when the system is non-minimum phase, while Francis [6] dealt with perfect tracking as an LQR cheap control problem. In the last few years it has been shown by several authors that the perfect tracking problem is solvable if the signal to be tracked is previewed by a significant amount of time and, in the continuous-time case, if the signal to be tracked is smooth enough. See, for instance, Devasia, Chen and Paden [4], Hunt, Meyer and Su [8], and for the SISO discrete-time case, Gross and Tomizuka [7] and Marro and Fantoni [9].

In the present paper the multivariable discrete-time right inversion or perfect tracking problem is solved by using a strictly geometric-type mathematical background, like that popularized by Wonham [14] and by Basile and Marro [2]. The solution is completely constructive. It can be considered to be the extension to the MIMO case of the results in [9]. It is shown that the tracking problem can be perfectly solved by convolution of the signal to be tracked with a suitable profile defined on the whole time axis, from minus infinity to plus infinity. However, technical practice imposes that the time interval to be considered is finite. This leads to the implementation of the compensator as a finite-impulse-response (FIR) system which enables almost perfect tracking with arbitrary accuracy to be achieved.

2. SOME NOTATION AND RECALLS

Throughout this paper we shall refer to the standard three-map discrete-time dynamical system

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k)\,, \\ y(k) &= C\,x(k)\,, \end{aligned} \tag{1}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^p$ and controlled output $y \in \mathbb{R}^q$. Matrices B and C are assumed to be full column rank and full row rank, respectively. We shall denote the image of the input matrix B ($\mathrm{im}\, B$) with $\mathcal{B}$ and the null space of the output matrix C ($\ker C$) with $\mathcal{C}$, the maximum $(A,\mathcal{B})$-controlled invariant contained in $\mathcal{C}$ by $\mathcal{V}^* = \max \mathcal{V}(A,\mathcal{B},\mathcal{C})$ and the minimum $(A,\mathcal{C})$-conditioned invariant containing $\mathcal{B}$ by $\mathcal{S}^* = \min \mathcal{S}(A,\mathcal{C},\mathcal{B})$, and by $\mathcal{R}_{\mathcal{V}^*}$ the reachable set on $\mathcal{V}^*$, computable as $\mathcal{V}^* \cap \mathcal{S}^*$. The symbol $A^{\#}$ denotes the pseudoinverse of A.

Let us provide extensions to the discrete-time systems of some definitions and properties that are well known and standard in the continuous-time case.

Definition 1 (Functional Controllability or Right-Invertibility and Output Relative Degree) *System (1) is said to be functionally controllable (or right-invertible) if, starting from the zero state and after some delay, arbitrary functions can be reproduced at the outputs, by means of suitable control inputs. The minimum delay for any output to be reproduced will be referred to as the output relative degree of the system.*

The following properties are easily derived as extensions of similar results for the continuous-time case.

Proposition 1 *System (1) is functionally controllable if and only if one of the following relations holds*

$$C\,\mathcal{S}^* = \mathbb{R}^q\,, \tag{2}$$

$$\mathcal{S}^* + \mathcal{C} = \mathbb{R}^n\,, \tag{3}$$

$$\mathcal{S}^* + \mathcal{V}^* = \mathbb{R}^n\,. \tag{4}$$

Proof: Condition (2) is simply the discrete-time counterpart of Theorem 4 in [1]. Since C is full row rank, (2) and (3) are equivalent (recall that $C\mathcal{S}^*$ is equal to $CP\mathcal{S}^*$ where P denotes the orthogonal projection on $\mathrm{im}\, C^{\mathrm{T}}$ along $\ker C$). Equivalence of (3) and (4) follows as the dual of the well known property

$$\mathcal{V}^* \cap \mathcal{B} = \{0\} \quad \Leftrightarrow \quad \mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^* = \{0\}\,, \tag{5}$$

first proved by Morse (Lemma 1.1 in [10]). ■

Proposition 2 *System (1) is left-invertible if and only if one of conditions (5) holds. Hence it is both right and left-invertible if and only if*

$$\mathcal{V}^* \oplus \mathcal{S}^* = \mathbb{R}^n \tag{6}$$

Theorem 1 *Let us assume that system (1) is right-invertible. Its output relative degree is the least integer ρ such that one of the following relations holds*

$$C\,\mathcal{S}_{\rho-1} = \mathbb{R}^q\,, \tag{7}$$

$$\mathcal{S}_{\rho-1} + \mathcal{C} = \mathbb{R}^n\,, \tag{8}$$

$$\mathcal{S}_{\rho-1} + \mathcal{V}^* = \mathbb{R}^n\,, \tag{9}$$

where $\mathcal{S}_i$ is provided by the standard conditioned invariant algorithm

$$\begin{aligned} \mathcal{S}_0 &:= \mathcal{B}\,, \\ \mathcal{S}_i &:= A\,(\mathcal{S}_{i-1} \cap \mathcal{C}) + \mathcal{B}\,, \quad i = 1, 2, \ldots . \end{aligned} \tag{10}$$

Proof: Eq. (7) is an obvious consequence of Definition 1 and Proposition 1, eq. (2). Equivalence of (7) and (8) comes straightforwardly. Equivalence of (8) and (9) can be shown by slightly extending the proof of the Morse Theorem as described below. The core of the proof consists of showing by induction that the sum of $\mathcal{S}_{\rho-1}$ with $\mathcal{V}^*$ is equal to the sum of $\mathcal{S}_{\rho-1}$ with the last term of a sequence $\mathcal{V}'_i$ $(i = 1, 2, \ldots)$ whose terms are equal to $\mathcal{C}$ for any i. In more detail, it consists in the following. Let $\mathcal{V}'^* = \max \mathcal{V}(A, \mathcal{S}_{\rho-1}, \mathcal{C})$, i.e. the last term of the sequence

$$\begin{aligned} \mathcal{V}'_0 &:= \mathcal{C}\,, \\ \mathcal{V}'_i &:= A^{-1}\left(\mathcal{V}'_{i-1} + \mathcal{S}_{\rho-1}\right) \cap \mathcal{C}\,, \quad i = 1, 2, \ldots\,, \end{aligned}$$

satisfying the condition $\mathcal{V}'_i \subset \mathcal{V}'_{i-1}$. By construction, $\mathcal{V}'_0 = \mathcal{C} = \mathcal{V}_0$, which implies

$$\mathcal{S}_{\rho-1} + \mathcal{V}'_0 = \mathcal{S}_{\rho-1} + \mathcal{C} = \mathcal{S}_{\rho-1} + \mathcal{V}_0\,. \tag{11}$$

Furthermore, if $\mathcal{S}_{\rho-1} + \mathcal{V}'_{i-1} = \mathcal{S}_{\rho-1} + \mathcal{V}_{i-1}$ holds for an integer i, then it implies $\mathcal{S}_{\rho-1} + \mathcal{V}'_i = \mathcal{S}_{\rho-1} + \mathcal{V}_i$ for the same i. In fact,

$$\begin{aligned} &\mathcal{S}_{\rho-1} + \mathcal{V}'_i \\ &\quad = \mathcal{S}_{\rho-1} + \left(A^{-1}(\mathcal{V}'_{i-1} + \mathcal{S}_{\rho-1}) \cap \mathcal{C}\right) \\ &\quad = \mathcal{S}_{\rho-1} + \left(A^{-1}(\mathcal{V}_{i-1} + \mathcal{S}_{\rho-1}) \cap \mathcal{C}\right) \\ &\quad = \mathcal{S}_{\rho-1} + \left(A^{-1}(\mathcal{V}_{i-1} + (A\,(\mathcal{S}_{\rho-2} \cap \mathcal{C}) + \mathcal{B})) \cap \mathcal{C}\right) \\ &\quad = \mathcal{S}_{\rho-1} + \left(\left(A^{-1}(\mathcal{V}_{i-1} + \mathcal{B}) + (\mathcal{S}_{\rho-2} \cap \mathcal{C})\right) \cap \mathcal{C}\right) \\ &\quad = \mathcal{S}_{\rho-1} + \left(\left(A^{-1}(\mathcal{V}_{i-1} + \mathcal{B}) \cap \mathcal{C}\right) + (\mathcal{S}_{\rho-2} \cap \mathcal{C})\right) \\ &\quad = \mathcal{S}_{\rho-1} + \mathcal{V}_i\,. \end{aligned} \tag{12}$$

Eqs. (11) and (12) imply that eq. (12) holds for all $i = 0, 1, \ldots$, which implies also

$$\mathcal{S}_{\rho-1} + \mathcal{V}'^* = \mathcal{S}_{\rho-1} + \mathcal{V}^*\,. \tag{13}$$

By definition of ρ,

$$\mathcal{V}'^* = \mathcal{V}'_0 = \mathcal{C}\,. \tag{14}$$

In fact,

$$\begin{aligned} \mathcal{V}'_1 &:= A^{-1}\left(\mathcal{V}'_0 + \mathcal{S}_{\rho-1}\right) \cap \mathcal{C} \\ &= A^{-1}\left(\mathcal{C} + \mathcal{S}_{\rho-1}\right) \cap \mathcal{C} = \mathcal{C}\,. \end{aligned}$$

Eq. (14) is equivalent to

$$\mathcal{S}_{\rho-1} + \mathcal{V}'^* = \mathcal{S}_{\rho-1} + \mathcal{C} = \mathbb{R}^n\,, \tag{15}$$

and, by virtue of eq. (13), is equivalent to eq. (9). ■

Corollary 1 *If system (1) is both right and left-invertible, then*

$$\mathcal{S}_{\rho-1} = \mathcal{S}^*\,, \tag{16}$$

i.e., ρ is the number of steps for evaluating $\mathcal{S}^*$.

Definition 2 (Invariant Zeros) *The invariant zeros of system (1) are the internal unassignable eigenvalues of $\mathcal{V}^*$, defined by*

$$\mathcal{Z} = \sigma(A + BF)_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}\,, \tag{17}$$

where F denotes any matrix such that $(A + BF)\,\mathcal{V}^* \subseteq \mathcal{V}^*$.

3. BASIC RESULTS

If a system is functionally controllable (or right-invertible), it is possible to reproduce at the outputs arbitrary reference trajectories, provided they are known in advance by ρ instants of time. Hence, perfect tracking *per se* should only require a preview equal to ρ. However, if the system is non-minimum phase, both the state x and the control input u, will exponentially diverge. This is not acceptable in technical practice, and is avoided by using *preaction*, i.e., action in advance on the control input, that is possible if a preview of the reference input significantly greater than the maximum time constant of the inverses of unstable zeros is available. In this section the concept of preaction is introduced on the basis of some fundamental results of the geometric approach.

Let us refer to system (1) and assume it to be square, right and left-invertible, asymptotically stable[1], with no invariant zeros on the unit circle and with (A,B) controllable. The symbol ρ denotes, as in the previous section, the output relative degree of the system.

Lemma 1 *Let us assume that A is nonsingular and denote with (A_r,B_r) the pair characterizing the reverse dynamics of system (1), i.e. $A_r := A^{-1}$, $B_r := -A^{-1}B$. The subspace $\mathcal{S}^*\cap\mathcal{C}$ is an $(A_r,\mathcal{B}_r)$-controlled invariant and all its internal unassignable eigenvalues are equal to zero.*

Proof: The $(A_r,\mathcal{B}_r)$-controlled invariance of $\mathcal{S}^*\cap\mathcal{C}$ is proved by

$$\begin{aligned} A_r\,(\mathcal{S}^*\cap\mathcal{C}) &\subseteq A_r\mathcal{S}^*\cap A_r\mathcal{C} \\ &= A_r\,(A\,(\mathcal{S}^*\cap\mathcal{C})+\mathcal{B})\cap A_r\mathcal{C} \\ &= \left(\mathcal{S}^*\cap\mathcal{C}+A^{-1}\mathcal{B}\right)\cap A^{-1}\mathcal{C} \\ &\subseteq \mathcal{S}^*\cap\mathcal{C}+\mathcal{B}_r\,. \end{aligned} \tag{18}$$

For any given state $x_f\in(\mathcal{S}^*\cap\mathcal{C})$, one and only one trajectory belonging to $\mathcal{S}^*\cap\mathcal{C}$ exists along which the state is driven from the origin to x_f. In fact, if there were two, their difference, still belonging to $\mathcal{S}^*\cap\mathcal{C}$, would lead the state back to the origin and this is against the hypothesis of left-invertibility of the system. The same trajectory, followed backward according to the reverse system dynamics, leads the state from x_f to the origin. This implies that all the internal eigenvalues of $\mathcal{S}^*\cap\mathcal{C}$ as an $(A_r,\mathcal{B}_r)$-controlled invariant are equal to zero. By the left-invertibility assumption, these eigenvalues are all unassignable. ■

Algorithm 1 *Computation of the control sequence $u(0),u(1),\ldots,u(\rho-2)$, which drives the state from $x(0)=0$ to $x(\rho-1)=\bar{x}\in\mathcal{S}^*\cap\mathcal{C}$, along a trajectory belonging to $\mathcal{S}^*\cap\mathcal{C}$.* Let $\bar{x}$ be any state belonging to $\mathcal{S}^*\cap\mathcal{C}$. Eqs. (2) and (10) imply that $\bar{x}$ can be reached from the origin in $\rho-1$ steps. By virtue of Lemma 1, the trajectory along which the state is driven from $x(\rho-1)=\bar{x}$ to the origin according to the dynamics of the closed-loop reverse system is given by

$$x(k-1) = (A_r+B_rF_r)\;x(k)\,,\; k=\rho-1,\ldots,1\,, \tag{19}$$

where F_r is such that $\mathcal{S}^*\cap\mathcal{C}$ is an invariant in $A_r+B_rF_r$. The relation

$$u(k) = F_r\,x(k+1)\,,\quad k=0,1,\ldots,\rho-2\,, \tag{20}$$

gives the corresponding control input sequence. ■

Remark 1 *By means of a simple contrivance, Algorithm 1 can also be used if A is singular. In this case, a suitable pole placement can be performed, since (A,B) is controllable. Let H be such that $\bar{A}:=A+BH$ is nonsingular and denote by $\bar{u}(k)$ the control sequence obtained for the triple $(\bar{A},B,C)$. Since the algorithm also provides the state $x(k)$, the control for (A,B,C) is given by*

$$u(k) = \bar{u}(k) - H\,x(k)\,,\quad k=0,1,\ldots,\rho-2\,. \tag{21}$$

Theorem 2 *For any given output $y_f\in\mathbb{R}^q$, a control input sequence $u(0),u(1),\ldots,u(\rho-1)$ exists, which drives the state from $x(0)=0$ to $x(\rho)=x_f$, where x_f is such that $C\,x_f=y_f$, along a trajectory belonging to $\mathcal{S}^*\cap\mathcal{C}$ (therefore invisible at the output) until the last step but one.*

Proof: The right invertibility of the system implies eq. (2). Therefore, for any given $y_f\in\mathbb{R}^q$, $x_f\in\mathcal{S}^*$ exists such that $C\,x_f=y_f$. Since $\mathcal{S}^*=A\,(\mathcal{S}^*\cap\mathcal{C})+\mathcal{B}$, $\bar{x}\in(\mathcal{S}^*\cap\mathcal{C})$ and $\mu\in\mathbb{R}^p$ exist such that $x_f=A\,\bar{x}+B\,\mu$. Algorithm 1 provides the control input sequence that drives the state from the origin to $\bar{x}$ along a trajectory belonging to $\mathcal{S}^*\cap\mathcal{C}$, while the control input that drives the state from $\bar{x}$ to x_f is $u(\rho-1)=\mu$. Let V_r be a basis matrix of $\mathcal{S}^*\cap\mathcal{C}$, so that $\bar{x}=V_r\,\beta$. Relation

$$\begin{bmatrix}\beta\\ \mu\end{bmatrix} = \left(C\;[AV_r\;B]\right)^{\#}\;y_f \tag{22}$$

provides β and μ. ■

Let us note that Theorem 2 solves the problem of "structural" perfect tracking (or right-inversion), i.e. the problem (considered in the earliest investigations on system invertibility) that aims at reproducing a given output trajectory with no care for possible divergence of the state. In fact, any output can be imposed with an invisible preaction consisting of $\rho-1$ samples. Then, by taking into account at every instant of time the output produced by the previous actions, it is possible to impose any output trajectory. However, in order to guarantee internal stability, it is necessary not only to impose the generic output y_f at the time instant ρ, but also to impose zero output at subsequent times, while maintaining the state bounded. The following theorem and corollaries provide the insight which is necessary to take into account also the internal stability constraint.

[1] Stability is often ensured by feedback — see the next section.

Theorem 3 *Let us assume that the state of system (1) is forced to a given $x_f \in \mathbb{R}^n$ at the time instant ρ by an external event. A control input sequence $u(-\infty),\ldots,u(0),\ldots,u(\infty)$ exists which both nulls the effect of the state x_f on the output and avoids state divergence.*

Proof: By the assumption of right and left-invertibility of the system, x_f can be decomposed as $x_f = \bar{x}_{\mathcal{V}^*} + \bar{x}_{\mathcal{S}^*}$, where $\bar{x}_{\mathcal{V}^*} \in \mathcal{V}^*$ and $\bar{x}_{\mathcal{S}^*} \in \mathcal{S}^*$. In the proof of Theorem 2 it has been shown that a control input sequence, $u_1(0), u_1(1), \ldots, u_1(\rho-1)$, exists which drives the state from $x(0)=0$ to $x(\rho) = -\bar{x}_{\mathcal{S}^*}$, thus nulling the effect of $\bar{x}_{\mathcal{S}^*}$ on the output from the time instant ρ on. The component $\bar{x}_{\mathcal{V}^*}$ is forced to remain on $\mathcal{V}^*$, decomposed into two trajectories converging asymptotically to the origin, one for k approaching infinity, by virtue of the postaction control sequence, the other for k coming from minus infinity, by virtue of the preaction control sequence. Let $\mathcal{V}_S$ and $\mathcal{V}_U$ be (A,B)-controlled invariants, strictly stable and strictly antistable respectively, such that $\mathcal{V}^* = \mathcal{V}_S \oplus \mathcal{V}_U$ (this decomposition is possible by left invertibility). Then $\bar{x}_{\mathcal{V}^*}$ can be decomposed as $\bar{x}_{\mathcal{V}^*} = \bar{x}_{\mathcal{V}_S} + \bar{x}_{\mathcal{V}_U}$, where $\bar{x}_{\mathcal{V}_S} \in \mathcal{V}_S$ and $\bar{x}_{\mathcal{V}_U} \in \mathcal{V}_U$. Let F be such that $\mathcal{V}^*$, $\mathcal{V}_S$ and $\mathcal{V}_U$ are $A+BF$ invariants. Since all the internal eigenvalues of $\mathcal{V}_S$ in $A+BF$ are stable, a control input sequence, $u_2(\rho), u_2(\rho+1), \ldots, u_2(\infty)$, exists which drives $\bar{x}_{\mathcal{V}_S}$ asymptotically to the origin, along a trajectory belonging to $\mathcal{V}_S$. Since all the internal eigenvalues of $\mathcal{V}_U$ in $A+BF$ are antistable, let us first compute the trajectory that, followed backwards, would lead $-\bar{x}_{\mathcal{V}_U}$ asymptotically to the origin and the corresponding control input sequence. By applying it backwards in time, $u_3(-\infty), \ldots, u_3(\rho-2), u_3(\rho-1)$, it is possible to null $\bar{x}_{\mathcal{V}_U}$ at the time instant ρ, thus avoiding exponential divergence of the state. In the most general case, by applying the sum of the previously defined control sequences (each assumed equal to zero wherever not explicitly defined), the target specified in the statement is achieved. ■

Remark 2 *The above proof of Theorem 3 points out that, if the system has no invariant zeros, only the relative-degree preaction, $u_1(k)$ $(k=0,\ldots,\rho-1)$ has to be computed, if the invariant zeros are all stable, also the infinite-horizon postaction, $u_2(k)$ $(k=\rho,\ldots,\infty)$ has to be applied, while the infinite-horizon preaction, $u_3(k)$ $(k=-\infty,\ldots,\rho-1)$ has to be added only in the nonminimum-phase case.*

Algorithm 2 *Details on the computation of the control sequences $u_1(k)$, $u_2(k)$ and $u_3(k)$.* Let V and V_r be basis matrices of $\mathcal{V}^*$ and $\mathcal{S}^* \cap \mathcal{C}$, respectively. Then $x_f \in \mathbb{R}^n$ can be decomposed as $x_f = V\alpha + AV_r\beta + B\mu$ where $\alpha \in \mathbb{R}^s$ with $s := \dim \mathcal{V}^*$, $\beta \in \mathbb{R}^t$ with $t := \dim(\mathcal{S}^* \cap \mathcal{C})$ and $\mu \in \mathbb{R}^p$ are given by

$$\begin{bmatrix} \alpha \\ \beta \\ \mu \end{bmatrix} = [V \; AV_r \; B]^{\#} \, x_f \, . \qquad (23)$$

The component $\bar{x}_{\mathcal{S}^*} = AV_r\beta + B\mu$ can be canceled by reaching its opposite as specified in Theorem 2. The relative-degree control sequence $u_1(k)$ is so obtained. The component $\bar{x}_{\mathcal{V}^*}$ can be managed by applying the control input sequences resulting from the procedure described below. Let us perform the state space basis transformation $T := [V \; S]$, where S is a basis matrix of $\mathcal{S}^*$. The matrices A', B', C' in the new basis have the following structures:

$$\begin{aligned} A' &:= T^{-1}AT = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix}, \\ B' &:= T^{-1}B = \begin{bmatrix} 0 \\ B'_2 \end{bmatrix}, \\ C' &:= CT = \begin{bmatrix} 0 & C'_2 \end{bmatrix}, \end{aligned} \qquad (24)$$

where $A'_{11} \in \mathbb{R}^{s\times s}$ and each of the other submatrices has accordingly defined dimensions. Let us consider a state feedback matrix $F' := [F'_1 \; 0]$, where $F'_1 := -(B'_2)^{\#} A'_{21}$. Then the closed-loop system matrix is

$$A'_F := A' + B'F' = \begin{bmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{bmatrix}, \qquad (25)$$

where $A'_{11} \in \mathbb{R}^{s\times s}$ is the restriction of A'_F to $\mathcal{V}^*$. By performing a further basis transformation $T' \in \mathbb{R}^{s\times s}$ separating the stable and unstable invariant subspaces of A'_{11} it follows

$$A''_{11} := (T')^{-1} A'_{11} T' = \begin{bmatrix} A_S & 0 \\ 0 & A_U \end{bmatrix}. \qquad (26)$$

The corresponding F''_1 of the restriction F'_1 of F' to $\mathcal{V}^*$ can be accordingly partitioned as $F''_1 = [F_S \; F_U]$. Let

$$\begin{bmatrix} \bar{x}_S \\ \bar{x}_U \end{bmatrix} := (T')^{-1} \alpha \, , \qquad (27)$$

the postaction state trajectory in the new basis is computed from the initial condition $x_2(\rho) = \bar{x}_S$ by the recursive formula

$$x_2(k+1) = A_S \, x_2(k) \, , \quad k = \rho, \rho+1, \ldots \, , \qquad (28)$$

while the corresponding control input sequence is

$$u_2(k) = F_S \, x_2(k) \, , \quad k = \rho, \rho+1, \ldots \, . \qquad (29)$$

The preaction state trajectory is similarly computed from the initial condition $x_3(\rho) = -\bar{x}_U$ by the recursive formula

$$x_3(k-1) = (A_U)^{-1} \, x_3(k) \, , \; k = \rho, \rho-1, \ldots \, , \qquad (30)$$

while

$$u_3(k) = F_U \, x_3(k) \, , \; k = \ldots, -1, 0, \ldots, \rho-1 \, . \qquad (31)$$

gives the corresponding control input sequence. ■

Corollary 2 *For any given $y_f \in \mathbb{R}^q$, a control input sequence reproducing y_f at the output at the time instant ρ exists, nulling the output elsewhere and maintaining the state bounded.*

Proof: This control input sequence can be obtained by combining the control input sequence leading the state from $x(0)=0$ to $x(\rho)=x_f$, where x_f is such that $C\,x_f=y_f$, along a trajectory belonging to $\mathcal{S}^*\cap\mathcal{C}$ until the time instant $\rho-1$ (Theorem 2) and the sequence nulling the effect on the output of the state $A\,x_f$ that is produced at the time instant $\rho+1$ by the previous sequence (Theorem 3). ■

A generic input component computed as specified in Corollary 2 can be represented, as a function of time, by a profile like that shown in Figure 1. The multi-

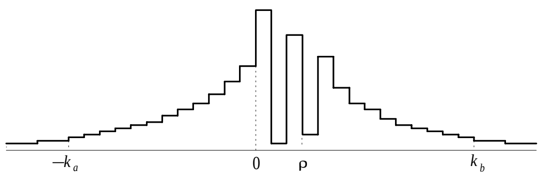

Fig. 1: A typical input component generating the impulse y_f at the time instant ρ.

variable convolution profile is obtained by considering, instead of a single output vector y_f, the identity matrix I_q. The corresponding state and input become matrices as well. In this case (22) is replaced by

$$\left[\begin{array}{c}\beta\\ \mu\end{array}\right]=(C\;[AV_r\;B])^{\#}\;I_q \tag{32}$$

and, having set $X_f=A\,V_r\,\beta+B\,\mu$, (23) by

$$\left[\begin{array}{c}\alpha\\ \beta\\ \mu\end{array}\right]=[V\;AV_r\;B]^{\#}\;X_f\,. \tag{33}$$

The control input sequence results in a sequence of matrices $H(k)$ ($k=-\infty,\ldots,0,1,\ldots,\infty$). The convolution

$$u(k)=\sum_{\ell=-\infty}^{\infty}H(\ell)\,r(k-\ell)\,, \tag{34}$$

provides perfect tracking of an arbitrary signal $r(k)$ with a delay of ρ samples.

4. THE RIGHT INVERSION WITH A FIR SYSTEM

We refer to the block diagram shown in Fig. 2, where a suitable digital processor delivers to a standard discrete-time control loop both the signal y_d (the output to be reproduced, equal to the reference input r suitably delayed) and an input-correction signal u_2. We suppose that the closed-loop system is stable and that the regulator possibly contains some internal model to perform asymptotic perfect tracking of standard signals like steps, ramps and so on. The purpose of the digital processor is to compute the correction signal $u_2(k)$ that realizes almost perfect tracking up to an arbitrary accuracy. As shown in the previous section, in order to obtain perfect tracking, it is necessary to perform the convolution (34) with profiles of the

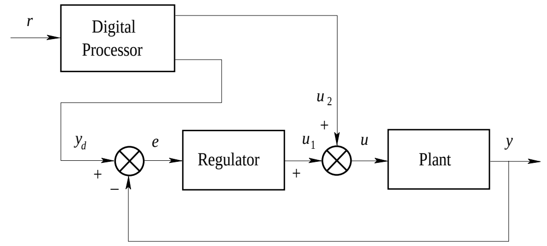

Fig. 2: The control system considered.

type shown in Fig. 1. This would require infinite preaction time (if the plant is nonminimum-phase) and infinite postaction time. However, in practice, due to exponential convergence to zero it is possible to truncate the signal at a finite time k_b towards $+\infty$ and at a finite time $-k_a$ towards $-\infty$ with negligible error (or, in any case, when the value of the signal is close to that of the computer relative accuracy). Such a behavior can be derived from a FIR system of the type

$$\left\{\begin{array}{rcl}u_2(k)&=&\displaystyle\sum_{\ell=-k_a}^{k_b}H(\ell)\,r(k-\ell-k_a)\\ y_d(k)&=&r(k-k_a-\rho)\,,\end{array}\right. \tag{35}$$

where the matrices $H(\ell)$ ($\ell=-k_a,\ldots,k_b$) are $p\times q$ in the MIMO case. Of course, when the plant is minimum-phase, in (35) the preaction time k_a should be set equal to zero. In some cases (e.g., tracking a profile with a machine-tool) the preaction time is arbitrarily large, while in other cases preaction time is related to a preview time, that may be variable during operation (e.g., tracking a route with an aircraft), or preaction time can be fixed (receding-horizon almost perfect tracking).

Truncation produces a tracking error that is managed by the regulator. As far as truncation at $-k_a$ is concerned, the error for an impulse to be tracked can be evaluated as due to an initial state of the overall system, which is equal to the opposite of the state reached at $-k_a$ by following backward the preaction state trajectory. Truncation at k_b has a similar effect.

5. AN ILLUSTRATIVE EXAMPLE

As an illustrative example, let us consider the computation of the convolution profiles for the noncausal inversion of the system (A,B,C) where

$$A=\left[\begin{array}{cccc}0.5&1&-0.4&0\\ 0.1&0.7&0&-0.5\\ 0&0&0.4&0\\ 0&0&0&0.6\end{array}\right],\;B=\left[\begin{array}{cc}1&0\\ 0&1\\ 1&0\\ 0&1\end{array}\right],$$

$$C=\left[\begin{array}{cccc}1&0&0&0\\ 0&1&0&0\end{array}\right].$$

The relative degree is equal to 1, the invariant zeros are 0.8 and 1.1. Finite preaction and postaction are assumed, consisting of 40 samples and 20 samples, respectively. Fig. 3 shows the convolution profiles corresponding to the first component of the output, i.e. the control input sequences to be applied in order to obtain an impulse at the time instant ρ on the first output, while maintaining the second identically zero. Preaction, postaction and dead-beat-like control can easily be recognized.

Fig. 4 shows the corresponding outputs. Note that the truncation error can be considered as negligible if compared with the impulse amplitude.

6. CONCLUSION

A complete computational setting in the geometric approach framework[2] has been provided for evaluating convolution profiles that guarantee almost perfect tracking, hence decoupling, in the multivariable case. When the controlled system is both right and left invertible, the same profiles can be used for left inversion, where a possibly delayed knowledge replaces action in advance. The optimization of the truncation error deriving from feasibility constraints is out of the aim of this work and left for future investigation.

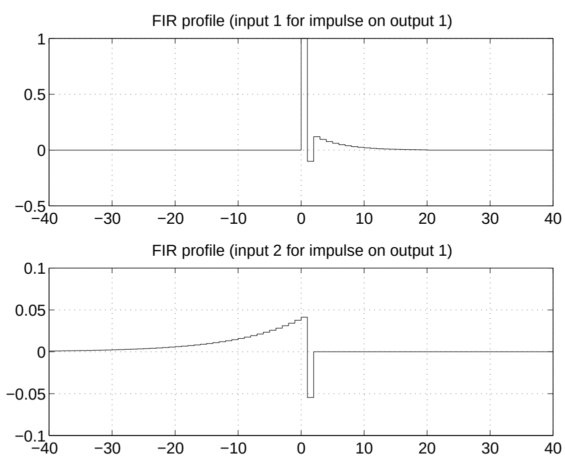

Fig. 3: Convolution profiles corresponding to the first output.

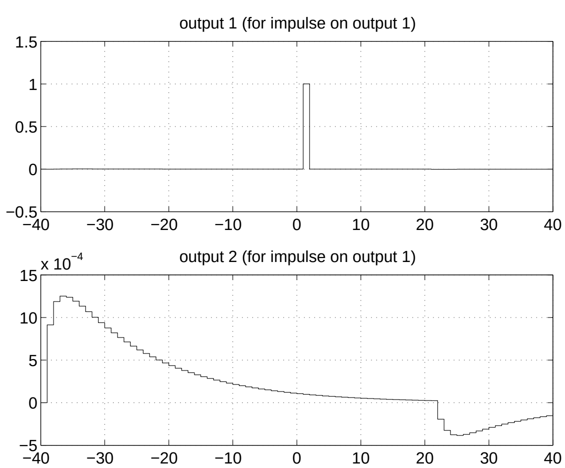

Fig. 4: Impulse and negligible errors at the outputs.

[2] This software is available for free downloading on the web site http://www.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm

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