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DISCRETE-TIME LINEAR-QUADRATIC OUTPUT VERSUS STATE REGULATOR

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Abstract. An augmented and minimal realization state space models are proposed for direct implementation of the discrete-time linear-quadratic regulator (DLQR) with measured not all the state variables but only the output of the plant. Both the models are related by means of original transformation with a rectangular matrix. Using this transformation it is shown that the resulting closet-loop (CL) system with dynamic output feedback regulator (DOFR) has the same stable roots of its characteristic equation as the CL system with state feedback and DLQR; the additional zero roots of the first CL system generated by DOFR do not change its properties, essentially. It is also shown that the CL system with DOFR realizes the optimal control with feedback from an augmented state, resulting from solving an appropriate DLQR problem.

Key Words. Linear-quadratic regulator; discrete-time systems; state space models; observers.

1. INTRODUCTION

The linear quadratic regulator (LQR) both in continuous- and discrete-time versions in now a classical problem being the subject of many papers and books. Remind here only two early papers of Kalman [7] and Letov [8] and several contemporary books [1], [3], [4]. The latter book, though it has an introductory character, but it contains a compact recapitulation of the state of art concerning this problem.

The solution of discrete-time linear quadratic regulator (DLQR) problem takes the form of state feedback control law. To implement such a control all the state variables must be measured, which usually is not possible. When only the output of the plant is measured, the state control law may be utilized if an appropriate state observer is included to the system [9].

The theory of observers is well elaborated. One possibility is a deterministic approach in which the observer is based on the model of the plant and some appropriate feedback from measurements. It is known that such an observer has the property that it does not change the roots of the characteristic equation of the closed-loop (CL) system with state feedback. The additional roots introduced by the observer may be freely prescribed [9]. However, it may be shown that such an observer is non optimal since its non zero initial conditions usually increase the optimal value of the performance index used for derivation of the state feedback law.

In the present paper the model of the regulator-observer results directly from solving an appropriate DLQR problem in which an essential role plays the original state determination. The obtained solution determines the dynamic output feedback regulator (DOFR) usable when only the plant output is measured. It is shown that the dynamics of the regulator-observer completes the roots of the characteristic equation of the closed-loop (CL) system only by additional zero roots. The remaining roots are the same as those of the CL system with state feedback control law resulting from DLQR problem. It is also shown that the derived from the solution of DLQR problem the dead-beat [2] observer is optimal, because it does not increase the value of the performance index of the CL system with state feedback. In the comparison of the properties of both the systems an important role plays the original state transformation based on the subspaces of the system modes, introduced in the paper.

The contribution of the paper is partially in the state proposals which together with DLQR technique makes it possible to derive the DOFR containing the optimal dead-beat observer and partially in showing, by means of original state transormation, that the CL system with DOFR has the same stable roots of its characteristic equation as the CL system with state feedback LQ regulateor.

2. AN AUGMENTED STATE SPACE MODEL

Consider the discrete-time plant described by the transfer function (TF)

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_{n-l} z^l + b_{n-l+1} z^{l-1} + ... + b_n}{z^n + a_1 z^{n-1} + ... + a_n} \qquad (1)$$

where $l < n$, $Y(z) = \mathcal{Z}[y(t)]$, $U(z) = \mathcal{Z}[u(t)]$, $\mathcal{Z}$ is the symbol of the Z-transform; $y(t)$ and $u(t)$ are the output and input signals and $t = 0, 1, 2...$ is the discrete time. Assume that the numerator and denominator of (1) are relatively prime polynomials. Determine the state variables in the form

$$\begin{aligned} &\hat{x}_1(t) = y(t + n - m - 1), \\ &\hat{x}_2(t) = y(t + n - m - 2), ..., \\ &\hat{x}_{n-m}(t) = y(t), \\ &\hat{x}_{n-m+1}(t) = y(t-1), \\ &\hat{x}_{n-m+2}(t) = y(t-2), ..., \\ &\hat{x}_n(t) = y(t-m), \\ &\hat{x}_{n+1}(t) = u(t-1), \\ &\hat{x}_{n+2}(t) = u(t-2), ..., \\ &\hat{x}_{n+m}(t) = u(t-m) \end{aligned} \qquad (2)$$

where m is appropriately chosen so that $l \leq m \leq n-1$.

Replacing t in (2) by $t+1$, using notation (2) as well as resulting from (1) equation

$$\begin{aligned} y(t+n-m) + ... + a_{n-m} y(t) + a_{nm+1} y(t-1) + \\ ... + a_n y(t-m) = \\ = b_{n-l} u(t+l-m) + b_{n-l+1} u(t+l-m-1) + \\ ... + b_l u(t-m) \end{aligned} \qquad (3)$$

we obtain the state space model, n-dimensional, in the form

$$\hat{x}(t+1) = \hat{A}\hat{x}(t) + \hat{B}u(t), \qquad y(t) = \hat{C}\hat{x}(t) \qquad (4)$$

where $\hat{x}(t) = [\hat{x}_1(t), \hat{x}_2(t), ..., \hat{x}_{n+m}(t)]^T$ is $(n + m)$–dimensional augmented state. State equations (4) for particular components $\hat{x}_i$ take the form

$$\begin{aligned} &\hat{x}_1(t+1) = -a_1\hat{x}_1(t) - a_2\hat{x}_2(t) - ... - a_n\hat{x}_n(t) + \\ &+ b_{n-m+1}\hat{x}_{n+1}(t) + ... + b_n\hat{x}_{n+m}(t) + b_{n-m}u(t) \\ \\ &\hat{x}_i(t+1) = \hat{x}_{i-1}(t), \text{ for } 2 \leq i \leq n, \ i \neq n+1 \\ &\hat{x}_i(t+1) = u(t), \quad \text{for} \quad i = n+1 \end{aligned} \qquad (5)$$

The first equation of (5) results from accounting (2) in (3); the remaining equations of (5) result directly from determination (2). The matrices $\hat{A}$, $\hat{B}$ result from (5); the elements of the row vector $\hat{C}$ are determined by

$$\hat{C}_{n-m} = 1 \quad \text{and} \quad \hat{\mathrm{C}}_{\mathrm{i}} = 0 \quad \text{for} \quad \mathrm{i} \neq \mathrm{n} - \mathrm{m} \qquad (6)$$

Note that appearing in (5) $b_{n-j} = 0$ for $l < j \leq m$

Taking into account the form of the matrix $\hat{A}$ resulting from (5) one can show that

$$\det[z\hat{I} - \hat{A}] = z^m \det[z\check{I} - \bar{A}] \qquad (7)$$

where $\bar{A}$ is the $n \times n$ matrix resulting from $\hat{A}$ by cancelling the last m rows and last m columns, while $\hat{I}$ and $\check{I}$ are unit matrices $(n + m) \times (n + m)$ and $n \times n$ dimensional, respectively; $\bar{A}$ is the companion matrix related to the denominator polynomial of (1).

Corollary 1. The matrix $\hat{A}$ has n eigenvalues covering with the poles of the TF (1) and additionally one m-multiple zero eigenvalue.

Thus, the TF describing the model (4) has the same factor z^m in numerator and denominator. Therefore the state space model (4) is neither controllable nor observable, but for $m = n - 1$ it is reconstuctable. The reconstructability results directly from determination (2) of the state variables.

3. THE SUBSPACE OF m-MULTIPLE ZERO-MODE

In literature the notion of the mode has some different interpretations. Here by the notion of λ_i-mode for the discrete-time system we understand the function related with eigenvalue λ_i and appearing in the fundamental (or transition) matrix $\hat{A}^t$. For example the mode related with a single real eigenvalue λ_i is determined by λ_i^t. By the subspace of λ_i-mode we understand the appropriate subspace of initial states $\hat{x}_0$ which excites the λ_i-mode only. For example for the single real λ_i the subspace of λ_i-mode is the one dimensional subspace spanned on the eigenvector related with λ_i. The latter statement may be justified transforming the model (4) to the canonical form by means of linear transformation defined by the matrix T composed of eigenvectors of $\hat{A}$ [10].

Our model (4) has m-multiple zero-mode; further on, the subspace for this mode will be determined. It is known that in this case the matrix Λ of the canonical form has in the last m rows and columns the following $m \times m$ dimensional Jordan block

$$J = \begin{bmatrix} 0, & 1 & 0, & ..., & 0 \\ 0, & 0 & 1, & ..., & 0 \\ ... & ... & ... & ... & ... \\ 0, & 0, & 0, & ..., & 1 \\ 0, & 0, & 0, & ..., & 0 \end{bmatrix} \qquad (8)$$

Let S_m denote a m-dimensional subspace of m-multiple zero mode J^t. The vectors $v_1, v_2, ..., v_m$, creating the basis for this subspace may be obtained from the following equations [10]

$$\hat{A}v_1 = 0, \quad \hat{A}v_2 = v_1, \quad ..., \quad \hat{A}v_m = v_{m-1} \qquad (9)$$

which implies

$$\hat{A}v_1 = 0, \quad \hat{A}^2 v_2 = 0, \quad ..., \quad \hat{A}^m v_m = 0 \qquad (10)$$

The equations (9) may be derived using the dependence $\hat{A}T = T\Lambda$ resulting from determination of the eigenvectors, invariant subspaces and matrices T and Λ. Note, that the zero-mode appears only in the subspace S_m. This means that the zero-mode disappears in the n-dimensional subspace $S_n^{\perp}$ orthogonal to S_m.

4. TRANSFORMATION OF THE STATE

Introduce the state transformation in the form

$$\check{x} = H\hat{x} \tag{11}$$

where H is a rectangular $n \times (n+m)$ dimensional matrix with full rank equal to n, while $\check{x}$ is the new n-dimensional state.

Theorem 1. Assume that the rows of the matrix H create a basis of the subspace $S_n^{\perp}$. Then

1. There exist the solutions $\check{A}$ and $\check{C}$ of the equations

$$\check{A}H = H\hat{A}, \qquad \check{C}H = \hat{C} \tag{12}$$

2. The equations (4) may be transformed, using (11), (12), to the new minimum realization n-dimensional state space model in the form

$$\check{x}(t+1) = \check{A}\check{x}(t) + \check{B}u(t), \qquad y(t) = \check{C}\check{x}(t) \tag{13}$$

where

$$\check{B} = H\hat{B} \tag{14}$$

3. Both the models (4) and (13) are described by the same TF (1).

□

Proof. 1. It may be noted that the solution $\check{A}$ of the first equation (12) exists iff all the row-vectors of the matrix $\bar{H} = H\hat{A}$ are orthogonal to S_m (or – which is equivalent – belong to $S_n^{\perp}$). To show the latter statement consider any $(n+m)$ dimensional column vector $v \in S_m$. Since $v = c_1v_1 + c_2v_2 + ... + c_mv_m$, $c_i \in R, i = 1, 2, ..., m$, then accounting (9) we obtain $\hat{A}v \in S_m$. Therefore $H\hat{A}v = \bar{H}v = 0$, which means that all the rows of $\bar{H}$ are orthogonal to S_m. The solution $\check{C}$ of the second equation (9) exist iff $\hat{C} \in S_n^{\perp}$. To show this, note that the $(n-m+i)$ - row of the matrix $\hat{A}^i$, $i = 1, 2, ..., m$ covers with $\hat{C}$, which results from the form of $\hat{A}$. Accounting (10) we see that $\hat{C}$ is orthogonal to v_i, $i = 1, 2, ..., m$ and to S_m, therefore $\hat{C} \in S_n^{\perp}$.

2. Multiplying both sides of the state equation (4) by H from left hand side we have

$$H\hat{x}(t+1) = H\hat{A}\hat{x}(t) + H\hat{B}u(t), \qquad y(t) = \hat{C}\hat{x}(t) \tag{15}$$

Accounting (11), (12) and (14) in (15) we obtain (13). Since $\dim \check{x} = n$ is equal to the order of denominator of (1) the equations (13) describe the minimum realization state space model.

3. Applying the Z-transform to both sides of the equations (4) (under the assumption of zero initial condition) we obtain

$$(z\hat{I} - \hat{A})\hat{X}(z) = \hat{B}U(z), \qquad Y(z) = \hat{C}\hat{X}(z) \tag{16}$$

where $\hat{X}(z) = \mathcal{Z}[\hat{x}(t)]$, $U(z) = \mathcal{Z}[u(t)]$, $Y(z) = \mathcal{Z}[y(t)]$, $\mathcal{Z}$-denotes the Z-transform. Thus, the TF of the model (4) is

$$G(z) = \frac{Y(z)}{U(z)} = \hat{C}(z\hat{I} - \hat{A})^{-1}\hat{B} \tag{17}$$

Multiplying both sides of the first equation of (16) by H from the left hand side we obtain

$$H(z\hat{I} - \hat{A})\hat{X}(z) = H\hat{B}U(z) \tag{18}$$

Accounting that $Hz\hat{I} = z\check{I}H$ as well as (12), (14) and the dependence $\check{X}(z) = H\hat{X}(z)$ we have

$$(z\check{I} - \check{A})\check{X}(z) = \check{B}U(z), \qquad Y(z) = \check{C}\check{X}(z) \tag{19}$$

and

$$G(z) = \frac{Y(z)}{U(z)} = \check{C}(z\check{I} - \check{A})^{-1}\check{B} \tag{20}$$

□

Corollary 2. Both the models (4) and (15) are equivalent from the point o view of input-output (TF) description. Both have the same nonzero modes. The m-multiple zero-mode plying no role in TF description disappears in model (15).

Note that the different but equivalent minimum state space realizations may be obtained by using any transformation $\bar{x} = P\check{x}$ with non-singular $n \times n$ matrix P. Note also that the realization with the new state $\bar{x}$ may be directly obtained using the transformation (11) with H replaced by $\bar{H} = PH$.

5. LINEAR-QUADRATIC REGULATOR

Consider the quadratic performance index for the system (4) in the form

$$\hat{J} = \sum_{t=0}^{N} [\hat{x}^T(t+1)\hat{Q}\hat{x}(t+1) + ru^2(t)], \qquad N \to \infty \tag{21}$$

where $\hat{Q}$ is a $(n+m)\times(n+m)$, symmetric, semipositive weighting matrix of the state and r is a small positive number. The steady-state solution of discrete-time linear-quadratic regulator DLQR problem (4), (21) in the form of the state feedback law takes the form

$$u = -\hat{k}\hat{x} = -\hat{k}_1\hat{x}_1 - \hat{k}_2\hat{x}_2 - ... - \hat{k}_{n+m}\hat{x}_{n+m} \tag{22}$$

Here $\hat{k}$ is $(n+m)$-dimensional row-vector with constant components $\hat{k}_i$, $i = 1, 2, ..., n+m$ [3].

Substituting to (22) the state components (2) and bringing the terms containing input u to the left hand side we obtain the difference equation relating u to y or, after applying the Z-transform, the TF $R(z)$ describing the regulator-observer in the form

$$\hat{R}(z) = -\frac{U(z)}{Y(z)} = -\frac{\hat{k}_1 z^{n-1} + \hat{k}_2 z^{n-2} + ... + \hat{k}_n}{z^m + \hat{k}_{n+1} z^{m-1} + ... + \hat{k}_{n+m}} \tag{23}$$

Note that for $m = n-1$ the TF $\hat{R}(z)$ is proper with $(n-1)$-th order polynomials in numerator and denominator. The regulator-observer (23) has some dynamics but it may be implemented when only the output y is available.

Corollary 3. For $m = n-1$ the two implementable models describing the same closed-loop (CL) system of $(2n-1)$-th order have been obtained: the state space model described by (4) and (22) and the TF model described by (1) and (23).

Really, the augmented state space model (4) of the plant has TF (1) and for the state determined by (2) the TF (23) is equivalent to (22).

Let the quadratic performance index for the plant model (15) be

$$\check{J} = \sum_{t=0}^{N} [\check{x}^T(t+1)\check{Q}\check{x}(t+1) + ru^2(t)], \qquad N \to \infty \quad (24)$$

where $\check{Q}$ is a $n \times n$ symmetric, semipositive weighting matrix of the state and r is a small positive number. From solving the DLQR problem (13), (24) we obtain the following state feedback law valid in steady state

$$u = -\check{k}\check{x} = -\check{k}_1\check{x}_1 - \check{k}_2\check{x}_2 - \ldots - \check{k}_n\check{x}_n \quad (25)$$

where $\check{x} = [\check{x}_1, \check{x}_2, ..., \check{x}_n]$ and $\check{k}$ is n-dimensional vector with constant components $\check{k}_i$, $i = 1, 2, ..., n$. The CL system composed of the plant (13) and state feedback (25) is optimal in steady state and has the characteristic equation of n-th order. However state feedback (25) may be implemented when all the state components $\check{x}_i$, $i = 1, 2, ..., n$ are available for measure, which usually is not possible. Therefore, in the following the case when only the output y is available will be considered.

To derive from (25) the TF of the regulator-observer we substitute (11) and (2) to (25) and bring the terms containing u to the left hand side of the equation. After applying Z-transform and rearranging the TF of the regulator-observer takes the form

$$\check{R}(z) = -\frac{\check{p}_1 z^{n-1} + \check{p}_2 z^n + \ldots + \check{p}_n}{z^m + \check{p}_{n+1} z^{m-1} + \ldots + \check{p}_{n+m}} \quad (26)$$

where $\check{p}_i = \check{k}h_i$, h_i - are the columns of the matrix H, $i = 1, 2, ..., n+m$ (i.e. $H = [h_1, h_2, ..., h_{n+m}]$). The TF $\check{R}(z)$ (26) has similar form as $\hat{R}(z)$ (23); it is proper and implementable when $m = n-1$.

Note that the CL system composed of plant (1) and regulator (26) is of $(n+m)$-th order. Thus, in comparison to the CL system (13), (25) with state feedback, the output feedback causes the increase of the order of the CL system by m.

Note also that it is possible to calculate the state $\check{x}$ appearing in (25), using (11) and (2), but then the static state feedback law (25) after substituting (11), (2) becomes a dynamic one. Really the static state feedback (25) takes then the form of a dynamic output feedback described by the TF (26). The CL system composed of plant (1) and regulator (26) has the characteristic equation of $(n+m)$-th order. Therefore there arises the question: which is the relation between the CL system described by (13), (25) and (1), (26)? The first system is optimal in steady state; whether and in which sense the second system is optimal? The response to these questions will be given further on.

6. DISCUSSION OF THE RESULTS

Let $\hat{k} = [\hat{k}_1, \hat{k}_2, ..., \hat{k}_{n+m}]$ and $\check{k} = [\check{k}_1, \check{k}_2, ..., \check{k}_n]$ be the gains appearing in (22) and (25), respectively. Let $\hat{S}$ and $\check{S}$ be the solutions of the algebraic Riccati equations for DLQR problems (4), (21) and (13), (24), respectively. The $n \times n$ matrix $\check{S}$ fulfils the following algebraic Riccati equation

$$\check{S} = \check{Q} + \check{A}^T\check{S}\check{A} - \check{A}^T\check{S}\check{B}(r + \check{B}^T\check{S}\check{B})^{-1}\check{B}^T\check{S}\check{A}, \quad (27)$$

while the gain $\check{k}$ is determined by

$$\check{k} = -(r + \check{B}^T\check{S}\check{B})^{-1}\check{B}^T\check{S}\check{A} \quad (28)$$

The $(n+m)\times(n+m)$ matrix $\hat{S}$ and gain $\hat{k}$ fulfil similar equations resulting appropriately from (27) and (28) by replacing the sign "ˇ" with "ˆ".

The characteristic equation of the CL system (4), (22) takes the form

$$\det\left[z\hat{I} - \hat{A} + \hat{B}\hat{k}\right] = 0 \quad (29)$$

where on the left hand side of (29) appears the so called characteristic polynomial. The characteristic equation of the CL system (13), (25) results from (29) by replacing "ˆ" with "ˇ"; $\hat{I}$ and $\check{I}$ denote $(n+m) \times (n+m)$ and $n \times n$ unit matrix, respectively.

Let

$$\hat{x}(t+1) = \hat{A}\hat{x}(t) \quad (30)$$

be the system with $(n+m)$-dimensional vector $\hat{x}$. Consider the other system

$$\check{x}(t+1) = \check{A}\check{x}(t) \quad (31)$$

with n-dimensional vector $\check{x}$ which has the modes appearing also in the system (30). The modes of (30) which appear also in (31) are called the preserved modes, while the modes of (30) which do not appear in (31) are called the neglected modes.

Lemma 1. The transformation (11) converts the system (30) into the system (31) iff the n-dimensional subspace $S_n^{\perp}$ spanned on the row-vectors of the matrix H is orthogonal to m-dimensional subspace S_m of modes neglected in (31). The modes of (30) for which the subspaces projections on $S_n^{\perp}$ are nonzero are preserved in (31), while the modes, the subspaces of which belong to S_m are neglected in (31).

□

Proof. If H is orthogonal to S_m then in accordance with Theorem 1, there exists the matrix $\check{A}$ which fulfils the first equation of (12). Multiplying both sides of (30) by H from LHS we obtain $H\hat{x}(t+1) = H\hat{A}\hat{x}(t)$. Accounting (12) and (11) gives (31). On the other hand the n-dimensional system (31) can not have all the modes of the $(n+m)$-dimensional system (30). From (11) it results that all the row-vectors of H must be orthogonal to S_m. The last statement of the Lemma 1 is the consequence of the transformation (11) with the matrix H having row-vectors orthogonal to S_m.

□

Theorem 2. Assume that H is as in Theorem 1 and the matrices $\hat{Q}$ and $\check{Q}$ appearing in the indices (21) and (24) fulfil the relation

$$\hat{Q} = H^T \check{Q} H \tag{32}$$

Then:

1. The steady state solutions $\hat{S}$ and $\check{S}$ of the algebraic Riccati equations for the DLQR problems (4), (21) and (13), (24), respectively, fulfil the relation

$$\hat{S} = H^T \check{S} H \tag{33}$$

2. The corresponding gains $\hat{k}$ and $\check{k}$ are related with

$$\hat{k} = \check{k} H \tag{34}$$

3. The characteristic polynomials of the closed loop systems (4), (22) and (13), (25) fulfil the dependence

$$\det[z\hat{I} - \hat{A} + \hat{B}\hat{k}] = z^m \det[z\check{I} - \check{A} + \check{B}\check{k}] \tag{35}$$

□

Proof. Multiplying both sides of equation (27) by H^T from LHS and by H from RHS and using (12) and (14) we prove validity of (33). The proof of (34) results from multiplying both sides of (28) by H from RHS with accounting (12) and (14).

To prove (35) write the state equation of the CL system (4), (22)

$$\hat{x}(t+1) = (\hat{A} - \hat{B}\hat{k})\hat{x}(t) \tag{36}$$

Multiplying (36) by H from LHS and accounting (12), (14), (34) and (11) gives

$$H\hat{x}(t+1) = (H\hat{A} - H\hat{B}\hat{k})\hat{x}(t) = (\check{A}H - \check{B}\check{k}H)\hat{x}(t) = \\ = (\check{A} - \check{B}\check{k})\check{x}(t)$$

and finally

$$\check{x}(t+1) = (\check{A} - \check{B}\check{k})\check{x}(t) \tag{37}$$

Then the transformation (11) has transferred (36) to (37) therefore from Lemma 1 it results that the row-vectors of H are orthogonal to the subspace of neglected modes of (36). Since from (34) it results that $\hat{k}$ is orthogonal to S_m then the subspace of m-multiple zero mode is not changed by the CL control (22). Therefore S_m creates also the subspace of m-multiple zero mode for the CL system (36). Thus from Lemma 1 it results that the modes corresponding to nonzero eigenvalues of (36) are preserved in (37), which proves (35).

□

Corollary 4. Under assumption (32) the TF-s of regulator-observers (23) and (26) are the same. The CL system (1), (26) with dynamic output feedback (26) realizes the static state feedback law (22) resulting from steady state solution of optimal DLQR problem (4), (21). Using other words, the modes of both the CL system (4), (22) and (1), (26) are the same.

Note, that the assumption (32) means that the states $\hat{x}$ belonging to S_m do not cost, while the relation (33) means that the optimal values of the performance indices (21) and (24) are the same (for the initial states fulfilling (11)). The latter property together with (35) and Corollary 4 denotes that the regulator (26) contains the optimal dead-beat observer [2].

Since the equations (13) determine a minimum realization state space model then the system (13) is controllable and observable. From this [3] and from (35) it results an important property formulated in the following.

Corollary 5. The nonzero stable roots of the characteristic equations both, of the CL system (13), (25) of n-th order with static state feedback (25) and of the CL system (1), (26) of $(n+m)$-th order with dynamic output feedback (26) are the same. The characteristic equation of the latter system has additionally m-multiple zero root caused by the optimal dead-beat observer appearing in the regulator (26) and realizing the dynamic output feedback.

It is known that the CL system with state feedback law resulting from solving LQR problem is robust [4]. From Corollary 5 it results that the CL system with LQ regulator and dynamic output feedback has similar properties, since additional m-multiple zero root does not change the properties of the CL system, essentially.

7. EXAMPLE

Consider the discrete-time model $G(z)$ composed of sampler with sampling period $h = 0.1$, zero-order hold and continuous-time plant $K(s)$ with

$$K(s) = \frac{1}{s^2 + 2s + 3}, \qquad G(z) = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2}, \tag{38}$$

where $b_1 = 0.0047$, $b_2 = 0.0044$, $a_1 = -1.7916$, $a_2 = 0.8187$. Determine the augmented state with $m = 1$ in the form $x_1(t) = y(t)$, $x_2(t) = y(t-1)$, $x_3 = u(t-1)$. Then the matrices $\hat{A}$ and $\hat{B}$ of the model (4) are

$$\hat{A} = \begin{bmatrix} -a_1, & -a_2 & b_2 \\ 1, & 0 & 0 \\ 0, & 0, & 0 \end{bmatrix}, \qquad \hat{B} = \begin{bmatrix} b_1 \\ 0 \\ 1 \end{bmatrix} \tag{39}$$

The one dimensional subspace of zero mode of $\hat{A}$ is spanned on the vector v fulfilling $\hat{A}v = 0$, thus $v = [0, 1, a_2/b_2]^T$ and H takes the form

$$H = \begin{bmatrix} 1, & 0, & 0 \\ 0, & -a_2 & b_2 \end{bmatrix} \tag{40}$$

The matrices $\check{A}$ and $\check{B}$ calculated from (12) and (14) are

$$\check{A} = \begin{bmatrix} -a_1, & 1 \\ -a_2, & 0 \end{bmatrix}, \qquad \check{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{41}$$

Assuming $f = [1, 0]$, $\check{Q} = f^T f$ and $r = 0.001$ we obtain from *dlqr* MATLAB function

$$\check{k} = [65.4283, -45.3770] \tag{42}$$

$$\lambda_{12} = 0.6219 \pm j\, 0.2694 \tag{43}$$

Thus $\hat{k} = \check{k}H = [65.4283, -45.3770, 0.2422]$ and the regulator-observer is described by

$$\hat{R}(z) = \check{R}(z) = -\frac{65.4283z - 45.3770}{z + 0.2422} \tag{44}$$

It is easy to check that the characteristic equation of the CL system $1 + \bar{G}(z)\check{R}(z) = 0$ has two roots determined by (43) and one zero root generated by the regulator- observer.

8. CONCLUSIONS

The proposed approach makes it possible to derive the regulator-observer transfer function implementable in feedback control when not all the state variables but only the plant output is measured.

Usual solution of the DLQR problem with minimum realization of the plant state space model determines the static state feedback law; the corresponding CL system is of n-th order equal to that of the plant. When only the plant output is available the derived regulator-observer realizes the dynamic output feedback and the corresponding CL system has the order increased by m.

It is important that the CL system with the dynamic output feedback, determined by the derived regulator with the optimal dead-beat observer, remains the same good properties as the CL system with static state feedback. Really, from Theorem 2 it results that the first system has the same nonzero, stable roots of its characteristic equation as the second one. Though the first system has the order higher by m, but related with this the m-multiple zero root, generated by the observer, does not change its good properties essentially.

On the other hand the CL system with derived regulator-observer realizes the static state feedback law resulting from solving the DLQR problem with augmented state $\hat{x}$. It is known that the CL system with the state feedback control resulting from solving a DLQR problem is robust. Thus, the CL system with the proposed regulator-observer has the same property.

The results have been obtained owing to the introduced transformation $\hat{x} = H\check{x}$ of the state $\hat{x}$ of higher dimension to the state $\check{x}$ of lower dimension. In these considerations the important meaning plays the notion of the mode subspace. The transformation $\check{x} = H\hat{x}$ has been created in this manner that m-multiple zero mode appearing in the augmented state model (4) is preserved in the CL system. This transformation makes it possible to find the minimal state space realization of the plant. Using the transformation it is proved that the obtained dead-beat observer is optimal, since it does not increase the value of the performance index evaluated in the system with state feedback. The transformation determines the equivalent DLQR problems in the state spaces with different dimensions.

Similar approach may be applied for continuous-time system. The difference is in the state variables proposal, in which in the place of output and control variables evaluated at times back shifted, some appropriate integrals of output and control should appear.

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