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Spectral Characterization of d.z. of Linear Systems

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Abstract

The classical result of Gilbert on the testing of controllability based on the Jordan canonical description is extended here by providing a new characterization of input decoupling zeros based on the properties of appropriate Piecewise Arithmetic Progression Sequences defined on spectral matrices determined from the Jordan canonical description. For any eigenvalue for which there is loss of modal controllability the degrees of the corresponding decoupling zeros are defined. The results given here for input decoupling zeros have their equivalent statement for the case of output decoupling zeros.

1 INTRODUCTION

The classical results on the spectral characterization of controllability and observability [1], [2] provide tests for controllability and observability of every mode; however, in case where we have uncontrollability and/ or unobservability, the structure of the emerging decoupling zeros [3] is not predicted by the existing tests. The only available means for determining the structure of decoupling zeros is to resort to the computation of the corresponding Smith forms, or Kronecker forms [3], [5] of the appropriate matrix pencils [4]. The characterization of elementary divisors of matrix pencils in terms of properties of sequences derived from the Kernel properties of Toeplitz matrices has been introduced in [6]; these results provide a number theoretic characterization of divisors and they are used here to extend the spectral theory of controllability, observability to the characterization of decoupling zeros.

In this paper, the classical spectral analysis for controllability are extended to characterize the set of decoupling zeros. This is achieved by introducing a set of spectral matrices, the rank properties of which define a special sequence, which is shown to be a Piecewise Arithmetic Progression [6]; such sequences have singular points and the structure of such singular points define the Segré characteristic of the corresponding decoupling zeros. The results are presented for the case of controllability, i.e. for the input decoupling zeros and may be easily translated by duality to the case of output decoupling zeros. The current results compliment those in [7] on the spectral analysis of controllable, observable spaces and provide a unifying treatment of the spectral analysis of the controllability and observability properties.

2 PROBLEM STATEMENT AND BACKGROUND RESULTS

We consider the linear system described by the state space model $S(A, B, C, D)$:

$$\underline{\dot{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \tag{1}$$

$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t) \tag{2}$$

Where $A \in R^{nxn}, B \in R^{nxl}, C \in R^{mxn}, D \in R^{lxm}$ and $u(t)$ is the $l \times 1$ input vector, $\underline{y}(t)$ is the $m \times 1$ output vector and $\underline{x}(t)$ is the $n \times 1$ state variable vector. If the characteristic polynomial of a matrix $A \in \mathbb{R}^{n\times n}$ is:

$$\phi(s) = \det(A - sI) = (s-\lambda_1)^{\pi_1}(s-\lambda_2)^{\pi_2}\ldots(s-\lambda_f)^{\pi_f} \tag{3}$$

Where $\lambda_1, \lambda_2, ..., \lambda_f \in \mathbb{C}$ are all the distinct eigenvalues of A and $\pi_1, \pi_2, ..., \pi_f$ are their corresponding algebraic multiplicities, with $\pi_1 + \pi_2 + ... + \pi_f = n$.

Definition 1 *We define as the Segré Characteristic of A at λ_i, the set of the degrees of the elementary divisors of A at λ_i: $\wp_{\lambda_i}(A) = \{\tau_{i\nu_i} \geq ... \geq \tau_{ik} \geq ... \geq \tau_{i1} > 0\}$.* □

If $U = V^{-1}$ is the matrix defined by the chains of eigenvectors of A, then the matrix A is similar to the Jordan matrix $J : A = UJU^{-1} = UJV$, where,

$$J = \text{block diag}\{J(\lambda_1), J(\lambda_2), ..., J(\lambda_i), ..., J(\lambda_f)\} \tag{4}$$

and $J(\lambda_i)$ is the block diagonal matrix of all Jordan blocks associated with the distinct eigenvalue λ_i :

$$J(\lambda_i) = \text{block diag}\{J_{i1}, ..., J_{ik}, ...J_{i\nu_i}\} \qquad (5)$$

and where J_{ik} is the $\tau_{ik} \times \tau_{ik}$ Jordan diagonal block.

The Jordan canonical description of the system $S_J(J, \mathcal{B}, \Gamma, \Delta)$ is given by the equations

$$\underline{\dot{z}}(t) = J\underline{z}(t) + \mathcal{B}\underline{u}(t) \qquad (6)$$

$$\underline{y}(t) = \Gamma\underline{z}(t) + \Delta\underline{u}(t) \qquad (7)$$

where :$\underline{z}(t) = U\underline{x}(t)$, $J = U^{-1}AU = VAU$, $\mathcal{B} = U^{-1}B$, $\Gamma = CU$, $\Delta = D$. From the block diagonal structure of J it follows that the $\mathcal{B}$, Γ, matrices have the following form:

$$\mathcal{B} = \begin{bmatrix} V(\lambda_1) \\ ... \\ V(\lambda_i) \\ ... \\ V(\lambda_f) \end{bmatrix} B = \begin{bmatrix} \mathcal{B}_1 \\ ... \\ \mathcal{B}_i \\ ... \\ \mathcal{B}_f \end{bmatrix}, \mathcal{B}_i = \begin{bmatrix} V_{i1} \\ ... \\ V_{ik} \\ ... \\ V_{i\nu_i} \end{bmatrix} B = \begin{bmatrix} \mathcal{B}_{i1} \\ ... \\ \mathcal{B}_{ik} \\ ... \\ \mathcal{B}_{i\nu_i} \end{bmatrix} \qquad (8)$$

whereas

$$\Gamma = C[U(\lambda_1)...U(\lambda_i)...U(\lambda_f)] = f[\Gamma_1...\Gamma_i...\Gamma_f],$$
$$\Gamma_i = C[U_{i1}...U_{ik}...U_{i\nu_i}] = [\Gamma_{i1}...\Gamma_{ik}...\Gamma_{i\nu_i}] \qquad (9)$$

and

$$\mathcal{B}_{ik} = \begin{bmatrix} \underline{v}_{ik_1}^\top \\ \underline{v}_{ik_2}^\top \\ ... \\ \underline{v}_{ik_{\tau_{ik}}}^\top \end{bmatrix} B = \begin{bmatrix} \underline{\beta}_{ik_1}^\top \\ \underline{\beta}_{ik_2}^\top \\ ... \\ \underline{\beta}_{ik_{\tau_{ik}}}^\top \end{bmatrix} \qquad (10)$$

$$\Gamma_{ik} = C\left[\underline{u}_{ik_1}, \underline{u}_{ik_2}, ..., \underline{u}_{ik_{\tau_{ik}}}\right] = \left[\underline{\gamma}_{ik_1}, \underline{\gamma}_{ik_2}, ..., \underline{\gamma}_{ik_{\tau_{ik}}}\right] \qquad (11)$$

Definition 2 *(a) The i-th spectrum controllability matrix $\mathcal{B}_i^S$ is the $l \times \nu_i$ matrix formed by the ν_i rows of $\mathcal{B}_i$ corresponding to the last rows of the Jordan blocks associated with the eigenvalue λ_i:*

$$\mathcal{B}_i^S = \begin{bmatrix} \underline{\beta}_{i1_{\tau_{i1}}}^\top \\ ... \\ \underline{\beta}_{ik_{\tau_{ik}}}^\top \\ ... \\ \underline{\beta}_{i\nu_{i,\tau_{i\nu_i}}}^\top \end{bmatrix} \qquad (12)$$

(b) The i-th spectrum observability matrix Γ_i^F is the $\nu_i \times l$ matrix formed by the ν_i rows of $\Gamma_i^\top$ corresponding to the first columns of the Jordan blocks associated with the eigenvalue λ_i:

$$\Gamma_i^F \triangleq \left[\underline{\gamma}_{i1_1}, ..., \underline{\gamma}_{ik_1}, ..., \underline{\gamma}_{i\nu_{i,1}}\right] \qquad (13)$$

□

Some basic result needed for the subsequent development are summarized below [2],[3],[8]:

Theorem 1 *(a) The mode $(\lambda_i, U(\lambda_i), V(\lambda_i))$ is controllable if and only if the rows of the i-th spectral controllability matrix $\mathcal{B}_i^S$ are linearly independent over the field of complex numbers.*

(b) The mode $(\lambda_i, U(\lambda_i), V(\lambda_i))$ for $i = 1, 2, ..., f$ is controllable, if and only if the rows of the pencil $[sI - A, B]$ are linearly independent over the field of complex numbers.

(c) Furthermore if the mode $(\lambda_i, U(\lambda_i), V(\lambda_i))$ is uncontrollable then it is, rank $[\lambda_i I - A, B] < n$ *and λ_i is an input decoupling zero of S.* □

The pencil $[sI - A, B]$ is defined as the input state pencil [4]. The roots of the e.d. of the pencil $[sI - A, B]$ are defined as the input decoupling zeros (i.d.z.) of the system S. The dual results for the case of observability follow by duality based on the pencil $\left[sI - A^\top, C^\top\right]^\top$; this is known as the output state pencil and the roots of the e.d. of the pencil are defined as the output decoupling zeros (o.d.z.) of the system S.

Under the transformation of the system $S(A, B)$ to the Jordan equivalent $S_J(J, \mathcal{B})$ the controllability matrix Q is also equivalent to the matrix Q_J i.e.: $Q \sim Q_J \triangleq \left[\mathcal{B}, J\mathcal{B}, ..., J^{n-1}\mathcal{B}\right]$ and Q_J is also partitioned according to (4) : $Q_J = [Q_{J_1}, ..., Q_{J_i}, ..., Q_{J_f}]^\top$ then from (8) and using only column operations on the above matrix we have,

$$Q_{J_i} \sim Q_{H_i} \triangleq \left[\mathcal{B}_i\, H_i\mathcal{B}_i \,...\, (H_i)^{n-1}\,\mathcal{B}_i\right] \qquad (14)$$

Remark 1 *For the above matrix Q_{H_i} we have the properties [7],[10],[11] that provide a spectral characterization of the controllable and by duality observable spaces of the system.* □

Let $\theta_{i1}, \theta_{i2}, ..., \theta_{i\nu_i}$ be the numbers denoting the orders of the above defined rows into each one block of Q_{H_i} and let the blocks be rearranged from top to bottom in a way such that : $\theta_{i1} \geq \theta_{i2} \geq ... \geq \theta_{i\nu_i} \geq 0$.

Definition 3 *The set of the above numbers is defined as the set of the i-th spectrum row controllability indices (r.c.i.) of A, B : $\Theta(A, B)_{\lambda_i} = \{\theta_{i1} \geq \theta_{i2} \geq ... \geq \theta_{i\nu_i} \geq 0\}$.* □

The spectral results in [7],[10] provide a characterization of the modal controllability, observability and indicate the existence of i.d.z., o.d.z.; however, such results do not predict the degrees of the decoupling zeros which may exist at $s = \lambda_i$. The problem we consider here is the use of the spectral matrices defined for every mode (eigenvalue) to compute the Segré characteristics of the decoupling zeros. This is enabled by using results on the characterization of elementary divisors of matrix pencils by the properties of sequences defined by the Kernels of Toeplitz matrices. The characterization of e.d. of the matrix pencils based on the properties of Piecewise Arithmetic Progression Sequences (PAPS) [6] is summarised next.

3 SEQUENCE CHARACTERIZATION OF ELEMENTARY DIVISORS OF MATRIX PENCILS

We consider the set of ordered pairs of matrices (F,G) and the associated pencils denoted as : $\mathcal{L}_{p,m} \triangleq W = [(F,G) : F,G \in \mathbb{R}^{p\times m}]$ and $\mathcal{L}_{p,m}(s,w) \triangleq [W(s,w) = sF - wG,\ W = (F,G) \in \mathcal{L}_{p,m}]$ where (s,w) is an ordered pair of indeterminate. The pair $W = (F,G)$ is called right regular, if $\mathcal{N}_{\mathrm{r}\mathbb{R}(s,w)}(sF - wG) = \{0\}$. The subset of $\mathcal{L}_{p,m}$ which is made up from all right regular pairs will be denoted by $\mathcal{L}^{\mathrm{rr}}_{p,m}$ and the corresponding set of pencils will be denoted by $\mathcal{L}^{\mathrm{rr}}_{p,m}(s,w)$. The set $\mathcal{L}^{\mathrm{lr}}_{p,m}$ of all left regular pairs is defined in a similar manner. It is clear that a necessary and sufficient condition for $W \in \mathcal{L}^{\mathrm{rr}}_{p,m}$ is that $W_{p,m}(s,w)$ has full rank over $\mathbb{R}(s,w)$ and $p \geq m$. We use the notation for elementary divisors (e.d.) and associated Segré characteristic for the pencil $sF - G$ at $s = a$ in the same way, as done for the structure of eigenvalues (e.d. of $sI - A$ pencil). The characterization of e.d. of $sF - G$ (right or left regular) is described by the following results [6],[9]:

Theorem 2 *Let $W = (G,F) \in \mathcal{L}^{\mathrm{rr}}_{p,m}(p \geq m)$. The pencil $W_{p,m}(s) = sF - G$ has an e.d. $(s-a)^{r_i}$, $a \in \mathbb{C}$, if and only if there exists a maximal chain of linearly independent vectors $\{\underline{x}_1, \underline{x}_2, ..., \underline{x}_{r_i}\} \in \mathbb{C}^p$ such that*

$$[G - aF]\,\underline{x}_j = F\underline{x}_{j-1},\ \underline{x}_0 = 0,\ j = 1,2,...,r_i$$

□

The above motivates the definition of the following sequence of Toeplitz matrices, defined for $\forall a \in \mathbb{C}$:

$$T^1_a \triangleq G - aF$$

$$\cdots\cdots\cdots \tag{15}$$

$$T^i_a \triangleq \begin{bmatrix} G-aF & 0 & \dots 0 & 0 \\ F & G-aF & \dots 0 & 0 \\ \cdot & \cdot & \cdots\cdot & \cdot \\ \cdot & \cdot & \cdots\cdot & \cdot \\ \cdot & \cdot & \cdots\cdot & \cdot \\ 0 & 0 & \dots F & G-aF \end{bmatrix} \in \mathbb{C}^{ip\times im},$$

for all $i = 1,2,...$

The matrices T^i_a, are referred to as *i-th* order *a*-Toeplitz matrices of $(G,F,)$ and we denote by :

$$N^k_a \triangleq \mathcal{N}_{\mathrm{r}}\left\{T^k_a\right\},\quad \tilde{N}^k_a \triangleq \mathcal{N}_{\mathrm{l}}\{T^k_a\},\ \forall a \in \mathbb{C},\ k = 1,2,... \tag{16}$$

For all the pairs $W = (G,F)$ and $b \in \mathbb{C} \cup \{\infty\}$ we define the sequences,

$$J^{\mathrm{r}}_b(G,F) \triangleq \{\eta^b_k : \eta^b_0 = 0, \eta^b_k = \dim N^k_b; k \geq 1\} \tag{17}$$
$$J^{\mathrm{l}}_b(G,F) \triangleq \{\vartheta^b_k : \vartheta^b_0 = 0, \vartheta^b_k = \dim \tilde{N}^k_b; k \geq 1\} \tag{18}$$

$J^{\mathrm{r}}_b(G,F)$, $J^{\mathrm{l}}_b(G,F)$ will be referred to as the right b-(G,F), left b-(G,F)-sequence of the pair (G,F). A sequence $J^{\mathrm{r}}_b(G,F)$, $J^{\mathrm{l}}_b(G,F)$ will be called neutral, if its elements are zero for all k : $k = 1,2,....$ We shall denote by $\Phi(F,G)$ the set of distinct zeros of the pencil.

Theorem 3 *[6]Let $(G,F) \in \mathcal{L}^{\mathrm{rr}}_{p,m}, a \in \Phi(G,F)$, Segré characteristic $\wp_a(G,F) = \{(d_i,\sigma_i), i \in \boldsymbol{\rho},\ 0 < d_1 < ... < d\rho$ and $\wp_a(G,F) = \{d_i : 0 < d_1 < ... < d\rho\}$. The sequence $J^{\mathrm{r}}_a(G,F)$ is nondecreasing and satisfies the following condition $nk \geq (n_{k-1} + n_{k+1})/2$ $k = 1,2,....$ In particular, we have that:*

(a) Strict inequality holds, if and only if $k = d_i \in \wp_a(G,F)$; in this case $\delta_k = 2n_k - n_{k-1} - n_{k+1} = \sigma_i$.

(b) Equality holds, if and only if $k \notin \wp_a(G,F)$. □

The sequence $J^{\mathrm{r}}_a(G,F)$ is known as a Piecewise Arithmetic Progression (PAP) sequence and its properties define $\wp_a(G,F)$ by deploying Ferrer's type diagrams [6]. Given that $\wp_a(G,F)$ is bounded we may define for every $a \in \Phi(G,F)$, the smallest integer τ_a for which $n_{\tau_a} = n_{\tau_a+1}, n_k \in \wp_a(G,F)$; this is called the index of annihilation of (G,F) at $s = a$.

Remark 2 *For every $a \in \Phi(G,F)$, $\tau_a = d\rho$, where $d\rho = \max\{d_i, d_i \in \wp_a(G,F)$, where $\wp_a(G,F)$ denotes all distinct degrees of e.d. at $s = a$.* □

Definition 4 *The set of the first non-zero successive differences in $J^r_a(G,F)$ is defined as the Weyr characteristic of (G,F) at a and it is denoted by $\mathcal{W}_a$. Clearly is given by :*

$$\mathcal{W}_a \triangleq \{\gamma_1 = \eta_1 - \eta_0,\ \gamma_2 = \eta_2 - \eta_1, ...,\ \gamma_k = \eta_k - \eta_{k-1}\}$$

□

4 SPECTRAL CHARACTERIZATION OF INPUT DECOUPLING ZEROS

The result of the previous section, together with the Jordan description of the state equations are used next for the derivation of a new characterization of i.d.z. Let the input state pencil of the equivalent system in Jordan form $S(J,\mathcal{B})$ be, $[sI-J,\mathcal{B}]=s[I,0]-[J,-\mathcal{B}]\in\mathcal{L}^{\mathrm{lr}}_{n,n+l}$. Consequently the structure of i.d.z. of the system $S(A,B)$ is determined equivalently by the root range of the input state pencil. For $S(J,\mathcal{B})$ and a i.d.z. we define :

$$T_a^1=\begin{bmatrix}J-aI\;\mathcal{B}\end{bmatrix}\in\mathbb{C}^{n\times(n+l)},$$
$$\dots\dots\dots\dots\dots\dots \tag{19}$$
$$T_a^j=\begin{bmatrix}J-aI\,\mathcal{B} & I & 0\,0\dots & 0 & 0\\ 0 & 0\,J-aI\,\mathcal{B}\,I\dots & & 0 & 0\\ . & . & . & \dots & .\\ 0 & 0 & 0 & 0\,0\dots\,J-aI\,\mathcal{B}\end{bmatrix}\in\mathbb{C}^{jn\times j(n+l)}$$

The properties of the above sequence will be considered next and the proof of the results is given in [11].

Proposition 1 *For* $\forall a\in\mathbb{C}:a\notin\Phi(A)$ *the matrix* T_a^j *has full rank.* □

Proposition 2 *Let* $a=\lambda_i\in\Phi(A)$ *and express,* $T^1_{\lambda_i}=\begin{bmatrix}H_i & 0 & \mathcal{B}_i\\ 0 & T' & \mathcal{B}'\end{bmatrix}$ *where* $H_i=J(\lambda_i)-\lambda_iI\in\mathbb{R}^{\pi_i\times\pi_i}$ *is nilpotent,* $T'\in\mathbb{C}^{(n-\pi_i)\times(n-\pi_i)}$ *is full rank,* $\mathcal{B}_i\in\mathbb{C}^{\pi_i\times l}$ *is (as defined in (8)) the matrix block of* $\mathcal{B}$ *corresponding to* $J(\lambda_i)$*. Then the left nullity of the matrix* $T^j_{\lambda_i}$ *is defined by the left nullity of the matrix* $\check{T}^j_{\lambda_i}$*, where,*

$$\check{T}^j_{\lambda_i}\triangleq\left.\begin{bmatrix}H_i\,\mathcal{B}_i & I & 0\,0\dots & 0 & 0\\ 0\;\;0 & H_i\,\mathcal{B}_i\,I\dots & & 0 & 0\\ . & . & .\dots & . & .\\ 0\;\;0 & 0 & 0\,0\dots & I & 0\\ 0\;\;0 & 0 & 0\,0\dots & H_i & \mathcal{B}_i\end{bmatrix}\right\}j\text{-blocks} \tag{20}$$

□

Remark 3 *From the above we conclude that only the numbers* $\lambda_i\in\Phi(A)$ *are candidate for i.d.z. We can study the sequence of left nullities and the corresponding tests, for determining the degree of i.d.z. by considering the case of each eigenvalue* λ_i *for which* $\check{T}^j_{\lambda_i}$ *is rank deficient.* □

From the above we conclude that any $\lambda_i\in\Phi(A)$ is a candidate decoupling zero. If the A-Segré Characteristic at λ_i is $\wp_{\lambda_i}(A)$, then the matrix $\mathcal{B}_i$ can be partitioned according to $\wp_{\lambda_i}(A)$ as in (8) and the nilpotent matrix H_i can also be represented according to the eigenstructure of A at λ_i.

Definition 5 *The* t-th *reduced matrix of* $\mathcal{B}^{t)}_{ik}$ *is defined as the matrix derived from* $\mathcal{B}_{ik}$ *as indicated below:*

$$\mathcal{B}^{t)}_{ik}=\begin{bmatrix}0\\ \dots\\ 0\\ \underline{\beta}^{\top}_{ik_t}\\ \dots\\ \underline{\beta}^{\top}_{ik_{\tau_{ik}}}\end{bmatrix}\in\mathbb{C}^{\tau_{ik}\times l},\; t=1,2,\dots,\tau_{ik} \tag{21}$$

and where $\mathcal{B}^{t)}_{ik}\triangleq\mathcal{B}_{ik}$ *for* $\forall t=0,-1,-2,\dots$ □

The same notation can be applied to any other matrix. Thus if I_{ik} is the $\tau_{ik}\times\tau_{ik}$ identity matrix, then $I^{t)}_{ik}$ is also a $\tau_{ik}\times\tau_{ik}$ matrix,

$$I^{t)}_{ik}=\begin{bmatrix}0\dots000\dots0\\ .\dots\dots\dots.\\ 0\dots000\dots0\\ 0\dots010\dots0\\ .\dots\dots\dots.\\ 0\dots000\dots1\end{bmatrix}\longleftarrow t\text{-}th\text{ row} \tag{22}$$

For any $p\geq 1,\; p\in\mathbb{Z}$ we define the $*$ operation on $\mathcal{B}_{ik}\in\mathbb{C}^{\tau_{ik}\times l}$ by :

$$p*\mathcal{B}_{ik}\triangleq\mathcal{B}^{*p}_{ik}\triangleq\mathcal{B}^{\tau_{ik}+1-p)}_{ik},\; p=1,2,\dots \tag{23}$$

Let $\mathcal{B}_i$ be partitioned into blocks as in (8), then we define the $*$ operation on $\mathcal{B}_i$ by some $p\in\mathbb{Z}$ as :

$$p*\mathcal{B}_i\triangleq\mathcal{B}^{*p}_i\triangleq\begin{bmatrix}p*\mathcal{B}_{i1}\\ \dots\\ p*\mathcal{B}_{ik}\\ \dots\\ p*\mathcal{B}_{i\nu_i}\end{bmatrix}=\begin{bmatrix}\mathcal{B}^{*p}_{i1}\\ \dots\\ \mathcal{B}^{*p}_{ik}\\ \dots\\ \mathcal{B}^{*p}_{i\nu_i}\end{bmatrix}=\begin{bmatrix}\mathcal{B}^{\tau_{i1}+1-p)}_{i1}\\ \dots\\ \mathcal{B}^{\tau_{ik}+1-p)}_{ik}\\ \dots\\ \mathcal{B}^{\tau_{i\nu_i}+1-p)}_{i\nu_i}\end{bmatrix} \tag{24}$$

Using the above notation we may simplify the computation of nullities of $\check{T}^j_{\lambda_i}$ using simpler matrices.

Proposition 3 *The above defined* j-th *left Toeplitz matrix* $\check{T}^j_{\lambda_i}$ *is equivalent over* $\mathbb{C}$ *by elementary column operations to the following form :*

$$\tilde{T}^j_{\lambda_i}\triangleq\left.\begin{bmatrix}H_i\,\mathcal{B}^{*1}_i\,I^{*1}_i & 0 & 0\,0\dots & 0 & 0\\ 0\;\;0\;\;H_i & \mathcal{B}^{*2}_i & I^{*2}_i\,0\dots & 0 & 0\\ .\;\;.\;\;. & . & .\;.\dots & . & .\\ 0\;\;0\;\;0 & 0 & 0\,0\dots & I^{*(j-1)}_i & 0\\ 0\;\;0\;\;0 & 0 & 0\,0\dots & H_i & \mathcal{B}^{*j}_i\end{bmatrix}\right\}j\text{-blocks}$$

□

Remark 4 *The form* $\tilde{T}^j_{\lambda_i}$ *is column equivalent to* $T^j_{\lambda_i}$ *and* $\check{T}^j_{\lambda_i}$ *and the left null-space of* $\check{T}^j_{\lambda_i}$ *may be studied by using* $\tilde{T}^j_{\lambda_i}$ *since the two are column space equivalent.* □

Proposition 4 *Let $\underline{y} \in \mathbb{C}^{jn}$ and be partitioned as, $\underline{y}^\top = [\underline{y}_1^\top, \underline{y}_2^\top, ..., \underline{y}_{j-1}^\top, \underline{y}_j^\top]$ then $\underline{y}^\top \in \mathcal{N}_l\{\check{T}_{\lambda_i}^j\}$ where $\check{T}_{\lambda_i}^j \in \mathbb{C}^{jn \times j(n+l)}$ if and only if the following conditions are satisfied,*

$$\left\{\begin{array}{l} \underline{y}_1^\top H_i = \underline{0} \\ \underline{y}_2^\top H_i = -\underline{y}_1^\top I_i^{*1} \\ \quad \cdots \\ \underline{y}_j^\top H_i = -\underline{y}_{j-1}^\top I_i^{*(j-1)} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} \underline{y}_1^\top B_i^{*1} = 0 \\ \underline{y}_2^\top B_i^{*2} = 0 \\ \quad \cdots \\ \underline{y}_j^\top B_i^{*j} = 0 \end{array}\right. \tag{25}$$

□

The first set of equations (25) are referred to as the *left recurrent equations* and the second are called the *left Kernel equations.*

Remark 5 *Let $\underline{y}_i^\top$ be partitioned according to Segré characteristic (Definition 1) as, $\underline{y}_i^\top = [\underline{y}_{\tau_{i1}}^\top, ..., \underline{y}_{\tau_{ik}}^\top, ..., \underline{y}_{\tau_{i\nu_i}}^\top]$, $i = 1, 2, ..., f$ and from the block diagonal structure of H_i and I_i^{*j} we have that the set of the recurrent equations is equivalent to,*

$$\left\{\begin{array}{ll} \underline{y}_{\tau_{ik}}^{\top 1} H_{ik} = \underline{0}^\top & (1) \\ \underline{y}_{\tau_{ik}}^{\top 2} H_{ik} = -\underline{y}_{\tau_{ik}}^{\top 1} I_{ik}^{*1} & (2) \\ \quad \cdots\cdots\cdots & \cdots \\ \underline{y}_{\tau_{ik}}^{\top j} H_{ik} = -\underline{y}_{\tau_{ik}}^{\top j-1} I_{ik}^{*j-1} & (\mathrm{j}) \end{array}\right. \tag{26}$$

where τ_{ik} takes values from the set of $\wp_{\lambda_i}(A)$. Equations (26) will be called the basic recurrent equations. □

Lemma 1 *For any $\tau_{ik} \geq 1$ the solution of the basic recurrent equation (26) is given by :*

(a) for $j \leq \tau_{ik}$:

$$\underline{y}_{\tau_{ik}}^{\top j} = [\underbrace{0, ..., 0}_{\tau_{ik}-j}, (-1)^{j-1} c_{\tau_{ik}}^1, (-1)^{j-2} c_{\tau_{ik}}^2, ... \\ ..., -c_{\tau_{ik}}^{j-1}, c_{\tau_{ik}}^j] \tag{27}$$

where $c_{\tau_{ik}}^1, c_{\tau_{ik}}^2, ..., c_{\tau_{ik}}^j$ arbitrary,

(b) for $j > \tau_{ik}$:

$$\left\{\begin{array}{l} \underline{y}_{\tau_{ik}}^{\top 1} = \underline{y}_{\tau_{ik}}^{\top 2} = ... = \underline{y}_{\tau_{ik}}^{\top j-\tau_{ik}} = \underline{0}^\top \\ \underline{y}_{\tau_{ik}}^{\top j-\tau_{ik}+1} = [0, ...0, c_{\tau_{ik}}^{j-\tau_{ik}+1}] \\ \cdots\cdots\cdots\cdots \\ \underline{y}_{\tau_{ik}}^{\top j} = [(-1)^{\tau_{ik}-1} c_{\tau_{ik}}^{j-\tau_{ik}+1}, ..., -c_{\tau_{ik}}^{j-1}, c_{\tau_{ik}}^j] \end{array}\right. \tag{28}$$

where $c_{\tau_{ik}}^{j-\tau_{ik}+1}, ..., c_{\tau_{ik}}^j$ arbitrary. □

From the above it is clear that, we may define for every $s = \lambda_i$ $\lambda_i \in \Phi(A)$ sets of matrices corresponding to the partition of $\mathcal{B}$ to $\mathcal{B}_i$ and then $\mathcal{B}_{ik}$ blocks, as shown before. In fact:

Definition 6 *The j-th input spectral Toeplitz matrix is defined in a row block partition form as shown below, $Q_{\lambda_i}^j \triangleq \left[Q_{\tau_1}^j, ..., Q_{\tau_{ik}}^j, ..., Q_{\tau_{i\nu_i}}^j\right]^\top$ where:*

(a) For $\forall j \leq \tau_{i\nu_i}$:

a.1 if $j \leq \tau_{ik}$,

$$Q_{\tau_{ik}}^j \triangleq \begin{bmatrix} \underline{\beta}_{\tau_{ik}+1-j,\tau_{ik}}^\top & \cdots & \underline{\beta}_{\tau_{ik}-1,\tau_{ik}}^\top & \underline{\beta}_{\tau_{ik},\tau_{ik}}^\top \\ \underline{\beta}_{\tau_{ik}-j,\tau_{ik}}^\top & \cdots & \underline{\beta}_{\tau_{ik},\tau_{ik}}^\top & 0 \\ . & \cdots & . & . \\ \underline{\beta}_{\tau_{ik}-1,\tau_{ik}}^\top & \cdots & 0 & 0 \\ \underline{\beta}_{\tau_{ik},\tau_{ik}}^\top & \cdots & 0 & 0 \end{bmatrix} \tag{29}$$

a.2 if $j > \tau_{ik}$,

$$Q_{\tau_{ik}}^j \triangleq \begin{bmatrix} \underline{\beta}_{1,\tau_{ik}}^\top & \cdots & \underline{\beta}_{\tau_{ik},\tau_{ik}}^\top & 0 \ldots 0 \\ \underline{\beta}_{2,\tau_{ik}}^\top & \cdots & 0 & 0 \ldots 0 \\ . & \cdots & . & . \ \ldots \ . \\ \underline{\beta}_{\tau_{ik}-1,\tau_{ik}}^\top & \cdots & 0 & 0 \ldots 0 \\ \underline{\beta}_{\tau_{ik},\tau_{ik}}^\top & \cdots & 0 & 0 \ldots 0 \end{bmatrix} \tag{30}$$

(b) For $\forall k > \tau_{i\nu_i}$: $Q_{\tau_{ik}}^j = Q_{\tau_{ik}}^{\tau_{i\nu_i}}$ □

Proposition 5 *For any set of indices $\{\tau_{i\nu_i} \geq ... \geq \tau_{ik} \geq ... \geq \tau_{i1} > 0\}$ the solution of the set of equations (25) is determined by the vectors $\underline{c}_j^\top$, where,*

$$\underline{c}_j^\top Q_i^j = \underline{0} \tag{31}$$

□

Remark 6 *The degrees of freedom of the set of equations (25) are determined by $\dim \mathcal{N}_l \left\{Q_i^j\right\}$. Furthermore, for all $j > \tau_{i\nu_i}$: $Q_{\tau_{ik}}^j = Q_{\tau_{ik}}^{\tau_{i\nu_i}}$ there are no more degrees of freedom to the solution of equations (25).*□

Proposition 6 *Consider the system $S(A, B)$ with $\wp_{\lambda_i}(A) = \{\tau_{i\nu_i} \geq ... \geq \tau_{ik} \geq ... \geq \tau_{i1} > 0\}$ and let $S(J, \mathcal{B})$ be the corresponding Jordan normal description. If $T_{\lambda_i}^j$ is the j-th, λ_i-characteristic Toeplitz matrix of $S(J, \mathcal{B})$ and $Q_{\lambda_i}^j$ is the j-th input spectral matrix of the system, then, $\mathcal{N}_l \left\{T_{\lambda_i}^j\right\} = \mathcal{N}_l \left\{Q_{\lambda_i}^j\right\}$.* □

Definition 7 *Using the $Q_{\lambda_i}^j$, $j = 1, 2, ...$ input spectral matrices we define the λ_i-input spectral sequence as $J_{\lambda_i}^l \triangleq \{n_j^{\lambda_i} : n_0^{\lambda_i} = 0,\ n_j^{\lambda_i} = \dim \mathcal{N}_l(Q_{\lambda_i}^j); j \geq 1\}$.*□

Theorem 4 *The sequence $J_{\lambda_i}^l$ is piecewise arithmetic progression satisfying, the condition*

$$n_j^{\lambda_i} \geq \frac{n_{j-1}^{\lambda_i} + n_{j+1}^{\lambda_i}}{2}, \ j = 1, 2, ...$$

In particular we have that strict inequality holds if $j = \mu$ is the degree of an input decoupling zero of the $S(A, B)$ pair. In this case the multiplicity of the degree $j = \mu$ is, $\sigma = 2n_j^{\lambda_i} - n_{j-1}^{\lambda_i} - n_{j+1}^{\lambda_i}$. □

Some further results on the characterization of i.d.z. are given below.

Remark 7 *From the Definition 6 it is directly concluded that:*

(a) The matrix $Q^1_{\lambda_i}$ coincides with the i-th spectrum controllability matrix $\mathcal{B}_i^S$ defined in (12): $Q^1_{\lambda_i} \equiv \mathcal{B}_i^S$.

(b) The matrix $Q_i^{\tau_{i\nu_i}}$ coincides with the matrix Q_{H_i} defined in (14): $Q_i^{\tau_{i\nu_i}} \equiv Q_{H_i}$. □

Consider the set of r.c.i. $\Theta(A, B)_{\lambda_i}$. Let the set of $\Theta(A, B)_{\lambda_i}$ be rearranged such that the index $\acute{\theta}_{ik}$, $k = 1, 2, ..., \nu_i$ be the r.c.i. which corresponds to the block $\mathcal{B}_{ik}$ of $\mathcal{B}$. Then this is denoted as, $\Theta'(A, B)_{\lambda_i} = \{\acute{\theta}_{i1}, \acute{\theta}_{i2}, ..., \acute{\theta}_{ik}, ..., \acute{\theta}_{i\nu_i}\}$, $\acute{\theta}_{ik} \geq 0$, $k = 1, 2, ..., \nu_i$. Consider now the set of differences $\Sigma'(A, B)_{\lambda_i} \triangleq \left\{q'_{i_1}, q'_{i_2}, ..., q'_{ik}, ..., q'_{i_{\nu_i}}\right\}$ between the corresponding elements of the two sets, $\Theta'(A, B)_{\lambda_i}$ and the set of the Segré characteristic of A at λ_i, $\wp_{\lambda_i}(A)$:

$$\begin{aligned} &\tau_{i1} - \acute{\theta}_{i1} \triangleq q'_{i1}, \tau_{i2} - \acute{\theta}_{i2} \triangleq q'_{i2}, ... \\ &\tau_{ik} - \acute{\theta}_{ik} \triangleq q'_{ik}, \tau_{i\nu_i} - \acute{\theta}_{i\nu_i} \triangleq q'_{i\nu_i} \end{aligned} \tag{32}$$

and let $\Sigma(A, B)_{\lambda_i}$ be the set of the non zero values of the above differences, described by the ordered set of integers, $\Sigma(A, B)_{\lambda_i} \triangleq \{(\mu_i^j, \sigma_i^j); \mu_i^{s_i} \geq ... \geq \mu_i^2 \geq \mu_i^1 > 0\}$, $i = 1, 2, ..., f$. Where σ_i^j is the multiplicity of μ_i^j ($j = 1, 2..., s_i$). Then we have the following result :

Theorem 5 *The degrees of the input decoupling zeros of a system $S(A, B)$ at $s = \lambda_i$ are defined by the set of indices $\Sigma(A, B)_{\lambda_i}$ (or the $\Sigma'(A, B)_{\lambda_i}$).* □

Corollary 1 *The sum of the degrees of i.d.z. at $s = \lambda_i$ is given as,*

$$q'_{i_1} + q'_{i_2} + ... + q'_{ik} + ... + q'_{i_{\nu_i}} = \pi_i - r_i$$

where π_i is the algebraic multiplicity of λ_i and r_i is the dimension of the controllable subspace $\mathcal{R}_i$. □

5 CONCLUSIONS

The results of this section provide an extension of the classical spectral analysis for the characterization of controllability and observability results (Gilbert results [1]) to the characterization of degrees of divisors associated with the input and output decoupling zeros of a continuous system. The results presented here provide a unifying framework for the study of effects of sampling on the controllability and observability properties of a discretized model [10]. In fact under the special values where collapsing of eigenvalues of discretized model occurs, (irregular sampling), the current analysis provides means for studying effects such as emergence of new decoupling zeros and transformation of indices. The results presented here for the case of input decoupling zeros have their equivalent to the case of output decoupling zeros and the analysis follows along similar lines.

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