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# A STABILIZATION METHODOLOGY FOR INVARIANT SYSTEMS ON LIE GROUPS

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**Abstract.** This paper deals with the stabilizability of invariant control systems defined on Lie groups. A stabilization technique is presented which under certain hypotheses can lead to a criterion assuring the existence of a feedback controller which steers every initial condition to a specified target point of the state space of these systems.

**Key words:** Invariant systems, stabilization, Lie groups.

## 1. INTRODUCTION

This paper deals with the stabilization of invariant control systems on Lie groups. By stabilizability it is meant that a feedback controller which steers every initial state to a specified target point exists. The stabilizing controller that will be proposed uses only a finite set of values of the control parameter.

Let $G$ be a real analytic and simply connected Lie group of dimension $n$ and $Lie(G)=L$ be the corresponding Lie algebra of left invariant vector fields on $G$. $G$ is considered as the state space of the systems occurring in the sequel. Consider also the following control system on $G$:

$$\dot{x} = f(x,u) = f^u(x)$$

where $x \in G$ is the state of the system, and $u$ is the control parameter taking values in a subset $U_a$ of the control space which is an analytic manifold $U$. Finally $f\colon G \times U \to TG$ (where $TG$ is the tangent bundle of $G$) is an analytic mapping. It is noted that the set $U_a$ of the acceptable control values can be much "smaller" than the control space $U$, for example a discrete submanifold. The system described above is called invariant if the vector fields $f^u$ on $G$ are left invariant for every constant $u \in U_a$. An invariant system on $G$ will be identified with the subset $\Gamma = \{f^u, u \in U_a\}$ of $L$. It is always assumed that $\Gamma$ contains the zero vector field. We write exp$tX$, for the integral curve of $X \in L$ passing through $e$ (the identity element of $G$) at $t=0$. The integral curve of $X$ through $x \in G$ is then $x$exp $tX$. Furthermore if $W \subseteq R$ we set exp$WX$= $\cap\{\exp tX, t \in W\}$. A point $y \in G$ is accessible from $x \in G$ for $\Gamma$ if: $y = x\exp(t_1 X_1)\ldots\exp(t_k X_k)$ with, $t_i > 0$, $X_i \in \Gamma$, $i=1,..,k$ (the reader can find more details about these Lie theoretic preliminaries in Helgason 1978, Varadarajan 1984). The Lie algebra of is the smallest Lie subalgebra of $L$ which contains $\Gamma$. This is denoted by $Lie(\Gamma)$.

Let us now proceed to the stabilization problem (treatments of various aspects of this problem can be found in Isidori 1989, Sontag 1990, Tsinias 1989). An invariant system on $G$ is called stabilizable if there exists a feedback control law $u=\phi(x)$ such that the emerging closed loop vector field $V$ with: $V_x = f(x,\phi(x))$ steers every initial state to a specified target point (in the sequel this target point will be the identity element but a modification of the results that will be presented later, covers the general case). Thus stabilization means the construction of a vector field $V$ on $G$ such that for every $x \in G$ there exists some $X \in \Gamma$ with $V_x = X_x$.

## 2. THE CASE OF SOLVABLE STATE SPACE

In this section a stabilization technique is presented for the case where the state space $G$ is a solvable Lie group. This technique is based on an appropriate decomposition of the state space (technical notions and facts can also be found in Varadarajan 1984). Let us first give the definition of a solvable Lie algebra. This definition requires the concept of derived subalgebras. The derived subalgebra $DL$ of a Lie algebra $L$ is defined as follows: $DL=[L,L]$. The $k$' th derived subalgebra of $L$ is then:

$$D^k L = D(D^{k-1}L), D^0 L = L$$

**Definition 1**. A Lie algebra $L$ is called solvable if $D^k L = 0$ for some $k \geq 1$.

Observe that if $L$ is solvable and $D^m L \neq 0$ then $D^{m+1}L \subset D^m L$ thus $D^k L$ strictly decrease until they become zero. A Lie group $G$ is called solvable if its Lie algebra is solvable.

**Lemma 2**. (Varadarajan 1984, Lemma 3.18.5) Let $G$ be a real analytic and simply connected Lie group. Suppose that $L_1, L_2, \ldots, L_r$ are subalgebras of $L$ such that:
i) $L = L_1 \oplus \ldots \oplus L_r$
ii) if $W_k = L_1 + \ldots + L_k$ then $W_k$ is a subalgebra of $L$ and an ideal of $W_{k+1}$ for every $k$.
Let also $G_{.1}, \ldots, G_r$ be the respective analytic subgroups corresponding to $L_1, L_2, \ldots, L_r$. Then the $G_i$' s are all closed and simply connected and the mapping: $(x_1, \ldots, x_r) \to x_1 \ldots x_r$, $x_i \in G_i$ *i=1,...,r* is an analytic diffeomorphism of $G_1 \times \ldots \times G$ onto *G*.

From Lemma 2 above it is immediate that every $x \in G$ has a unique expression of the form: $x = x_1 \ldots x_r$ where $x_i \in G_i$.

Now the main result, which is essentially a stabilizability criterion for a control system $\Gamma$, can be stated and proved.

**Theorem 3.** Let $G$ be a real analytic, simply connected and solvable Lie group with Lie algebra $L$ and a control system on $G$. If $\Gamma = -\Gamma$ (symmetry) and $Lie(\Gamma \cap D^k L) = D^k L$ for *k=0,1,...* then is stabilizable.

In the proof of Theorem 3 a preliminary technical lemma is used (see also the similar well known result in Varadarajan 1984, Corollary 3.7.5). Let $L$ be solvable and $d_k = \dim D^k L$. Since the derived subalgebras strictly decrease there exists a basis

$\{X_1,\ldots,X_n\}$ of $L$ such that the first $d_k$ vectors constitute a basis of $D^kL$ for every $k$. For this kind of basis one has the following:

**Lemma 4**. Let $L$ be a solvable Lie algebra. Let also $\{X_1,\ldots,X_n\}$ be a basis of $L$ of the form previously described. Then $M_i = span\{X_1,\ldots,X_i\}$ is a subalgebra of $L$ and an ideal of $M_{i+1}$ for *i=1,...,n-1*.

*Proof*. By construction of the basis for every $i$ there exists a maximal $k$ such that $M_i \subseteq D^kL$. $M_i$ is a subalgebra of $L$ since $[M_i,M_i]\subseteq D^{k+1}L \subset M_i$. If $M_i \neq D^kL$ then $M_{i+1} \subseteq D^kL$ and $[M_{i+1},M_i]\subseteq D^{k+1}L \subset M_i$. If $M_i = D^kL$ then $M_{i+1} \subseteq D^{k-1}L$ (=*L* if *k=0*) and $[M_{i+1},M_i]\subseteq D^kL = M_i$. In both cases $M_i$ is an ideal of $M_{i+1}$ and the proof is complete.•

*Proof of Theorem* 3. We shall first prove that $\Gamma$ contains a basis of $L$ of the form previously described. Indeed $\Gamma$ contains a basis of $D^kL$ for every *k=0,1,...* such that $D^kL$ is nontrivial. To see this let $m$ be the maximal integer such that $D^mL \neq 0$. Since $D^mL$ is abelian and $Lie(\Gamma \cap D^mL)= D^mL$ it is clear that $\Gamma$ contains a basis of $D^mL$. Now, if $\Gamma$ contains a basis of $D^kL$ for some $1\leq k \leq m$ then $\Gamma$ also contains a basis of $D^{k-1}L$. If this is not the case then $span(\Gamma \cap D^{k-1}L)\neq D^{k-1}L$ and

$$[\Gamma \cap D^{k-1}L,\Gamma \cap D^{k-1}L]\subseteq D^kL \subseteq span(\Gamma \cap D^{k-1}L)\neq D^{k-1}L$$

which contradict the hypothesis $Lie(\Gamma \cap D^{k-1}L)= D^{k-1}L$. Thus one can choose a basis of *Lie(G)* contained in $\Gamma$ which is of the wanted form. Let $\{X_1,\ldots,X_n\}$ be such a basis. Now we are going to construct a feedback controller for $\Gamma$ which is a piecewise left-invariant vector field *V*. It will then be proved that *V* steers every initial state *x* to *e*. From Lemma 4 it is clear that Lemma 2 applies, thus:

$$G = G_1\ldots G_n, G_i = \exp RX_i$$

For *k=1,...,n* define the following subsets of *G*:

$$S_k = G_1\ldots G_k$$

$$S_k^+ = G_1\ldots G_{k-1}\exp R^+X_{ki}$$

$$S_k^- = G_1\ldots G_{k-1}\exp R^-X_k$$

$$S_k^0 = G_1\ldots G_{k-1} = S_{k-1}$$

Observe that $S_n = G$, $S_1^0 =\{e\}$. Furthermore the sets $S_1^0, S_1^\pm, S_2^\pm,\ldots,S_n^\pm$ are pairwise disjoint and cover *G*. On each of these sets define the vector field *V* as follows: $V = -X_k$ on $S_k^+$ and $V = X_k$ on $S_k^-$ since (from symmetry) $-X_i \in \Gamma$ for *i=1,...,n*. On $S_1^0 =\{e\}$ we naturally define *V=0*. Observe also that for $x \in S_k$ then $x\exp tV \in S_k$ for every $t$ such that $-\varepsilon < t < \varepsilon$ for some $\varepsilon > 0$. This fact ensures that *V* is well defined as far as existence and uniqueness of integral curves are concerned. It is easy now to see that *V* steers every initial state *x* to *e*.

Consider any initial $x \in G$. Let for example $x \in S_k^+$. There exists some $t_k > 0$ such that: $x = x_1\ldots x_{k-1}\exp t_kX_k$ for some $x_i$'s in $G_i$, *i=1,...,k-1*. Thus the application of $-X_k$ leads the state on the subset $S_{k-1}$ within the finite time $t_k$ since

$$x\exp t_k(-X_k)= x_1\ldots x_{k-1}\exp t_kX_k\exp t_k(-X_k) = x_1\ldots x_{k-1} \in S_{k-1}$$

Inductively one can see that the state eventually reaches $S_1^0 =\{e\}$ and remains there under the application of the zero vector field. Thus *V* is a stabilizing feedback controller for and the proof is complete.•

**Remark 5**. If a feedback controller which steers every initial state to a point $x \neq e$ is wanted then the stabilizing vector field $V$ is defined as follows: $V = \mp X_k$ on $K_k^{\pm} = xS_k^{\pm}$

**Remark 6**. The stabilizing vector field constructed as before depends on the basis of $L$ which is contained in $\Gamma$. Thus the stabilizing feedback controller is not unique and depends on the choice of a particular basis of $L$ contained in $\Gamma$.

**Remark 7**. Since the proposed feedback control law is piecewise constant and incorporates only a finite number of values of the control parameter it can be used in the case where the control parameter is restricted to belong to a discrete subset of the control space.

**Remark 8**. As shown in the proof of Theorem 3 the conditions of this theorem imply that $span(\Gamma)= L$. This is a strong assumption and in fact stronger than the accessibility condition $Lie(\Gamma)= L$. In order however to construct the presented stabilizing control law this assumption is necessary.

## 3. THE CASE OF SEMI-SIMPLE STATE SPACE

In this section the case where $G$ is semisimple and noncompact is examined. Some mathematical preliminaries which are also treated in much greater detail in Helgason 1978 are given below.

Let $L$ be a Lie algebra. The radical of $L$, *rad(L)* is the maximal solvable ideal of $L$. Now one can define:

**Definition 9**. A Lie algebra L is called semisimple if *rad(L)=0.*

A Lie group $G$ is called semisimple if the corresponding Lie algebra is semisimple. Let $L$ be a semisimple Lie algebra and consider the Cartan decomposition $L = L_1 \oplus P$ where $L_1$ is a compact subalgebra (that is there exists a compact Lie group with Lie algebra isomorphic to $L_1$) of $L$ and $P$ a subspace of $L$. Let also $A$ be a maximal abelian subspace of $P$ (all subspaces of this kind have the same dimension). An element $\alpha$ belonging to the dual space $A^*$ of $A$ is called a restricted root of *(L, A)* if:

$$L(\alpha)=\{X \in L:[Y,X]= \alpha(Y)X, \forall Y \in A\}\neq 0$$

Fixing a Weyl chamber one can order the roots and determine the set of positive roots $\Sigma^+$. Then

$$L = L_1 \oplus A \oplus L^+$$

where $L^+ = span\{L(\alpha), \alpha \in \Sigma^+\}$ and $L_2 = A \oplus L^+$ is solvable. The above decomposition is known as the Iwasawa decomposition. Now the following theorem can be stated:

**Theorem 10**. Let $G$ be a real analytic, connected and semisimple Lie group with Lie algebra $L$. Consider the Iwasawa decomposition

$$L = L_1 \oplus A \oplus L^+$$

Let $H_1, H_A, H^+$ be the Lie subgroups of $G$ corresponding to $L_1, A, L^+$. Then the mapping $(x_1, x_A, x^+) \rightarrow x_1 x_A x^+$ is an analytic diffeomorphism of $H_1 \times H_A \times H^+$ onto $G$. Furthermore $H_A, H^+$ are simply connected.

A proof of this theorem can be found in Helgason 1978, Chapter VI, Theorem 5.1. One can express the Iwasawa decomposition in a different form as follows:

$$L = L_1 \oplus L_2, G = H_1 H_2$$

where $L_2 = A \oplus L^+$ and $H_2 = H_A H^+$. Next we impose hypothesis (H) on *G*:

(H) Let *G* be a Lie group with Lie algebra *L*. If $L = L_1 \oplus L_2$ is an Iwasawa decomposition of *L* then $L_1$ is solvable.

Hypothesis (H) implies that $L_1$ is in fact abelian. Since $L_1$ is compact it follows that (cf. Helgason 1978, Chapter II, Proposition 6.6) $L = center(L_1) \oplus DL_1$ with $DL_1$ semisimple. But $DL_1$ has also to solvable as a subalgebra of the solvable algebra $L_1$. This means that $DL_1 = 0$ and $L_1$ is abelian. The presented stabilization technique can be applied in the case of Lie groups satisfying (H). Taking into account the classification of semisimple Lie algebras (cf. Sagle and Walde 1973) it follows that *L* satisfies (H) iff it is of the form

$$L = sl(2,R) \oplus ... \oplus sl(2,R) \;\; l \text{ times}$$

where $sl(2,R)$ consists of the $2\times 2$ real matrices of zero trace. In the following theorem the slightly modified decomposition is used in order to examine the stabilizability of a control system on a semisimple Lie group.

**Theorem 11**. Assume that *G* satisfies (H) and let $\Gamma$ be a control system on *G*. If $\Gamma = -\Gamma$, $Lie(\Gamma \cap L_1) = L_1$ and $Lie(\Gamma \cap D^k L_2) = D^k L_2$ for *k=0,1,2,...* then $\Gamma$ is stabilizable.

*Proof*. Since *G* is simply connected $H_1$ is also simply connected. $\Gamma$ contains a basis $\{X_1,...,X_\mu,Y_1,...Y_v\}$ of *L* such that $\{X_1,...,X_\mu\}$, $\{Y_1,...,Y_v\}$ are bases of $L_1, L_2$ of the form described in Lemma 4. This is true because $L_1$ as observed before is abelian and $L_2$ is solvable. Now we can write

$$G = \exp(RX_1)...\exp(RX_\mu)$$
$$\exp(RY_1)...\exp(RY_v)$$

In a manner similar to the one showed in the proof of Theorem 3 one can construct a vector field *V* which steers every initial condition to *e*. Thus $\Gamma$ is stabilizable and the proof is complete.•

## 4. THE GENERAL CASE

In the previous sections the cases where the state space was solvable or semisimple were treated. These special cases are very important because every Lie algebra can be decomposed into a solvable and a semisimple subalgebra. Let *G* be any real analytic and simply connected Lie group with Lie algebra *L*. Let us now remind what the Levi decomposition of a Lie algebra is (see also Varadarajan 1984). If $L_r = rad(L)$ then the quotient algebra $L/L_r$ is semisimple. A Lie subalgebra $L_m$ of *L* is called a Levi subalgebra if $L = L_m \oplus L_r$. A Levi subalgebra of *L* is isomorphic to $L/L_r$ so it is semisimple. Now one has the following theorems (for a proof see Varadarajan 1984, Theorem 3.14.1 and Theorem 3.18.13).

**Theorem 12.** Any Lie algebra admits Levi subalgebras.

Let *G* be a real analytic and simply connected Lie group with Lie algebra *L*. Let also $L = L_m \oplus L_r$ be a Levi decomposition of *L* and $G_m, G_r$ the Lie subgroups corresponding to $L_m, L_r$. Then $G_m, G_r$ are closed and the map

$(x_m, x_r) \to x_m x_r$ is an analytic diffeomorphism of $G_m \times G_r$ onto $G$.

From this theorem it is immediate that $G_m, G_r$ are simply connected. Since $G_m$ is semisimple it follows that $G_m$ admits the Iwasawa decomposition. Hence one can write:

$$L_m = L_1 \oplus L_2, G_m = G_1 G_2$$

where $L_1, L_2, G_1, G_2$ are as defined in Section 3. We let for notational simplicity $L_3 = L_r, G_3 = G_r$. Now the stabilizability of a control system on $G$ is examined by stating and proving the following theorem.

**Theorem 13.** Let *G,L* be as before. Suppose that $L_m$ is any Levi subalgebra satisfying the hypothesis (H). Let $\Gamma$ be a control system on *G*. If $\Gamma = -\Gamma$, $Lie(\Gamma \cap L_1) = L_1$ and $Lie(\Gamma \cap D^k L_i) = D^k L_i$ for *i=2,3* and *k=0,1,...* then $\Gamma$ is stabilizable.

*Proof*. It is evident that $G = G_1 G_2 G_3$. Using the same arguments as in Theorems 3 and 11 it can be concluded that $\Gamma$ contains a basis $\{X_1, ..., X_\lambda, Y_1, ..., Y_\mu, Z_1, ..., Z_v\}$ of $L$ such that $\{X_1, ..., X_\lambda\}, \{Y_1, ..., Y_\mu\}, \{Z_1, ..., Z_v\}$ are bases of $L_1, L_2, L_3$ respectively of the form described in Lemma 4. Hence we have that:

$$G = \exp(RX_1)...\exp(RX_\lambda)$$
$$\exp(RY_1)...\exp(RY_\mu)$$
$$\exp(RZ_1)...\exp(RZ_v)$$

In a manner similar to that introduced in the proof of Theorem 3 one can construct a vector field which steers every initial condition to the identity element and the proof is complete.•

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