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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO A NONLINEAR NONLOCAL SECOND ORDER INITIAL-BOUNDARY VALUE PROBLEM

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Abstract. In this short note we announce an existence and uniqueness result for solutions to a class of second order distributed parameter systems with sudden changes in the input term. Such systems are often encountered in flexible structures and structure-fluid interaction systems that utilize smart actuators.

1. INTRODUCTION

We study the nonlinear, nonlocal partial differential equation

$$\begin{aligned} w_{tt} + \kappa_1 w_{xxxx} + \kappa_2 w_{xxxxt} &= [\beta(x,t)g(y)]_{xx} \\ &\quad + f(x,t), \end{aligned} \tag{1}$$

with boundary and initial conditions given by

$$\begin{aligned} &w_x(0,t) = w(0,t) = 0, \quad w_x(1,t) = w(1,t) = 0, \\ &w(\cdot,0) = w_0 \in H_0^2(0,1),\ w_t(\cdot,0) = w_1 \in L^2(0,1). \end{aligned} \tag{2}$$

In equation (1) the function y satisfies

$$y(t) = \int_0^1 k_s \chi_{[x_1,x_2]}(x) w_{xxt}(x,t)dx \tag{3}$$

where $\chi_{[x_1,x_2]}$ denotes the characteristic function of the interval $[x_1, x_2]$, with $0 \le x_1 < x_2 \le 1$. The constants κ_1, κ_2 and k_s are positive and g is a Lipschitz continuous function.

Equation (1) is a general form of the model developed in (Demetriou and Polycarpou, 1997*a*; Demetriou and Polycarpou, 1997*b*). Indeed, in the context of the flexible structure encountered in (Demetriou and Polycarpou, 1997*a*), κ_1 denotes the *stiffness* parameter, κ_2 the *damping* parameter and k_s the sensor *piezoceramic* constant; see (Banks *et al.*, 1996) and (Dosch *et al.*, 1992). When the actuator (input) failure term $\beta(x,t)g(y)$ is written as

$$\beta(x,t)g(y) = \beta_1(t)\left(k_a \chi_{[x_1,x_2]}(x)\epsilon(t)\right) g(y) \tag{4}$$

with the *time profile* (Polycarpou and Helmicki, 1995) of the failure given by

$$\beta_1(t) = \begin{cases} 0 & \text{if } t < T_f \\ 1 - e^{-\lambda(t-T_f)} & \text{if } t \ge T_f \end{cases}, \quad \lambda > 0, \tag{5}$$

and the nominal forcing (*actuator*) term given by

$$f(x,t) = [k_a \chi_{[x_1,x_2]}(x)\epsilon(t)]_{xx}, \qquad k_a > 0, \tag{6}$$

then equation (1) has exactly the same form as the beam equation considered in (Demetriou and Polycarpou, 1997*a*). The time T_f denotes the unknown instance of the failure occurrence and the signal ϵ denotes the input voltage to the patch. Similarly, k_a denotes the actuator *piezoceramic* constant; see above describe the dynamics of a flexible cantilevered beam before $(t < T_f)$ and after $(t \ge T_f)$ the occurrence of an anticipated actuator failure commencing at an unknown time T_f. In view of the above, the plant equation (1) can now be written as follows:

$$\begin{aligned} w_{tt} + \kappa_1 w_{xxxx} + \kappa_2 w_{xxxxt} &= \left[k_a \chi_{[x_1,x_2]}(x)\epsilon(t)\right]_{xx} \\ &\quad + \beta_1(t)\left[k_a \chi_{[x_1,x_2]}(x)\epsilon(t) g(y(t))\right]_{xx}. \end{aligned} \tag{7}$$

Our goal here is to announce a recent existence and uniqueness result for the solution of (1)-(2).

2. EXISTENCE AND UNIQUENESS

We begin this section by letting $H = L^2(0,1)$, $V = H_0^2(0,1)$ and $V^* = H^{-2}(0,1)$, so we have the Gelfand triple

$$V \hookrightarrow H \hookrightarrow V^*.$$

We denote by $\langle \cdot, \cdot \rangle$ the inner product in H, while $\langle \cdot, \cdot \rangle_{V^*,V}$ stands for the usual duality product. Let $|| \cdot ||$, $|| \cdot ||_V$, and $|| \cdot ||_{V^*}$ denote the norms of the spaces H, V, and V^*, respectively. We impose the following assumptions on the parameters in (1)-(2):

(A_β) *The function β satisfies*

$$\beta \in L^\infty(0,T,H), \; \|\beta\|_{L^\infty(0,T;H)} \leq L. \quad (8)$$

(A_g) *The nonlinear function g satisfies the following Lipschitz condition:*

$$|g(\xi) - g(\chi)| \leq \frac{\widetilde{C}_1}{k_s} |\xi - \chi|, \quad \text{for all } \xi, \chi \in \mathbb{R}, (9)$$

where $\widetilde{C}_1 < \kappa_2 / L$.

(A_f) *The forcing term f satisfies*

$$f \in L^2(0,T;V^*). \quad (10)$$

To establish the existence-uniqueness of solutions we use a Galerkin approach which is comparable to the one employed in the study of well-posedness for other second order (in time) evolution equations (see, e.g., (Banks *et al.*, 1995*a*; Banks *et al.*, 1997; Banks *et al.*, 1995*b*; Dautray and Lions, 1993; Lions, 1971; Lions and Magenes, 1972)). To this end, we define the space of solutions to be

$$\mathcal{U}(0,T) = \left\{ u \;\middle|\; \begin{array}{l} u \in L^2(0,T;V), u_t \in L^2(0,T;V), \\ u_{tt} \in L^2(0,T;V^*) \end{array} \right\}$$

with norm

$$\begin{aligned} ||u||_{\mathcal{U}(0,T)} = (||u||^2_{L^2(0,T;V)} + ||u_t||^2_{L^2(0,T;V)} \\ + ||u_{tt}||^2_{L^2(0,T;V^*)})^{1/2}. \end{aligned} \quad (11)$$

We now define the concept of a weak solution to the problem (1)-(2).

Definition 1. We say that a function $w \in \mathcal{U}(0,T)$ is a weak solution of (1)-(2) if it satisfies

$$\begin{aligned} &\langle w_{tt}(t), \phi \rangle_{V^*,V} + \kappa_1 \langle w_{xx}(t), \phi_{xx} \rangle \\ &+ \kappa_2 \langle w_{xxt}(t), \phi_{xx} \rangle = \langle \beta(t) g(y(t)), \phi_{xx} \rangle \quad (12) \\ &+ \langle f(t), \phi \rangle_{V^*,V}, \quad \forall \phi \in V \end{aligned}$$

and

$$w(0) = w_0 \in V, \qquad w_t(0) = w_1 \in H. \quad (13)$$

Next we state the existence and uniqueness theorem, the proof of which can be found in (Ackleh *et al.*, 2000).

Theorem 2. The problem (1)-(2) has a unique weak solution.

Remark 3. The strong convergence of the Galerkin approximations to the unique weak solution of (1)-(2) is established in (Ackleh et al., 2000). Furthermore, numerical results supporting the theory are also presented in that paper.

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3. REFERENCES

Ackleh, A.S., S. Aizicovici, M.A. Demetriou and S. Reich (2000). Existence and uniqueness of solutions to a second order nonlinear nonlocal hyperbolic equation. in preparation.

Banks, H. T., D. S. Gilliam and V. I. Shupov (1995a). Well-posedness for a one dimensional nonlinear beam. In: Computation and control, IV (Bozeman, MT, 1994), Progr. Systems Control Theory. Vol. 20. Birkhäuser Boston. Boston, MA. pp. 1–21.

Banks, H. T., D. S. Gilliam and V. I. Shupov (1997). Global solvability for damped abstract nonlinear hyperbolic systems. Differential Integral Equations **10**(2), 309–332.

Banks, H. T., K. Ito and Y. Wang (1995b). Well posedness for damped second order systems with unbounded input operators. Differential Integral Equations **8**, 587–606.

Banks, H. T., R. C. Smith and Y. Wang (1996). Smart Material Structures: Modeling, Estimation and Control. Wiley-Masson. New York.

Dautray, R. and J. L. Lions (1993). Mathematical Analysis and Numerical Methods for Science and Technology, Volume 5, Evolution Problems I. Springer. New York.

Demetriou, M. A. and M. M. Polycarpou (1997a). Fault accommodation of output-induced actuator failures for a flexible beam with collocated input and output. In: Proceedings of the 5th IEEE Mediterranean Conference on Control and Systems. Phaethon Beach Hotel, Paphos, Cyprus.

Demetriou, M. A. and M. M. Polycarpou (1997b). Fault diagnosis of output-induced actuator failures for a flexible beam with collocated input and output. In: Proceedings of the IFAC Symposium on Fault Detection, Supervision and Safety for Technical Processes:SAFEPROCESS'97. University of Hull, Hull United Kingdom.

Dosch, J., D. J. Inman and E. Garcia (1992). A self-sensing piezoelectric actuator for collocated control. Journal of Intelligent Material Systems and Structures **3**, 166–185.

Lions, J. L. (1971). Optimal Control of Systems Governed by Partial Differential Equations. Springer-Verlag. New York.

Lions, J. L. and E. Magenes (1972). Non-Homogeneous Boundary Value Problems, I. Springer-Verlag. New York.

Polycarpou, M. M. and A. J. Helmicki (1995). Automated fault detection and accomodation: A learning systems approach. IEEE Trans. on Systems, Man and Cybernetics **25**, 1447–1458.