REACHABILITY/CONTROLLABILITY PROPERTIES OF INTERNALLY PROPER PMDs

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ABSTRACT

A recently developed algorithm for obtaining an "equivalent" state-space representation of a linear multivariable system whose dynamics are expressed by internally proper Polynomial Matrix Descriptions (PMDs) is used to analyze and examine certain pseudo-state and output controllability properties of the associated PMD .

1 INTRODUCTION

Let us consider a linear multivariable system ($\sum$) described by a polynomial matrix model (PMD) $\sum$:

$$A(\rho)\beta(t) = B(\rho)u(t) \qquad (1)$$

$$y(t) = C(\rho)\beta(t) + D(\rho)u(t) \qquad (2)$$

where $\rho \doteq \frac{d}{dt}$ the differential operator , $A(\rho) = \sum_{i=0}^{q_1} A_i \in \Re[\rho]^{r\times r}$, $q_1 \geq 1$, with $\mathrm{rank}_{\Re} A_{q_1} \leq r$, $B(\rho) = \sum_{i=0}^{\sigma} B_i \in \Re[\rho]^{r\times m}$, $\sigma \geq 0, C(\rho) = \sum_{i=0}^{\sigma_1} C_i \in \Re[\rho]^{p\times r}$, $\sigma_1 \geq 0$, $D(\rho) = \sum_{i=0}^{\sigma_2} D_i \in \Re[\rho]^{p\times m}$, $\sigma_2 \geq 0$ and $\beta(t) : [0^-,\infty) \to \Re^r$ the pseudo state , $u(t) : [0^-,\infty) \to \Re^m$ the input vector and $y(t) : [0^-,\infty) \to \Re^p$ the output vector of ($\sum$).

The main objective of the paper is the analysis of the reachability/controllability properties of the Fuhrmann equivalent state space system $\sum_1$

$$\rho x(t) = Ax(t) + Bu(t) \qquad (3)$$

$$y(t) = Cx(t) + Du(t) \qquad (4)$$

where $x(t) \in \Re^n$ is the state vector of ($\sum_1$) , $n = \deg|A(s)|$, and A,B,C,D are constant matrices of appropriate dimensions. Equivalency between the PMD $\sum$ and State Space System $\sum_1$means preservation of significant information such as several structural properties as transfer function matrix, system poles & zeros, input/output decoupling zeros. We study the reachability properties in the case when the PMD $\sum$ is **Internally Proper** [Kucera(1983), Callier and Desoer(1982)].The internal properness of a PMD is associated with the **Behavior of the pseudo state $\beta(t)$ and the output $y(t)$ at t = 0** and is characterized by the absence from the rational matrices $A^{-1}(s), A^{-1}(s)B(s)$, $C(s)A^{-1}(s)$, $C(s)A^{-1}(s)B(s) + D(s) \doteq H(s)$ of poles at s = ∞ :

Theorem 1 *[Callier and Desoer (1982)]*

The PMD $[A(\rho), B(\rho), C(\rho), D(\rho)]$ in (1)-(2) is internally proper iff the following four conditions are all satisfied

:

$$\begin{array}{c} A^{-1}(s) \in \Re_{pr}^{r\times r}(s) \text{ i.e.} \\ A^{-1}(s) \text{ has no poles at s} = \infty \\ A^{-1}(s)B(s) \in \Re_{pr}^{r\times m}(s) \\ C(s)A^{-1}(s) \in \Re_{pr}^{p\times r}(s) \\ C(s)A^{-1}(s)B(s) + D(s) \doteq \\ H(s) \in \Re_{pr}^{p\times m}(s) \end{array} \qquad (5)$$

There are several advantages of using Internally Proper PMDs. First of all the system has no infinite zeros and that means many application to areas as composite system studies , system invertibility and minimality of system descriptions. Secondly the system has no infinite input/output decoupling zeros and infinite system zeros, hence the transformation has to preserve the basic properties of the PMD only in the finite frequencies .Finally we have to remark that using internally proper PMDs we have a "natural" transformation from systems of the form $\sum$ to systems of the form $\sum_1$, whose the input-output and reachability/controllability properties , as well as , the feedback theory are well known over many years .

2 BACKGROUND

Theorem 2 *[Vardulakis et al. (1982)]* *Let* $A(s) = A_0 + A_1 s + ... + A_{q_1} s^{q_1} \in \Re^{r\times r}[s]$, $rank_{\Re(s)} A(s) = r$, $q_1 \geq 1$. *We can write*

$$\mathbf{A}^{-1}(\mathbf{s}) = \mathbf{H}_{pol}(\mathbf{s}) + \mathbf{H}_{spr}(\mathbf{s}) \qquad (6)$$

where $H_{pol}(s) \in \Re^{r\times r}[s]$ *and* $H_{spr}(s) \in \Re_{pr}^{r\times r}(s)$ *is strictly proper* , $n = deg\ |A(s)| = \delta_M(H_{spr}(s))$, $\mu = \sum_{i=k+1}^{r}(\widehat{q}_i + 1)$,*where* $\widehat{q}_i$ $i = k+1 ,...,r$ *are the orders of the zeros at infinity of* $A(s)$. *Let* $C_f \in \Re^{r\times n}, J_f \in \Re^{n\times n}, B_f \in \Re^{n\times r}$ *be a minimal realization of* $H_{spr}(s)$ *and* $C_\infty \in \Re^{r\times\mu}, J_\infty \in \Re^{\mu\times\mu}, B_\infty \in \Re^{\mu\times r}$ *be a minimal realization of* $H_{pol}(s)$. *Then* C_f, J_f *is a* **Finite Jordan pair** *of A(s) and* C_∞, J_∞ *is an* **Infinite Jordan Pair** *of A(s). Furthermore* $A^{-1}(s)$ *can be written :*

$$\mathbf{A}^{-1}(\mathbf{s}) = \begin{bmatrix} C_f & C_\infty \end{bmatrix} \times \begin{bmatrix} sI_n - J_f & 0_{n,\mu} \\ 0_{\mu,n} & I_\mu - sJ_\infty \end{bmatrix}^{-1} \begin{bmatrix} B_f \\ B_\infty \end{bmatrix} \qquad (7)$$

The PMD $[A(\rho), B(\rho), C(\rho), D(\rho)]$ is **internally proper** iff the matrices $A^{-1}(s)$, $A^{-1}(s)B(s)$, $C(s)A^{-1}(s)$, $C(s)A^{-1}(s)B(s) + D(s) \doteq H(s)$ are **all** proper.

The condition $A^{-1}(s) \in \Re_{pr}^{r\times r}(s)$ is satisfied iff $\widehat{q}_r = 0$, where $\widehat{q}_r$ is the **zero at infinity** of $A(s)$ with maximum order. In that case $J_\infty = 0$ and the matrix $A^{-1}(s)$ can be written as:

$$\mathbf{A}^{-1}(\mathbf{s}) = \mathbf{C}_f\left[sI_n - J_f\right]^{-1}\mathbf{B}_f + \mathbf{C}_\infty\mathbf{B}_\infty \qquad (8)$$

Proposition 3
Let $A(s), B(s), C(s), D(s)$ *be polynomial matrices as in the definition of the Internally Proper PMD in (1)-(2) , with* $A^{-1}(s)$ *as in (8). Then the* **TRANSFER FUNCTION MATRIX** *of the PMD (1) - (2) has in general the form :*

$$\mathbf{H}(\mathbf{s}) = \widetilde{C}[\mathbf{sI}_n - \widetilde{J}]^{-1}\widetilde{\Omega} + \widetilde{E} \qquad (9)$$

where :

$$\widetilde{C} = \left[C_0 C_f + C_1 C_f J_f + \ldots + C_{\sigma_1} C_f J_f^{\sigma_1}\right] \in \Re^{p\times n} \qquad (10)$$

$$\widetilde{J} = \mathbf{J}_f \in \Re^{n\times n} \qquad (11)$$

$$\widetilde{\Omega} = \mathbf{J}_f^{\sigma}\mathbf{B}_f\mathbf{B}_\sigma + \mathbf{J}_f^{\sigma-1}\mathbf{B}_f\mathbf{B}_{\sigma-1} + \ldots + \mathbf{B}_f\mathbf{B}_0 \in \Re^{n\times m} \qquad (12)$$

$$\widetilde{E} = \left[\ C_0, C_1, ..., C_{\sigma_1}\ \right] \times \begin{bmatrix} C_\infty B_\infty B_0 + C_f B_f B_1 + \ldots + C_f J_f^{\sigma-1} B_f B_\sigma \\ C_f B_f B_0 + \ldots + C_f J_f^{\sigma} B_f B_\sigma \\ \vdots \\ C_f J_f^{\sigma_1 - 1} B_f B_0 + \ldots + C_f J_f^{\sigma_1 + \sigma - 1} B_f B_\sigma \end{bmatrix} + \mathbf{D}_0 \in \Re^{p\times m} \qquad (13)$$

3 MAIN RESULTS

Consider again the linear multivariable system $\sum$ whose dynamical behavior is described by the PMD :

$$\mathbf{A}(\rho)\beta(\mathbf{t}) = \mathbf{B}(\rho)\mathbf{u}(\mathbf{t}) \qquad (14)$$

$$\mathbf{y}(\mathbf{t}) = \mathbf{C}(\rho)\beta(\mathbf{t}) + \mathbf{D}(\rho)\mathbf{u}(\mathbf{t}) \qquad (15)$$

If we denote by $\beta^c(t; 0^-, \beta^c(0^-), u(t))$ the **Complete Solution** of the nonhomogeneous matrix d.e. (14) for u(t) : $(0^-, \infty) \to \Re^m$, then :

$$\beta^{r}(\mathbf{t}) = C_f \left[\begin{array}{c} e^{J_f t} x_s(0^-) \\ + \int_0^t e^{J_f(t-\tau)} \Omega u(\tau) d\tau \\ + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u^{(i)}(t) \end{array} \right] + C_\infty \left[\begin{array}{c} - \sum_{i=1}^{\widehat{q}_r} \delta^{(i-1)}(t) J_\infty^i x_f(0^-) \\ + \sum_{i=0}^{\widehat{q}_r} J_\infty^i \overline{\Omega} u^{(\sigma+i)}(t) \\ + \sum_{i=0}^{\sigma-1} Z_i u^{(i)}(t) \end{array} \right] \tag{16}$$

where (i) means distributional derivative of i-th order and

$$\Omega = \sum_{i=0}^{\sigma} J_f^i B_f B_i, \Phi_j = \sum_{i=0}^{\sigma-j} J_f^i B_f B_{i+j} \text{ for j } = 1,2,\ldots,\sigma \tag{17}$$

$$\overline{\Omega} = \sum_{i=0}^{\sigma} J_\infty^i B_\infty B_{(\sigma-i)}, Z_{(\sigma-j)} = \sum_{i=0}^{\sigma} J_\infty^i B_\infty B_{(\sigma-j)-i} \text{ for j } = 1,2,\ldots,\sigma \tag{18}$$

with $B_k \equiv 0$ for k< 0. We assume that the PMD $\sum$ is **Internally Proper.** Then the transfer function matrix is given by $\mathbf{H}(s) = \widetilde{C}[s\mathbf{I}_n - \widetilde{J}]^{-1}\widetilde{\Omega} + \widetilde{E}$ the **complete solution** $\beta(t)$ can be written:

$$\beta(\mathbf{t}) = \mathbf{C}_f \mathbf{e}^{J_f t} \mathbf{x}_s(\mathbf{0}^-) + \mathbf{C}_f \int_0^t \mathbf{e}^{J_f(t-\tau)} \boldsymbol{\Omega} \mathbf{u}(\tau) \mathbf{d}\tau + \mathbf{E}\mathbf{u}(\mathbf{t}) \tag{19}$$

where:

$$E = [C_f, C_\infty] \times \begin{bmatrix} J_f^{\sigma-1} B_f, J_f^{\sigma-2} B_f, \cdots, B_f & 0_{n,r} \\ 0_{\mu,\sigma} & B_\infty \end{bmatrix} \times \begin{bmatrix} B_\sigma \\ B_{\sigma-1} \\ \vdots \\ B_1 \\ B_0 \end{bmatrix} \in \Re^{r \times m} \tag{20}$$

We define:

$$x(\mathbf{t}) = \mathbf{e}^{J_f t} \mathbf{x}_s(\mathbf{0}^-) + \int_0^t \mathbf{e}^{J_f(t-\tau)} \boldsymbol{\Omega} \mathbf{u}(\tau) \mathbf{d}\tau \in \Re^{n \times 1} \tag{21}$$

Then $\beta(t)$ can be written :

$$\beta(\mathbf{t}) = \mathbf{C}_f \mathbf{x}(\mathbf{t}) + \mathbf{E}\mathbf{u}(\mathbf{t}) \tag{22}$$

Denoting with $\rho \doteq \frac{d}{dt}$ the differential operator :

$$\begin{aligned} \rho x(t) &= J_f e^{J_f t} x_s(0^-) + e^{J_f(t-t)} \Omega u(t) + \int_0^t \rho \left[e^{J_f(t-\tau)} \Omega u(\tau) d\tau \right] = \\ &= J_f \left[e^{J_f t} x_s(0^-) + \int_0^t e^{J_f(t-\tau)} \Omega u(\tau) d\tau \right] + \Omega u(t) \\ &= J_f x(t) + \Omega u(t) \end{aligned} \tag{23}$$

Substituting $J_f = \widetilde{J}$ and $\widetilde{\Omega} = \Omega$ we obtain :

$$\begin{aligned} \rho\mathbf{x}(\mathbf{t}) = \widetilde{J}\mathbf{x}(\mathbf{t}) + \widetilde{\Omega}\mathbf{u}(\mathbf{t}) &\Leftrightarrow \rho\mathbf{x}(\mathbf{t}) - \widetilde{J}\mathbf{x}(\mathbf{t}) = \widetilde{\Omega}\mathbf{u}(\mathbf{t}) \\ &\Leftrightarrow \left[\rho I_n - \widetilde{J} \right] \mathbf{x}(\mathbf{t}) = \widetilde{\Omega}\mathbf{u}(\mathbf{t}) \\ &\Leftrightarrow \mathbf{x}(\mathbf{t}) = \left[\rho I_n - \widetilde{J} \right]^{-1} \widetilde{\Omega}\mathbf{u}(\mathbf{t}) \end{aligned} \tag{24}$$

The output of the PMD $\sum$ becomes :

$$\begin{aligned} y(t) &= C(\rho)\beta(t) + D(\rho)u(t) = \\ &\left[C(\rho)A^{-1}(\rho)B(\rho) + D(\rho) \right] u(t) = H(\rho)u(t) \\ &= \left[\widetilde{C} \left[\rho I_n - \widetilde{J} \right]^{-1} \widetilde{\Omega} + \widetilde{E} \right] u(t) = \\ &\widetilde{C} \left[\rho I_n - \widetilde{J} \right]^{-1} \widetilde{\Omega} u(t) + \widetilde{E} u(t) \overset{(24)}{\Rightarrow} \\ y(t) &= \widetilde{C} x(t) + \widetilde{E} u(t) \end{aligned} \tag{25}$$

Therefore in the case the PMD (14) - (15) is **internally proper** through the quadruple of constant matrices $[\widetilde{C}, \widetilde{J}, \widetilde{\Omega}, \widetilde{E}]$ We can form a state space system $\sum_1$:

$$\rho x(t) = \widetilde{J} x(t) + \widetilde{\Omega} u(t) \tag{26}$$

$$y(t) = \widetilde{C} x(t) + \widetilde{E} u(t) \tag{27}$$

where $x(t) \in \Re^{n \times 1}$ defined in (21) plays the role of the **state vector** of $\sum_1$.

Definition 4 *The vector $x(t) \in \Re^n$ defined in (21) is called* **the state vector** *of the internally proper PMD (14)-(15) .*

In distinction $\beta(t)$ is termed as the **pseudostate** of the PMD (14)-(15) . Define the polynomial matrix $M(s)$

$$\mathbf{M(s)} = \mathbf{M}_0 + \mathbf{M}_1\mathbf{s} + \ldots + \mathbf{M}_{q_1-1}\mathbf{s}^{q_1-1} \in \Re^{r\times n}[\mathbf{s}] \tag{28}$$

where the coefficient matrices are given by the formula :

$$M_i = -\sum_{j=0}^{i} A_j C_f J_f^{-[(i+1)-j]} \quad \text{for } i = 0, 1, \ldots, q_1 - 1 \tag{29}$$

and the matrix $X(s) \in \Re^{p\times n}[s]$:

$$\mathbf{X(s)} = -[C_1, C_2, \ldots, C_{\sigma_1}] \times \begin{bmatrix} C_f & 0 & \cdots & 0 \\ C_f\widetilde{J} & C_f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_f\widetilde{J}^{\sigma_1-1} & C_f\widetilde{J}^{\sigma_1-2} & \cdots & C_f \end{bmatrix} \times \begin{bmatrix} I_n \\ sI_n \\ \vdots \\ s^{\sigma_1-1}I_n \end{bmatrix} \tag{30}$$

Theorem 5 *Consider a linear multivariable system* $\sum$ *described by an* **INTERNALLY PROPER** *polynomial matrix model PMD (14)-(15) with* X_u *the* **solution set** *for some input* $u(t)$. *Its* **transfer function matrix** *can be written:*

$$\mathbf{H(s)} = \widetilde{C}[\mathbf{sI}_n - \widetilde{J}]^{-1}\widetilde{\Omega} + \widetilde{E} \tag{31}$$

with $n = \deg|A(s)|$ *and* $\widetilde{C}, \widetilde{J}, \widetilde{\Omega}, \widetilde{E}$ *as in (10)-(12) ,(13). From the quadruple of matrices* $\left[\widetilde{J}, \widetilde{\Omega}, \widetilde{C}, \widetilde{E}\right]$ *we can form a (regular)* **state space system** $\sum_1$ *as in (26)-(27) with* X_u^1 **as solution set** *and* $x(t) \in \Re^n$ *as* **state vector** .

Then

i) There always exists a bijective mapping $g : X_u^1 \rightarrow X_u$ *having the form*

$$\beta(\mathbf{t}) = \mathbf{C}_f\mathbf{x}(\mathbf{t}) + \mathbf{Eu}(\mathbf{t}) \tag{32}$$

for some constant matrices C_f *and* E *of appropriate dimensions . Its inverse mapping* $g^{-1} : X_u \rightarrow X_u^1$ *exists and has the form :*

$$x(t) = Q(\rho)\beta(t) + L(\rho)u(t) \tag{33}$$

for some polynomial matrices $Q(\rho), L(\rho)$.

Furthermore the mapping g *in (32) is such that the following diagram :*

$$\begin{tikzcd} X_u^1 \arrow[r, "(27)"] \arrow[d, "(32)"'] & Y_u \\ X_u \arrow[ur, "(15)"'] & \end{tikzcd}$$

Diagram 4

commutes , with Y_u *equal to the set of outputs .*

ii) The two systems $\sum$ *and* $\sum_1$ *are* **Fuhrmann system equivalent** *and are related by*

$$\begin{bmatrix} M(\rho) & 0_{r,p} \\ X(\rho) & I_p \end{bmatrix} \begin{bmatrix} \rho I_n - \widetilde{J} & \widetilde{\Omega} \\ -\widetilde{C} & \widetilde{E} \end{bmatrix} = \begin{bmatrix} A(\rho) & B(\rho) \\ -C(\rho) & D(\rho) \end{bmatrix} \begin{bmatrix} C_f & -E \\ 0_{m,n} & I_m \end{bmatrix} \tag{34}$$

$$M(\rho), A(\rho) \qquad \textit{left coprime} \tag{35}$$

$$(\rho I_n - \widetilde{J}), C_f \qquad \textit{right coprime} \tag{36}$$

We denote by $\beta(t; 0^-, \beta(0^-), u(t))$ the corresponding solution of (14) with initial time constant 0^-, initial condition $\beta(0^-)$ and input function $u(t)$. From (32) we have (in light of $x_s(0^-) = x(0^-) = x_0$) :

$$\beta(t) = \beta(t; 0^-, \beta(0^-), u(t)) = C_f e^{\widetilde{J}t}x(0^-) + C_f \int_0^t e^{\widetilde{J}(t-\tau)}\widetilde{\Omega}u(\tau)d\tau + Eu(t) \tag{37}$$

We also denote by $x(t; 0^-, x(0^-), u(t))$ the corresponding solution of (26) with initial time constant 0^-, initial condition $x(0^-)$ and input function $u(t)$. From (21) we have :

$$x(t) = x(t; 0^-, x(0^-), u(t)) = e^{\widetilde{J}t}x(0^-) + \int_0^t e^{\widetilde{J}(t-\tau)}\widetilde{\Omega}u(\tau)d\tau \tag{38}$$

According to Theorem 5 there exists a bijective mapping between the pseudostate $\beta(t)$ of $\sum$ and the state $x(t)$ of $\sum_1$ such that :

$$\beta(t) = C_f x(t) + Eu(t) \tag{39}$$

For $t = 0^-$ the above relation becomes :

$$\beta(0^-) = C_f x(0^-) + Eu(0^-) \qquad (40)$$

We state :

Definition 6 *Given a point $\beta_0 = \beta(0^-) \in \Re^r$ as in (40), we say that another point $\beta_T \in \Re^r$ is pseudostate - reachable from β_0 if there exists an input $u(t)$, $T > 0$ such that the complete solution of (14)-(see also (37)) $\beta(T) = \beta(T; 0^-, \beta_0, u(t)) = \beta_T$. If $\beta_0 = \beta(0^-) \neq 0$ and $\beta(T) = \beta_T = 0 \in \Re^r$ i.e. if the origin $0 \in \Re^r$ is pseudostate - reachable from β_0, then we say that the point β_0 is pseudostate - controllable.*

Given a point $x_0 = x(0^-) \in \Re^n$, we say that another point $x_T \in \Re^n$ is state - reachable from x_0 if there exists an input $u(t)$, $T > 0$ such that the complete solution of (26)-(see also (38)) $x(T) = x(T; 0^-, \beta_0, u(t)) = x_T$. If $x_0 = x(0^-) \neq 0$ and $x(T) = x_T = 0 \in \Re^n$ i.e. if the origin $0 \in \Re^n$ is state - reachable from x_0, then we say that the point x_0 is state - controllable.

Due to the Definition 6 $x(t)$ is termed also as *the state of the PMD $\sum$*. To connect the definition of state - reachability/controllability to the complete solution $\beta(t)$ of (14) we consider the following relationship between $x(t)$ and $\beta(t)$ (see (33) in Theorem 5) :

$$x(t) = Q(\rho)\beta(t) + L(\rho)u(t) \qquad (41)$$

In light of the above an alternative definition of state - reachability/controllability is the following:

Definition 7 *Given a point $x_0 = x(0^-) \in \Re^n$, we say that another point $x_T \in \Re^n$ is state - reachable from x_0 if there exists an input $u(t)$, $T > 0$ such that the complete solution $\beta(t)$ of (14) is such that $(x(T)) \equiv Q(\rho)\beta(T) + L(\rho)u(T) = x_T$ (where $x(0^-), \beta(0^-)$ are related through (40)). If $x_0 = x(0^-) \neq 0$ and $x(T) = x_T = 0 \in \Re^n$ i.e. if the origin $0 \in \Re^n$ is state - reachable from x_0, then we say that the point x_0 is state - controllable.*

Definition 8 *a) The state space system $\sum_1$ is said to be :*

i) state - reachable iff for every pair of points $(w_1, w_2) \in \Re^n \times \Re^n$, w_2 is state - reachable from w_1. ii) state - controllable iff every point $w \in \Re^n$, $w \neq 0$, is state - controllable.

b) The PMD $\sum$ is said to be :

i) pseudostate - reachable iff for every pair of points $(z_1, z_2) \in \Re^r \times \Re^r$, z_2 is pseudostate - reachable from z_1. ii) pseudostate - controllable iff every point $z \in \Re^r$, $z \neq 0$, is pseudostate - controllable. iii) state - reachable iff for every pair of points $(w_1, w_2) \in \Re^n \times \Re^n$, w_2 is state-reachable from w_1. iv) state - controllable iff every point $w \in \Re^n$, $w \neq 0$, is state - controllable.

It is well known:

Theorem 9 *Let $R(x_0)$ be the set of all points $x_T \in \Re^n$ that are state - reachable from $x_0 = x(0^-) \in \Re^n$. Then for the state space system of the form of $\sum_1$ we can describe $R(x_0)$ directly in terms of $\tilde{J}, \tilde{\Omega}$:*

$$R(x_0) \doteq <\tilde{J}/Im\tilde{\Omega}> = Im\tilde{\Omega} + \tilde{J}Im\tilde{\Omega} + \ldots + \tilde{J}^{n-1}Im\tilde{\Omega} \qquad (42)$$

Definition 10 *The set $R(x_0) \subset \Re^n$ as in (42) is called the state - reachable subspace of $\sum_1$. $R(x_0)$ can be termed also as state-reachable subspace of the PMD $\sum$.*

The state-reachable subspace $R(x_0)$ is spanned by the linearly independent columns of the matrix :

$$Q_1 \doteq \left[\tilde{\Omega}, \tilde{J}\tilde{\Omega}, \ldots, \tilde{J}^{n-1}\tilde{\Omega}\right] \in \Re^{n \times (nm)} \qquad (43)$$

which is called the *state -reachability matrix* of $\sum_1$ (and of $\sum$ also).

We can state the obvious :

Theorem 11 *The PMD $\sum$ (the state space system $\sum_1$) is state - reachable iff*

$$R(x_0) \equiv \Re^n \Leftrightarrow rank_{\Re}[Q_1] = n \qquad (44)$$

Let now $R_{pse}(\beta_0)$ be the set of all points $\beta_T \in \Re^r$ that are pseudostate - reachable from $\beta_0 = \beta(0^-) \in \Re^r$. Following the procedure in the proof of Theorem 4.6 in [Fragulis (1994)] we have:

Theorem 12

$$\begin{aligned} R_{pse}(\beta_0) \doteq\ & C_f < \tilde{J}/Im\tilde{\Omega} > +ImE = \\ & C_f Im\tilde{\Omega} + C_f\tilde{J}Im\tilde{\Omega} + \dots \\ & + C_f\tilde{J}^{n-1}Im\tilde{\Omega} + ImE \end{aligned} \tag{45}$$

Definition 13 *The set $R_{pse}(\beta_0) \subset \Re^r$ as in (45) is called the pseudostate - reachable subspace of the PMD $\sum$. The pseudostate - reachable subspace $R_{pse}(\beta_0)$ is spanned by the linearly independent columns of the matrix :*

$$Q_2 \doteq \left[C_f\tilde{\Omega}, C_f\tilde{J}\tilde{\Omega}, \dots, C_f\tilde{J}^{n-1}\tilde{\Omega}, E\right] \in \Re^{r\times(n+1)m} \tag{46}$$

which is called the pseudostate - reachability matrix of $\sum$.

We can state the obvious :

Theorem 14 *The PMD $\sum$ is pseudostate - reachable iff*

$$R_{pse}(\beta_0) \equiv \Re^r \Leftrightarrow rank_{\Re}[Q_2] = r \tag{47}$$

From Theorems 11 and 14 we have that state - reachability of $\sum$ implies pseudostate - reachability of $\sum$ in the case the matrix $C_f \in \Re^{r\times n}$ has full row rank i.e. $rank[C_f] = r$. Hence :

Corollary 15 *The PMD $\sum$ is pseudostate - reachable if :*

a) $\sum$ is state - reachable and b) $rank[C_f] = r$.

It is well known (Kailath (1980)) that for continuous-time state-space systems such as $\sum_1$, since the transition matrix $e^{\tilde{J}t}$ is always nonsingular , the concepts of state-reachability and state-controllability are identical . Hence :

Corollary 16 *The state space system $\sum_1$ is state-reachable iff $\sum_1$ is state-controllable.*

We can prove the following :

Theorem 17 *The internally proper PMD $\sum$ is state-reachable iff $\sum$ is state-controllable .*

Let now $C_{pse}(\beta_0)$ denotes the set of all points $\beta_0 \in \Re^r$ that are pseudostate-controllable . It is found (Fragulis (1990)) that $C_{pse}(\beta_0)$ can be written :

$$C_{pse}(\beta_0) \doteq [C_f, C_\infty] \times \begin{bmatrix} < \tilde{J}/Im\tilde{\Omega} > + \sum_{i=0}^{\sigma-1} Im\Phi_{i+1} \\ < J_\infty/Im\overline{\Omega} > + \sum_{i=0}^{\sigma-1} ImZ_i + KerJ_\infty \end{bmatrix} \tag{48}$$

The internal properness of the PMD $\sum$ implies that $J_\infty \equiv 0$. Hence we obtain that for an internally proper PMD its *pseudostate-controllable subspace* $C_{pse}(\beta_0)$ can be written:

$$\begin{aligned} C_{pse}(\beta_0) \doteq C_f < \tilde{J}/Im\tilde{\Omega} > +ImE = \\ C_f Im\tilde{\Omega} + C_f\tilde{J}Im\tilde{\Omega} + \dots + C_f\tilde{J}^{n-1}Im\tilde{\Omega} + ImE \\ \equiv R_{pse}(\beta_0) \end{aligned} \tag{49}$$

i.e. the subspaces $C_{pse}(\beta_0)$ and $R_{pse}(\beta_0)$ are identical . Hence :

Corollary 18 *The internally proper PMD $\sum$ is pseudostate-reachable iff $\sum$ is pseudostate-controllable .*

The relation between left coprimeness of the polynomial matrices $A(s), B(s)$ and state (pseudostate) controllability of the associated internally proper PMD $\sum$ become clear in the sequel. First of all because of the Fuhrmann equivalence between $\sum$ and $\sum_1$ the following relations hold true: $(sI_n - \tilde{J}) \overset{s}{\sim} A(s)$, $\left[(sI_n - \tilde{J}), \tilde{\Omega}\right] \overset{s}{\sim} [A(s), B(s)]$, $\begin{bmatrix} (sI_n - \tilde{J}) \\ \tilde{C} \end{bmatrix} \overset{s}{\sim} \begin{bmatrix} A(s) \\ C(s) \end{bmatrix}$ where the superscript "s" means *Smith equivalence* or in other words preservation of the non unity invariant polynomials (see e.g. Kailath (1980),Theorem 8.2.3) . The above relationships will be used to study the controllability properties of the PMD $\sum$ via those of the state-space $\sum_1$.

Corollary 19 *a)The internally proper PMD $\sum$ is state-controllable if and only if* **at least one** *of the following equivalent conditions hold : i)* $A(s), B(s)$ *are left coprime i.e.* $rank_{\Re}[A(s), B(s)] = r$, $\forall s \in \Re$. *ii)* $(sI_n - \widetilde{J}), \widetilde{\Omega}$ *are left coprime i.e.* $rank_{\Re}\left[(sI_n - \widetilde{J}), \widetilde{\Omega}\right] = n$, $\forall s \in \Re$. *iii)* $rank_{\Re}[Q_1] = rank_{\Re}\left[\widetilde{\Omega}, \widetilde{J}\widetilde{\Omega}, \ldots, \widetilde{J}^{n-1}\widetilde{\Omega}\right] = n$.

b)The internally proper PMD $\sum$ *is state-controllable if and only if the state-space system* $\sum_1$ *is state-controllable .*

c)The internally proper PMD $\sum$ *as in (14)-(15) is* **state-controllable** *if and only if the polynomial matrices* $A(s), B(s)$ *are left coprime .*

d)The internally proper PMD $\sum$ *as in (14)-(15) is* **pseudostate-controllable** *if the polynomial matrices* $A(s), B(s)$ *are left coprime and* $rank_{\Re}[C_f] = r$

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