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REACHABILITY AND CONTROLLABILITY OF 2D POSITIVE LINEAR SYSTEMS WITH STATE FEEDBACKS

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Abstract. It is shown that the reachability and controllability of positive 2D linear systems are not invariant under the state-feedbacks. By suitable choice of the state-feedbacks the unreachable positive 2D Roesser model can be made reachable and the controllable positive 2D Roesser model can be made uncontrollable.

Key Words: reachability, controllability, positive 2D Roesser model, state-feedback.

1. INTRODUCTION

The reachability and controllability are the basic concepts of modern control theory [15,5,6,14]. An overview of recent developments in reachability and controllability of 2D linear systems can be found in [16,13,12,9]. The positive (non-negative) 2D Roesser type model has been introduced in [7] and its reachability and controllability has been considered in [7-9]. The spectral and combinatorial structure and asymptotic behaviour of 2D positive system has been investigated in [3,19] and recent developments in 2D positive system theory are given in [4].
It is well-known [6] that the reachability and controllability of the standard linear systems are invariant under the state-feedbacks. Similar results are also valid for standard 2D linear systems [11]. It has been shown [10] that the reachability and controllability of positive linear 1D systems are not invariant under the state-feedbacks. To the best author's knowledge the reachability and controllability of positive 2D linear systems with state feedbacks have been not considered yet. In this paper it will be shown that the reachability and controllability of the positive 2D linear systems described by the Roesser type model are not invariant under the state-feedbacks.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR THE REACHABILITY AND CONTROLLABILITY OF POSITIVE 2D LINEAR SYSTEMS

Let $Z_+ := \{0,1,2,\dots\}$ and $R_+^{n\times m}$ be the set of real matrices of the dimension $n\times m$ with nonnegative entries ($R_+^n := R_+^{n\times 1}$).

Consider the 2D Roesser model [18]

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij} \tag{1a}$$

$$y_{ij} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + Du_{ij}, \quad i,j \in Z_+ \tag{1b}$$

where $x_{ij}^h \in R^{n_1}$ and $x_{ij}^v \in R^{n_2}$ are the horizontal and vertical state vectors at the point (i,j), respectively, $u_{ij} \in R^m$ is the input vector, $y_{ij} \in R^p$ is the output vector and $A_{kl} \in R^{n_k \times n_l}, B_k \in R^{n_k \times m}, C_k \in R^{p\times n_k}$, $k,l = 1,2$ $D \in R^{p\times m}$.

The model (1) is called internally positive (shortly positive) if for all boundary conditions

$$x_{0j}^h \in R_+^{n_1}, j \in Z_+ \quad \text{and} \quad x_{i0}^v \in R_+^{n_2}, i \in Z_+ \tag{2}$$

and all $u_{ij} \in R_+^m$, $i,j \in Z_+$ we have $x_{ij} = \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} \in R_+^n$, $n = n_1 + n_2$ and $y_{ij} \in R_+^p$ for all $i,j \in Z_+$.

It is easy to show [7] that the model (1) is positive if and only if

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\in R_+^{n\times n}, B=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\in R_+^{n\times m}, \quad C=[C_1\ C_2]\in R_+^{p\times n}, D\in R_+^{p\times m} \tag{3}$$

The transition matrix T_{ij} for (1) is defined as follows [18,5]

$$T_{ij}=\begin{cases} I_n & for \quad i=j=0 \\ T_{10}T_{i-1,j}+T_{01}T_{i,j-1} & for \quad i,j\geq 0 (i+j\neq 0) \\ T_{ij}=0 & for \quad i<0 \quad or/and j<0 \end{cases} \tag{4}$$

where

$$T_{10}:=\begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, T_{01}:=\begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

From (4) it follows that the transition matrix T_{ij} of the positive model (1) is a positive matrix, $T_{ij}\in R_+^{n\times n}$ for all $i,j\in Z_+$.

Definition 1. The positive model (1) is called reachable for zero boundary conditions (2) (ZBC) at the point $(h,k),(h,k\in Z_+, h,k>0)$, if for every $x_f\in R_+^n$ there exists a sequence of inputs $u_{ij}\in R_+^m$ for $(i,j)\in D_{hk}$ such that $x_{hk}=x_f$, where

$$D_{hk}:=\{(i,j)\in Z_+\times Z_+ : 0\leq i\leq h, 0\leq j\leq k,\ i+j\neq h+k\} \tag{5}$$

Definition 2. The positive model (1) is called controllable to zero (shortly controllable) at the point $(h,k),(h,k\in Z_+, h,k>0)$ if for any nonzero boundary conditions

$$x_{0j}^h\in R_+^{n_1},\ 0\leq j\leq k \quad \text{and} \quad x_{i0}^v\in R_+^{n_2},\ 0\leq i\leq h \tag{6}$$

there exists a sequence of inputs $u_{ij}\in R_+^m$ for $(i,j)\in D_{hk}$ such that $x_{hk}=0$.

A matrix $A\in R^{n\times n}$ is called the generalised positive permutation matrix (GPPM) or monomial matrix if and only if it has only one positive entry in each row and column and the remaining entries are equal zero.

In [7-9] the following necessary and sufficient conditions for the reachability and controllability have been proved.

Theorem 1. The positive model (1) is reachable for ZBC at the point (h,k) if and only if there exists a GPPM R_n consisting of n linearly independent columns of the reachability matrix

$$R_{hk}:=[M_{hk}, M_{h-1,k}, M_{h,k-1},\dots, M_{01}, M_{10}] \tag{7}$$

where

$$M_{ij}:=T_{i-1,j}\begin{bmatrix} B_1 \\ 0 \end{bmatrix}+T_{i,j-1}\begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{8}$$

and T_{ij} is defined by (4).

Theorem 2. The positive model (1) is controllable if and only if the matrix A is nilpotent matrix, i.e.

$$\det\begin{bmatrix} I_{n_1}z_1-A_{11} & -A_{12} \\ -A_{21} & I_{n_2}z_2-A_{22} \end{bmatrix}=z_1^{n_1}z_2^{n_2} \tag{9}$$

3. REACHABILITY OF POSITIVE LINEAR SYSTEMS WITH FEEDBACKS

To simplify the notation we assume that $m=1$ (the single-input systems) and the matrices A and B of the positive model (1) have the canonical form [5,9]

$$A_{11}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_{n_1} \end{bmatrix}, A_{12}=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \hdashline 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}, B_1=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{10}$$

$$A_{21}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n_1} \\ a_{21} & a_{22} & \cdots & a_{2n_1} \\ \hdashline a_{n_21} & a_{n_22} & \cdots & a_{n_2n_1} \end{bmatrix}, A_{22}=\begin{bmatrix} b_1 & 1 & 0 & \cdots & 0 & 0 \\ b_2 & 0 & 1 & \cdots & 0 & 0 \\ \hdashline b_{n_2-1} & 0 & 0 & \cdots & 0 & 1 \\ b_{n_2} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, B_2=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n_2-1} \\ b_{n_2} \end{bmatrix}$$

where $a_l\geq 0, a_{kl}\geq 0, b_k\geq 0$ for $k=1,\dots,n_2$, $l=1,\dots,n_1$.

Consider the system (1) with the state-feedback

$$u_{ij}=v_{ij}+K\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix},\ i,j\in Z_+. \tag{11}$$

where $K=[K_1, K_2], K_1\in R^{1\times n_1}, K_2\in R^{1\times n_2}$ and $v_{ij}\in R^m$ is a new input vector.

Substitution of (11) into (1a) yields

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = A_c \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + Bv_{ij} \tag{12}$$

where

$$A_c = A + BK = \begin{bmatrix} A_{11} + B_1K_1, & A_{12} + B_1K_2 \\ A_{21} + B_2K_1, & A_{22} + B_2K_2 \end{bmatrix} \tag{13}$$

The standard closed-loop system (12) is reachable (controllable) if and only if the standard 2D Roesser model (1) is reachable (controllable) [11].

It is easy to show that if at least one of $a_l \neq 0, l = 1,..., n_1$ or $b_k \neq 0, k = 1,..., n_2$ then the condition of theorem 1 is not satisfied and the positive model (1) is not reachable at the point (n_1, n_2). To simplify the calculations let us assume that $n_1 = 3$ and $n_2 = 2$. In this case using (10), (4) and (8) we obtain

$$T_{10} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ a_1 & a_2 & a_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, T_{01} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & b_1 & 1 \\ a_{21} & a_{22} & a_{23} & b_2 & 0 \end{bmatrix}$$

$$T_{11} = T_{10}T_{01} + T_{01}T_{10} =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & b_1 & 1 \\ a_1a_{13} & a_{11} + a_{13}a_2 & a_{12} + a_{13}a_3 & a_{13} & 0 \\ a_1a_{23} & a_{21} + a_{23}a_2 & a_{22} + a_{23}a_3 & a_{23} & 0 \end{bmatrix}$$

$$M_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, M_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_1 \\ b_2 \end{bmatrix},$$

$$M_{11} = T_{01}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} + T_{10}\begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ b_1 \\ a_{13} \\ a_{23} \end{bmatrix}$$

$$M_{20} = T_{10}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ a_3 \\ 0 \\ 0 \end{bmatrix}, M_{02} = T_{01}\begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_1^2 + b_2 \\ b_1b_2 \end{bmatrix}$$

and

$$[M_{10}, M_{01}, M_{11}, M_{20}, M_{02}, ...] = \\ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 1 & 0 & b_1 & a_3 & 0 & \cdots \\ 0 & b_1 & a_{13} & 0 & b_1^2 + b_2 & \cdots \\ 0 & b_2 & a_{23} & 0 & b_1b_2 & \cdots \end{bmatrix} \tag{14}$$

It is easy to see that the matrix (14) does not satisfy the condition of theorem 1 if $a_l \neq 0, l = 1,2,3$.

Let the positive system (1) with (10) be unreachable at the point (n_1, n_2). It will be shown that there exists a state-feedback gain matrix K such the closed-loop system (12) is reachable at the point (n_1, n_2).

Let

$$K = [-a_1, -a_2, ..., -a_{n_1}, -1, 0, ..., 0] \tag{15}$$

For (10) and (15) the matrix (13) has the form

$$A_c = A + BK = \begin{bmatrix} \bar{A}_{11}, & \bar{A}_{12} \\ \bar{A}_{21}, & \bar{A}_{22} \end{bmatrix} \tag{16a}$$

where

$$\bar{A}_{11} = A_{11} + B_1K_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{n_1} \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [-a_0, -a_1, ..., -a_{n_1}] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\bar{A}_{12} = A_{12} + B_1K_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \hdashline 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix} + \tag{16b}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [-1 \quad 0 ... 0] = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \hdashline 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\overline{A}_{21} = A_{21} + B_2K_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n_1} \\ a_{21} & a_{22} & \cdots & a_{2n_1} \\ \hdashline a_{n_21} & a_{n_22} & \cdots & a_{n_2n_1} \end{bmatrix} +$$

$$+\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n_2} \end{bmatrix} [-a_0, -a_1, \dots, -a_{n_1-1}] = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n_1} \\ \overline{a}_{21} & \overline{a}_{22} & \cdots & \overline{a}_{2n_1} \\ \hdashline \overline{a}_{n_21} & \overline{a}_{n_22} & \cdots & \overline{a}_{n_2n_1} \end{bmatrix}$$

$$\overline{A}_{22} = A_{22} + B_2K_2 = \begin{bmatrix} b_1 & 1 & 0 & \cdots & 0 & 0 \\ b_2 & 0 & 1 & \cdots & 0 & 0 \\ \hdashline b_{n_2-1} & 0 & 0 & \cdots & 0 & 1 \\ b_{n_2} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} +$$

$$+\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n_2} \end{bmatrix} [-1\ 0 \cdots 0] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

If the assumptions of the canonical form are satisfied [5,9] then it can be shown that $\overline{a}_{kl} \geq 0$ for $k = 1,\dots,n_2$, $l = 1,\dots,n_1$. Now we shall show that the closed-loop system with (16b) and $b_1 = b_2 = \cdots = b_{n_2-1} = 0, b_{n_2} \neq 0$ is reachable at the point (n_1, n_2). Using (16), (4) and (8) we obtain $M_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = e_{n_1}$ (n_1 - th column of the $n \times n$ identity matrix)

$$M_{20} = T_{10} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = e_{n_1-1}, \dots, M_{n_10} = T_{10}^{n_1-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = e_1 \, , \tag{17}$$

$$M_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = b_{n_2} e_n, M_{02} = T_{01} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = b_{n_2} e_{n-1},$$

$$\dots, M_{0n_2} = T_{01}^{n_2-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = b_{n_2} e_{n_1+1}$$

Note that in this case the matrix

$$[M_{10}, M_{20}, \dots, M_{n_10}, M_{01}, M_{02}, \dots, M_{0n_2}] =$$
$$= [e_{n_1}, e_{n_1-1}, \dots, e_1, b_{n_2} e_n, b_{n_2} e_{n-1}, \dots, b_{n_2} e_{n_1+1}]$$

is GPPM and by the theorem 1 the positive system (1) with (17) and $b_1 = b_2 = \cdots = b_{n_2-1} = 0, b_{n_2} \neq 0$ is reachable at the point (n_1, n_2). In the case when $b_k \neq 0$ for $k = 1,\dots,n_2$ the calculations in the proof are more complicated. Therefore, the following theorem has been proved.

Theorem 3. Let the positive system (1) with (10) is unreachable at the point (n_1, n_2). Then the closed-loop system (13) with (16) is reachable at the point (n_1, n_2) if the state-feedback gain matrix K has the form (15).

From theorem 3 we have the following important collorary.

Collorary 1. The reachability of positive system (1) with (10) is not invariant under the state-feedback (11).

Example 1. Consider the positive 2D Roesser model (1) with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ \hdashline 2 & 3 & 1 & 0 \\ 3 & 4 & 2 & 1 \end{array}\right], B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \left[\begin{array}{c} 0 \\ 1 \\ \hdashline 1 \\ 2 \end{array}\right] \tag{18}$$

which is unreachable at the point (2,2).

In this case $n_1 = n_2 = 2$, $m = 1$ and using (15) and (13) we obtain

$$K = [-1, -2, -1, 0] \tag{19}$$

and

$$A_c = A + BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 4 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} [-1, -2, -1, 0] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

Using (4), (8) and (20) we calculate

$$M_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, M_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix},$$

$$M_{11} = T_{01} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + T_{10} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$M_{20} = T_{10} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, M_{02} = T_{01} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

Hence the matrix

$$[M_{10}, M_{11}, M_{20}, M_{02}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

is GPPM and by the theorem 1 the closed-loop system with (19) is reachable at the point $(n_1, n_2) = (2,2)$.

4. CONTROLLABILITY OF POSITIVE LINEAR SYSTEMS WITH FEEDBACKS

Consider the positive single-input model (1) with (10) and the state-feedback (11). According the theorem 2 the positive system is controllable (to zero) if and only if the matrix A is nilpotent. It is said that the state-feedback (11) violetes the nilpotency of` A if and only if the closed-loop matrix (13) is not nilpotent. From theorem 2 the following theorem follows.

Theorem 4. The closed-loop system (12) is uncontrollable at the point (n_1, n_2) if the state-feedback (11) violetes the nilpotency of A.

Collorary. The controllability of positive system (1) is not invariant under the state-feedback (11).

Example 2. Consider the positive model (1) with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc:c} 0 & 1 & 1 \\ 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 \end{array}\right], \; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \left[\begin{array}{c} 0 \\ 1 \\ \hdashline 1 \end{array}\right] \qquad (21)$$

In this case $n_1 = 2,\ n_2 = 1$ and

$$\det \begin{bmatrix} I_{n_1} z_1 - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} z_2 - A_{22} \end{bmatrix} = \begin{vmatrix} z_1 & -1 & -1 \\ 0 & z_1 & -1 \\ 0 & 0 & z_2 \end{vmatrix} = z_1^2 z_2$$

Using (4) we obtain

$$T_{10} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, T_{01} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$T_{20} = T_{10}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, T_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{for } \begin{cases} i > 2 \ \ and \ \ j = 0 \\ i = 0 \ \ and \ \ j > 1 \\ i > 0 \ \ and \ \ j > 0 \end{cases}$$

and the component of x_{ij} caused by nonzero boundary conditions (2) is [18,5,15]

$$x_{bc}(i,j) = \sum_{k=0}^{i} T_{i-k,j} \begin{bmatrix} 0 \\ x_{k0}^v \end{bmatrix} + \sum_{l=0}^{j} T_{i,j-l} \begin{bmatrix} x_{0l}^h \\ 0 \end{bmatrix} = 0 \text{ for } i>2, j>1$$

and any x_{k0}^v and x_{0l}^h.

Therefore, the system can be transfered to zero by zero input sequence for any boundary conditions (2) and arbitrary matrix B.

Note that if the matrix B has the form (21) then any nonzero gain matrix $K = [k_1, k_2, k_3]$ violetes the nilpotency of the matrix A given by (21) since

$$A + BK = \begin{bmatrix} 0 & 1 & 1 \\ k_1 & k_2 & k_3 + 1 \\ k_1 & k_2 & k_3 \end{bmatrix}$$

If $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ then we have $A + BK = \begin{bmatrix} k_1 & 1 + k_2 & 1 + k_3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

and for $k_1 = 0$ the nilpotency of A is not violeted.

5. EXTENSIONS AND CONCLUDING REMARKS

It has been shown that the reachability and controllability of positive 2D Roesser type model are not invariant under the state-feedbacks. By suitable choice of the state-feedbacks the unreachable positive 2D Roesser type model can be made reachable and the controllable positive 2D Roesser model can be made uncontrollable.

With slight modifications the presented above considerations can be extended for multi-input positive 2D Roesser type model and positive nD (n>2) Roesser type models. It is well known [5] that the first Fornasini-Marchesini model [1] can be recasted in the 2D Roesser model. Therefore, the considerations can be immediately extended for the positive first Fornasini-Marchesini model. Extensions of the considerations for the positive second Fornasini-Marchesini model [2] and general 2D model [17] are also possible. An open problem is an extension of the considerations for singular 2D linear systems [6].

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