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DESIGN FOR WELL CONDITIONING OF PROGENITOR MODELS BY INPUT, OUTPUT REDUCTION

N. KARCANIAS †, K. G. VAFIADIS†

† City University, Department of Electrical, Electronic and Information Engineering, Control Engineering Centre, Northampton Square, London EC1V 0HB, United Kingdom, N.Karcanias@city.ac.uk, K.Vafiadis@city.ac.uk

ABSTRACT. Transfer function models used for early stages of design are large dimension models containing all possible physical inputs, outputs. Such models may be badly conditioned and possibly degenerate. The problem considered here is the selection of maximal cardinality subsets of the physical input, output sets, such as the resulting model is nondegenerate. This problem is part of the early design task of selecting well- conditioned progenitor models on which successive design has to be carried out. The established conditions for different type of degeneracy are used to define necessary and sufficient conditions required to guarantee nondegeneracy. A simple redesign procedure that guarantees transfer function and input, output nondegeneracy is suggested and parameterisation of all such solutions is given. The results provide the basis for selection of well-conditioned transfer function models, which may be used for subsequent control design.

Key Words. Integrated design, process control, system structure, linear systems.

1.INTRODUCTION

The derivation of models that can be used for early design stages studies of processes requires the use of the system interconnection graph, the availability of simple models describing the fundamental dynamics of subprocesses and the selection of control (input) and measurement (output) variables. Before we embark on the investigation of the properties of the resulting model it is useful to include all possible inputs and outputs; at a later stage we can then determine the effective subsets of inputs, outputs using different "controllability", "operability" criteria. Such models corresponding to all possible inputs and all possible outputs will be referred to as **progenitor models** [7]. Progenitor models are derived on the basis that possible inputs, outputs are selected using heuristics, physical arguments and thus the resulting transfer function may be of large dimensions and possibly not well behaved. The essential feature of such models is that the input, output variables are physical variables, on which specifications may be imposed. Transfer functions corresponding to subsets of the potential input and output sets are referred to as **effective models** and are submatrices of the progenitor transfer function. Different families of effective models may be defined. Characterising such families of models, in terms of a range of important properties, is an important part of the "process controllability" studies.

This paper deals with a specific problem within the general area of selecting effective models, when we use as criteria the nondegeneracy of the effective transfer function and the nonredundancy of the instrumentation schemes i.e. independence of selected sensors and actuators. Nondegeneracy is a fundamental property for the effective model, since it is linked to the output function controllability [14], and thus to the solvability of a number of control problems. Conditions for the characterisation of system degeneracy and redundancy of the input, output structure of the system have been derived in [9]; these conditions indicate the criteria required to guarantee nondegeneracy and input, output scheme nonredundancy. Using these conditions simple sufficient conditions are given, which guarantee nondegeneracy and nonredundancy of the resulting effective model. The selection of maximal dimension effective models, which have both of the previous properties, is then considered using different criteria and parameterisation issues are discussed. The approach suggested here leads to a parameterisation of all maximal dimension effective models, which are nondegenerate and input, output nonredundant. The elements of this set may then be used for the selection of models having additional desirable properties. A more detailed exposition and proof of the results is given in [9].

2.STATEMENT OF THE PROBLEM AND PRELIMINARY RESULTS

The development of models, which may be used for evaluation of alternatives is an integral part of the Early Process Design of process plants [13]. Such models are usually developed for the entire plant, are based on the selected process flowsheet (interconnection graph and involve the use of simple models of the subprocesses). As such, they are large dimension models and their final structure is determined when the control structure is decided. The selection of control structures is a topic of strong interest within the process control area ([3], [4], [8], [11], [13] and references there in). This problem involves a number of key subproblems [8], which are: (i) The classification of process variables into potential inputs, outputs and referred to as **Model Orientation Problem** (MOP). (ii) Specification of effective sets of inputs, outputs on an oriented model and referred to as **Model Projection Problem** (MPP). (iii) Deciding on the way we couple effective inputs and outputs for control design purposes and referred to as **Input – Output Coupling Problem** (I-O.C.P.). Most of the attention so far has been focused on I-O.C.P., when heuristics and diagnostic indicators have been used. For the first two problems, less attention has been given, especially from the Control Theory viewpoint; with the exception of the work in [3], [6], [8], [11] on some specific problems. In this paper we are concerned with the selection of the effective sets of inputs, outputs on a system, in order to satisfy criteria for the system nondegeneracy and the nonredundancy of the input, output scheme for the resulting effective model.

We assume that we are given an oriented linearised model that includes all possible variables that can be used for control and measurement; these inputs, outputs are referred to as **potential** sets. The model that corresponds to the potential inputs, outputs provides the basis for deriving all subsequent models based on **effective** input, output sets and it is thus referred to as the **progenitor model** and all inputs and outputs are physical variables. Given that the classification of internal variables into inputs, outputs has been done mainly with physical, process based criteria, a progenitor model may not be well behaving. That is the transfer function may be degenerate, there may be redundancy in the input, output schemes and a number of other fundamental properties may not have good values (i.e. condition numbers etc.). System models, which are degenerate, do not satisfy the basic condition of the output function controllability. It is thus desirable to select subsets of the potential inputs and outputs such that the resulting transfer function is "well- conditioned" in some sense. Amongst the basic criteria we can use are the properties of nondegeneracy of the system model and nonredundancy of the input and output scheme. Any submodel that satisfies the above three properties and has maximal cardinality for the input and output set will be called a **normal progenitor model**; clearly, a system may have more than one such models. The problem we consider here is the parameterisation and systematic construction of the family of normal progenitor models.

We will assume that the progenitor model is described by the state space equations:

$$\underline{\dot{x}} = A\,\underline{x} + B\,\underline{u}\ ,\ \underline{y} = C\,\underline{x} + D\,\underline{u} \tag{2.1}$$

where

$A \in \Re^{n\times n}$, $B \in \Re^{n\times r}$, $C \in \Re^{q\times n}$, $D \in \Re^{q\times r}$
or equivalently by the Rosenbrock's System Matrix Pencil [14] P(s), and has a transfer function $H(s) = C\,(s\,I - A)^{-1}\,B + D \quad \in \Re^{q\times r}(s)$, where $\rho = \mathrm{rank}_{\Re(s)}\,\{H(s)\}$. Clearly $\rho \le \min(q, r)$ and whenever strict inequality holds, then the system is called **degenerate**; when equality holds the system is called **nondegenerate**.

Remark (2.1) [14]: ρ defines the maximal number of output variables that may be controlled independently, as well as the minimal number of independent inputs required to control ρ outputs.

For the system $S(A, B, C, D)$ for which $r, q \le n$, we define the numbers:

$$\tau_r \overset{\Delta}{=} \mathrm{rank}\,\{[B^t, D^t]\} \le r,\ \tau_l = \mathrm{rank}\,\{[C, D]\} \le q \tag{2.2}$$

If $\tau_r < r, (\tau_l < r)$ the system will be said to have **input** (**output**) redundancy; otherwise, i.e. if $\tau_r = r\,(\tau_l = r)$, then it will be said to be **regular**. Regularity of the model is clearly equivalent to nonredundancy of both sensor and actuator schemes.

PROBLEM: Given the progenitor model described by H(s), or S(A,B,C,D) define:
(i) A maximal cardinality subset of the potential input and output sets such as that the resulting transfer function is nondegenerate, has the maximal possible normal rank and it is also regular.
(ii) Parameterise all solutions with the properties described above.

The above problem will be referred to as **well-conditioning of Progenitor models** and part (ii) describes the parameterisation of solutions. Different types of criteria may be used for selecting the models. The well conditioning of progenitor models using as criterion the system nondegeneracy and input, output regularity is considered here. For the system described by S(A,B,C,D) we shall denote by $Z_r \overset{\Delta}{=} N_r\,\{P(s)\}, Z_l \overset{\Delta}{=} N_l\,\{P(s)\}$ the right, left null spaces of P(s). A pair of polynomial vectors $\underline{x}(s) \in \Re^n[s], \underline{u}[(s] \in \Re^r(s)$ will be said to be a **right pair** if for the composite vector $\underline{\zeta}(s) = [\underline{x}(s)^t, \underline{u}(s)^t]$, $P(s)\,\underline{\zeta}(s) = \underline{0}$. We introduce in a similar way the notion of a left pair as any pair of polynomial vectors $\underline{y}(s) \in \Re^n[s], \underline{v}(s) \in \Re^q[s]$ such that for $\underline{\xi}(s)^t = [\underline{y}(s)^t, \underline{v}(s)^t]$, $\underline{\xi}(s)^t\,P(s) = \underline{0}$. Note that for any right pair $(\underline{x}(s), \underline{u}(s))$, left pair $(\underline{y}(s), \underline{v}(s))$ we have that $\partial[\underline{u}(s)] = \partial[\underline{x}(s)] + 1$, $\partial[\underline{v}(s)] = \partial[\underline{y}(s)] + 1$.

Lemma (2.1): For the system S(A,B,C,D), let $\eta = \dim N_r\{P(s)\}$, $\theta = \dim N_l\{P(s)\}$,
$\tau = \mathrm{rank}_{\Re(s)}\{P(s)\}$ and $\rho = \mathrm{rank}_{\Re(s)}\{H(s)\}$. Then,

$$\tau = n + \rho, \quad \eta = \dim N_r\{P(s)\} = \dim N_r\{H(s)\} = r - \rho$$

and $\theta = \dim N_l\{P(s)\} = \dim N_l\{H(s)\} = q - \rho$.

Remark (2.2): The system is degenerate, if and only if $\tau = \mathrm{rank}_{\Re(s)}\{P(s)\} < \min(n + r, n + q)$. That is we can use either P(s) or H(s) for characterisation of the property. Some relationships between degeneracy and input, output loss of regularity are described below and this provides some classifications of the different types of degeneracy.

Proposition (2.1): For the system S(A,B,C,D) the following properties hold true:
(i) If q ≥ r (q ≤ r) and the system is not input (output) regular, then it is degenerate.
(ii) If a system is not input or output regular, then it is degenerate.
(iii) Let $\tau_l = \mathrm{rank}\,[C, D], \tau_r = \mathrm{rank}\begin{bmatrix} B^t & D^t \end{bmatrix}$. Then, if q ≥ r and $\tau_l < r$, the system is degenerate; if q ≤ r and $\tau_r < q$, then the system is degenerate.

For the pencil P(s), the null spaces N_r {P(s)}, N_l {P(s)} are characterised by a set of column, row minimal indices (cmi, rmi) [2], which are also referred to as **right**, **left indices** of P(s) [1] and are denoted by $I_p{}^c = \{\varepsilon_i : i = 1, \ldots\ldots\ldots \eta = n - \rho\}$,
$I_p{}^r = \{\mu_j : j = 1, \ldots\ldots\ldots \theta = q - \rho\}$. Such sets may have t_r zero cmi and t_l zero rmi; in fact,

$$t_r = r - \mathrm{rank}\left\{\begin{bmatrix} B^t, D^t \end{bmatrix}^t\right\} = r - \tau_r \le r - \rho,$$

$$t_l = q - \mathrm{rank}\,\{[C, D]\} = q - \tau_l \le q - \rho \qquad (2.3)$$

The numbers t_r, t_l which characterise 0 – cmi, 0 – rmi respectively, express the order of input, output redundancy and are referred to as **input-**, **output – redundancy index** correspondingly.

Proposition (2.2): The numbers τ_r = rank {[B^t, D^t]} and τ_l = rank {[C, D]} provide bounds for $\rho = \mathrm{rank}_{\Re(s)}\{H(s)\}$ and in particular $\rho \le \min(\tau_r, \tau_l)$. The case of $\rho = \min(\tau_r, \tau_l)$ implies:
(a) If $\tau_r \le \tau_l$, then all indices in $I_p{}^c$ are zero, or the set is empty; in particular, if $r > \tau_r$, then all cmi are zero and if $r = \tau_r$, then $I_p{}^c$ is empty and the system is nondegenerate.
(b) If $\tau_l \le \tau_r$, then all indices in $I_P{}^r$ are zero, or the set is empty; in particular, if $q > \tau_l$ then all indices in $I_P{}^r$ are zero and if $q = \tau_l$, then $I_P{}^r$ is empty and the system is nondegenerate.
(c) If $\rho = \tau_r = \tau_l$ and at least one of r, q is equal to ρ, then clearly we have nondegeneracy and redundancy for the index that is greater than ρ. If r, q > ρ, then we have both degeneracy and input, output degeneracy.

The case where $t_r = r - \rho$ ($t_l = q - \rho$) is referred to as **total input – (output-) irregularity**. When at least one such condition holds true, that implies that degeneracy of the transfer function may be removed by eliminating redundancy in the corresponding part of the instrumentation map. The type of system degeneracy inferred from the input, output redundancy is called **simple**. Another type of degeneracy that is linked to properties of the internal mechanism and is referred to as **strong degeneracy** [9] and considered next.

3. STRONG SYSTEM DEGENERACY AND SUFFICIENT CONDITIONS FOR NONDEGENERACY

In the previous section we examined issues of degeneracy and input, output redundancy, which are linked to zero values of cmi, rmi. Here we will consider the case of nonzero indices. Results describing strong degeneracy are described [9] and this leads to sufficient conditions for nondegeneracy and input, output regularity. The sets of indices [1] I_p^c, I_p^r may contain nonzero indices and this is characterised by the following result.

Proposition (3.1): For any system S(A, B, C, D) with r inputs, q outputs, transfer function H(s) and $\rho = \mathrm{rank}_{\Re(s)}\{H(s)\}$ the following properties hold true:
(a) The numbers ρ, τ_r, τ_l, r, q satisfy the conditions: $\rho \le \tau_r \le r$ and $\rho \le \tau_l \le q$.
(b) The system has $\tau_r - \rho$ nonzero cmi, if and only if $\rho < \tau_r \le r$ and all such indices are nonzero, if $\tau_r = r$.
(c) The system has $\tau_l - \rho$ nonzero rmi, if and only if $\rho < \tau_l \le q$ and all such indices are nonzero, if $\tau_l = q$.

Proposition (3.2): The system S(A,B,C,D) with $q \ge r$ and $\rho < r$ has a right index with value k, if and only if there exists a set of vectors $\{\underline{u}_0, \underline{u}_1, \dots, \underline{u}_k\}$ such that the following conditions are satisfied:

$$\begin{bmatrix} A^k B & A^{k-1}B & \dots & AB & B \\ CA^{k-1}B & CA^{k-2}B & \dots & CB & D \\ CA^{k-2}B & CA^{k-3}B & \dots & D & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ CAB & CB & \dots & 0 & 0 \\ CB & D & \dots & 0 & 0 \\ C & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{u}_k \\ \underline{u}_{k-1} \\ \underline{u}_{k-2} \\ \vdots \\ \underline{u}_2 \\ \underline{u}_1 \\ \underline{u}_0 \end{bmatrix} = \underline{0} \quad (3.1)$$

For the given system, we define the following set of matrices:

$$M_0 = \begin{bmatrix} B \\ D \end{bmatrix}, M_1 = \begin{bmatrix} AB & B \\ CB & D \\ D & 0 \end{bmatrix}, M_2 = \begin{bmatrix} A^2B & AB & B \\ CAB & CB & D \\ CB & D & 0 \\ D & 0 & 0 \end{bmatrix}, \dots,$$

$$M_k = \begin{bmatrix} A^k B & \dots & AB & B \\ CA^{k-1}B & \dots & CB & D \\ CA^{k-2}B & \dots & D & 0 \\ \vdots & & \vdots & \vdots \\ CAB & \dots & 0 & 0 \\ CB & \dots & 0 & 0 \\ D & \dots & 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} A^k B..B & B \\ \hline N_k & 0 \end{array}\right] \quad (3.2)$$

In terms of the above matrices, we may state some tests for nondegeneracy as shown below [9].

Remark (3.1): If $q \ge r$, then the maximal possible value of right index of P(s) is:
(i) If $D \ne 0$ and rank $(D)=\delta$, then $\varepsilon_{max} = n - q + 2\delta - 1$.
(ii) If $D = 0$, then $\varepsilon_{max} = n - q - 1$.

Theorem (3.1): For the system S(A,B,C,D) with $q \ge r$, the following properties hold true:
(i) If D has full rank, then the system has no right indices of any value and it is thus non-degenerate.
(ii) If $D \ne 0$ and rank $(D) = \delta < r$, then the system is non-degenerate, if and only if the matrix M_r is full rank, where $\tau = n - q + 2\delta - 1$.

Corollary (3.1): For the system S(A,B,C,D) with $q \ge r$, the following properties hold true:
(i) If CB is full rank, then the system has no right indices and the system is non-degenerate.
(ii) The system with CB rank deficient is non-degenerate, if and only if the matrix $\tilde{M}_{\tau'}$ is full rank, where $\tau' = n - q - 1$.

The results in this section characterise a type of degeneracy, which depends on the models inner structure and will be referred to as **strong degeneracy**. The distinction between the simple and strong type is the nature of associated indices, that is zero and non-zero respectively.

4. WELL CONDITIONING OF TRANSFER FUNCTIONS: SELECTION PROCEDURES AND PARAMETERISATIONS

The results in the previous sections provide criteria for selecting subsystems of H(s), or P(s) which satisfy the input, output regularity requirements and the conditions for non-degeneracy. Although, input, output redundancy may imply degeneracy, input, output regularity does not guarantee non-degeneracy. Guaranteeing non-degeneracy may be achieved by using the sufficient conditions based on the D, CB matrices, or testing selections using the full rank tests based on $M_\tau, \widetilde{M}_\tau$, matrices; which however are not easy to use for making initial selections. Two different strategies for model selection can be made, and these are referred to as Direct and Indirect methods.

4.1 Direct Method for Well-conditioning

We assume that $q \geq r$ and that the S(A,B,C,D) model is degenerate. If $D \neq 0$, then degeneracy implies that D is rank deficient and if $D = 0$, then necessarily CB has to be rank deficient.

Remark (4.1): If $q \geq r$, S(A,B,C,D) is degenerate, a redesign leading to $\widetilde{S}(A,\widetilde{B},C,\widetilde{D})$ with $\widetilde{D}$ full rank leads to a system which is non-degenerate and has full rank input and output structure. If $D = 0$, a redesign procedure leading to $\widetilde{S}(A,\widetilde{B},\widetilde{C})$ with $\widetilde{C}\widetilde{B}$ full rank yields a system which is non-degenerate and has full rank input and output structure.

The meaning of redesign of D, or CB is that we aim to define a maximal subset of the columns of D, or CB that guarantee the maximal full rank property. This procedure is clearly sufficient, but not necessary and leads to a system of smaller dimensions, as far as input, output structure is concerned. Note that we would like to achieve this selection without transforming the matrices D, CB, since it is desirable to keep the physical variables involved in the original model. It is clear that if the transformation of the input, output structure is allowed, this problem is trivial.

Definition (4.1): Let $T = [\underline{t}_1, \underline{t}_2, ..., \underline{t}_r] \in \Re^{q \times r}$, $\{\|\underline{t}_i\| = 1, i \in \underline{r}\}$, $q \geq r$ with rank $(T) = \rho < \min(q, r)$. Any ρ-subset of the set $\{\underline{t}_i, i \in \underline{r}\}$ of columns that is linearly independent is said to form a **natural basis** for the space colsp{T}, has a measure of orthogonality σ (Grammian, condition number etc) and is referred to as a **σ-natural basis**. The natural basis with the highest degree of orthogonality will be called a **proper basis** of colsp {T}. The selection of a proper basis for a set of vectors has been previously addressed in [10] as a problem of selection of "best uncorrupted base" and an algorithm for achieving this is introduced here using the notion of the Grammian [2].

Definition (4.2) [13]: Let $\underline{x}_1, \underline{x}_2, ..., \underline{x}_m \in \Re^n$, $\|\underline{x}_i\| = 1$. The matrix:

$$G = \begin{bmatrix} (\underline{x}_1, \underline{x}_1) & (\underline{x}_1, \underline{x}_2) & \cdots & (\underline{x}_1, \underline{x}_m) \\ (\underline{x}_2, \underline{x}_1) & (\underline{x}_2, \underline{x}_2) & \cdots & (\underline{x}_2, \underline{x}_m) \\ \vdots & \vdots & & \vdots \\ (\underline{x}_m, \underline{x}_1) & (\underline{x}_m, \underline{x}_{21}) & \cdots & (\underline{x}_m, \underline{x}_m) \end{bmatrix} \quad (4.1)$$

where (,) denotes inner product, is called the Gram matrix of the set and $G_m = G(\underline{x}_1, \underline{x}_2, ..., \underline{x}_m) = |G|$ is called their **Grammian**.

Remark (4.2). Note that $0 \leq G_m \leq 1$ and this characterises the degree of orthogonality. An alternative test for closeness to normality of a normalised selected set with a basis matrix A, can be the condition number of the corresponding matrix.

We will use the Grammian as the criterion for selection of natural bases with degree of orthogonality greater than a given number $\underline{\sigma}\,(0 < \sigma \leq 1)$. The set of all natural bases with orthogonality σ : $\underline{\sigma} \leq \sigma \leq 1$ may be defined as follows:

Direct Method for Well-conditioning: Let $T = [\underline{t}_1, \underline{t}_2, ..., \underline{t}_r] \in \Re^{q \times r}, q \geq r$ be a matrix that may represent D, or CB, ρ = rank {T} and assume all its columns to be normalised. The selection of the well-conditioned model involves:

STEP (1): Select an acceptable order of orthogonality $\underline{\sigma}$ and using the Grammian test we define the set of all column submatrices $\{T_a\}$, of T which correspond to index sets $a = (i_1, i_2, \ldots, i_\rho)$ and have orthogonality degree $\sigma \geq \underline{\sigma}$.

STEP (2): For every set of indices $a = (i_1, i_2, ..., i_\rho)$ associated with $\{T_a\}$, define the subsystems $\{H, \underline{\sigma}\} = \{H_a$: $a = i_1, i_2, \ldots, i_\rho)$ having as inputs those corresponding to the set $a = i_1, i_2, \ldots, i_\rho)$ of indices defined before. This procedure leads to a set of systems $\langle S, \underline{\sigma} \rangle = \{H_a(s), \underline{\sigma}\}$ for which D_a, or CB_a is a matrix with orthogonality order at least $\underline{\sigma}$.

The above procedure produces submodels, which are always non-degenerate and are input, output regular. However, it may lead to systems with unnecessarily small numbers of inputs (outputs), if rank of D, CB are small. The second approach aims at avoiding such problems.

4.2 Indirect Method for Well-conditioning

The second approach is based on the selection and parameterisation of all subsets of inputs and outputs for which input and output regularity is guaranteed and then testing for non-degeneracy using the tests derived before. We rely on the selection of natural bases for selecting the suitable input, output sets of variables. For the progenitor model S(A,B,C,D) we denote by:

$$F=\begin{bmatrix} B \\ D \end{bmatrix}=[\underline{f}_1,\underline{f}_2,\ldots,\underline{f}_r],\ G=[C,D]=\begin{bmatrix} \underline{g}_1^{\,t} \\ \vdots \\ \underline{g}_q^{\,t} \end{bmatrix} \qquad (4.2)$$

and let $\operatorname{rank}(F)=\tau_r \le r$, $\operatorname{rank}(G)=\tau_l \le q$ and $\operatorname{rank}_{\Re(s)}\{H(s)\}=\rho$. Without loss of generality we may also assume that the columns of F and the rows of G are normalised.

Definition (4.3): For the matrices F, G we shall denote by:

$$\{F\}=\left\{F_\beta,\ \beta=(j_1,\ldots,j_{\tau_r})\right\},\ \{G\}=\left\{G_\gamma,\ \gamma=(l_1,\ldots,l_{\tau_r})\right\}$$

the set of all submatrices of F, G which correspond to the natural bases of F, G and associated with set of indices β, γ respectively. The subsets of {F}, {G}, which have a degree of orthogonality greater or equal to some value $\{\sigma\}$, will be denoted by $\{F\}_{\underline{\sigma}},\{G\}_{\underline{\sigma}}$ correspondingly. We denote by:
$\Omega_F = \{\forall\beta: \beta = (j_1, j_2,\ldots, j_{\tau_r})\}$,
$\Omega_F = \{\forall\gamma: \gamma = (l_1, l_2,\ldots, l_{\tau_l})\}$, the set of sequences characterising the natural bases of F, G respectively. For every $\beta\in\Omega_F$ and $\gamma\in\Omega_G$ we shall denote by $S_{\beta,\gamma}(A, B, C, D)$ the subsystem of S(A,B,C,D) corresponding to the β set of inputs and γ set of outputs.

Remark (4.3): For proper systems S(A,B,C,D), D≠0, the subsystem $S_{\beta,\gamma}(A,B,C,D)$ that corresponds to some $\beta\in\Omega_F$ and $\gamma\in\Omega_G$ is not necessarily input and output regular. This implies that the process of selecting sets $\beta\in\Omega_F$ and $\gamma\in\Omega_G$ to guarantee input and output regularity are not always independent. In fact, although we can always make the system input regular with cardinality τ_r, or output regular with cardinality τ_l, achieving both may not be possible.

The above indicates that progenitor models may be classified as shown below:

Definition (4.4): Given a system S(A,B,C,D) we say that:
(i) It is **input-output independent**, if **any** selection of the maximal τ_r number of independent inputs does not affect the selection of the maximal number τ_l of independent outputs and vice versa; otherwise, it is called **input-output dependent**.
(ii) It is called **input-output regularisable,** if for at least a $\beta\in\Omega_F$ there is a $\gamma\in\Omega_G$ such that the subsystem $S_{\beta,\gamma}(A,B,C,D)$ is input, output regular; otherwise, it is called **input-output non-regularisable**.

Proposition (4.1): The system S(A,B,C,D) is input-output independent if at least one of the following conditions holds true: $\operatorname{rank}[C,D]=\operatorname{rank}[C]$ and/or $\operatorname{rank}[B^t,D^t]=\operatorname{rank}[B^t]$.

Remark (4.4): A strictly proper system S(A,B,C) is always an input-output independent system. Furthermore, any input-output independent system is always input-output regularisable.

Remark (4.5): For a model S(A,B,C,D) the maximal number of inputs and outputs required for input and output regularity is τ_r, τ_l respectively. These values can always be achieved for input-output independent systems, but not necessarily for the case of input-output dependent, where they act as upper bounds.

The problem of determining the maximal values of cardinality of inputs, outputs, as well as the parameterisation of the corresponding family of systems is considered below in an algorithmic manner. The overall family of such systems will be denoted by $\langle f\rangle$ and every subfamily, with (r',q') input, output cardinality (which is input-output regular) will be denoted by $\langle f\rangle_{r',q'}$. $\langle f\rangle$ will be referred to as the **input-output regular** family and can always be partitioned as a union of subsets with different indices (r',q').

Searching Algorithm for determining the input-output regular family <f>:

Consider the progenitor model S(A,B,C,D) and let $\tau_r = \operatorname{rank}\ [B^t, D^t] = \tilde{r}$, $\tau_l = \operatorname{rank}[C,D]=\tilde{q}$ and assume for simplicity that $\tilde{r}<\tilde{q}$. Defining $\langle f\rangle$ and the corresponding indices (r',q') involves the following:

CASE (I): Input – Output Independent Systems

For this case the maximal cardinality is $(\tilde{r}, \tilde{q})$ and the family of $\langle f \rangle_{\tilde{r}, \tilde{q}}$ systems is constructed as:

Maximal Cardinality Family: Consider the sets of indices $\Omega_F = \{\beta = (j_1, ..., j_{\tilde{r}})\}$, $\Omega_G = \{\gamma = (l_1, ..., l_{\tilde{q}})\}$. If B_β, C_γ, $D_{\beta,\gamma}$ denote the submatrices corresponding to these indices then for $\forall \beta \in \Omega_F$ and $\forall \gamma \in \Omega_G$ the subsystem S(A, B_β, C_γ, $D_{\beta,\gamma}$) is a maximal cardinality $(\tilde{r}, \tilde{q})$ input-output regular subsystem.

CASE (II): Input – Output Dependent Systems

For this case the search involves a number of steps:
STEP (1): For all $\beta \in \Omega_F$ define the submatrices D_β corresponding to the set β of columns, q_β = rank [C, D_β], and let $q_1 = \max\{q_\beta, \forall \beta \in \Omega_F\}$.
(a) $\underline{q_1 = \tilde{q}}$: Then the search stops and the maximal number of inputs, outputs that guarantee regularity is $(\tilde{r}, \tilde{q})$ and the system is input-output regularisable. The parameterisation of the family is done as follows:

Maximal Cardinality Family: Let Ω'_F be the subset of sequences of Ω_F for which $q_\beta = \tilde{q}$. For every such $\beta \in \Omega'_F$ we shall denote by $\{\gamma(\beta)\}$ all sequences in Ω_G, which correspond to natural bases of G row space. Thus, we define the set of sequences $\Omega_{F,G} \overset{\Delta}{=} \{(\beta, \gamma) \forall \beta \in \Omega'_F \text{ and } \gamma \in \gamma\{\beta\}\}$ and for all $(\beta, \gamma) \in \Omega_{F,G}$ the maximal cardinality $(\tilde{r}, \tilde{q})$ regular family is defined by S(A, B_β, C_γ, $D_{\beta,\gamma}$).
(b) $\underline{q_1 < \tilde{q}}$: Then the system is not input, output regularisable and $(\tilde{r}, q_1)$ is a maximal number of inputs solution. The corresponding family of solutions with $(\tilde{r}, q_1 < \tilde{q})$ cardinality is constructed as before.

If a reduced input cardinality and increased output cardinality is desirable, then we proceed to the following step.

STEP (2): For the matrix F, define all sets of $\tilde{r} - 1$ independent vectors of the columns of F (lexicographically ordered, denote this set by $\{F\}_1$ and let the corresponding set of indices be $\Omega^1_F = \{\beta^1 : \beta^1 = (j_1, ..., j_{\tilde{r}-1})\}$. For the set Ω^1_F repeat STEP (1) and this leads to a new solution pair $(\tilde{r} - 1, q_2)$ where $q_2 \geq q_1$. The construction of the corresponding family of subsystems follows along the lines described in STEP (1).

The above algorithmic procedure defines the maximal cardinality for input, output regularity, as well as producing a parameterisation of $\langle f \rangle_{\tilde{r}, \tilde{q}}$ family, as well as families with orders less than $(\tilde{r}, \tilde{q})$. We can now proceed to the description of the overall methodology for well-conditioning using the Indirect Method.

Indirect Method for Well-conditioning

For the system S(A, B, C, D) we define the maximal cardinality pair $(\tilde{r}, \tilde{q})$ for which input, output regularity is guaranteed and let $\langle f \rangle_{\tilde{r}, \tilde{q}}$ be the corresponding family of input, output regular models parameterised by pairs of sequences $(\beta, \gamma) \in \Omega_{F,G}$ with $\beta = (j_1, ..., j_{\tilde{r}})$, $\gamma = (l_1, ..., l_{\tilde{q}})$. The general element of this family is denoted by $S_{\beta,\gamma} \overset{\Delta}{=} S(A, B_\beta, C_\gamma, D_{\beta,\gamma})$. For each $S_{\beta,\gamma}$ we proceed with testing as follows:

STEP (1): If $D_{\beta,\gamma} \neq 0$ and rank $(D_{\beta,\gamma})$ = min $(\tilde{r}, \tilde{q})$ or $\tilde{D}_{\beta,\gamma} = 0$ and $\text{rank}(C_\gamma B_\beta) = \min(\tilde{r}, \tilde{q})$, then system is non-degenerate and search stops.
STEP (2): If $D_{\beta,\gamma} \neq 0$ and rank $(D_{\beta,\gamma})$ < min $(\tilde{r}, \tilde{q})$, or $\tilde{D}_{\beta,\gamma} = 0$ and $\text{rank}(C_\gamma B_\beta) < \min(\tilde{r}, \tilde{q})$, then test for full rank of the Toeplitz matrix M_τ (Theorem (3.1)), or respectively Toeplitz matrix $\tilde{M}_{\tau'}$ (Corollary (3.1)). If M_τ, $\tilde{M}_{\tau'}$ are full rank, then the system is nondegenerate and the search stops. Otherwise, the system is degenerate and we proceed to the testing of another $S_{\beta,\gamma}$ subsystem.
STEP (3): If all elements of $\langle f \rangle_{\tilde{r}, \tilde{q}}$ have been tested for degeneracy and there is no element, which is nondegenerate, repeat the analysis of Steps (1), (2) for the smaller order family $\langle f \rangle_{\tilde{r}-1, \tilde{q}}$ etc. The overall procedure always leads to a nondegenerate system.

The system of $(\tilde{r}, \tilde{q})$-maximal cardinality subsystems, which are input-output regular and nondegenerate, will be denoted by $\langle f\rangle^{o}{}_{\tilde{r},\tilde{q}}$ and $\Psi_{F,G}$ will denote the corresponding pairs of (β, γ) sequences.

5. CONCLUSIONS

The problem of selecting subsystems of a progenitor model S(A,B,C,D), or H(s), which have maximal input and output cardinality, are input-output regular and are nondegenerate has been considered in detail. We have given criteria for the presence of input, output redundancy and system degeneracy, and suggested procedures for how we can avoid such properties. The results lead to parameterisation of all subsystems, which are input-output regular and nondegenerate and have maximal cardinality $(\tilde{r},\tilde{q})$, and leads to the family $\langle f\rangle^{o}{}_{\tilde{r},\tilde{q}}$. Every system in $\langle f\rangle^{o}{}_{\tilde{r},\tilde{q}}$ has $\tilde{r}$-inputs and $\tilde{q}$-outputs and it is parameterised by a set of sequences (β, γ) ∈ $\Psi_{F,G}$ defining the subsets of inputs and outputs that has to be considered. Every element $S(A,B_{\beta},C_{\gamma},D_{\beta,\gamma})$ ∈ $\langle f\rangle^{o}{}_{\tilde{r},\tilde{q}}$ does not necessarily have a structure that is desirable, as far as other properties. In fact, $S_{\beta,\gamma}$ may be either uncontrollable, and/or unobservable and other properties may not hold true. This family $\langle f\rangle^{o}{}_{\tilde{r},\tilde{q}}$ may then be used as the starting point for additional investigations.

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