Relay and time delay in set point control of drift-free systems. *

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Abstract

The paper demonstrates the usefulness of standard control mechanisms such as the relay, zero hold and time delay for feedback control of drift-free systems. A novel switching feedback strategy is formulated which accomplishes set point control for such systems. The strategy can be treated as a new type of sliding mode control ; however it does not exhibit chattering. An advantage of the strategy is that it can handle constraints on the controls and seems easy to implement.

Keywords: feedback control, sliding mode control, nonholonomic systems.

1 Introduction

The purpose of this paper is to describe a novel procedure for the construction of feedback set point control for controllable systems without drift.

In principle, the procedure is applicable to control systems of the type

$$\dot{x} = \sum_{i=1}^{m} f_i(x)u_i, \tag{1}$$

where $f_1, ..., f_m$ are linearly independent, smooth vector fields in $I\!R^n$ with $m < n$, and u_i are Lebesgue integrable control functions on the interval $[0, \infty)$. A admissible trajectory for system (1) is an absolutely continuous function $t \to x^u(t) \in I\!R^n$ which satisfies (1) almost everywhere and corresponds to an admissible control $u \stackrel{def}{=} (u_1, ..., u_m)$.

We will limit our attention to real analytic systems, i.e. systems for which the vector fields $f_i, i = 1, ..., m$ are real analytic. Morover, we will assume that the system is completely controllable, i.e. that for every pair of points x_1 and x_2 there exists an admissible control which steers the system from x_1 to x_2.

For real analytic systems, complete controllability is equivalent to the well known LARC (Lie algebraic rank condition): if $L(f_1, ..., f_m)$ denotes the Lie algebra of vector fields generated by $f_1, ..., f_m$, and $L(f_1, ..., f_m)(x) \stackrel{def}{=} \{f(x) : f \in L(f_1, ..., f_m)\}$, then $L(f_1, ..., f_m)(x)$ must span $I\!R^n$ for all $x \in I\!R^n$.

Although the LARC guarantees the existence of an admissible control which steers the system from any point x_1 to any point x_2, it is not obvious how to construct such a control explicitly. Here, two approaches are possible: control in open loop (requiring the construction of a control function $t \to u(t), t \in [0, T]$), or in a feedback loop (requiring the construction of a feedback strategy $x \to u(x), x \in I\!R^n$ both of which are supposed to accomplish the task of steering the system from point x_1 to point x_2 . A number of papers have been devoted to this problem [7, 8, 9] where var-

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ious constructions 'in open loop' have been proposed. Relatively fewer papers are concerned with the construction of feedback controls [3, 4, 5, 6].

In this paper we construct a feedback strategy which steers system (1) from any given intial point x_1 to any desired set point x_2 with bounded feedback controls $u(x) \in [-1,1]$, $x \in I\!R^n$. Without the loss of generality, the desired set point is taken to be $x_2 = 0$. The proposed approach employs a quadratic 'guiding function', $V(x)$:

$$V(x) \stackrel{def}{=} (1/2)x^T x \tag{2}$$

(This choice is of course arbitrary; other types of guiding functions can be considered provided they satisfy a given set of assumptions).

Although V is positive definite, proper and decrescent, it is not a Lyapunov function for (1) as, clearly, there might exist points $x \in I\!R^n$ $x \neq 0$ at which the standard inequality

$$\dot{V}(x) = \sum_{i=1}^{m} x^T f_i(x) u_i < 0 \tag{3}$$

cannot be satisfied by any $u_1, ..., u_m$. Such points are members of the set S

$$S \stackrel{def}{=} S_1 \cap S_2 \cap ... \cap S_m \tag{4}$$

where, under given assumptions on the system and the guiding function, $S_i, i \in \{1, ..., m\} \stackrel{def}{=} \underline{m}$ are hypersurfaces

$$S_i \stackrel{def}{=} \{x \in I\!R^n : x^T f_i(x) = 0\} \quad i \in \underline{m} \tag{5}$$

such that f_i is never tangent to S_i, $i \in \underline{m}$. To ensure the latter, other types of guiding functions can also be considered .

Exterior to the set S, a standard feedback control can be used:

$$u_i(x) \stackrel{def}{=} -sign(x^T f_i(x)), \text{ for } x \in I\!R^n \setminus S \quad i \in \underline{m} \tag{6}$$

If the system never traverses the set S then (3) and (6) imply that $\dot{V}(x) < 0$ along the trajectory of the system , and hence that $x^u(t) \to 0$ as $t \to \infty$, by virtue of standard properties of the function V.

Once $x \in S$, the above strategy fails in that $\dot{V}(x) = 0$ regardless of the controls.

At this point, the proposed strategy is based on considering the 'Lie bracket extension' of system (1), i.e. the system

$$\dot{x} = \sum_{k=1}^{r} f_k(x) v_k, \tag{7}$$

where the first m vector fields $f_1, ..., f_m$ are the same as in (1), and the new directions of instantaneous motion $f_{m+1}, ..., f_r$ are Lie brackets of the $f_i, i \in \underline{m}$. The new $f_i, i \in \underline{m}$ are selected in such a way that, for all x in some region G of interest,

$$span\{f_i, i \in \underline{r}\} = I\!R^n. \tag{8}$$

(Such a choice is always possible, if the region G is bounded and if the LARC holds. In many cases of interest, e.g. when the f_i are polynomial vector fields, G can be taken to be equal to $I\!R^n$.)

Inequality (3) along a trajectory of the extended system takes the form

$$\dot{V}_e(x) = \sum_{i=1}^{r} x^T f_i(x) v_i < 0 \tag{9}$$

By virtue of (8), the controls $v_i \stackrel{def}{=} -sign(x^T f_i(x))$, $i \in \underline{r}$ yield $\dot{V}_e(x) = \sum_{k=1}^{r} |x^T f_k(x)| < 0$ whenever $x \neq 0$, forcing x to converge to the origin. However, there are no controls u which produce velocity of system (1) in any of the directions $f_{m+1}, ..., f_r$.

Despite this obvious fact, our strategy selects a vector field f_s which corresponds to the lagest 'coefficient' $|x^T f_s(x)|$ $s \in \{m+1, ..., r\}$ in $\dot{V}_e(x)$.

A special feedback switching strategy is then introduced which involves only control elements such as the relay, time delay and zero hold, which produces an 'average' direction of motion of the system (defined later) which is $-f_s(x)$ if $x^T f_s(x) > 0$, and $f_s(x)$ if $x^T f_s(x) < 0$.

For simplicity of exposition, we discuss here only the case when a current, desired direction of motion f_s is a first order Lie bracket $[f_i, f_j]$, but the strategy can be extended to cover the case of more complex Lie bracket directions. Hence, we further limit our attention to systems in which $m = n - 1$.

The switching strategy is based on the action of relay elements, associated with each of the surfaces $S_i, i \in \underline{m}$, and the fact that f_i is never tangent to S_i.

The state of a given relay element i further depends on the position of the system state with respect to the 'switching surfaces' S_i over a finite time interval $[t-\Delta_i, t]$ in the past with respect to the current time t. The 'time delay' Δ_i may be interpreted as the time needed for a relay element to change state after it is activated. The action of the individual relay elements is then synchronized centrally, in a well specified way, to produce the desired average motion, as can readily be verified by employing the Campbell-Baker-Hausdorff formula .

As a result the system 'slides' on the S surface or rather remains in some finite ϵ-neighbourhood of it.

Since the motion of the system in the direction $-f_s(x)$ (or else in the direction $f_s(x)$) decreases V_ϵ , then the 'average motion' in the same direction also causes an 'average' decrease in the guiding function V along the trajectory of system (1).

The switching strategy is finally incorporated into a global feedback strategy in which the average motions take place in the directions which agree with the largest coefficients $|x^T f_s(x)|$ in $\dot{V}_\epsilon(x)$ for $x \in I\!R^n$.

If the switching delays Δ_i are reduced to zero as $x^u(t) \to 0$, e.g. as functions of some power $p \in (0,1]$ of the actual value of the guiding function $V(x^u(t))$, then it can be shown that $V(x^u(t)) \to 0$, implying asymptotic convergence of the controlled trajectories to the origin, $x^u(t) \to 0$ as $t \to \infty$.

In the feedback strategy outlined above controls switch values at finite intervals of time, which are bounded from below by the smallest delay Δ_i $i \in \underline{m}$. Hence, there is no chattering and the existence of solutions of (1) is not endangered.

As will become clear, the time delays on the relay elements need not be small. Large delays which are equivalent to large deviations of the system's state from the switching surfaces are allowed as long as the guiding function decreases 'on average'. The latter can be verified on line, hence allowing for delay adjustment.

The strategy has the advantage of using uniformly bounded controls , is particularly straightforward, and seems simple to implement.

2 The control problem and assumptions

An ϵ-neighbourhood of the surface S is denoted by $S + B(0;\epsilon)$, i.e.

$$S + B(0;\epsilon) \stackrel{def}{=} \{x \in I\!R : \exists\, p \in S \text{ s.t. } ||x-p|| < \epsilon\} \tag{10}$$

For a general smooth vector field f on $I\!R^n$, we let $exp[tf](p)$ denote the solution, at time t, of $\dot{x}(t) = f(x(t))$ with initial condition $x(0) = p$.

We also use the symbol $x^u(t; x_0, t_0)$ (or shortly $x^u(t)$) to denote the trajectory of the controlled system (1) passing through the point (x_0, t_0).

The set point control problem

SPC: Find a feedback control strategy in terms of uniformly bounded controls

$$u_i(x^u : (-\infty, t]) \in [-1, 1] \quad i \in \underline{m} \tag{11}$$

in which $u_i(x^u : (-\infty, t])$ signifies a possible dependence of the controls on the state of the controlled system x^u in the past (with reference to the moment t at which the controls are evaluated) such that:

for any given pair of points $x_1 \in I\!R^n$ and $x_2 \in I\!R^n$,

$$x^u(t; x_1, 0) \to x_2 \tag{12}$$

as $t \to \infty$.

Without the loss of generality, we assume that $x_2 = 0$ (or else, the coordinate system can be translated as necessary). We need the following assumptions:

Controllability :

A1.

$$span\{f_i(x), [f_i, f_j](x),\ i,j \in \underline{n-1}\} = I\!R^n \tag{13}$$

for all $x \in I\!R^n$;

Regularity of the surfaces S_i

A2.

$$span\{d[x^T f_i(x)],\ i \in \underline{m}\} = I\!R^m \qquad (14)$$

Additionally,

$$< f_i(x), d[x^T f_i(x)] > \neq 0,\ \ i \in \underline{m}, x \in I\!R^n \qquad (15)$$

for all $x \in I\!R^n$, where $m = n - 1$.

Assumption A2 guarantees the existence of a tangent plane (or tangent vector) to S at every $x \in I\!R^n$ and ensures that f_i is not tangent to S_i.

3 The switching strategy

The aim in this section is to determine a 'local' feedback control strategy which produces an 'average' motion of the system in a direction which is not directly admissible.

We will say that the 'average' motion of system (1) in the direction $+[f_i, f_j]$ (or in the direction $-[f_i, f_j]$ respectively) is T-nonzero at a point p, if for a given constant $T \in (0, \infty)$

$$\int_0^T [x^u(t)]^T [f_i, f_j](x^u(t))dt > 0 \qquad (16)$$

(or < 0, respectively), where $x^u(t)$ denotes the trajectory of the controlled system (1) emanating from point p.

Let $G(x)$ denote a matrix whose rows are $d[x^T f_i(x)], i \in \underline{n-1}$, and $F(x)$ denote a matrix whose columns are $f_i(x), i \in \underline{n-1}$. Before we specify the switching strategy let us note the following

Proposition 1

- *The controls*

$$u_i(x) \stackrel{def}{=} -sign(x^T f_i(x)), i \in \underline{n-1} \qquad (17)$$

steer the system (1) to a point $x \neq 0$, $x \in S$ in finite time, or else the controlled trajectory converges asymptotically to the origin.

- *Sliding motion on S (in the sense defined by Filippov [2] or in the sense of equivalent controls [1]) does not occur (when $G(x)F(x)$ is nonsingular), or else is not defined uniquely (when $G(x)F(x)$ is singular).*

We also note the following immediate consequence of the definition of 'average' motion on S, the definition of the Lie bracket extension of (1), and inequality (9).

Proposition 2 *Suppose that a point $p \in S, p \neq 0$ is the initial state of system (1) at time $\tau \geq 0$ and the indices $i, j \in \underline{n-1}$ are such that*

$$|p^T[f_i, f_j](p)| = max\{|p^T[f_a, f_b](p)|,\ a, b \in \underline{n-1}\} \qquad (18)$$

Further suppose that system (1) is controlled in such a way that $u_k = 0$ for $k \neq i, j$, and

- *the motion of system (1) at p is T-nonzero in the direction $+[f_i, f_j]$ if $p^T[f_i, f_j](p) < 0$,*
- *or else, the motion of system (1) at point p is T-nonzero in the direction $-[f_i, f_j]$ if $p^T[f_i, f_j](p) > 0$.*

Under these conditions, there exists an $\epsilon > 0$ such that if the controlled trajectory of system (1) remains in $S + B(0; \epsilon)$ over the time interval $[\tau, \tau + T]$, then the guiding function V exhibits an 'average' T-nonzero decrease along the controlled system trajectory in the sense that

$$\int_\tau^{\tau+T} \dot{V}(x^u(t))dt = V(x^u(T)) - V(p) < -0.5M(p) \qquad (19)$$

where $x^u(t) \stackrel{def}{=} x^u(t; p, \tau)$ and

$$M(p) \stackrel{def}{=} |\int_\tau^{\tau+T} [x^u(t)]^T [f_i, f_j](x^u(t))dt| \qquad (20)$$

For $l \in \underline{n-1}$ let S_l^+ and S_l^- denote the two open domains to the left and right of each surface S_l, i.e. $S_l^+ \stackrel{def}{=} \{x \in I\!R^n : x^T f_l(x) > 0\}$ and $S_l^- \stackrel{def}{=} \{x \in I\!R^n : x^T f_l(x) < 0\}$.

The switching strategy described below is initialized at a point $p \in S$, when the trajectory of system (1) with $m = n-1$, using the control (17), first traverses the surface S.

Suppose that to each surface $S_l, l \in \underline{n-1}$ there is attached a relay element of a special characteristic and the same time delay, Δ, whose task is to implement the control u_l. The switching feedback strategy consists of time intervals $[t_s, t_{s+1}], s \in I\!N$ (called switching cycles) at the beginning of which a relay element is allowed to change state. The duration of the switching cycles $[t_s, t_{s+1}]$ is bounded from below by the constant Δ, and determined by a synchronizing supervisory controller whose action will be made precise later.

Further, suppose that each of the relay elements obeys the same rule for change of state at time t_s.

$$u_l(x^u(t_s+)) = 1 \quad \text{if } x^u(t_{s-1}+) \in S_l^- \quad (21)$$
$$u_l(x^u(t_s+)) = -1 \quad \text{if } x^u(t_{s-1}+) \in S_l^+ \quad (22)$$

where $x^u(t_s+)$ denotes the value of the controlled trajectory x^u to the right of the point t_s, i.e. at the beginning of the current switching cycle, and , consequently, $x^u(t_{s-1}+)$ denotes the value of the state at the beginning of the previous switching cycle. (The above rule is well defined since the state x^u cannot remain on S_l if $u_l \neq 0$.)

The value of u_l, determined at the beginning of each switching cycle, is maintained until time t_s^* at which u_l is temporarily set to zero. The time t_s^* is determined as follows

$$\begin{aligned} t_s^* &= \Delta \\ &\quad \text{if } u_l > 0 \text{ and } x^u(t_s+) \in S_l^+, \\ &\quad \text{or else if } u_l < 0 \text{ and } x^u(t_s+) \in S_l^- \quad (23) \\ t_s^* &= \text{time } t \text{ at which } x^u(t) \text{ returns to } S_l \\ &\quad \text{if } u_l < 0 \text{ and } x^u(t_s+) \in S_l^+, \\ &\quad \text{or else if } u_l > 0 \text{ and } x^u(t_s+) \in S_l^-. \quad (24) \end{aligned}$$

The end of the switching cycle is then determined as the time t_{s+1} at which all controls in the system are on hold (temporarily zero), after which the next switching cycle begins.

Now, suppose that at a time $\tau \geq 0$ the system (1) reaches a point $p \in S$, $p \neq 0$. Then, by virtue of the controllability assumption, there exist indices $i, j \in \underline{n-1}$ such that $|p^T[f_i, f_j](p)| = max\{|p^T[f_k, f_l](p)|, k, l \in \underline{n-1}\} > 0$ and V decreases in the direction $+[f_i, f_j](p)$ or else in the direction $-[f_i, f_j](p)$.

An average motion of the system in the desired direction and the desired orientation is implemented as follows. First, all the controls except u_i and u_j are set to zero: i.e. $u_k = 0, k \in \underline{n-1}, k \neq i, j$. Before initiating the switching feedback strategy described above in the controls u_i and u_j, an additional, preliminary switching cycle is executed in which the controls u_i and u_j take the following values :

If, prior to time instant τ, the surfaces S_i and S_j are approached by the system from opposite sides (i.e. $x^u(\tau-) \in S_i^- \cap S_j^+$ or $x^u(\tau-) \in S_i^+ \cap S_j^-$), then

$$\begin{aligned} u_i &= 0 \text{ for } t \in (\tau, \tau+\Delta] \quad (25) \\ &\quad \text{if } p^T[f_i, f_j](p) > 0 \\ &\text{or else} \\ u_j &= 0 \text{ for } t \in (\tau, \tau+\Delta] \quad (26) \\ &\quad \text{if } p^T[f_i, f_j](p) < 0 \end{aligned}$$

while the value of the remaining control u_j, or else u_i, over the same interval of time $(\tau, \tau+\Delta]$ is determined by (21) or (22). Similarly, if, prior to time instant τ, the surfaces S_i and S_j are approached by the system from the same sides (i.e. $x^u(\tau-) \in S_i^- \cap S_j^-$ or $x^u(\tau-) \in S_i^+ \cap S_j^+$) then the roles of the controls u_i and u_j, as determined by (25) and (26), is reversed.

The switching strategy is then started, for $s = 1, 2, \ldots, t_1 = \tau+\Delta$, as described in (21)-(24). The position of the system with respect to a surface S_i , at time t_0+ (at the beginning of the switching cycle corresponding to $s = 0$), is understood to be the position of the system with respect to this surface at the beginning of the preliminary cycle (26) if $u_i \neq 0$ for $t < \tau+\Delta$, or else it is the position of the system with respect to S_i prior to time τ (in the case when $u_i = 0$ for $t < \tau+\Delta$).

By virtue of Proposition 1 , it is possible to show that

Proposition 3 *For any $\epsilon > 0$ there exists a time delay $\Delta > 0$ such that*

- *the time t_s^* is finite with respect to both controls u_i and u_j (the controlled trajectory returns to $S_i \cap S_j$).*
- *the trajectory of system (1) with the switching control defined in (21)-(26) remains in $S + B(0;\epsilon)$ for all times $t \geq \tau$.*

It is easy to see that the above strategy can produce an average motion in the directions $\pm[f_i, f_j]$ at p. If, for example, the position of the system with respect to the surfaces S_i, S_j, prior to time τ at which $x(\tau) = p \in S$ is : $p \in S_i^+ \cap S_j^+$ and $p^T[f_i, f_j](p) < 0$, then, it is easy to verify that the switching strategy (with its corresponding initialization) produces the following sequence of controls u_i, u_j :

$$\begin{bmatrix} u_i \\ u_j \end{bmatrix} = \begin{matrix} -1, & 1, & 1, & -1, & -1, & 1, & 1, & ... \\ 0, & -1, & 1, & 1, & -1, & -1, & 1, & ... \end{matrix}$$

over a corresponding sequence of switching cycles $[t_s, t_{s+1}]$, $s = 0, 1, 2, ...$

If the state of the system at time $\tau + \Delta$ (after the initialization step) is p_1 and the delay Δ is sufficiently small, then the state of the system, say p_5, at time t_5 can be approximated by

$$p_5 \approx exp[\Delta(f_i - f_j)] \circ exp[\Delta(f_i + f_j)] \circ exp[\Delta(-f_i + f_j)] \circ exp[\Delta(-f_i - f_j)]p_1 \quad (27)$$

(Of course, the time intervals in which u_i or u_j are nonzero are not exactly equal to Δ , but for small Δ this approximation is reasonable and provides an immediate intuitive insight).

Applying the Campbell-Baker-Hausdorff for the product of exponentials in (27) it is easy to verify that

$$p_5 \approx exp[(\Delta)^2[f_i, f_j]]p_1 \quad (28)$$

which motivates our claim that the above switching strategy produces average motions in the directions associated with the first order Lie bracket $[f_i, f_j]$.

In fact, more rigourous analysis confirms that,

Proposition 4 *Regardless to the initial state of the system prior to time τ at which $x^u(\tau) = p \in S$, and under the assumption that the projection of x onto the Lie bracket $x^T[f_i, f_j](x)$ does not change sign along the controlled system trajectory, the above switching strategy, together with its initialization, produces a T-nonzero motion :*

- *- in the direction $+[f_i, f_j]$ at p if $p^T[f_i(p), f_j(p)] < 0$* (29)

or else

- *- in the direction $-[f_i, f_j]$ at p if $p^T[f_i(p), f_j(p)] > 0$* (30)

with $T \geq 5\Delta$.

As a consequence, there exists a Δ sufficiently small to guarantee that the guiding function V exhibits a T-nonzero decrease along the controlled trajectory.

Some comments are in place to finalize this section.

During application of the switching strategy it may happen that the projection of the state of the system x onto the Lie bracket direction $x^T[f_i, f_j](x)$ changes sign. When this happens, in order to maintain the 'average' decrease of the guiding function V, the switching strategy should be restarted with the initialization step as in (25)-(26).

To guarantee asymtotic convergence of V to zero, the amplitudes of the controls must decrease to zero as V approaches the origin. This can easily be provided for, for example, by scaling the amplitude of the controls by a factor proportional to some nonzero power of the current value of V.

The 'correct' value of the delay Δ need not be known a priori. The delay can be adjusted on line in such a way that an average decrease in the value of V can be observed.

4 The feedback law and its properties

The discussion of the previous section justifies the introduction of the following feedback law, whose aim is to solve the SPC problem:

Set point control feedback strategy (SPCF)

Data: time delay $\Delta > 0$, $n \in I\!N$, $n \geq 3$.

•1 If $p \in I\!R^n \setminus S$, apply the controls

$$u_i(x) = -sign(x^T f_i(x)),\ \ i \in \underline{n-1} \quad (31)$$

•2 If $p \in S$,

- **•2a** Select control variables u_i and u_j satisfying (18) and set $u_k = 0$ for all $k \neq i, j$.
- **•2b** Initialize the switching strategy for the controls u_i and u_j as in (25)-(26) and apply it as in (21)- (24).
- **•2c** Interrupt the switching strategy if any of the following events occur:
  - **(i)** the indices i, j no longer satisfy the condition in (18), in which case : repeat starting from Step 2a;
  - **(ii)** the projection of the current value of the state of the controlled system x onto the Lie bracket $[f_i, f_j](x)$, $x^T[f_i, f_j](x)$, changes sign; in which case : rescale the amplitude of the controls by a factor proportional to the current value of the guiding function V and reinitialize the switching strategy, i.e. repeat from Step 2b;
  - **(iv)** the state of the controlled system does not return to the surface S over an interval of duration $n\Delta$ after the last switch, or else, the value of the guiding function V does not decrease periodically (with a period n switching cycles); in which case set $\Delta = 0.5\Delta$ and restart Step 1.

Propositions 2-4 provide a good basis to prove the following

Theorem 1 *The feedback strategy SPCF is well defined and solves the set point control problem.*

5 Example

The above feedback law has been applied to the Reeds-Shepp convexified car model :

$$\dot{x}(t) = f_1(x(t))u_1 + f_2(x(t))u_2 \quad (32)$$

where

$$\begin{aligned} x(t) &\stackrel{def}{=} [x_1(t), x_2(t), x_3(t)]^T \in I\!R^n \\ f_1(x) &= [1, 0, 0]^T \\ f_2(x) &= [0, sin(x_1), cos(x_1)]^T \end{aligned} \quad (33)$$

The SPC problem was stated as the one to drive the model from initial condition $[x_1, x_2, x_3](0) = [0.4, 1., 0.8]$ to the origin. Figure 1. shows the state variables versus time. It is visible that the surface $S = S_1 \cap S_2 = \{x \in I\!R^n : x_1 = 0\} \cap \{x \in I\!R^n : x_2 sin(x_1) + x_3 cos(x_1) = 0\} = \{x \in I\!R^n : x_2 = 0\}$ is reached at about $t = 5$. The feedback strategy then enters its second stage of subsequent switches in the neighbourhood of S. These switches cause sliding along the x_2 axis which is the direction $+[f_1, f_2]$ at any point $(0, x_2, 0)$, $x_2 > 0$. (The desired direction of average motion for the case when $x_2 < 0$ would clearly be $-[f_1, f_2]$, which could be observed if the initial condition was opposite in sign.) This sliding motion produces an average decrease in the guiding function V whose plot is shown in Figure 4. The control magnitudes are reduced by a factor of $V(x)^{0.25}$ which can be seen from Figure 2. Figure 3 shows the actual trajectory of the car's centre of mass. The cusps in the plot correspond to reversing of the car. The S 'surface' which in this case is the x_2-axis, clearly consists of points at which the car is positioned sideways to its goal.

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Figure 2. Plot of control variables u(t).

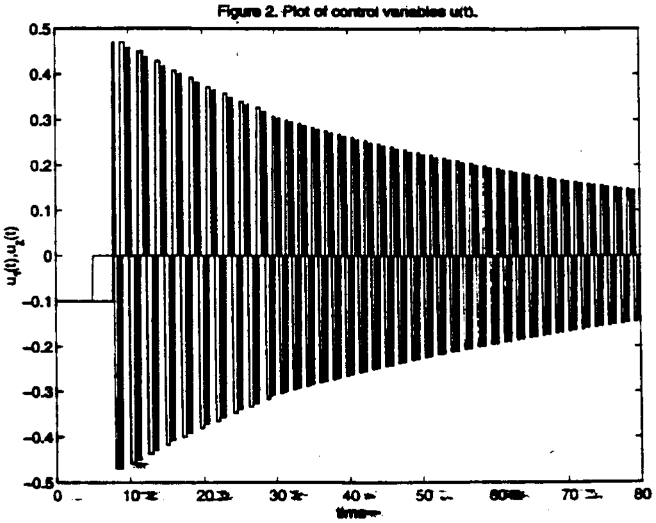

Figure 3. Plot of car trajectory.

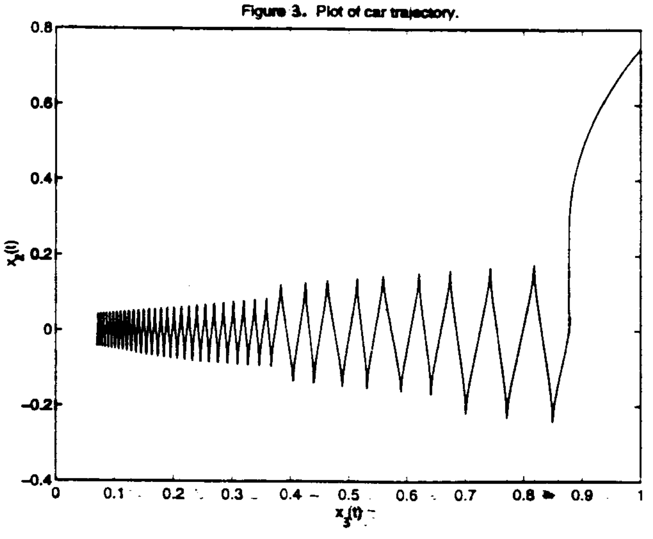

Figure 1. Plot of state variables x(t).

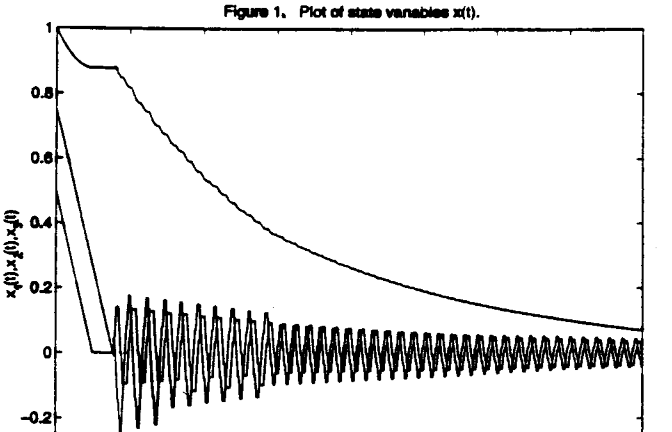

Figure 4. Plot of guiding function v(t).