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ROBUST DESIGN WITH STABILITY PRESERVING MAPS

THEODORE E. DJAFERIS[1]

Department of Electrical and Computer Engineering, University of Massachusetts, Amherst, MA 01003, USA, djaferis@ecs.umass.edu

Abstract. It has recently been shown that matrix Stability Preserving Maps (SPMs) play an important role in robust stabilization. They can be used to provide a different characterization for the existence of a fixed order controller that robustly stabilizes a plant family. Not only do these maps give a different perspective into the problem, but stability preserving map tests have been developed that form the basis of robust controller design procedures. Results have been reported for plant families that either consist of a finite number of systems or can be expressed by transfer function models that involve real parameter uncertainty. In this paper we develop additional stability preserving map tests. We also demonstrate how these tests lead to robust stabilization techniques and apply the methodology to a number of examples.

Key Words. Linear systems, stability, robust design

1 Introduction

The ability to robustly stabilize a family of linear time invariant plants is a topic of great importance in automatic control research and practice. Given a plant family one would like to be able to answer the question of whether a robustly stabilizing controller exists and have a method for constructing it. Classes of plant families that have been studied include those that consist of a finite number of systems and those that can be expressed in terms of a transfer function model with real parameter uncertainty. We know that the problem of simultaneous stabilization of a finite number of plants is a difficult one. In fact, in the general case elegant and computationally attractive solutions do not exist [2, 11]. For systems with real parameters one may be able to express plant uncertainty using a special structure (e.g., multiplicative uncertainty) for which a solution can be given using H_∞ techniques [7]. However, such an approach can lead to conservative designs. A complete solution exists for the case of a single (affine) uncertainty [7] but the techniques cannot be generalized. Other approaches that have been suggested for systems with real parameter uncertainty include Parameter Space methods, [1], the QFT framework [9] and Finite Inclusions Theorem (FIT) design [5].

The notion of a matrix stability preserving map has recently been introduced [3, 4] and used in the formulation of robust stabilization problems. We have demonstrated that the concept can be used for scalar as well as multivariable systems. In the literature one can find the notion of a *stability preserving mapping* defined in the context of Lyapunov based analysis of dynamical systems [10]. However, the matrix stability preserving maps discussed here appear to be different. Following the exposition in [10], stability preserving mappings map families of motions of a dynamical system to families of motions of another dynamical system (i.e., from one state space to another in the time domain). They are used to establish qualitative equivalences between dynamical systems. In contrast, matrix SPMs map polynomial coefficients to polynomial coefficients (i.e., frequency domain) and can be defined without any reference to dynamical systems as simply matrix properties. Furthermore, we use SPMs to develop methods for robust controller design/synthesis. Nevertheless, it would be worthwhile to investigate possible connections that may exist between the two concepts.

In our earlier work we began the development of the theory of stability preserving maps and presented a number of SPM tests. One of these, is the fact that an upper triangular matrix with "ones" on the main diagonal does generate a SPM. Another, is that a ma-

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trix with one stable row generates a SPM. In section 2 of this paper we review some of the pertinent definitions and in section 3 continue the development of the theory by stating new SPM tests. We will confine our discussion to scalar maps but in view of [4], extensions to the multivariable case are immediate. In section 4 we apply these results to a number of examples and show how to design controllers for robust stabilization.

2 Stability Preserving Maps

First, let us recall the definition of a scalar stability preserving map [3]. With $\mathbf{R}$ denoting the reals and $\mathbf{R}^n$ the space of real n-vectors, let A be an $n \times m$ matrix with elements in $\mathbf{R}$, $\phi \in \mathbf{R}^n, \psi \in \mathbf{R}^m$ and $f_A : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ the function defined by:

$$f_A(\phi) = \phi * A = \psi$$

Here "$*$" indicates vector-matrix multiplication. For notational simplicity the "$*$" will not be explicitly shown. The n-vector $\chi = [\chi_{n-1}, \chi_{n-2}, \ldots, \chi_0]$ represents the coefficients of the degree $n-1$ polynomial $\chi(s) = \chi_{n-1}s^{n-1} + \chi_{n-2}s^{n-2} + \ldots + \chi_0$. The vector χ is called *stable* if the corresponding polynomial $\chi(s)$ has all its $n-1$ roots in the left half complex plane (LHCP).

Definition 1 *The function f_A is called a Stability Preserving Map (SPM) if there exists some stable n-vector ϕ that is mapped to a stable m-vector $\psi = f_A(\phi) = \phi A$.*

We say that A *generates* the SPM f_A. It is apparent that the notion of a SPM is simply a matrix property that need not have any connection to dynamical systems. However, we have shown in [3] that this concept plays an important role in robust stabilization. Consider the feedback system shown in Figure 1. The transfer function $P_i(s), i \in \{1, 2, \ldots, N\}$ is one of a finite number of order $\bar{n}$, strictly proper, single-input, single-output plants given by:

$$P_i(s) = \frac{n_{pi}(s)}{d_{pi}(s)} \tag{1}$$

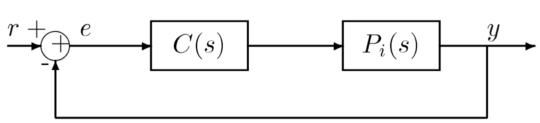

Figure 1: Unity Feedback Configuration

where $n_{pi}(s) = n_{i\bar{n}-1}s^{\bar{n}-1} + n_{i\bar{n}-2}s^{\bar{n}-2} + \ldots + n_{i0}$ and $d_{pi}(s) = s^{\bar{n}} + d_{i\bar{n}-1}s^{\bar{n}-1} + \ldots + d_{i0}$ are coprime for all values of i. The controller given by $C(s) = \frac{n_c(s)}{d_c(s)}$ is proper and degree $\bar{n}-1$, with $n_c(s) = y_{\bar{n}-1}s^{\bar{n}-1} + y_{\bar{n}-2}s^{\bar{n}-2} + \ldots + y_0$ and $d_c(s) = s^{\bar{n}-1} + x_{\bar{n}-2}s^{\bar{n}-2} + \ldots + x_0$. The closed loop characteristic polynomials are of degree $2\bar{n}-1$ and given by:

$$\begin{aligned} \phi_i(s) &= d_c(s)d_{pi}(s) + n_c(s)n_{pi}(s) \\ &= s^{2\bar{n}-1} + \phi_{i2\bar{n}-2}s^{2\bar{n}-2} + \ldots + \phi_{i0} \end{aligned} \tag{2}$$

In the simultaneous stabilization problem we want to know if a single controller exists that makes the polynomials $\phi_i(s)$ stable for all i. It is well known how this relationship can be expressed in coefficient space in terms of the Sylvester resultant matrix. Let us first collect the controller coefficients in a single vector $x = [1 \ y_{\bar{n}-1} \ x_{\bar{n}-2} \ y_{\bar{n}-2} \ x_{\bar{n}-3} \ \ldots \ x_0 \ y_0]$. The family of $\bar{n}^{th}$ order Sylvester resultants are $2\bar{n} \times 2\bar{n}$ matrices given by:

$$\underbrace{\begin{bmatrix} 1 & d_{i\bar{n}-1} & d_{i\bar{n}-2} & \ldots & d_{i0} & 0 & \ldots & 0 \\ 0 & n_{i\bar{n}-1} & n_{i\bar{n}-2} & \ldots & n_{i0} & 0 & \ldots & 0 \\ 0 & 1 & d_{i\bar{n}-1} & \ldots & d_{i1} & d_{i0} & \ldots & 0 \\ 0 & 0 & n_{i\bar{n}-1} & \ldots & n_{i1} & n_{i0} & \ldots & 0 \\ & & & \vdots & & & & \\ 0 & 0 & & 1 & d_{i\bar{n}-1} & d_{i\bar{n}-2} & \ldots & d_{i0} \\ 0 & 0 & & 0 & n_{i\bar{n}-1} & n_{i\bar{n}-2} & \ldots & n_{i0} \end{bmatrix}}_{S_{\bar{n}}(d_{pi}(s), n_{pi}(s))}$$

With this notation we can rewrite (2) as:

$$xS_{\bar{n}}(d_{pi}(s), n_{pi}(s)) = \phi_i, \qquad i \in \{1, \ 2, \ \ldots, \ N\}$$

Suppose for the moment that $N = 2$ and that a single controller $C(s)$ (represented by x) exists that makes $\phi_1(s)$ and $\phi_2(s)$ stable. This implies:

$$\begin{aligned} xS_{\bar{n}}(d_{p1}(s), n_{p1}(s)) &= \phi_1 \\ xS_{\bar{n}}(d_{p2}(s), n_{p2}(s)) &= \phi_2 \end{aligned}$$

Since the two resultants are invertible this means that we must have:

$$\phi_1 \underbrace{S_{\bar{n}}(d_{p1}(s), n_{p1}(s))^{-1} \ S_{\bar{n}}(d_{p2}(s), n_{p2}(s))}_{A} = \phi_2 \tag{3}$$

or that the function f_A is a SPM! The converse is also true, for if the matrix "A" defined in (3) generates a stability preserving map then the corresponding two plants are simultaneously stabilizable. Constructing a simultaneously stabilizing controller is simply done by computing $x = \phi_1 S_{\bar{n}}(d_{p1}(s), n_{p1}(s))^{-1}$. The concept of stability preserving maps thus leads very naturally to robust controller synthesis/design procedures. In the case of more than two plants a set of "A" matrices are generated along with the corresponding set of f_A maps. Specifically, for N plants the matrices $A_i = S_{\bar{n}}(d_{p1}(s), n_{p1}(s))^{-1} \ S_{\bar{n}}(d_{pi}(s), n_{pi}(s)), 2 \leq i \leq N$. For simultaneous stabilization one needs to find a single stable polynomial that is mapped to stable polynomials by the corresponding maps. We then say that

f_A is a *family* of SPM maps. The situation is similar if we have a plant transfer function which contains a (vector) parameter $\mathbf{a}$. We form the set of matrices: $A(\mathbf{a}) = S_{\bar{n}}(d_{p0}(s), n_{p0}(s))^{-1} S_{\bar{n}}(d_p(s,\mathbf{a}), n_p(s,\mathbf{a})), \mathbf{a} \in \Omega_a$, and where "0" indicates a nominal plant. For robust stabilization one needs to find a single stable polynomial that is mapped to stable polynomials by the corresponding maps (i.e., f_A is a *family* of SPM maps). To make this theory useful in design, tests are needed for checking whether appropriate matrices generate stability preserving maps. Several such tests have been reported in [3]. In the next section we continue the development by reporting a number of additional tests.

3 More SPM Tests

In [3] we proved that if A is an $n \times n$ which has some row that is a stable (degree $n-1$) polynomial, the f_A it generates is a SPM. In this section we will prove several more results. Other results are also reported in [6].

Lemma 1 *Let A be $n \times n$ with two consecutive rows whose sum is a degree $n-1$ stable polynomial. The corresponding f_A is a SPM.*

Proof: We consider two cases. In Case 1, the two rows in question are the first two rows of the matrix and in Case 2, any other two consecutive rows.
Case 1: Suppose that the sum of rows 1 and 2 of matrix A forms a degree $n-1$ stable polynomial ψ. We need to construct a stable degree $n-1$ polynomial that gets mapped by f_A to some stable degree $n-1$ polynomial. Consider the polynomial:

$$\phi(s) = (s + \frac{1}{q})^{n-3}(qs^2 + qs + 1)$$

where q is positive, real and "large." It is the product of stable polynomials and thus is stable. After expanding the product it can be seen that the coefficients of the terms s^{n-1} and s^{n-2} are q and $q+n-3$ respectively. Lower powers of s have coefficients that include terms $(1/q)^j$ where $j \geq 0$. When ϕA is formed it generates a polynomial which can be expressed as: $q\psi + \chi$ where χ has entries that include terms $(1/q)^j$ where $j \geq 0$. By making q "large" the roots of ϕA can be made to lie arbitrarily close to the roots of ψ which is stable.
Case 2: Suppose that the sum of rows $i-1$ and i $(3 \leq i \leq n)$ of matrix A forms a degree $n-1$ stable polynomial ψ. We need to construct a stable degree $n-1$ polynomial that gets mapped by f_A to some stable degree $n-1$ polynomial. Consider the polynomial:

$$\phi(s) = (s+q)^{j-1}(s + \frac{1}{q})^{n-2-j}(s^2 + qs + q)$$

where q is positive, real and "large," and $1 \leq j \leq n-2, i = j+2$. It is a product of stable polynomials and thus is stable. Consider first the product $g(s) = (s+q)^{j-1}(s+\frac{1}{q})^{n-2-j}$ and focus on the coefficient of s^{n-2-j}. For large enough q it will be dominated by the term q^{j-1}. Furthermore, all other coefficients of $g(s)$ in this product will contain terms which are lower powers of q (and can be negative). When $g(s)$ is multiplied by $s^2 + qs + q$ to form $\phi(s)$, the coefficients of the terms s^{n-1-j} and s^{n-2-j} will include the same q^j term (call it $\alpha(q)$), and all other coefficients will have terms with lower powers of q. When ϕA is formed it generates a polynomial which can be expressed as: $\alpha(q)\psi + \chi$ where χ has entries that include terms with lower powers of q. By making q "large" the roots of ϕA can be made to lie arbitrarily close to the roots of ψ which is stable. $\square$

It is clear that the idea behind the proof can be extended to other "sums of rows." In fact this is a special case of a more general result:

Lemma 2 *Let A be $n \times n$ with k consecutive rows that form the $k \times n$ submatrix A_1 which generates the SPM f_{A_1}. Then f_A is a SPM.*

Proof: We consider two cases. In Case 1, the submatrix A_1 is formed by the first k rows of the matrix and in Case 2, any other k consecutive rows.
Case 1: Since A_1 is a SPM there exists some degree $k-1$ stable polynomial $\alpha(s)$ that is mapped to a stable degree $n-1$ polynomial ψ. We need to construct a stable degree $n-1$ polynomial that gets mapped by f_A to some stable degree $n-1$ polynomial. Consider the polynomial:

$$\phi(s) = (s + \frac{1}{q})^{n-k-1}(qs\alpha(s) + 1)$$

where q is positive, real and "large." It is the product of stable polynomials and thus is stable. The fact that polynomial $qs\alpha(s)+1$ is stable follows from root locus arguments. After expanding the product it can be seen that the coefficients of the terms $s^{n-1}, s^{n-2}, \ldots, s^{n-k}$ are dominated by the term q. Lower powers of s have coefficients that include terms $(1/q)^j$ where $j \geq 0$. When ϕA is formed it generates a polynomial which can be expressed as: $q\psi + \chi$ where χ has entries that include terms $(1/q)^j$ where $j \geq 0$. By making q "large" the roots of ϕA can be made to lie arbitrarily close to the roots of ψ which is stable.
Case 2: Suppose that A_1 is the submatrix formed by rows $i-k+1, i-k+2, \ldots, i$ $(k+1 \leq i \leq n)$ of matrix A and suppose that $\alpha(s)$ is a degree $k-1$ polynomial that is mapped by A_1 to the degree $n-1$ stable polynomial ψ. We need to construct a stable degree $n-1$ polynomial that gets mapped by f_A to some stable degree $n-1$ polynomial. Consider the polynomial:

$$\phi(s) = (s+q)^{j-1}(s + \frac{1}{q})^{n-k-j}(s^k + q\alpha(s))$$

where q is positive, real and "large," and $1 \leq j \leq n-k, i = j+k$. It is a product of stable polynomials and thus is stable. Consider first the product $g(s) = (s+q)^{j-1}(s+\frac{1}{q})^{n-k-j}$ and focus on the coefficient of s^{n-k-j}. For large enough q it will be dominated by the term q^{j-1}. Furthermore, all other coefficients of $g(s)$ in this product will contain terms which are lower powers of q (and can be negative). When $g(s)$ is multiplied by $s^k + q\alpha(s)$ to form $\phi(s)$, the coefficients of the terms $s^{n-1-j}, \ldots, s^{n-k+1-j}, s^{n-k-j}$ will include a q^j term, and all other coefficients will have terms with lower powers of q. In fact, we can write $\phi(s) = q^j s^{n-k-j}\alpha(s) + f(s,q)$ where $f(s,q)$ has coefficients with lower powers of q. When ϕA is formed it generates a polynomial which can be expressed as: $q^j\psi + \chi$ where χ has entries that include terms with lower powers of q. By making q "large" the roots of ϕA can be made to lie arbitrarily close to the roots of ψ which is stable. □

Other results along the same theme can also be stated. One possibility is expressed in the next lemma.

Lemma 3 *Let A be $n \times n$ matrix where the sum of rows $i-2$ and i, $3 \leq i \leq n$ is a degree $n-1$ stable polynomial. The corresponding f_A is a SPM.*

Proof: The method of proof is similar to that of Lemma 1. Suppose that the sum of rows $i-2$ and i ($3 \leq i \leq n$) of matrix A forms a degree $n-1$ stable polynomial ψ. We need to construct a stable degree $n-1$ polynomial that gets mapped by f_A to some stable degree $n-1$ polynomial. Consider the polynomial:

$$\phi(s) = (s+q)^{j-1}(s+\frac{1}{q})^{n-2-j}(qs^2+s+q)$$

where q is positive, real and "large," and $1 \leq j \leq n-2, i = j+2$. It is a product of stable polynomials and thus is stable. Consider first the product $g(s) = (s+q)^{j-1}(s+\frac{1}{q})^{n-2-j}$ and focus on the coefficient of s^{n-2-j}. For large enough q it will be dominated by the term q^{j-1}. Furthermore, all other coefficients of $g(s)$ in this product will contain terms which are lower powers of q (and can be negative). When $g(s)$ is multiplied by qs^2+s+q to form $\phi(s)$, the coefficients of the terms s^{n-j} and s^{n-2-j} will include the same q^j term (call it $\alpha(q)$), and all other coefficients will have terms with lower powers of q. When ϕA is formed it generates a polynomial which can be expressed as: $\alpha(q)\psi + \chi$ where χ has entries that include terms with lower powers of q. By making q "large" the roots of ϕA can be made to lie arbitrarily close to the roots of ψ which is stable. □

One can also state the following result which is an immediate consequence of the Hermite-Biehler theorem.

Lemma 4 *Let A be $n \times n$ matrix where the submatrix A_1 consisting of all the even (or all the odd) rows is a SPM which in addition maps a stable polynomial with distinct real roots to a stable polynomial. The corresponding f_A is a SPM.*

Proof: We know that the polynomial $\phi(s) = h(s^2) + sg(s^2)$ is stable if and only if $h(u)$ and $g(u)$ form a positive pair [8]. Clearly, if some polynomial $\phi(s)$ is stable and $q > 0$, then polynomials and $\phi_{oq} = h(s^2) + sqg(s^2)$ are also stable. Let n be even ($n-1$ is then odd) and A_1 be the matrix consisting of all the even rows. By assumption it is a SPM which in addition maps some stable polynomial h with distinct real roots to the stable degree $n-1$ polynomial ψ. It is always possible to construct a polynomial $g(u)$ such that $h(u)$ and $g(u)$ form a positive pair and $h(s^2) + sg(s^2)$ is stable degree $n-1$. The polynomial $\phi(s) = qh(s^2) + sg(s^2)$ will be stable for all $q > 0$. The polynomial ϕ will be mapped by A to the polynomial $q\psi + \chi$, where χ does not involve q and for large q its roots lie arbitrarily close to those of ψ. If A_1 is the submatrix of all odd rows for which a stable polynomial g exists with distinct real roots that is mapped to a stable polynomial, $h(u)$ can be constructed that makes $h(u)$ and $g(u)$ a positive pair and $h(s^2) + sg(s^2)$ a stable degree $n-1$ polynomial. The required $\phi(s) = h(s^2) + sqg(s^2)$. A similar argument can be made for the case when n is odd. □

An immediate consequence of this result is that any order Sylvester resultant of a proper (not strictly proper) degree $\bar{n}$ plant which is minimum phase is a SPM.

It should be clear that many more results can be formulated that make use of the ideas expressed in the above proofs. It would be great if one could provide a complete characterization of when a given matrix generates a SPM. This would immediately lead to robust synthesis procedures. No such characterization exists at the present time. However, we are able to make definitive statements for specific system classes. In particular, we can identify system classes for which the corresponding A matrix has special structure (e.g., the sum of two consecutive rows is a stable polynomial). For these system classes are then able to carry out robust controller synthesis. For other system classes we gain insight for robust design.

4 Robust Controller Design

In the previous section we developed a number of tests for stability preserving maps. In this section we show how these results can form the basis for robust controller design procedures. Here we will confine our discussion to plant families with parameter uncertainty where $\mathbf{a} \in \Omega_a$. In the SPM approach the first step is to use Sylvester Resultants to construct the matrix maps. Specifically, for a plant family $P(s, \mathbf{a})$ consisting of order $\bar{n}$ plants a controller of order $\bar{n}-1$ is used and the matrix $A(\mathbf{a}) = S_{\bar{n}}(d_{p0}(s), n_{p0}(s))^{-1} S_{\bar{n}}(d_p(s,\mathbf{a}), n_p(s,\mathbf{a}))$ is formed.

One then proceeds to check whether the map $f_{A(\mathbf{a})}$ generated is a SPM family for all $\mathbf{a} \in \Omega_a$. To accomplish this one uses available SPM tests. In particular, one can check: i) if one row is a stable polynomial, ii) if the sum of two consecutive rows is a stable polynomial, or iii) if the sum of rows $i-2$ and i form a stable polynomial for some $i \in 3 \leq i \leq n$. The procedure then continues by identifying a polynomial ϕ, with the right structure, which is mapped by the matrix family to stable polynomials. Finally, the stabilizing controller is computed as $x = \phi S_{\bar{n}}(d_{p0}(s), n_{p0}(s))^{-1}$.

It is important to point out a salient feature of this robust stabilization procedure. In view of the structure of the generated family of $A(\mathbf{a})$ matrices, the identity matrix will also be included for some value of the uncertain parameter, typically when $\mathbf{a} = 0$. Clearly, for this value no row (or sum of rows) of the corresponding A matrix forms a stable polynomial. This would imply that the required "stability properties" are not present for the entire parameter range. However, the identity matrix is a SSPM which means that it maps *any* stable polynomial to a stable polynomial. Therefore, one can argue that the stability properties of the A matrix would point to ϕ-polynomials that possess a promising "structure." One then checks polynomials of this structure to identify (if possible) an appropriate one. The approach will be demonstrated through a number of examples and can be summarized as follows:

Robust Controller Design

- For the specific plant family generate the appropriate matrix $A(\mathbf{a})$ and check if this matrix generates a stability preserving map family f_A. Specifically, check if the A matrix has the appropriate stability properties. This will identify a promising structure of stable ϕ-polynomials that are parameterized by q.

- Check polynomials with this structure and identify (if possible) one that is mapped by A to stable polynomials.

- Construct the robustly stabilizing controller $x = \phi S_{\bar{n}}(d_{p0}(s), n_{p0}(s))^{-1}$.

Example 1

Consider the plant family below with a single parameter uncertainty.

$$P(s, a_1) = \frac{s - 1 + a_1(3s+2)}{s^2 - s - 2 + a_1(4s-1)} \tag{4}$$

where $a_1 \in \Omega_a = [0, 10]$. With a proper order 1 controller $C(s) = \frac{y_1 s + y_0}{s + x_0}$ the closed loop characteristic polynomial becomes:

$$\begin{aligned}\phi(s, a_1) = {} & (y_1 s + y_0)(s - 1 + a_1(3s+2)) \; + \\ & (s + x_0)(s^2 - s - 2 + a_1(s-1))\end{aligned}$$

Setting up the appropriate Sylvester Resultant we have:

$$S(a_1) = \begin{bmatrix} 1 & 4a_1 - 1 & -a_1 - 2 & 0 \\ 0 & 3a_1 + 1 & 2a_1 - 1 & 0 \\ 0 & 1 & 4a_1 - 1 & -a_1 - 2 \\ 0 & 0 & 3a_1 + 1 & 2a_1 - 1 \end{bmatrix} \tag{5}$$

The nominal plant corresponds to $a_1 = 0$ and the plant family will be robustly stabilized [3] if and only if the matrix $A(a_1) = S(0)^{-1} S(a_1)$ generates a SPM family for all $a_1 \in \Omega_a$:

$$A(a_1) = \begin{bmatrix} 1 & 23/2a_1 & 7a_1 & 15/2a_1 \\ 0 & 9/2a_1 + 1 & 4a_1 & 5/2a_1 \\ 0 & 3/2a_1 & 2a_1 + 1 & 5/2a_1 \\ 0 & 3/2a_1 & -a_1 & 1/2a_1 + 1 \end{bmatrix}$$

It is easy to verify that the first row of this matrix is a stable polynomial for all values of $a_1 > .09317$. This would imply that the "promising polynomial" structure has the form: $(s + 1/q)^3$. On the other hand for small values of a_1 the $A(a_1)$ matrix tends to the identity which maps *any* stable polynomial to stable polynomials. Clearly, this indicates that it is likely that a polynomial with this structure may be found. Indeed, if $q = 3$ we have that $[1 \;\; 1 \;\; 1/3 \;\; 1/27]$ gets mapped to:

$$[1 \;\; 149/9a_1 + 1 \;\; 314/27a_1 + 1/3 \;\; 293/27a_1 + 1/27]$$

One can easily verify that this polynomial is stable for all nonnegative values of a_1. The controller which accomplishes this task is computed as:

$$\begin{aligned}[1 \;\; y_1 \;\; x_0 \;\; y_0] &= [1 \;\; 1 \;\; 1/3 \;\; 1/27] S(0)^{-1} \\ &= [1 \;\; 113/27 \;\; -59/27 \;\; 13/3]\end{aligned}$$

Remark In essence, the SPM design methodology provides insight on how to answer the following question: For the plant family under consideration where should the nominal characteristic polynomial be placed so that robust stability is guaranteed? From plant family data we conclude that a particular polynomial structure is very promising. In this example this insight leads to a solution of the problem.

Example 2

For this example we modify the parameter uncertainty structure and consider the following plant family:

$$P(s, a_1) = \frac{s - 1 + a_1(s+1)}{s^2 - s - 2 + a_1(16/5s - 1)} \tag{6}$$

where $a_1 \in \Omega_a = [0, 10]$. With a proper order 1 controller $C(s) = \frac{y_1 s + y_0}{s + x_0}$ the closed loop characteristic polynomial becomes:

$$\phi(s, a_1) = (y_1 s + y_0)(s - 1 + a_1(s + 1)) + (s + x_0)(s^2 - s - 2 + a_1(16/5s - 1))$$

Setting up the appropriate Sylvester Resultant we have:

$$S(a_1) = \begin{bmatrix} 1 & 16/5a_1 - 1 & -a_1 - 2 & 0 \\ 0 & a_1 + 1 & a_1 - 1 & 0 \\ 0 & 1 & 16/5a_1 - 1 & -a_1 - 2 \\ 0 & 0 & a_1 + 1 & a_1 - 1 \end{bmatrix}$$

The nominal plant corresponds to $a_1 = 0$ and the plant family will be robustly stabilized [3] if and only if the matrix $A(a_1) = S(0)^{-1} S(a_1)$ generates a SPM family for all $a_1 \in \Omega_a$:

$$A(a_1) = \begin{bmatrix} 1 & 57/10a_1 & -3/10a_1 & 9/2a_1 \\ 0 & 3/2a_1 + 1 & 9/10a_1 & 3/2a_1 \\ 0 & 1/2a_1 & -1/10a_1 + 1 & 3/2a_1 \\ 0 & 1/2a_1 & -11/10a_1 & 1/2a_1 + 1 \end{bmatrix}$$

Note that the first row of $A(a_1)$ is no longer a stable polynomial for any positive value of a_1. However, it is easy to verify that the sum of the first rows of this matrix is a stable polynomial for all values of $a_1 > 1.25$. From Lemma 1 we know that a promising structure for a ϕ-polynomial is: $(s+1/q)(qs^2+qs+1)$. Furthermore, as a_1 approaches zero the $A(a_1)$ matrix tends to the identity which maps *any* stable polynomial to stable polynomials. This suggests that a polynomial with this structure may be found. Indeed, if $q = 3$ we have that $[3 \;\; 4 \;\; 2 \;\; 1/3]$ gets mapped to:

$$[3 \;\; 364/15a_1 + 4 \;\; 32/15a_1 + 2 \;\; 68/3a_1 + 1/3]$$

One can easily verify that this polynomial is stable for all nonnegative values of a_1. The controller which accomplishes this task is computed as:

$$\begin{aligned} [1 \;\; y_1 \;\; x_0 \;\; y_0] &= [3 \;\; 4 \;\; 2 \;\; 1/3] S(0)^{-1} \\ &= [3 \;\; 44/3 \;\; -23/3 \;\; 15] \end{aligned}$$

Remark Even though the plant families considered in these two examples are nonminimum phase and unstable, the examples are clearly "academic," and are only intended to demonstrate the SPM approach. The methodology has also been applied to more realistic systems [6].

5 Conclusions

In this paper we have continued the development of the theory of matrix stability preserving maps. We first presented more tests for checking whether some matrix A generates a stability preserving map. These tests were then used to develop robust controller design procedures for plant families with parameter uncertainty. The methodology was demonstrated on a number of examples. In this work the order of the controller is fixed in a manner that guarantees the invertibility of the corresponding Sylvester Resultant. In other work [6] we develop more SPM tests and extend the formulation to cover the case of controllers of arbitrary order. We believe that these results provide more insight for robust controller design and lead to new robust design tools.

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