On maximal invariant sets for discrete time linear systems with disturbances

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1 Introduction

In the last years the study of invariant sets in the case of dynamical systems in presence of unknown input disturbances and/or parametric uncertaintis in the dynamic matrices of the system has had some attention in literature. We cite as an example the papers [8], [9], [10], [12], [18], [20] and [14]. See in this context also the papers [7] and [13], where the robust controlled invariant problem was addressed, when the involved sets are linear subspaces.

It is evident that the analysis of invariant sets is a subproblem of the more general problem of constrained system control. This paper is mostly based in fact on results and methodologies developed by the same author on this matter.

More in particular, we are concerned with the study of maximal sets of initial states, contained in a given set, starting from which a pointwise in time constraint on the state is satisfied, for any unknown disturbance in a given set.

The approach is based on the evolution, backward in time, of the set of admissible states. This backward evolution was already studied in [6] (see also [2] and [4] where the same result is given in an earlier version), to solve the general problem of deriving closed loop optimal controller for linear dynamical systems with linear constraints and linear functional. Moreover a similar approach was also used in [5] to introduce a generalization of positive systems.

Some authors (see [11] and [19]) solved problems in linearly state-control constrained systems with a similar backward recursion. I think that some results in this paper (theorems 2 and 3, section 3) can even be derived from results in [11] as a particular case.

The main advantage of our technique (see [6]) is that we do not use projection algorithms, as done in [11] and [19], so that we give an explicit expression of all the involved polyhedra. Thanks to this explicit expression, in this paper we are able to give necessary and sufficient conditions for the existence of a ball in the disturbance space exists such that the given constraints on the state are satisfied for all time (section 4). If such a ball exists, we can compute also its maximum radius. Moreover we give necessary and sufficient conditions such that a bounded set, positively invariant with respect to a given but arbitrary set of disturbances, exists (section 5).

Another advantage is that, from a computational point of view, in front of a good deal of numerical effort, required off line, the computations to be performed on line to solve the problem of determining the maximal positively invariant set in a given set, with respect to a set in disturbance space, are very simple (no more than a constant number of inner products at each step). The problem of considering also the control (problem of controlled invariance) is an easy generalization, we do not perform here for space reasons.

All the results in this paper are stated both in general, and in the case of polyhedral sets.

See [16] for a more complete discussion and examples. In [17] the problem of controlled invariance in discrete time linear systems with disturbances and parametric uncertainties is addressed.

2 Notations and problem definition

Let us consider the system

$$x(t+1) = Ax(t) + D\delta(t) \tag{1}$$

with the constraint

$$x(t) \in \Sigma \quad \forall \delta(t) \in \Delta,\ t_0 \le t \le T \tag{2}$$

where $x(t) \in R^n$ is the state, $\delta(t) \in R^d$ is a disturbance and Δ is a set in disturbance space. Somewhere in the sequel the dependence of δ on time is omitted, to simplify notations. The horizon T may be finite or infinite.

The symbols $\mathcal{N}(F)$ and $\mathcal{R}(F)$ denote the null space and the range of some matrix F. The symbols $\mathcal{L}(A)$, $\mathcal{C}(A)$ and $\mathcal{C}o(A)$ denote respectively the linear, convex and conical hull of some set A. We denote the interior and relative interior of some set A respectively by $int(A)$ and $rint(A)$. Given a matrix F the symbol F_i denotes the ith row vector of F and given a vector f the symbol f_i denotes the ith component of f. The symbol $f \geq 0$ means that all the components of the vector f are nonnegative. If C is a convex cone, the polar cone to C is the set $C^p = \{y : (y,x) \leq 0, \ \forall x \in C\}$ The set $\mathcal{R}(F)^{\perp} \cap P$, $F \in R^{m \times n}$, being P the nonnegative orthant of the space R^m, is a pointed polyhedral cone. A set of vectors formed taking a nonzero vector from each of its extreme ray is called minimal generating set of $\mathcal{R}(F)^{\perp} \cap P$ and is denoted by $gen(F)$. We denote by $Q(F) \in R^{m \times n_g}, n_g = card(gen(F))$, the matrix whose column vectors are the elements of $gen(F)$. In [3] an efficient algorithm is given for the computation of $gen(F)$.

We make use of a result which will be recalled now for readers convenience.

It was proved in [1] that a polyhedron $P(F,v) = \{x : Fx \leq v\}$ is nonvoid if and only if

$$Q(F)'v \geq 0. \tag{3}$$

We give the following definition:

DEFINITION 1: Assume $D\Delta$ bounded. The system (1) is Δ- stable if $\forall x(t_0) \in R^n$ a neighborhood $N_{x(t_0)}$ (possibly depending on $x(t_0)$) of the origin exists such that $x(t) \in N_{x(t_0)}$, $\forall t > t_0$, $\forall \delta(t) \in \Delta$.

The first problem we address in this paper is the following:

P1: Find the maximal set X_{t_0} of initial states, such that the constraint (2) is satisfied. If such a set of initial states is nonvoid, we say that problem P1 has solution.

The second problem is the following:

P2: Let us assume that the set Δ is a ball in some norm. Given the set Σ, find the maximal ball in disturbance space such that problem P1 has solution.

The definition of positive invariance with respect to some disturbance set Δ is the following:

DEFINITION 2: A set X is Δ-positive invariant (shortly Δ-p.i.) if $AX + D\Delta \subseteq X$.

In the infinite horizon case, we can reformulate problem P1 as:

P1': Given the sets Σ and Δ, find the maximal Δ-positively invariant set contained in Σ. Denote this last set by X^*.

Another question that seems us interesting is that of characterize a system, stating if a bounded nonvoid Δ-positively invariant set exists or not (problem P3). We perform our analysis for some given but arbitrary bounded disturbance set Δ. Notice that in the undisturbed case this problem has always solution, for any system, because the set $\{0\}$ is surely positively invariant, and moreover it is contained in any other positively invariant set.

We study all the above problems both in general, with no assumption on the involved sets, both in the case of polyhedral sets. In this last case we give computable conditions.

A final remark is on order. We refer to systems in the form (1), where output variables are not defined. This is not a limitation, because a constraint on the output implies a constraint on the state, even in case of additive output disturbances. This is evident in the polyhedral case. In fact if we consider the system

$$x(t+1) = Ax(t) + D\delta(t)$$

$$y(t) = Cx(t) + E\pi(t)$$

where $\pi \in \Pi = \{\pi : M\pi \leq m\}$, with the constraint:

$$y(t) \in Y = \{y : Sy \leq s\}, \ 0 \leq t \leq T$$

we can say that $y(t) \in Y$ if and only if $x(t) \in \Sigma = \{x : SCx \leq s - q\}$ where $q_i = \max_{\pi \in \Pi}(SE)_i\pi$, $t_0 \leq t \leq T$.

3 Problem P1

We proved in [14] that the maximal Δ-p.i. set contained in a given set Σ is well defined and unique, if Σ is convex. In the following theorem we state some other properties of Δ-p.i. sets, which are useful to deduce properties of maximal Δ-p.i. set, contained in a given set.

THEOREM 1. *Assume $D\Delta$ convex and let X be a nonvoid Δ-p.i. set.*

i) The set $\mathcal{C}(X)$ is Δ-p.i.

ii) If $D\Delta$ is a cone, $\mathcal{C}o(X)$ is Δ-p.i.

iii) If $D\Delta$ is a subspace, $\mathcal{L}(X)$ is Δ-p.i.

iv) The set $\bar{X}$ is Δ-p.i.

v) If $0 \in D\Delta$, the set αX is Δ-p.i., for all $\alpha \geq 1$.

vi) Let us assume $int(D\Delta) \neq \emptyset$, $0 \in int(D\Delta)$ and $D\Delta$ symmetric. If a bounded nonvoid Δ-p.i. set X exists, the system (1) is Δ- stable.

PROOF:

i) Given two arbitrary points x_1 and x_2 in X, we can write:

$$Ax_1 + D\Delta \subseteq X$$

$$Ax_2 + D\Delta \subseteq X$$

and therefore, given arbitrary $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, from the convexity of $D\Delta$ it follows that

$$A(\alpha_1 x_1 + \alpha_2 x_2) + D\Delta \subseteq (\alpha_1 X + \alpha_2 X)$$

and finally

$$Az + D\Delta \subseteq \mathcal{C}(X) \quad \forall z \in \mathcal{C}(X)$$

ii) If X is Δ-p.i. it follows that $AX + D\Delta \subseteq X$. Therefore $\alpha AX + D\Delta \subseteq \alpha X$, for any nonnegative $\alpha \in R$. Moreover if $Ax_1 + D\Delta \subseteq X$, $x_1 \in X$ and $Ax_2 + D\Delta \subseteq X$, $x_2 \in X$ it follows that $A(x_1 + x_2) + D\Delta \subseteq (X + X)$ and hence $\mathcal{C}o(X)$ is Δ- p.i..

iii) The proof is straightforward. Apply the same technique of the point ii) above.

iv) If $AX + D\Delta \subseteq X$ it is also true that $\overline{AX + D\Delta} \subseteq \bar{X}$ and hence $A\bar{X} + D\Delta \subseteq \bar{X}$.

v) If X is Δ-p.i. it follows that $AX + D\Delta \subseteq X$. Therefore $\alpha AX + \alpha D\Delta \subseteq \alpha X$, for any $\alpha \geq 1$. Moreover if $0 \in D\Delta$ we have that $D\Delta \subseteq \alpha D\Delta$ and hence $\alpha AX + D\Delta \subseteq \alpha X$, for any $\alpha \geq 1$.

vi) If $D\Delta$ is symmetric, and a Δ-p.i. set X exists, we have that $AX + D\Delta \subseteq X$ and also $-AX + D\Delta \subseteq -X$. Therefore, applying statement i) of this theorem, the set $Z = \mathcal{C}(X \cup -X)$ is a Δ-p.i. set. It is easy to see that $0 \in Z$ and from $AZ + D\Delta \subseteq Z$ we deduce that $D\Delta \subseteq Z$. Therefore, because $int(D\Delta) \neq \emptyset$, it follows that $int(Z) \neq \emptyset$ and $0 \in int(Z)$, because $0 \in int(D\Delta)$. This means that for any initial state x_0, or x_0 belongs to Z or a value $\alpha_{x_0} \geq 1$ exists such that $x_0 \in \alpha_{x_0} Z$. In statement v) of this theorem we have stated that αZ, $\alpha \geq 1$, is Δ-p.i., if Z is Δ-p.i. and if $0 \in D\Delta$. Therefore, in either cases above, $x(t)$ belongs to a neighborhood of the origin, $\forall x_0$, $\forall t$ and for any disturbance, and hence, by definition, the system (1) is Δ-stable. □

Consider the sequence of sets:

$$\{\Sigma_t = \Gamma_t \cap \Sigma \quad t = t_0 \dots T-1\}$$

$$\Gamma_t = \{x : Ax + B\Delta \subseteq \Sigma_{t+1}\} \quad t = t_0 \dots T-1 \tag{4}$$

$$\Sigma_T = \Sigma$$

We can state the following theorem:

THEOREM 2.

i) Problem P1 has solution if only if the set Σ_t is nonvoid for any t, $t_0 \leq t < T$.

ii) $X_{t_0} = \Sigma_{t_0}$.

iii) The set X^* is given by $\lim_{t_0 \to -\infty} \Sigma_{t_0}$

PROOF: It suffices to note that Γ_{T-1} is the set of all and nothing but the states admissible with respect to the constraint at T, i.e. the states starting from which the constraint at T is satisfied. The set Σ_{T-1} is the set of states that enjoy the above properties and moreover satisfy the state constraint at $T-1$. This last set Σ_{T-1} plays the role of state constraining set at $T-1$ for the second step of the recursion. Generalizing at a generic $\bar{t}$, we can say that $\Sigma_{\bar{t}}$ is the set of states admissible with respect to the constraints at $t \geq \bar{t}$, and hence the statements follow. □

From the above proof, it is evident that, because the admissible states with respect to the constraints at $t \geq t_1$, $t < T$, are also admissible with respect to the constraints at $t \geq t_2$, $t < T$, $t_2 \geq t_1$, the sequence $\{\Sigma_t\}$ is monotone decreasing, i.e. $\Sigma_T \supseteq \Sigma_{T-1} \supseteq \dots \Sigma_t \supseteq \dots \supseteq \Sigma_{t_0}$.

We can formulate the following corollaries of theorem 2, which consider three relevant particular cases.

The first one states a necessary and sufficent condition such that the set Σ is Δ-p.i., so that the backward recursion defined in theorem 2 reduces to only 1 step.

The second one states a necessary and sufficient condition such that the backward recursion defined in (4) becomes finite, even in the infinite horizon case.

The third one states a condition such that it is possible to remove the constraints $x(t) \in \Sigma$, $t = t_0, \dots \bar{t}$, for some $\bar{t}$. The statements are self-evident and therefore the proofs are omitted. (See also [5] for details on this matter).

COROLLARY 1. *$X^* = \Sigma$ if and only if $\Sigma \subseteq \Gamma_{T-1}$.*

COROLLARY 2. *$X^* = \Sigma_{\bar{t}}$ for some $\bar{t}$ if and only if $\Sigma_{\bar{t}-1} = \Sigma_{\bar{t}}$.*

COROLLARY 3. *We can remove the constraint $x(t) \in \Sigma$, $t = t_0, \dots \bar{t}$, obtaining the same set X_{t_0} as in the original problem, if and only if $\Gamma_{\bar{t}} \subseteq \Sigma$.*

3.1 The polyhedral case

In this subsection, as in the subsequent subsections where the polyhedral case is analyzed, we assume that the sets Σ and Δ are polyhedral, i.e.:

$$\Sigma = \{x : Gx \leq v, G \in R^{g\times n}\} \tag{5}$$

$$\Delta = \{\delta : F\delta \leq h, F \in R^{s\times d}\} \tag{6}$$

With these assumptions, theorem 2 is reformulated in the following

THEOREM 3.
i) Problem P1 has solution if and only if the polyhedron

$$\Sigma_t = \{x : G(t)x \leq g(t)\} \tag{7}$$

where

$$G(T) = G$$

$$G(t) = \begin{pmatrix} G(t+1)A \\ G \end{pmatrix} \quad t = t_0, \dots T-1 \tag{8}$$

$$g(T) = v$$

$$g(t) = \begin{pmatrix} g(t+1) - \bar{g}(t) \\ v \end{pmatrix} \quad t = t_0, \dots T-1 \tag{9}$$

$$\bar{g}(t)_i = \max_{\delta\in\Delta}[G(t+1)D]_i\delta \tag{10}$$

is nonvoid for any t in the set $[t_0, T]$.
ii) In the finite horizon case the maximal set X_{t_0} is a convex polyhedron and coincides with the set Σ_{t_0}.
iii) The set X^* is convex and is given by $\lim_{t_0\to-\infty}\Sigma_{t_0}$.

PROOF: We only outline the proof. In this case $\Gamma_{T-1} = \{x : G(T)Ax \leq g(T) - \bar{g}(T-1)\}$. The set $\Sigma_t = \Gamma_t \cap \Sigma$ can be obtained generalizing the above expression of Γ_{T-1} to arbitrary t.□

Performing the substitutions in formulas (8) and (9), it is easy to give the explicit expression of $G(t)$ and $\bar{g}(t)$ as follows:

$$G(t) = \begin{pmatrix} GA^{T-t} \\ GA^{T-t-1} \\ \vdots \\ GA \\ G \end{pmatrix} \quad t = t_0, \dots T-1 \tag{11}$$

$$\bar{g}(t)_i = \max_{\delta\in\Delta}\left[\begin{pmatrix} GA^{T-t-1} \\ GA^{T-t-2} \\ \vdots \\ GA \\ G \end{pmatrix} D\right]_i \delta \tag{12}$$

or, equivalently

$$\bar{g}(t)_i = \max_{\gamma\in D\Delta}\begin{bmatrix} GA^{T-t-1} \\ GA^{T-t-2} \\ \vdots \\ GA \\ G \end{bmatrix}_i \gamma \tag{12'}$$

REMARK 1: It is useful to remark that if Δ is simmetric, it follows that the vector $\bar{g}(t)$ is nonnegative for any t.

Theorems 4 and 5 below are the reformulations of corollaries 1 and 3, in the polyhedral case.

THEOREM 4. *The set Σ is Δ-p.i. if and only if*

$$\max_{x\in\Sigma}(GA)_i x \leq v_i - \max_{\delta\in\Delta}(GD)_i\delta, \quad i = 1\dots g \tag{13}$$

PROOF: Apply corollary 1, remembering the expression of the set Γ_{T-1} given in theorem 3 and noticing that $\Sigma \subseteq \Gamma_{T-1}$ if and only if conditions (13) are verified.□

THEOREM 5. *If for some $\bar{t}$, $t_0 < \bar{t} < T$, $\max_{x\in\Gamma_{\bar{t}}} G_i x \leq v_i$, $i = 1\dots g$, then X_{t_0} has the following expression:*

$$X_{t_0} = \{x : G(\bar{t})A^{\bar{t}-t_0}x \leq g(\bar{t}) - z\}$$

$$z_i = \max_{\delta\in\Delta}[G(\bar{t})D]_i\delta + \max_{\delta\in\Delta}[G(\bar{t})AD]_i\delta + \dots + $$

$$+ \max_{\delta\in\Delta}[G(\bar{t})A^{\bar{t}-t_0-1}D]_i\delta. \tag{14}$$

PROOF: It is easy to verify that $\Gamma_{\bar{t}} \subseteq \Sigma$ if $\max_{x\in\Gamma_{\bar{t}}} G_i x \leq v_i$, $i = 1\dots g$. The set $\Sigma_{\bar{t}}$ has the expression

$$G(\bar{t})x(\bar{t}) \leq g(\bar{t}).$$

Substituting now for $x(\bar{t})$ the dynamical equation (1), we have the inequality

$$G(\bar{t})(Ax(\bar{t}-1) + D\delta(\bar{t}-1) \leq g(\bar{t}) \;\; \forall\delta(\bar{t}-1) \in \Delta$$

which have a solution if and only if the following inequality has a solution

$$G(\bar{t})Ax(\bar{t}-1 \leq g(\bar{t}) - d$$

where

$$d_i = \max_{\delta \in \Delta}[G(\bar{t})AD]_i\delta.$$

The last inequality describes the set $\Sigma_{\bar{t}} - 1$. We can now perform another substitution for $x(\bar{t}-1)$, and so on, obtaining the expression (14) for the set $X_{t_0} = \Sigma_{t_0})$ at $t = t_0$.□

REMARK 2: In the statements of the above theorems it is required to solve g linear programming problems at each step of time, being g the number of row of the matrix G. In all these problems the constraining set is the same, while the functional to be maximized is different for each problem. This task become trivial if we apply the method developed in [1]. In fact, as it was analyzed in [15], if a linear programming problem is reformulated in dual form, the dependence of the minimum or the maximum of the problem from the linear functional can be explicitated. This means that a more heavy computation, not depending on the functional, may be performed off-line, while for each problem the minimum or the maximum can be easily found, given the functional.

4 Problem P2

In this section we denote by $\mathbf{\Delta}$ a unitary ball in the disturbance space, and $\Delta = \rho\mathbf{\Delta}$, with $\rho \geq 0$ to be determined. The set Σ is given.

The sets defined in (4) become:

$$\Sigma_{t,\rho} = \Gamma_{t,\rho} \cap \Sigma \quad t = t_0, \ldots T-1$$

$$\Gamma_{t,\rho} = \{x : Ax + \rho D\mathbf{\Delta} \subseteq \Sigma_{t+1,\rho}\} \quad t = t_0 \ldots T-1 \quad (15)$$

$$\Sigma_{T,\rho} = \Sigma$$

We can state the following theorem:

THEOREM 6. *Given a norm, a ball in the disturbance space such that problem P1 has a solution exists if and only if the problem:*

$$\min_{t\in[0,T)}(\max \rho : \Gamma_{t,\rho} \textit{ is nonvoid}, \; t_0 \leq t < T)$$

has a nonnegative solution ρ^*. If such $\rho^* \geq 0$ exists, it is the radius of the maximal disturbance ball.

4.1 The polyhedral case

In this subsection we show that applying condition (3) we can explicitly solve problem P2.

From equation (9) we have

$$g_\rho(T) = v$$

$$g_\rho(t) = \begin{pmatrix} g(t+1) - \rho\bar{g}(t) \\ v \end{pmatrix} \quad t = t_0, \ldots, T-1 \quad (16)$$

$$\bar{g}(t)_i = \max_{\delta\in\mathbf{\Delta}}[G(t+1)D]_i\delta \quad (17)$$

It is easy to compute that the bound vector $g_\rho(t) \in R^{g(T-t+1)}$ has the structure

$$g_\rho(t) = g_1(t) - \rho g_2(t) \quad (18)$$

where $g_1(t)$ and $g_2(t)$ don't depend on the parameter ρ and have the form:

$$g_1(t) = \begin{pmatrix} v \\ v \\ \vdots \\ v \end{pmatrix} \quad (19)$$

$$g_2(t) = \begin{pmatrix} \bar{g}(T-1) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} \bar{g}(T-2) \\ \vdots \\ 0 \end{pmatrix} + \ldots + $$

$$+ \begin{pmatrix} \bar{g}(T-t) \\ 0 \end{pmatrix} \quad (20)$$

where $\bar{g}(T-i) \in R^{gi}$, $i = 1 \ldots t$.

Let us define now the set

$$\Lambda_t = \{\rho : Q(G(t))'[g_1(t) - \rho g_2(t)] \geq 0]\} \quad (21)$$

where $G(t)$ was defined in (8) and (11).

REMARK 3: Notice that in [3] it was showed that $Q(G(t+1))$ is a submatrix of $Q(G(t))$. More in particular, $Q(G(t)) = \begin{pmatrix} Q(G(t+1)) & Q_{12}(t) \\ 0 & Q_{22}(t) \end{pmatrix}$, where $Q_{12}(t)$ and $Q_{22}(t)$ are matrices to be determined.

Applying now the nonvoidness condition (3) to polyhedron defined in (7) with the position $g(t) = g_\rho(t)$ we can state the following theorem:

THEOREM 7. *A ball in the disturbance space such that problem P1 has a solution exists if and only if the problem:*

$$\min_{t\in[t_0,T)} \max_{\rho\in\Lambda_t} \rho$$

has a nonnegative solution ρ^*. If such $\rho^* \geq 0$ exists, it is the radius of the maximal disturbance ball.

PROOF: From theorem 3 and condition (3) we have that Σ_t is nonvoid if and only if $Q(G(t))'g_\rho(t) \geq 0$. Substituting for $g_\rho(t)$, this imply $\rho \in \Lambda_t$. Therefore problem P2 has solution if and only if $\rho \in \Lambda_{t_0} \cap \Lambda_1 \cap \ldots \cap \Lambda_{T-1} \neq \emptyset$. The statement follows. □

REMARK 4: Because $\mathbf{\Delta}$ is a ball, it is obvious that $g_2(t) \geq 0$, $\forall t$, and hence $Q(G(t))'g_2(t) \geq 0$, $\forall t$. Therefore the above theorem 7 is satisfied if and only if $g_1(t) \in -(\mathcal{R}(G(t))^\perp \cap P)^p$ or equivalently if and only if $g_1(t) \in \mathcal{R}(G(t)) + P$. In particular this last condition is true if $v \in P$, i.e. if $0 \in \Sigma$. Therefore we have deduced, as a particular case, the well known property that problem P1 has always solution if $0 \in \Sigma$ and $\Delta = 0$.

5 Problem P3

The idea is that of solving problem P1, being the constraining set a ball in the state space, with its radius ρ as parameter, i.e. $\Sigma_\rho = \rho\mathbf{\Sigma}$, where $\mathbf{\Sigma}$ is a unitary ball in state space.

The sets defined in (4) becomes:

$$\Sigma_{t,\rho} = \Gamma_{t,\rho} \cap \rho\mathbf{\Sigma} \quad t = t_0 \ldots T-1$$

$$\Gamma_{t,\rho} = \{x : Ax + B\Delta \subseteq \Sigma_{t+1,\rho}\} \quad t = t_0 \ldots T-1 \quad (22)$$

$$\Sigma_{T,\rho} = \rho\mathbf{\Sigma}$$

We can state the theorem

THEOREM 8. *Given a system (1) a bounded Δ-positively invariant set exists if and only if a radius $\rho \geq 0$ exists such that the set $\Sigma_{t,\rho}$ is nonvoid for any $t \in (-\infty, T]$.*

It is evident that $\rho = 0$ is a feasible solution of the problem only if $B\Delta = \{0\}$.

5.1 The polyhedral case

Let $\mathbf{\Sigma}$ be a unitary polytopic ball in the state space, so that

$$\mathbf{\Sigma} = \{x : Gx \leq \mathbf{1}\} \quad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad (23)$$

$$\Sigma_\rho = \{x : Gx \leq \rho\mathbf{1}\}$$

It is clear that the bound vector in (9) depends on the parameter ρ.

$$g_\rho(T) = \rho\mathbf{1}$$

$$g_\rho(t) = \begin{pmatrix} g_\rho(t+1) - \bar{g}(t) \\ \rho\mathbf{1} \end{pmatrix} \quad t = t_0, \ldots T-1 \quad (24)$$

$$\bar{g}(t)_i = \max_{\delta\in\Delta}[G(t+1)D]_i\delta.$$

It can be computed that

$$g_\rho(t) = \rho\mathbf{1}(t) - r(t) \quad (25)$$

where $\mathbf{1}(t)$ and $r(t) \in R^{g(T-t+1)}$, $r(t)$ doesn't depend on ρ and has the same expression of vector $g_2(t)$ in (19), i.e.:

$$r(t) = \begin{pmatrix} \bar{g}(T-1) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} \bar{g}(T-2) \\ \vdots \\ 0 \end{pmatrix} + \ldots + \begin{pmatrix} \bar{g}(T-t) \\ 0 \end{pmatrix} \quad (26)$$

The matrix $G(t)$ is defined as in (8).

We are now in position to prove the following

THEOREM 9. *Given a system (1) a bounded Δ-positively invariant set exists if and only if a bounded $\rho \geq 0$ exists such that*

$$Q(G(t))'r(t) \leq \rho Q(G(t))'\mathbf{1}(t)$$
$$t_0 \leq t \leq T,\ t_0 \to -\infty. \quad (27)$$

PROOF: Apply condition (3) to $P(G(t), g_\rho(t)$ and substitute for $g_\rho(t)$ the expression (25). □

With some assumptions on the disturbance set, this last theorem particularizes in the following

COROLLARY 4. *Assume Δ symmetric.*

i) Theorem 9 holds if and only if each component of $\mathbf{r}(t)$ *is bounded,* $t \to \infty$.

ii) Assume $D\Delta$ *bounded. A bounded* Δ*-positively invariant set exists if* $\sum_{k=0}^{\infty} \|A^k\| \leq M$*, for some bounded* M.

PROOF:

i) In inequality (27) we can assure that the vector $Q(G(t))'\mathbf{1}$ is strictly positive for all t, thanks to the definition of $Q(G(t))$. Therefore for a fixed t we can always compute a nonnegative ρ such that inequality (27) is satisfied. The problem is that the value of ρ might become arbitrarily large. But this may happen only if some component of $Q(G(t))'r(t)$ tends to infinity.

Because Σ_ρ is symmetric, it follows that the matrix G has the structure $G = \begin{pmatrix} H \\ -H \end{pmatrix}$, and hence the subspace $\mathcal{R}(G)$ is strictly tangent to P, i.e. $\mathcal{R}(G) \cap P = 0$. It follows that also $\mathcal{R}(G(t))$ is strictly tangent to P, and therefore the matrix $Q(G(t))$ cannot have zero rows, for any t, because $\mathcal{R}(G)^\perp$ intersects the interior of P. Moreover the ith component of the vector $r(t)$ is given by:

$$r_i(t) = \max_{\delta\in\Delta}(GD)_j\delta + \max_{\delta\in\Delta}(GAD)_j\delta + \max_{\delta\in\Delta}(GA^2D)_j\delta +$$

$$+ \ldots + \max_{\delta\in\Delta}(GA^{t-i}D)_j\delta$$

$$j = i - int(i/g) * g$$

which is equivalent to

$$r_i(t) = \max_{\delta\in\Delta} G_j D\delta + \max_{\delta\in\Delta} G_j AD\delta + \max_{\delta\in\Delta} G_j A^2 D\delta +$$

$$+ \ldots + \max_{\delta\in\Delta} G_j A^{t-i} D\delta. \qquad (28)$$

Thanks to the assumption on Δ, recalling remark 1, we know that all the elements of the last sum are nonnegative, and therefore we can assure that $(Q(G(t))'r(t))_i$ is bounded if and only if $r_i(t)$ is bounded.

ii) If in (28) we make the position $D\delta = \gamma$, $\gamma \in \Gamma = D\Delta$, from the boundedness of Γ it follows that

$$\forall\gamma \in \Gamma,\ \exists R : \|\gamma\| \leq R$$

and hence

$$r_i(t) \leq \|G_j\|R + \|G_jA\|R + \|G_jA^2\|R + \ldots + \|G_jA^{t-i}\|R \leq$$

$$R\|G_j\|[1 + \|A\| + \|A^2\| + \ldots + \|A^{t-i}\|$$

Therefore if $\sum_{k=0} \infty \|A^k\| \leq M$, it follows that

$$\lim_{t\to\infty} r_i(t) \leq MR\|G_j\|$$

and hence theorem 9 holds. □

References

[1] P. d'Alessandro et al., "Techniques of Linear Programming Based on the Theory of Convex Cones", *Optimization*, 20, pp. 761-777, 1989.

[2] P. d'Alessandro, E. De Santis, "Conical Theory of Feasibility and Optimality for Dynamical Linear Systems", *Reseach Report*, no. 40, Department of Electrical Engineering, University of L'Aquila, 1990.

[3] P. d'Alessandro, "The conical approach to linear programming, *Research Report* 43, Department of Electrical Engineering, University of L'Aquila, 1991 (submitted).

[4] P. d'Alessandro, E. De Santis, "Conical Approach to Linear Dynamic Optimization", in P. Gritzmann, R. Hettich, R. Horst, E. Sachs Eds, *Operations Reserach '91*, Physica Verlag Heidelberg, 1992.

[5] P. d'Alessandro, E. De Santis, "Positiveness of Dynamic Systems with Non Positive Coefficient Matrices", *IEEE Trans. on Automatic Control*, AC-39, pp. 131-134, 1994.

[6] P. d'Alessandro, E. De Santis, "General closed loop optimal solutions for linear dynamic systems with linear constraints and functional", to appear on *J. of Math. Syst., Estim. and Contr.*, 1995.

[7] G. Basile, G. Marro, "On the robust controlled invariant", *Syst. Contr. Lett.*, 9, pp. 191-195, 1987.

[8] G. Bitsoris, "On the positive invariance of polyhedral sets for discrete time systems", *Syst. Contr. Lett*, vol. 11, pp. 243-248, 1988.

[9] G. Bitsoris, "Positively invariant polyhedral sets for discrete-time linear systems", *Int. J. Contr.*, vol. 47, no. 6, pp. 1713-1726, 1988.

[10] F. Blanchini, "Feedback control for linear time-invariant systems with state and control bounds in the presence of disturbance", *IEEE trans. Automat. Contr.*, vol. 35, no. 11, pp. 1231-1234, 1990.

[11] F. Blanchini, "Ultimate Boundedness Control for Uncertain Discrete-Time Systems via Set-Induced Lyapunov Functions", *IEEE Trans. Automat. Contr.*, vol. 39, no. 2, pp. 428-433, 1994.

[12] C. Burgat, A. Benzaouia and S. Tarbouriech, "Positively invariant sets of discrete time systems

with constrained inputs", *Int. J. Syst. Sci*, vol. 21, no. 7, pp. 1249-1271, 1990.

[13] G. Conte, A.M. Perdon, G. Marro, "Computing the maximum robust controlled invariant", *Syst. Contr. Lett.*, 17, pp. 131-135, 1991.

[14] E. De Santis, "On Positively Invariant Sets for Discrete-Time Linear Systems with Disturbance: An Application of Maximal Disturbance Sets", *IEEE Trans. Automat. Contr.*, vol. 39, no. 1, pp. 245-249, 1994.

[15] E. De Santis, "A conical approach to perturbed and uncertain linear programming problems".*Research Report.* Department of Electrical Engineering, University of L'Aquila. January 1995.

[16] E. De Santis, "On maximal invariant sets for discrete time linear systems with disturbances". *Research Report.* Department of Electrical Engineering, University of L'Aquila. February 1995.

[17] E. De Santis, "On controlled invariant sets for discrete time linear systems with disturbances and parametric uncertainties. *Research Report.* Department of Electrical Engineering, University of L'Aquila. February 1995.

[18] E.G. Gilbert and K.T. Tan, "Linear systems with state and control constraints: the theory and application of maximal output admissible sets", *IEEE Trans. Automat. Contr.*, vol. 36, no. 9, pp. 1008-1020, 1991.

[19] S.S. Keerthi, E.G. Gilbert, "Computation of Minimum-Time Feedback Control Laws for Discrete-Time Systems with State- Control Constraints", *IEEE Trans. on Automatic Control,* AC-32, pp. 432-435, 1987.

[20] S. Tarbouriech and C. Burgat, "Positively invariant sets for constrained continuous-time systems with cone properties", in *Proc. 30th IEEE Conf. Decision Contr.*, vol. 2, pp. 1748-1754, 1991.