# The Square Root of Linear Time Varying Systems With Applications

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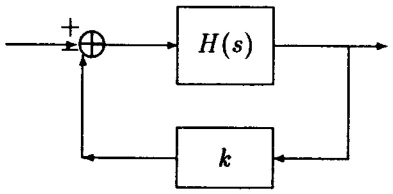


Figure 1: A closed-loop configuration

## Abstract

This paper considers the extension of a number of passive multiplier theory based results, previously known only for linear time invariant scalar systems, to time varying multivariable settings. The extensions obtained here have important applications to the stability of both adaptive systems and linear systems in general. We demonstrate in this paper that at the heart of the extensions carried out here lies the result that if a stable multivariable, linear time varying system is stable under all scalar constant, positive feedback gains, then it has a well defined square root. The existence of this square root is demonstrated through a constructive Newton-Raphson based algorithm. The various extensions provided here though different in form from their linear time invariant scalar counterparts, do recover these as a special case.

## 1 Introduction and Problem Motivation

This paper is concerned with finding time-varying, multivariable generalizations of some multiplier theory results involving Strictly Positive Real (SPR) functions.

The following is a well known result in linear systems theory [1]. Consider an asymptotically stable linear time invariant (LTI), single input single output (SISO) system with a strictly proper transfer function $H(s)$. Then the system in Fig. 1 is asymptotically stable for all

$$0 \leq \quad k \leq 1, \tag{1.1}$$

if, and only if, there exists a SPR scalar operator Z(s), such that

$$Z(s)(1 + H(s)) \tag{1.2}$$

is SPR. The concept of a SPR operator is defined as follows.

**Definition 1.1** *A real, square matrix transfer function Z(s) is Positive Real (PR) if:*

1. *Z(s) is analytic in the right half plane; and*
2. *for all $Re[s] \geq 0$,*

$$Z(s) + Z^H(s) \geq 0$$

*where the superscript H denotes the Hermitian transpose.*

*We say Z(s) is SPR if for some $\alpha > 0$, $Z(s - \alpha)$ is PR.*

From this result spring a number of other important results of which two are cited below: The first states that two scalar polynomials of equal degree $p_1(s)$ and $p_2(s)$, have the property that $p_1(s) + kp_2(s)$ is Hurwitz (i.e. has roots in the open left half plane) for all k as in (1.1) iff there exists an asymptotically stable minimum phase $G(s)$, such that $G(s)(p_1(s) + kp_2(s))$ is Strictly Positive Real (SPR) for all $k$ as in (1.1); in turn, this holds iff there exists an asymptotically stable minimum phase $G(s)$ such that $G(s)p_1(s)$ and $G(s)(p_1(s) + p_2(s))$ are SPR. As will be evident in a later section of this paper, this has an important application in certain adaptive systems problems involving a single unknown parameter.

The second result concerns the stability of a class of linear time varying (LTV) systems. Specifically, suppose that the configuration in Fig. 1 is stable with ***a degree of stability*** $\alpha$ for all $k$ as in (1.1). Now consider the LTV systems obtained in Fig. 1, when the feedback gain $k(t)$ is allowed to be time varying while obeying

$$0 < \quad k(t) \quad < 1. \tag{1.3}$$

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Then it has been shown in [2, 3] that the closed loop retains stability whenever, there exist $T$ and $\delta \in (0, \alpha)$ for which

$$\sup_{t \geq 0} \frac{1}{T} \int_t^{t+T} \left[ \frac{d}{d\tau} ln \frac{k(\tau)}{1-k(\tau)} \right]^+ d\tau < 2(\alpha - \delta) \tag{1.4}$$

where

$$[a]^+ = \begin{cases} a; & a \geq 0 \\ 0; & a < 0 \end{cases}$$

See [4] for an association between the result of [1] and that of [2, 3], using tools that include the Popov-Kalman-Yakubovic (PKY) Lemma.

The question addressed in this paper is: *to what extent do these results extend to systems that are LTV or for that matter multiple input multiple output (MIMO) LTI?* The ability to answer this question depends critically on the existence of the square root of certain LTV systems. This can be understood by noting that the result of [1] can itself be viewed in the following terms. The stability of the closed loop of Fig. 1 for all $k \in [0, 1]$ is equivalent to the transfer function $1 + H(s)$ having a phase function that lies in $(-\pi, \pi)$, [5]. Accordingly, there is a well defined square root of $1 + H(s)$. Then a $Z(s)$ chosen as a suitable approximant of the inverse of this square root will be SPR, as indeed will be the product in (1.2).

Having dispensed with some preliminaries in Section 2, the first question we ask concerns square, LTV, strictly causal continuous operators $H$ (observe this class obviously includes square MIMO, LTI operators) such that both $H$ and $[I + kH]^{-1}$ are stable for all $k$ as in (1.1). Observe, that this corresponds to a stable closed loop of the form of Fig. 1, with the LTV operator $H$ occupying the position of $H(s)$. Then, using a Newton-Raphson technique, we demonstrate in Section 3, that in such a case $I + H$ does indeed have a square root. Of course $H$ is presumed to be the operator relating inputs and outputs of a strictly causal system. Stability corresponds to the boundedness of the operator given suitable input and output norms.

Sections 4 and 5 respectively provide the analogs of the result of [1] and its first consequence mentioned in the foregoing. Both these results assume that $H$ is finite dimensional, i.e. has a finite dimensional state variable description. Section 5 also discusses the application of this latter result to certain types of adaptive identification algorithms involving MIMO, LTI systems. The results of this Section also resolve an open problem presented in [6]. Section 6 derives the analog of the [2, 3] result. Each of the results in Sections 4 through 6, though different from their SISO, LTI counterparts, capture these as special cases. Section 7 is the conclusion. Most proofs are omitted due to space constraints. They can be found in [15].

## 2 Preliminaries

All systems in this paper will be represented by square, LTV, real, continuous operators mapping $L_2$ to $L_2$. Consider such an operator $G$. Then $G^a$ will denote the *Adjoint* of $G$, i.e. if $G$ has impulse response $g(t, \tau)$ then $G^a$ has the impulse response $g'(\tau, t)$. For an input signal $x(t)$, $Gx$ will denote the corresponding output, i.e. if $g(t, \tau)$ is the impulse response of $G$ then

$$[Gx](t) = \int_{-\infty}^{\infty} g(t, \tau) x(\tau) d\tau. \tag{2.1}$$

This operator is *causal* if $g(t, \tau) = 0 \ \forall t < \tau$. In this case the upper limit in the integral of (2.1) can be replaced by $t$. The norm of $G$ will be the induced $L_2$ operator norm. In the sequel we will use the terms bounded and stable interchangeably to signify operators that have a finite norm. Moreover, the operator $G^n$ for a positive integer $n$ will designate the combined operator obtained by a cascade of $n$ operators $G$. A bounded operator $R : L_2 \rightarrow L_2$ will be called the inverse of $G$ if $GR = RG = I$. In such a case we denote $R = G^{-1}$ and note that the existence of $G^{-1}$ automatically signifies its stability. Further, $G$ will be called *symmetric* or *self adjoint* if

$$G^a = G. \tag{2.2}$$

Every symmetric operator $G$ can in turn be expressed as:

$$G = G_c + G_{ac} \tag{2.3}$$

where $G_c$ is causal and called the *causal part* of $G$; $G_{ac}$ is anticausal and called the *anticausal part* of $G$; and together they obey

$$G_c^a = G_{ac}. \tag{2.4}$$

If a term such as $\alpha I$ appears in $G$ then it will be shared equally between $G_c$ and $G_{ac}$; i.e. each of $G_c$ and $G_{ac}$ will get $0.5\alpha I$.

Two assumptions are needed.

**Assumption 2.1** *The operator $H : L_2 \rightarrow L_2$ is causal and $[I + kH]^{-1} : L_2 \rightarrow L_2$ is invertible and causal for all $k \in [0, 1]$. Further, there exist numbers $M_1$ and $M_2$ such that:*

$$\|[I + kH]^{-1}\| \leq M_1 \qquad \forall k \in [0, 1]; \tag{2.5}$$

*and*

$$\|H\| \leq M_2. \tag{2.6}$$

*Moreover, the impulse response $h(t, \tau)$ of $H$ is finite for all finite $t$ and $\tau$.*

**Remark 2.1** *The boundedness assumption on $h(t, \tau)$ precludes the presence of impulse functions in $h(t, \tau)$.*

To provide the assumption on the state variable realization (SVR) of $H$ we introduce the following notation. For a given continuous square matrix function $A(t)$ we designate

$$A_\alpha(t) = \alpha I + A(t). \tag{2.7}$$

We will be concerned with the notion of degree of stability of operators such as $H$. To this end we introduce $H_\alpha$ having SVR:

$$\{A_\alpha(t), B(t), C(t)\}, \tag{2.8}$$

where each of $A_\alpha(t), B(t), C(t)$ is a bounded, continuous function of time. We also make the following definition.

**Definition 2.1** *The matrix $A(t)$ is exponentially asymptotically stable with degree of stability $\alpha > 0$ ($\alpha$-eas) if for the LTV system*

$$\dot{x}(t) = A(t)x(t) \tag{2.9}$$

*$\exists c, \gamma > 0$ such that for all $x(t_0)$ and $t \geq t_0$,*

$$||x(t)||e^{\alpha(t-t_0)} \leq c||x(t_0)||e^{-\gamma(t-t_0)}. \tag{2.10}$$

*If $\alpha = 0$, we simply say that $A(t)$ is eas. Further, we will call a system with SVR, $\{A(t), B(t), C(t), D(t)\}$, all matrices bounded and continuous, $\alpha$-eas ( resp. eas) if $A(t)$ is $\alpha$-eas ( resp. eas).*

Then we have the following assumption.

**Assumption 2.2** *The system $H_\alpha$ has an SVR*

$$\{A_\alpha(t), B(t), C(t)\},$$

*such that $[A_\alpha(t), B(t)]$ is uniformly completely controllable (u.c.c.), [7], $[A_\alpha(t), C(t)]$ is uniformly completely observable (u.c.o.), [7], and both the systems $H_\alpha$ and*

$$\{A_\alpha(t) - kB(t)C'(t), B(t), -kC(t), I\}$$

*are eas, for all $k \in [0, 1]$.*

**Remark 2.2** *If $H_\alpha$ satisfies Assumption 2.2 then it also satisfies Assumption 2.1.*

The LTV analog of SPR is Strict Passivity, in turn equivalent to the concept of a positive operator.

**Definition 2.2** *An operator $P : L_2 \to L_2$ is called Strictly Positive ($P \geq \epsilon I > 0$) if for all $x$ in $L_2$*

$$< x, Px > \quad \geq \quad \epsilon < x, x >, \tag{2.11}$$

*where $<,>$ denotes the norm in $L_2$.*

In Section 4 we will need the concept of *Spectrum* of a LTV operator.

**Definition 2.3** *The resolvent set $\rho(H)$ of an operator $H : L_2 \to L_2$ is the set of all complex numbers $\lambda$ such that $[\lambda I - H]^{-1} : L_2 \to L_2$ exists. The complement of all $\rho(H)$ in the complex plane is called the spectrum of $H$ and is denoted $\sigma(H)$.*

## 3 Existence of the Square Root

The principal contribution of this Section is:

1. to demonstrate that subject to Assumption 2.1, I+H has a square root and
2. to give an algorithm for constructing this square root.

In the sequel, we say that $G : L_2 \to L_2$ is the square root of $I + F$ with $F : L_2 \to L_2$ if

$$G^2 = I + F. \tag{3.1}$$

To compute the square root we propose the following Newton-Raphson based algorithm,

$$G_{i+1} = \frac{1}{2}[(I + F)G_i^{-1} + G_i]. \tag{3.2}$$

When initiated with $G_0 = I$, it will be shown that the successive $G_i$ are rational in $F$, that $G_i$ and $\Delta G$ commute, and that under suitable assumptions on $F$, $G_i$ is invertible for all $i$. Observe if $G_i = G_{i+1}$ then $G_i^2 = I + F$.

The global convergence of (3.2) is difficult to demonstrate. On the other hand as will be evident in the sequel, it is possible to determine a number $\epsilon$ such that whenever

$$||F|| \leq \epsilon, \tag{3.3}$$

(3.2) converges uniformly whenever it is initiated by $G_0 = I$, i.e. $NR(I + F, I)$ exists. To circumvent the apparent difficulty inherent in the restriction (3.3), we will adopt a nested Newton-Raphson strategy for determining the square root of $I + H$. Specifically, we will select suitably small $\delta_1$ and $\delta_2$, so that (see (2.5,2.6)),

$$\delta_1 M_2 \leq \min \ \{\epsilon, 1\}, \tag{3.4}$$

and

$$\delta_2 M_1 M_2 \leq \min \ \{\epsilon, 1\}. \tag{3.5}$$

Further, choose $(1 - \delta_1)/\delta_2$ as an *integer* (N.B. this can always be done without violating (3.4,3.5)), and define $N$:

$$N = \frac{1 - \delta_1}{\delta_2}. \tag{3.6}$$

Note

$$\delta_1 + N\delta_2 = 1. \tag{3.7}$$

Then the nested approach first divides the interval $[0, 1]$ into intervals $[0, \delta_1]$, $[\delta_1, \delta_1 + \delta_2]$, $[\delta_1 + \delta_2, \delta_1 + 2\delta_2]$, etc. up to and including $[\delta_1 + (N - 1)\delta_2, 1]$ (see (3.7)). It then uses (3.2) to compute the square root of $[I + \delta_1 H]$. Because of (2.6), (3.4) and the definition of $\epsilon$, this is possible. It then uses the square root of $[I + \delta_1 H]$ to compute the square root of $[I + (\delta_1 + \delta_2)H]$, etc. until eventually the square root of $[I + H]$ has been obtained. More precisely the nested algorithm proceeds as follows.

1. If $N = 0$, i.e. $\delta_1 = 1$, then the definition of $\epsilon$ and (2.6,3.4) assure that $NR(I+H, I)$ exists. Thus, from here onwards assume $N > 0$.

2. Find
$$U_0 = NR(I + \delta_1 H, I). \quad (3.8)$$

3. For all $1 \leq m \leq N$, determine, should it exist:
$$V_m = NR(I + \delta_2 U_{m-1}^{-1} H U_{m-1}^{-1}, I) \quad (3.9)$$
and
$$U_m = U_{m-1} V_m. \quad (3.10)$$

**Remark 3.1** *As will become evident in the sequel, for each $m \geq 0$, $U_m$ represents the square root of $[I + (\delta_1 + m\delta_2)H]$, and $V_m$ the square root of $[I + \delta_2 U_{m-1}^{-1} H U_{m-1}^{-1}]$. Moreover, $V_m$ exists because of (3.5,3.4) which together will be shown to force*
$$\|\delta_2 U_{m-1}^{-1} H U_{m-1}^{-1}\| \leq \epsilon. \quad (3.11)$$
*Notice also, (3.7), that $U_N$ is the square root of $[I + H]$.*

The following Theore demonstrates the convergence of (3.2) under small perturbations.

**Theorem 3.1** *Let $F : L_2 \to L_2$ be bounded and causal, $[I + kF]^{-1}$ exist and be causal for all $k \in [0,1]$ and let $G_i$, be the sequence of operators defined by (3.2) and $G_0 = I$. Then, there exists an $\epsilon > 0$ such that whenever (3.3) holds, so do the following for all $k \in [0,1]$: (i) there exists bounded $G(kF) = \lim_{i\to\infty} G_i(kF) : L_2 \to L_2$; (ii) $G(kF)^{-1} : L_2 \to L_2$ exists and both $G(kF)$ and $G(kF)^{-1}$ are causal; (iii) $G(kF)$ and $G(kF)^{-1}$ commute with any operator that commutes with $F$; (iv) $G(0) = I$; (v) $G(kF)$ varies continuously with $k$; and (vi) $G^2(kF) = I + kF$.*

**Remark 3.2** *We note that the convergence rate is faster than exponential.*

Using this Theorem we now prove the convergence of the nested algorithm.

**Theorem 3.2** *Consider the nested Newton-Raphson algorithm, i.e. (3.3-3.10), with $NR(I + F, I)$ the convergent point (should it exist) of (3.2) initiated with $G_0 = I$. Suppose Assumption 2.1 holds as do (3.4),(3.5) and (3.7) and that $\epsilon$ is as in Theorem 3.1. Then the bounded operators $U_N : L_2 \to L_2$ and $U_N^{-1} : L_2 \to L_2$ exist and*
$$U_N^2 = I + H.$$

We will prove this Theorem using an inductive argument. To this end the following Lemma helps sustain the induction.

**Lemma 3.1** *With the conditions of Theorem 3.2 in force suppose that for some $0 < m \leq N$ the following hold.*

1. *The bounded operators $U_{m-1}$ and its inverse exist; and both commute with any operator that commutes with $H$.*

2. $$U_{m-1}^2 = [I + (\delta_1 + (m-1)\delta_2)H]. \quad (3.12)$$

*Then the bounded operators $U_m$ and its inverse exist; both commute with any operator that commutes with $H$; and*
$$U_m^2 = [I + (\delta_1 + m\delta_2)H]. \quad (3.13)$$

**Proof:** First observe that because of item 1 of the Lemma statement
$$\begin{aligned} \|\delta_2 U_{m-1}^{-1} H U_{m-1}^{-1}\| &= \|\delta_2 H U_{m-1}^{-2}\| \\ &= \|\delta_2 H [I + (\delta_1 + (m-1)\delta_2)H]^{-1}\| \\ &\leq \delta_2 M_1 M_2 \\ &\leq \epsilon. \end{aligned} \quad (3.14)$$
where the last two inequalities arise consequent to (2.5,2.6,3.5,3.4, 3.7). Thus, from Theorem 3.1 and (3.9), $V_m$ and its inverse exist and commute with any operator that commutes with $U_{m-1}^{-1} H U_{m-1}^{-1}$. Then because of the commutativity and invertibility hypothesized in item 1 of the Lemma statement, $U_{m-1}^{-1} H U_{m-1}^{-1}$ commutes with any operator that commutes with $H$. Hence $V_m$ and its inverse also commute with any operator that commutes with $H$. Thus, from (3.10) $U_m^{-1}$ exists and together with $U_m$ satisfies the required commutativity property.

Because of (3.9)
$$V_m^2 = I + \delta_2 U_{m-1}^{-1} H U_{m-1}^{-1}. \quad (3.15)$$
Further, the commutativity property established on $V_m$ assures that $V_m$ commutes with both $U_{m-1}$ and $H$.

Thus, from (3.10) (3.12) and (3.15) we obtain,
$$\begin{aligned} [I + (\delta_1 + m\delta_2)H] &= [I + (\delta_1 + (m-1)\delta_2)H] + \delta_2 H \\ &= U_{m-1}[I + \delta_2 U_{m-1}^{-1} H U_{m-1}^{-1}] U_{m-1} \\ &= U_{m-1} V_m^2 U_{m-1} \\ &= (U_{m-1} V_m)^2 \\ &= U_m^2. \end{aligned} \quad (3.16)$$
This completes the proof. ■

Theorem 3.2 then follows readily from Lemma 3.1, and the fact that that (2.6) and (3.4) ensure that $U_0$ in (3.8), together with its inverse, exists and commutes with any operator that commutes with $H$.

It is clear from (3.8), (3.9) and Theorem 3.1 that each of the operators $U_0$ and $V_m$, $m = 1, \cdots, N$ are obtained as the limit point of *uniformly convergent* sequences. Thus, by running each procedure implicit in their determination for a finite but arbitrarily large number of iterations, one can obtain a $\hat{X}$ such that with $X$ the square root of $I+H$, $\|\hat{X}-X\|$ is arbitrarily small. Since the convergence rate is greater than exponential one can expect the number of iterations needed to secure an acceptable tolerance to be small. Also note that not only are $X$ and $\hat{X}$ causal stable, but they also have inverses that are causal stable.

In Section 4 we will be concerned with the continuity of the square root $X(kH)$ of $I + kH$ as $k$ varies continuously in $[0,1]$. To obtain $X(kH)$ for $k < 1$ one can introduce an obvious modification of the nested Newton-Raphson Algorithm under study. Specifically, define integer $\mu_1$ and real $0 \leq \mu_2 < \delta_2$ such that for some integer $0 \leq m < N$,

$$k = \delta_1 + m\mu_1 + \mu_2.$$

Select

$$V_{m+1}(\mu_2) = NR(I + \mu_2 U_m^{-1} H U_m^{-1}, I).$$

Then

$$X(kH) = U_m V_{m+1}(\mu_2).$$

**Remark 3.3** *Indeed, from (iv,v) of Theorem 3.1 one can readily deduce that $X(0) = I$ and $X(kH)$ varies continuously with $k$.*

In Section 6, in our search for the analog of the result of [2], we are concerned with the square roots of systems $H$ having an SVR and comparing them with the square root of $H_\alpha$. Observe that while in this case $X$ may not have a SVR, its approximant $\hat{X}$ will. In order to ensure that the SVR of the approximate square root of $H$ is related to that of $H_\alpha$ in a manner that facilitates future analysis we first present the following Lemma which follows easily from a Bounded Real Lemma type result deducable from a result in [9].

**Lemma 3.2** *Suppose for some $\alpha > 0$, $H_\alpha$ has SVR $\{A_\alpha(t), B(t), C(t)\}$, (see (2.7)) and $H$ has SVR $\{A(t), B(t), C(t)\}$. Then*

$$\|H_\alpha\| < M_2$$

*implies that*

$$\|H\| < M_2.$$

In a similar vein $\|[I + kH_\alpha]^{-1}\| < M_1$ implies $\|[I + kH]^{-1}\| < M_1$. Thus, define $M_i$ according to $\|[I + kH_\alpha]^{-1}\| < M_1$ and $\|H_\alpha\| < M_2$ and select the parameters $\delta_i$ in (3.4,3.5) with these $M_i$. Then one can operate successfully the Nested Newton-Raphson iterations for both $I + H$ and $I + H_\alpha$ using this same pair of $\delta_i$. What is more, if $H$ has degree of eas $\alpha$ then so does $\hat{X}$. Moreover, if one carries the nested Newton-Raphson iterations for the same number of times with respect to both $H$ and $H_\alpha$, and if $\hat{X}$ so determined for $H$ has SVR $\{A_x(t), B_x(t), C_x(t), I\}$, then the corresponding approximate square root of $H_\alpha$ can be shown to have SVR

$$\{\alpha I + A_x(t), B_x(t), C_x(t), I\} = \{A_{x\alpha}(t), B_x(t), C_x(t), I\}. \tag{3.17}$$

Observe also that the SVR corresponding to the inverse of this approximate square root will be eas.

# 4 Existence of Passive Multipliers

Having demonstrated the existence of the square root of $I+H$, we now generalize the result of [1] and its first implication discussed in the Introduction. Instead of focussing on PR type properties, we will consider SPR type (or strict passivity) properties. This is simply a matter of minor technicality in an attempt to avoid having to deal with singular situations.

In the spirit of [1] the principal result we derive takes the following form: Under Assumption 2.2 there exist operators, $X_{1\alpha}$ and $X_{2\alpha}$, both eas and having eas inverses, for which:

1. $X_{1\alpha}X_{2\alpha}$ is Strictly Positive.
2. $X_{1\alpha}[I + H_\alpha]X_{2\alpha}$ is Strictly Positive.

Observe that, since in the LTI, SISO case all operators in question are mutually commutative, this directly reduces to one direction of the Brockett and Willems result. The other direction will be discussed later. In keeping with the requirements of the next Section, in our discussion here, we will pay special attention to degree of stability considerations.

In view of the results of Section 3, the starting point of our development here will be that there exists an $X$ which is causal stable and has a causal stable inverse such that

$$X^2 = I + H_\alpha. \tag{4.1}$$

Hence,

$$X = [I + H_\alpha]X^{-1} = X^{-1}[I + H_\alpha]. \tag{4.2}$$

We need to concern ourselves with the Spectrum of $X$, specifically that it is in the open right half plane.

**Lemma 4.1** *Suppose $H_\alpha$ satisfies Assumption 2.2 and $X$ is obtained as the convergent point of the Nested Newton-Raphson Algorithm of Section 3. Then, there exists an $\epsilon > 0$ such that for every $\lambda \in \sigma(X)$*

$$\Re(\lambda) > \epsilon. \tag{4.3}$$

*where $\Re$ denotes the real part.*

It is easy to see that $X^{-1}$ also obeys (4.3). Thus, in light of (4.2), we have proved that $[I + H_\alpha]$ can be multiplied from the right by an operator with spectrum in the open right half plane, to yield another that too has similar spectral properties. A SISO, LTI operator with such a spectrum is necessarily SPR. But of course a general linear, even MIMO, LTI operator, need not be. Herein lies the need for finding a combination of left and right multipliers.

In light of the discussion at the end of Section 3, and the closed and bounded nature of $\sigma(X)$ it follows that an $\hat{X}_\alpha$ obtained by utilizing a sufficient number of iterations in the underlying nested Newton-Raphson algorithm, will have the property that both $[I + H_\alpha]\hat{X}_\alpha^{-1}$ and $\hat{X}_\alpha$ have spectra confined to the open right half plane. In other words the spectral confinement property is essentially *robust*.

The question remains as to how one can convert the spectral confinement requirement to a strict positivity requirement. According to [8], the open right half plane spectral confinement of the two operators mentioned in the foregoing suffices for the existence of a *symmetric* operator $P = P^a$, such that

$$X^a P + PX > 0; \tag{4.4}$$

and, in view of (4.2)

$$[(I + H_\alpha)X^{-1}]^a P + P[(I + H_\alpha)X^{-1}] > 0. \tag{4.5}$$

Observe also that as $X$ is causal, stable invertible, by post and pre-multiplying (4.4) by $X^{-1}$ and $[X^a]^{-1}$, respectively, we obtain:

$$[X^a]^{-1}P + PX^{-1} > 0; \tag{4.6}$$

in other words both $P[(I + H_\alpha)X^{-1}]$ and $PX^{-1}$ are strictly positive. Moreover in view of the robustness of the strict positivity property, we can further state that in all these expressions, one can replace $X$ by $X_\alpha$ as long as $X_\alpha$ is obtained by carrying the various stages of the nested Newton-Raphson algorithm through to a sufficiently large number of iterations. Consequently we have obtained the existence of a symmetric P, for which:

$$[(I + H_\alpha)X_\alpha^{-1}]^a P + P[(I + H_\alpha)X_\alpha^{-1}] > 0. \tag{4.7}$$

and

$$[X_\alpha^a]^{-1}P + PX_\alpha^{-1} > 0; \tag{4.8}$$

simultaneously hold.

In other words the objectives stated at the beginning of this Section are apparently met with associations between $P$ and $X_1$ and $X_\alpha^{-1}$ and $X_2$.

However, there are several respects in which (4.7) and (4.8) need to be developed further. To begin with the symmetric nature of $P$ implies that it is noncausal, thus making its implementation practically infeasible. Moreover, keeping the goals of Section 5 in mind, it is desirable to obtain multipliers that are both finite dimensional and reflect in an appropriate way information concerning the degree of stability of $H$. Finally, given that part of our objective is to provide implementable algorithms, it behooves us to go beyond the mere existence of $P$ and to enunciate implementable algorithms for its construction. To achieve these objectives we first provide an algorithm for computing $P$ by employing an obvious analog of the Cayley transform.

**Lemma 4.2** *Adopt the hypothesis of Lemma 4.2, and let $X_\alpha$ be an approximation to the operator $X$ such that (4.5, 4.6) hold for a symmetric $P$. Define*

$$\Gamma_\alpha = [(I + H_\alpha)X_\alpha^{-1}] \tag{4.9}$$

*Then $\Gamma_\alpha + I$ has an inverse (by defintion bounded) and with*

$$\Omega_\alpha = [\Gamma_\alpha - I](\Gamma_\alpha + I)^{-1} \tag{4.10}$$

*$P$ is given by the following uniformly convergent series:*

$$P = \sum_{i=0}^{\infty} [\Omega_\alpha^a]^i \Omega_\alpha^i.$$

*Further for every $n \geq 1$, the causal part of $P_n$ below can be realized by some eas SVR of the form $\{\alpha I + A_{pn}(t), B_{pn}(t), C_{pn}(t), D_{pn}\}$, with all matrices continuous:*

$$P_n = \sum_{i=0}^{n} [\Omega_\alpha^a]^i \Omega_\alpha^i. \tag{4.11}$$

Observe also that the uniform convergence of the power series realizing $P$ and the robustness of the strict positivity property together assure that for sufficiently large $n$, $P_n$ obeys both:

$$[(I + H_\alpha)X_\alpha^{-1}]^a P_n + P_n[(I + H_\alpha)X_\alpha^{-1}] > 0. \tag{4.12}$$

and

$$[X_\alpha^a]^{-1}P_n + P_n X_\alpha^{-1} > 0. \tag{4.13}$$

Of course $P_n$ is also symmetric. It follows from [10] that $P_n$ has a spectral factorization of the form:

$$P_n = W^a W, \tag{4.14}$$

and that $W$ can be chosen to have the same $A$ and $C$ matrices as the causal part of $P_n$. Thus in fact $W$ is eas and has a SVR of the form:

$$\{\alpha I + A_w(t), B_w(t), C_w(t), I\}, \tag{4.15}$$

with all matrices continuous, [10]. Further, from [11], $W^{-1}$ can be chosen to be eas as well (see [12], for a Newton-Raphson based algorithm for computing $W$).

We then have the following Theorem that captures one direction of the result of [1].

**Theorem 4.1** *Under Assumption 2.2, there exist eas and eas invertible operators $X_\alpha^{-1}$ and $W$, with SVR of the form above, such that*

$$[W(I+H_\alpha)X_\alpha^{-1}W^{-1}]^a + W(I+H_\alpha)X_\alpha^{-1}W^{-1} > 0. \quad (4.16)$$

*and*

$$[WX_\alpha^{-1}W^{-1}]^a + [WX_\alpha^{-1}W^{-1}] > 0. \quad (4.17)$$

**Proof:** Follows from (4.14) and the multiplication of (4.12) and (4.13) by $[W^a]^{-1}$ from the left and $W^{-1}$ from the right. ∎

**Remark 4.1** *This theorem also says that there is a causal operator $WX_\alpha^{-1}W^{-1}$ that is strictly positive (i.e. (4.17) holds) and such that the product of this operator with $W(I+H_\alpha)W^{-1}$ is also positive (i.e. 4.16) holds). Because in the time-invariant scalar case the various operators commute $W$ drops out of the picture. This difference reappears in the next section when we generalize the result of [2].*

Before discussing the second direction of the [1] result we turn now to the following Corollary.

**Corollory 4.1** *Under Assumption 2.2, there exist eas and eas invertible operators $X_\alpha^{-1}$ and $W$, with SVR of the form above, such that for all $k \in [0,1]$*

$$[W(I+kH_\alpha)X_\alpha^{-1}W^{-1}]^a + W(I+kH_\alpha)X_\alpha^{-1}W^{-1} > 0.$$

**Proof:** Follows from the fact that the above equation holds for $k=0$ and $k=1$ and the fact that Positivity is convex property. ∎

Observe eas Strictly Positive operators have an inverse that is eas (a fact easily proved from the PKY Lemma). Thus as long as $H_\alpha$ is eas and one can find eas and eas invertible operators $W$, $X_\alpha$ such that (4.16) and (4.17) hold, then the operator $(I+H_\alpha)^{-1}$ must be eas for all $k \in [0,1]$. Thus the analog of the reverse direction of the [1] result also holds.

## 5 Solution to a Problem Posed in [7]

Motivated by adaptive systems problems, [6] had posed the following question: Suppose the following set of square Matrix Polynomials:

$$\{A_1(s) + kA_2(s) | k \in [0,1]\} \quad (5.1)$$

has all its members Hurwitz (i.e. the determinant is Hurwitz). Does there exist a single LTI operator $Z(s)$ such that all members of the set

$$\{[A_1(s) + kA_2(s)]Z(s) | k \in [0,1]\}$$

are SPR. The next Theorem shows that such construction of SPR products is possible provided one allows multiplication from both sides.

**Theorem 5.1** *There exist, square, stable minimum phase matrix transfer functions $Z_1(s)$ and $Z_2(s)$ with the former strictly proper and the latter biproper, such that with $A_1(s)$ and $A_2(s)$ two square matrix polynomials, and $A_1^{-1}A_2$ strictly proper, all members of the set*

$$\{Z_1(s)[A_1(s) + kA_2(s)]Z_2(s) | k \in [0,1]\}$$

*are biproper and SPR, iff all members of the set (5.1) are Hurwitz.*

The main application of this result is in output error adaptive identification [13]. Consider the identification of the proper MIMO plant:

$$[A_1(s) + kA_2(s)]Y(s) = [B_1(s) + kB_2(s)]U(s) \quad (5.2)$$

with $k$ a scalar unknown parameter and $u(t)$ and $y(t)$, the input and output of the plant. To identify the plant generally, one performs state variable filtering to avoid explicit differentiation of the various signals. This requires rewriting of the model as

$$Z_1(s)[A_1(s)+kA_2(s)]Y(s) = Z_1(s)[B_1(s)+kB_2(s)]U(s) \quad (5.3)$$

such that $Z_1(s)[A_1(s) + kA_2(s)]$ is biproper. Then, for exponential convergence of the underlying identification algorithm, one requires that $Z_1(s)[A_1(s) + kA_2(s)]$ be SPR. This can be seen readily from the result of [14] which treats the SISO case. As $k$ is unknown the underlying SPR condition is difficult to ensure. However, suppose *a priori* bounds are available for $k$. In fact without sacrificing generality, assume that $k \in [0,1]$. Then as long as $[A_1(s) + kA_2(s)]$ is Hurwitz for all $k \in [0,1]$, one can choose square, stable, minimum phase matrix transfer functions $Z_1(s)$ and $Z_2(s)$ such that the requirements of Theorem 5.1 are satisfied. Then, noting that $Z_2(s)$ is biproper, one can rexpress the plant as

$$Z_1(s)[A_1(s) + kA_2(s)]Z_2(s)\bar{Y}(s)$$

$$= Z_1(s)[B_1(s) + kB_2(s)]U(s)$$

where

$$\bar{Y}(s) = Z_2^{-1}(s)Y(s)$$

acts as the converted output. Observe it can be constructed from $Y(s)$ without any explicit differentiation. Further, as $Z_1(s)[A_1(s) + kA_2(s)]Z_2(s)$ is SPR, the output error identification algorithm for this redefined system will be exponentially convergent.

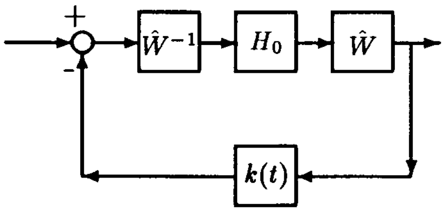


Figure 2: Closed-loop under time varying feedback.

## 6 Generalization of the Freedman Zames Result

In this Section we generalize the second consequence of the result of [1] namely that of [2]. To this end the principal result to be derived is as follows:
**Theorem 6.1** *Suppose Assumption 2.2 holds. Then there exists an eas and eas invertible operator* $\hat{W}$ *(independent of k) such that, the operator* $[I + k(t)\hat{W}H_0\hat{W}^{-1}]^{-1}$ *is eas provided (1.3) and (1.4) hold.*

Essentially, it states that provided the closed loop of Fig. 1 (with $H(s)$ replaced by $H_0$) is $\alpha$-eas for all time invariant feedback gains in the open interval [0,1], then under a logarithic time variation bound as in ([2]), by suitable pre and post filtering of $H_0$, the closed loop in Fig. 2 is also stable. Observe, if $H(s)$ is scalar LTI, then the underlying commutativity recovers the result of [2]. Moreover, the fact that the pre and post filters $\hat{W}^{-1}$ and $\hat{W}$ are independent of the particular trajectory that the time-varying feedback gain follows simplifies their selection.

The proof appeals to the time-varying version of the Positive Real Lemma and Lyapunov techniques developed in [4].

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