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SYNTHESIS OF A NON-INTEGER CONTROL LOOP USING PERFORMANCE CONTOURS

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Abstract: The article proposes a method to design a controller ensuring dynamic behavior of a closed-loop control. Dynamic performance is, in the time domain, the first overshoot of the step response, and the damping ratio and the natural frequency of its dominant oscillatory mode. Dynamic performance is quantified, in the frequency domain, by two contours called "performance contours" and the open-loop gain crossover frequency. The first contour is the Nichols chart magnitude contour which can be considered as an iso-overshoot contour. The second contour, whose construction and analytic expression are given in this article, is a new contour defined in the Nichols plane and parameterized by the damping ratio. The proposed method uses complex non-integer (or fractional) differentiation to compute a transfer function whose open-loop Nichols locus tangents both performance contours, thus ensuring stability margins (or stability degree).

Key words: Dynamic behavior, Stability margins, Overshoot, Damping ratio, Fractional differo-integration

1. INTRODUCTION

Any control loop structure can be reduced to the series connection of a controller and a plant with disturbed output (Fig.1). This configuration permits the definition of two transfer functions: one for the output disturbance, one for the loop reference input. The transfer function relative to the output disturbance and defining the regulation function, is sensitivity function $S(s) = (1+\beta(s))^{-1}$ where $\beta(s)$ is the open-loop transfer function. The transfer function relative to the loop reference input and defining the tracking function, is complementary sensitivity function $T(s) = 1-S(s) = (1+\beta(s))^{-1}\beta(s)$. Step responses in regulation mode and in tracking mode have thus the same dynamics: same first overshoot, same natural frequency and same damping ratio of the oscillatory mode.

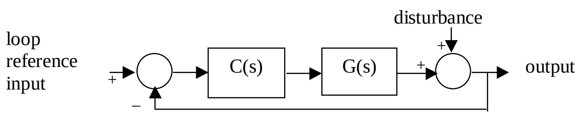

Fig.1. Elementary control loop structure

Although the stability of a closed-loop is easily found by the sign of the real part of closed-loop poles, it is more difficult to give a quantitative value to the concept of stability degree. Its value in both time and frequency domains, is given by dynamic performance which must be selected.

In the time domain, the stability degree is quantified in the tracking or in the regulation mode by either the first overshoot O'_1 of the step response, or by the damping ratio ζ. This damping ratio characterizes the decrease rate of the overshoots of the step response oscillatory mode, and is defined as the damping coefficient of this mode.

In the frequency domain, the stability degree can be characterized by the distance from the Nichols locus to point (-1,0). This distance is commonly quantified by stability margins (gain and phase), or by a Nichols magnitude contour whose graduation M_t gives the magnitude peak of the tracking transfer function, or by a magnitude contour, defined notably by Oustaloup [3], whose graduation M_r gives the magnitude peak of the regulation transfer.

To design a control system in the frequency domain, frequency dynamic performance indicating time dynamic performance must be selected, as imposed specifications are usually time-type. The magnitude peak, M_t, correlates strongly with the first overshoot [6]. A Nichols chart magnitude contour can thus be considered as an iso-overshoot contour. On the contrary, the peak magnitude M_r does not correlate with any time dynamic performance. For example, although M_r parameterizes a circle centered on the point (-1,0) of the Nyquist plane, measuring thus a true distance from this point, M_r does not correlate with the first overshoot or with the damping ratio.

For the quantification of the stability degree in the time domain to have an equivalent in the frequency domain, a new definition of the measurement of the distance from the critical point needs to be given. A new contour whose graduation is the damping ratio ζ has been determined in the Nichols plane and is thus an isodamping contour [5].

Both contours, the magnitude contour and the isodamping contour, define performance contours. To ensure dynamic performance, this paper introduces a method to compute a transfer function whose open-loop Nichols locus tangents both performance contours. To simplify computation, we use complex fractional (or non-integer) order transfer function which can be defined with few parameters.

Section 2 introduces the transfer function of a complex non-integer integrator defining a generalized template which is considered as part of the open-loop Nichols locus [4][1] and which is used in the geometrical construction method for a network of isodamping contours [5].

Section 3 shows how to use the networks of performance contours and defines the open-loop transfer.

Section 4 describes a two-step method to compute an open-loop transfer whose Nichols locus tangents both performance contours and an example is given to validate the whole of the proposed approach.

2. COMPLEX NON-INTEGER INTEGRATION AND ISODAMPING CONTOURS

2.1. Generalized template and non-integer integration

A vertical template [3] is achieved in the Nichols plane using a real non-integer integration order, n, which defines its phase placement at crossover frequency ω_{cg}, $-n90°$. The vertical template (Fig.2) is described by the transfer function of a real non-integer (or fractional) integrator :

$$\beta(s) = \left(\frac{\omega_{cg}}{s}\right)^n \text{ for } \omega \in \left[\omega_A, \omega_B\right],\ n \in \mathbb{R}. \qquad (1)$$

From the extension of the description of the vertical template, the generalized template can then be characterized by a complex non-integer integration order, n. The real part defines its phase placement at ω_{cg}, $-\mathcal{Re}(n)90°$, and the imaginary part defines its angle to the vertical (Fig.2). The generalized template is thus described by the transfer function [4][1]:

$$\beta(s) = \left(\cosh\left(b\frac{\pi}{2}\right)\right)^{\mathrm{sign}(b)} \left(\frac{\omega_{cg}}{s}\right)^a \left(\mathrm{Re}_{/i}\left[\left(\frac{\omega_{cg}}{s}\right)^{ib}\right]\right)^{-\mathrm{sign}(b)}. \qquad (2)$$

The imaginary unit i of the integration order n ($n = a + \mathrm{i}b$) is independent of the imaginary unit j of the variable s ($s=\sigma+\mathrm{j}\omega$).

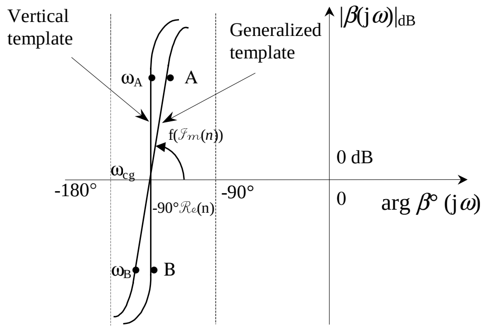

Fig.2. Representation of the vertical template and of the generalized template in the Nichols chart

2.2. Generalized template envelope as isodamping contour [5][6]

The easiest geometrical way to construct an isodamping contour is to use an envelope technique. The contour is then defined as the envelope tangented by a set of segments (Fig.3). In the Nichols plane, each segment of the set can be considered as the rectilinear part of an open-loop Nichols locus that ensures the closed-loop damping ratio corresponding to the contour. This rectilinear part around gain crossover frequency, ω_{cg}, is the "generalized template" defined above.

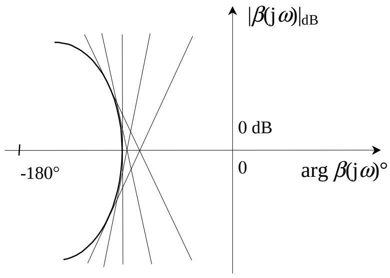

Fig.3. Envelope defining an isodamping contour in the Nichols plane

3. ANALYTIC STUDY OF PERFORMANCE CONTOURS AND OF THE OPEN-LOOP TRANSFER

3.1. Magnitude contours

The analytic expression of a magnitude contour Γ_{Mt} is determined from a point M of the Nichols plane P, with X and Y, expressed in degrees and in decibels, the cartesian coordinates of M. Γ_{Mt} is defined by :

$$\Gamma_{Mt} = \left\{ M(X,Y) \in P, \frac{10^{\frac{Y}{10}}}{1 + 2.10^{\frac{Y}{20}}\cos\left(\frac{\pi}{180}X\right) + 10^{\frac{Y}{10}}} = M_t^{\,2} \right\}. \qquad (3)$$

The equation of the tangent to Γ_{Mt} at point (X_i, Y_i) is deduced from relation (3) and can be written:

$$Y = \alpha_1 X + \beta_1, \qquad (4)$$

with:

$$\alpha_1 = \frac{-\pi}{9\ln(10)} \frac{\sin\left(\frac{\pi}{180} X_\mathrm{i}\right)}{\cos\left(\frac{\pi}{180} X_\mathrm{i}\right) + 10^{-\frac{Y_\mathrm{i}}{20}}} \tag{5}$$

and

$$\beta_1 = Y_\mathrm{i} - \frac{\pi}{9\ln(10)} \frac{\sin\left(\frac{\pi}{180} X_\mathrm{i}\right)}{\cos\left(\frac{\pi}{180} X_\mathrm{i}\right) + 10^{-\frac{Y_\mathrm{i}}{20}}} X_\mathrm{i} \,. \tag{6}$$

3.2. Isodamping contours

Isodamping contours can be defined analytically using a polynomial equation determined by interpolation of graphical data of each contour[3]. To use the same syntax as for a magnitude contour [3], a contour Γ_ζ is thus defined by:

$$\Gamma_\zeta = \left\{ M(X,Y) \in P,\ X - \sum_{j=0}^{2} f_j(\zeta) Y^{2j} . = 0 \right\}, \tag{7}$$

with : $f_j(\zeta) = \sum_{k=0}^{3} a_{jk} \zeta^k$, (8)

X and Y being the coordinates, always expressed in degrees and in decibels, and a_{jk} the coefficients given in table 1.

The equation of the tangent to Γ_ζ at point (X_i, Y_i) is deduced from relation (7) and can be written:

$$Y = \alpha_2 X + \beta_2 , \tag{9}$$

with :

$$\alpha_2 = \frac{1}{2Y_\mathrm{i}\left(\sum_{k=0}^{3} a_{1k}\zeta^k\right) + 4Y_\mathrm{i}^3\left(\sum_{k=0}^{3} a_{2k}\zeta^k\right)} \tag{10}$$

and

$$\beta_2 = Y_\mathrm{i} - \frac{1}{2Y_\mathrm{i}\left(\sum_{k=0}^{3} a_{1k}\zeta^k\right) + 4Y_\mathrm{i}^3\left(\sum_{k=0}^{3} a_{2k}\zeta^k\right)} X_\mathrm{i} \,. \tag{11}$$

Table 1. Values of coefficients a_{jk}

j/ k	0	1	2	3
0	-180.36	117.7	-74.316	40.376
1	-1.1538	3.8888	-5.2999	2.5417
2	-0.0057101	0.0080962	-0.0060354	0.0016158

3.3. Open-loop transfer including generalized template

The aim of this section is to describe analytically, for the nominal plant, the open-loop behavior which takes into account:

- the accuracy specifications at low frequencies;
- the generalized template around frequency ω_cg;
- the plant behavior at high frequencies in accordance with input sensitivity specifications for these frequencies.

For stable minimum-phase plants, this behavior can be described by the following transfer function (Fig.4):

$$\beta(s) = \beta_\mathrm{l}(s)\beta_\mathrm{m}(s)\beta_\mathrm{h}(s) \,. \tag{12}$$

- $\beta_\mathrm{m}(s)$, based on complex non-integer integration, is the transfer function describing the band-limited generalized template [6]:

$$\beta_\mathrm{m}(s) = \mathrm{K}\left(\frac{1+\frac{s}{\omega_\mathrm{h}}}{1+\frac{s}{\omega_\mathrm{l}}}\right)^{a} \left[\mathrm{Re}_{/\mathrm{i}}\left\{\left(\left[\frac{1+\left(\frac{\omega_\mathrm{cg}}{\omega_\mathrm{l}}\right)^2}{1+\left(\frac{\omega_\mathrm{cg}}{\omega_\mathrm{h}}\right)^2}\right]^{\frac{1}{2}} \left(\frac{1+\frac{s}{\omega_\mathrm{h}}}{1+\frac{s}{\omega_\mathrm{l}}}\right)\right)^{ib'}\right\}\right]^{-q'\mathrm{sign}'b')} , \tag{13}$$

q' being the smallest integer such that b' verifies $|b'| < \min(|b_1|,|b_2|)$ with :

$$|b_1| = \pi / \left| \ln\left(\left(1 + \left(\frac{\omega_\mathrm{cg}}{\omega_\mathrm{l}}\right)^2\right) / \left(1 + \left(\frac{\omega_\mathrm{cg}}{\omega_\mathrm{h}}\right)^2\right) \right) \right| \tag{14}$$

and $|b_2| = \pi / \left| \ln\left[\left(\left(1 + \left(\frac{\omega_\mathrm{cg}}{\omega_\mathrm{l}}\right)^2\right) / \left(1 + \left(\frac{\omega_\mathrm{cg}}{\omega_\mathrm{h}}\right)^2\right) \left(\frac{\omega_\mathrm{l}}{\omega_\mathrm{h}}\right)^2 \right) \right] \right|$,(15)

and K being computed to get a gain of 0 dB at ω_cg.

- $\beta_\mathrm{l}(s)$ is the transfer function of a order n_l proportional-integrator, whose corner frequency equals the low corner frequency of $\beta_\mathrm{m}(s)$, so that joining $\beta_\mathrm{l}(s)$ and $\beta_\mathrm{m}(s)$ does not introduce extra parameters. $\beta_\mathrm{l}(s)$ is defined by:

$$\beta_\mathrm{l}(s) = \left(1 + \frac{\omega_\mathrm{l}}{s}\right)^{n_\mathrm{l}} . \tag{16}$$

If n_pl is the order of asymptotic behavior of the plant in low frequency ($\omega << \omega_\mathrm{l}$), order n_l is given by $n_\mathrm{l} \geq 1$ if $n_\mathrm{pl} = 0$, and $n_\mathrm{l} \geq n_\mathrm{pl}$ if $n_\mathrm{pl} \geq 1$, with $n_\mathrm{l}=1$ canceling the position error and $n_\mathrm{l}=2$ canceling the velocity error.

- $\beta_\mathrm{h}(s)$ is the transfer function of a order n_h low-pass filter, whose corner frequency equals the high corner frequency of $\beta_\mathrm{m}(s)$, so that joining $\beta_\mathrm{h}(s)$ and $\beta_\mathrm{m}(s)$ does not introduce extra parameters. $\beta_\mathrm{h}(s)$ is defined by:

$$\beta_\mathrm{h}(s) = 1/\left(1 + \frac{s}{\omega_\mathrm{h}}\right)^{n_\mathrm{h}} . \tag{17}$$

If n_ph is the order of asymptotic behavior of the plant in high frequency ($\omega >> \omega_\mathrm{h}$), order n_h is given by $n_\mathrm{h} \geq n_\mathrm{ph}$, with $n_\mathrm{h} = n_\mathrm{ph}$ ensuring invariability of the input sensitivity function with the frequency, and $n_\mathrm{h} > n_\mathrm{ph}$ ensuring decrease.

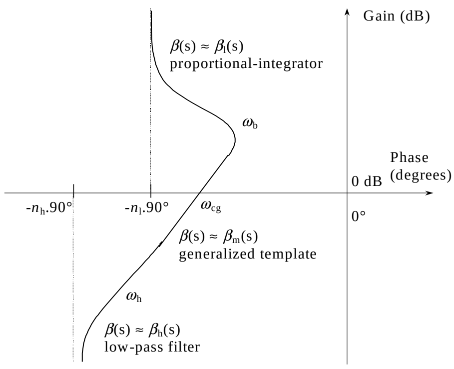

Fig.4. Different parts of the open-loop Nichols locus

♦ The modulus and the argument of the open-loop frequency response are expressed respectively by:

$$\left|\beta(\mathrm{j}\omega)\mathrm{dB}\right| = 20\log K + 20n_{\mathrm{l}}\log\rho_{\mathrm{l}} - 20n_{\mathrm{h}}\log\rho_{\mathrm{h}} + 20\,a\log\left(\frac{\rho_{\mathrm{h}}}{\rho_{\mathrm{l}}}\right) - 10q'\mathrm{sign}(b')\log\left(\cosh^2\left(b'\left(\theta_{\mathrm{h}}-\theta_{\mathrm{l}}\right)\right)\right) \qquad (18)$$

and

$$\arg\beta^{\circ}\left(\mathrm{j}\omega\right) = \frac{180}{\pi}\left[-n_{\mathrm{l}}\frac{\pi}{2} + (n_l - a)\theta_{\mathrm{l}} + (a - nh)\theta_{\mathrm{h}}\right] + \frac{180}{\pi}q'\mathrm{sign}(b')\arctan\left[\tan\left(b'\ln\left(C_0\frac{\rho_{\mathrm{h}}}{\rho_{\mathrm{l}}}\right)\right)\tanh\left[b'\left(\theta_{\mathrm{h}}-\theta_{\mathrm{l}}\right)\right]\right], \qquad (19)$$

with :

$$\rho_{\mathrm{l}} = \sqrt{1+\left(\frac{\omega}{\omega_{\mathrm{l}}}\right)^2}, \quad \rho_{\mathrm{h}} = \sqrt{1+\left(\frac{\omega}{\omega_{\mathrm{h}}}\right)^2}, \quad \theta_{\mathrm{l}} = \arctan\left(\frac{\omega}{\omega_{\mathrm{l}}}\right) \text{ and } \theta_{\mathrm{h}} = \arctan\left(\frac{\omega}{\omega_{\mathrm{h}}}\right). \qquad (20)$$

♦ The expression of the slope of the tangent to the open-loop Nichols locus is given by:

$$\Delta(\omega) = \frac{d\left|\beta\left(j\omega\right)\right|dB}{d\arg\beta^{\circ}\left(j\omega\right)} = \frac{c_0 + c_1 f(b')}{d_0 + d_1 f(b')}, \qquad (21)$$

with:

$$\omega'_{\mathrm{l}} = \frac{\omega}{\omega_{\mathrm{l}}}, \quad \omega'_{\mathrm{h}} = \frac{\omega}{\omega_{\mathrm{h}}}, \qquad (22)$$

$$c_0 = -20n_{\mathrm{b}}\frac{1}{1+\omega'^2_{\mathrm{l}}} - 20a\frac{\omega'^2_{\mathrm{l}}}{1+\omega'^2_{\mathrm{l}}} + 20\left(a-n_{\mathrm{h}}\right)\frac{\omega'^2_{\mathrm{h}}}{1+\omega'^2_{\mathrm{h}}}, \qquad (23)$$

$$c_1 = \frac{\sin\left[2b'\ln\left(C_0\frac{\rho_{\mathrm{h}}}{\rho_{\mathrm{l}}}\right)\right]\left(\frac{\omega'^2_{\mathrm{h}}}{1+\omega'^2_{\mathrm{h}}} - \frac{\omega'^2_{\mathrm{l}}}{1+\omega'^2_{l}}\right)}{\cos^2\left[b'\ln\left(C_0\frac{\rho_{\mathrm{h}}}{\rho_{\mathrm{l}}}\right)\right] + \sinh^2\left[b'\left(\theta_{\mathrm{h}}-\theta_{\mathrm{l}}\right)\right]} - \frac{\sinh\left[2b'\left(\theta_{\mathrm{h}}-\theta_{\mathrm{l}}\right)\right]\left(\frac{\omega'_{\mathrm{h}}}{1+\omega'^2_{\mathrm{h}}} - \frac{\omega'_{\mathrm{l}}}{1+\omega'^2_{\mathrm{l}}}\right)}{\cos^2\left[b'\ln\left(C_0\frac{\rho_{\mathrm{h}}}{\rho_{\mathrm{l}}}\right)\right] + \sinh^2\left[b'\left(\theta_{\mathrm{h}}-\theta_{\mathrm{l}}\right)\right]}, \qquad (24)$$

$$d_0 = \frac{180}{\pi}\ln(10)\left(\left(a-n_{\mathrm{h}}\right)\frac{\omega'_{\mathrm{h}}}{1+\omega'^2_{\mathrm{h}}} + \left(n_{\mathrm{l}}-a\right)\frac{\omega'_{\mathrm{l}}}{1+\omega'^2_{\mathrm{l}}}\right), \qquad (25)$$

$$d_1 = \frac{180\ln(10)}{\pi}\frac{\tan\left[b'\ln\left(C_0\frac{\rho_{\mathrm{h}}}{\rho_l}\right)\right]\left[1-\tanh^2\left[b'\left(\theta_{\mathrm{h}}-\theta_{\mathrm{l}}\right)\right]\right]\left(\frac{\omega'_{\mathrm{h}}}{1+\omega'^2_{\mathrm{h}}} - \frac{\omega'_{\mathrm{l}}}{1+\omega'^2_{\mathrm{l}}}\right)}{1+\tan^2\left[b'\ln\left(C_0\frac{\rho_{\mathrm{h}}}{\rho_{\mathrm{l}}}\right)\right]\tanh^2\left[b'\left(\theta_{\mathrm{h}}-\theta_{\mathrm{l}}\right)\right]} + \frac{180\ln(10)}{\pi}\frac{\tanh\left[b'\left(\theta_{\mathrm{h}}-\theta_{\mathrm{l}}\right)\right]\left[1+\tan^2\left[b'\ln\left(C_0\frac{\rho_{\mathrm{h}}}{\rho_{\mathrm{l}}}\right)\right]\right]\left(\frac{\omega'^2_{\mathrm{h}}}{1+\omega'^2_{\mathrm{h}}} - \frac{\omega'^2_{\mathrm{l}}}{1+\omega'^2_{\mathrm{l}}}\right)}{1+\tan^2\left[b'\ln\left(C_0\frac{\rho_{\mathrm{h}}}{\rho_{\mathrm{l}}}\right)\right]\tanh^2\left[b'\left(\theta_{\mathrm{h}}-\theta_{\mathrm{l}}\right)\right]}, \qquad (26)$$

and

$$f(b') = q'b'\mathrm{sign}(b'). \qquad (27)$$

♦ At frequency ω_{cg}, relations (18) and (19) become:

$$\left|\beta(\mathrm{j}\omega_{\mathrm{cg}})\right|\mathrm{dB} = 0, \qquad (28)$$

and

$$\arg\beta^{\circ}(\mathrm{j}\omega_{\mathrm{cg}}) = \frac{180}{\pi}\left[n_{\mathrm{l}}\left(\theta_{\mathrm{cgl}} - \frac{\pi}{2}\right) - n_{\mathrm{h}}\theta_{\mathrm{cgh}} + a\left(\theta_{\mathrm{cgh}} - \theta_{\mathrm{cgl}}\right)\right] \qquad (29),$$

with

$$\theta_{\mathrm{cgl}} = \arctan\left(\frac{\omega_{\mathrm{cg}}}{\omega_{\mathrm{l}}}\right) \text{ and } \theta_{\mathrm{cgh}} = \arctan\left(\frac{\omega_{\mathrm{cg}}}{\omega_{\mathrm{h}}}\right). \qquad (31)$$

The equation of the tangent to the Nichols locus at this frequency is then given by:

$$Y = \alpha_{\mathrm{BO}}\left(X - X_{\mathrm{BO}}\right), \qquad (32)$$

with :

$$\alpha_{\mathrm{BO}} = \left.\frac{d\left|\beta\left(j\omega\right)\right|dB}{d\arg\beta^{\circ}(j\omega)}\right|_{\omega=\omega_{\mathrm{cg}}} \text{ and } X_{\mathrm{BO}} = \arg\beta^{\circ}(\mathrm{j}\omega_{\mathrm{cg}}). \qquad (33)$$

4. TANGENCY OF THE OPEN-LOOP NICHOLS LOCUS TO TWO PERFORMANCE CONTOURS

4.1. First step

The principle of the first step is conditioned by the hypothesis that the corner frequencies, ω_{l} and ω_{h}, must be far enough from each other so that the rectilinear part of the open-loop Nichols locus (which defines the generalized template) is long enough to tangent both a magnitude contour and an isodamping contour (Fig.5). In this study context, it is possible to interpret the generalized template as a part of the common tangent to both contours.

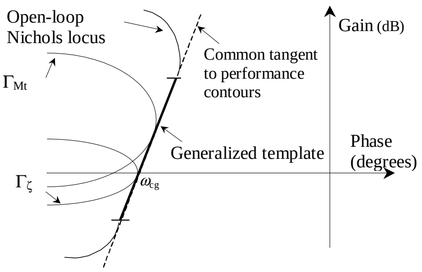

Fig.5. Illustration of the first step

The first step requires the determination of:
- the equation of the common tangent to performance contours (Fig.6);
- the parameters of the open-loop transfer whose rectilinear part of the Nichols locus belongs to this common tangent, or, in other words, whose tangent to the Nichols locus at frequency ω_{cg} is the same as the common tangent (Fig.5).

As the common tangent equation is characterized by only two parameters, only two parameters of the open-loop transfer function can be determined using the equality of this common tangent with the tangent to the open-loop Nichols locus. The others parameters need to be fixed:
- the gain crossover frequency , ω_{cg}
- the orders of the transfer functions β_l(p) and β_h(p), n_l and n_h
- the corner frequencies, ω_l and ω_h.

It is then possible to determine *a* and *b'*.

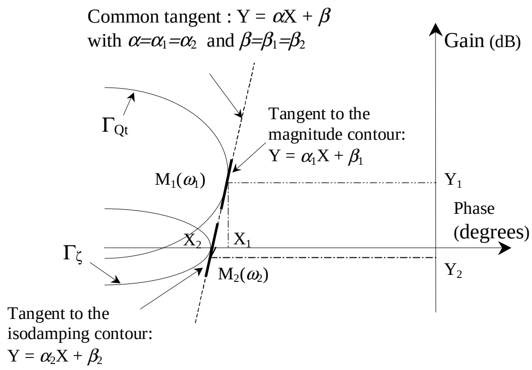

Fig.6. Common tangent to performance contours

The determination of the common tangent which tangents both a given magnitude contour and a given isodamping contour, requires the analytic expressions of each contour and of the equations of their tangent (relations 3, 7, 4, 9). This common tangent can be defined by two points, $M_1(X_1,Y_1)$ and $M_2(X_2,Y_2)$, and thus by four coordinates which are the solutions of the set of equations :

$$\frac{10^{\frac{Y_1}{10}}}{1+2.10^{\frac{Y_1}{20}}\cos(\frac{\pi}{180}X_1)+10^{\frac{Y_1}{10}}} = M_t^2,\; X_2 = \sum_{j=0}^{2} f_j(\zeta) Y_{2\,dB}^{2j} \tag{34}$$

$$\alpha_1 = \alpha_2 \text{ and } \beta_1 = \beta_2 .$$

The first two equations express that the points M_1 and M_2 belong respectively to the contours Γ_{Mt} and Γ_ζ (Figure 6), while the last two equations express the equality of the tangents to these contours (relations 4 and 9).

From the four solutions of (34), that is to say X_1, Y_1, X_2 and Y_2, the equation of the common tangent can be written:

$$Y = \alpha_T\left(X - X_T\right), \tag{35}$$

with:

$$\alpha_T = \frac{Y_2 - Y_1}{X_2 - X_1} \quad \text{and} \quad X_T = X_1 - \frac{X_2 - X_1}{Y_2 - Y_1} Y_1 . \tag{36}$$

Once the equation of the common tangent to both performance contours is obtained (relation 35), it is rendered equal to the tangent to the open-loop Nichols locus at frequency ω_{cg} (relation 32). Gain crossover frequency ω_{cg}, orders n_l and n_h of the transfer functions β_l(s) and β_h(s), and corner frequencies ω_l and ω_h, must all be fixed previously. The others parameters of the transfer function (*a* and *b'*) can then be computed.

a is then given by the relation:

$$a = \frac{1}{\theta_{cgh} - \theta_{cgl}}\left[\frac{\pi}{180} X_T + n_l\left(\frac{\pi}{2} - \theta_{cgl}\right) + n_h \theta_{cgh}\right] \tag{37}$$

while *b'* is the solution of equation:

$$\alpha_T = \left.\frac{d\left|\beta\left(j\omega\right)\right|dB}{d \arg \beta^\circ(j\omega)}\right|_{\omega=\omega_{cg}} . \tag{38}$$

Finally, *K*, computed to guarantee a gain of 0 dB at frequency ω_{cg}, is defined by:

$$20\log K = 10a\log\left(1+\left(\frac{\omega_{cg}}{\omega_l}\right)^2\right) - 10n_l \log\left(1+\left(\frac{\omega_l}{\omega_{cg}}\right)^2\right)$$
$$- 10(a-n_h)\log\left(1+\left(\frac{\omega_{cg}}{\omega_h}\right)^2\right) + 10q' sign(b')\log\left(\cosh^2\left(b'\left(\theta_{cgl}-\theta_{cgb}\right)\right)\right). \tag{39}$$

4.2. Second step

Although the first step has the advantage of being programmed easily while giving satisfactory results, it has the disadvantage of requiring an arbitrary choice of the corner frequencies ω_l and ω_h which may be incompatible with the performance specifications of the control (input sensitivity, perturbation rejection,…).

Also, even if the open-loop Nichols locus tangency with performance contours is well-ensured around ω_{cg}, outside this zone the Nichols locus may curve back across the contours .

Therefore, the second step aims to compensate these disadvantages. It uses results from the first step, notably the coordinates of tangency points M_1 and M_2.

To avoid an arbitrary choice of ω_l and ω_h, the problem is set differently by considering other study frequencies: tangency frequencies ω_1 and ω_2, and no longer gain crossover frequency ω_{cg}, are taken into account (Fig.6). Instead of trying to render equal the tangent to open-loop Nichols locus at ω_{cg} and the common tangent to performance contours (first step), we can try to render equal tangents to open-loop Nichols locus at ω_1 and ω_2 and tangents to performance contours at M_1 and M_2.

As M_1 and M_2 are both on performance contours and on the open-loop Nichols locus, the coordinates in decibels and in degrees of these points are equal to moduli and arguments of the open-loop frequency response at ω_1 and ω_2. Thus:

$$Y_1 = |\beta(j\omega_1)|,\ Y_2 = |\beta(j\omega_2)|,\ X_1 = \arg\beta(j\omega_1),\ X_2 = \arg\beta(j\omega_2). \quad (40)$$

Also, as the open-loop Nichols locus tangents the performance contours at these points, the slopes α of the tangents to the magnitude and isodamping contours are equal to the slopes $\Delta(\omega_1)$ and $\Delta(\omega_2)$ of the tangents to the open-loop Nichols locus at frequencies ω_1 and ω_2, thus:

$$\alpha = \Delta(\omega_1) \text{ and } \alpha = \Delta(\omega_2)\ . \quad (41)$$

Relations (40) and (41) constitute a set of six non linear equations. As gain crossover frequency ω_{cg} and orders n_l and n_h of transfer functions $\beta_l(s)$ and $\beta_h(s)$ are fixed, the four parameters a, b', ω_l and ω_h must be determined to characterize completely the open-loop transfer. As frequencies ω_1 and ω_2 must also be computed, a set of six non linear equations with six unknowns is to be solved.
The non linearity of the set of equations making it difficult to solve, optimization must be used. We have chosen a method which minimizes a cost function under equality constraints. As the aim is to guarantee equality, or near equality, of slopes to ensure open-loop Nichols locus tangency to both contours, the chosen cost function is:

$$J = (\Delta(\omega_1) - \alpha)^2 + (\Delta(\omega_2) - \alpha)^2, \quad (42)$$

and equality constraints are defined by:

$$|\beta(\omega_1)| = Y_1,\ |\beta(\omega_2)| = Y_2,\ \arg\beta(\omega_1) = X_1,\ \arg\beta(\omega_2) = X_2\ . \quad (43)$$

Remark: To guarantee that the open-loop Nichols locus does not curve back across the magnitude contour and the isodamping contour (that it must only tangent), inequality constraints are added to the algorithm which computes the open-loop transfer function parameters.

4.3. Example

The experiment plant consists in a DC motor. The transfer function of the plant whose output signal is the velocity expresses:

$$G(s) = \frac{9092}{(1+0.0047s)(1+96s)}\ . \quad (44)$$

The control system must be computed so that the dynamic specifications in the time domain are:
- a first overshoot O'_1 around 20 to 25%.
- a damping factor ζ around 0.7, with a tolerance of ±5%.

The control law being designed in the frequency domain, such specifications must be converted into two performance contours.
For the first overshoot O'_1, the value of the peak magnitude M_t must be determined so that a Nichols contour of parameter M_t may be an iso-overshoot contour of parameter O'_1. Thus, a result to parameter a Nichols magnitude contour by the corresponding first overshoot is used [6]. A 20 to 25% overshoot corresponds to a value of M_t of 2dB (the exact value of O'_1 for M_t equal to 2dB is 22.14%).
For the damping factor ζ, we only have to consider the isodamping contour of parameter ζ.
Fig.7 shows the step-response to a reference input signal of magnitude 20,000. Measured value of the first overshoot is 25%. The specification concerning a required first overshoot of 20 to 25% is thus respected.
The damping factor is evaluated from the highest half-angle at the origin formed by the pair of dominant complex poles. Computation of complex poles leads to ζ = 0.72, thus a relative error of 2.85% compared to designed damping factor, ζ = 0.7.

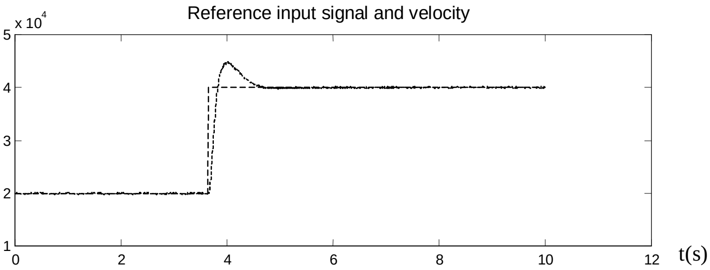

Fig.7. Experimental results

5. CONCLUSION

The first part of this article (section 2) introduces the generalized template based on complex non-integer integration and recalls the method for construction of isodamping contours by the envelope technique. This technique uses segments obtained using complex non-integer integration.
Section 3 gives the formalism used in the design of the control loop, in particular the equations of the tangents to the performance contours and to the open-loop Nichols locus.
Section 4 defines the design method using tangency relations between the performance contours and the open-loop Nichols locus. The first step of this method is conditioned by a constraint on the open-loop behavior at low and high frequencies. The second step of the method relaxes this constraint which can be prejudicial to performance at low and high frequencies. The final example shows the validity of the method.

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