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QUADRATIC CHARACTERIZATION AND USE OF OUTPUT STABILIZABLE SUBSPACES

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Abstract. The study is based on the characterization of Output Stabilizable (C,A,B)-invariant subspaces through two coupled quadratic stabilization conditions. The paper shows the equivalence between the existence of a solution to this set of conditions and the possibility to stabilize the system by output feedback. An algorithm and a numerical example are provided to illustrate the approach.

Key Words. Output feedback, geometric approach, quadratic stabilizability, LMIs.

1 INTRODUCTION

The stabilization of linear systems by output feedback is recognized as an important and still open problem in control theory. A review of existing approaches and techniques to treat different versions of this problem can be found in [10]. Other recent results not covered by this survey paper are found, for instance, in [1], [7], [4] and [6].

A commom feature shared by different methods (Lyapunov, Riccati, LMI or Eigenstructure Assignment) is that the output feedback stabilization problem is equivalent to obtaining the solution of a coupled set of matrix equations. In particular, through the use of coupled Sylvester equations [8] [9], the output feedback control problem can be decomposed into two stages :

- determination of a (C,A)-outer detectable subspace
- inner stabilization of this subspace.

This paper shows that this geometric approach based on the solution of coupled Sylvester equations has a quadratic counterpart, so that coupled Lyapunov like equations can also be used for construction of an output stabilizable (C, A, B)-invariant subspace as an intermediate mechanism in the process of designing a static output feedback controller. In particular, we show that solutions for the first stage can be obtained as solutions of a reduced-order Lyapunov equation.

Furthermore, the quadratic characterization of both stages by Lyapunov equations provides a convenient framework for the numerical resolution of the problem (through LMIs for instance) and for the integration of additional performance requirements, such as regional pole placement, robustness to structured parametric uncertainties and disturbance attenuation.

The second section of the paper introduces the key notion of *Output Stabilizable* (C, A, B)-invariant subspace and the basic equivalence between existence of such a subspace and static output stabilizability of the system. The third and fourth sections provide a generic formulation of the output stabilization problem in the form of coupled Lyapunov equations. An orthogonal decomposition technique and an algorithm to solve the problem are also proposed in section 4. This approach is illustrated in section 5 through a numerical example and some concluding remarks are finally presented.

2 PRELIMINARIES

Linear time-invariant systems are considered :

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

$$y(t) = Cx(t) \quad (2)$$

where : $x \in \mathcal{X} \subset \Re^n$, $u \in \mathcal{U} \subset \Re^m$, $y \in \mathcal{Y} \subset \Re^p$. B is assumed full column-rank, C full row-rank and (C, A, B) stabilizable and detectable. The studied problem is to find a static output feedback control law

$$u(t) = Ky(t) \quad (3)$$

such that $\sigma(A+BKC) \in \mathcal{C}^-$, or equivalently, the closed-loop system is asymptotically stable.

As in [9], the mathematical analysis is based on some known concepts and definitions from the geometric control theory [11]. It is first recalled that a subspace $\mathcal{V} \subset \mathcal{X}$ is (A,B)-invariant if there exists $F : \mathcal{X} \longrightarrow \mathcal{Y}$ such that $(A+BF)\mathcal{V} \subset \mathcal{V}$, or equivalently , $A\mathcal{V} \subset \mathcal{V} + Im\,B$. In a dual way, a subspace $\mathcal{T} \subset \mathcal{X}$ is (C,A)-invariant if there exists $L : \mathcal{Y} \longrightarrow \mathcal{X}$ such that $(A+LC)\mathcal{T} \subset \mathcal{T}$, or equivalently, $\mathcal{T} \supset A(\mathcal{T} \cap Ker\,C)$.

Definition 2.1 *[9] : A v-dimensional subspace $\mathcal{V} \subset \mathcal{X}$ is (C,A,B)-invariant if $\mathcal{V}$ is both (A,B)-invariant and (C,A)-invariant.*

Let $V \in \Re^{v\times v}$ be such that $span(V) = \mathcal{V}$ and $T \in \Re^{(n-v)\times v}$ be a left annihilator of V, *i.e*: $Ker\,T = Im\,V$. Thus, definition 2.1 is equivalent to the existence of matrices ($H_V \in \Re^{v\times v}$, $W \in \Re^{m\times v}$) and ($H_T \in \Re^{(n-v)\times(n-v)}$, $U \in \Re^{(n-v)\times p}$) solutions to the following *coupled-Sylvester equations*:

$$AV - VH_V = -BW \tag{4}$$
$$TA - H_T T = -UC \tag{5}$$
$$TV = 0 \tag{6}$$

Definition 2.1 and equations (4),(5) and (6) play a fundamental role in treating the problem of control by static output feedback, mainly by eigenstructure assignment [8] [9]. This study rather focuses on the properties of stabilizability and detectability. It is recalled that:
(i) a (A,B)-invariant subspace $\mathcal{V}$ is (A,B)-inner stabilizable if there exists F such that $(A+BF)|\mathcal{V}$ is (asymptotically) stable; and
(ii) a (C,A)-invariant subspace $\mathcal{V}$ is (C,A)-outer detectable if there exists L such $(A+LC)|\mathcal{X}/\mathcal{V}$ is (asymptotically) stable.

Definition 2.2 *[9] : A v-dimensional subspace $\mathcal{V}$ is an Output Stabilizable (C,A,B)-invariant subspace if $\mathcal{V}$ is (A,B)-inner stabilizable and (C,A)-outer detectable.*

Thus, a necessary and sufficient condition for $\mathcal{V} = Im(V)$ to be an *Output Stabilizable* (or simply, OS)(C,A,B)-invariant subspace is that (4), (5), and (6) hold true for some H_V, H_T, W and U, with the additional stability conditions:

$$\sigma(H_V) \in \mathcal{C}^- \tag{7}$$
$$\sigma(H_T) \in \mathcal{C}^- \tag{8}$$

The following theorem relates the concept of $OS(C,A,B)$-invariant subspaces to the existence of a stabilizing output feedback control law (3).

Theorem 2.1 *: There exists a static output feeback matrix $K : \mathcal{Y} \longrightarrow \mathcal{U}$ such that $\sigma(A+BKC) \in \mathcal{C}^-$ if and only if there exists a matrix $V \in \Re^{n\times v}$ spanning a v-dimensional subspace $\mathcal{V} = Im\,V$ that is $OS(C,A,B)$-invariant and such that*

$$Ker\,CV \subseteq Ker\,W \tag{9}$$
$$Ker\,B'T' \subseteq Ker\,U' \tag{10}$$

It is important to note that the above result has been presented and exploited under different forms in the literature related to the eigenstructure assignment by output feedback. The above statement is essentially equivalent to the one of theorem 3.2 stated in [9]. In the sequel, we shall assume that v, the dimension of $\mathcal{V}$, is p. This assumption has been also adopted in many works that, explicitly or implicitly, use the coupled-Sylvester equations for eigenstructure assignment by output feedback [1] [8] [5]. Furthermore, (9) and (10) can then be equivalently replaced by

$$KCV = W \tag{11}$$
$$TBK = U \tag{12}$$

In particular, if $CV \in \Re^{p\times p}$ is full rank, $Ker\,CV = \{0\}$ is always included in $Ker\,W$ and K verifies:

$$K = W(CV)^{-1} \tag{13}$$

It is also easy to verify that $Ker\,CV = \{0\}$ if and only if $Ker\,T \cap Ker\,C = \{0\}$, or, equivalently, that $rank(CV) = p$ if and only if $rank\left(\begin{bmatrix} T \\ C \end{bmatrix}\right) = n$. For algorithmic purposes, this condition on T may be considered, for instance, in the solution of the second Sylvester equation and shall lead, subsequently, to the solutions of (4) such that (13) holds true [8].

3 A QUADRATIC STABILIZABILITY CHARACTERIZATION

A quadratic characterization of $OS(C,A,B)$-invariant subspaces can be obtained from definition 2.2 by replacing the (A,B)-inner stabilizability condition (7) and the (C,A)-outer detectability condition (8) by the two equivalent quadratic Lyapunov stability conditions [3]:

$$\sigma(H_V) \in \mathcal{C}^- \iff \exists\,\Pi = \Pi' > 0 \text{ such that}$$
$$\Pi H_V' + H_V \Pi = -Q_V, \forall\, Q_V = Q_V' > 0 \tag{14}$$
$$\sigma(H_T) \in \mathcal{C}^- \iff \exists\,\Gamma = \Gamma' > 0 \text{ such that}$$
$$H_T'\Gamma + \Gamma H_T = -Q_T, \forall\, Q_T = Q_T' > 0 \tag{15}$$

Theorem 3.2 *: Let $V \in \Re^{n\times p}$ be a basis of a p-dimensional subspace $\mathcal{V} \subset \mathcal{X}$ and $T \in \Re^{(n-p)\times n}$ be such that $TV = 0$. Then, $\mathcal{V}$ is an $OS(C,A,B)$-invariant subspace if and only if :*

(i) $\forall\ Q_V = Q_V' > 0,\ Q_V \in \Re^{p\times p}$, *there exist* $P = P' \geq 0,\ P \in \Re^{n\times n}$ *and* $Y \in \Re^{m\times n}$ *such that*

$$AP + PA' + BY + Y'B' = -VQ_VV' \quad (16)$$

$$V'PV > 0 \quad ; \quad TPT' = 0 \quad (17)$$

$$Y = W_\Pi V' \text{ for some } \quad W_\Pi \in \Re^{m\times p} \quad (18)$$

(ii) $\forall\ Q_T = Q_T' > 0,\ Q_T \in \Re^{n-p\times n-p}$, *there exist* $S = S' \geq 0,\ S \in \Re^{n\times n}$ *and* $Z \in \Re^{n\times p}$ *such that*

$$A'S + SA + C'Z' + ZC = -T'Q_TT \quad (19)$$

$$TST' > 0 \quad ; \quad V'SV = 0 \quad (20)$$

$$Z = T'U_\Gamma \text{ for some } \quad U_\Gamma \in \Re^{n-p\times p} \quad (21)$$

Proof (outline):
Necessity: Consider that $\mathcal{V} \subset \mathcal{X}$ is an $OS(C, A, B)$-invariant subspace and, hence, that the coupled Sylvester equations (4), (5) and (6) are verified.

Let us first show the necessity of part (i). For any $Q_V = Q_V' > 0$, the quadratic stability condition given by (14) holds true, and we obtain :

$$\begin{aligned} V(\Pi H_V' + H_V\Pi)V' &= V\Pi H_V'V' + VH_V\Pi V' \\ &= -VQ_VV' \leq 0 \end{aligned} \quad (22)$$

From (4) we also have $AV + BW = VH_V$, which can be used in (22) to obtain

$$\begin{aligned} V\Pi V'A' + V\Pi W'B' + AV\Pi V' + BW\Pi V' = \quad (23) \\ -VQ_VV' \quad (24) \end{aligned}$$

Thus, by setting $P = P' = V\Pi V'$ and $Y = W\Pi V'$, and by considering that $rank(V) = p$ and that $\Pi > 0 \Longrightarrow \begin{cases} V'V\Pi V'V > 0 \\ T'V\Pi V'T = 0 \end{cases}$, (24) can be equivalently replaced by (16), (17) and (18).

Using similar arguments, we can show the necessity of part (ii). Thus, from (5) and (15), for any $Q_T = Q_T' > 0$ we obtain:

$$\begin{aligned} T'(H_T'\Gamma + \Gamma H_T)T = A'T'\Gamma T + C'U'\Gamma T \quad (25) \\ +T'\Gamma TA + T'\Gamma UC = -T'Q_TT \quad (26) \end{aligned}$$

By setting $S = T'\Gamma T$ and $Z = T'\Gamma U$, (26) can replaced by (19) and (21). Furthermore, since $rank(T) = n - p$ and $\Gamma > 0$, we also have $\begin{cases} TT'\Gamma TT' > 0 \\ V'T'\Gamma TV = 0 \end{cases}$.

Sufficiency: By using arguments from the geometric theory, it can be shown that the verification of condition (i) implies that $\mathcal{V} = Im\,V$ is a (A, B)-inner stabilizable invariant subspace. Similar arguments can be used to show that $\mathcal{V} = Ker\,T$, which is implied by the coupling condition, is also a (C, A)-outer detectable subspace.
□
The coupling condition between parts (i) and (ii) of theorem 3.2 can be restated as follows:

Corollary 3.1 *If $\mathcal{V} \subset \mathcal{X}$ is an $OS(C, A, B)$-invariant subspace, then any pair of matrices (P, S) solutions to parts (i) and (ii) of theorem 3.2 verifies $Ker\ S = Im\ \mathcal{V} = Im\ P$, or, equivalently:*

$$SP = 0 \quad (27)$$

4 OUTPUT FEEDBACK STABILIZATION

The quadratic conditions of theorem 3.2, which could be applied to verify whether an arbitrary subspace $\mathcal{V} \subset \mathcal{X}$ is an $OS(C, A, B)$-invariant subspace or not, will now be used to reformulate the necessary and sufficient condition, given in theorem 2.1, for the existence of a stabilizing static output feedback.

Theorem 4.3 : *There exists an output feedback $K : \mathcal{Y} \longrightarrow \mathcal{U}$ such that $\sigma(A + BKC) \in \mathcal{C}^-$ if and only if there exist positive semi-definite matrices $P = P' \geq 0$, Y, $S = S' \geq 0$ and Z such that the coupled quadratic stabilization conditions of theorem 3.2 hold true for some matrices V and T verifying $TV = 0$, and such that:*

$$Ker\,CP \subseteq Ker\,Y \quad (28)$$

$$Ker\,B'S \subseteq Ker\,Z' \quad (29)$$

Proof: It remains to show that (28) and (29) are, respectively, equivalent to conditions (9) and (10) of theorem 2.1. Thus, from theorem 3.2, (28) can be equivalently replaced by

$$\exists\ K \text{ such that } KCV\Pi V' = W\Pi V' \quad (30)$$

where $\Pi = \Pi' > 0$ and V' has full row rank. Hence, relation (30) (or, equivalently, (28)), is verified if and only (9) holds. Similar arguments show the equivalence between (29) and (10). □

The previous theoretical results allow to readily associate a quadratic Lyapunov function to the closed-loop system.

Corollary 4.2 *Consider that P, S, V and T have been found such that theorem 4.3 holds true and let K be the corresponding stabilizing output feedback matrix. Define matrices $\bar{T} \in \Re^{p\times n}$ and $\bar{V} \in \Re^{n\times(n-p)}$ as:*

$$\bar{T} = \overbrace{(V'V)^{-1}V'}^{V^\dagger} + D_lT \quad ; \quad \bar{V} = \overbrace{T'(TT')^{-1}}^{T^\dagger} + VD_l \quad (31)$$

where $D_l \in \Re^{p\times(n-p)}$ is such that:

$$D_lH_T - H_VD_l = -V^\dagger(A + BKC)T^\dagger \quad (32)$$

Then, $v(x) = x'\mathcal{S}x$, *where* (33)

$$\begin{aligned} \mathcal{S} &= \begin{bmatrix} \bar{T}' & T' \end{bmatrix} \begin{bmatrix} \Pi^{-1} & 0 \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} \bar{T} \\ T \end{bmatrix} \\ &= \begin{bmatrix} V & \bar{V} \end{bmatrix} \begin{bmatrix} \bar{P}^{-1} & 0 \\ 0 & \bar{S} \end{bmatrix} \begin{bmatrix} V' \\ \bar{V}' \end{bmatrix} \end{aligned}$$

is strictly decreasing along the trajectories of the closed-loop system $\dot{x}(t) = (A + BKC)x(t)$.

Proof: From the definition of matrices $\bar{T}$ and $\bar{V}$, it can be verified that

$$\begin{bmatrix} \bar{T} \\ T \end{bmatrix} \begin{bmatrix} V & \bar{V} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_{n-p} \end{bmatrix}. \tag{34}$$

Furthermore, by considering the equivalence between theorems 4.3 and 2.1, we also get

$$\begin{bmatrix} \bar{T} \\ T \end{bmatrix} (A+BKC) \begin{bmatrix} V & \bar{V} \end{bmatrix} = \begin{bmatrix} H_V & H_{12} \\ 0 & H_T \end{bmatrix} \tag{35}$$

where, from (32), $H_{12} = D_l H_T - H_V D_l + V^\dagger (A + BKC) T^\dagger = 0$. Thus, relation (35) can be rewritten:

$$\begin{bmatrix} \bar{T} \\ T \end{bmatrix} (A+BKC) = \begin{bmatrix} H_V & 0 \\ 0 & H_T \end{bmatrix} \begin{bmatrix} \bar{T} \\ T \end{bmatrix} \tag{36}$$

From (14) and (15), we also have:

$$\begin{bmatrix} H_V' & 0 \\ 0 & H_T' \end{bmatrix} \begin{bmatrix} \Pi^{-1} & 0 \\ 0 & \Gamma \end{bmatrix} + \begin{bmatrix} \Pi^{-1} & 0 \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} H_V & 0 \\ 0 & H_T \end{bmatrix} =$$

$$\begin{bmatrix} -Q_{\bar{V}} & 0 \\ 0 & -Q_T \end{bmatrix} \text{ where: } Q_{\bar{V}} = Q_{\bar{V}}' = \Pi^{-1} Q_V \Pi^{-1} > 0 \,. \tag{37}$$

Thus, by taking into account (36), left and right multiplications of (37) by respectively $\begin{bmatrix} \bar{T}' & T' \end{bmatrix}$ and $\begin{bmatrix} \bar{T} \\ T \end{bmatrix}$, give: $(A+BKC)' \begin{bmatrix} \bar{T}' & T \end{bmatrix} \begin{bmatrix} \Pi^{-1} & 0 \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} \bar{T} \\ T \end{bmatrix} +$ $\begin{bmatrix} \bar{T}' & T \end{bmatrix} \begin{bmatrix} \Pi^{-1} & 0 \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} \bar{T} \\ T \end{bmatrix} (A+BKC) < 0$. Hence, $\dot{v}(x) = x'((A+BKC)'\mathcal{S} + \mathcal{S}(A+BKC))x < 0 \ \forall\ x \neq 0$. Furthermore, $\mathcal{S}$ can also be written as:

$$\begin{aligned} \mathcal{S} &= \begin{bmatrix} V & T' \end{bmatrix} \begin{bmatrix} \bar{P}^{-1} & 0 \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} V' \\ T \end{bmatrix} \\ &= \begin{bmatrix} V & \bar{V} \end{bmatrix} \begin{bmatrix} \bar{P}^{-1} & 0 \\ 0 & \bar{S} \end{bmatrix} \begin{bmatrix} V' \\ \bar{V}' \end{bmatrix}. \end{aligned}$$

□

4.1 Algorithm

Theorem 4.3 gives a necessary and sufficient condition for the existence of a stabilizing output feedback. Unfortunately, this result involves equations that are nonlinear in the considered unknows. However the quadratic characterization of theorem 3.2 may be used to adequately construct $OS(C, A, B)$-invariant subspaces that lead to stabilizing output feedback matrices K. This can be done, for instance, by taking into account the coupling requirement and solving subsequently part (ii) and part (i) of theorem 3.2.

Before presenting a procedure based on this last comment, a result will be formulated allowing the use of a reduced-order generalized Lyapunov equation both to solve part (ii) and to consider the inclusion (9). The proof given for the next lemma sets into evidence some connections between the adopted quadratic approach and the Sylvester equations approach. The choice of using orthogonal transformations is due both to their known numerical interests and simplicity for deriving the theoretical results.

Lemma 4.1 *Under the assumption that (C, A) is detectable, there always exist $S = S' \geq 0$ and Z verifying (19), (20) and (21) for some $T \in \Re^{n-p \times n}$, with rank $(T) = n - p$, such that $Ker\,T = \mathcal{V}$ is (C, A)-outer detectable and*

$$Ker\,T \cap Ker\,C = \{0\}$$

Proof: Consider the following change of basis :

$$x = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{38}$$

where $[M_1 \ \ M_2]$ is an orthogonal matrix such that:

$$C \begin{bmatrix} M_1 & M_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 \end{bmatrix} \tag{39}$$

with $C_1 \in \Re^{p \times p}$ and $rank(C_1) = p$.

In this basis, the open-loop system takes the form

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{40}$$

$$y = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{41}$$

where the involved matrices have appropriate dimensions.

As shown in [3], the detectability of the pair (C, A) implies the detectability of the pair (A_{12}, A_{22}). Thus, for any given matrix $Q_T = Q_T' > 0$, let $S_{22} \in \Re^{n-p \times n-p}$, with $S_{22} = S_{22}' > 0$ and $S_{21} \in \Re^{n-p \times p}$ be solutions to the following generalized reduced-order Lyapunov equation:

$$A_{22}' S_{22} + S_{22} A_{22} + A_{12}' S_{21}' + S_{21} A_{12} = -Q_T \tag{42}$$

Let $L_2 = \Re^{n-p \times p}$ be such that

$$S_{22} L_2 = S_{21} \tag{43}$$

and consider also a Choleski decomposition of $S_{22} = S_{22}' > 0$ given by:

$$S_{22} = T_2' T_2 \tag{44}$$

Thus, from (42), (43) and (44), we get

$$T_2' T_2 \left(A_{22} + L_2 A_{12}\right) + \left(A_{22}' + A_{12}' L_2'\right) T_2' T_2 = -Q_T$$

where, by construction, $\sigma\left(A_{22} + L_2 A_{12}\right) \in \mathcal{C}^-$.

Since T_2 is, by definition, non-singular, we can define the asymptotically stable matrix $H_T \in \Re^{n-p \times n-p}$ from the similarity relation

$$T_2 \left(A_{22} + L_2 A_{12}\right) = H_T T_2 \tag{45}$$

Furthermore, since C_1 is invertible, a matrix $U \in \Re^{n-p \times p}$ can always be computed from:

$$U C_1 = -\left(T_2 L_2 A_{11} + T_2 A_{21} - H_T T_1\right) \tag{46}$$

Thus, by defining $T_1 = T_2 L_2$, (45) and (46) can be replaced by $\begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - H_T \begin{bmatrix} T_1 & T_2 \end{bmatrix} = -U \begin{bmatrix} C_1 & 0 \end{bmatrix}$.

Hence, $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}$ is such that;

- $Ker\,T$ is (C, A)-outer detectable, since $H_T = (A + LC)|\mathcal{X}/\mathcal{V}$ is such that $\sigma(H_T) \in \mathcal{C}^-$ for $L = T'(TT')^{-1}U$;
- $Ker\,T \cap Ker\,C = \{0\}$, since the invertibility of T_2 implies $rank\left(\begin{bmatrix} T \\ C \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}\right) = n$.

Furthermore, let $\Gamma = \Gamma' > 0$ be the unique solution of $H_T'\Gamma + \Gamma H_T = -Q_T$. Thus, the following matrices $S = S' \geq 0$ and Z verify (19), (20) and (21) for T defined above:

$$S = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} S_{11} & S_{21}' \\ S_{21} & S_{22}' \end{bmatrix} \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}, \tag{47}$$

with $\begin{bmatrix} S_{11} & S_{21}' \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} T_1' \\ T_2' \end{bmatrix} \Gamma \begin{bmatrix} T_1 & T_2 \end{bmatrix}$

$$\text{and } Z = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} T_1' \\ T_2' \end{bmatrix} U\Gamma \tag{48}$$

□

From the proof given above, it can also be verified that $V = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ such that $Im\,V = Ker\,T$ can be obtained as a basis of the null-space of $\mathcal{T} = \begin{bmatrix} S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}$:

$$\mathcal{T}V = 0 \tag{49}$$

Based on these results, and in the light of the algorithm proposed in [9], the following basic procedure is proposed to compute a stabilizing output feedback when $m + p > n$[1]:

Step 1: Find the orthogonal decomposition $C = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}$ and set $\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = \begin{bmatrix} M_1' \\ M_2' \end{bmatrix} AM_2$;

Step 2: Solve the *reduced-order* generalized Lyapunov equation (42) to find S_{21} and S_{22};

Step 3: Compute V from (49) as an orthogonal basis of $Ker\,\mathcal{T}$, *i.e*: $V'V = I_p$;

Step 4: Solve equations (16), (17) and (18) to find P, Y and W_Π;

Step 5: Compute the stabilizing output feedback matrix as the unique solution of

$$KCP = Y \iff KCV\bar{P} = W_\Pi \text{ , since } V'V = I_p \text{ .}$$

[1] If $m + p = \bar{\nu} \leq n$, a dynamic compensator of order $\nu > n - \bar{\nu}$ can be considered to recover this condition.

The steps of this procedure have a correspondence with those of the algorithm proposed in [8] to compute an output feedback matrix K that assigns stable closed-loop eigenvalues. An example in the next section shows that even in the case $m+p > n$ the basic procedure may fail and some iterations can be necessary to adequately define $\mathcal{V}$.

Although in this study, stability is the only closed-loop requirement, the degrees-of-freedom existing in the quadratic approach given here could be exploited to consider other numerical and closed-loop performance requirements. In particular, additional requirements may be associated to the Lyapunov function defined in corollary 4.2. Also, Q_V and Q_T may be let as free variables in steps 2 and 4, respectively, and convex optimization techniques may be used to solve the corresponding equations [2].

5 EXAMPLES

Consider the following data [5]:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Two different output matrices C are considered. In both cases, the corresponding triples (C, A, B) are controllable and observable and $m + p = 5 > n$. Also, to obtain closed-loop eigenvalues with real parts less than $-\alpha = -2$, computations are done using $(A + \alpha I)$ instead of A. Standard convex programming techniques are applied to find feasible solutions for the coupled quadratic equations, without additional requirements that could improve the numerical solution or the closed-loop performance.

1^{st} *Case*: Consider first the following output matrix: $C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Using an orthogonal matrix M to perform step 1, a matrix $\mathcal{T} = \begin{bmatrix} S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}$ is readily found in step 2:

$$\mathcal{T} = \begin{bmatrix} 0.0006 & 0 & -4.1142 & 1 \end{bmatrix} \begin{bmatrix} -0.5257 & 0 & 0 & -0.8507 \\ 0 & 0 & 1 & 0 \\ 0.8507 & 0 & 0 & -0.5257 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$= \begin{bmatrix} -3.5001 & 1 & 0 & 2.1625 \end{bmatrix}$.

It implies, in step 3: $V = \begin{bmatrix} 0.5107 & 0 & 0.2362 \\ -0.0660 & 0 & 0.9695 \\ 0 & 1 & 0 \\ 0.8572 & 0 & -0.0660 \end{bmatrix}$.

A feasible solution for step 4 is then found:

$$P = \begin{bmatrix} 19753.561 & -41669.85 & 6418.2843 & 51240.969 \\ -41669.85 & 237613.71 & -6863.8855 & -177323.06 \\ 6418.2843 & -6863.8855 & 3104.1283 & 13562.213 \\ 51240.969 & -177323.06 & 13562.213 & 164934.04 \end{bmatrix},$$

$Y = \begin{bmatrix} -51685.095 & -747434.01 & -14350.967 & 261981.08 \\ -33408.348 & 639608.61 & -4267.0209 & -349845.47 \end{bmatrix}$

$W_{\Pi} = \begin{bmatrix} 247529.57 & -14350.967 & -754116.8 \\ -359186.68 & -4267.0209 & 635288.97 \end{bmatrix}$.

The corresponding stabilizing output feedback $K = \begin{bmatrix} -67.1094 & 56.3478 & 84.9136 \\ 34.6173 & -26.0568 & -45.3505 \end{bmatrix}$ gives: $\sigma(A+BKC) = \{ \ -2.5001 \ ; \ -2.1333 \ ; \ -2.5499 \pm 4.4622j \ ; \}$ where the eigenvalue -2.5001 corresponds to step 2, through (43).

2^{nd} *Case*: Consider now the output matrix used in [5]: $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

In this example, convex programming techniques do not produce a feasible solution in step 4, due to the particular matrices $\mathcal{T}$ selected in step 2. However, a stabilizing solution exists. It has been found in [5] by eigenstructure assignment. Thus, the left eigenvector of this solution is selected as a candidate T:
$T = [\ 35 \quad -7 \quad -28 \quad -10\ ]$ so that $\lambda = -4$ must be assigned as a closed-loop eigenvalue, and:

$$V = \begin{bmatrix} 0.2153 & 0.6027 & 0.1507 \\ -0.0185 & -0.0518 & 0.9871 \\ -0.0740 & 0.7928 & -0.0518 \\ 0.9736 & -0.0740 & -0.0185 \end{bmatrix}.$$

Hence, a feasible solution for step 4 can now be obtained:

$P = \begin{bmatrix} 40952.514 & -88879.744 & 14627.136 & 164593.46 \\ -88879.744 & 601592.13 & -19920.323 & -676416.69 \\ 14627.136 & -19920.323 & 6758.6188 & 46215.07 \\ 164593.64 & -676416.69 & 46215.07 & 920167.23 \end{bmatrix}$

$W_{\Pi} = \begin{bmatrix} 603289.9 & -79801.264 & -1989176.3 \\ -1990617.4 & -25183.375 & 2876237.8 \end{bmatrix}$;

$Y = \begin{bmatrix} -217972.83 & -1970444.2 & -4872.8597 & 630050.05 \\ -10282.026 & 2877121.4 & -21648.917 & -1989355.1 \end{bmatrix}$

The corresponding stabilizing output feedback $K = \begin{bmatrix} -116.2499 & 158.3979 & 13.5233 \\ 97.4749 & -122.0786 & -13.4663 \end{bmatrix}$ gives: $\sigma(A + BKC) = \{ \ -4.0 \ ; \ -2.2319 \ ; \ -3.1172 \pm 6.6548j \}$.

6 CONCLUDING REMARKS

The concept of (C, A, B)-invariance has been revisited to obtain a quadratic characterization of the so-called *Output Stabilizable* (C, A, B)-invariant subspaces, which have a fundamental role for output feedback stabilization in linear systems. The paper has focused on both theoretical and algorithmic aspects, and some links with the coupled Sylvester equations approach have been stressed. In particular, a necessary and sufficient condition for the existence of a stabilizing output feedback has been obtained in terms of two coupled-quadratic equations. An algorithm has also been proposed to compute a stabilizing output feedback matrix when $m + p > n$.

Among the remaining open questions for future studies, one may consider the use of the existing degrees of freedom to include additional control requirements, the possible extension to the less restricitive case $mp \geq n$ [1], and numerical and theoretical comparisons with other quadratic approaches.

REFERENCES

[1] A.T. Alexandridis and P.N. Parakevopoulos : *A new approach to Eigenstructure Assignment by Output Feedback.* IEEE Trans. Automatic Control, Vol. 41, No 7, pp. 1046-1050, July 1996.

[2] S.P. Boyd, L. El Ghaoui, E. Feron and V. Balakrishhnan : *Linear Matrix Inequalities in System and Control Theory.* Philadelphia, PA: SIAM, 1994.

[3] C.T. Chen : Linear system theory and design, Holt, Rinehart and Winston, INC, New York, 1984.

[4] C.A.R. Crusius and A. Trofino : *Sufficient LMI Conditions for Output Feedback.* IEEE Trans. Automatic Control, Vol. 44, No 5, pp. 1053-1057, May 1999.

[5] L.R. Fletcher, J. Kautsky, G.K.G Kolka and N.K Nichols : *Some necessary and sufficient conditions for eigenstructure assignment.* Int. J. Control, Vol. 42, No 6, pp. 1457-1468, 1985.

[6] J.C. Geromel, C.C. de Souza and R.E. Skelton : *Static Output Feedback Controllers: Stability and Convexity.* IEEE Trans. Automatic Control, Vol. 43, No 1, pp. 120-125, January 1998.

[7] L. El Ghaoui, F. Oustry and M. AitRami : *A Cone Complementarity Linearization Algorithm for Static Output-Feedback and Related Problems.* IEEE Trans. Automatic Control, Vol. 42, No 8, pp. 1171-1176, August 1997.

[8] V.L. Syrmos and F.L. Lewis : *Output Feedback Eigenstructure Assignment Using Two Sylvester Equations.* IEEE Trans. Automatic Control, Vol. 38, No 3, pp. 495-499, March 1993.

[9] V.L. Syrmos and F.L. Lewis : *A Bilinear Formulation for the Output Feedback Problem in Linear Systems.* IEEE Trans. Automatic Control, Vol. 39, No 2, pp. 410-414, February 1994.

[10] V.L. Syrmos, C.T. Abdallah and K. Grigoriadis : *Static Output Feedback - A Survey.* Automatica, Vol. 33, No 3, pp. 125-137, 1997.

[11] W.M. Wonham: *Linear Multivariable Control, a Geometric Approach*, Springer Verlag, New York, Heidelberg, Berlin, 1979.