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GUARANTEED COST AND POSITIVE REAL UNCERTAINTY

J.BERNUSSOU#, G.GARCIA# and J.C.GEROMEL&

#L.A.A.S - C.N.R.S,7, Avenue du Colonel Roche, 31077 Toulouse cedex 4, France.
E-mail: tarbour@laas.fr - Fax: +33 (0) 561 33 69 69

& : LAC-DT / School of Electrical Engineering, UNICAMP,
CP 6101, 13081 - 970, Campinas, SP, Brazil,
e-mail: bernusou@laas.fr or garcia@laas.fr

Abstract. This paper presents a kind of overview of the quadratic guaranteed cost problem in face of positive real uncertainty. Continuous as well as discrete time systems are discussed to point out the main discrepancies and difficulties arising when considering the different stability conditions associated with the positive real and strongly positive real conditions. The state feedback control problem is only addressed.

1 Introduction

There is a fairly extensive literature dealing with positive real systems [1], [2], [5] and, more recently, their application in the area of robust control. Practical interest to deal with positive real uncertainty stands on the fact that it enables to directly take into account phase information [12], [11], [8]. For practical and theoretical motivations on this topic, see [9], [6] and the references therein.

In the robust control area, working with positive real uncertainty, most of the results are related to the so called extended (or strongly) positive real systems [12], [3], [13]. All the cases are revisited here restricting ourselves to the state feedback control problem which is achieved with the following performances

- it robustly stabilizes the system with unbounded positive nonlinear uncertainty and consequently gives to the linear part the strict positive real property,
- It solves a guaranteed cost problem corresponding to minimize an upper bound of the worst case performance with respect to system uncertainty.

The paper contains mainly two parts. Part one is devoted to continuous time dynamical systems and three different cases of positive realness are discussed and solved. Part two corresponds to the discrete time case which exhibits a high complexity for its solution. A brief conclusion is given to point out future directions.

2 Continuous time systems

Let the uncertain dynamical system (S) be described by

$$\begin{array}{rcl} \dot{x} & = & Ax + B_1 w + B_2 u \\ y_1 & = & C_1 x + D_{11} w + D_{12} u \\ y_0 & = & C_0 x + D_0 u \end{array} \qquad (1)$$

where $x \in R^n$, $u \in R^m$, $y_0 \in R^p$ are respectively the state, control and controlled output vectors. The uncertainty is modelled as a negative nonlinear feedback of the $y_1 \in R^m$ output vector

$$w = -f(y_1) \qquad (2)$$

where $f(.)$ satisfies the positivity condition

$$f(y) \in \mathcal{F} = \{f(.) \ : \ f(0) = 0, \ y' f(y) \geq 0\} \qquad (3)$$

The general robust guaranteed cost problem is stated as :

$$\min_u \max_{f \in \mathcal{F}} \int_0^\infty y_0' y_0 \ dt \qquad (4)$$

The restrictions adopted in this paper are i - $u = Kx$, the state feedback control problem is considered, the aim being that of pointing out the main difficulties encountered in the different cases : positive, strictly positive, strongly positive cases

ii - Instead of (4), a kind of sufficient version of the guaranteed cost control problem is addressed where an upper bound of the integral cost (4) is going to be minimized. This is a classical procedure which has been settled in the pioneering paper [10] and then followed in other ones [4] and from which comes the word "guaranteed"

A necessary and sufficient condition of the uncertain system (1) is given in the positive real lemma (also termed as Kalman-Yakubovitch Bellman lemma). Furthermore these conditions are the ones for the strict positive real property of the square transfer matrix (in open loop; $u = 0$)

$$T(s) = C_1(sI - A)^{-1}B_1 + D_{11} \tag{5}$$

Lemma 1 *$T(s)$ is strictly positive real (and asymptotically stable) if and only if there exist $P = P' > 0$, L and W matrices such that*

$$\begin{cases} A'P + PA + L'L < 0 \\ B_1'P - C_1 + W'L = 0 \\ D_{11} + D_{11}' - W'W = 0 \end{cases} \tag{6}$$

From these conditions, a guaranteed cost state feedback control can readily be stated as

$$\begin{array}{l} \min\limits_{P,K,\epsilon>0} \ \epsilon^{-1}x_0'Px_0 \\ \text{subjected to} \\ \begin{cases} (A + B_2K)'P + P(A + B_2K) + L'L + \epsilon Q_0 < 0 \\ B_1'P - C_1 - D_{12}K + W'L = 0 \\ D_{11} + D_{11}' - W'W = 0 \end{cases} \\ \text{where } Q_0 = (C_0 + D_0K)'(C_0 + D_0K) \end{array} \tag{7}$$

Proof : The condition given in (7) is still a necessary and sufficient condition for the existence of a feedback matrix K which renders the closed-loop matrix $T(s)$ strictly positive real. Indeed, whenever (6)) is fullfilled, it does exist a non empty interval $[0, \bar{\epsilon} > 0]$ for ϵ such that (7) holds. Using the candidate Lyapunov function $v(x) = x'Px$, one gets :

$$\begin{array}{rcl} \dot{v}(x) & = & 2x'[(A + B_2K)'P]x + 2x'PB_1w \\ & < & -\epsilon x'Q_0x - x'L'Lx - \\ & & 2x'(C_1 + D_{12}K - W'L)'f(y_1) \\ & = & -\epsilon x'Q_0x - x'L'Lx - 2y_1'f(y_1) \\ & & +2x'L'Wf(y_1) - f(y_1)W'Wf(y_1) \\ & = & -\epsilon x'Q_0x \\ & & -(Lx - Wf(y_1))'(Lx - Wf(y_1)) \\ & \leq & -\epsilon x'Q_0x \end{array}$$

so that

$$\epsilon \int_0^\infty x'Q_0x \ dt \leq x_0'Px_0$$

Assuming x_0 a random vector with zero mean and covariance $E(x_0x_0'] = B_0B_0'$ (E expectation operator) and averaging the cost, one has

$$E[\int_0^\infty x'Q_0xdt] \leq \epsilon^{-1} \ Trace[B_0'PB_0]$$

and finally the guaranteed cost control problem to be solved

$$\begin{array}{l} \min\limits_{P,K} \ \epsilon^{-1}Trace[B_0'PB_0] \\ \text{subjected to} \\ \begin{cases} (A + B_2K)'P + P(A + B_2K) + L'L + \epsilon Q_0 < 0 \\ B_1'P - C_1 - D_{12}K + W'L = 0 \\ D_{11} + D_{11}' - W'W = 0 \end{cases} \\ \text{where } Q_0 = (C_0 + D_0K)'(C_0 + D_0K) \end{array} \tag{8}$$

2.1 Case $D_{11} = 0$

Let us first consider the case when $T(s)$ is strictly proper. Obviously $W = 0$ and although non absolutely necessary, the choice $L = 0$ is made since it corresponds to the less stringent inequality condition in (8). Let the matrix U be such that Although non absolutely necessary, the obvious choice $W = 0$ is made and subsequently $L = 0$ which then correspond to the less stringent inequality condition in (8).

$$U \geq \epsilon^{-1}B_0'PB_0$$

and

$$Y = \epsilon P^{-1}$$

The use of Shur complement formula enables to write the guaranteed cost control problem (8) in a LMI-LME (linear matrix inequality- linear matrix equality) formulation

$$\begin{array}{l} \min\limits_{(Y,R)} \ Trace[U] \\ \text{subjected to} \\ \begin{pmatrix} Y & B_0 \\ B_0' & U \end{pmatrix} \geq 0 \\ \begin{pmatrix} AY + YA' + B_2R + R'B_2' & (C_0Y + D_0R)' \\ (C_0Y + D_0R) & -I \end{pmatrix} < 0 \\ \epsilon B_1' = C_1Y + D_{12}R \end{array} \tag{9}$$

The state feedback is given by $K = RY^{-1}$. Problem (9) can be solved using almost standard LMI and LME software.

2.2 Case $D_{11} + D_{11}' > 0$

We now move to the other extreme case termed strongly or extended positive real in the literature [13]. By elimination of the W and L unknown matrices in (8) an equivalent formulation of these constraints can easily be given in terms of a Riccati matrix inequality :

$$\begin{array}{l} (A+B_2K)'P + P(A+B_2K) + \\ (B_1'P - C_1 - D_{12}K)'(D_{11}+D_{11}')^{-1}(B_1'P - C_1 - D_{12}K) + \\ \epsilon Q_0 < 0 \end{array} \tag{10}$$

or similarly (Schur Cohn complement)

$$\begin{pmatrix} (A+B_2K)'P + P(A+B_2K) + \epsilon Q_0 & (B_1'P - C_1 - D_{12}K)' \\ (B_1'P - C_1 - D_{12}K) & -(D_{11}+D_{11}') \end{pmatrix} \tag{11}$$

Finally, with $Y = P^{-1}$ and $\lambda = \epsilon^{-1}$, the guaranteed cost control problem is written

$$\begin{array}{l} \min\limits_{(Y,R,\lambda)} \quad Trace[\lambda U] \\ \text{subjected to} \\ \begin{pmatrix} Y & B_0 \\ B_0' & U \end{pmatrix} \geq 0 \\ \begin{pmatrix} \begin{array}{c} AY + YA' \\ +B_2R + R'B_2' \end{array} & (C_0Y + D_0R)' & \begin{array}{c} (B_1' - C_1Y \\ -D_{12}R)' \end{array} \\ (C_0Y + D_0R) & -\lambda I & 0 \\ (B_1' - C_1Y - D_{12}R) & 0 & -(D_{11}+D_{11}') \end{pmatrix} < 0 \end{array} \tag{12}$$

the constraints are linear with respect to the unknowns, but due to the cost (bilinear) the "LMI machinery" cannot be directly applied to (12). There are however several possible ways to overcome this difficulty. The first one, which would always provide a solution (if any), is to perform a one line search along the scalar λ, performing iteratively LMI optimizations with respect to Y and R. Another way is by an iterative linearization procedure leading to solve, at iteration n the following optimization problem

$$\min_{Y,R,\lambda} \; trace \; [\lambda_{n-1}U + \lambda U_{n-1}]$$

where λ_{n-1}, U_{n-1} are the solutions of the $(n-1)$ previous iterations. This kind of algorithm has proven to be efficient to solve some hard problems such as the static output control one and the decentralized control one [7].

2.3 Case $Case D_{11} \neq 0$ and $D_{11} + D_{11}' \geq 0$

Trying to proceed as far as possible towards an LMI-LME formulation, with $Y = \epsilon P^{-1}$, $S = LY$, $\lambda = \epsilon^{-1}$, the guaranteed cost control problem can be written as

$$\begin{array}{l} \min\limits_{(Y,R,S,W,\lambda)} \quad Trace[U] \\ \text{subjected to} \\ \begin{pmatrix} Y & B_0 \\ B_0' & U \end{pmatrix} \geq 0 \\ \begin{pmatrix} AY + YA' + B_2R + R'B_2' & (C_0Y + D_0R)' & S' \\ (C_0Y + D_0R) & -I & 0 \\ S & 0 & -\lambda I \end{pmatrix} < 0 \\ \lambda B_1' - C_1Y - D_{12}R + W'S = 0 \\ D_{11} + D_{11}' - W'W = 0 \end{array} \tag{13}$$

In this formulation, the difficulty for solution comes from the two last equations with the quadratic and bilinear terms $W'W$ and $W'S$. With the choice $S = 0$,the problem (13) reduces to the one presented in section 2.1 ($D_{11} = 0$) and in fact, the strictly positive real condition with $D_{11} = 0$ is a sufficient condition for the case $D_{11} + D_{11}' \geq 0$.

A more intuitively appealing way consists in fixing first a W matrix satisfying the last equality constraint in (13) and then solving a true LMI optimization problem in the unknowns Y, R, S, λ where the S matrix is a degree of freedom to be used to fulfill the strictly positive real condition. It is conjectured that this is a reasonable and efficient way to solve this general problem which has not yet received a complete solution as stated in [5].

3 Discrete time systems

Let us now consider the discrete time uncertain dynamical system described by :

$$\begin{array}{rcl} x_{k+1} & = & Ax_k + B_1w_k + B_2u_k \\ y_{1k} & = & C_1x_k + D_{11}w_k + D_{12}u_k \\ y_{0k} & = & C_0x_k + D_0u_k \end{array} \tag{14}$$

where $x_k \in R^n$, $u_k \in R^m$, $y_{0k} \in R^p$ are respectively the state, control and controlled output vectors at time k. The uncertainty is modelled as a negative nonlinear feedback of the $y_{1k} \in R^m$ output vector

$$w_k = -f(y_{1k}), \text{ with, } f \in \mathcal{F} \tag{15}$$

As in continuous time case, the discrete positive real lemma provides a necessary and sufficient condition for (14,15) to be asymptotically stable (in open loop, $u_k = 0$), namely there exist $P = P' > 0$ and L, W matrices such that

$$\left\{ \begin{array}{l} A'PA - P + L'L < 0 \\ B_1'PA - C_1 + W'L = 0 \\ D_{11} + D_{11}' - B_1'PB_1 - W'W = 0 \end{array} \right. \tag{16}$$

This condition is also a necessary and sufficient condition for the square transfer matrix

$$T(z) = C_1(zI - A)^{-1}B_1 + D_{11} \tag{17}$$

to be strictly positive real , i.e.

$$Real[T(z] > 0, \; \forall z \; : \; ||z|| = 1$$

From the last equality constraint (16), it is clear that

$$D_{11} \neq 0 \tag{18}$$

As previously, considering the cost

$$J = \sum_{k=0}^{\infty} y_{0k}' y_{0k}$$

a general guaranteed cost control problem with state feedback can be stated as

$$\begin{array}{l}
\min\limits_{P,K} \ \epsilon^{-1} Trace[B_0' P B_0] \\
\text{subjected to} \\
\left\{ \begin{array}{l}
(A+B_2K)'P(A+B_2K) - P + L'L + \epsilon Q_0 < 0 \\
B_1'P(A+B_2K) - C_1 - D_{12}K + W'L = 0 \\
D_{11} + D_{11}' - B_1'PB_1 - W'W = 0
\end{array} \right. \\
\text{where } Q_0 = (C_0 + D_0K)'(C_0 + D_0K)
\end{array} \tag{19}$$

Proof : The proof is done using the candidate Lyapunov function $v(x_k) = x_k' P x_k$. We have

$$\begin{aligned}
\Delta_k v &= x_{k+1}' P x_{k+1} - x_k' P x_k \\
&= x_k'(A+B_2K)'P(A+B_2K)x_k - x_k' P x_k + \\
&\quad 2w_k' B_1' P(A+B_2K)x_k + w_k' B_1' P B_1 w_k \\
&< -\epsilon x_k' Q_0 x_k - x_k' L' L x_k - \\
&\quad 2f(y_{1k})'(z_{1k} - D_{11}w_k - W'L) + \\
&\quad f(y_{1k})' B_1' P B_1 f(y_{1k}) \\
&= -\epsilon x_k' Q_0 x_k - x_k' L' L x_k - \\
&\quad 2f(y_{1k})' y_{1k} + 2f(y_{1k})' W' L x_k - \\
&\quad f(y_{1k})' W' W f(y_{1k}) \\
&= -\epsilon x_k' Q_0 x_k - 2f(y_{1k})' y_{1k} - \\
&\quad (Lx_k - Wf(y_{1k}))'(Lx_k - Wf(y_{1k})) \\
&\le -\epsilon x_k' Q_0 x_k
\end{aligned}$$

Then

$$\sum_{k=0}^{\infty} \Delta_k v(x_k) = -x_0' P x_0 \le -\epsilon \sum_{k=0}^{\infty} x_k' Q_0 x_k$$

Assuming x_0 a random vector with zero mean and covariance $E[x_0 x_0'] = B_0 B_0'$ and averaging the cost, one has

$$E[\sum_{k=0}^{\infty} x_k' Q_0 x_k] \le \epsilon^{-1} \ Trace[B_0' P B_0]$$

3.1 Case $D_{11} + D_{11}' - B_1' P B_1 > 0$

In this case (strongly or extended positive real) proceeding as in the continuous time case, one can write, condition (19) in a discrete Riccati inequality form with $A_c = A + B_2K$, $C_{1c} = C_1 + D_{12}K$, $C_{0c} = C_0 + D_0K$

$$\begin{array}{l}
A_c' P A_c - P + \\
(B_1' P A_c - C_{1c})'(D_{11} + D_{11}' - B_1' P B_1)^{-1}(B_1' P A_c - C_{1c}) \\
+\epsilon C_{0c}' C_{0c} < 0
\end{array} \tag{20}$$

or similarly

$$\begin{pmatrix} A_c' P A_c - P + \epsilon C_{0c}' C_{0c} & (B_1' P A_c - C_{1c})' \\ (B_1' P A_c - C_{1c}) & -(D_{11} + D_{11}' - B_1' P B_1) \end{pmatrix} < 0 \tag{21}$$

After some tedious but straightforward developments using Schur Cohn complement formula, it is possible to get the guaranteed cost control problem as

$$\begin{array}{l}
\min\limits_{(Y,R,\lambda)} \ Trace[\lambda U] \\
\text{subjected to} \\
\begin{pmatrix} Y & B_0 \\ B_0' & U \end{pmatrix} \ge 0 \\
\begin{pmatrix}
-Y & (AY + B_2R)' & -(C_1Y + D_{12}R)' & (C_0Y + D_0R)' \\
(AY + B_2R) & -Y & B_1 & 0 \\
-(C_1Y + D_{12}R) & B_1' & -(D_{11} + D_{11}') & 0 \\
(C_0Y + D_0R) & 0 & 0 & -\lambda I
\end{pmatrix} < 0
\end{array} \tag{22}$$

where $Y = P^{-1}$ and $\lambda = \epsilon^{-1}$. The problem presents the same degree of difficulty as its counterpart in the continuous time case. The inequalities are linear in the unknowns, the cost is bilinear but with a scalar factor which enables to undertake efficient numerical algorithms for solution.

3.2 Case $D_{11} + D_{11}' - B_1' P B_1 \ge 0$

In the discrete time case, the solution in this case is much more involved than the one corresponding to the continuous time one where a practical procedure had been exhibited. Here there are two major difficulties. The first one comes from the $P(A+B_2K)$ term which cannot be brought into a linear form. The second one is put by the dependence between the P and W matrices due to the last equality.

4 Conclusion

In this paper, the quadratic guaranteed cost control problem has been quite fully investigated for the various cases of positive real conditions in the continuous as well as the discrete time problems. The state feedback control has been given either exact or workable efficient solutions in almost all the situations except in the non strongly positive real discrete case which deserves more investigations.

The dynamic ouput control which has been already addressed in some papers needs to be more fully developed and it is another prospective for such a work.

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