ROBUST DISCRETE-TIME H_∞-OPTIMAL TRACKING WITH PREVIEW

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ABSTRACT

The problem of robust, finite-time, H_∞–tracking for linear, discrete, time -varying uncertain systems is considered from the game theory point of view. No *a priori* knowledge of the dynamic model of the reference signal to be tracked is assumed, and the parameters of the system are not completely known. Two tracking problems are investigated, depending on whether the reference signal is perfectly known in advance, or previewed in a fixed-interval of time ahead.

A zero-sum game is defined where the controller plays against nature that may choose any initial condition, any bounded energy disturbance input and measurement noise, and any set of the plant parameters in a prescribed bounded region.

Conditions for the existence of a saddle-point solution to this game are not easy to find. Therefore, an augmented state-space description that converts the parameter uncertainty to bounded energy signal is used. An augmented tracking game is then defined and solved. It is shown that its saddle-point equilibrium, if it exists, guarantees a prescribed H_∞-norm performance of the tracker in the original system, for all possible parameters.

1. INTRODUCTION

Tracking given or measured signals is one of the main objectives of control theory and practice. While optimal tracking methods were based in the past on L_2-norm minimization techniques [1], [2], they were susceptible to uncertainty in the plant parameters and they required a full knowledge of the statistics of the exogenous disturbance and noise signals.

This shortcoming has been partially alleviated by attacking the tracking problem from the H_∞-norm minimization point of view. Tracking can be expressed as a special case of the standard problem in the H_∞ control theory [3]. It has lately been shown that an improvement on the standard problem approach to tracking can be achieved by using the information that is gathered on the reference signal during the operation of the controler [4].

The H_∞ methods are insensitive to the exact knowledge of the statistics of the exogenous signals, and they may also reduce the design sensitivity to plant parameter uncertainty. These methods may fail, however, to produce an acceptable tracking performance in presence of large uncertainties. A method that incorporates the uncertainty description in the tracking performance index has been suggested in [5], in the continuous-time case, where the system parameters reside in a prescribed bounded region. If some sufficient conditions are met there, this method guarantees a prescribed tracking performance in spite of the uncertainty. In the present paper we adopt the approach of [5] and extend the results that have been lately derived for the discrete-time H_∞-optimal tracking for systems that are completely known [6], also to cases with parameter uncertainty, where the uncertain part of the plant state-space matrices is known to be norm-bounded and may be time-varying.

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2. PROBLEM FORMULATION

Consider the following linear, discrete time-varying system (Σ_1):

$$x_{k+1} = (A_k + \Delta A_k)x_k + B_{1,k}w_k + (B_{2,k} + \Delta B_k)u_k + B_{3,k}r_k$$

$$z_k = \begin{bmatrix} C_{1,k}x_k + D_{13,k}r_k \\ D_{12,k}u_k \end{bmatrix}$$

$$y_k = (C_{2,k} + \Delta C_k)x_k + (D_{22,k} + \Delta D_k)u_k + D_{21,k}v_k \,, \qquad (2.1\text{a}-\text{c})$$

where $x_k \in \boldsymbol{R}^n$ is the state vector, x_o is an unknown initial state, $u_k \in \boldsymbol{R}^l$ is the control input, $y_k \in \boldsymbol{R}^m$ is the measurement, $r_k \in \boldsymbol{R}^r$ is the measurable reference signal, z_k is the signal we want to minimize, $w_k \in \boldsymbol{R}^p$ is the disturbance input, and $v_k \in \boldsymbol{R}^v$ is the measurement noise. We assume that the uncertainty matrices ΔA_k and ΔB_k are given by:

$$\begin{bmatrix} \Delta A_k & \Delta B_k \\ \Delta C_k & \Delta D_k \end{bmatrix} = \begin{bmatrix} H_{1,k} \\ H_{2,k} \end{bmatrix} F_k \begin{bmatrix} E_{1,k} & E_{2,k} \end{bmatrix}, \qquad (2.2\text{a}-\text{c})$$

with

$$F_k^T F_k \le I \quad , \forall k \in [0 \;\; N]. \qquad (2.3)$$

We make the following assumptions:

$$R_{1,k} \triangleq D_{12,k}^T D_{12,k} > 0 \;\text{ and }\; R_{2,k} \triangleq D_{21,k} D_{21,k}^T > 0.$$

We also assume that the measurement starts at $k = 1$ and that $C_{1,0} = 0$ and $D_{13,0} = 0$.

In the present paper we consider cases where $\{r_i\}$ is not necessarily known *a priori* for the whole time interval. We assume then that at any instant of time, the unknown part of $\{r_i\}$ is a white noise discrete time process of zero mean.

We want $-C_{1,k}x_k$ to follow $D_{13,k}r_k$ by considering also a weighted sum of the control effort u_k. We consider, therefore, for a given $\gamma > 0$, the following performance index:

$$J_E(r,u,w,v,x_0,F) = -\gamma^2 x_0^T R^{-1} x_0 + \Sigma_{k=0}^N E_{\bar{R}_{k+h}}\{||z_k||^2 - \gamma^2(||w_k||^2 + ||v_k||^2)\} \,, \qquad (2.4)$$

where $E_{\bar{R}_{k+h}}$ means an expectation over $\bar{R}_{k+h}$, $h \ge 0$ being the preview length of r, and where $\bar{R}_j$ denotes the future information on r from time j on, i.e. $\bar{R}_j = \{r_i, \; j < i \le N\}$. We are looking for u^* and $\{w^*, v^*, x_0^*, F^*\}$ that achieve a saddle-point equilibrium, where u^* is the minimizing tracking strategy.

This problem is very difficult to solve. We therefore consider the following auxiliary system for a given $\{\epsilon_k\}$, $\epsilon_i \ne 0 \;\; \forall i \in [0 \;\; N]$:

$$(\Sigma_2): \quad \xi_{k+1} = A_k\xi_k + [B_{1,k} \;\; \frac{\gamma}{\epsilon_k}H_{1,k}]w_{a,k} + B_{2,k}u_{a,k} + B_{3,k}r_k$$

$$z_{a,k} = \bar{C}_{1,k}\xi_k + \bar{D}_{12,k}u_{a,k} + \bar{D}_{13,k}r_k$$

$$y_{a,k} = C_{2,k}\xi_k + [0 \;\; \frac{\gamma}{\epsilon_k}H_{2,k}]w_{a,k} + D_{21,k}v_{a,k} + D_{22,k}u_{a,k} \,, \qquad (2.5\text{a}-\text{c})$$

where

$$\bar{C}_{1,k} \triangleq \begin{bmatrix} C_{1,k} \\ 0 \\ \epsilon_k E_{1,k} \end{bmatrix}, \quad \bar{D}_{12,k} \triangleq \begin{bmatrix} 0 \\ D_{12,k} \\ \epsilon_k E_{2,k} \end{bmatrix},$$

$$\text{and } \bar{D}_{13,k} \triangleq \begin{bmatrix} D_{13,k} \\ 0 \\ 0 \end{bmatrix}. \qquad (2.6\text{a}-\text{c})$$

We also define the following index of performance:

$$J_a(r,u_a,w_a,v_a,\xi_0,\epsilon) = -\gamma^2\xi_0^T R^{-1}\xi_0 + \Sigma_{k=0}^N E_{\bar{R}_{k+h}}\{||\bar{C}_{1,k}\xi_k + \bar{D}_{13,k}r_k + \bar{D}_{12,k}u_{a,k}||^2 - \gamma^2(||w_{a,k}||^2 + ||v_{a,k}||^2)\}. \qquad (2.7)$$

We prove the following lemma:

Lemma 2.1 *Consider the systems (Σ_1) and (Σ_2), together with the performance indices (2.4) and (2.7), respectively. Assume that $\{u_k\}$ and $\{u_{a,k}\}$ are generated by the same controller, namely, $u_k = \mathcal{G}_y \cdot y_k + \mathcal{G}_r \cdot r_k$ and $u_{a,k} = \mathcal{G}_y \cdot y_{a,k} + \mathcal{G}_r \cdot r_k$. Then, the following holds:*

$$\sup_{x_0,w,v,F} J_E(r,u,w,v,x_0,F) \le \sup_{\xi_0,w_a,v_a} J_a(r,u_a,w_a,v_a,\xi_0,\epsilon)$$

for all admissible $\{\epsilon_k\}$.

Proof: For any given x_0, $\{w_k\}$, $\{v_k\}$, $\{F_k\}$ and $\{r_k\}$ in (2.4), for (Σ_1), and any admissible$\{\epsilon_k\}$ we take

$$\xi_0 = x_0, \;\; w_{a,k} = \begin{bmatrix} w_k \\ \frac{\epsilon_k}{\gamma}F_k(E_{1,k}\xi_k + E_{2,k}u_{a,k}) \end{bmatrix},$$

$$v_{a,k} = v_k .$$

Then, we find that for all $k \in [0 \; N-1]$,

$$\xi_{k+1} = x_{k+1} \; , \;\; y_{a,k} = y_k \; , \;\; u_{a,k} = u_k \; ,$$

which means that

$$J_a(r, u_a, w_a, v_a, \xi_0, \epsilon) = J_E(r, u, w, v, x_0, F)$$
$$+\Sigma_{k=0}^{N} E_{R_{k+h}} \{ \| \epsilon_k (E_{1,k} x_k + E_{2,k} u_k) \|^2 -$$
$$\| \epsilon_k F_k (E_{1,k} x_k + E_{2,k} u_k) \|^2 \} \geq J_E(r, u, w, v, x_0, F).$$

□

The problem of finding the saddle-point equilibrium solution to the zero-sum game that is described by (Σ_2) and (2.6 a-c),(2.7) has been solved in [6], for the special case where

$$D_{22,k} = 0 \text{ and } \begin{bmatrix} \bar{C}_{1,k}^T \\ \bar{D}_{13,k}^T \end{bmatrix} \bar{D}_{12,k} = 0 \; , \; \forall k \in [0 \; N].$$

In order to use the latter solution, we apply the following transformation

$$\bar{u}_{a,k} = u_{a,k} + \tilde{R}_{1,k}^{-1} [\bar{D}_{12,k}^T \bar{C}_{1,k} \xi_k + \bar{D}_{12,k}^T \bar{D}_{13,k} r_k], \quad (2.8)$$

where $\tilde{R}_{1,k} = \bar{D}_{12,k}^T \bar{D}_{12,k}$. We obtain then that

$$\xi_{k+1} = (A_k - B_{2,k} \tilde{R}_{1,k}^{-1} \bar{D}_{12,k}^T \bar{C}_{1,k}) \xi_k + B_{2,k} \bar{u}_{a,k}$$
$$+ [B_{1,k} \;\; \frac{\gamma}{\epsilon_k} H_{1,k}] w_{a,k} + (B_{3,k} - B_{2,k} \tilde{R}_{1,k}^{-1} \bar{D}_{12,k}^T \bar{D}_{13,k}) r_k, \quad (2.9)$$

$$J_a(r, u_a, w_a, v_a, \xi_0, \epsilon) = \Sigma_{k=0}^{N} E_{R_{k+h}} \{ \| \bar{D}_{12,k} \bar{u}_{a,k} \|^2$$
$$+ \| (I - \bar{D}_{12,k} \tilde{R}_{1,k}^{-1} \bar{D}_{12,k}^T)^{\frac{1}{2}} (\bar{C}_{1,k} \xi_k + \bar{D}_{13,k} r_k) \|^2$$
$$- \gamma^2 (\| w_{a,k} \|^2 + \| v_{a,k} \|^2) \} - \gamma^2 \xi_0^T R^{-1} \xi_0 . \quad (2.10)$$

3. THE ROBUST STATE-FEEDBACK TRACKING PROBLEM

We use the representation of (2.9) with the performance index of (2.10) and we assume access to the state ξ_k. Since, in this case, $\{v_k\}$ does not affect the controller, nature will take $v_k^* \equiv 0$.

We obtain from the solution in [6] that the problem stated in (2.5a-c) and (2.7) has a saddle-point equilibrium iff $\exists \, P_i > 0 \; \forall i \in [0 \;\; N]$ that solves

$$P_k = \tilde{A}_k^T M_{k+1} \tilde{A}_k + \tilde{C}_{1,k}^T \tilde{C}_{1,k} \; , \;\; P(N) = \tilde{C}_{1,N}^T \tilde{C}_{1,N}, \quad (3.1)$$

and satisfies

$$\gamma^2 I - \bar{B}_{1,k}^T P_{k+1} \bar{B}_{1,k} > 0 \;\; k \in [0 \;\; N-1],$$

and

$$\gamma^2 R^{-1} - P_0 > 0 \; , \quad (3.2\text{a, b})$$

where

$$M_{k+1} \triangleq Q_{k+1} [I + B_{2,k} \tilde{R}_{1,k}^{-1} B_{2,k}^T Q_{k+1}]^{-1},$$
$$Q_{k+1} \triangleq P_{k+1} [I - \gamma^{-2} \bar{B}_{1,k} \bar{B}_{1,k}^T P_{k+1}]^{-1} \quad (3.3\text{a, b})$$
$$\tilde{A}_k \triangleq A_k - \epsilon_k^2 B_{2,k} (R_{1,k} + \epsilon_k^2 E_{2,k}^T E_{2,k})^{-1} E_{2,k}^T E_{1,k} \; ,$$
$$\tilde{C}_{1,k}^T \tilde{C}_{1,k} = \bar{C}_{1,k}^T (I - \bar{D}_{12,k} \tilde{R}_{1,k}^{-1} \bar{D}_{12,k}^T) \bar{C}_{1,k}$$
$$= C_{1,k}^T C_{1,k} + \epsilon_k^2 E_{1,k}^T (I + \epsilon_k^2 E_{2,k} R_{1,k}^{-1} E_{2,k}^T)^{-1} E_{1,k} \; ,$$

and

$$\bar{B}_{1,k} \triangleq [B_{1,k} \;\; \frac{\gamma}{\epsilon_k} H_{1,k}] \, .$$

The saddle-point strategy is given by

$$\xi_0^* = (\gamma^2 R^{-1} - P_0)^{-1} \theta_0 \; , \quad (3.4)$$

$$w_{a,k}^* = \gamma^{-2} \begin{bmatrix} B_{1,k}^T \\ \frac{\gamma}{\epsilon_k} H_{1,k}^T \end{bmatrix} (I - \gamma^{-2} P_{k+1} \bar{B}_{1,k} \bar{B}_{1,k}^T)^{-1}$$
$$[\theta_{k+1} + P_{k+1} (\tilde{A}_k \xi_k + B_{2,k} \bar{u}_{a,k} + B_{3,k} r_k)], \quad (3.5)$$

and

$$\bar{u}_{a,k}^* = - \tilde{R}_{1,k}^{-1} B_{2,k}^T M_{k+1} [\tilde{A}_k \xi_k + B_{3,k} r_k + P_{k+1}^{-1} \theta_{k+1}^c] \; , \quad (3.6)$$

where $\theta_{k+1}^c = E_{R_{k+h}} \{\theta_{k+1}\}$, and where θ_k satisfies

$$\theta_k = \bar{A}_k \theta_{k+1} + \bar{B}_k r_k \; , \; \theta_N = C_{1,N}^T D_{13,N} r_N, \quad (3.7)$$

with

$$\bar{A}_k = \tilde{A}_k^T M_{k+1} P_{k+1}^{-1} \text{ and } \bar{B}_k = \tilde{A}_k^T M_{k+1} B_{3,k} + C_{1,k}^T D_{13,k} \, . \quad (3.8)$$

The resulting saddle-point control is

$$u_k^* = \bar{u}_{a,k}^* - \tilde{R}_{1,k}^{-1} \epsilon_k^2 E_{2,k}^T E_{1,k} \xi_k$$
$$= - \tilde{R}_{1,k}^{-1} B_{2,k}^T M_{k+1} (B_{3,k} r_k + P_{k+1}^{-1} \theta_{k+1}^c)$$
$$- \tilde{R}_{1,k}^{-1} (B_{2,k}^T M_{k+1} \tilde{A}_k + \epsilon_k^2 E_{2,k}^T E_{1,k}) \xi_k . \quad (3.9)$$

We distinguish between two information patterns of $\{r_k\}$:

i. **Model Following**:

In this case, $\{r_k\}$ is known *a priori* for all $k \in [0,\ N]$. Then, θ^c_{k+1} in (3.9) is obtained by:

$$\theta^c_{k+1} = \Phi_{k+1}\theta_N + \Sigma_{j=1}^{N-k-1}\Psi_{k+1,j}\bar{B}_{N-j}r_{N-j}, \qquad k \le N-2 \qquad (3.10)$$

where

$$\Phi_{k+1} \triangleq \bar{A}_{k+1}\bar{A}_{k+2}......\bar{A}_{N-1}$$

and

$$\Psi_{k+1,j} = \begin{cases} \bar{A}_{k+1}\bar{A}_{k+2}......\bar{A}_{N-j-1} \\ \qquad j < N-k-1 \\ I \qquad j = N-k-1 \end{cases} \qquad (3.11\text{a, b})$$

ii. **Preview tracking**

Given that at time k, r_i is known for $i \le \ min(N, k+h)$, where h is the preview length. Then, the control law is given by (3.9), where θ^c_{k+1} satisfies the following:

$$\theta^c_{k+1} = \begin{cases} \Sigma_{j=1}^{h}\bar{\psi}_{k+1,j}\bar{B}_{k+h+1-j}r_{k+h+1-j} \\ \qquad k+h \le N-1 \\ \Phi_{k+1}\theta_N + \Sigma_{j=1}^{h-1}\bar{\psi}_{k+1,j}\bar{B}_{N-j}r_{N-j} \\ \qquad k+h = N \end{cases} \qquad (3.12)$$

where $\bar{\psi}_{k+1,j}$ is obtained from (3.11b) by replacing N by $k+h+1$.

Note that on-line tracking occurs for $h = 0$. In this case we get $E_{R_k}\{r_i\} = 0$ for $i \ge k+1$, and $\theta^c_{k+1} = E_{R_k}\{\theta_{k+1}\} = 0$. As a result, the on-line optimal control law is given by

$$u^*_k = -\tilde{R}^{-1}_{1,k}(B^T_{2,k}M_{k+1}\tilde{A}_k + \epsilon^2_k E^T_{2,k}E_{1,k})\xi_k - \tilde{R}^{-1}_{1,k}B^T_{2,k}M_{k+1}B_{3,k}r_k \ .$$

4. THE ROBUST OUTPUT TRACKING FEEDBACK

In the case where the information on the state-vector of (2.9) is obtained through noisy measurements, nature can affect the outcome of the auxiliary tracking game by selecting a nonzero measurement noise sequence. We assume that the measurement is given by (2.5c) and we look for the control sequence $\{u_k\}$ that minimizes the auxiliary performance index of (2.10), based on the information that is available on $\{r_k\}$.

We start by considering the model following case, where $\{r_k\}$ is known em a priori for all $k \in [0\ N]$. We denote:

$$\tilde{Q}_{k+1} \triangleq \tilde{R}_{1,k} + B^T_{2,k}Q_{k+1}B_{2,k} \quad \text{and} \quad \tilde{P}_{k+1} \triangleq \gamma^2 I - \bar{B}^T_{1,k}P_{k+1}\bar{B}_{1,k} \qquad (4.1\text{a, b})$$

Using the results of [6] we obtain the following result.

Lemma 4.1 *The performance index of (2.10) is given by*

$$J_a(r, \bar{\bar{u}}_a, \bar{w}_a, v_a, \xi_0, \epsilon) = -\gamma^2\|\xi_0 - \xi_0^*\|^2_{R^{-1}-\gamma^{-2}P_0} + \Sigma_{k=0}^{N}E_{R_{k+h}}\{\|\bar{\bar{u}}_{a,k} + \hat{C}_{1,k}\xi_k\|^2\} - \gamma^2\Sigma_{k=0}^{N}(\|\bar{w}_{a,k}\|^2 + \|v_{a,k}\|^2) + \bar{J}(r)\ , \qquad (4.2)$$

where $\bar{J}(r)$ *is a term that depends only on* r_i, $i \in\ [1\ \ N]$,

$$\hat{C}_{1,k} \triangleq \tilde{Q}^{\frac{1}{2}}_{k+1}\tilde{R}^{-1}_{1,k}(B^T_{2,k}M_{k+1}\tilde{A}_k + \epsilon^2_k E^T_{2,k}E_{1,k})\ ,$$

$$\bar{w}_{a,k} \triangleq \gamma^{-1}\tilde{P}^{\frac{1}{2}}_{k+1}w_{a,k} - \gamma^{-1}\tilde{P}^{-\frac{1}{2}}_{k+1}\bar{B}^T_{1,k}\left[P_{k+1}(A_k\xi_k + B_{2,k}u_{a,k} + B_{3,k}r_k) + \theta_{k+1}\right]\ ,$$

and

$$\bar{\bar{u}}_{a,k} \triangleq \tilde{Q}^{\frac{1}{2}}_{k+1}u_{a,k} + \tilde{Q}^{-\frac{1}{2}}_{k+1}B^T_{2,k}Q_{k+1}\left[B_{3,k}r_k + P^{-1}_{k+1}\theta_{k+1}\right]\ .$$

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We thus represent the system by:

$$\xi_{k+1} = \hat{A}_k\xi_k + \hat{B}_{1,k}\bar{w}_{a,k} + \hat{B}_{2,k}\bar{\bar{u}}_{a,k} + \hat{B}_{3,k}r_k + \hat{B}_{4,k}\theta_{k+1}\ ,$$

$$y_{a,k} = C_{2,k}\xi_k + D_{22,k}u_{a,k} + [0\ \ \frac{\gamma}{\epsilon_k}H_{2,k}]\gamma\tilde{P}^{-\frac{1}{2}}_{k+1}\bar{w}_{a,k} + D_{21,k}v_{a,k} + [0\ \ \frac{\gamma}{\epsilon_k}H_{2,k}]\tilde{P}^{-1}_{k+1}\bar{B}^T_{1,k}\left[P_{k+1}(A_k\xi_k + B_{2,k}u_{a,k} + B_{3,k}r_k) + \theta_{k+1}\right]\ , \qquad (4.3\text{a, b})$$

where

$$\hat{A}_k = (I - \gamma^{-2}\bar{B}_{1,k}\bar{B}^T_{1,k}P_{k+1})^{-1}A_k\ , \quad \hat{B}_{1,k} = \gamma\bar{B}_{1,k}\tilde{P}^{-\frac{1}{2}}_{k+1}$$

$$\hat{B}_{2,k} = P^{-1}_{k+1}Q_{k+1}B_{2,k}\tilde{Q}^{-\frac{1}{2}}_{k+1}, \quad \hat{B}_{3,k} = P^{-1}_{k+1}M_{k+1}B_{3,k}\ ,$$

and $\hat{B}_{4,k} = P_{k+1}^{-1}(M_{k+1}P_{k+1}^{-1} - I)$. (4.4a – e)

Introducing

$$\bar{y}_{a,k} = y_{a,k} - [0 \;\; \frac{\gamma}{\epsilon_k} H_{2,k}]\tilde{P}_{k+1}^{-1}\bar{B}_{1,k}^T [P_{k+1}(B_{2,k}u_{a,k} + B_{3,k}r_k) + \theta_{k+1}] - D_{22,k}u_{a,k} \qquad (4.5)$$

we obtain

$$\bar{y}_{a,k} = \hat{C}_{2,k}\xi_k + [0 \;\; \frac{\gamma^2}{\epsilon_k} H_{2,k}]\tilde{P}_{k+1}^{-\frac{1}{2}}\bar{w}_{a,k} + D_{21,k}v_{a,k} , \qquad (4.6)$$

where

$$\hat{C}_{2,k} \triangleq C_{2,k} + [0 \;\; \frac{\gamma}{\epsilon_k} H_{2,k}]\tilde{P}_{k+1}^{-1}\bar{B}_{1,k}^T P_{k+1}A_k .$$

Defining the augmented noise vector

$$\tilde{w}_k \triangleq \begin{bmatrix} \bar{w}_{a,k} \\ v_{a,k} \end{bmatrix} ,$$

(4.3) and (4.6) are rewritten as

$$\xi_{k+1} = \hat{A}_k\xi_k + \tilde{B}_k\tilde{w}_k + d_k, \text{and } \bar{y}_{a,k} = \hat{C}_{2,k}\xi_k + \tilde{D}_k\tilde{w}_k, \qquad (4.7\text{a, b})$$

where

$$\tilde{B}_k = \begin{bmatrix}\hat{B}_{1,k} & 0\end{bmatrix} , \; \tilde{D}_k = \begin{bmatrix}[0 \;\; \frac{\gamma^2}{\epsilon_k} H_{2,k}]\tilde{P}_{k+1}^{-\frac{1}{2}} & D_{21,k}\end{bmatrix} ,$$

and $d_k = \hat{B}_{2,k}\bar{u}_{a,k} + \hat{B}_{3,k}r_k + \hat{B}_{4,k}\theta_{k+1}$.

It follows from (4.7a,b) that the problem becomes an H_∞-filtering game problem with correlated measurement and process noises. In order to obtain an equivalent system with uncorrelated noises, such that we could apply the results of the *a posteriori* filtering as in [6], we decompose:

$$\tilde{w}_k = (I - \tilde{D}_k^T(\tilde{D}_k\tilde{D}_k^T)^{-1}\tilde{D}_k)^{\frac{1}{2}} w_k^{(1)} + \tilde{D}_k^T(\tilde{D}_k\tilde{D}_k^T)^{-\frac{1}{2}} w_k^{(2)},$$

and get $w_k^{(2)} = (\tilde{D}_k\tilde{D}_k^T)^{-\frac{1}{2}}(\bar{y}_{a,k} - \hat{C}_{2,k}\xi_k)$. Denoting $\hat{\hat{A}}_k = \hat{A}_k - \tilde{B}_k\tilde{D}_k^T(\tilde{D}_k\tilde{D}_k^T)^{-1}\hat{C}_{2,k}$ the equivalent filtering game is then the following:

Given the system

$$\xi_{k+1} = \hat{\hat{A}}_k\xi_k + \tilde{B}_k(I - \tilde{D}_k^T(\tilde{D}_k\tilde{D}_k^T)^{-1}\tilde{D}_k)^{\frac{1}{2}} w_k^{(1)} + \tilde{d}_k, \qquad (4.8)$$

and

$$\bar{y}_{a,k} = \hat{C}_{2,k}\xi_k + (\tilde{D}_k\tilde{D}_k^T)^{\frac{1}{2}} w_k^{(2)} , \qquad (4.9)$$

with $\tilde{d}_k = d_k + \tilde{B}_k\tilde{D}_k^T(\tilde{D}_k\tilde{D}_k^T)^{-1}\bar{y}_{a,k}$. Find a minimizing estimation strategy for the game that is defined by the following performance index:

$$J_a(r, \bar{\bar{u}}_a, \tilde{w}_a, \xi_0, \epsilon) = -\gamma^2\|\xi_0 - \xi_0^*\|^2_{R^{-1}-\gamma^{-2}P_0} + \Sigma_{k=0}^N E_{R_{k+h}}\{\|\bar{\bar{u}}_{a,k} + \hat{C}_{1,k}\xi_k\|^2\} - \gamma^2\Sigma_{k=0}^N(\|w_k^{(1)}\|^2 + \|w_k^{(2)}\|^2) + \bar{J}(r) . \qquad (4.10)$$

Assuming that (3.1) has a solution $P_k > 0$ over $[0, \; N]$, satisfying (3.2a,b), we consider the following Riccati equation:

$$\Sigma_k = Z_k[I + (\hat{C}_{2,k}^T(\tilde{D}_k\tilde{D}_k^T)^{-1}\hat{C}_{2,k} - \gamma^{-2}\hat{C}_{1,k}^T\hat{C}_{1,k})Z_k]^{-1} \qquad (4.11)$$

with

$$Z_{k+1} - \hat{\hat{A}}_k\Sigma_k\hat{\hat{A}}_k^T - \tilde{B}_k(I - \tilde{D}_k^T(\tilde{D}_k\tilde{D}_k^T)^{-1}\tilde{D}_k)\tilde{B}_k^T = 0$$

$$\Sigma_0 = [R^{-1} - \gamma^{-2}(P_0 + \hat{C}_{1,0}^T\hat{C}_{1,0})]^{-1} . \qquad (4.12\text{a, b})$$

The solution of the latter estimation game is stated in the following theorem, for the *a posteriori* case, where u_k can use the information on $\{\bar{y}_{a,i}, \;\; 0 \le i \le k\}$:

Theorem 4.1 *Consider the system (2.1a) with the measured output (2.1c). Assume that $\{r_k\}$ is known a priori for all $k \le N$. Then, the auxiliary tracking problem with output-feedback has a saddle-point solution iff*

i. $\exists \; P_i > 0 \; \forall i \in [0 \;\; N]$ *that solves (3.1) and satisfies (3.2a,b), and*

ii. $\exists \; \Sigma_i \ge 0 \; \forall i \in [0 \;\; N]$ *that solves (4.11)-(4.12).*

Under the latter, a saddle-point tracking strategy is given by:

$$u_k^* = -\tilde{Q}_{k+1}^{-\frac{1}{2}}\hat{C}_{1,k}\hat{\xi}_k - \tilde{R}_{1,k}^{-1}B_{2,k}^T M_{k+1}[B_{3,k}r_k + P_{k+1}^{-1}\theta_{k+1}] , \qquad (4.13)$$

where θ_{k+1} satisfies (3.7). The 'estimate' $\hat{\xi}_k$ is defined by

$$\hat{\xi}_{k+1} = (I - L_{k+1}X_{k+1}^{(1)})^{-1}[\hat{\hat{A}}_k\hat{\xi}_k + L_{k+1}\{y_{k+1} - \hat{C}_{2,k+1}(\hat{\hat{A}}_k\hat{\xi}_k - \tilde{d}_k) + X_{k+1}^{(2)}(B_{3,k+1}r_{k+1} + P_{k+2}^{-1}\theta_{k+2})\} + \tilde{d}_k] ,$$

$$\hat{\xi}_0 = \xi_0^* , \qquad (4.14\text{a, b})$$

with

$$L_k \triangleq Z_k\hat{C}_{2,k}^T(\tilde{D}_k\tilde{D}_k^T + \hat{C}_{2,k}Z_k\hat{C}_{2,k}^T)^{-1} , \qquad (4.15)$$

where ξ_0^* *is defined in (3.4), and* $X_{k+1}^{(1)}$ *and* $X_{k+1}^{(2)}$ *are given by*

$$X_k^{(1)} = (\epsilon_k^{-2} H_{2,k} H_{1,k}^T Q_{k+1} B_{2,k} + D_{22,k}) \tilde{Q}_{k+1}^{-\frac{1}{2}} \hat{C}_{1,k}$$

and

$$X_k^{(2)} = (D_{22,k} \tilde{R}_{1,k}^{-1} B_{2,k}^T - \epsilon_k^{-2} H_{2,k} H_{1,k}^T) M_{k+1}.$$

$\nabla\nabla\nabla$

The above theorem addresses the case where $\{r_k\}$ is *a priori* known for the whole time interval. Using the reasoning of [6] we obtain that in the case where the reference signal is measured on-line, or with a preview $h > 0$, nature's saddle-point strategy, that is a function of the 'error' $\xi_k - \hat{\xi}_k$, will not change. However, the saddle-point control strategy of the auxiliary tracking problem can no longer be given by (4.13) since θ_{k+1} is not known. The control strategy in this case satisfies

$$E_{\bar{R}_{k+h}}\{\bar{\bar{u}}_{a,k} + \hat{C}_{1,k}\hat{\xi}_k\} = 0 \, .$$

We thus readily obtain the following result, by applying the operator $E_{\bar{R}_{k+h}}$ on u_k^* of (4.13) and on $\hat{\xi}_k$ of (4.14).

Corollary 4.1 *Given that at time* k*,* r_i *is known for* $i \le k+h$*. Then, the saddle-point control strategy of the auxiliary tracking game is given by*

$$u_k^* = -\tilde{Q}_{k+1}^{-\frac{1}{2}} \hat{C}_{1,k} \hat{\hat{\xi}}_k$$
$$- \tilde{R}_{1,k}^{-1} B_{2,k}^T M_{k+1} (B_{3,k} r_k + P_{k+1}^{-1} \theta_{k+1}^c) \qquad (4.16)$$

where

$$\hat{\hat{\xi}}_{k+1} = (I - L_{k+1} X_{k+1}^{(1)})^{-1} \{\hat{A}_k \hat{\hat{\xi}}_k + L_{k+1}[y_{k+1}$$
$$-\hat{C}_{2,k+1} \hat{A}_k \hat{\hat{\xi}}_k + X_{k+1}^{(2)} (B_{3,k+1} r_{k+1} + P_{k+2}^{-1} \theta_{k+2}^c)$$
$$-\hat{C}_{2,k+1} \hat{d}_k] + \hat{d}_k\},$$
$$\hat{\hat{\xi}}_0 = \gamma^{-2} \Sigma_0 \theta_0^c, \qquad (4.17)$$

and where

$$\hat{d}_k = P_{k+1}^{-1} Q_{k+1} [B_{2,k} u_k + B_{3,k} r_k$$
$$+ \gamma^{-2} \bar{B}_{1,k} \bar{B}_{1,k}^T E_{\bar{R}_{k+h+1}} \{\theta_{k+1}\}$$
$$+ \frac{\gamma^2}{\epsilon_k^2} H_{1,k} H_{2,k}^T (\tilde{D}_k \tilde{D}_k^T)^{-1} \hat{y}_{a,k}],$$

$$\hat{y}_{a,k} = y_k - \epsilon_k^{-2} H_{2,k} H_{1,k}^T Q_{k+1} (B_{2,k} u_k + B_{3,k} r_k$$
$$+ P_{k+1}^{-1} E_{\bar{R}_{k+h+1}} \{\theta_{k+1}\}) - D_{22,k} u_k \, . \qquad (4.18\text{a, b})$$

The vector θ_i^c *is given by (3.12) and* $E_{\bar{R}_{k+h+1}}\{\theta_{k+1}\}$ *is computed from (3.10) by replacing* N *by* $k+h+1$ *and by taking* $\theta_{k+h+2} = 0$.

When r_k *is measured on-line,* $h = 0$ *and the optimal control strategy is given by (4.16) with* $\theta_{k+1}^c = 0$. *The estimate* $\hat{\hat{\xi}}_k$ *is given by (4.17) with* $\hat{\hat{\xi}}_0 = \bar{B}_0 r_0$*,* $\theta_{k+2}^c = 0$*, and* $E_{\bar{R}_{k+1}}\{\theta_{k+1}\} = \bar{B}_{k+1} r_{k+1}$ *in (4.18a,b).*

The above results have been derived for the *a posteriori* case, where the measurement of y_k (and r_k) is available for the control u_k. In the *a priori* case, we assume that u_k can only be based on y_i, $i < k$. The solution to the estimation game that is described by the cost function (4.10) is given then by the following theorem.

Theorem 4.2 *Consider the system (2.1a) with the measured output (2.1c). Assume that* $\{r_k\}$ *is known a priori for all* $k \le N$*. Then, the a priori auxiliary tracking problem with output-feedback has a saddle-point solution iff*

i. $\exists\ P_i > 0\ \forall i \in [0\ \ N]$ *that solves (3.1) and satisfies (3.2a,b), and*

ii. $\exists\ \Sigma_i$ *that solves (4.11)-(4.12) so that* $\tilde{D}_i \tilde{D}_i^T - \hat{C}_{2,i} \Sigma_i \hat{C}_{2,i}^T \le 0\ \ \forall i \in [0\ \ N-1]$.

Under the latter, a saddle-point tracking strategy is given by (4.13) where $\hat{\xi}_k$ *is replaced by*

$$\hat{\xi}_{k+1} = \hat{A}_k \hat{\xi}_k + \bar{L}_k [\bar{y}_{a,k} - \hat{C}_{2,k} \hat{\xi}_k] + \tilde{d}_k \ , \ \ \hat{\xi}_0 = \xi_0^* \ , \ (4.19)$$

where $\bar{L}_k = \hat{A}_k \Sigma_k \hat{C}_{2,k}^T (\tilde{D}_k \tilde{D}_k^T)^{-1}$.

The result for the case where the reference signal is measured either on-line or by a preview is stated as follows:

Corollary 4.2 *When, at time* k*,* r_i *is known for* $i \le k+h-1$ *the minimizing tracking strategy is given by*

$$u_k^* = -\tilde{Q}_{k+1}^{-\frac{1}{2}} \hat{C}_{1,k} \hat{\xi}_k$$
$$- \tilde{R}_{1,k}^{-1} B_{2,k}^T M_{k+1} (B_{3,k} r_k + P_{k+1}^{-1} E_{\bar{R}_{k+h-1}} \{\theta_{k+1}\}) \qquad (4.20)$$

where $\hat{\xi}_k$ satisfies (4.19), with $\hat{\xi}_0 = \gamma^{-2}E_{\bar{R}_{h-1}}\{\theta_0\}$, with $\bar{y}_{a,k}$ replaced by

$$\hat{y}_{a,k} = y_k - (\epsilon_k^{-2}H_{2,k}H_{1,k}^TQ_{k+1}B_{2,k} + D_{22,k})u_k$$

$$-\epsilon_k^{-2}H_{2,k}H_{1,k}^TQ_{k+1}(B_{3,k}r_k + P_{k+1}^{-1}\theta_{k+1}^c), \qquad (4.21)$$

and with

$$\tilde{d}_k = P_{k+1}^{-1}Q_{k+1}[B_{2,k}u_k + B_{3,k}r_k + P_{k+1}^{-1}\theta_{k+1}^c$$

$$+\frac{\gamma^2}{\epsilon_k^2}H_{1,k}H_{2,k}^T(\tilde{D}_k\tilde{D}_k^T)^{-1}\hat{y}_{a,k}], \qquad (4.22)$$

where θ_{k+1}^c is given in (3.12) and $E_{\bar{R}_{k+h-1}}\{\theta_{k+1}\}$ is computed from (3.10) and (3.11a,b) by replacing N by $k+h$ and by taking $\theta_{k+h} = 0$.

When r_k is measured on-line, namely, when r_i is known only for $i < k$ and $h = 0$, the corresponding minimizing control strategy is given by

$$u_k^* = -\tilde{Q}_{k+1}^{-\frac{1}{2}}\hat{C}_{1,k}\hat{u}_k^* = -\tilde{Q}_{k+1}^{-\frac{1}{2}}\hat{C}_{1,k}\hat{\xi}_k\,, \qquad (4.23)$$

with $\theta_{k+1}^c = 0$ and $E_{\bar{R}_{k-1}}\{\theta_{k+1}\} = 0$ in (4.21) and (4.22).

5. EXAMPLE

We consider the system of (2.1a) with the following objective function:

$$J = \Sigma_{k=0}^N E_{R_{k+h}}\{\|C_1x_k - r_k\|^2 + 0.04\|u_k\|^2\}$$

$$-\gamma^2\Sigma_{k=0}^N\|w_k\|^2$$

where N is assumed to be very large, and where :

$$A_k = \begin{bmatrix} 0 & 1 \\ -0.8 & 1.6+\delta_k \end{bmatrix},\ B_{1,k} = \begin{bmatrix} 0.25 \\ 0.5 \end{bmatrix}$$

$$B_{2,k} = \begin{bmatrix} 0 \\ -1+5\delta_k \end{bmatrix},\ B_{3,k} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } C_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

We assume that $\|\delta_k\| \le 0.04$, $\forall\ k$. We also define the uncertainty matrices:

$$E_1 = [\ 0 \quad .04\ ],\ E_2 = .2,\ H_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } H_2 = 0.$$

We first design a controller for the nominal system, i.e $F = 0$. Using the H_∞ standard model, r_k is considered as a disturbance so that the augmented disturbance vector is now $\begin{bmatrix} w_k^T & r_k^T \end{bmatrix}^T$. Using the notation of the standard problem, we define

$$B_1 = \begin{bmatrix} 0.25 & 1 \\ 0.5 & 1 \end{bmatrix},\ D_{11} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$$D_{21} = 0_{2\times2},\ D_{22} = 0_{2\times1}.$$

We then get that the minimum possible value of γ is near $\gamma_s = 2.61$. The obtained control law is then $u_k = K_sx_k$ with $K_s = [-1.0991\ \ 2.0395]$, and the resulting closed-loop transfer function from r_k to C_1x_k is $G_s = (2z + 1.7386)(z^2 + 0.4395z - 0.2991)^{-1}$.

We apply the results obtained in [6], without preview, i.e. $h = 0$, for the lowest value of γ. For $\gamma = 0.86$, a value very close to the lowest achievable value of γ, the solution of the Riccati equation and the resulting state-feedback gain are, respectively,

$$P = \begin{bmatrix} 1.0256 & 0.9281 \\ 0.9281 & 1.7649 \end{bmatrix} \text{ and } K_0 = [0.7999\ \ 2.2459].$$

The closed-loop transfer function from r_k to C_1x_k is then $G_0 = (0.354z - .0002)(z^2 + .6459z + .0001)^{-1}$.

We finally apply the results of the robust design given in Section 3 and obtain that the minimal value of γ is close to 2.5 for $\epsilon = 10.4$. The solution of the Riccati equation and the resulting state-feedback gain are, respectively,

$$P_r = \begin{bmatrix} 3.3705 & -3.5951 \\ -3.5951 & 12.4684 \end{bmatrix}$$

and

$$K_r = [\ -0.6786 \quad 1.1173\ ].$$

The resulting closed-loop transfer function from r_k to C_1x_k, for $F = 0$, is $G_r = (1.3616z - 0.2425)(z^2 - 0.4827z + 0.1214)^{-1}$.

The results, for the nominal case, of the standard problem, [6] and the new robust design, are computed for $r_k = sin(0.5k)$ and $w_k = 0$, and are compared for $N = 50$ in Figure 6. The tracking error obtained using the standard method is 2.6682. The one that is achieved by the results of [6] is 0.3089, and the result for the robust design is 0.4623. As expected, the method of [6] yields the best result since it is aimed at the case where the system parameters are given without any uncertainty .

We apply the results obtained by the three methods on the actual system with $F = -1$. Unfortunately, the designs obtained by the standard method and the optimal H_∞ tracking method of [6] lead to unstable closed-loop systems. The closed-loop transfer function of the system found by the standard method is $G_s(z) = (2z + 2.4063)(z + 1.2897)^{-1}(z - 0.4023)^{-1}$, and the one found by the method of [6] is $G(z) = (0.0248z + 0.3199)(z + 1.26)^{-1}(z - 0.126)^{-1}$. As expected, the closed loop with $F = -1$ remains stable when we use the control law that is obtained by the new robust method. The corresponding transfer function is given by $G_r(z) = (1.2339z + 0.029)(z^2 - 0.2192z - 0.0144)^{-1}$.

Finally we apply the results of the new H_∞ robust tracking of $r_k = sin(0.5k)$,for $F = -1$, with various lengths of preview. The tracking error power for the different preview lengths are depicted in Figure 6. Note that a drop in the error norm occurs at $h = 1$ (namely a preview of 0.5 sec). This is due to a small increase in the power of the control effort, that is normally low, at this preview length. It is shown in Figure 6 how the increase in the preview length reduces the tracking error.

6. CONCLUSIONS

The robust discrete-time tracking problem is solved in this paper by defining, and solving, an auxiliary tracking game problem. The minimizing saddle-point control strategy of this game guarantees a prescribed tracking performance of the actual controlled system, for all plant parameters in a given bounded set. Solutions to the model following case, and to the case where the reference signal is measured on-line or with a preview, are obtained. These solutions entail an overdesign that stems from the inherent nature of the auxiliary game whose performance index provides only an upper bound to the performance index of the original problem. These solutions are also sensitive to the selection of the parameter sequence $\{\epsilon_k\}$ that determines how, in the auxiliary game, the uncertainty based disturbances split between the system input and the objective function.

In the present paper, we have assumed that in cases where the reference signal is not ***a priori*** known for the whole time interval, its future unknown part behaves like a random white process. The theory developed can, however, deal also with cases where some ***a priori*** information is available on the reference signal. This information should not necessarily be accurate, and it may involve some parameter uncertainty. The uncertain model that is assumed to produce the signal can be incorporated into the state-space description of the system, together with its parameter uncertainty, and the method of the present paper can be applied on the augmented system.

In the present paper, we have considered the finite-horizon time-varying problem. A question may arise what happens in the time-invariant case when the horizon tends to infinity. The answer to this may be found from the fact that the resulting feedback loop of our solution is identical to the one obtained in [8]. It is shown in [8] that under mild assumptions this loop is robust in the limiting infinite-horizon case.

The theory of this paper can also be applied in cases of measurable disturbances. The measurable part of the disturbance, that can be obtained on-line or by a fixed preview, can be considered as a part of the reference signal. This part will drive the system but it will not appear in the controlled output.

7. REFERENCES

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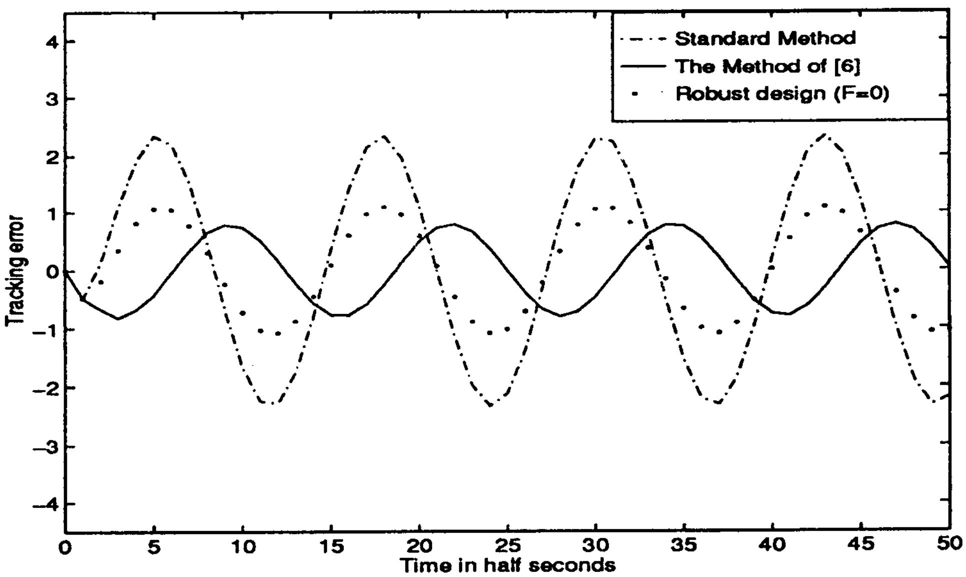

Figure 5.1: Comparison between the tracking errors obtained by the different methods for the nominal system for $r_k = sin(0.5k)$, measured on-line

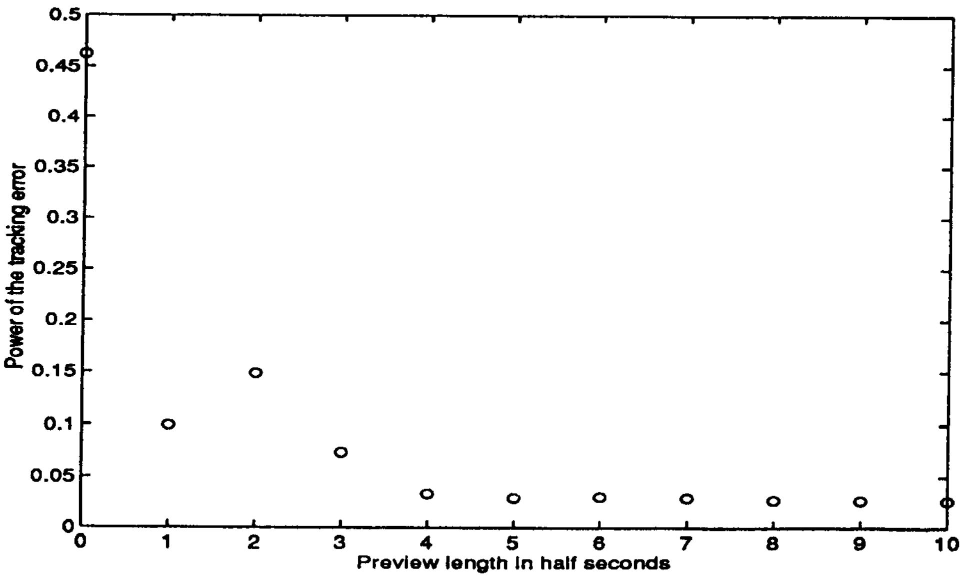

Figure 5.2: The power of the tracking error as a function of the preview length