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Lotstreaming Single Product in 3-Machine No-Wait Flow=Shops

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Abstract *We consider the problem of minimizing makespan in a no-wait flowshop with three machines. A lot consists of many identical items of the same product. Lot streaming (lot sizing) is the process of creating sublots to move the completed portion of a production sublot to downstream machines so that operations will be overlapped. for each product, we first consider the number of sublots as given, and we obtain optimal continuous-sized sublots. Some general results are proved for the general no-wait flow-shop with an arbitrary number of machines. These results are then applied to the 3-machine case, to find the optimal size sublots. Then, using this solution, and assuming there is a setup time between the transfer of batches, the optimal number of sublots is derived for the 2-machine no-wait flow-shop. For the 3-machine cases, we distinguish two situations. A closed form formula is proved in the first, and a lower bound is provided in the second.*

Key words and phrases: *No-wait flowshop, lot streaming, batches.*

1 Introduction

We consider the problem of lot streaming of a single product through 3 machines in no-wait flowshops in order to minimize the makespan. A lot consists of many identical items of a given product. Lot streaming is the process of creating sublots to move the completed portion of a production sublot to the next machines. The creation of sublots permits the overlapping of different operations on the same product and may therefore reduce the makespan. Summaries of literature on lot streaming are given by Baker (1990, 1995), Potts and Van Wassenhove (1992), Trietsch and Baker (1993), and Vickson (1995).

Szendrovits (1975) models a single product with equal-sized sublots requiring treatments in a multi-stage flowshop production system where the processing sublots are *continuous* in the sense that each product must be processed continuously on each machine, so that there is no idle time between its sublots. But there may be idle times between the production runs of different products. Goyal (1976) studies the problem of finding optimal sublot sizes for the above model. In our flowshop model, processing of sublots is *no-wait*: the processing of a sublot on a machine is started as soon as the processing of the sublot on the preceding machine has ben completed. This may lead to machine idle times between successive sublots. Lotsizing and scheduling problems in no-wait flowshops arise in chemicals processing and petro-chemical production environments. Another example of the no-wait situation arises in hot metal rolling industries where the metals have to be processed continuously at high temperature. In recent years, a considerable amount of interest has arisen in no-wait scheduling problems. This interest appears to be motivated as much by applications as by questions of research interest (refer to the survey papers on no-wait scheduling by Hall and Sriskandarajah 1996, Goyal and Sriskandarajah 1988).

Potts and Baker (1993) deal with the lot streaming problem of one product in a general flowshop while Glass, Gupta and Potts (1994) deal with only the three-machine flowshop. Baker (1993) and Vickson (1995) deal with the problem of lot streaming and scheduling multiple products in the two-machine flowshop where the buffer between machines is unlimited. In this paper, we consider two-machine flowshop with no-wait constraint. We study the problem allowing continuous-sized sublots.

In our model, we first assume that the number of sublots for each product is fixed, then we show how the optimal number of sublots may be computed, or, in the worst case, bounded from below, when setups are assumed inbetween the processing of sublots are assumed.

The paper is organized as follows. In Section 2, we state some general. results for the m-machine case. In particular, we show that, between the processing of any two consecutive sublots, at least two of

the machines never remain idle. Also, we show that neither of these need to be slowest machine. This provides in particular a new elementary proof of the result in Sriskandarajah and Wagneur (1999) for the continuous sublots sizes. A closed form formula for the optimal number of sublots is proved for the 2-machine case.
In Section 3, we analyze the 3-machine environment, and, in the first of two cases, we give a closed form fomula for the continuous sublots sizes, while in the second case, we provide an algorithm which yields the continuous sublots sizes in at most $n-1$ steps.
A short discussion then conclude the paper.

1.1 Notations

X_j : the total number of items (or units) in the sublot j; X_j is a rational number.
ξ_j ; the ratio X_j/X_{j-1} of two consecutive sublots.
a_i : the unit processing time on machine i.
n : the number of sublots for the product considered.
$W = \sum_{j=1}^{n} X_j$: the total number of items (units) demanded for the product.
M: makespan for the single product problem.
I_j^k : machine k idle time prior to the processing of sublot j.
δ_j^k : I_j^k/X_{j-1} "normalized" machine k idle time.

1.2 Assumptions

1. All W units of the product are available at time zero.
This assumption limits the analysis to the static no-wait flowshop for which the quantity of the product to be produced for a planning horizon is known in advance.
2. The product can be treated as infinitely divisible.
3. The processing of sublots is *no-wait.*
4. *Consistent* sublots are used to ensure no-wait processing of sublots.
This assumption means that the sublot sizes remain the same on all machines.
5. The processing of a sublot is proportional to its size, i.e, the processing time of sublot j on machine i is a_iX_j .

1.3 Motivation for the No-Wait Lot Streaming Model

Our motivation in studying this problem is derived from the following real world problem that arises in a manufacturing system called "anodizing line" which is a flowshop made up of a series of chemical processing tanks (Hsu and Stein 1991). The line produces many types of products for the commercial and automotive industries such as pipes, trim and truck grilles. The objective here is to produce the daily production requirements of various products in the shortest possible time. In front of the line is a racking area, where items of a product are loaded onto racks prior to chemical processing. Since the shapes of products are different, each product has its own racks. The number of racks available for a product (n_j) is limited. The products need to be processed as no-wait for two reasons : (i) there is no buffer between tanks (ii) once a rack exits a tank it must immediately enter the next one or the products will be ruined due to deterioration of the items while exposed to the atmosphere. Since it is a chemical process, the processing time of a rack (or a sublot) in a tank depends on the total surface area. Hence the processing time is proportional to the number of items in a rack.

2 Basic Formulas for the m-machine Lot Streaming Problem

We consider here the problem of lot streaming for a single product. We assume that the number of sublots for the product is fixed. In other words, we have to determine the optimal value of $X_j, j = 1, 2, \ldots, n$, so that the makespan is minimized. We seek for a closed form formula, giving the optimal size, when the product is considered as infinitely divisible.
We define a sequence S as the ordered set of the sublot indices: $S = \{1, 2, \ldots, n\}$. The Gantt's chart of the schedule obtained from the sequence S is shown in Figure 1 and Figure 2.
We write I_j^k for the idle time on machine k inbetween the processing of sulots $j-1$ and j. The reader may refer to Section 1.1 for the description of the notations used.
Makespan may be computed on any of the machines, in particular, for machine ℓ $(1 \leq \ell \leq m)$, we have :

$$M = X_1 \sum_{i=1}^{\ell-1} a_i + a_\ell W + X_n \sum_{i=\ell+1}^{m} a_i + \sum_{j=2}^{n} I_j^\ell \quad (1)$$

where we define $\sum_{i=1}^{0} a_i = \sum_{i=m+1}^{m} a_i = 0$.

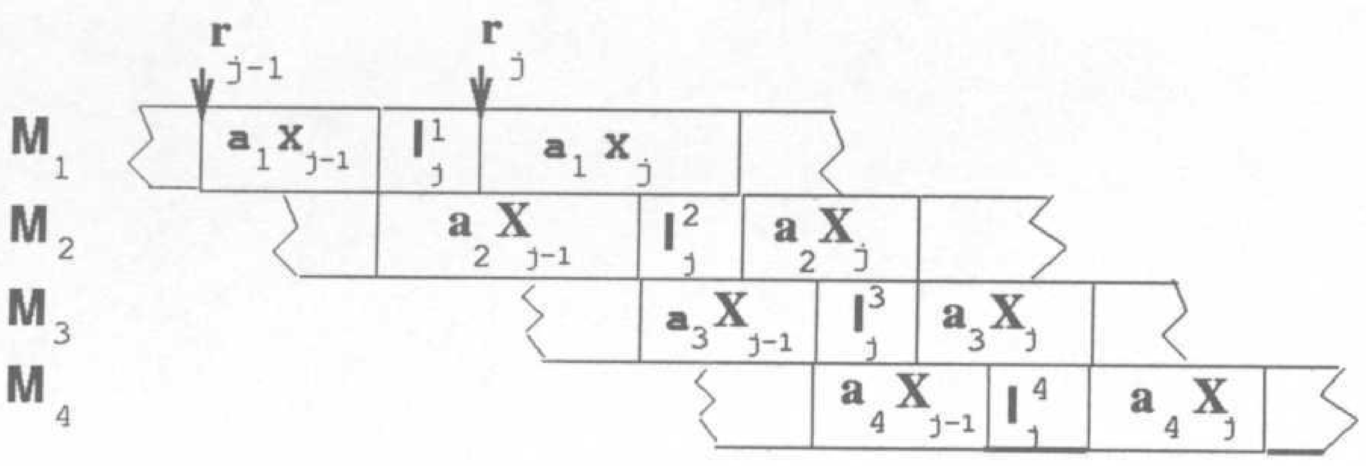

Figure 1: Sublots release and processing times

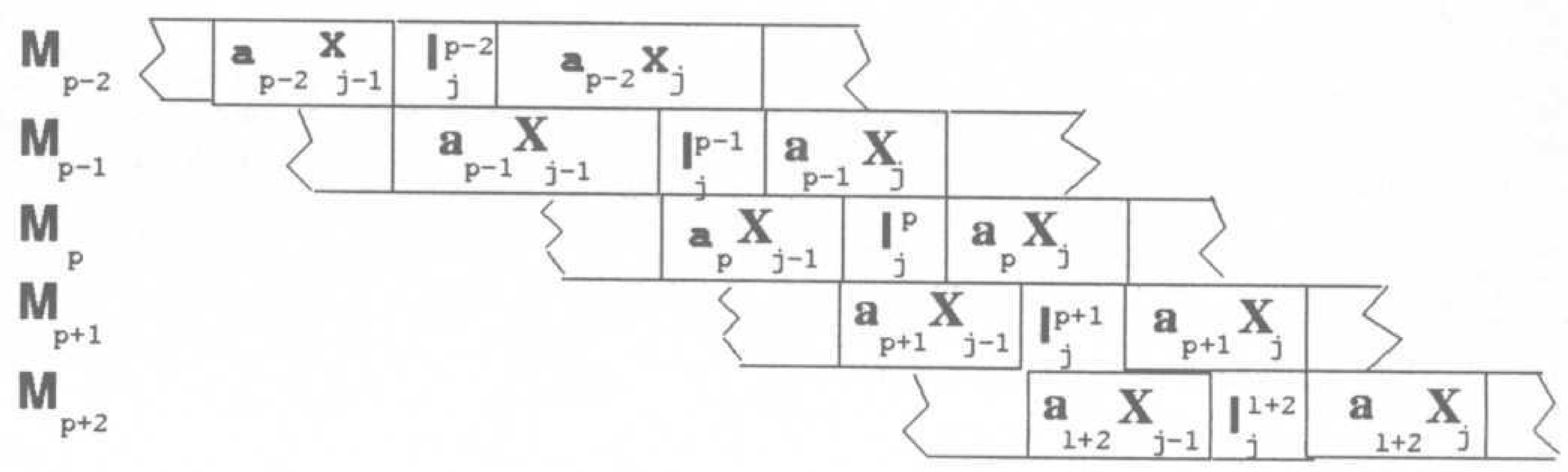

Figure 2: A schedule for the sequence of sublots

Let r_j stand for the release time of sublot j (Figure 1). We have $r_1 = 0$, and, as a consequence of the no-wait constraint, for $j = 2, \ldots, n$, we must have :
$r_j \geq r_{j-1} + a_1 X_{j-1}$
$r_j + a_1 X_j \geq r_{j-1} + a_1 X_{j-1} + a_2 X_{j-1}$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$r_j + (\sum_{i=1}^{k-1} a_i) X_j \geq r_{j-1} + a_1 X_{j-1} (\sum_{i=2}^{k} a_i) X_{j-1}$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$r_j + (\sum_{i=1}^{m-1} a_i) X_j \geq r_{j-1} + a_1 X_{j-1} + (\sum_{i=2}^{m} a_i) X_{j-1}$.
Hence $r_j \geq r_{j-1} + a_1 X_{j-1} +$
$\max_{2 \leq k \leq m} \{0, X_{j-1} \sum_{i=2}^{k} a_i - X_j \sum_{i=1}^{k-1} a_i\}$.
Since we want to minimize makespan, and there is no other constraint on r_j, the sublots must be released as soon as possible, i.e.
$r_j = a_1 \sum_{i=1}^{j-1} X_i + \max_{2 \leq k \leq m} \{0, X_{j-1} \sum_{i=2}^{k} a_i - X_j \sum_{i=1}^{k-1} a_i\}$.
Also, since $r_j - r_{j-1} + a_1 X_i = I_j^1$, we get:
$I_j^1 = \max_{2 \leq k \leq m} \{0, X_{j-1} \sum_{i=2}^{k} a_i - X_j \sum_{i=1}^{k-1} a_i\}$.
Let ξ_j stand for the ratio of two consecutive sublots : $X_j = \xi_j X_{j-1}$, Then
$I_j^1 = \max_{2 \leq k \leq m} \{0, X_{j-1} (\sum_{i=2}^{k} a_i - \xi_j \sum_{i=1}^{k-1} a_i)\} =$
$X_{j-1} \max_{2 \leq k \leq m} \{0, (\sum_{i=2}^{k} a_i - \xi_j \sum_{i=1}^{k-1} a_i)\}$, $j = 2, \ldots n$.
Writing $\delta_j^1 = \frac{I_j^1}{X_{j-1}}$, we get :

$$\delta_j^1 = \max_{2 \leq k \leq m} \{0, \sum_{i=2}^{k} a_i - \xi_j \sum_{i=1}^{k-1} a_i\}, \; j = 2, \ldots, n, \quad (2)$$

Let $A_j^k = \sum_{i=2}^{k} (a_i - \xi_j a_{i-1}), k = 2, \ldots, m$, and $A_j^1 = 0$, $j = 2, \ldots, n$. Then (2) becomes:

$$\delta_j^1 = \max_{1 \leq k \leq m} A_j^k, \quad (3)$$

Consider a rectangle involving M_{p-1}, M_p (Figure 2), and sublots X_{j-1}, X_j. We have :
$I_j^{p-1} + a_{p-1} X_j = a_p X_{j-1} + I_j^p$
equivalently $\delta_j^{p-1} + \xi_j a_{p-1} = a_p + \delta^p$

Similarly: $I_j^{p-2} + a_{p-2} X_j = a_{p-1} X_{j-1} + I_j^{p-1}$, and $\delta_j^{p-2} + \xi_j a_{p-2} = a_{p-1} + \delta_j^{p-1}$.
Eliminating δ_j^{p-1} between these equations yields:

$$\delta_j^{p-2} + \xi_j (a_{p-2} + a_{p-1}) = a_{p-1} + a_p + \delta_j^p$$

More generally, considering M_k, M_{k+1}, $\ldots$, M_p, we have,
$\forall\, k, p, \; 1 \leq k < p \leq m$, and $j = 2, \ldots, n$:

$$\delta_j^k + \xi_j \sum_{i=k}^{p-1} a_i = \sum_{i=k+1}^{p} a_i + \delta_j^p \quad (4)$$

ie

$$\delta_j^p = \delta_j^k - [\sum_{i=k+1}^{p} a_i - \xi_j \sum_{i=k}^{p-1} a_i]$$

Since (4) holds for all j, we may temporarily drop the subscript. For $k = 1$, this yields :

$$\delta^p = \delta^1 - A^p, \; p = 1, \ldots, m \quad (5)$$

We have the following statement.

Lemma 1

$$\min_{1 \leq p \leq m} \delta^p = 0$$

Proof
Let ℓ $(1 \leq \ell \leq m)$ be such that $\delta^1 = A^\ell$. Then $\delta^\ell = 0$. $\square$

As a straightforward consequence of Lemma 1, we have the following statement.

Theorem 1
For any sublot j there is a machine which remains active between the processing of sublots $j-1$ and j. $\square$

Note that Theorem 1 holds independently of the sublot sizes. Also, this statement comforts our intuition from Figure 1 that r_j can be "pushed to the left" until one of the I_k^j vanishes.

We introduce dummy machines M_o, M_{m+1} with unit processing times $a_o = a_{m+1} = 0$. The following statement may be called the Fundamental Theorem of lotstreaming in no-wait flowshops.

Theorem 2

For $p = 1, \ldots, m$, we have : $\delta^p = 0 \iff$

$$\max_{p+1 \le \ell \le m+1} \left\{ \frac{\sum_{i=p+1}^{\ell} a_i}{\sum_{i=p}^{\ell-1} a_i} \right\} \le \xi \le \min_{0 \le k \le p-1} \left\{ \frac{\sum_{i=k+1}^{p} a_i}{\sum_{i=k}^{p-1} a_i} \right\} \quad (6)$$

Proof

For $p = 1$, $\delta^1 = 0 \iff A_\ell \le 0$, $\ell = 2, \ldots m \iff$

$\sum_{i=2}^{\ell} a_i - \xi \sum_{i=1}^{\ell-1} a_i \le 0$, $\ell = 2, \ldots m \iff$

$\frac{\sum_{i=2}^{\ell} a_i}{\sum_{i=1}^{\ell-1} a_i} \le \xi$, $\ell = 2, \ldots m \iff$

$$\max_{2 \le \ell \le m+1} \left\{ \frac{\sum_{i=2}^{\ell} a_i}{\sum_{i=1}^{\ell-1} a_i} \right\} = \max_{2 \le \ell \le m} \left\{ \frac{\sum_{i=2}^{\ell} a_i}{\sum_{i=1}^{\ell-1} a_i} \right\} \le \xi .$$

Now $\min_{0 \le k \le p-1} \left\{ \frac{\sum_{i=k+1}^{p} a_i}{\sum_{i=k}^{p-1} a_i} \right\} = \frac{a_1}{a_o} = +\infty \ge \xi$.

Clearly, for $1 < p < m$, $\delta^p = 0 \Leftrightarrow \delta^1 = A^p \Leftrightarrow$ $\forall \ell \; (1 \le \ell \le m)$, $A^\ell \le A^p \iff$

$\forall \ell$, $\sum_{i=2}^{\ell} a_i - \xi \sum_{i=1}^{\ell-1} a_i \le \sum_{i=2}^{p} a_i - \xi \sum_{i=1}^{p-1} a_i \iff$

$\forall \ell > p$, $\sum_{i=p+1}^{\ell} a_i \le \xi \sum_{i=p}^{\ell-1} a_i$ and $\forall \ell < p$, $\xi \sum_{i=\ell}^{p-1} a_i \le \sum_{i=\ell+1}^{p} a_i \iff$

$\forall \ell > p$, $\frac{\sum_{i=p+1}^{\ell} a_i}{\sum_{i=p}^{\ell-1} a_i} \le \xi$ and $\forall \ell < p$, $\xi \le \frac{\sum_{i=\ell+1}^{p} a_i}{\sum_{i=\ell}^{p-1} a_i} \iff$

$$\max_{p+1 \le \ell \le m} \frac{\sum_{i=p+1}^{\ell} a_i}{\sum_{i=p}^{\ell-1} a_i} \le \xi \le \min_{1 \le \ell \le p-1} \frac{\sum_{i=\ell+1}^{p} a_i}{\sum_{i=\ell}^{p-1} a_i} .$$

But $\min_{1 \le \ell \le p-1} \frac{\sum_{i=\ell+1}^{p} a_i}{\sum_{i=\ell}^{p-1} a_i} \ge 0$ is finite, hence it is also equal to $\min_{0 \le \ell \le p-1} \frac{\sum_{i=\ell+1}^{p} a_i}{\sum_{i=\ell}^{p-1} a_i}$.

Clearly, $\max_{p+1 \le \ell \le m} \frac{\sum_{i=p+1}^{\ell} a_i}{\sum_{i=p}^{\ell-1} a_i} = \max_{p+1 \le \ell \le m+1} \frac{\sum_{i=p+1}^{\ell} a_i}{\sum_{i=p}^{\ell-1} a_i}$.

For $p = m$, the inequality $\xi \le \min_{0 \le \ell \le p-1} \frac{\sum_{i=\ell+1}^{p} a_i}{\sum_{i=\ell}^{p-1} a_i}$ is straightforward, and

$$\max_{p+1 \le \ell \le m+1} \frac{\sum_{i=p+1}^{\ell} a_i}{\sum_{i=p}^{\ell-1} a_i} = \frac{a_{m+1}}{a_m} = 0 \le \xi . \qquad \square$$

It is a well-known result in flow-shop scheduling theory that a necessary condition for optimality is that the slowest machine is never idle. It is of primary importance to know wether this also holds for the lotstreaming problem considered here. For in this case, the minimum makespan problem would become much more easy to solve. Indeed, if M_ℓ is the slowest machine, then, setting $\delta_j^\ell = 0$, $j = 2, \ldots, n$ yields, by (1):

$$M = X_1 \sum_{i=1}^{\ell-1} a_i + a_\ell W + X_n \sum_{i=\ell+1}^{m} a_i \qquad (7)$$

and we would just have to minimize

$X_1 \sum_{i=1}^{\ell-1} + X_n \sum_{i=\ell+1}^{m} a_i$, which is relatively easy.

Now $X_j = \xi_j X_{i-1}$, $j = 2, \ldots n \Rightarrow$

$X_j = X_1 \prod_{i=2}^{j} \xi_i$ $j = 2, \ldots n$. Let $\xi_1 = 1$. Then

$W = X_1 \sum_{j=1}^{n} \prod_{i=1}^{j} \xi_i$, and

$$X_j = \frac{\prod_{i=1}^{j} \xi_i}{\sum_{j=1}^{n} \prod_{i=1}^{j} \xi_i}, \; j = 1, \ldots, n. \qquad (8)$$

Hence we would have to minimize $F: \mathbb{R}^{n-1} \to \mathbb{R}$, defined by $F(\xi_2, \ldots, \xi_n) = \frac{\sum_{i=1}^{\ell-1} a_i + \sum_{i=\ell+1}^{m} a_i \prod_{j=1}^{n} \xi_j}{\sum_{j=1}^{n} \prod_{i=1}^{j} \xi_i}$.

However, for the lotstreaming problem in the no-wait environment it is generally not true that to keep the slowest machine always active is optimal. This is stated in Theorem 3 below.

Theorem 3
In an optimal solution, the slowest machine need not be always active.

Proof
The statement will be proved by assuming the first machine is the slowest and is never idle, and, for an instance of the problem, we will compute the optimal sublots sizes, and the minimum makespan M_1 compatible with this condidion. Then we provide another set of sublots for this instance, and we show that the makespan M_2 for these sublots satisfies $M_2 < M_1$.
Instance : $m = 3$, $n = 3$. The machine processing times are $a = 5$, $b = 4$, $c = 3$, and
$M = aW + (b+c)X_3$. Lot size is W.
We have: $X_1 = \frac{W}{\sum_{j=1}^{3}\prod_{i=1}^{j}\xi_i}$, $X_2 = \frac{\xi_2 W}{\sum_{j=1}^{3}\prod_{i=1}^{j}\xi_i}$
$X_3 = \frac{\xi_2\xi_3 W}{\sum_{j=1}^{3}\prod_{i=1}^{j}\xi_i}$, and $F(\xi) = \frac{(b+c)\xi_2\xi_3}{\sum_{j=1}^{3}\prod_{i=1}^{j}\xi_i}$.
Then $\frac{\partial F}{\partial \xi_2} = \frac{(b+c)\xi_3}{(\sum_{j=1}^{3}\prod_{i=1}^{j}\xi_i)^2}[\sum_{j=1}^{3}\prod_{i=1}^{j}\xi_i - \xi_2(1+\xi_3)] = \frac{(b+c)\xi_3}{(\sum_{j=1}^{3}\prod_{i=1}^{j}\xi_i)^2} > 0$ (note that this holds independently of the particular values of $a > 0, b > 0$, and $c > 0$). Hence F is an increasing function of ξ_2. It follows that ξ_2 must be chosen at its lower bound, i.e. $\xi_2 = \max\{b/a, (b+c)/(a+b)\} = b/a = 4/5$.
Similarly, $\frac{\partial F}{\partial \xi_3} = \frac{(b+c)\xi_2(1+\xi_2)}{(\sum_{j=1}^{3}\prod_{i=1}^{j}\xi_i)^2} > 0$, and $\xi_3 = 4/5$.
Then $X_1 = 25W/61$, $X_2 = 20W/61$
$X_3 = 16W/61$, $M_1 = 5W + 112W/61 = 417W/61$.
On the other hand, let
$\tilde{X}_1 = 5W/12$, $\tilde{X}_2 = W/3$, $\tilde{X}_3 = W/4$. Then
$M_2 = 5W + 7W/4 = 27W/4 < 417W/61$. □

Remark 2.1 In the counterexample above, we have $I_3^1 = b\tilde{X}_2 - a\tilde{X}_3 = W/12$. So we "pay" $W/12$ time units of machine 1 idle time, and we get a bonus of $7(X_3 - \tilde{X}_3) = 21W/244$ time units from the reduced size of X_3. Thus $\tilde{C}_{max} = M + I_3^1 - 7(X_3 - \tilde{X}_3) = M + W/12 - 21W/244 = M - W/366$.
More generally, from $M = aW + (b+c)X_n$ we see that a decrease of one unit of X_3 yields a decrease of $a+b$ times units of M. The cost is measured in terms of machine 1 idle time, whose cost per unit is a. Although the correspondance $\Delta X_n \mapsto I_j^1$ is not straightforward, the intuition behind the result is that if $a < b+c$, then it may pay-off to redcue X_n at the cost of some extra machine 1 idle time.
Clearly, the same holds true for any number of machines $m > 2$.

By Theorem 3, we cannot assume that the slowest machine should always remain busy.
It is easy to see that the $\delta_j^p = \delta_j^p(\xi_j)$ are piecewise linear functions of ξ_j. Let J_ξ stand for a maximal interval where $\delta_j^p(\xi_j)$ is linear. Then M (resp $\frac{\partial M}{\partial \xi_j}$) is a piecewise differentiable (resp continuous) function of ξ_j on J_ξ.
From (1), wa have :

$$M = a_1 W + X_n \sum_{i=2}^{m} a_i + \sum_{j=2}^{n} I_j^1 =$$
$$a_1 W + \frac{W}{\sum_{j=1}^{n}\prod_{i=1}^{j}\xi_i}\left[\prod_{j=1}^{n}\xi_j + \sum_{j=2}^{n}(\prod_{i=1}^{j-1}\xi_i)\delta_j^1\right].$$

The following statement holds for $\ell = 2, \ldots, n$, and any maximal interval of linearity of the δ_ℓ^p.

Lemma 2.2
The sign of $\frac{\partial M}{\partial \xi_\ell}$ is constant on J_{ξ_ℓ}.

Proof
Let $u = \prod_{j=1}^{n}\xi_j + \sum_{j=2}^{n}(\prod_{i=1}^{j-1}\xi_i)\delta_j^1$, $v = \sum_{j=1}^{n}\prod_{i=1}^{j}\xi_i$. Then $M = W(a_1 + \frac{u}{v})$. It is enough to show that $\frac{\partial u}{\partial \xi_\ell}v - \frac{\partial v}{\partial \xi_\ell}u$ is independent of ξ_ℓ.
We have $u = \sum_{j=2}^{\ell-1}(\prod_{i=1}^{j-1}\xi_i)\delta_j^1 + (\prod_{i=1}^{\ell-1} xi_i)\delta_\ell^1$
$+\xi_\ell\left(\sum_{j=\ell+1}^{n}(\prod_{i=1,i\neq\ell}^{j-1} xi_i)\delta_j^1 + \prod_{i=1,i\neq\ell}^{n}\xi_i\right)$.
Hence u is an affine function of ξ_ℓ in J_{ξ_ℓ}, which may be written as $u = \alpha_1 + \beta_1\xi_\ell$, where δ_ℓ^1 splits into α_1 and into $\beta_1\xi_\ell$, with α_1, β_1 independent of ξ_ℓ. Similarly $v = \sum_{j=1}^{\ell-1}\prod_{i=1}^{j}\xi_i + \xi_\ell(\prod_{i=1}^{\ell-1}\xi_i)(1 + \sum_{j=\ell+1}^{n}\prod_{i=\ell+1}^{j}\xi_i) = \alpha_2 + \beta_2\xi_\ell$ is also an affine function of ξ_ℓ in J_{ξ_ℓ} (with α_2, β_2 independent of ξ_ℓ). □

Let $\ell = \arg\max_q\{\frac{\sum_{i=p+1}^{q} a_i}{\sum_{i=p}^{q-1} a_i}$, $L_B(p) = \frac{\sum_{i=p+1}^{\ell} a_i}{\sum_{i=p}^{\ell-1} a_i}$, and

$k = \arg\min_q \frac{\sum_{i=q+1}^{p} a_i}{\sum_{i=q}^{p-1} a_i}$, $U_B(p) = \frac{\sum_{i=k+1}^{p} a_i}{\sum_{i=k}^{p-1} a_i}$. Write ξ_j^* for the optimal value of ξ_j. From Theorem 2 and

Lemma 2.2, we have the following statement.

Corollary 2.3
For $j = 2, \ldots, n$, $\frac{\partial M}{\partial \xi_\ell} \neq 0$:

$$\begin{aligned} \delta_j^1 = 0 &\iff \xi_j^* = U_B(1) \\ \delta_j^p = 0 &\iff \xi_j^* \in \{L_B(p),\ U_B(p)\},\ 1 < p < m \quad (9) \\ \delta_j^p = 0 &\iff \xi_j^* = L_B(m) \end{aligned}$$

Proof
By Lemma 2.2, M is an increasing or a decreasing function of ξ_j. Thus, ξ_j^* must be selected at one of the bounds of the interval in (6).
Moreover, for $p = 1$ (resp $p = m$), M is an increasing (resp decreasing) function of ξ_ℓ. □

Remark 2.4
If $\frac{\partial M}{\partial \xi_\ell} = 0$, in J_{ξ_ℓ} then we may choose ξ_ℓ so that (9) holds.
We now state our third fundamental theorem of lot-streaming in no-wait flow-shops.

Theorem 4
For the optimal policy the following condition holds:

$$\forall j,\ \exists k, p \text{ such that } \delta_j^k = \delta_j^p = 0 .$$

Theorem 4 states that, for the optimal policy, we have to operate the shop in such a way that there are always two machines which remain busy between the processing of two consecutive sublots.

Proof
Let $p = \arg\min_i \{\delta_j^i\}$. Then $\delta_j^p = 0$, by Theorem 1. If $\frac{\partial M}{\partial \xi_\ell} \neq 0$, then by Corollary 2.3, we must have $\xi_j^* = L_B(p)$ or $\xi_j^* = U_B(p)$.

If $\xi_j^* = L_B(p) = \frac{\sum_{i=p+1}^{\ell} a_i}{\sum_{i=p}^{\ell-1} a_i}$, we have $\sum_{i=p+1}^{\ell} a_i = \xi_j \sum_{i=p}^{\ell-1} =$.

$\sum_{i=2}^{\ell} a_i - \sum_{i=2}^{p} a_i = \xi_j \sum_{i=1}^{\ell-1} a_i - \xi_j \sum_{i=1}^{p-1} a_i$, i.e. $A_\ell = A_p$.
Now $\delta_j^p = 0 \iff \delta_j^1 = A_p = A_\ell \iff \delta_j^\ell = 0$.
The case $\xi_j^* = U_B(p)$ is similar. We leave it to the reader.
If $\frac{\partial M}{\partial \xi_\ell} \neq 0$, then we can always choose $\xi_\ell^* = L_B(1) > 0$, or $\xi_\ell^* = U_B(p) < \infty$. □
The following statement is a straightforward consequence of Theorem 4.

Corollary 2.5 of Theorem 4 (Sriskandarajah & Wagneur, 1999)
For $m = 2$, $\delta_j^1 = \delta_j^2 = 0$, and
$\xi_j^* = \frac{a_2}{a_1} W$, $X_j = \frac{(a_2/a_1)^{j-1}}{\sum_{i=0}^{n-1} (a_2/a_1)^i}$, $j = 2, \ldots, n$. □

Theorem 5

$$\forall p\,(1 \leq p \leq m),\ U_B(p) < L_B(p) \Rightarrow \delta_j^p > 0\ j = 2, \ldots, m$$

in particulat :

$$\frac{a_{p+1}}{a_p} < \frac{a_p}{a_{p-1}} \Rightarrow \delta_j^p > 0\ j = 2, \ldots, m \qquad (10)$$

Proof
The inequality (5) is a necessary and sufficient condition for $\delta_j^p = 0$. □

In the next Section, we show how to compute the optimal lots sizes X_j in the case of 3 machines.

3 The 3-machine case

We assume in this section that $m = 3$. Also, we write $a_1 = a$, $a_2 = b$, $a_3 = c$.
We study the maps $\delta_j^k(\xi_j)$, $k = 1, 2, 3$, and distinguish two cases :
1) $\frac{b}{a} < \frac{c}{b}$, and 2) $\frac{c}{b} \leq \frac{b}{a}$.
We have $\delta_j^1 = \max\{0, b - \xi_j a, b + c - \xi_j(a+b)\}$
$\delta_j^2 = \max\{0, \xi_j a - b, c - \xi_j b\}$,
$\delta_j^3 = \max\{0, \xi_j b - c, \xi_j(a+b) - (b+c)\}$.

Case 1)
By (10), we have: $\delta_j^2 > 0 \,\forall j$.
By Theorem 4, we then have $\delta_j^1 = \delta_j^3 = 0$, hence $\xi_j^* = \frac{b+c}{a+b}$, $j = 2, \ldots n$, and we have :

$$F(\xi) = \frac{(a+b) \prod_{i=1}^{n} \xi_i}{\sum_{j=1}^{n} \prod_{i=1}^{j} \xi_i}, \text{ and } \min F(\xi), \text{ s.t. } \delta_j^1 = \delta_j^3 = 0$$

(or $\min F(\xi)$, s.t. $\xi_j = \frac{b+c}{a+b}$), $j = 2, \ldots n$, yields :

$$M = W\left(a + \frac{(b+c)^n}{\sum_{i=0}^{n-1} (a+b)^{n-1-i} (b+c)^i}\right) = W\left(\frac{(a+b)^n}{\sum_{i=0}^{n-1} (a+b)^{n-1-i} (b+c)^i} + c\right) \qquad (11)$$

We have shown the following statement.

Theorem 6
If $\frac{b}{a} < \frac{c}{b}$, then the minimum makespan is given by choosing $\xi_2^* = \ldots = \xi_n^* = \frac{b+c}{a+b}$. □

Case 2)

We have $\delta_j^1 = \begin{cases} b+c-\xi_j(a+b) & , \xi_j \in [0, \frac{c}{b}] \\ b-\xi_j a & , \xi_j \in [\frac{c}{b}, \frac{b}{a}] \\ 0 & , \xi_j \geq \frac{b}{a} \end{cases}$,

$\delta_j^2 = \begin{cases} c-\xi_j b & , \xi_j \in [0, \frac{c}{b}] \\ 0 & , \xi_j \in [\frac{c}{b}, \frac{b}{a}] \\ \xi_j a - b & , \xi_j \geq \frac{b}{a} \end{cases}$, and

$\delta_j^3 = \begin{cases} 0 & , \xi_j \in [0, \frac{c}{b}] \\ \xi_j b - c & , \xi_j \in [\frac{c}{b}, \frac{b}{a}] \\ \xi_j(a+b)-(b+c) & , \xi_j \in [\frac{b}{a}, \infty[\end{cases}$.

It follows that δ_j^1 and δ_j^3 never vanish simultaneously. Therefore, by Theorem 4, we must have $\delta_j^2 = 0$, $j = 2, \ldots, n$.

By Theorem 2, this means that $\xi_j \in [\frac{c}{b}, \frac{b}{a}]$. By (7), $M = W(b + aX_1 + cX_n)$, and

$$F(\xi) = aX_1 + cX_n = \frac{a + c\prod_{i=1}^{n} \xi_i}{\sum_{j=1}^{n}\prod_{i=1}^{j} \xi_i} .$$

The program becomes

$\min F(\xi)$ s.t. $\xi_j \in [\frac{c}{b}, \frac{b}{a}]$, $j = 2, \ldots, n$, or

$$\min_{\xi \in [\frac{c}{b}, \frac{b}{a}]^{n-1}} \frac{a + c\prod_{i=1}^{n} \xi_i}{\sum_{j=1}^{n}\prod_{i=1}^{j} \xi_i} .$$

Note that in order to minimize $F(\xi)$, we have to take both X_1 and X_n as small as possible. Hence, raughly speaking, we must first increase the ξ_j's, and then, decrease them. This means we have to take ξ_j at the upper bound first, and then at the lower bound. The problem will then be to determine exactly when to change from the upper to the lower bound. This is stated in Theorem 7 below.

Theorem 7

If $\frac{c}{b} \leq \frac{b}{a}$, then the optimal policy is given by

$\xi^* = (\underbrace{\frac{b}{a}, \ldots, \frac{b}{a}}_{k-1}, \underbrace{\frac{c}{b}, \ldots, \frac{c}{b}}_{n-k})$ for some k , $1 \leq k \leq n$.

Remark 3.1

1. For $k = 1$: $\xi^* = (\frac{c}{b}, \ldots, \frac{c}{b})$.
For $k = n$, $\xi^* = (\frac{b}{a}, \ldots, \frac{b}{a})$.

Proof of Theorem 7

We prove the statement by induction as follows:

Initial step:

If $c \leq a$, then $\xi_2^* = \frac{b}{a}$. If $a \leq c$, then $\xi_n^* = \frac{c}{b}$.

Induction step :

Suppose $\xi_i = \frac{b}{a}$, $i = 2, \ldots, j-1$, and $\xi_i = \frac{c}{b}$, $i = \ell+1, \ldots, n$, where $j \leq \ell$. We show that :
$\xi_j^* = \frac{b}{a}$, or $\xi_\ell^* = \frac{c}{b}$.

Initial step:

Let $N(\frac{\partial F}{\partial \xi_2})$ stand for the numerator of $\frac{\partial F}{\partial \xi_2})$. We have $N(\frac{\partial F}{\partial \xi_2}) = \prod_{i=3}^{n} \xi_j(c-a) - a(1 + \sum_{j=3}^{n-1}\prod_{i=3}^{j})$, hence
$c \leq a \Rightarrow \frac{\partial F}{\partial \xi_2} < 0 \Rightarrow \xi_2^* = \frac{b}{a}$.

Similarly: $N(\frac{\partial F}{\partial \xi_n}) = \left(\prod_{i=1}^{n-1} \xi_i\right)(c\sum_{j=1}^{n-1}\prod_{i=1}^{j} \xi_i - a) = \left(\prod_{i=1}^{n-1} \xi_i\right)(c - a + c\sum_{j=2}^{n-1}\prod_{i=2}^{j} \xi_i)$.

Hence $c \geq a \Rightarrow \frac{\partial F}{\partial \xi_n} > 0 \Rightarrow \xi_n^* = \frac{c}{b}$.

Induction step :

Let $\beta_i = (\frac{b}{a})^{i-2}$, $B_j = \sum_{i=2}^{j} \beta_i$, $\gamma_i = (\frac{c}{b})^i$, and $C_\ell = \sum_{i=0}^{n-\ell} \gamma_i$. Then:

$$F(\xi) = \frac{a + c(\frac{b}{a})^{j-2}(\frac{c}{b})^{n-\ell}\prod_{i=j}^{\ell} \xi_i}{\sum_{i=0}^{j-2}(\frac{b}{a})^i + \xi_j\left(1 + \sum_{k=j+1}^{\ell-1}\prod_{i=j+1}^{k} \xi_i + \prod_{i=j+1}^{\ell} \xi_i \sum_{i=0}^{n-\ell}(\frac{c}{b})^i\right)} .$$

$$= \frac{a + c\beta_j\gamma_{n-\ell}\prod_{i=j}^{\ell} \xi_i}{B_j + \xi_j\left(1 + \sum_{k=j+1}^{\ell-1}\prod_{i=j+1}^{k} \xi_i + C_\ell \prod_{i=j+1}^{\ell} \xi_i\right)} .$$ We have:

$$\begin{aligned} N(\tfrac{\partial F}{\partial \xi_j}) &= c\beta_j\gamma_{n-\ell}B_j \prod_{i=j+1}^{\ell} \xi_i \\ &\quad - a(1 + \sum_{k=j+1}^{\ell-1}\prod_{i=j+1}^{k} \xi_i + C_\ell \prod_{i=j+1}^{\ell} \xi_i) \\ &= (\prod_{i=j+1}^{\ell} \xi_i)\{\underbrace{c\beta_j\gamma_{n-\ell}B_j - aC_\ell}_{\Gamma_{j,\ell}}\} \\ &\quad - a(1 + \sum_{k=j+1}^{\ell-1}\prod_{i=j+1}^{k} \xi_i) . \end{aligned}$$

Similarly:

$$\begin{aligned} N(\tfrac{\partial F}{\partial \xi_\ell}) &= \\ c\beta_j\gamma_{n-\ell}&\prod_{i=j}^{\ell-1} \xi_i \Big(B_j + \xi_j(1 + \sum_{k=j+1}^{\ell-1}\prod_{i=j+1}^{k} \xi_i)\Big) - aC_\ell \prod_{i=j}^{\ell-1} \\ &= (\prod_{i=j}^{\ell-1} \xi_i)\Big[\Gamma_{j,\ell} + c\beta_j\gamma_{n-\ell}\xi_j(1 + \sum_{k=j+1}^{\ell-1}\prod_{i=j+1}^{k} \xi_i)\Big] . \end{aligned}$$

Clearly:

$c\beta_j\gamma_{n-\ell}B_j \leq aC_\ell$ (ie $\Gamma_{j,\ell} \leq 0$) $\Rightarrow \frac{\partial F}{\partial \xi_k} < 0 \Rightarrow \xi_j^* = \frac{b}{a}$

$c\beta_j\gamma_{n-\ell}B_j \geq aC_\ell$ ($\Gamma_{j,\ell} \geq 0$) $\Rightarrow \frac{\partial F}{\partial \xi_\ell} > 0 \Rightarrow \xi_\ell^* = \frac{c}{b}$.

After at most $n-1$ steps, all the ξ_i^*'s have been determined. □

Remark 3.2
We have $\Gamma_{2,n} = c - a$.

Remark 3.3
Our solution is also optimal for the unconstrained 3-machine flow-shop.

4 Conclusions and further research

In this paper, we have shown how to optimally process a single product in a no-wait flow-shop when lotstreaming is allowed. We deal with the continuous sublots sizes only.

It is clear that, in the case where no setups are necessary inbetween the processing of sublots, then the optimal solution is to take each sublot of size 1. This solution ensures the maximum overlapping of activities in the shop. However, this solution ceases to be optimal when sum setups (e.g. cleaning) are required inbetween the processing of consecutive batches. Researchers in the field usually consider the number of sublots as given. In order to determine the optimal number of sublots in case setups are required inbetween the batches, a hierarchical optimisation could be performed:
- First, as we did in this paper, consider the number n of sublots as given, and find the optimal sublots sizes $X_j^*(n)$, $j = 1, \ldots, n$,
- Then use the solutions $X_j^*(n)$, $j = 1, \ldots, n$, with n a decision variable, and find the optimal n.

Although our closed form results are limited to the 3-machine case, we get a good insight into the general case, through our Theorems 1-3, which apply to an arbitrary number of machines. however, this problem is combinatorially explosive, since we have to investigate $(m-1)!$ cases.
Due to space limitations, we did not investigate here the multiple products problem, where the products must also be scheduled. We have interesting preliminary results in this direction, using the heuristics extension of the Gilmore and Gomory algorithm due to Röck and Smith (1983).
Further research should be pursued for the discrete (e.g. integer) sublots sizes problem.
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