Proceedings of the 8th IEEE Mediterranean Conference
on Control and Automation (MED 2000)
Rio, Patras, GREECE 17-19 July, 2000

ROBUST TRIANGULAR DECOUPLING VIA MEASUREMENT OUTPUT FEEDBACK

F. N. KOUMBOULIS , M. G. SKARPETIS

University of Thessaly, Department of Mechanical & Industrial Engineering, 383 34 Pedion Areos, Volos, Greece.

Abstract. The problem of robust triangular decoupling (RTD), via measurement output feedback (MOF), is solved. The general analytical expressions of the feedback matrices, are derived. The stability properties of the resulting RTD closed loop system are proven to be analogous to those in RTD via pure state feedback. All above results are successfully applied to control the steering dynamics of cars with four wheels steering (4WS) and with no measurement of the lateral acceleration .

Key Words. Robust control, I/O Triangular Decoupling, Measurement Output Feedback, Linear Systems.

1. INTRODUCTION

The problem of robust triangular decoupling (RDT) via state feedback, has been defined in [1]-[3]. For linear perturbations and using state feedback, the RTD problem has been solved in [4]. For system with nonlinear uncertain structure the RTD problem, via static state feedback, has been solved in [5]. Here, the problem of robust triangular decoupling via static measurement output feedback (RTDMOF) is solved. The general category of uncertain linear systems of nonlinear uncertain structure is considered

$$\dot{x}(t) = A(q)x(t) + B(q)u(t)\ , \quad y_M(t) = M(q)x(t)\ , \qquad y(t) = C(q)x(t) \tag{1.1}$$

where $x \in \Re^n$, $u \in \Re^m$, $y \in \Re^p$ and $y_M \in \Re^\mu$. $A(q) \in [\wp(q)]^{n\times n}, B(q) \in [\wp(q)]^{n\times m}, C(q) \in [\wp(q)]^{p\times n}$ and $M(q) \in [\wp(q)]^{\mu\times n}$ are function matrices depending upon the uncertainty vector $q = [q_1 \cdots q_l] \in Q$ that is independent of time. $\wp(q)$ is the set of scalar nonlinear functions of q.With regard to the structure of $A(q), B(q)$ $C(q)$ and $M(q)$, no limitations or specifications (continuity, boundness, smoothness, etc.) are required. The vector $y_M(t)$ denotes the measurement part of the state vector $x(t)$ while the vector $y(t)$ is the performance output vector. The description (1.1) covers all cases of linear time-invariant systems with uncertain structure. The RTDMOF problem is studied by using a regular static measurement output feedback law of the form

$$u(t) = Ky_M(t) + G\omega(t) \tag{1.2}$$

where $\omega(t) \in \Re^m$ is an external input vector. According to [1-8] and the controller to be physically implementable, K and G are required to be independent of q. Thus, the RTDMOF problem can be interpreted as in Fig. 1 and can be formulated as in definition 1.1.

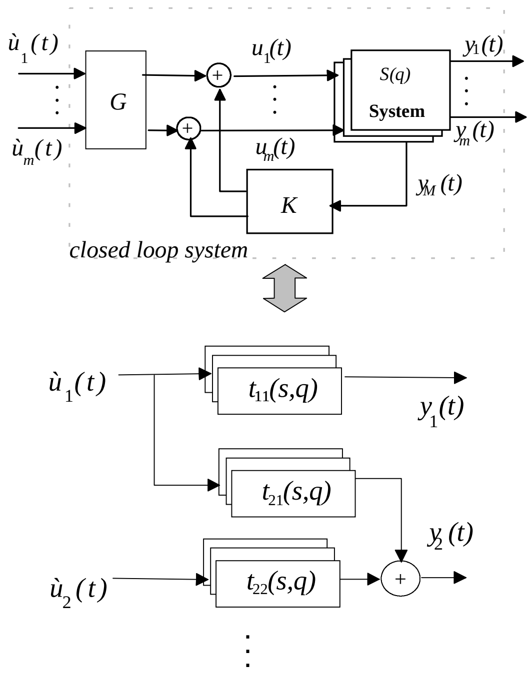


Fig 1. RTDMOF configuration

Definition 1.1. The RTDMOF problem is solvable if there exist independent of q matrices K and G, such that

$$C(q)\left[sI_n - A(q) - B(q)KM(q)\right]^{-1}B(q)G = T(s,q) \ , \qquad \forall q \in \mathcal{Q} \quad (1.3)$$

where $T(s,q)$ is an invertible (for every q) matrix having the form $T(s,q) = \text{triang}\{t_{ij}(s,q)\}$, $t_{ii}(s,q) \neq 0 \ (\forall q \in \mathcal{Q})$, where $\text{triang}\{\cdot\}$ denotes a triangular matrix (let lower triangular) and $t_{ij}(s,q)$ stands for the (i,j) element of $T(s,q)$. z

In this paper, the necessary and sufficient conditions for the RTDMOF problem to have a solution, are established. The class of all the independent of the uncertainties RTDMOF feedback matrices, is determined. The general form of the closed loop system, as well as the respecting robust stability properties are shown to be recasted to those in [5]. All above results are successfully applied to control 4WS cars in case where the measurement of the lateral acceleration is not available. This case appears to be of great practical importance.

It is mentioned that, for the special case where $M(q) = C(q) \ (\forall q \in \mathcal{Q})$, i.e. the case of performance output feedback, the RTD problem has been solved in [8].

2. PRELIMINARY RESULTS

The equation (1.3), formulating the RTDMOF problem can be rewritten as

$$C(q)\left[sI_n - A(q)\right]^{-1}B(q)\times \left\{I_m - KM(q)\left[sI_n - A(q)\right]^{-1}B(q)\right\}^{-1}G = T(s,q) \ , \ \forall q \in \mathcal{Q} \qquad (2.1)$$

From (2.1) as well as the invertibility of the matrix $T(s,q)$ and assumption that the matrix G (map matrix) is invertible, it is readily observed that for the RTDMOF problem to be solvable, for every $q \in \mathcal{Q}$, it is necessary that $p = m$ and the following conditions are necessary to be satisfied

$$\det\left\{C(q)\left[sI_n - A(q)\right]^{-1}B(q)\right\} \neq 0 \ , \ \forall q \in \mathcal{Q} \qquad (2.2)$$

$$\det G \neq 0 \qquad (2.3)$$

Obviously, for the measurement output feedback case to be solvable it is necessary for the respective state feedback problem to be solvable. According to [5] (RTD via state feedback) the following condition is necessary to be satisfied

$$\text{rank}_{\mathbb{R}}\left[C_i^*(q)B(q)\right]^T = i \quad , \quad i = 1,\ldots,m \qquad (2.4)$$

where the operator $\text{rank}_{\mathbb{R}}[\cdot]$ denotes the rank of an uncertain matrix on the field of real numbers (see f.e. [5-8]) and where $C_i^*(q) = \begin{bmatrix} c_1^*(q) \\ \vdots \\ c_i^*(q) \end{bmatrix}$, $c_i^*(q) = c_i^{(\rho_i(q))}(q)$ $(i = 1,\ldots,m)$. The matrix $C_i^*(q)$ is analytically determined in the Appendix.

As proved in [5] condition (2.4) is equivalent to the existence of functions $\upsilon_i(q) \in \wp(q) \quad (i = 1,\ldots,m)$, $(\upsilon_i(q) \neq 0, \forall q \in \mathcal{Q})$ as well as independent of the uncertainties vectors $b_i^* \in \mathbb{R}^{1\times(m-i+1)} (i = 1,\ldots,m)$ such that

$$c_i^*(q)B(q)\prod_{j=0}^{i-1}[b_j^*]^\perp = \upsilon_i(q)b_i^* \quad , \quad i = 1,\ldots,m \qquad (2.5)$$

where $c_i^*(q)$ is the i-th row of $C^*(q) = C_m^*(q)$ and $[b_j^*]^\perp$ is an $(m-j+1)\times(m-j)$ full column rank matrix being orthogonal to the $1\times(m-j+1)$ vector b_j^*, $([b_0^*]^\perp = I_m)$. Based on (2.5) the vectors b_i^* can constructively be computed (see [5]).

If condition (2.4) is satisfied then the precompensator G must be of the following form [5]

$$G = J_{m+1}(P^*)^{-1} \quad ; \quad P^* = \text{triang}\left\{(p_i^*)_j\right\} ,$$
$$J_i = \prod_{j=0}^{i-1}\begin{bmatrix} I_{j-1} & 0 \\ 0 & [b_j^*]^+ \end{bmatrix} \qquad (2.6)$$

where $[b_j^*]^+ = \left[[b_j^*]^T(b_j^*[b_j^*]^T)^{-1} \middle| [b_j^*]^\perp \right]$ and $[b_0^*]^+ = I_m$, while I_{-1} and I_0 are of zero dimension, and where $(p_i^*)_j$ are arbitrary parameters. Expression (2.6) is the general solution of G, for the case of state feedback. Clearly, the set of precompansators solving the RDT problem via MOF is a subset of the precompansator characterisation in (2.6).

3. SOLUTION OF THE PROBLEM

Define

$$\Delta(q) = B(q)[C^*(q)B(q)]^{-1} \ ,$$
$$A_c(q) = A(q) - \Delta(q)C^*(q)A(q) \ ,$$
$$L_i(q) = \left[\Delta_i(q) \quad A_c(q)\Delta_i(q)\cdots \ [A_c(q)]^{2n-1}\Delta_i(q)\right]$$
$$\Delta_i(q) = \begin{cases} \left[\delta_{i+1}(q) \mid \cdots \mid \delta_m(q)\right], & i = 1,\ldots,m-1 \\ 0_{n\times 1}, & i = m \end{cases} \ ;$$
$$\delta_i(q): i\text{-th column of } \Delta(q)$$

Theorem 3.1. The necessary and sufficient conditions for the solvability of the RTD problem for systems with nonlinear uncertain structure, via static measurement output feedback, are $p = m$ and

$$\det\left\{C(q)\left[sI_n - A(q)\right]^{-1}B(q)\right\} \neq 0 \quad , \quad \forall q \in \mathcal{Q} \qquad (3.1)$$

$$\text{rank}_{\mathbb{R}}\left[C_i^*(q)B(q)\right]^T = i \quad , \quad i = 1,\ldots,m \qquad (3.2)$$

$$\text{rank}\begin{bmatrix} M(q)L_i(q) \\ n_i(q)C^*(q)A(q)L_i(q) \end{bmatrix} = \text{rank}_{\Re}[L_i(q)], \quad i = 1, \ldots, m-1 \tag{3.3}$$

where $n_i(q)$ is the i-th row of the matrix $J_{m+1}^{-1}[C^*(q)B(q)]^{-1}$.
Proof: see Appendix ■

4. GENERAL FORM OF THE FEEDBACK MATRICES

Define $\eta_i = \left\langle -n_i(q)C^*(q)A(q)L_i(q) \;\middle\backslash\; M(q)L_i(q) \right\rangle$, where the notation $\langle \cdot \backslash \cdot \rangle$ denotes the projection (in the field of real numbers) of an uncertain vector to the subspace defined by the rows of the uncertain matrix ([5]-[8]). Also define $\left[M(q)L_i(q)\right]^{\perp}$ to be a $\xi_i \times n$ independent of q matrix being orthogonal over the field of real numbers to $M(q)L_i(q)$. The following theorem regarding the derivation of the feedback matrices is presented.

Theorem 4.1. Assume that system (1.1) satisfies the conditions of Theorem 3.1. The general analytical expressions of the independent of the uncertainties measurement output feedback matrices G and K, yielding RTD, are

$$G = J_{m+1}(P^*)^{-1}, \quad K = J_{m+1}[R + \Lambda S];$$
$$R = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_m \end{bmatrix}, \; S = \begin{bmatrix} \left[M(q)L_1(q)\right]^{\perp} \\ \vdots \\ \left[M(q)L_m(q)\right]^{\perp} \end{bmatrix} \tag{4.1}$$

where $P^* = \text{triang}\{p_{i,j}^*\}$ and $\Lambda = \text{diag}\{\tilde{\tau}_i\}$ are arbitrary matrices ($\tilde{\tau}_i = \left[(\tilde{\tau}_i)_1 \cdots (\tilde{\tau}_i)_{\xi_i}\right]$, $\xi_i = \text{rank}\left[M(q)L_i(q)\right]$)
Proof: see Appendix ■

From relation (A.10b) and Theorem 4.1, it is observed that the closed loop system transfer function and the characteristic polynomial are in forms analogous to those in RTD via state feedback [5]. Hence, these establishment in [5] can readily be extended to cover the present case.

6. FOUR WHEEL CAR STEERING

A simplified linear single track model of a 4WS car with arbitrary mass distribution is (see [1-3] and [5])

$$\begin{bmatrix} \dot{a}_f \\ \dot{r} \\ \dot{\delta}_f \end{bmatrix} = \begin{bmatrix} d_{11} & d_1 & 0 \\ d_{21} & d_{22} & d_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_f \\ r \\ \delta_f \end{bmatrix} + \begin{bmatrix} d_1 & 0 \\ 0 & b_{22} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_{cf} \\ \delta_{cr} \end{bmatrix}, \quad \begin{bmatrix} a_f \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_f \\ r \\ \delta_f \end{bmatrix}$$

where $d_{11} = \frac{-l_w c_f}{mul_r}$, $d_1 = \frac{l_w c_f}{ml_r}$, $d_{21} = \frac{(l_f c_f - l_r c_r)}{l_f c_f l_w}$, $d_{22} = \frac{-l_w c_r}{mul_f}$, $d_{23} = \frac{c_r}{ml_f}$, $b_{22} = \frac{-c_r}{ml_f}$ and where a_f is the lateral acceleration of the front axle of the car, $r(t)$ is the yaw rate, while $\delta_{cf}(t)$ and $\delta_{cr}(t)$ are the driver commands to the front and rear steering angles $\delta_f(t)$ and $\delta_r(t)$ ($\delta_r(t) = \delta_{cr}(t)$). m and u are the vehicle's mass and velocity. l_w is the length of the wheelbase ($l_w = l_f + l_r$) with l_f and l_r the distance between the center of gravity and the front and rear axle. c_f and c_r are the front and the rear "cornering stiffnesses", depending on several uncertain parameters like normal force, tire pressure, tire temperature and most importantly on the adhesion coefficient μ between the road surface and the tire. The model uncertainties are u, m, the tire side force characteristics (c_f, c_r) and l_f (for arbitrary mass distribution). Clearly $l_r = l_w - l_f$.

The design requirement is RTDMOF between lateral acceleration and yaw rate. The feedback law is proposed to be $[\delta_{cf}\; \delta_{cr}]^T = K[r \;\; \delta_f]^T + G[\omega_1 \;\; \omega_2]^T$, where ω_1 and ω_2 are external inputs (ω_1 to command a_f). The required sensors are a gyro for r, and a potentiometer for the steering angle δ_f. It is noted that the model has been studied under the assumption of using an accelerometer at the front axle for a_f [1-3], [5]. In practice, the later sensors appears not to be cost efficient and rather sensitive to road disturbances. The model is of *nonlinear uncertain structure* with respect to $q = (u, m, c_f, c_r, l_f)$, i.e. it is of the form (1.1).The uncertainties vary over finite domains, i.e. $u \in \left[u^-, u^+\right]$ $m \in \left[m^-, m^+\right]$, $c_f \in \left[c_f^-, c_f^+\right]$, $c_r \in \left[c_r^-, c_r^+\right]$ and $l_f \in \left[l_f^-, l_f^+\right]$. Note that $u^- > 0$, $m^- > 0$, $c_f^- > 0$ and $c_r^- > 0$.
From Theorem 3.1 it is concluded that for the 4WS car RTDMOF can be achieved. From Theorem 4.1, we get $G = \begin{bmatrix} (p_1^*)_1 & 0 \\ (p_2^*)_1 & (p_2^*)_2 \end{bmatrix}^{-1}$, $K = \begin{bmatrix} -1 & 0 \\ (\lambda_2)_1 & (\lambda_2)_2 \end{bmatrix}$ where $(p_1^*)_1$, $(p_2^*)_1$, $(p_2^*)_2$, $(\lambda_1)_1$, $(\lambda_2)_1$ and $(\lambda_2)_2$ are arbitrary parameters with $(p_1^*)_1 \neq 0$ and $(p_2^*)_2 \neq 0$. G and F are independent of l_r, l_f and l_w. The RTDMOF transfer function is $H_{cl}(s,q) = \begin{bmatrix} t_{11}(s,q) & 0 \\ t_{21}(s,q) & t_{22}(s,q) \end{bmatrix}$, where

$$t_{11}(s,q) = \frac{c_f l_w u (p_1^*)_1^{-1}}{s(l_r mu) + c_f l_w},$$
$$t_{22}(s,q) = -\frac{s c_r u (p_2^*)_2^{-1}}{s^2 l_f mu + s c_r\left[(\lambda_2)_1 u + l_w\right] - c_r u[(\lambda_2)_2 - 1]}$$
$$t_{21}(s,q) = \frac{u(h_{21})_2 s^2 + u(h_{21})_1 s + u(h_{21})_0}{(p_1^*)_1 (p_2^*)_2 \left\{s(l_r mu) - c_f l_w[(\lambda_1)_1 u - 1]\right\}\left\{s^2 l_f mu + s c_r\left[(\lambda_2)_1 u + l_w\right] - c_r u[(\lambda_2)_2 - 1]\right\}}$$

and where $(h_{21})_0 = c_f c_r l_w (p_2^*)_2 - [(\lambda_2)_2 - 1]$, $(h_{21})_2 = c_r l_r mu (p_2^*)_1$, and $(h_{21})_1 = -c_r l_r mu (p_2^*)_2 (\lambda_2)_2 + c_f\{c_r l_w (p_2^*)_1 + l_f mu (p_2^*)_2\}$.

The closed-loop characteristic polynomial, resulting after achieving RTDMOF, is the product of the denominators of $h_{11}(s,q)$ and $h_{22}(s,q)$. For $(\lambda_2)_1 > \frac{-l_w}{u}$, $(\lambda_2)_2 < 1$ the closed loop system becomes stable. Thus, choosing $(\lambda_2)_3 < 0$ and $(\lambda_2)_2 > 0$ the closed loop system is Hurwitz invariant. Hence, RTDMOF with simultaneous robust stability can be satisfied.

Consider the case of the city bus O-305 with [1]:

$$l_f = 3.67[\text{m}],\ l_r = 1.93[\text{m}],\ m = 10000[\text{Kg}],$$
$$c_f = 198000[\text{N/rad}],\ c_r = 470000[\text{N/rad}],u = 10[\text{m/sec}]$$

The uncertain parameters are limited to be the virtual mass *m*, the velocity *u*, c_f and c_r. Choose $(\lambda_2)_1 = 1.0017$, $(\lambda_2)_2 = -6.80851$, $(p_1^*)_1 = 10$, $(p_2^*)_1 = 0$ and $(p_2^*)_2 = 0.0987$ and use the commands $\omega_1(t) = 0$ and $\omega_2(t) = 0.4\sin(2t)$. The trajectories of the response of the resulting robust closed loop system are illustrated in Fig. 2 for nominal values of the uncertainties (continuous lines), $+10\%$ deviation from the nominal values (dashed lines) and -10% deviation from the nominal values (dotted lines). According to Fig. 2 the performance of the closed loop system appears to be satisfactory.

9. CONCLUSIONS

The problem of robust triangular decoupling for systems with nonlinear uncertain structure, via static measurement output feedback, has extensively been solved. The necessary and sufficient conditions for the problem to have a solution, have been established. The class of all independent of the uncertainties controller matrices solving the problem has explicitly been characterized via an analytic formula. The results have successfully been applied to control 4WS cars.

10. REFERENCES

[1] J. Ackermann, A. Barlett, D. Kaesbauer, W. Sienel and R. Steinhouser, **Robust Control Systems with uncertain physical parameters**, *Springer-Verlag*, London, 1993.

[2] J. Ackermann, "Robust decoupling, ideal steering dynamics and yaw stabilization of 4WS cars", *Automatica*, vol. 30, pp. 1761-1768, 1994.

[3] J. Ackermann and W. Sienel, "Robust yaw damping of cars with front and rear wheel steering", *IEEE Trans. on Control System Technology*, vol. 1, pp. 15-20, 1993.

[4] F.N. Koumboulis, "Input - output triangular decoupling and data sensitivity", *Automatica*, Vol. 32, pp. 569-573, 1996.

[5] F. N. Koumboulis and M.G.Skarpetis, "Robust triangular decoupling with application to 4WS cars", *IEEE Transaction of Automatic Control*, vol. 45, pp. 344-352, 2000.

[6] F.N. Koumboulis, and M.G. Skarpetis, "Input output decoupling for linear systems with nonlinear uncertain structure", *J. Franklin Institute*, vol. 333(B), pp. 593-624, 1996.

[7] F.N. Koumboulis, and M.G. Skarpetis, "Robust disturbance rejection with simultaneous robust input output decoupling", *Automatica* vol. 33, pp. 1415-1421, 1997.

[8] F.N. Koumboulis, and M.G. Skarpetis, "Robust triangular decoupling via output feedback", *J. Franklin Institute*, vol. 337, pp. 11-20, 2000.

APPENDIX

A. Construction of $C_i^*(q)$

The vectors $c_i^{(\rho_i(q))}$, are determined by the following recursive definitions [4 and 5]:

$$\rho_1(q) = \min\left\{\ j\ :\ c_1^{(j)}(q)B(q) \neq 0\ ,\ \ j = 0,\ldots,n-1\ \right\} \quad ; \quad c_1^{(j)}(q) = c_1(q)\left[A(q)\right]^j \qquad \text{(A.1)}$$

$$\rho_i(q) = \min\left\{\ j\text{: rank}\begin{bmatrix} C_{i-1}^*(q)B(q) \\ c_i^{(j)}(q)B(q)\end{bmatrix} = i, j = 0,..,n-1\ \right\} \quad i = 2,\ldots,m \qquad \text{(A.2)}$$

$$c_i^{(0)}(q) = c_i(q)\ (i = 1,\ldots,m),$$
$$c_i^{(j)}(q) = c_i^{(j-1)}(q)\left[I - B(q)C_{i-1}^+(q)C_{i-1}^*(q)\right]A(q) \quad (j = 1,\ldots,n-1,\ i = 2,\ldots,m) \qquad \text{(A.3)}$$

where

$$C_{i-1}^+(q) = \left[C_{i-1}^*(q)B(q)\right]^T \times \left\{\left[C_{i-1}^*(q)B(q)\right]\left[C_{i-1}^*(q)\left[B(q)\mid D(q)\right]\right]^T\right\}^{-1}$$

and $c_i(q)$ is the *i*-th row of $C(q)$. Since the rational matrix $C(q)\left[sI - A(q)\right]^{-1}B(q)$ is of full raw rank, the above definitions are well stated, i.e. there always exist integers $j_1(q)$, $j_i(q)$ ($j_1(q), j_i(q) \in \{0,...,n-1\}$) such that: $c_1(q)\left[A(q)\right]^{j_1}B(q) \neq 0$, $\text{rank}\begin{bmatrix} C_{i-1}^*(q)B(q) \\ c_i^{(j_i)}(q)B(q)\end{bmatrix} = i$, for $i = 2,...,m$. Thus, it holds

$$\text{rank}\left[C_i^*(q)B(q)\right] = i \quad , \quad \forall q \in Q \quad , \quad i = 1,\ldots,m \qquad \text{(A.4)}$$

$$c_i^{(j-1)}(q)B(q) = \tilde{c}_i^{(j-1)}(q)C_{i-1}^*(q)B(q)\ ,\ \ j = 1,\ldots,\rho_i(q)\ , \quad i = 2,\ldots,m \qquad \text{(A.5)}$$

where $\tilde{c}_i^{(j-1)}(q) = c_i^{(j-1)}(q)B(q)C_{i-1}^+(q)$, $j = 1,\ldots,\rho_i(q)$. The vector $\tilde{c}_i^{(j-1)}(q)$ $(j = 1,...,\rho_i(q))$ is the vector involving the coefficients that satisfy the dependence relation (A.5).

B. Proof of Theorem 3.1

Proof: The necessity of $p = m$ and of (3.1 and 2) has already been proven and a precompansator is derived in relation (2.6). In order to derive the rest of the solvability conditions, define the interactor of the open loop system, let $L^I(s,q)$. The interactor is a polynomial matrix with respect to *s*, with coefficients depending possibly nonlinearly upon the uncertainty vector *q*. The polynomial matrix $L^I(s,q)$ is invertible for every particular q and has the property

$$L^I(s,q)C(q)[sI_n - A(q)]^{-1}B(q) = C^*(q)[sI_n - A(q)]^{-1}B(q) \quad ; \quad \operatorname{rank}\left[C^*(q)B(q)\right] = m \ \ (\forall q \in \mathcal{Q}) \qquad \text{(A.6)}$$

The interactor is of the form (see [4] and [5])

$$L^I(s,q) = \prod_{i=1}^{m} \left\{ \prod_{j=0}^{\rho_i(q)-1} J^I_{i,j}(s,q) \right\} \quad ;$$

$$J^I_{i,j}(s,q) = \begin{bmatrix} I_{i-1} & 0 & 0 \\ -s\tilde{c}_i^{(j)}(q) & s & 0 \\ 0 & 0 & I_{m-i-1} \end{bmatrix} \quad , \quad J^I_{i,-1}(s,q) = I_m$$

According to (A.6), the equation (2.1) takes on the form

$$sP(s,q)C^*(q)\left[sI_n - A(q)\right]^{-1}B(q) = G^{-1} - G^{-1}KM(q)\left[sI_n - A(q)\right]^{-1}B(q) \qquad \text{(A.7)}$$

where

$$P(s,q) = \operatorname{triang}\{p_{ij}(s,q)\} = s^{-1}\left[T(s,q)\right]^{-1}\left[L^I(s,q)\right]^{-1} \qquad \text{(A.8)}$$

Expand both sides of (2.1) in negative power series of s, to yield

$$s[P_0(q)s^0 + P_1(q)s^{-1} + \cdots] \times [C^*(q)B(q)s^{-1} + C^*(q)A(q)B(q)s^{-2} + \cdots] = G^{-1} - \Phi M(q)B(q)s^{-1} - \Phi M(q)A(q)B(q)s^{-2} - \cdots$$

where $\Phi = G^{-1}K$. The later equation is satisfied if and only if the coefficients of like powers of s in both sides are equal. Clearly, it suffices only the first $2n+1$ coefficients to be equal. So, the equation is reduced to the following set of algebraic equations

$$P_0(q) = P^* J_{m+1}^{-1}[C^*(q)B(q)]^{-1} \qquad \text{(A.9a)}$$

$$\left\{\Phi M(q) + P_0(q)C^*(q)A(q)\right\} \times \left[B(q) \quad A(q)B(q) \cdots [A(q)]^{2n-1}B(q)\right] = -\left[P_1(q) \middle| \ P_2(q) \middle| \cdots \middle| P_{2n}(q)\right]K^*(q) \qquad \text{(A.9b)}$$

where (2.6) has been used and where

$$K^*(q) = \begin{bmatrix} C^*(q)B(q) & C^*(q)A(q)B(q) & \cdots & C^*(q)[A(q)]^{2n-1}B(q) \\ 0 & C^*(q)B(q) & \cdots & C^*(q)[A(q)]^{2n-2}B(q) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C^*(q)B(q) \end{bmatrix}$$

Equation (A.9a) is always solvable for $P_0(q)$. Postmultiplication and premultiplication of equation (A.9b) by $[K^*(q)]^{-1}$ and $[P^*]^{-1}$, respectively, the following design equation is derived

$$\left\{[P^*]^{-1}\Phi M(q) + [N_m(q)]^{-1}C^*(q)A(q)\right\} \times \left[\Delta(q) \quad A_c(q)\Delta(q) \cdots [A_c(q)]^{2n-1}\Delta(q)\right] = \left[\hat{P}_1(q) \quad \hat{P}_2(q) \cdots \hat{P}_{2n}(q)\right]$$

where $N_m(q) = C^*(q)B(q)J_{m+1}$ and $\hat{P}_j(q) = -[P^*]^{-1}P_j(q)$

Define

$$\hat{P}(s,q) = \hat{P}_0(q)s^0 + \hat{P}_1(q)s^{-1} + \ldots = -[P^*]^{-1}P(s,q)$$

Since $\hat{P}(s,q)$ is a lower triangular matrix the i-th row of $\hat{P}_j(q)$, let $\hat{p}_{i,j}(q)$, is of the form

$$\hat{p}_{i,j}(q) = \left[\ \tilde{p}_{i,j}(q) \ \middle| \ 0_{1\times(m-i)}\ \right]$$

where

$$\tilde{p}_{i,j}(q) = e_i \hat{P}_j \begin{bmatrix} I_i \\ 0 \end{bmatrix} \quad (j = 0, \ldots, 2n)$$

and e_i is the $1\times m$ unity vector having the unity in its i-th position. According to the above definition, the design equation can be broken down into the following two equations

$$\tilde{\phi}_i M(q)L_i(q) + n_i(q)C^*(q)A(q)L_i(q) = 0 \qquad \text{(A.10a)}$$

$$\left[\tilde{p}_{i,1}(q) \cdots \tilde{p}_{i,2n}(q)\right] = \left[\tilde{\phi}_i M(q) + n_i(q)C^*(q)A(q)\right] \times \left[\left[\delta_1(q)\middle|\cdots\middle|\delta_i(q)\right] \middle| \cdots \middle| [A_c(q)]^{n-1}\left[\delta_1(q)\middle|\cdots\middle|\delta_i(q)\right]\right] \qquad \text{(A.10b)}$$

where $n_i(q)$ is the i-th row of $[N_m(q)]^{-1}$ and $\tilde{\phi}_i$ is the i-th row of $[P^*]^{-1}\Phi$. The equation (A.10a) governs the general form of $\tilde{\phi}_i$, while (A.10b) gives the parameterization of the RTDMOF closed loop system. Equation (A.10a) is a linear non homogeneous uncertain equation. According to [5-8], the equation (A.10a) is solvable for $\tilde{\phi}_i$ (with $\tilde{\phi}_i$ independent of q), if and only if (3.3) is satisfied.

C. Proof of Theorem 4.1

According to [5-8] the general form of all independent of q vectors $\tilde{\phi}_i$, solving a linear equation of the form of (3.8a), is

$$\tilde{\phi}_i = \tilde{\tau}_i\left[M(q)L_i(q)\right]^{\perp}_{\mathcal{Q}} + \eta_i$$

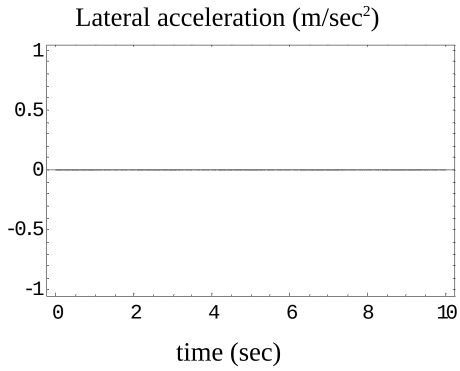
where $\tilde{\tau}_i = \left[(\tilde{\tau}_i)_1 \cdots (\tilde{\tau}_i)_{\xi_i}\right]$ is an independent of q arbitrary vector with $\xi_i = \operatorname{rank}_{\mathcal{Q}}\left[M(q)L_i(q)\right]$. Since

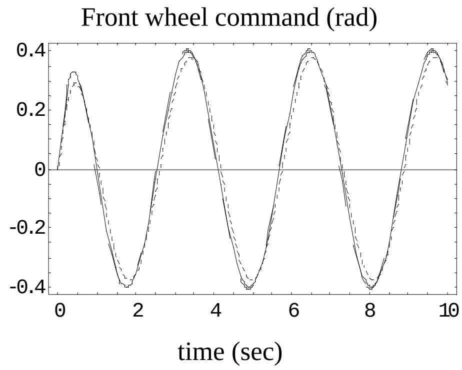
$$\begin{bmatrix} \tilde{\phi}_1 \\ \vdots \\ \tilde{\phi}_m \end{bmatrix} = (P^*)^{-1}\Phi$$

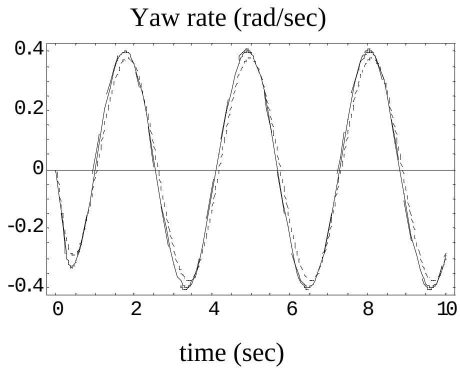
it holds that

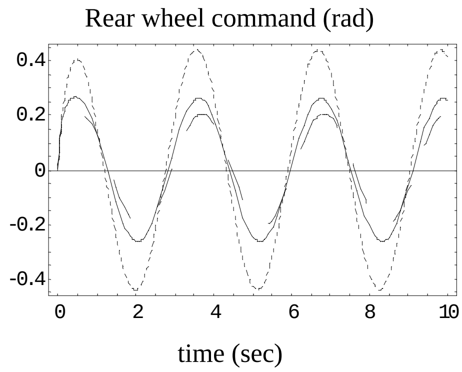
$$\Phi = P^* \begin{bmatrix} \tilde{\tau}_1\left[M(q)L_1(q)\right]^{\perp}_{\mathcal{Q}} \\ \vdots \\ \tilde{\tau}_m\left[M(q)L_m(q)\right]^{\perp}_{\mathcal{Q}} \end{bmatrix} + S$$

Using the relation $K = G\Phi$, relation (4.1) is derived.









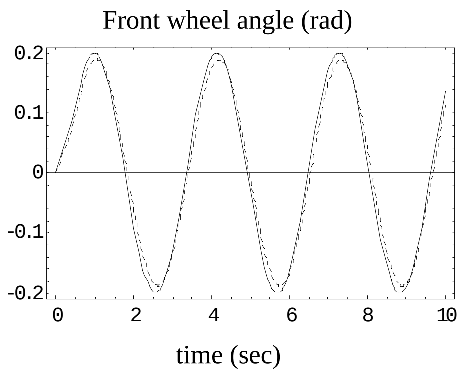


Fig. 2. Trajectories of the response of the RTD closed loop system