Proceedings of the 8th IEEE Mediterranean Conference
on Control and Automation (MED 2000)
Rio, Patras, GREECE 17-19 July, 2000

A DIFFERENTIAL GAME WITH GRAPH CONSTRAINED SWITCHING STRATEGIES

JOÃO BORGES DE SOUSA AND FERNANDO LOBO PEREIRA

DEEC, Faculdade de Engenharia, R. dos Bragas, 4050-123 Porto, Portugal
{jtasso, flp}@fe.up.pt

Abstract

In this paper paper we discuss a two-person zero-sum differential game of finite horizon with graph-constrained control strategies. Both players are constrained to use piecewise constant controls, where the control switches are selected from a finite control graph. The control graph is a directed graph where the vertices define pairs of control values for both players, and the edges define the allowable control switches. Each edge represents a positive switching cost. The payoff function includes the sum of the switching costs associated with each player.
We prove the existence of value and state optimality conditions in terms of value functions that solve a coupled system of quasi-variational inequalities. Optimal strategies are then derived in terms of these value functions.

Keywords: Hybrid systems, graph games, differential games, switching strategies, non-smooth analysis and control.

1 INTRODUCTION

In this paper we introduce a two-person zero-sum differential game of finite horizon where both players are constrained to use piecewise constant controls and the control switches are selected from a finite control graph. The control graph is a directed graph where the vertices define pairs of control values for both players, and the edges define the allowable control switches. Whenever each player decides to switch its control, he/she pays a strictly positive cost. The payoff function includes the sum of the switching costs associated with each player.

We state the existence of value for this game in terms of the value functions that solve a coupled system of quasi-variational inequalities which express the optimality conditions. Optimal controls are derived from the solution of the system of quasi-variational inequalities. The problem is that, generally, value functions are not continuously differentiable. In order to overcome this difficulty, several notions of weak solutions which, under some assumptions are equivalent, were introduced in the literature (see [4], [8], [1]). The value functions solve the quasi-variational inequalities in this weak sense.

The preliminary results presented in this paper constitute part of a research effort that aims at bringing together techniques from differential games [5] and graph games [7], non-smooth analysis and control [4] and partial differential equations ([8],[1]) under a framework of hybrid systems.

The problem of controlling an ordinary differential equation subject to positive switching costs was addressed in [3], where it was proved that value functions are viscosity solutions to dynamic programming quasi-variational inequalities. The corresponding differential game formulation was introduced in [10] as a zero-sum differential game where both players select their switching controls independently and incur positive switching costs. Graph and differential games were introduced in the field of hybrid systems. Hybrid systems and discrete games on graphs are discussed in [7]. In [6], a game theoretic approach to hybrid systems control design is proposed. However, a formulation of differential games with graph constrained switching strategies is not encompassed by any of these approaches. The graph game context is also absent from [2], in which systems with switched controls are addressed, again in the context of hybrid systems.

In this paper, we extend the results from [10] by considering not only a different set of more easily verifiable hypotheses, but also graph constrained switching strategies.

This paper is organized as follows. In Section 2 we formulate this differential game. In Section 3 we state and briefly discuss optimality conditions, the existence of value and control synthesis. In Section 4 we draw some conclusions and discuss future work.

2 PROBLEM FORMULATION

We consider a zero-sum differential game between players A and B having, respectively, at their disposal piecewise constant functions, $u(\cdot)$ and $v(\cdot)$, to control the dynamic system:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t), v(t)), \qquad [0,T]\text{-a.e.} \\ x(0) &= x_0 \end{aligned}$$

where

- $x \in X$ and X is a finite-dimensional Euclidean space.
- $u(t) \in U = \{u_1, u_2, ..., u_N\}$ and $v(t) \in V = \{v_1, v_2, ..., v_M\}$.
- The switching times for $u(\cdot)$ and $v(\cdot)$ are respectively $\{\tau_i\}$ and $\{\sigma_j\}$:
  - $T \geq \tau_i > \tau_{i-1} \geq 0$, $i = 1, \ldots, \bar{N}$ for some $\bar{N} < \infty$.
  - $T \geq \sigma_j > \sigma_{j-1} \geq 0$, $j = 1, \ldots, \bar{M}$ for some $\bar{M} < \infty$.
  - $u(t)$ and $v(t)$ are selected from the control graph $\mathcal{G}$, that is defined next.

Except for the control graph constraint, this formulation is essentially the same as the one reported in [10]. By constraining the control pairs (u(t), v(t)) to evolve in a directed graph, where the edges define the allowable control switches, we intend to capture the modeling features of discrete graph games [7], now in a differential game setting.

Definition 1 (Control graph $\mathcal{G}$) $\mathcal{G} := (\mathcal{N}, \mathcal{E})$, *where:*

- *the set of nodes is* $\mathcal{N} := \{n_p : p = 1, ...P\}$ *with* $n_p := (u_k, v_l)$ *for some* $k = 1, ..., N$*; and* $l = 1, ..., M$*, and*
- *the arcs* $\mathcal{E}$ *define the allowable control switches. The set of arcs is generated by the set-valued function* $e : \mathcal{N} \to 2^{\mathcal{N}}$*, where the edges out of the vertex* (u, v) *are all the pairs of the form* $((u, v), (\bar{u}, \bar{v}))$ *where* $(\bar{u}, \bar{v}) \in e(u, v)$*.*

The payoff function for this zero-sum game is:

$$J_\eta(u(\cdot), v(\cdot)) = h(x(T)) + \int_s^T g(t, x(t), u(t), v(t))dt + \sum_{i\geq 1} k(u_{i-1}, u_i) - \sum_{i \geq 1} l(v_{i-1}, v_i)$$

where:

- $\eta := (\bar{u}, \bar{v}, s, \bar{x}) = (u(s), v(s), s, x(s))$
- $k(a, b)$, where $a, b \in U$, and $l(c, d)$, where $c, d \in V$, are the switching cost functions.

We consider the following standing hypotheses on the data of this problem:

(H1) h is Lipschitz continuous with constant K_h.

(H2) f and g are continuous in u and v and Lipschitz continuous in x with constants K_f and K_g respectively. f is measurable in t and g continuous in t.

(H3) U and V are finite bounded sets.

(H4) k and l are bounded $\forall (u, v) \in U \times V$.

Hypotheses (H1-2) are related to the existence and uniqueness of trajectories for the dynamic system under consideration. Hypotheses (H3-4) are used to establish that the pay-off function is well defined.

Next we define the class of strategies available to both players in this differential game.

Let $n_p = (u_k, v_l)$ and consider the projection operators, $Proj_U(n_p) = u_k$ and $Proj_V(n_p) = v_l$ and the concatenation operator $\Pi_{i=1}^n u_i = u_1 u_2 \cdots u_n$. A feasible control sequence of length L starting at node $\bar{q} \in \mathcal{N}$ and denoted as $G_{\bar{q}}^L$ is given by

$$G_{\bar{q}}^L := \{\Pi_{j=0}^{L-1} q_j : q_j \in e(q_{j-1}) j \geq 1 \; q_0 = \bar{q}\}$$

A feasible control function $u(\cdot)$ is built from this sequence by timing the control switches that occur in the corresponding sequence of projections $Proj_U(q_0)$, ..., $Proj_U(q_{L-1})$. Hence, the following definition of the corresponding feasible control sets $\mathcal{U}^{\bar{u},s}$ and $\mathcal{V}^{\bar{v},s}$ for players A and B, respectively.

Definition 2 (Feasible control sets) [1]*:*

a) $\mathcal{U}^{\bar{u},s} := \{u(\cdot) : [\tau_0, T] \to U|$
$\exists L, \tau_L \leq T, \alpha \in G_{\bar{q}}^L, (\bar{q} = (\bar{u}, v) \in \mathcal{N}),\ \tau_0 = s,$
with $u(\cdot) = \sum_{j=1}^L u_{n_{j-1}} \chi_{[\tau_{j-1}, \tau_j)}(\cdot)\}$

b) $\mathcal{V}^{\bar{v},s} := \{v(\cdot) : [\sigma_0, T] \to V|$
$\exists F, \sigma_F \leq T, \alpha \in G_{\bar{q}}^F, (\bar{q} = (u, \bar{v}) \in \mathcal{N}),\ \sigma_0 = s,$
with $v(\cdot) = \sum_{j=1}^F v_{m_{j-1}} \chi_{[\sigma_{j-1}, \sigma_j)}(\cdot)\}$.

An arbitrary selection of a pair of feasible control functions $u(.)$ and $v(.)$ is not, in terms of the control graph, necessarily admissible. In fact, the control graph imposes joint constraints on feasible control functions.

[1] $\chi_A(t)$ is the indicator function of the set A, i.e., $\chi_A(t) = 1$ if $t \in A$ and $\chi_A(t) = 0$ otherwise.

Definition 3 (Compatible control functions) *Compatible control functions $u(\cdot)$ and $v(\cdot)$ for both players must satisfy the additional constraint*

$$\forall t \geq s; \quad \textit{either } (u(t), v(t)) \in (e(u(t^-), v(t^-)) \\ \textit{or } (u(t), v(t)) = (u(t^-), v(t^-)),$$

where $(u(s), v(s)) = (\bar{u}, \bar{v}) \in \mathcal{N}$

The definition of the admissible strategies $\alpha^{\bar{u},s}$ and $\beta^{\bar{v},s}$ for both players is in order.

Definition 4 (Admissible strategies) *: Let $\alpha^{\bar{u},s}$ and $\beta^{\bar{v},s}$ be defined as follows:*

$$\forall s \in [0,T), \bar{u} \in U, \alpha^{\bar{u},s} : \bigcup_{\bar{v} \in V : (\bar{u},\bar{v}) \in \mathcal{N}} \mathcal{V}^{\bar{v},s} \to \mathcal{U}^{\bar{u},s},$$

$$(\forall s \in [0,T), \bar{v} \in V, \beta^{\bar{v},s} : \bigcup_{\bar{u} \in U : (\bar{u},\bar{v}) \in \mathcal{N}} \mathcal{U}^{\bar{u},s} \to \mathcal{V}^{\bar{v},s})$$

$\alpha^{\bar{u},s}$ ($\beta^{\bar{v},s}$) is a feasible strategy for player A (B) if:

$$v(t) = \tilde{v}(t)(u(t) = \tilde{u}(t)), \forall t \in [s, \tilde{s}]$$

implies that, for such t,

$$\alpha^{\bar{u},s}[v(\cdot)](t) = \alpha^{\bar{u},s}[\tilde{v}(\cdot)](t)$$

$$(\beta^{\bar{v},s}[u(\cdot)](t) = \beta^{\bar{v},s}[\tilde{u}(\cdot)](t)).$$

It is now possible to define the sets of admissible strategies $\Gamma^u[s,T]$ and $\Gamma^v[s,T]$ for this differential game.

Definition 5 (Sets of admissible strategies) *$\Gamma^u[s,T]$ (where $\Gamma^u[T,T] = \{u\}$) and $\Gamma^v[s,T]$ (where $\Gamma^v[T,T] = \{v\}$) are the sets of all admissible strategies for players A and B respectively.*

We consider the following additional hypotheses:

(H5) Let $n = (u,v), \bar{n} = (\bar{u},\bar{v}), \tilde{n} = (\tilde{u},\tilde{v})$. Then, $\forall (n,\tilde{n}), (n,\bar{n}), (\bar{n},\tilde{n}) \in \mathcal{E}$, the following holds:

- $k(u,\tilde{u}) \leq k(u,\bar{u}) + k(\bar{u},\tilde{u})$,
- $l(v,\tilde{v}) \leq l(v,\bar{v}) + l(\bar{v},\tilde{v})$,
- $k(u,\tilde{u}) \geq 0$, $l(v,\tilde{v}) \geq 0$, with $k(u,\tilde{u}) = 0$ and $l(v,\tilde{v}) = 0$ only if $u = \tilde{u}$ and $v = \tilde{v}$, respectively.

(H6) There are constants $\bar{K}$ and $\bar{L}$ such that, for any integer J, $J \geq 1$ and feasible sequence $\{e_{i+j}\}_{j=0}^{J}$, the following inequalities are satisfied:

$$\sum_{j=0}^{J-1} k(u_{n_{i+j}}, u_{n_{i+j+1}}) \leq \bar{K}(\tau_{n_i} - \tau_{n_{i+J}})$$

$$\sum_{j=0}^{J-1} l(v_{m_{i+j}}, v_{m_{i+j+1}}) \leq \bar{L}(\sigma_{m_i} - \sigma_{m_{i+J}})$$

(H7) The players select controls one at a time and in turns. This means that for each i, either $u_{n_i} = u_{n_{i+1}}$ or $v_{m_i} = v_{m_{i+1}}$.

(H8) For any loop $\{q_1, ..., q_S\}$ in the graph, it holds that

$$\sum_{j=1}^{S} k(u_j, u_{j+1}) - \sum_{j=1}^{S} l(v_j, v_{j+1}) \neq 0, \forall s \in [0,T]$$

Notice that, under the hypotheses (H5-8), it can be shown that the minimum time interval between two consecutive jumps is strictly positive. Hence, the number of switches for each player in each finite time interval is finite but not defined apriori, i.e., it will follow from the choice of the control strategy.

Under the above hypotheses there exists a unique solution $x(\cdot)$ to the problem:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t), v(t)), \qquad [s,T]\text{-a.e.} \\ x(s) &= x_0 \end{aligned}$$

where $(\bar{u},\bar{v}) \in \mathcal{N}, x \in X$ and u(.),v(.) are compatible control functions. Then, the payoff functional is well defined.

3 OPTIMALITY CONDITIONS

In the sequel we present a sequence of results which lead to the statement of the existence of value for this game. We have adapted the proof techniques from [3] and [10] in order to take into account the graph constrained switching strategies. Hypothesis H6 was crucial in the process of adapting these proofs, namely in what concerns uniqueness of jumps and to prevent the occurrence of instantaneous loops in the control graph.

Let $x_{s,\bar{x}}(t)$ denote the state trajectory at time $t > s$ when $x(s) = \bar{x}$.

The lower (V) and upper (U) value functions for this game are defined next.

Definition 6 (Lower and upper value functions)

$$\begin{aligned} V(\eta) &= \inf_{\alpha \in \Gamma^{\bar{u}}[s,T]} \sup_{v(\cdot)\in \mathcal{V}^{\bar{v},s}} J_\eta(\alpha(\cdot), v(\cdot)) \\ U(\eta) &= \sup_{\beta \in \Gamma^{\bar{v}}[s,T]} \inf_{u(\cdot)\in \mathcal{U}^{\bar{u},s}} J_\eta(u(\cdot), \beta(\cdot)) \end{aligned}$$

with

$$J_\eta|_{s=T}(u(\cdot), v(\cdot)) = V(\eta)|_{s=T} = U(\eta)|_{s=T} = h(\bar{x}).$$

Straightforward arguments reveal that both the lower and the upper value functions satisfy an optimality principle. Below, we only state the one for the former.

Theorem 1 *For any feasible pair $(\bar{u}, \bar{v}) \in \mathcal{N}$, for all $x \in X$ and $0 \le s < \tilde{s} \le T$, we have*

$$\begin{aligned} &V(\bar{u}, \bar{v}, s, \bar{x}) = \\ &\inf_{\alpha \in \Gamma^{\bar{u}}[s,T]} \sup_{v(\cdot)\in \mathcal{V}^{\bar{v},s}} \Big\{ \int_s^{\tilde{s}} g(t, x_{s,\bar{x}}(t), \alpha[v(\cdot)](t), v(t)) dt \\ &\qquad + \sum_{i\ge 1} k(u_{i-1}, u_i) - \sum_{j \ge 1]} l(v_{j-1}, v_j) \\ &\qquad + V(\alpha[v(\cdot)](\tilde{s}^+), v(\tilde{s}^+), \tilde{s}, x_{s,\bar{x}}(\tilde{s})) \Big\}. \end{aligned}$$

In order to state the main results it is convenient to introduce the following operators.

Definition 7 (Obstacle operators)

$$\begin{aligned} &M^+[V](u, v, s, x) = \\ &\qquad \min_{\overline{u}\neq u, \overline{u} \in Proj_U(e(u,v))} \{V(\overline{u}, v, s, x) + k(u, \overline{u})\} \end{aligned}$$

$$\begin{aligned} &M^-[V](u, v, s, x) = \\ &\qquad \max_{\overline{v}\neq v, \overline{v} \in Proj_V(e(u,v))} \{V(u, \overline{v}, s, x) - l(v, \overline{v})\} \end{aligned}$$

Let $H(\bar{u}, \bar{v}, s, x, p) = \langle p, f(s, x, \bar{u}, \bar{v})\rangle + g(s, x, \bar{u}, \bar{v})$

Theorem 2 *The lower value function $V(\cdot)$ and the upper value function $U(\cdot)$ are viscosity solutions of the quasi-variational inequality with bilateral obstacles:*

$$M^-[V](\eta) \le V(\eta) \le M^+[V](\eta)$$

on the set $\{(s, x) \in [0, T] \times X : M^-[V](\eta) < V(\eta)\}$,

$$V_s(\eta) + H(\eta, V_x(\eta)) \ge 0$$

on the set $\{(s, x) \in [0, T] \times X : M^+[V](\eta) > V(\eta)\}$,

$$V_s(\eta) + H(\eta, V_x(\eta)) \le 0$$

with terminal condition $V(\bar{u}, \bar{v}, T, x) = h(x)$

Theorem 3 *Under the above hypotheses the upper and lower value functions coincide and the value of the differential game exists.*

The optimal controls can be derived from these optimality conditions. Let $\tau_0 = 0$, $\sigma_0 = 0$, $(\bar{u}, \bar{v}) = (u_0, v_0) \in \mathcal{N}$ and $x(0) = \bar{x}$, then:

1. The <u>optimal switching times</u> τ_i, σ_j are defined by the following expressions:

$$\tau_i = \begin{cases} \inf\{t > \tau_{i-1} : V(u(t^-), v(t^-), 0, x_{0,\bar{x}}(t)) \\ \qquad = M^+[V](u(t^-), v(t^-), 0, x_{0,\bar{x}}(t))\} \\ T^+ \quad \text{if the above set is empty} \end{cases}$$

$$\sigma_j = \begin{cases} \inf\{t > \sigma_{j-1} : V(u(t^-), v(t^-), 0, x_{0,\bar{x}}(t)) \\ \qquad = M^-[V](u(t^-), v(t^-), 0, x_{0,\bar{x}}(t))\} \\ T^+ \quad \text{if the above set is empty} \end{cases}$$

2. The corresponding <u>optimal controls</u> u(.) and v(.) are defined by:

$$u_i = \begin{cases} min\{u \in Proj_U(e(u(\tau_i^-), v(\tau_i^-))) : \\ \quad u \ne u_{i-1} \text{ and} \\ \quad M^+[V](u(\tau_i^-), v(\tau_i^-), 0, x_{0,\bar{x}}(\tau_i)) = \\ \quad V(\bar{u}, \bar{v}, 0, x_{0,\bar{x}}(\tau_i)) + k(u(\tau_i^-), u)\} \\ \qquad\qquad \text{if } \tau_i \le T \\ u_{i-1} \qquad\qquad \text{if } \tau_i > T \end{cases}$$

$$v_j = \begin{cases} max\{v \in Proj_V(e(u(\sigma_i^-), v(\sigma_i^-))) : \\ \quad v \ne v_{i-1} \text{ and} \\ \quad M^-[V](u(\sigma_i^-), v(\sigma_i^-), 0, x_{0,\bar{x}}(\sigma_i)) = \\ \quad V(\bar{u}, \bar{v}, 0, x_{0,\bar{x}}(\sigma_i)) - l(v(\sigma_i^-), v)\} \\ \qquad\qquad \text{if } \sigma_i \le T \\ v_{i-1} \qquad\qquad \text{if } \sigma_i > T \end{cases}$$

4 CONCLUSIONS

This paper reports preliminary results concerning differential games with graph constrained strategies. The novelty of the approach consists in constraining the switching strategies to a finite graph. Future work involves extending this framework in order to encompass more general transition systems, to incorporate vector-valued controls and interpreting the results in terms of impulsive control methods [9].

Ackowledgments

This work was partially funded by Fundacao Luso-Americana para o Desenvolvimento.

References

[1] Martino Bardi and Italo Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.* Systems & control. Birkhauser, 1997.

[2] Michael Branicky. *Studies in Hybrid Systems: Modeling, Analysis and Control.* PhD thesis, MIT, 1995.

[3] I. Capuzzo Dolcetta and L. C. Evans. Optimal switching for ordinary differential equations. *SIAM J. Control and Opt.*, 22(1):143–161, 1984.

[4] F. H. Clarke et. al. *Nonsmooth Analysis and Control Theory.* Graduate texts in mathematics. Springer, 1998.

[5] N.N. Krasovskii and A.I. Subbotin. *Game-theoretical control problems.* Springer-Verlag, 1988.

[6] J. Lygeros, Datta N. Godbole, and Shankar Sastry. A game theoretic approach to hybrid system design. Technical Report UCB/ERL M95/77, University of California, Berkeley. Electronics Research Laboratory, 1995.

[7] Anuj Puri. *Theory of hybrid systems and discrete event systems.* PhD thesis, University of California at Berkeley, 1995.

[8] A.I. Subbotin. *Generalized solutions of first-order PDEs : the dynamical optimization perspective.* Systems & control. Birkhauser, 1995.

[9] R. B. Vinter and F. L. Pereira. A maximum principle for optimal processes with discontinuous trajectories. *SIAM J. Control and Opt.*, pages 205–229, 1988.

[10] Jiogmin Yong. A zero-sum differential game in a finite duration with switching strategies. *SIAM J. Control and Opt.*, 28(5):1234–1250, 1990.