Proceedings of the 8th IEEE Mediterranean Conference
on Control and Automation (MED 2000)
Rio, Patras, GREECE 17-19 July, 2000

FUNCTIONAL CONTROLLABILITY AND RIGHT INVERTIBILITY FOR SYSTEMS OVER RINGS

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Abstract: This paper is devoted to study the structure of linear systems with coefficients in a ring in connection with the so-called Functional Controllability Property and the Right Inversion Problem. A geometric characterization of Functional Controllability is given and a related notion of Relative Degree is introduced. Applications to delay-differential systyems are considered.

Keywords: Linear systems over ring, output reproducibility, delay-differential systems

1. INTRODUCTION

Systems over rings have been introduced for dealing with families of discrete-time, parameter-dependent systems as well as for modeling integer coefficients systems. Their study is receiving an increasing attention as they prove to be an efficient tool for investigating and solving design problems concerning continuous-time, delay-differential systems (see [6] and the references therein).
In recent time, many efforts, in particular, have been made to extend the geometric approach to the class of systems over rings (see [8] and the references therein). The difficulties one face in this case are essentially due to the difference between the geometric notions of invariance and the related dynamic notions of feedback invariance. In addition, basic properties of ring and module algebra complicate the construction of algorithmic procedures for checking geometric relations and finding specific submodules. Nevertheless, using geometric methods and tools it is possible to give complete solutions to many noninteracting control problems for systems over rings, which in particular can be interpreted in a delay-differential framework, as well as to provide algorithms for their practical computation (see [2], [5], [6], [8]).
The aim of this paper is to investigate, using the geometric approach, the notion of Functional Controllability and the related notion of Right Invertibility for dynamical systems with coefficients in a ring.
After defining Functional Controllability and the related notion of Weak Functional Controllability, we give a geometric characterization of both, in terms of solvability of a suitable Disturbance Decoupling Problem. If a given system is Functionally Controllable, feedback solutions of the above

mentioned DDP are shown to provide a right inverse of the system itself and therefore allow one to solve the problem of following an arbitrary trajectory. Related to the notion of Functional Controllability is that of Relative Degree, which also turns out to be useful in trajectory following problems. The Relative Degree is first defined in terms of functional controllability and then characterized in geometric terms.

2. FUNCTIONAL CONTROLLABILITY

Given a commutative ring with identity R, a dynamical system Σ over R is described by a set of equations of the form

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where x belongs to the free state module $\mathcal{X} = R^n$, u belongs to the free input module $\mathcal{U} = R^m$, y belongs to the free output module $\mathcal{Y} = R^p$ and A, B, C are matrices of suitable dimensions with entries in R.
For continuous time, linear, dynamical systems with real coefficients the notion of *Functional Controllability*, as described in [4], denotes the possibility, starting from zero initial state, of imposing any sufficiently smooth output function by a suitable choice of the input function
A notion of Functional Controllability for systems over a ring can be defined as follows.

Definition 1. The system Σ, defined by equations (1), is said *functionally controllable* if any arbitrary sequence $\{y_t,\ y_t \in \mathcal{Y},\ t \geq 0\}$ can be obtained as output of Σ, starting from zero initial state, for $t \geq n$, namely

$$y(t+n) = y_t \quad for\ t \geq 0,$$

by means of a suitable control input $\{u(t),\ t \geq 0\}$.

Definition 2. The system Σ is said *weakly functionally controllable* if there exists a nonsingular $p \times p$ matrix K with entries in R, such that any sequence $\{Ky_t,\ y_t \in \mathcal{Y},\ t \geq 0\}$ can be obtained as output of Σ, starting from zero initial state, for $t \geq n$, namely

$$y(t+n) = Ky_t \quad for\ t \geq 0,$$

by means of a suitable control input $\{u(t),\ t \geq 0\}$.

Example 1. The simple scalar system Σ defined by

$$\begin{cases} x(t+1) = u(t) \\ y(t) = cx(t) \end{cases} \tag{2}$$

is functionally controllable if c is an invertible element of R. It is weakly functionally controllable if c is nonzero and noninvertible and it is neither functionally controllable nor weakly functionally controllable if $c = 0$.
In particular, if $R = \mathcal{R}[\Delta]$ and $c = \Delta$, Weak Functional Controllability of Σ can be viewed as representing the fact that the delay-differential system Σ_d, described by

$$\begin{cases} \dot{x}(t) = u(t) \\ y(t) = x(t-\tau) \end{cases} \tag{3}$$

and associated to Σ as described in [6], can reproduce any arbitrary differentiable function only after an initial delay equal to τ.

If a system Σ is functionally controllable it can reproduce, after n time instants or, possibly, a shorter delay, arbitrary reference trajectories. I n other terms, it admits a right inverse. In particular, functionally controllable systems can be forced to follow, after time n, trajectories having all the output components y_j, for $j \neq i$, identically zero and the output component y_i equal to any arbitrary sequence.

3. GEOMETRIC CHARACTERIZATION

As remarked in [13] in the context of continuous time systems with coefficients in a field, Functional Controllability of a given system Σ is equivalent to the solvability of a specific Disturbance Decoupling Problem for a suitable extension of Σ. To show an analogous result for systems over a ring, let us consider the extended system Σ_e of dimension $n + p(n+1)$, defined by the equations

$$\begin{cases} x_e(t+1) = A_e x_e(t) + B_e u(t) + D_e q(t) \\ y(t) = C_e x_e(t) \end{cases} \tag{4}$$

where q is a disturbance and the matrices A_e, B_e, C_e, D_e are defined as follows: $A_e = \begin{bmatrix} A & 0 & \dots & 0 \\ 0 & S & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S \end{bmatrix}$,

$B_e = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $C_e = \begin{bmatrix} C_1 & Q & 0 & \dots & 0 \\ C_2 & 0 & Q & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_p & 0 & 0 & \dots & Q \end{bmatrix}$, $D_e = \begin{bmatrix} 0 & 0 & \dots & 0 \\ R & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R \end{bmatrix}$, C_i being the i-th row of C while the $(n+1) \times (n+1)$ matrix S, the $(n+1)$-dimensional row vector Q and the $(n+1)$-dimensional column vector R are defined respectively as $S = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$, $Q^T = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \\ 0 \end{bmatrix}$, $R = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}$. The system $\Sigma_e = (A_e, B_e, C_e, D_e)$ is the parallel composition of the system Σ, described by equations (1), together with n chains of delay operators. Each output e_i, for $i = 1, \dots, p$, is the sum of y_i and the output of one chain of delay operators. It is clear that the system Σ is Functionally Controllable, namely it can reproduce at the output any arbitrary sequence after n instant, starting from zero initial conditions, by means of a suitable control input, if and only if the disturbance q can be decoupled from the output of Σ_e. Therefore, we can state the following result.

Proposition 1. The system Σ, defined by equations (1), is functionally controllable if and only if the Disturbance Decoupling Problem for the system Σ_e is solvable.

Remark that in some case the length of the chains of delay operators can be reduced, leading to a simpler formulation of the above result (see the following Examples). For systems over a field, denoting by V_e^* the maximum controlled invariant subspace contained in $Ker\ C_e$, the Disturbance Decoupling Problem is solvable for Σ_e, with a static state feedback, if and only if

$$Im\ D_e \subseteq V_e^* + Im\ B_e. \tag{5}$$

The same condition characterizes solvability of the DDP also in the case of systems over a ring, in case dynamic feedback are allowed (see [5]). If Σ is Functional Controllable, feedback solutions of the DDP considered above provide a right inverse of the system.

Example 2. Let us consider the system Σ over the ring of integers $\mathcal{Z}$, described by the equations

$$\begin{cases} x_1(t+1) = 2x_1(t) + x_2(t) + u_1(t) \\ x_2(t+1) = -x_2(t) + u_1(t) + u_2(t) \\ y_1(t) = 2x_1(t) - x_2(t) \\ y_2(t) = x_1(t) - x_2(t) \end{cases} \tag{6}$$

and the extended system Σ_e, obtained as explained above by adding chains of delay operators of reduced length, described by the equations

$$\begin{cases} x_1(t+1) = 2x_1(t) + x_2(t) + u_1(t) \\ x_2(t+1) = -x_2(t) + u_1(t) + u_2(t) \\ z_1(t+1) = d_1(t) \\ z_2(t+1) = d_2(t) \\ y_1(t) = 2x_1(t) - x_2(t) - z_1(t) \\ y_2(t) = x_1(t) - x_2(t) - z_2(t) \end{cases} \tag{7}$$

Computation of V_e^* shows that the DDP concerning Σ_e is solvable and a solution is e.g. given by the feedback

$$\begin{cases} u_1(t) = -2x_1(t) - x_2(t) + d_1(t) + d_2(t) \\ u_2(t) = 2x_1(t) + 2x_2(t) - d_2(t) \end{cases} \tag{8}$$

Feeding Σ by the above input, one gets the system described by the equations

$$\begin{cases} x_1(t+1) = d_1(t) - d_2(t) \\ x_2(t+1) = 2d_2(t) + d_1(t) \\ y_1(t) = 2x_1(t) - x_2(t) \\ y_2(t) = x_1(t) - x_2(t) \end{cases} \tag{9}$$

whose output verifies, for $t \geq 1$,

$$\begin{cases} y_1(t) = d_1(t-1) \\ y_2(t) = d_2(t-1) \end{cases} \tag{10}$$

Example 3. Similar conclusions hold for the system Σ over the ring of real polynomials $\mathcal{R}[\Delta]$, described by the equations

$$\begin{cases} x_1(t+1) = \Delta x_1(t) + x_2(t) + u_1(t) \\ x_2(t+1) = -x_2(t) + u_2(t) \\ y_1(t) = \Delta x_1(t) + (\Delta - 1)x_2(t) \\ y_2(t) = x_1(t) + x_2(t) \end{cases} \tag{11}$$

A solution to the DDP is in this case given e.g. by the fedback

$$\begin{cases} u_1(t) = \Delta x_1(t) + x_2(t) + d_1(t) - (\Delta - 1)d_2(t) \\ u_2(t) = -x_2(t) - d_1(t) + \Delta d_2(t) \end{cases} \tag{12}$$

Interpreting Σ as the system associated, in the way explained in [6], to the delay-differential system Σ_d, described by the equations

$$\begin{cases} \dot{x}_1(t) = x_1(t-\tau) + x_2(t) + u_1(t) \\ \dot{x}_2(t) = -x_2(t) + u_2(t) \\ y_1(t) = x_1(t-\tau) - x_2(t) + x_2(t-\tau) \\ y_2(t) = x_1(t) + x_2(t) \end{cases} ,(13)$$

the procedure outlined above allows one to derive a rigth inverse of Σ_d in the original delay-differential context.

Remark that the solvability conditions (5) can be practically checked by the algorithms described in [1] and [3].

4. RELATIVE DEGREE

If a system Σ is functionally controllable, it may happen that, at some output channel, an arbitrary reference trajectory can be reproduced after k instants with $k < n$. The notion of relative degree of the i-th output deals with this aspect.

Definition 3. Assume that the system Σ described by equations (1) is functionally controllable. Then, the relative degree of the i-th output is the minimum r_i, $r_i \leq n$, such that any sequence can be obtained at the i-th output channel after r_i instants, while all the other output components are kept equal to zero for $t \geq n$.

A notion of relative degree for the overall system Σ described by equations (1) can now be given as follows.

Definition 4. Assume that the system (1) is functionally controllable. Then its r*elative degree* is the vector $r_\Sigma = (r_1, r_2, \ldots, r_p)$, where r_i is the relative degree of the i-th output.

Example 4. Consider the system Σ described over the ring $\mathcal{R}[\Delta]$ by the equations

$$\begin{cases} x_1(t+1) = u_2(t) \\ x_2(t+1) = x_3(t) + u_1(t) \\ x_3(t+1) = \Delta u_2(t) \\ y_1(t) = x_1(t) \\ y_2(t) = x_2(t) \end{cases} \qquad (14)$$

It is not difficult to see that, starting from $x = 0$, the output of Σ verifies the following equality

$$\begin{aligned} y_1(t+1) &= u_2(t) \\ y_2(t+1) &= \Delta u_2(t-1) + u_1(t) \end{aligned} \qquad (15)$$

and, therefore, $y_1(t)$ can be forced to follow any arbitrary reference trajectory for $t \geq 1$, while keeping $y_1(t) = 0$ for $t \geq 2$. In turn, $y_2(t)$ can be forced to follow any arbitrary reference trajectory for $t \geq 1$, while keeping $y_1(t) = 0$ for $t \geq 1$. As a consequence, the Relative Degree of Σ is in this case $r_\Sigma = (1, 1)$.

Assume that the system Σ, defined by equations (1) over the ring R, is functionally controllable and that the relative degree of the i-th output y_i is lower then n. As a consequence of Proposition 1, and (5), the relative degree can be computed as follows.

Computation of the relative degree

- **Step 1**
  Modify the i-th column D_i of the matrix D_e by substituting to the column vector R the column vector R_k, whose components are all equal to 0, except the $n+1-k$-th component, which is equal to 1
- **Step 2**
  Find the least value $\bar{k}$ of k for which the DDP concerning Σ_e, with the modified matrix D_e is solvable. Define $r_i = \bar{k}$.

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