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DIRECT SOLUTION OF CONTINUOUS TIME SINGULAR SYSTEMS BASED ON THE FUNDAMENTAL MATRIX

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Abstract. The fundamental and the transition matrices of the continuous-time generalized state-space system are defined. Then the solution of the generalized state-space model is given in terms of the fundamental matrix directly for the given system, without applying any decomposition in fast and slow subsystems. The proposed solution is actually a generalization of the solution of the regular state-space equation and provides insight for the particular properties of the generalized systems. The set of admissible initial conditions is directly determined from the solution and the decomposition of the solution into two orthogonal subspaces easily results by applying orthogonal operators. The fundamental and the transition matrices may be calculated in terms of the system's matrices via algebraic recursive algorithms.

Key Words. Singular systems, generalized systems, fundamental matrix, transition matrix.

1. INTRODUCTION

Consider the singular or generalized continuous time, linear time-invariant (LTI) dynamical-algebraic system [1],[2]

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),\ t \geq t_0^-,\ \ \mathbf{x}(t_0^-) = \mathbf{x}_0 \quad (1.1a)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \quad (1.1b)$$

where $\mathbf{E}, \mathbf{A} \in \Re^{nxn}, \mathbf{B} \in \Re^{mxn}, \mathbf{C} \in \Re^{nxm}$ and rank$\mathbf{E} \leq n$. Moreover the term $\mathbf{D}\mathbf{x}$(t) is absent in (1.1b), without loss of generality, since (1.1) can incorporate a direct-feed term, due to the singularity of **E**. The given system is assumed regular, or equivalently the pencil $(s\mathbf{E}-\mathbf{A})$ solvable, i.e. $\det(s\mathbf{E}-\mathbf{A}) \neq 0$. Note also that in (1.1) $\dot{\mathbf{x}}(t)$ is the distributional derivative, since the initial condition is t_0^-, in contrary to the regular systems, where the ordinary derivative (right-hand derivative at t_0) is used. For any piece-wise continuous distribution **x**(t), $\dot{\mathbf{x}}(t)$ is related to the regular derivative $\mathbf{x}'(t)$ by the relation [3]

$$\dot{\mathbf{x}}(t) = \mathbf{x}'(t) + \delta(t-t_0)\mathbf{x}(t_0^-),\ \ t \in [t_0, \infty) \quad (1.2a)$$

$$\dot{\mathbf{x}}(t) = \delta(t-t_0)\mathbf{x}(t_0^+) - \delta(t-t_0)\mathbf{x}(t_0^-), t \in [t_0^-, t_0^+] \quad (1.2b)$$

Singular systems find applications in engineering systems (electrical circuits, interconnected systems, robotics, systems with derivative feedback), economic and biological systems, time series analysis and singularly perturbed systems. Singular systems (also called generalized or semi-state or descriptor variable systems) are governed by singular differential equations that consist of algebraic, as well as first order differential equations, which may result to impulsive behavior of the system. The impulsive behavior causes a number of special features that do not appear in classical systems, such as impulsive terms and input derivatives in the state response, nonproperness of transfer matrix, nondifferentiable functions, noncausality between input and state or output and consistency of initial conditions. The singular systems may be considered as a generalization of the classical systems, since they are reduced to them in the case where **E** is a nonsingular square matrix. Although in the state space equations $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ the analysis may be accomplished in terms of **A**, it was not possible to carry out the analysis in the singular case $\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ in terms of **A** and **E** using existing techniques (Drazin inverse [2], Weierstrass form [4], [5], deflating subspaces [5],[6], shuffle algorithm [7]). However the derivation of the analytic expressions of the fundamental and transitional matrix sequences of singular systems [8],[9] permits the complete analysis of solutions and properties of $\mathbf{E}\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)$ and finally of (1.1), in terms of **A** and **E**. Thus the use of the fundamental matrix (FM) leads to the development of an algebraic methodology for the analysis and design of generalized state-space systems, which is called the *Fundamental Matrix Approach (FMA)*. The FMA constitutes an efficient methodology for the generalization of the whole theory of linear systems, using computationally attractive algebraic techniques.

The FM sequence of singular systems is calculated using recursive algebraic relations in terms of the coefficients of the adjoint and the characteristic polynomial of the pencil $(z\mathbf{E}-\mathbf{A})$ and finally in terms of **A** and **E** [8]. Based on FMA, complete solutions of forward, backward and symmetric cases of discrete singular systems have been derived [9]. Moreover, various analysis problems (controllability and observability criteria [10], stability [11], minimal realization using generalized Hankel matrix [12]), as well as synthesis problms (P-D feedback for decoupling and pole assignment of singular systems [13],[14]) have been studied, using the FMA. The extension to the 2-D generalized state-space systems is currently under consideration.

In this paper the general expressions of the fundamental and transition matrices of the continuous time singular systems and the complete analytic zero state and zero input solutions are derived.

2. FUNDAMENTAL MATRIX

For the derivation of the fundamental matrix of the generalized systems, we need to apply generalized or distributional derivatives of functions with "jump" discontinuity" at t_0 that contain Dirac operators. Consider the generalized continuous time, linear time-invariant (LTI) dynamical-algebraic system (1.1). Suppose that $\mathbf{x}(t),\ \dot{\mathbf{x}}(t),\ t\in[t_0^-,\infty)$ are continuous and of exponential growth and $\mathbf{x}(t)$ has a discontinuity at t_0 [2, vol. 2]. Therefore, $\dot{\mathbf{x}}(t), \in[t_0^-,\infty)$ is treated as the distributional derivative $\mathrm{D}\mathbf{x}(t)$. Then the Laplace transform L_- of $\dot{\mathbf{x}}(t),\ t\in[t_0^-,\infty)$, using (1.2a) and (1.2b) in the corresponding areas, gives

$$L_-[\dot{\mathbf{x}}(t)]=\mathbf{X}_-(s)=L_-\left[\mathbf{x}'(t)+\delta(t-t_0)\mathbf{x}(t_0^-)\right]=$$

$$L_+[\mathbf{x}'(t)]+L_-\left[\delta(t-t_0)\mathbf{x}(t_0^-)\right]=\int_{t_0^-}^{\infty}\dot{\mathbf{x}}(t)e^{-s(t-t_0^-)}dt=$$

$$=\int_{t_0^-}^{t_0^+}\dot{\mathbf{x}}(t)e^{-s(t-t_0^-)}dt+\int_{t_0^+}^{\infty}\mathbf{x}'(t)e^{-s(t-t_0^-)}dt=$$

$$=\int_{t_0^-}^{t_0^+}\left[\delta(t-t_0)\mathbf{x}(t_0^+)-\delta(t-t_0)\mathbf{x}(t_0^-)\right]e^{-s(t-t_0^-)}dt+$$

$$\int_{t_0^+}^{\infty}\mathbf{x}(t)e^{-s(t-t_0^+)}dt=\mathbf{x}(t_0^+)-\mathbf{x}(t_0^-)+\left[s\mathbf{X}(s)-\mathbf{x}(t_0^+)\right]=$$

$$=\Delta_0\mathbf{x}+L_+\left[\mathbf{x}'(t)\right]=s\mathbf{X}(s)-\mathbf{x}(t_0^-) \tag{2.1}$$

since $L_+\left[\mathbf{x}'(t)\right]=[s\mathbf{X}(s)-\mathbf{x}(t_0^+)]$, $L_-[\delta(t)]=1$ and $L_+[\delta(t)]=0$. In the case where $\mathbf{x}(t_0^-)=\mathbf{0}$, then $L_-[\dot{\mathbf{x}}(t)]=s\mathbf{X}(s)$ [2]. Equivalently, (2.1) may be derived considering that the restriction of $\mathbf{x}(t)$ to $[t_0^+,\infty)$ denoted by $\mathbf{x}_{[t_0^+,\infty)}$ [3]. Applying (2.1) on the homogeneous zero input equation $\mathbf{E}\dot{\mathbf{x}}_{zi}(t)=\mathbf{A}\mathbf{x}_{zi}(t)$, we obtain

$$(s\mathbf{E}-\mathbf{A})\mathbf{X}_{zi}(s)=\mathbf{E}\mathbf{x}(t_0^-) \tag{2.2}$$

The Laplace Transforms on $\mathbf{E}\dot{\mathbf{x}}_{zi}(t)=\mathbf{A}\mathbf{x}_{zi}(t)$ for restrictions $\mathbf{x}_{[t_0^+,\infty)}$, $\mathbf{x}_{[t_0^-,t_0^+]}$ are

$$(s\mathbf{E}-\mathbf{A})\mathbf{X}_{F,zi}(s)=\mathbf{E}\mathbf{x}(t_0^+) \tag{2.3}$$

$$(s\mathbf{E}-\mathbf{A})\mathbf{X}_{I,zi}(s)=\mathbf{E}\mathbf{x}(t_0^-)-\mathbf{E}\mathbf{x}(t_0^+)=-\Delta_0\mathbf{x} \tag{2.4}$$

respectively, using $L_+\left[\mathbf{x}'(t)\right]=[s\mathbf{X}(s)-\mathbf{x}(t_0^+)]$ and (2.1). Moreover, the Laplace *L*- transform on the nonhomogeneous equation (1.1) yields

$$(s\mathbf{E}-\mathbf{A})\mathbf{X}(s)=\mathbf{B}\mathbf{U}(s)+\mathbf{E}\mathbf{x}(t_0^-) \tag{2.5}$$

Definition 2.1

Any nonsingular matrix $\mathbf{\Phi}(t)$, with $\mathbf{\Phi}^{-1}(t)$ existing for all t, and satisfying the equations

$$\mathbf{E}\dot{\mathbf{\Phi}}(t)=\mathbf{A}\mathbf{\Phi}(t)-\mathbf{A}\mathbf{\Phi}_{-1}\delta(t-t_0),\ t\in[t_0^-,\infty) \tag{2.6a}$$

$$\det\mathbf{\Phi}(t)\neq 0,\quad t\in[t_0^-,\infty) \tag{2.6b}$$

is said to be a fundamental matrix $\mathbf{\Phi}(t)$ of the generalized system (1.1). Note that (2.6a) is written for $t\in[t_0^+,\infty)$ in the known form $\mathbf{E}\dot{\mathbf{\Phi}}(t)=\mathbf{A}\mathbf{\Phi}(t)$ that holds in the regular systems.

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The fundamental matrix $\mathbf{\Phi}(t)$ is not unique and therefore does not depend on t_0. Therefore, it results from (2.6a) that the Laplace Transform L_- of the $\mathbf{\Phi}(t)$ is determined by $(s\mathbf{E}-\mathbf{A})\mathbf{\Phi}(s)=\mathbf{Z}$, where $\mathbf{Z}\in\Re^{nxn}$ may be any constant square matrix.

Assuming that the given system is solvable and taking **Z** equal to the unity matrix, then $\mathbf{\Phi}(s)$ is selected to be the generalized resolvent matrix $(s\mathbf{E}-\mathbf{A})^{-1}$. The Laurent series expansion of $(s\mathbf{E}-\mathbf{A})^{-1}$ about infinity in the area of convergence $r<|s|<R$, where *R* may be any arbitrary great finite positive number

with the exception of R=∞ and r is the radius of the circle that includes all the finite eigenvalues (isolated regularities) of the matrix pencil $(s\mathbf{E}-\mathbf{A})$, is given by

$$\mathbf{\Phi}(s) = L_{-}[\mathbf{\Phi}(t)] = (s\mathbf{E}-\mathbf{A})^{-1} = \sum_{k=-\mu}^{\infty}\mathbf{\Phi}_k(\mathbf{E},\mathbf{A})s^{-k-1}$$

$$= \sum_{k=0}^{\infty}\mathbf{\Phi}_k s^{-k-1} + \sum_{k=0}^{\mu-1}\mathbf{\Phi}_{-k-1}s^{k} = \mathbf{\Phi}_F(s)+\mathbf{\Phi}_I(s) \quad (2.8)$$

where $\mathbf{\Phi}_i = \mathbf{\Phi}_i(\mathbf{E},\mathbf{A}) \in \Re^{nxn}$ is the (forward) fundamental matrix sequence [8],[9] and μ is the index of nilpotency of the pencil $(s\mathbf{E}-\mathbf{A})$. Note that μ is the maximum order of infinite zeros of $(s\mathbf{E}-\mathbf{A})$. In (2.8), $\mathbf{\Phi}_F(s)$ and $\mathbf{\Phi}_I(s)$ are the finite and impulse fundamental matrices respectively, i.e. the parts of $\mathbf{\Phi}(s)$ corresponding to the finite and infinite eigenspace of $(s\mathbf{E}-\mathbf{A})$ respectively. Moreover, $\mathbf{\Phi}_F(s)$ and $\mathbf{\Phi}_I(s)$ correspond to the strictly proper part and to the polynomial part of the associated transfer function respectively. The fundamental matrix $\mathbf{\Phi}(t)$ may be computed by applying the inverse Laplace transform L_{-}^{-1} to (2.8), as follows:

$$\mathbf{\Phi}(t) = L_{-}^{-1}[\mathbf{\Phi}(s)] = L_{-}^{-1}[(s\mathbf{E}-\mathbf{A})^{-1}] = \mathbf{\Phi}_F(t)+\mathbf{\Phi}_I(t) \quad (2.9)$$

The finite fundamental matrix $\mathbf{\Phi}_F(t)$ is given by

$$\mathbf{\Phi}_F(t) = L_{+}^{-1}[\mathbf{\Phi}_F(s)] = L_{+}^{-1}\left[\sum_{k=0}^{\infty}\mathbf{\Phi}_k s^{-k-1}\right] =$$

$$= \left[\sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{\Phi}_k t^k\right]1(t) = \left[\sum_{k=0}^{\infty}\frac{1}{k!}(\mathbf{\Phi}_o\mathbf{A})^k t^k\right]\mathbf{\Phi}_0\,1(t)$$

$$= e^{\mathbf{\Phi}_0\mathbf{A}t}\mathbf{\Phi}_0 1(t) = \mathbf{\Phi}_0 e^{\mathbf{\Phi}_0\mathbf{A}t}1(t) \quad (2.10)$$

where $1(t)$ denotes the unit step function and use of the property $\mathbf{\Phi}_i = (\mathbf{\Phi}_0\mathbf{A})^i\mathbf{\Phi}_0 = \mathbf{\Phi}_0(\mathbf{\Phi}_0\mathbf{A})^i,\ i \ge 0$, was made [9]. Moreover, the impulse fundamental matrix $\mathbf{\Phi}_I(t)$ results as the inverse generalized Laplace transform L_- of the polynomial part $\mathbf{\Phi}_I(s)$ of $\mathbf{\Phi}(s)$, which represents exactly the restriction $\mathbf{x}[t_0]$ of $\mathbf{x}[t_0,\infty)$ to t_0 and is given by

$$\mathbf{\Phi}_I(t) = L_{-}^{-1}[\mathbf{\Phi}_I(s)] = L_{-}^{-1}\left[\sum_{k=0}^{\mu-1}\mathbf{\Phi}_{-k-1}s^k\right]$$

$$= \sum_{k=0}^{\mu-1}\delta^{(k)}(t-t_0)\mathbf{\Phi}_{-k-1} \quad (2.11)$$

where $\delta^{(k)}(t)$ is the k^{th} distributional derivative of $\delta(t)$. Multiplying $\mathbf{\Phi}(s)$ from the left with $(s\mathbf{E}-\mathbf{A})$, it results that

$$(s\mathbf{E}-\mathbf{A})\mathbf{\Phi}(s) = (s\mathbf{E}-\mathbf{A})[\mathbf{\Phi}_F(s)+\mathbf{\Phi}_I(s)] =$$

$$= (s\mathbf{E}-\mathbf{A})\left[\sum_{i=0}^{\infty}\mathbf{\Phi}_i s^{-\iota-1} + \sum_{i=0}^{\mu-1}\mathbf{\Phi}_{-i-1}s^i\right]$$

$$= (\mathbf{E}\mathbf{\Phi}_{-\mu})s^{\mu} + \sum_{k=\mu-1}^{\infty}[\mathbf{E}\mathbf{\Phi}_{-k} - \mathbf{A}\mathbf{\Phi}_{-k-1}]s^k = \mathbf{I} \quad (2.12)$$

$$(s\mathbf{E}-\mathbf{A})\mathbf{\Phi}_F(s) = \mathbf{E}\mathbf{\Phi}_0 \quad (2.13a)$$

$$(s\mathbf{E}-\mathbf{A})\mathbf{\Phi}_I(s) = -\mathbf{A}\mathbf{\Phi}_{-1} \quad (2.13b)$$

$$\mathbf{E}\mathbf{\Phi}_k - \mathbf{A}\mathbf{\Phi}_{k-1} = \delta(k)\,\mathbf{I}, \quad k \ge -\mu \quad (2.13c)$$

Moreover, the multiplication of $\mathbf{\Phi}(s)$ from the right with $(s\mathbf{E}-\mathbf{A})$ gives:

$$\mathbf{\Phi}_F(s)(s\mathbf{E}-\mathbf{A}) = \mathbf{\Phi}_0\mathbf{E} \quad (2.14a)$$

$$\mathbf{\Phi}_I(s)(s\mathbf{E}-\mathbf{A}) = -\mathbf{\Phi}_{-1}\mathbf{A} \quad (2.14b)$$

$$\mathbf{\Phi}_k\mathbf{E} - \mathbf{\Phi}_{k-1}\mathbf{A} = \delta(k)\,\mathbf{I}, \quad k \ge -\mu \quad (2.14c)$$

It results from (2.13c), (2.14c) that for k=-μ, it holds $\mathbf{E}\mathbf{\Phi}_{-\mu} = \mathbf{\Phi}_{-\mu}\mathbf{E} = \mathbf{0}$.

Theorem 2.2

The fundamental matrix $\mathbf{\Phi}(t)$ of the generalized system (1.1) is given by

$$\mathbf{\Phi}(t) = \mathbf{\Phi}_F(t) + \mathbf{\Phi}_I(t) =$$

$$= e^{\mathbf{\Phi}_0\mathbf{A}t}\mathbf{\Phi}_0 1(t) + \sum_{=0}^{\mu-1}\mathbf{\Phi}_{-i-1}\delta^{(i)}(t-t_0),\ t \in [t_0^-,\infty) \quad (2.15)$$

Proof: The fundamental matrix $\mathbf{\Phi}(t)$ should satisfy (2.6a). Applying the ordinary derivative on (2.10), we obtain

$$\mathbf{E}\dot{\mathbf{\Phi}}_F(t) = \mathbf{E}(\mathbf{\Phi}_0\mathbf{A})e^{\mathbf{\Phi}_0 At}\mathbf{\Phi}_0 = (\mathbf{E}\mathbf{\Phi}_0)\mathbf{A}\mathbf{\Phi}_F(t)$$

$$= (\mathbf{I}-\mathbf{A}\mathbf{\Phi}_{-1})\mathbf{A}\mathbf{\Phi}_F(t) = \mathbf{A}\mathbf{\Phi}_F(t) - \mathbf{A}\mathbf{\Phi}_{-1}\mathbf{A}\mathbf{\Phi}_F(t)$$

$$= \mathbf{A}\mathbf{\Phi}_F(t),\quad t \in [t_0^+,\infty) \quad (2.16)$$

where use of (2.13a), (2.13c) was made. Equivalently, the above mathematical manipulations are derived using the fact that $(-\mathbf{A}\mathbf{\Phi}_{-1})$ is the projection on the infinite eigenspace AH_I along the finite eigenspace EH_F, hence $(-\mathbf{A}\mathbf{\Phi}_{-1})\mathbf{A}e^{\mathbf{\Phi}_0\mathbf{A}t}\mathbf{\Phi}_0 = \mathbf{0}$ (where H_F and H_I denote the finite and the infinite eigenspaces of the generalized pencil $(s\mathbf{E}-\mathbf{A})$ respectively, for which $H_F \oplus H_I = \Re^n$) [9]. Now, applying the distributional derivative formula on (2.11), we obtain

$$\mathbf{E}\dot{\mathbf{\Phi}}_I(t) = \mathbf{E}\left[\sum_{k=1}^{\mu}\delta^{(k)}(t-t_0)\mathbf{\Phi}_{-k}\right] =$$

$$= \mathbf{A}\left[\sum_{k=1}^{\mu}\delta^{(k)}(t-t_0)\mathbf{\Phi}_{-k-1}\right] \quad (\mathbf{E}\mathbf{\Phi}_k = \mathbf{A}\mathbf{\Phi}_{k-1},\ \text{for } k \ne 0)$$

$$= \mathbf{A}\left[\sum_{k=1}^{\mu-1}\delta^{(k)}(t-t_0)\mathbf{\Phi}_{-k-1}\right] \quad (\mathbf{\Phi}_{-\mu-1} = \mathbf{0})$$

$$= \mathbf{A}\mathbf{\Phi}_I(t) - \mathbf{A}\mathbf{\Phi}_{-1}\delta(t-t_0),\quad t \in [t_0^-,t_0^+] \quad (2.17)$$

The addition of (2.16) and (2.17) gives (2.6a).

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For regular systems where $\mathbf{E} = \mathbf{I}$, it is seen from (2.8) that $\mathbf{\Phi}_0 = \mathbf{I}$, and therefore the fundamental matrix is reduced to $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$. Note that in the regular state-space case there are not eigenvalues at infinity. The FM sequences $\mathbf{\Phi}_k, k = -\mu,-\mu+1,\ldots,-1$ and $\mathbf{\Phi}_k, k = 0,1,2,\ldots$ may be calculated only in terms of $\mathbf{\Phi}_0$ and $\mathbf{\Phi}_{-1}$ respectively. Moreover, $\mathbf{\Phi}_0$ and $\mathbf{\Phi}_{-1}$ are calculated using recursive algebraic relations in terms of the coefficients of the adjoint and the characteristic polynomial of the pencil $(z\mathbf{E}-\mathbf{A})$ and finally in terms of $\mathbf{A}$ and $\mathbf{E}$ [8],[16].

3. TRANSITION MATRIX

The forward transition matrix $\mathbf{\Psi}(t,t_0^-)$ of continuous linear time-invariant generalized systems satisfies the transition property

$$\mathbf{x}(t)=\mathbf{\Psi}(t,t_0^-)\mathbf{x}(t_0^-)=\mathbf{\Psi}(t-t_0^-)\mathbf{x}(t_0^-) \qquad (3.1)$$

since then the transition matrix is also time-invariant, i.e. $\mathbf{\Psi}(t,t_0^-)=\mathbf{\Psi}(t-t_0^-)$.

Definition 3.1

The forward transition matrix $\mathbf{\Psi}(t-t_0^-)$ of continuous linear time-invariant generalized systems is unique since it does depend on the initial time instant t_0^- and is defined by the equations:

$$\mathbf{E}\dot{\mathbf{\Psi}}(t-t_0^-)=\mathbf{A}\mathbf{\Psi}(t-t_0^-)-\mathbf{A}\mathbf{\Psi}_{-1}\delta(t-t_0),$$
$$t\in[t_0^-,\infty), \qquad (3.2a)$$

$$\mathbf{\Psi}(0)=\mathbf{I} \qquad (3.2b)$$

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The Laplace L_- transform of the distribution $\dot{\mathbf{\Psi}}(t,t_0^-)$, $t\in[t_0^-,\infty)$, in correspondence to (2.1), is given by

$$L_-[\dot{\mathbf{\Psi}}(t-t_0^-)]=L_+[\mathbf{\Psi}'(t-t_0^+)]-\mathbf{\Psi}'(t_0^- -t_0^-)$$
$$=s\mathbf{\Psi}(s)-\mathbf{I} \qquad (3.3)$$

taking into account the initial condition (3.2b). Now, applying (3.3) on (3.2a) we obtain:

$$L_-[\mathbf{E}\dot{\mathbf{\Psi}}(t,t_0^-)]=\mathbf{E}[s\mathbf{\Psi}(s)-\mathbf{I}]$$
$$=\mathbf{A}\mathbf{\Psi}(s)-\mathbf{A}\mathbf{\Psi}_{-1}1(t-t_0^-) \qquad (3.4)$$

Solving (3.4) for $\mathbf{\Psi}(s)$ and assuming that the given system is solvable, we obtain

$$\mathbf{\Psi}(s)=(s\mathbf{E}-\mathbf{A})^{-1}\left[\mathbf{E}-\mathbf{A}\mathbf{\Psi}_{-1}1((t-t_0^-)\right]=$$
$$=\left[\mathbf{\Phi}_F(s)+\mathbf{\Phi}_I(s)\right]\mathbf{E}-(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{A}\mathbf{\Psi}_{-1}1(t-t_0^-)$$
$$=\left[\mathbf{\Psi}_F(s)+\mathbf{\Psi}_I(s)\right]-(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{A}\mathbf{\Psi}_{-1}1(t-t_0^-)$$
$$=\left[\mathbf{\Psi}_F(s)+\mathbf{\Psi}_I(s)\right]-(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{A}\mathbf{\Phi}_{-1}\mathbf{E}1(t-t_0^-)$$
$$=\left[\mathbf{\Psi}_F(s)+\mathbf{\Psi}_I(s)\right]-\mathbf{\Phi}_I(s)\mathbf{A}\mathbf{\Phi}_{-1}\mathbf{E}1(t-t_0^-)$$
$$=\mathbf{\Psi}_F(s)+\mathbf{\Psi}_I(s)+\mathbf{\Psi}_I(s)\,1(t-t_0^-) \qquad (3.5)$$

It is known already from discrete-time generalized systems that $\mathbf{\Psi}_F(s)=\mathbf{\Phi}_F(s)\mathbf{E}$, $\mathbf{\Psi}_I(s)=\mathbf{\Phi}_I(s)\mathbf{E}$ and in general $\mathbf{\Psi}_i=\mathbf{E}\mathbf{\Phi}_i$, $i=-\mu,-\mu+1,\dots,-1,0,1,2,\dots$ [8],[9]. In (3.5) the relation $\mathbf{\Psi}_{-1}=\mathbf{E}\mathbf{\Phi}_{-1}$, as well as (2.11) and the property $\mathbf{\Phi}_i\mathbf{A}\mathbf{\Phi}_j=-\mathbf{\Phi}_{i+j+1}$, if $i<0, j<0$ [9] were used. The parts $\mathbf{\Psi}_F(s)$ and $\mathbf{\Psi}_I(s)$ of $\mathbf{\Psi}(s)$ corresponding to the finite and infinite eigenspaces of $(s\mathbf{E}-\mathbf{A})$, are given by

$$\mathbf{\Psi}_F(s)=\mathbf{\Phi}_F(s)\mathbf{E}=\sum_{i=0}^{\infty}\mathbf{\Phi}_i\mathbf{E}s^{-i-1} \qquad (3.6)$$

$$\mathbf{\Psi}_I(s)=\mathbf{\Phi}_I(s)\mathbf{E}=\sum_{i=0}^{\mu-1}\mathbf{\Phi}_{-i-1}\mathbf{E}s^{i} \qquad (3.7)$$

respectively. The transition matrix $\mathbf{\Psi}(t-t_0^-)$ may be computed by applying the inverse Laplace transform L_- to $\mathbf{\Psi}(s)$, using the expression (3.16), as follows:

$$\mathbf{\Psi}(t-t_0^-)=L_-^{-1}[\mathbf{\Psi}(s)]=L_-^{-1}[(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{E}]$$
$$=\mathbf{\Psi}_F(t-t_0^+)+\mathbf{\Psi}_I(t-t_0^-)+L_-^{-1}\left[\mathbf{\Psi}_I(s)1((t-t_0^-)\right] \qquad (3.8)$$

where $\mathbf{\Psi}_F(t-t_0^+)$ is the finite transition matrix, given by

$$\mathbf{\Psi}_F(t-t_0^+)=L_+^{-1}[\mathbf{\Phi}_F(s)\mathbf{E}]=\left[\sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{\Psi}_k(t-t_0^+)^k\right]1(t)$$
$$=\left[\sum_{k=0}^{\infty}\frac{1}{k!}(\mathbf{\Phi}_o\mathbf{A})^k(t-t_0^+)^k\right]\mathbf{\Phi}_0\mathbf{E}1(t-t_0^+)$$
$$=e^{\mathbf{\Phi}_0\mathbf{A}(t-t_0^+)}\mathbf{\Phi}_0\mathbf{E}1(t-t_0^+)=e^{\mathbf{\Phi}_0\mathbf{A}(t-t_0^+)}\mathbf{\Phi}_0\mathbf{E},\quad t\in[t_0^+,\infty)$$
$$(3.9)$$

and $\mathbf{\Psi}_I(t-t_0^-)$, $t\in[t_0^-,t_0^+]$ is the impulse transition matrix, given by

$$\mathbf{\Psi}_I(t-t_0^-)=L_-^1[\mathbf{\Phi}_I(s)\mathbf{E}]=\sum_{i=0}^{\mu-1}\delta^{(i)}(t-t_0)\mathbf{\Phi}_{-i-1}\mathbf{E} \qquad (3.10)$$

According to the initial condition (3.2b), at $t=t_0^-$, it should hold

$$\mathbf{\Psi}(t_0^- -t_0^-)=\mathbf{\Phi}_I(t_0^- -t_0^-)\mathbf{E}=\mathbf{\Phi}_I(0)\mathbf{E}$$
$$=\left[\sum_{i=0}^{\mu-1}\delta^{(i)}(0)\mathbf{\Phi}_{-i-1}\mathbf{E}\right]=\mathbf{I} \qquad (3.11)$$

Using (3.1) and (3.9),(3.10), the zero-input state vector $\mathbf{x}_{zi}(t)$ is calculated, given $\mathbf{x}(t_0^-)$, as follows:

$$\mathbf{x}_{zi}(t)=\mathbf{\Psi}_F(t-t_0^+)\mathbf{x}(t_0^+)+\mathbf{\Psi}_I(t-t_0^-)\mathbf{x}(t_0^-)$$
$$=e^{\mathbf{\Phi}_0\mathbf{A}(t-t_0^+)}\mathbf{\Phi}_0\mathbf{E}\mathbf{x}(t_0^+)1(t-t_0^+)+$$
$$\sum_{k=0}^{\mu-1}\delta^{(k)}(t-t_0)\mathbf{\Phi}_{-k-1}\mathbf{E}\mathbf{x}(t_0^-) \qquad (3.12)$$

The impulses in (3.12) are eliminated if

$$\mathbf{\Psi}_I(t-t_0^-)\mathbf{x}(t_0^-)=\left[\sum_{k=0}^{\mu-1}\delta^{(k)}(t-t_0)\mathbf{\Phi}_{-i-1}\mathbf{E}\right]\mathbf{E}\mathbf{x}(t_0^-)=\mathbf{0}$$
$$(3.13)$$

or equivalently if $\mathbf{x}_{zi}(t)$ is in the null space of $(\mathbf{\Phi}_{-1}\mathbf{E})$, $i=1,2,\dots,\mu$, i.e.

$$\mathbf{x}(t_0^-)\in N(\mathbf{\Phi}_{-1}\mathbf{E})\cap N(\mathbf{\Phi}_{-2}\mathbf{E})\cap\dots\cap N(\mathbf{\Phi}_{-\mu}\mathbf{E}) \quad (3.14)$$

which is the zero input admissible initial condition.

Considering the derivative of $\mathbf{\Psi}_F(t,t_0^+)$ we obtain:

$$\mathbf{E}\dot{\mathbf{\Psi}}_F(t-t_0^+)=(\mathbf{E}\mathbf{\Phi}_0)\mathbf{A}e^{\mathbf{\Phi}_0\mathbf{A}(t-t_0^+)}=\mathbf{A}\mathbf{\Psi}_F(t-t_0^+)$$
$$(3.15)$$

since $\mathbf{E}\mathbf{\Phi}_0$ is the projection on the finite eigenspace EH_F along the infinite eigenspace AH_I [9].

Considering now the distributional derivatives of $\mathbf{\Psi}_I(t,t_0^-)$ we obtain:

$$\mathbf{E}\dot{\mathbf{\Psi}}_I(t-t_0^-) = \mathbf{E}\left[\sum_{i=0}^{\mu}\delta^{(i)}(t-t_0)\mathbf{\Phi}_{-i}\right]\mathbf{E} =$$

$$\mathbf{A}\left[\sum_{i=0}^{\mu}\delta^{(i)}(t-t_0)\mathbf{\Phi}_{-i-1}\right]\mathbf{E} = \mathbf{A}\left[\sum_{i=0}^{\mu-1}\delta^{(i)}(t-t_0)\mathbf{\Phi}_{-i-1}\right]\mathbf{E}$$

$$= \mathbf{A\Phi}_I(t-t_0^-)\mathbf{E} - (\mathbf{A\Phi}_{-1})\mathbf{E}\delta(t-t_0)$$

$$= \mathbf{A\Psi}_I(t-t_0^-) - \mathbf{A\Psi}_{-1}\delta(t-t_0) \quad t \in [t_0^-, t_0^+] \qquad (3.16)$$

The summation of (3.15) and (3.16) gives

$$\mathbf{E}\dot{\mathbf{\Psi}}_F(t-t_0^+) + \mathbf{E}\dot{\mathbf{\Psi}}_I(t-t_0^-)$$

$$= \mathbf{A\Psi}_F(t-t_0^+) + \mathbf{A\Psi}_I(t-t_0^-) - \mathbf{A\Phi}_{-1}\mathbf{E}\delta(t-t_0)$$

$$= \mathbf{A\Psi}_F(t-t_0^+) + \mathbf{A\Psi}_I(t-t_0^-) - \mathbf{A\Psi}_{-1}\delta(t-t_0),$$

$$t \in [t_0^-, \infty) \qquad (3.17)$$

which is the desired equation (3.2a). Note that for $t \in [t_0^+, \infty)$, $\delta(t-t_0) = 0$ and therefore (3.17) is reduced to (3.15).

4. SOLUTION OF THE GENERALIZED STATE-SPACE MODEL

In this Section we derive the general solution of (1.1), using the algebraic fundamental matrix approach. The zero input solution has been derived in Section 3 and it is given by (3.12).

For the derivation of the zero state solution, we need to apply generalized or distributional derivatives of functions with “jump” discontinuity” at t_0 that contain Dirac operators. If the ordinary derivative $u^{(i)}(t)$ of the function $u(t)$ exists for all t, with the exception of the point t_0, and represents a locally integrable function, and if moreover, for both limiting processes $t \to t_0^-$ and $t \to t_0^+$ $u(t)$ converges, then the i^{th} distributional derivative $\mathrm{D}^{(i)}u(t)$ of $u(t)$ is given by [17],[18]

$$\mathrm{D}^{(i)}u(t) = u^{(i)}(t) +$$

$$\sum_{k=0}^{i-1}\left[u^{(i-k-1)}(t_0^+) - u^{(i-k-1)}(t_0^-)\right]\delta^{(k)}(t-t_0), \quad t \in [t_0^-, \infty)$$

$$(4.1)$$

Zero state solution

Theorem 4.1

The zero state solution of (1.1a) is given by

$$\mathbf{x}_{zs}(t) = \int_{t_0^-}^{t} e^{\mathbf{\Phi}_0\mathbf{A}(t-\tau)}\mathbf{\Phi}_0\mathbf{Bu}(t)dt + \sum_{i=0}^{\mu-1}\mathbf{\Phi}_{-i-1}\mathbf{Bu}^{(i)}(t)$$

$$+ \sum_{i=0}^{\mu-1}\mathbf{\Phi}_{-i-1}\sum_{j=0}^{i-1}\delta^{(i-j-1)}(t-t_0)\mathbf{Bu}^{(k)}(t_0^-) \qquad (4.2)$$

Proof: The zero state solution of (1.1a) may be determined from the inverse Laplace transform L_- of $(s\mathbf{E}-\mathbf{A})^{-1}\ \mathbf{BU}(s)$, as follows:

$$\mathbf{x}_{zs}(t) = L_-^{-1}[(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{BU}(s)]$$

$$= L_-^{-1}(s\mathbf{E}-\mathbf{A})^{-1} * L_-^{-1}(\mathbf{BU}(s)]$$

$$= L_-^{-1}[(\mathbf{\Phi}_I(s) + \mathbf{\Phi}_F(s)\mathbf{BU}(s)]$$

$$= L_-^{-1}[\mathbf{\Phi}_F(s) + \mathbf{\Phi}_I(s)] * L_-^{-1}[\mathbf{BU}(s)$$

$$= L_+^{-1}[\mathbf{\Phi}_F(s)] * L_+^{-1}[\mathbf{BU}(s)] + L_-^{-1}[\mathbf{\Phi}_I(s)] * L_-^{-1}[\mathbf{BU}(s)]$$

$$= \int_{t_0^+}^{t}\mathbf{\Phi}_F(t-\tau)\mathbf{B}u(\tau)d\tau +$$

$$+ [L^{-1}\mathbf{BU}(s)]\left[\sum_{i=0}^{\mu-1}[\mathbf{\Phi}_{-i-1}s^k]\mathbf{BU}(s)\right]$$

$$= \int_{t_0^+}^{t}\mathbf{\Phi}_F(t-\tau)\mathbf{B}u(\tau)d\tau + \int_{t_0^-}^{t_0^+}\sum_{k=1}^{\mu}\left[\mathbf{\Phi}_{-\kappa}\delta^{\kappa-1}(t-\tau)\right]\mathbf{B}u(\tau)d\tau$$

$$= \int_{t_0^+}^{t} e^{\Phi_0 A(t-\mathrm{T})}\mathbf{\Phi}_0\mathbf{Bu}(\tau)d\tau + \sum_{i=0}^{\mu-1}\mathbf{\Phi}_{-i-1}\mathbf{B}D^{(i)}\mathbf{u}(t)$$

$$= \int_{t_0^+}^{t} e^{\Phi_0 A(t-\mathrm{T})}\mathbf{\Phi}_0\mathbf{Bu}(\tau)d\tau + \sum_{i=0}^{\mu-1}\mathbf{\Phi}_{-i-1}\mathbf{Bu}^{(i)}(t)$$

$$+ \sum_{i=0}^{\mu-1}\mathbf{\Phi}_{-i-1}\sum_{j=0}^{i-1}\delta^{(i-j-1)}(t-t_0)\mathbf{Bu}^{(j)}(t_0^-) \qquad (4.3)$$

Note that the derivative of the first term of the right hand side of (4.3) is the ordinary derivative, since it is the right hand derivative at the origin.

Making use of (4.1) for the calculation of the distributional derivative $\mathrm{D}^{(i)}\mathbf{u}(t)$, we finally obtain

$$\mathbf{x}_{zs}(t) = \mathbf{x}_{F,zs}(t) + \mathbf{x}_{I,zs}(t) \qquad (4.4)$$

where

$$\mathbf{x}_{F,zs}(t) = \int_{t_0^+}^{t} e^{\mathbf{\Phi}_0\mathbf{A}(t-\tau)}\mathbf{\Phi}_0\mathbf{Bu}(\tau)d\tau =$$

$$e^{\mathbf{\Phi}_0\mathbf{A}\tau}\int_{t_0}^{t} e^{-\mathbf{\Phi}_0\mathbf{A}\tau}\mathbf{Bu}(\tau)d\tau \qquad (4.5)$$

$$\mathbf{x}_{I,zs}(t) = \sum_{i=0}^{\mu-1}\mathbf{\Phi}_{-i-1}\mathbf{Bu}^{(i)}(t) +$$

$$+ \sum_{i=0}^{\mu-1}\mathbf{\Phi}_{-i-1}\sum_{j=0}^{i-1}\delta^{(i-j-1)}(t-t_0)\mathbf{Bu}^{(j)}(t_0^-) \quad (4.6)$$

In the sequel we will verify that $\mathbf{x}_{zs}(t)$ satisfies (1.1a) for zero initial conditions $\mathbf{x}(t_0^-) = \mathbf{0}$. Indeed, applying the distributional derivative operator on $\mathbf{x}_{zs}(t)$ in (4.3), we obtain

$$\mathbf{E}\dot{\mathbf{x}}_{zs}(t) = \mathbf{E}\mathbf{\Phi}_0\mathbf{A}\int_{t_0^+}^{t} e^{\mathbf{\Phi}_0\mathbf{A}(t-\tau)}\mathbf{\Phi}_0\mathbf{B}\mathbf{u}(\tau)d\tau +$$

$$+\mathbf{E}e^{\mathbf{\Phi}_0\mathbf{A}t-\mathbf{\Phi}_0\mathbf{A}t}\mathbf{\Phi}_0\mathbf{B}\mathbf{u}(t) + \sum_{i=1}^{\mu}\mathbf{E}\mathbf{\Phi}_{-i}\mathbf{B}\mathbf{u}^{(i)}(t)$$

$$= \mathbf{A}\int_{t_0^+}^{t} e^{\mathbf{\Phi}_0\mathbf{A}(t-\tau)}\mathbf{\Phi}_0\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{E}\mathbf{\Phi}_0\mathbf{B}\mathbf{u}(t) +$$

$$+\sum_{i=1}^{\mu}\mathbf{A}\mathbf{\Phi}_{-i-1}\mathbf{B}\mathbf{u}^{(i)}(t)$$

$$= \mathbf{A}\int_{t_0^+}^{t} e^{\mathbf{\Phi}_o(\mathbf{A}(t-\tau)}\mathbf{\Phi}_0\mathbf{B}\mathbf{u}(\tau)d\tau + (\mathbf{A}\mathbf{\Phi}_{-1}+\mathbf{I})\mathbf{B}\mathbf{u}(t) +$$

$$+\mathbf{A}\sum_{i=1}^{\mu-1}\mathbf{\Phi}_{-i-1}\mathbf{B}\mathbf{u}^{(i)}(t)$$

$$= \mathbf{A}\left[\int_{t_0^+}^{t} e^{\mathbf{\Phi}_0\mathbf{A}(t-\tau)}\mathbf{\Phi}_0\mathbf{B}\mathbf{u}(\tau)d\tau + \sum_{i=0}^{\mu-1}\mathbf{\Phi}_{-i-1}\mathbf{B}\mathbf{u}^{(i)}(t)\right] +$$

$$+\mathbf{B}\mathbf{u}(t) = \mathbf{A}\mathbf{x}_{zs}(t) + \mathbf{B}\mathbf{u}(t) \qquad (4.7)$$

Q.E.D.

■

General state-apace solution

Theorem 4.2

The general state space solution of (1.1a) is given by

$$\mathbf{x}(t) = \mathbf{x}_{zi}(t) + \mathbf{x}_{zs}(t) =$$

$$= e^{\mathbf{\Phi}_0\mathbf{A}(t-\tau)}\mathbf{\Phi}_0\mathbf{E}\mathbf{x}(t_0)1(t-t_0^-) +$$

$$+\sum_{i=0}^{\mu-1}\delta^{(i)}(t-t_0)\mathbf{\Phi}_{-i-1}\mathbf{E}\mathbf{x}(t_0^-) +$$

$$+\int_{t_0^+}^{t} e^{\mathbf{\Phi}_0\mathbf{A}(t-\tau)}\mathbf{\Phi}_0\mathbf{B}\mathbf{u}(\tau)d\tau + \sum_{i=0}^{\mu-1}\mathbf{\Phi}_{-i-1}\mathbf{B}\mathbf{u}^{(i)}(t) +$$

$$+\sum_{i=0}^{\mu-1}\mathbf{\Phi}_{-i-1}\sum_{j=0}^{i-1}\delta^{(i-j-1)}(t-t_0)\mathbf{B}\mathbf{u}^{(j)}(t_0^-)$$

(4.8)

Proof: The general solution of the generalized system (1.1a) results from the summation of the zero input and the zero state solution, which are given by (6.4) and (6.11) respectively. The solution (4.8) may be derived by following the Laplace transform approach.

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