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The reduced-order Luenberger observer and its relation to the full-order state observer

A.T.Alexandridis
Department of Electrical and Computer Engineering
University of Patras
Rion 265 00, Patras, Greece

Abstract – The relation between the reduced-order Luenberger and the full-order state observer is considered. In particular, it is proved that for a completely observable linear system, one can always design a full-order state observer which is dynamically equivalent to any arbitrarily selected reduced-order Luenberger observer.

Keywords – Observers, Linear systems, State estimation.

1. Introduction

In systems analysis and design it is usually needed to know the response of all the states of a dynamic system. However, in many practical applications, it is not possible to measure all the states because some of them may not be physically accessible. In these situations, if the system is completely observable, it is possible to reconstruct the states by using just the output and the input of the system. The dynamic system that reconstructs the states of a system is called observer. In particular, for linear dynamic systems two types of observers have been proposed: the full-order state observer and the reduced-order Luenberger observer. [1-5]. Both of them are by themselves linear systems which reconstruct either the full-order state vector of a system or the unavailable states.

The full-order observer is represented by a dynamic linear system of the same order of the original system, while the reduced-order observer is represented by a dynamic linear system of the order of the unavailable states. In the literature many comments on the response of the two types of observers have been discussed; in general, it is considered that the response of the full-order observer additionally achieves a significant degree of filtering of the available states [3,4]. However, there is not exist any direct mathematical relation between the two kinds of observers which could be used as a comparison basis.

In this paper it is proved that for any completely observable linear dynamic system, there exists a full-order observer which has exactly the same dynamic performance as a reduced-order observer. Particularly, in this case, it is proved that the gain-matrix of the full-order observer is related by a simple algebraic expression to the gain-matrix of the reduced-order observer and both observers appear to be equivalent.

2. Main results

Consider the completely observable linear system

$$\dot{x} = Ax + Bu \quad (1)$$
$$y = Cx$$

where $x \in R^n, u \in R^m, y \in R^r$ and *A, B,* and *C* are constant matrices of appropriate dimensions with *rank(B)* = *m* and *rank(C)* = *r*. Define an arbitrary *(n-r)xn* constant full rank matrix C_c so that the inverse of $\begin{bmatrix} C \\ C_c \end{bmatrix}$ exists, i.e.

$$\begin{bmatrix} C \\ C_c \end{bmatrix}^{-1} = \begin{bmatrix} Q & Q_c \end{bmatrix} \quad (2)$$

where *Q* and Q_c are constant matrices with dimensions *nxr* and *nx(n-r)* respectively.

It is well known [3,4] that for a completely observable system by eq. (1), an asymptotic state observer is constructed by the following n-order dynamic system

$$\dot{x} = (A - LC)x + Bu + Ly \quad (3)$$

where x is the estimated state vector of *x* and *L* is the *nxr* gain-matrix of the state observer. In order to achieve fast reconstruction matrix *L* is selected in such a way that the eigenvalues of *A-LC* be located more to the left than the system eigenvalues- usually three to ten times. However, (3) provides an estimate of the *n*-order state vector. One may alternatively use a *(n-r)* reduced-order Luenberger observer. The reduced-order observer is given by

$$\dot{z} = (A_{22} - PA_{12})z + (B_2 - PB_1)u + Gy \quad (4)$$

where

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} CAQ & CAQ_c \\ C_c AQ & C_c AQ_c \end{bmatrix}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} CB \\ C_c B \end{bmatrix} \quad (5)$$

and

$$G = (A_{22} - PA_{12})P + A_{21} - PA_{11} \quad (6)$$

In equation (4), *P* is the *(n-r)xr* gain matrix of the reduced-order observer with state vector the *(n-r)*-order vector z, [4]:
$z = C_c x_r - PCx = C_c x_r - Py$.

Therefore, in this case, the estimated state vector x_r of system (1) is

$$x_r = \begin{bmatrix} Q & Q_c \end{bmatrix} \begin{bmatrix} y \\ z + Py \end{bmatrix} \quad (7)$$

Now, we are ready to propose the main result of the paper.

THEOREM 1. The full-order observer given by eq. (3) with gain-matrix $L = Q_c G$, is exactly equivalent to the reduced-order observer given by eq. (4).
Proof:

Let a reduced-order observer given by equations (4), (7), where *P* is arbitrarily selected. Consider the similarity transformation

$$M = \begin{bmatrix} I_r & 0 \\ -P & I_{n-r} \end{bmatrix} \begin{bmatrix} C \\ C_c \end{bmatrix} = \begin{bmatrix} C \\ C_c - PC \end{bmatrix} \quad (8)$$

Then, the inverse of *M* is

$$M^{-1} = \begin{bmatrix} Q & Q_c \end{bmatrix} \begin{bmatrix} I_r & 0 \\ P & I_{n-r} \end{bmatrix} = \begin{bmatrix} Q + Q_c P & Q_c \end{bmatrix} \quad (9)$$

Applying the coordinate transformation M on the full-order observer of eq. (3), one obtains

$$M\dot{x} = M(A-LC)M^{-1}M\,x + MBu + MLy \tag{10}$$

where for L = Q_cG :

$$ML = \begin{bmatrix} CL \\ (C_c - PC)L \end{bmatrix} = \begin{bmatrix} CQ_cG \\ C_cQ_cG - PCQ_cG \end{bmatrix} = \begin{bmatrix} 0 \\ G \end{bmatrix} \tag{11}$$

and

$$M(A-LC)M^{-1} = \begin{bmatrix} CAQ + CAQ_cP & CAQ_c \\ O & C_cAQ_c - PCAQ_c \end{bmatrix} \tag{12}$$

Substituting (11) and (12) into (10) and denoting $\begin{bmatrix} y \\ y_c \end{bmatrix} = Mx$, one arrives at

$$\begin{bmatrix} \dot{y} \\ \dot{y}_c \end{bmatrix} = \begin{bmatrix} CAQ + CAQ_cP & CAQ_c \\ O & A_{22} - PA_{12} \end{bmatrix}\begin{bmatrix} y \\ y_c \end{bmatrix} + \begin{bmatrix} C \\ C_c - PC \end{bmatrix}Bu + \begin{bmatrix} O \\ G \end{bmatrix}y \tag{13}$$

Equation (13) reveals that the dynamic response of y_c is completely independent from the dynamic response of y. Especially, taking into account (9), the first row of (13) obviously results in:

$$\dot{y} = CAM^{-1}\begin{bmatrix} y \\ y_c \end{bmatrix} + CBu = CAx + CBu \tag{14}$$

Therefore, y actually represents y, as given by $y = CAx + CBu$, while from the second row of (13), it is evident that y_c, is exactly z, as is given by eq. (4). Furthermore, by definition we have

$$\begin{bmatrix} y \\ y_c \end{bmatrix} = Mx \Rightarrow x = [Q \quad Q_c]\begin{bmatrix} y \\ y_c + Py \end{bmatrix} = [Q \quad Q_c]\begin{bmatrix} y \\ z + Py \end{bmatrix} \tag{15}$$

Comparing (15) with (7), one obviously ascertains that using the gain-matrix $L = Q_cG$, the states estimated by the full-order observer of eq. (3) are dynamically equivalent to the states estimated by the reduced-order observer of eq. (4). This completes the proof of the theorem.

3. Illustrative example

Consider the system

$$\dot{x} = \begin{bmatrix} 0 & -2 \\ 1 & -4 \end{bmatrix}x + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u$$
$$y = [0 \quad 1]x$$

Let

$$C_c = [1 \quad 0]$$

then, applying (4), a first-order Luenberger observer with a pole at –3 has a gain P=3.
Also, from (6), G=1.
Therefore, the "equivalent" full-order observer has a gain matrix

$$L = Q_cG = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

4. Concluding remarks

As shown by equation (13) which is a substantial point of the proof of the proposed theorem, i) the full-order state observer of eq.(3) results in the reduced-order observer of eq.(4) under the constraint $L = Q_cG$, ii) both, the reduced-order and the full-order observers have *n-r* eigenvalues assigned in common values.

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