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H^∞ APPROACH TO PRECISION MISSILE GUIDANCE

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Abstract. The paper addresses the precision missile guidance problem where the successful intercept criterion has been defined in terms of both minimizing the miss distance and controlling the missile body attitude with respect to the target at the terminal point. We show that the H^∞ control theory when suitably modified provides as effective framework for the precision missile guidance problem. Existence of feedback controllers (guidance laws) is investigated for the case of finite horizon and non-zero initial conditions. Both state feedback and output feedback implementations are explored.

Keywords. Precision missile guidance, robust control, H^∞ control, optimal control, Riccati equations

1. INTRODUCTION

This paper considers the formulation of the precision guidance control problem where the control objective is to minimize the target/interceptor miss distance and, in addition, satisfy the terminal constraint on the interceptor body attitude relative to the target. This latter requirement ensures that the warhead principle axis is pointed towards the target aim point and lies within the lethality cone about this point. The above two requirements, taken together define sufficient conditions for maximasing warhead effectiveness. The need for the precision missile guidance problem has been brought about as a result of recent developments in weapon system and sub-system technologies as well as a shift in guided weapon system deployment and operational philosophies.

In the past, due to real-time computing constraints, major simplification of engagement kinematics model, performance index and constraints had to be implemented in order to render the solution suitable for mechanization of a real system. These simplifications lead to relatively straightforward feedback guidance laws, such as "the optimum guidance law" or the "augmented proportional navigation" with a time-varying (time-to-go) parameter (see e.g. [2, 4, 7, 16]). The performance of the resulting systems does not meet the criterion that could be classed as "precision guidance". However, with recent technological advances, particularly in computing, the past constraints do not apply. It is now feasible to look at guidance strategies that are aimed at, more accurately, placing the interceptor (warhead) with respect to the target (aim point) in order to maximize warhead effectiveness.

Firstly, we formulate the precision missile guidance problem as a linear-quadratic optimal control problem. The associated performance index is defined in a way to that explicitly takes into account both the end-game relative target/interceptor requirements as well as missile acceleration requirements. Then the optimal controller can be obtained from the corresponding Riccati differential equation. However, this approach gives the optimal solution for the case of non-maneuvering targets. Moreover, a significant shortcoming of the optimal control approach is that all the states of the target/interceptor system are typically assumed to be precisely known. However, in all practical situations only some states of the system are available for measurements and even these measurements are subject to noise and uncertainties. In other words, the precision missile guidance problem is an output feedback control problem. Another shortcoming of the optimal control theory is its lack of concern for the issue of robustness. In the design of feedback control systems, robustness is a critical issue. This is, the requirement that the control system will maintain an adequate level of performance in the face of significant plant uncertainty. Such plant uncertainties may be due to variation in the plant parameters and the effects on nonlinearities and unmodeled dynamics which have not been included in the plant model. In fact, the requirement for robustness is one of the main reasons for using feedback in control system design. Furthermore, robustness is extremely important in the precision missile guidance problem because of possible unknown target maneuvers.

One of the most significant recent advances in the area of control systems was the theory of H^∞ control (see e.g. [1, 3, 14]). The use of H^∞ control methods has provided an important tool for the synthesis of robustly stable output feedback control systems (see e.g. [9–13]). In this paper, we show that the H^∞ control theory when suitably modified provide as effective framework for the precision missile guidance problem. Our computer simulations prove that in the precision missile guidance problem with disturbances, the H^∞ control guidance law gives a much better performance than the linear quadratic optimal guidance law.

2. TARGET - INTERCEPTOR KINEMATICS MODEL

In order to develop precision guidance laws, target/interceptor engagement kinematics need to be defined in terms of the relative target/interceptor variables (system states), including target aimpoint and warhead principle axes, and the interceptor steering commands (control inputs). Using these state variables, the guidance requirements may be implemented by defining a performance index that is optimized subject to state and control constraints.

We will assume that the target and the interceptor (missile) are moving in one plane. Let $x_T(t) \in \mathbf{R}^2$ and $x_M(t) \in \mathbf{R}^2$ be the coordinates of the target and the missile at time t, respectively. Furthermore, let $v_T(t)$ and v_M be their velocities, that is

$$\dot{x}_T(t) = v_T(t),$$
$$\dot{x}_M(t) = v_M(t).$$

Introduce the relative target/missile variables

$$x_R(t) := x_T(t) - x_M(t),$$
$$v_R(t) := v_T(t) - v_M(t).$$

Furthermore, let $a_M(t) \in \mathbf{R}^2$ be the missile acceleration at time t, and let $a_T(t) \in \mathbf{R}^2$ be the target acceleration at time t. Introduce a new state variable

$$\hat{x}(t) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \\ \hat{x}_4(t) \end{bmatrix} := \begin{bmatrix} x_R(t) \\ v_R(t) \end{bmatrix} \in \mathbf{R}^4.$$

Then, using the second Newton's law, we can describe the target/interceptor motion by the following state space equation

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B_1 a_M(t) + B_2 a_T(t) \qquad (1)$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \qquad (2)$$

Let T be the so-called "time-to-go". In these notations, our first control objective to minimize the miss distance at time T can be stated as follows

$$\hat{x}_1(T)^2 + \hat{x}_2(T)^2 \to \min. \qquad (3)$$

Furthermore, let

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \in \mathbf{R}^2$$

be the vector that describes the desired end-game missile/target geometry. Then, this objective can be formalized as

$$(\hat{x}_3(T) - V_1)^2 + (\hat{x}_4(T) - V_2)^2 \to \min. \qquad (4)$$

Finally, we would like to minimize the missile acceleration over the whole time interval $[0, T]$. This natural requirement can be interpreted as

$$\int_0^T \|a_M(t)\|^2 dt \to \min. \qquad (5)$$

Here $\|\cdot\|$ denotes the standard Euclidean norm.

3. OPTIMAL CONTROL APPROACH

In this section, we suppose that the plant is described by the following linear differential equation

$$\dot{x}(t) = Ax(t) + B_1 u(t) \tag{6}$$

where $x(t) \in R^n$ is the state, $u(t) \in \mathbf{R}^m$ is the control input. We assume that the initial condition of the system is given,

$$x(0) = x_0 \tag{7}$$

where $x_0 \in \mathbf{R}^n$ is a given vector.

With this system let us associate the performance index

$$\begin{aligned} J[x(\cdot), u(\cdot)] &:= \\ \frac{1}{2}(x(T) - h)' X_T (x(T) - h) &+ \\ \frac{\alpha}{2} \int_0^T \|u(t)\|^2 dt. \end{aligned} \tag{8}$$

Here $X_T \geq 0$ is a given matrix, $h \in \mathbf{R}^n$ is a given vector, and $\alpha > 0$ is a given constant.

The linear quadratic optimal control problem can be formulated as follows:

To find the minimum of the functional (8) over the set of all $[x(\cdot), u(\cdot)] \in \mathbf{L}_2[0, T]$ satisfying the equations (6) and (7),

$$J[x(\cdot), u(\cdot)] \to \min. \tag{9}$$

Introduce the following Riccati differential equation

$$\begin{aligned} -\dot{X}(t) &= A'S(t) + S(t)A \\ &- \frac{1}{\alpha} S(t) B_1 B_1' S(t), \\ S(T) &= X_T. \end{aligned} \tag{10}$$

Furthermore, introduce the following equations

$$\begin{aligned} -\dot{r}(t) &= (A - \frac{1}{\alpha} B_1 B_1' S(t))' r(t), \\ r(T) &= X_T h, \end{aligned} \tag{11}$$

$$u_{opt}(t) = -\frac{1}{\alpha} B_1' S(t) x_{opt}(t) + \frac{1}{\alpha} B_1' r(t), \tag{12}$$

$$-\dot{g}(t) = -\frac{1}{2\alpha} r(t)' B_1 B_1' r(t),$$

$$g(T) = \frac{1}{2} h' X_T h. \tag{13}$$

Now we are in a position to state the following theorem.

Theorem 1 Consider the linear quadratic optimal control problem (6), (7), (8), (9). Then, for any x_0, h, $X_T \geq 0$ and $\alpha > 0$, the following statements hold:

(i) The minimum in the linear quadratic optimal control problem (9) is achieved.

(ii) The Riccati differential equation (10) has a unique solution on the time interval $[0, T]$.

(iii) The optimal control law $[x_{opt}(\cdot), u_{opt}(\cdot)]$ is given by the equations (10), (11), (12).

(iv) The optimal cost in the problem (9) is

$$\frac{1}{2} x_0' S(0) x_0 - x_0' r(0) + g(0)$$

where $g(\cdot)$ is defined by (13).

Proof

See [6]. □

We now can apply Theorem 1 to our precision missile guidance problem. In this case, $u(\cdot) \equiv a_M(\cdot)$, $x(\cdot) \equiv \hat{x}(\cdot)$ and the equation (6) coincides with (1) for $a_T(\cdot) \equiv 0$. The coefficients of the system (6) is defined by (2). The control objectives (3), (4), (5) can be interpreted as the optimal control problem (9) with the cost function (8) where

$$X_T := I_4, \quad h := \begin{bmatrix} 0 \\ 0 \\ V_1 \\ V_2 \end{bmatrix}. \tag{14}$$

Here I_4 is the unity square matrix of order 4.

4. H^∞ CONTROL

In this section, we present some results on H^∞ control problem, that will be applied for the precision missile guidance problem.

The H^∞ control problem was originally introduced by Zames in 1981 [15] and has subsequently played a major role in the area of robust control theory. Given a linear time invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 u(t) + B_2 w(t), \\ z(t) &= C_1 x(t) + D_1 u(t), \\ y(t) &= C_2 x(t) + D_2 w(t), \end{aligned} \tag{15}$$

where $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^m$ is the control input, $w(t) \in \mathbf{R}^p$ is the disturbance input, $z(t) \in \mathbf{R}^q$ is the controlled output, and $y(t) \in \mathbf{R}^l$ is the measured output. A, B_1, B_2, C_1, D_1, C_2, D_2 are real constant matrices of appropriate dimensions. Suppose that the exogenous disturbance input is such that $w(\cdot) \in \mathbf{L}_2[0,\infty)$.

The control problem addressed in this section is that of designing a controller that minimizes the induced norm from the uncertainty inputs $w(\cdot)$ and the *initial conditions* x_0 to the controlled output $z(\cdot)$. This problem is referred to as a H^∞ control problem with transients. The results presented in this section are based on results obtained in reference [5]. The class of controllers considered in Reference [5] are *time-varying* linear output feedback controllers K of the form

$$\begin{aligned} \dot{x}_c(t) &= A_c(t)x_c(t) + B_c(t)y(t), \\ x_c(0) &= 0, \\ u(t) &= C_c(t)x_c(t) + D_c(t)y(t), \end{aligned} \tag{16}$$

where $A_c(\cdot), B_c(\cdot), C_c(\cdot)$ and $D_c(\cdot)$ are bounded piecewise continuous matrix functions. Note, that the dimension of the controller state vector x_c may be arbitrary.

In the problem of H^∞ control with non-zero initial conditions, the performance of the closed loop system consisting of the underlying system (15) and the controller (16), is measured with a *worst-case closed-loop performance measure* defined as follows. For a fixed time $T > 0$, a symmetric positive definite matrix P_0 and a nonnegative definite symmetric matrix X_T, the worst-case closed-loop performance measure is defined by

$$\Pi(K, X_T, P_0, T) := \sup \left\{ \frac{x(T)'X_T x(T) + \int_0^T \|z(t)\|^2 dt}{x(0)'P_0 x(0) + \int_0^T \|w(t)\|^2 dt} \right\}, \tag{17}$$

where the supremum is taken over all $x(0) \in \mathbf{R}^n$, $w(\cdot) \in \mathbf{L}_2[0,T]$ such that

$$x(0)'P_0 x(0) + \int_0^T \|w(t)\|^2 dt > 0.$$

From this definition, the performance measure $\Gamma(K, X_T, P_0, T)$ can be regarded as the induced norm of the linear operator which maps the pair $(x_0, w(\cdot))$ to the pair $(x(T), z(\cdot))$ for the closed loop system; see [5]. In this definition, T is allowed to be ∞ in which case $X_T := 0$ and the operator mentioned above is an operator mapping the pair $[x(0), w(\cdot)]$ to $z(\cdot)$. Another special case arises where $x(0) = 0$. In this case, the supremum on the right-hand side of (17) is taken over all $w(\cdot) \in \mathbf{L}_2[0,\infty)$, and the performance measure reduces to the standard H^∞ norm defined as

$$\Pi(K, T) := \sup \left\{ \frac{\int_0^T \|z(t)\|^2 dt}{\int_0^T \|w(t)\|^2 dt} \right\}.$$

The H^∞ control problem with non-zero initial conditions is now defined as follows. Let the constant $\gamma > 0$ be given.

Finite Horizon Problem Does there exist a controller of the form (16) such that

$$\Pi(K, X_T, P_0, T) < \gamma^2? \tag{18}$$

The solutions of the both stated H^∞ problems can be found in [5].

5. STATE FEEDBACK H^∞ MISSILE GUIDANCE

In this section, we apply the results of [5] to the precision missile guidance problem.

The missile/target dynamics is described by the equation (1) with the coefficients (2). In this case, the whole state vector $\hat{x}(t)$ is available for the measurement. Moreover, we assume that the measurements are "perfect" (contain no noise). Let x_0 be an estimate of the initial condition $\hat{x}(0)$. Firstly, we assume that $a_T(\cdot) \equiv 0$ and solve the optimal control problem (8), (9), (14) for the system (6), (2). Let $[x_{opt}(\cdot), u_{opt}(\cdot)]$ be the solution of this optimal control problem. Furthermore, let

$$\begin{aligned} x(t) &:= \hat{x}(t) - x_{opt}(t), \\ u(t) &:= a_M(t) - u_{opt}(t), \\ w(t) &:= a_T(t). \end{aligned}$$

Then, $x(\cdot), u(\cdot)$ and $w(\cdot)$ satisfy the first of the equations (15) with the coefficients (2). Furthermore, let

$$C_1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_1 = \sqrt{\frac{\alpha}{2}} I_2. \tag{19}$$

The main idea of our approach can be formulated as follows. At the first step, we find the solution $[x_{opt}(\cdot), u_{opt}(\cdot)]$ of the optimal control problem. Then, we design an H^∞ controller and use it to compensate the target maneuvers $a_T(\cdot)$ and keep the real trajectory $[\hat{x}(t), a_M(\cdot)]$ of the missile/target system as close as possible to the "perfect" trajectory $[x_{opt}(\cdot), u_{opt}(\cdot)]$. Here we treat the target acceleration $a_T(\cdot)$ as the disturbance input.

We can summarize our method as the following four step procedure:

Step 1. Applying Theorem 1, find the solution $[x_{opt}(\cdot), u_{opt}(\cdot)]$ of the linear quadratic optimal control problem (9) for the system (6), (2) with the cost function (8), (14).

Step 2. Applying the results of [5] on state feedback H^∞ control to the system (15), (2), (19) with $P_0 = I_4$ and X_T, h defined by (14), find sub-minimal γ_0 such that the state feedback H^∞ control problem (18) has a solution for $\gamma = \gamma_0$.

Step 3. For this sub-minimal γ_0, design the corresponding state feedback H^∞ controller $u(\cdot)$. Here $\hat{x}(t)$ is available for the measurement, and $x_{opt}(t)$ is pre-computed.

Step 4. The resulting control command $a_M(\cdot)$ in our state feedback precision missile guidance problem is given by the following equation

$$a_M(t) = u_{opt}(t) + u(t).$$

6. OUTPUT FEEDBACK H^∞ MISSILE GUIDANCE

In this section, we apply results on output feedback H^∞ control to the precision missile guidance problem.

As in the state feedback case, the missile/target dynamics is described by the equation (1) with the coefficients (2). However, we now consider the case when only the vector $x_R(t)$ is available for the measurement. Moreover, we assume that these measurements are affected by sensor noise. This can be expressed in a vector form as

$$\hat{y}(t) = C_2\hat{x}(t) + n(t).$$

Here $\hat{y}(t) \in \mathbf{R}^2$ is the measured output, $n(t) \in \mathbf{R}^2$ is the sensor noise, and

$$C_2 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{20}$$

We apply robust filtering methods from the book [8].

Let x_0 be an estimate of the initial condition $\hat{x}(0)$. Again, as in the state feedback case, at the first step, we assume that $a_T(\cdot) \equiv 0$ and solve the optimal control problem (8), (9), (14) for the system (6), (2). Let $[x_{opt}(\cdot), u_{opt}(\cdot)]$ be the solution of this optimal control problem. Furthermore, let

$$\begin{aligned} x(t) &:= \hat{x}(t) - x_{opt}(t), \\ u(t) &:= a_M(t) - u_{opt}(t), \\ w(t) &:= \begin{bmatrix} a_T(t) \\ n(t) \end{bmatrix}. \end{aligned}$$

Then, $x(\cdot), u(\cdot)$ and $w(\cdot)$ satisfy the equations (15) with the coefficients C_2 defined by (20), and A, B_1, B_2, D_2 defined by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{21}$$

Furthermore, it immediately follows from the above equations that

$$y(t) = \hat{y}(t) - C_1 x_{opt}(t). \tag{22}$$

Now let

$$C_1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_1 = \sqrt{\frac{\alpha}{2}} I_2. \tag{23}$$

The main idea of our method can be formulated as follows. At the first step, we find the solution $[x_{opt}(\cdot), u_{opt}(\cdot)]$ of the optimal control problem. Then, we design an H^∞ controller and use it to compensate the target maneuvers $a_T(\cdot)$ and keep the real trajectory $[\hat{x}(t), a_M(\cdot)]$ of the missile/target system as close as possible to the "perfect" trajectory $[x_{opt}(\cdot), u_{opt}(\cdot)]$.

In this case, the target acceleration $a_T(\cdot)$ and the sensor noise $n(\cdot)$ are treated as the disturbance input.

We can summarize our method as the following four step procedure:

Step 1. Applying Theorem 1, find the solution $[x_{opt}(\cdot), u_{opt}(\cdot)]$ of the linear quadratic optimal control problem (9) for the system (6), (2) with the cost function (8), (14).

Step 2. Applying the results of [5] on output feedback H^∞ control to the system (15), (20), (21) with $P_0 = I_4$ and X_T, h defined by (14), find sub-minimal γ_0 such that the output feedback H^∞ control problem (18) has a solution for $\gamma = \gamma_0$.

Step 3. For this sub-minimal γ_0, design the corresponding output feedback H^∞ controller $u(\cdot)$. Here $\hat{y}(t)$ is available for the measurement, and $x_{opt}(t)$ is pre-computed.

Step 4. The resulting control command $a_M(\cdot)$ in our state feedback precision missile guidance problem is given by the following equation

$$a_M(t) = u_{opt}(t) + u(t).$$

7. CONCLUSIONS

The precision missile guidance problem was considered. A mathematically rigorous statement of this problem has been given. We have compared optimal control approach and H^∞ control methods for this problem. It has been shown that the H^∞ control theory when suitably modified provide as effective framework for the precision missile guidance problem. Both state feedback and output feedback problems were considered.

8. ACKNOWLEDGEMENTS

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