Input-output relations of linear discrete-time periodic processes: state-space representation and causality[1]

Osvaldo Maria Grasselli*, Antonio Tornambè**, Paolo Valigi*

* Dipartimento di Ingegneria Elettronica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy

**Dipartimento di Meccanica e Automatica, Terza Università di Roma, via Segre 60, 00146, Roma, Italy

Abstract The problem here considered is that of obtaining a state-space representation of an input-output relation characterizing a process described by linear difference equations with periodic coefficients. For a given periodic process, under the assumption that it has no null characteristic multipliers, necessary and sufficient conditions are given for the existence of a linear periodic system whose input-output behaviour coincides with that of the original process, for all the initial times. For the general case (i.e., processes possibly with null characteristic multipliers), the causality property is studied giving sufficient conditions for a process described by linear difference equations with periodic coefficients to be causal.

1. Introduction

The study of the problem of finding a state-space representation of a process that can be modelled by linear difference (or differential) equations with constant coefficients (solved by Rosenbrock [1] via strict system equivalence) was extended in [2, 3] to processes that can be modelled by linear difference equations with periodic coefficients (whose period will be denoted by ω) of the following form:

$$\sum_{i=0}^{r} T_i(t)\xi(t+i) = \sum_{i=0}^{r} U_i(t)u(t+i), \quad (1a)$$

$$y(t) = \sum_{i=0}^{r} V_i(t)\xi(t+i) + \sum_{i=0}^{r} W_i(t)u(t+i), \quad (1b)$$

for some integer $r \geq 0$, where $t \in \mathbb{Z}, \xi(t+i) \in \mathbb{R}^m$ is the vector of the internal variables or pseudo-state, $u(t+i) \in \mathbb{R}^p$ is the input, $y(t) \in \mathbb{R}^q$ is the output, $T_i(t)$, $U_i(t)$, $V_i(t)$ and $W_i(t)$, $i = 0, ..., r$, are real periodic matrices of period ω (briefly ω-periodic), and the $T_i(t)$, $i = 0, ..., r$, are possibly square. Equations (1) were termed the *model* of the process under consideration.

In [2] conditions were given under which it is possible to obtain a description of (1) in state-space form, within the class of models that were called system equivalent to model (1) at some (initial) time t_0. In [3] weaker conditions were given under which it is possible to obtain a state-space representation that is "largely system equivalent" to model (1) at some (initial) time t_0.

The interest of obtaining a description of (1) in state-space form is motivated by the variety of processes that can be modelled by linear difference equations with periodic coefficients (see, e.g., [4, 5, 6]) and by the resulting attention devoted to linear periodic discrete-time systems, for which a control theory (based on a state-space description) is developing, including eigenvalue assignment, state and output dead-beat control, disturbance localization, model matching, robust tracking and regulation, and block decoupling [7]-[15]. Although [2, 3] gave state-space representations of the whole input-pseudostate-output behaviour of model (1) for the output and pseudostate responses beginning *at the same initial time* t_0 at which system equivalent or largely system equivalent state-space representations were obtained, the analysis was not complete for the output and pseudostate responses beginning at a different initial time. Then, the purpose of this paper is to focus the attention on the mere input-output relation specified by model (1), by looking for a state-space representation of it that gives rise to its output responses beginning *at any initial time*, and by studying its causality. At the best author's knowledge these problems have not yet been solved, although the different problem of finding a periodic realization of a periodic rational matrix or impulse response matrix was studied by several authors (see, e.g., [16]-[19]), the conditions for the causality of input-output maps only were given in [20], and in [2, 21] only the necessity of these conditions for the causality of model (1) was shown. In Sections 2 and 3 all the background material is recalled. In Section 4 it is shown that, under the same assumption considered in [2], an ω-periodic system that is system equivalent to model (1) at some time t_0 is also a state-space representation of the input-output relation specified by model (1), and, in addition, that the conditions given in [20] are sufficient for the causality of the input-output relation specified by model (1).

2. Basic notations and time-invariant characterization of periodic processes

Henceforth, the identity matrix of dimension ν will be denoted either by I_ν, or simply by I; Δ will denote the ω-steps forward-shift operator, Δ^{-1} its inverse. In addi-

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tion, let $R_\nu(\Delta)$, $\nu \in \mathbb{Z}^+$, be defined by:

$$R_\nu(\Delta) := \begin{bmatrix} 0 & I_{(\omega-1)\nu} \\ \Delta I_\nu & 0 \end{bmatrix}, \tag{2}$$

where $\mathbb{Z}^+$ is the set of positive integers.

Let a vector function $z(t) \in \mathbb{R}^\nu$ be given, with $t \in \mathbb{Z}$; for any $k \in \mathbb{Z}$, the *ω-stacked form of $z(t)$ at (the initial) time k* is defined by

$$z_k(h) := \left[z^T(k+h\omega) \ \dots \ z^T(k+h\omega+\omega-1)\right]^T, \ h \in \mathbb{Z}.$$

From now on, whenever the operator $R_\nu(\Delta)$ will be applied to $z_k(h)$, the operator Δ will have the meaning of an ω-steps forward-shift in the k variable, or, equivalently, a one-step forward-shift in the h variable. Let an ω-periodic matrix $H(t) \in \mathbb{R}^{\nu\times\mu}$ be given, with $t \in \mathbb{Z}$, representing the linear map $z(t) = H(t)w(t)$; for any $k \in \mathbb{Z}$, the *ω-stacked form of $H(t)$ at (the initial) time k* is defined by $\mathcal{H}_k := \operatorname{diag}\{H(k), H(k+1), ..., H(k+\omega-1)\}$, and represents the induced linear map between the ω-stacked forms at time k of the vector functions $z(t)$ and $w(t)$, i.e. $z_k(h) = \mathcal{H}_k w_k(h), h \in \mathbb{Z}$.

Lemma 1 *[2, 21]. For any vector function $z(t) \in \mathbb{R}^\nu$ and for any ω-periodic matrix $H(t) \in \mathbb{R}^{\nu\times\mu}$ ($t \in \mathbb{Z}$), the following relations hold for all $k \in \mathbb{Z}$:*

$$\begin{aligned} R_\nu(\Delta) z_k(h) &= z_{k+1}(h), & \text{(3a)} \\ R_\nu(\Delta)\mathcal{H}_k R_\mu^{-1}(\Delta) &= \mathcal{H}_{k+1}; & \text{(3b)} \end{aligned}$$

relation (3b) *still holds with Δ replaced by a scalar complex variable.*

By introducing the ω-stacked forms $\xi_{t_0}(h), u_{t_0}(h), y_{t_0}(h)$ at time $t_0, t_0 \in \mathbb{Z}$, of vectors $\xi(t), u(t), y(t)$ and the ω-stacked forms $\mathcal{T}_{i,t_0}, \mathcal{U}_{i,t_0}, \mathcal{V}_{i,t_0}$ and $\mathcal{W}_{i,t_0}$ at time t_0 of matrices $T_i(t)$, $U_i(t)$, $V_i(t)$, $W_i(t)$, $i = 0, \dots, r$, by Lemma 1 model (1) can be expressed in the following form, which is called the *ω-stacked form at (the initial) time t_0 of model* (1), or, briefly, *ω-stacked model at time t_0*:

$$\begin{aligned} \mathcal{T}_{t_0}(\Delta)\xi_{t_0}(h) &= \mathcal{U}_{t_0}(\Delta)u_{t_0}(h), & \text{(4a)} \\ y_{t_0}(h) = \mathcal{V}_{t_0}(\Delta)\xi_{t_0}(h) &+ \mathcal{W}_{t_0}(\Delta)u_{t_0}(h), & \text{(4b)} \end{aligned}$$

where $\mathcal{T}_{t_0}(\Delta) := \sum_{i=0}^{r} \mathcal{T}_{i,t_0} R_m^i(\Delta)$, $\mathcal{U}_{t_0}(\Delta) := \sum_{i=0}^{r} \mathcal{U}_{i,t_0} R_p^i(\Delta)$, $\mathcal{V}_{t_0}(\Delta) := \sum_{i=0}^{r} \mathcal{V}_{i,t_0} R_m^i(\Delta)$, $\mathcal{W}_{t_0}(\Delta) := \sum_{i=0}^{r} \mathcal{W}_{i,t_0} R_p^i(\Delta)$ [21]. The following polynomial matrix of Δ:

$$S_{t_0}^M(\Delta) := \begin{bmatrix} -\mathcal{T}_{t_0}(\Delta) & \mathcal{U}_{t_0}(\Delta) \\ \mathcal{V}_{t_0}(\Delta) & \mathcal{W}_{t_0}(\Delta) \end{bmatrix} \tag{5}$$

is termed the *ω-stacked system matrix at (the initial) time t_0 of model* (1), thus extending the time-invariant Rosenbrock system matrix [1]. The following assumption is justified by the subsequent Proposition 2, and will be assumed to hold throughout the paper.

Assumption 1 *The polynomial matrix $\mathcal{T}_{t_0}(\Delta)$ is square and nonsingular.*

Proposition 1 *[2, 22] If Assumption 1 holds for $t_0 = \bar{t}_0 \in \mathbb{Z}$, then it holds for any $t_0 \in \mathbb{Z}$, and the polynomial* $\det(\mathcal{T}_{t_0}(\Delta))$ *is independent of the initial time t_0.*

The following proposition clarifies that Assumption 1 is not restrictive. It is based on the following three types of elementary operations on the scalar rows of (4a): (i) multiply any row by a non-zero real constant c; (ii) interchange rows i and j; (iii) add a multiple, by a polynomial $d(\Delta)$ in Δ with real coefficients, of row j to row i.

Proposition 2 *[2, 21] If Assumption 1 does not hold, then one (or more) of the following situations occurs for equation* (4a)*: (α) by a finite sequence of elementary operations of types (i), (ii), and (iii) on the rows of equation* (4a)*, one of the scalar rows of the transformed equation of* (4a) *can be reduced to the trivial identity $0 = 0$; (β) there exists an ω-stacked input function $u_{t_0}(\cdot)$ for which* (4a) *admits no solution for $h \geq 0$; (γ) there exist solutions of* (4a) *for $h \geq 0$ and for any $u_{t_0}(\cdot)$, but they depend on an infinite number of arbitrary and independent initial conditions.*

If Assumption 1 holds, then, for each input function $u(\cdot)$, there exist solutions $\xi_{t_0}(\cdot), y_{t_0}(\cdot)$ of (4) *for $h \geq 0$, and they depend on arbitrary and independent initial conditions whose number is equal to the degree of* $\det(\mathcal{T}_{t_0}(\Delta))$.

If Assumption 1 holds, the degree of $\det(\mathcal{T}_{t_0}(\Delta))$ is called the *order of model* (1).

Under Assumption 1, for a fixed initial time t_0, the application of the z-transform to both sides of (4) — with all the "initial conditions" of $\xi_{t_0}(h)$ that make unique the solution of (4a) [21] put equal to zero, and a suitable number of initial values of $u_{t_0}(h)$ equal to zero — yields

$$y_{t_0}(z) = G_{t_0}^M(z)\, u_{t_0}(z), \tag{6}$$

where $u_{t_0}(z)$ and $y_{t_0}(z)$ are the z-transforms of $u_{t_0}(h)$ and $y_{t_0}(h)$, respectively, and $G_{t_0}^M(z) := \mathcal{V}_{t_0}(z)\mathcal{T}_{t_0}^{-1}(z)\mathcal{U}_{t_0}(z) + \mathcal{W}_{t_0}(z)$ is called the *ω-stacked transfer matrix of model* (1) *at (the initial) time t_0*. Lemma 1 yields [2, 21]:

$$G_{t_0+1}^M(z) = R_q(z) G_{t_0}^M(z) R_p^{-1}(z), \ \forall z \in \mathbb{C}, \ \forall t_0 \in \mathbb{Z}. \tag{7}$$

In view of the discussion in [2], under Assumption 1, the polynomial $\det(\mathcal{T}_{t_0}(z))$ was called the *characteristic polynomial of model* (1), and its zeros were called the *characteristic multipliers of model* (1) with corresponding *ordered sets of structural indices at time t_0* defined as their nondecreasing sequences of multiplicities as zeros of the invariant polynomials of $\mathcal{T}_{t_0}(z)$. In a similar way, under the same Assumption 1, the *invariant zeros, input decoupling zeros,* and *output decoupling zeros of model* (1) *at time t_0* were defined to be the zeros of the invariant polynomials of $S_{t_0}^M(z)$, $[-\mathcal{T}_{t_0}(z) \ \ \mathcal{U}_{t_0}(z)]$, $[-\mathcal{T}_{t_0}^T(z) \ \ \mathcal{V}_{t_0}^T(z)]^T$, respectively, with *ordered sets of structural indices* at the same time defined as their nondecreasing sequences of multiplicities as zeros of such polynomials. All types of zeros were shown to be independent of time t_0 (together with their ordered sets of structural indices), except for the null ones [2, 23].

3. System equivalence and large system equivalence: background material

In order to find a description of model (1) in state-space form that takes into account the whole input-pseudostate-output behaviour, two $(m\omega + q\omega) \times (m\omega + p\omega)$ polynomial system matrices $S^1(\Delta)$ and $S^2(\Delta)$ with real coefficients were said to be *strictly system equivalent* if a relation of the following form holds [2], thus extending the notion introduced in [1]:

$$S^2(\Delta) = \begin{bmatrix} M(\Delta) & 0 \\ Y(\Delta) & I_{q\omega} \end{bmatrix} S^1(\Delta) \begin{bmatrix} N(\Delta) & X(\Delta) \\ 0 & I_{p\omega} \end{bmatrix}, \quad (8)$$

where $M(\Delta), N(\Delta), X(\Delta)$ and $Y(\Delta)$ are polynomial matrices in Δ with real coefficients, and $M(\Delta), N(\Delta)$ are square and unimodular.

The following further operations on the ω-stacked form (4) at time t_0 of model (1) were considered in [2], thus extending the similar operations used in [1]:

(a) for each $l = 0, ..., \omega - 1$, add to the vector component $\xi(t_0 + h\omega + l)$ of $\xi_{t_0}(h)$, ν scalar components, $\nu \geq 0$, which are defined to be equal to zero for each $h \geq 0$;

(b) for each $l = 0, ..., \omega - 1$, remove from the vector component $\xi(t_0 + h\omega + l)$ of $\xi_{t_0}(h)$, ν scalar components, $0 \leq \nu \leq m$, if they are equal to zero for each $\ell = 0, ..., \omega - 1$, for each $h \geq 0$, for all input functions $u(\cdot)$ and for all admissible initial conditions.

Then, two ω-periodic models $\mathcal{M}_1$ and $\mathcal{M}_2$ of the type (1), satisfying Assumption 1 and having inputs and outputs of the same dimensions p and q, respectively, and corresponding ω-stacked models $\mathcal{M}^i_{t_0}$ of the form (4), $i = 1, 2$, at the same time t_0, were said to be *system equivalent at (the initial) time* t_0 [2] if there exist an operation of the type (a) or (b) to be carried out on $\mathcal{M}^1_{t_0}$ and an operation of the type (a) or (b) to be carried out on $\mathcal{M}^2_{t_0}$ such that the ω-stacked system matrices at time t_0 corresponding to the resulting ω-stacked models at time t_0 are strictly system equivalent.

The following extra operations on the ω-periodic model (1) were introduced in [3]:

(c) add the following vector equations to equations (1):

$$\begin{aligned} \zeta_i(t+1) &= \zeta_{i+1}(t), \quad i = 1, 2, \ldots, \omega - 2, & (9a) \\ \zeta_{\omega-1}(t+1) &= u(t), & (9b) \end{aligned}$$

so that, defining

$$\xi^L(t) := [\zeta_1^T(t) \; \zeta_2^T(t) \; ... \; \zeta_{\omega-1}^T(t) \; \xi^T(t)]^T, \quad (10)$$

a new model of the form (1) is obtained, with $\xi^L(t) \in \mathbb{R}^{m+(\omega-1)p}$ instead of $\xi(t)$;

(d) if vector $\xi(t)$ can be partitioned as follows:

$$\xi(t) := [\zeta_1^T(t) \; \zeta_2^T(t) \; ... \; \zeta_{\omega-1}^T(t) \; \xi^{0^T}(t)]^T, \quad (11)$$

so that $\zeta_i(t), i = 1, ..., \omega - 1$, satisfy (9) and $\xi^0(t)$ satisfies an $[m - (\omega - 1)p]$-dimensional vector equation of the form (1a), and a q-dimensional vector equation of the form (1b), with $\xi^0(t)$ instead of $\xi(t)$, then remove equations (9) from the given model.

The ω-stacked system matrix at time t_0 of the model obtained after that an operation of the type (c) has been carried out on model (1), is strictly system equivalent to the following one:

$$S^{ML}_{k_0}(\Delta) =$$

$$\left[\begin{array}{cccccc|c} -R_p(\Delta) & I_{\omega p} & ... & 0 & 0 & & 0 \\ 0 & -R_p(\Delta) & ... & 0 & 0 & & 0 \\ ... & ... & ... & ... & ... & & ... \\ 0 & 0 & ... & I_{\omega p} & 0 & & 0 \\ 0 & 0 & ... & -R_p(\Delta) & 0 & & I_{\omega p} \\ 0 & 0 & ... & 0 & -\mathcal{T}_{k_0}(\Delta) & & \mathcal{U}_{k_0}(\Delta) \\ \hline 0 & 0 & ... & 0 & \mathcal{V}_{k_0}(\Delta) & & \mathcal{W}_{k_0}(\Delta) \end{array}\right]$$

$$=: \begin{bmatrix} -\mathcal{T}^L_{k_0}(\Delta) & \mathcal{U}^L_{k_0}(\Delta) \\ \mathcal{V}^L_{k_0}(\Delta) & \mathcal{W}_{k_0}(\Delta) \end{bmatrix}, \quad (12)$$

having $\omega - 1$ block rows and columns in addition to $S^M_{t_0}(\Delta)$.

Then, two ω-periodic models $\mathcal{M}_1$ and $\mathcal{M}_2$ of the type (1), satisfying Assumption 1 and having inputs and outputs of the same dimensions p and q, respectively, were said to be *largely system equivalent at (the initial) time* t_0 [3] if there exist a finite number of operations of the type (c) or (d) to be carried out on $\mathcal{M}_1$ and a finite number of operations of the type (c) or (d) to be carried out on $\mathcal{M}_2$, such that the resulting models, $\overline{\mathcal{M}}_1$ and $\overline{\mathcal{M}}_2$, respectively, are system equivalent at time t_0.

Proposition 3 *[2, 3] The relation of system equivalence at time t_0 and the relation of large system equivalence at time t_0 between two ω-periodic models of the type (1) are equivalence relations.*

A special case of model (1) is that of a linear ω-periodic model in state-space form, i.e., a linear ω-periodic *system* described by:

$$\begin{aligned} x(t+1) &= A(t)\,x(t) + B(t)\,u(t), & (13a) \\ y(t) &= C(t)\,x(t) + D(t)\,u(t). & (13b) \end{aligned}$$

where $t \in \mathbb{Z}$, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the input, $y(t) \in \mathbb{R}^q$ is the output, and $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ are real ω-periodic matrices.

The system equivalence relation at time t_0 between two models of the form (1) (or, specifically, between a model of the form (1) and a system of the form (13)) was studied in [2]. The corresponding results are summarized by the following proposition and remark.

Proposition 4 *[2] Given two ω-periodic models $\mathcal{M}_1$ and $\mathcal{M}_2$ of the type (1), satisfying Assumption 1 and having inputs and outputs of the same dimensions p and q, respectively, pseudo-states of dimensions $m_i, i = 1, 2$, and the following ω-stacked system matrices at time t_0:*

$$S^M_{t_0,i}(\Delta) = \begin{bmatrix} -\mathcal{T}_{t_0,i}(\Delta) & \mathcal{U}_{t_0,i}(\Delta) \\ \mathcal{V}_{t_0,i}(\Delta) & \mathcal{W}_{t_0,i}(\Delta) \end{bmatrix}, \quad i = 1, 2, \quad (14)$$

if $\mathcal{M}_1$ and $\mathcal{M}_2$ are system equivalent at time t_0, then for each input function $u(t), t \geq t_0$, the solutions of $\mathcal{M}_1$ and $\mathcal{M}_2$ for $t \geq t_0$ in the pseudostates are biuniquely related, and their solutions for $t \geq t_0$ in the output are exactly the same.
In addition: (α) the matrices in each of the pairs $(S^M_{t_0,1}(z),\ S^M_{t_0,2}(z))$, $(\mathcal{T}_{t_0,1}(z),\ \mathcal{T}_{t_0,2}(z))$, $([-\mathcal{T}_{t_0,1}(z)\ \mathcal{U}_{t_0,1}(z)],\ [-\mathcal{T}_{t_0,2}(z)\ \mathcal{U}_{t_0,2}(z)])$, $([-\mathcal{T}^T_{t_0,1}(z)\ \mathcal{V}^T_{t_0,1}(z)]^T,\ [-\mathcal{T}^T_{t_0,2}(z)\ \mathcal{V}^T_{t_0,2}(z)]^T)$, have the same Smith form, apart from some unit invariant polynomials, equal in number to $\omega|m_1 - m_2|$; (β) the orders of $\mathcal{M}_1$ and $\mathcal{M}_2$ coincide; (γ) the ω-stacked transfer matrices of $\mathcal{M}_1$ and $\mathcal{M}_2$ at any initial time coincide; (δ) $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same nonzero invariant zeros, nonzero input (output) decoupling zeros at all times and the same corresponding ordered sets of structural indices (apart from $\omega|m_1 - m_2|$ null structural indices), the same characteristic multipliers at all times, and the same ordered sets of structural indices of their nonzero characteristic multipliers (apart from $\omega|m_1 - m_2|$ null structural indices).

Remark 1 By Proposition 4, if a system $\mathcal{M}_2$ of the form (13) is system equivalent at time t_0 to a given model $\mathcal{M}_1$, then its output responses from time t_0, and all the features of $\mathcal{M}_2$ that are listed in items (β), (γ) and (δ) of Proposition 4, are specified by the original model $\mathcal{M}_1$; in addition, the state response of $\mathcal{M}_2$ from time t_0 is biuniquely related with the pseudostate response of $\mathcal{M}_1$ from time t_0. Hence, such a system $\mathcal{M}_2$ is controllable [resp., reconstructible] if and only if $\mathcal{M}_1$ has no nonzero input [resp., output] decoupling zeros, it is stabilizable [resp., detectable], if and only if $\mathcal{M}_1$ has no input [resp., output] decoupling zeros outside the open disk of unit radius, it is reachable [resp. observable] at time t_0 if and only if $\mathcal{M}_1$ has no input [resp., output] decoupling zeros at time t_0 [23]; moreover, the order, the ω-stacked transfer matrix at any time k_0, the asymptotic stability [24], the rate of convergence of the free motions, all the characteristic multipliers of system $\mathcal{M}_2$, and even the number and the dimensions of the Jordan blocks corresponding to each nonzero characteristic multiplier, in the Jordan form of the monodromy matrix of system $\mathcal{M}_2$ [23] at any time k_0, are determined by the properties of the original model $\mathcal{M}_1$; in addition, $S^M_{k_0,2}(z)$ has full row-rank for any $k_0 \in \mathbf{Z}$ and for any nonzero $z \in \mathbb{C}$ if and only if $S^M_{t_0,1}(z)$ has full row-rank (it is recalled that such a condition on the stacked system matrix $S^M_{k_0,2}(z)$ of the ω-periodic system $\mathcal{M}_2$ described by equations of the form (13), is necessary and sufficient, together with stabilizability and detectability, for the existence of a solution of the robust tracking and regulation problem when the ω-stacked forms of reference signals and disturbance functions have a time dependence of the form $z^h, |z| \geq 1$ [12]). □

In view of the above discussion, it is reasonable to look for an ω-periodic system of the form (13) that is system equivalent at time t_0 to the given ω-periodic model (1). This probelm was solved in [2] under the following assumption, which implies Assumption 1.

Assumption 2 *The polynomial matrix $\mathcal{T}_{t_0}(\Delta)$ is square and such that $\mathcal{T}_{t_0}(z)|_{z=0}$ is nonsingular.*

Proposition 5 *[2] If Assumption 2 holds for $t_0 = \bar{t}_0 \in \mathbf{Z}$, then it holds for any $t_0 \in \mathbf{Z}$.*

Lemma 2 *[2] For the ω-periodic model (1) and its corresponding ω-stacked form (4) at time t_0, under Assumption 2, there exists an ω-periodic system of the form (13) that is system equivalent at time t_0 to model (1), if and only if its ω-stacked transfer matrix $G^M_{t_0}(z)$ satisfies the following conditions:*
(i) $G^M_{t_0}(z)$ is a proper rational matrix;
(ii) if $G^M_{t_0}(z)$ is rewritten as $G^M_{t_0}(z) = F_{t_0}(z) + Q_{t_0}$, with $F_{t_0}(z)$ strictly proper and Q_{t_0} constant, and Q_{t_0} is decomposed into blocks of dimensions $q \times p$, then Q_{t_0} is lower block triangular.
If conditions (i) and (ii) hold for $t_0 = \bar{t}_0, \bar{t}_0 \in \mathbf{Z}$, then (i) and (ii) hold for all $t_0 \in \mathbf{Z}$.

Lemma 2 allows to solve the problem of finding a system of the form (13) that is system equivalent to model (1) at time t_0 under the assumption $\det(\mathcal{T}_{t_0}(0)) \neq 0$. In [3] the general case was considered within large system equivalence, and the found results are summarized by the following proposition, remark and lemma.

Proposition 6 *[3] Given two ω-periodic models $\mathcal{M}_1$ and $\mathcal{M}_2$ of the type (1), satisfying Assumption 1 and having inputs and outputs of the same dimensions p and q, respectively, pseudo-states of dimensions $m_i, i = 1, 2$, and ω-stacked system matrices at time t_0 $S^M_{t_0,i}(\Delta), i = 1, 2$, if they are largely system equivalent at time t_0, then for each input function $u(t), t \geq t_0$, the solutions of $\mathcal{M}_1$ and $\mathcal{M}_2$ for $t \geq t_0$ in the output are exactly the same, and their solutions for $t \geq t_0 + \omega - 1$ in the pseudostates are biuniquely related.*
In addition: (α) the ω-stacked transfer matrices of $\mathcal{M}_1$ and $\mathcal{M}_2$ at any initial time coincide; (β) $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same nonzero characteristic multipliers, nonzero input decoupling zeros, and nonzero output decoupling zeros at all times and the same corresponding ordered sets of structural indices (apart from $\omega|m_1 - m_2|$ null structural indices); (γ) $S^M_{t_0,1}(\Delta)$ has full row-rank if and only if $S^M_{t_0,2}(\Delta)$ has full row-rank; (δ) if $S^M_{t_0,i}(\Delta)$, $i = 1, 2$, have full row-rank, then $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same nonzero invariant zeros at all times and the same corresponding ordered sets of structural indices (apart from $\omega|m_1 - m_2|$ null structural indices).

Remark 2 By Proposition 6, if a system $\mathcal{M}_2$ of the form (13) is largely system equivalent at time t_0 to a given model $\mathcal{M}_1$, then its output responses from time t_0, and all the features of $\mathcal{M}_2$ that are listed in item (β), (γ) and (δ) of Proposition 6, are specified by the original model $\mathcal{M}_1$. Hence, such a system $\mathcal{M}_2$ is controllable [resp., reconstructible] if and only if $\mathcal{M}_1$ has no nonzero input [resp., output] decoupling zeros, it is stabilizable [resp., detectable], if and only if $\mathcal{M}_1$ has no input [resp., output] decoupling zeros outside the open disk of unit radius [23]; moreover, the ω-stacked transfer matrix at any time k_0, the asymptotic stability [24], all the nonzero characteristic multipliers of system $\mathcal{M}_2$, and even the number and the dimensions of the Jordan blocks corresponding to each nonzero characteristic multiplier, in the Jordan form of the monodromy matrix of system $\mathcal{M}_2$ [23], at any time k_0, are determined by the properties of the original model $\mathcal{M}_1$; in addition, by the results in [2], $S^M_{k_0,2}(z)$ has full row-rank for any $k_0 \in \mathbb{Z}$ and for any nonzero $z \in \mathbb{C}$ if and only if $S^M_{t_0,1}(z)$ has full row-rank. On the contrary, the orders of two models $\mathcal{M}_1$ and $\mathcal{M}_2$ that are largely system equivalent at time t_0 do not coincide, in general, since (12) yields

$$\det \mathcal{T}^L_{t_0}(\Delta) = \Delta^{p(\omega-1)} \det \mathcal{T}_{t_0}(\Delta). \qquad (15)$$

Lemma 3 *[3] For the ω-periodic model* (1), *under Assumption 1, there exists an ω-periodic system of the form* (13) *that is largely system equivalent at time t_0 to model* (1), *if and only if the ω-stacked transfer matrix $G^M_{t_0}(z)$ satisfies conditions (i) and (ii) of Lemma 2.*

4. Main results

As it was previously recalled, the results in [2, 3] give solution to the problem of finding, for a fixed initial time t_0, a system of the form (13) that is system equivalent (or largely system equivalent) at time t_0, to a given model of the form (1). In particular, Lemma 2, under the assumption $\det \mathcal{T}_{t_0}(0) \neq 0$, allows to find a system of the form (13) whose output responses for $t \geq t_0$ coincide with those of model (1), and whose state responses for $t \geq t_0$ are biuniquely related with the pseudostate responses of model (1) for $t \geq t_0$ – so that several features and properties of it coincide with those of model (1) (see Proposition 4 and Remark 1) –. As far as the mere input-output behaviour is concerned, however, it seems worth to ask whether the found system is also a state-space representation of the input-output relations defined through model (1), not only for the fixed initial time t_0, but for any other initial time k_0.

In order to give an answer to this question, some further notations will be introduced. For each initial time $t_0 \in \mathbb{Z}$, let T_{t_0} be defined as $T_{t_0} := \{t \in \mathbb{Z} : t \geq t_0\}$, let $U^*_{t_0}$, $\Xi^*_{t_0}$, and $Y^*_{t_0}$ be the sets of all the input, pseudo-state, and output functions, respectively, over T_{t_0}, taking values on $\mathbb{R}^p$, $\mathbb{R}^m$, and $\mathbb{R}^q$, respectively, i.e. $U^*_{t_0} := \{u(\cdot) : u(t) \in \mathbb{R}^p, \forall t \in T_{t_0}\}$, $\Xi^*_{t_0} := \{\xi(\cdot) : \xi(t) \in \mathbb{R}^m, \forall t \in T_{t_0}\}$, $Y^*_{t_0} := \{y(\cdot) : y(t) \in \mathbb{R}^q, \forall t \in T_{t_0}\}$. The members of $U^*_{t_0}$, $\Xi^*_{t_0}$, and $Y^*_{t_0}$ will be generically denoted by $u(\cdot)$, $\xi(\cdot)$, and $y(\cdot)$, respectively.

For each initial time $t_0 \in \mathbb{Z}$, a subset $\mathcal{R}^M_{t_0}$ of the set $U^*_{t_0} \times Y^*_{t_0}$ can be associated with model (1), namely the subset $\mathcal{R}^M_{t_0}$ of $U^*_{t_0} \times Y^*_{t_0}$ consisting of all the pairs $(u(\cdot), y(\cdot)) \in U^*_{t_0} \times Y^*_{t_0}$, such that $u(\cdot)$ and $y(\cdot)$ satisfy equations (1) for some $\xi(\cdot) \in \Xi^*_{t_0}$, that is, relation $\mathcal{R}^M_{t_0}$ is the subset of all the pairs $(\bar{u}(\cdot), \bar{y}(\cdot))$, $\bar{u}(\cdot) \in U^*_{t_0}$, $\bar{y}(\cdot) \in Y^*_{t_0}$, such that, if the process described by equations (1) is subject to the input function $u(t) = \bar{u}(t)$ for all $t \geq t_0$, then $\bar{y}(t)$ is one of the possible output responses $y(t)$ satisfying equations (1) for all $t \geq t_0$, for some $\xi(\cdot) \in \Xi^*_{t_0}$. With this definition of $\mathcal{R}^M_{t_0}$, the family of relations

$$\mathcal{R}^M := \{\mathcal{R}^M_{t_0},\ t_0 \in \mathbb{Z}\}, \qquad (16)$$

wholly characterizes the input-output behaviour of the process described by equations (1), since, for each initial time t_0, it contains all pairs $(u(\cdot), y(\cdot)) \in U^*_{t_0} \times Y^*_{t_0}$ that satisfy equations (1). From now on, with an abuse of terminology, $\mathcal{R}^M$ will be called *the input-output relation of model* (1). It is worth to stress that the input-output relation $\mathcal{R}^M$ of model (1) enjoys the property of the closure with respect to the restriction of the input and output functions, namely, that the following condition holds:

$$(u(\cdot), y(\cdot)) \in \mathcal{R}^M_{t_0} \;\Rightarrow\; (u(\cdot)|_{T_{t_1}}, y(\cdot)|_{T_{t_1}}) \in \mathcal{R}^M_{t_1}, \quad \forall t_0, t_1 \in \mathbb{Z}, t_1 > t_0, \qquad (17)$$

where $u(\cdot)|_{T_{t_1}}$ and $y(\cdot)|_{T_{t_1}}$ denote, respectively, the restrictions of $u(\cdot)$ and $y(\cdot)$ over T_{t_1}.

In a similar way, for each initial time $t_0 \in \mathbb{Z}$, a subset $\mathcal{R}^S_{t_0}$ of the set $U^*_{t_0} \times Y^*_{t_0}$ can be associated with the linear ω-periodic system (13), namely the subset $\mathcal{R}^S_{t_0}$ of $U^*_{t_0} \times Y^*_{t_0}$ consisting of all the pairs $(u(\cdot), y(\cdot)) \in U^*_{t_0} \times Y^*_{t_0}$, such that $u(\cdot)$ and $y(\cdot)$ satisfy equations (13) for some initial state $x(t_0)$ at the initial time t_0. The family of relations

$$\mathcal{R}^S := \{\mathcal{R}^S_{t_0},\ t_0 \in \mathbb{Z}\} \qquad (18)$$

wholly characterizes the input-output behaviour of system (13), and trivially enjoys a property similar to (17) rewritten with $\mathcal{R}^S_{t_0}$ and $\mathcal{R}^S_{t_1}$ instead of $\mathcal{R}^M_{t_0}$ and $\mathcal{R}^M_{t_1}$, respectively.

Thus, the main problem studied here consists of finding, if any, a linear ω-periodic discrete-time system of the form (13) such that

$$\mathcal{R}^S = \mathcal{R}^M, \qquad (19)$$

i.e., such that

$$\mathcal{R}^S_{t_0} = \mathcal{R}^M_{t_0}, \quad \forall t_0 \in \mathbb{Z}, \qquad (20)$$

that is, such that, for all the initial times, the set of all the input-output pairs of model (1) coincide with the set of all the input-output pairs of the found system.

The problem thus defined will be called Problem 1, and an ω-periodic system of the form (13) satisfying condition (20) will be called a *state-space representation of the input-output relation $\mathcal{R}^M$ of model* (1).

It is stressed that, by the periodicity of the coefficients of both model (1) and system (13), for any $t_0 \in \mathbb{Z}$, the following conditions hold:

$$\mathcal{R}^S_{t_0+h\omega} = \mathcal{R}^S_{t_0}, \quad \forall h \in \mathbb{Z}, \qquad (21a)$$

$$\mathcal{R}^M_{t_0+h\omega} = \mathcal{R}^M_{t_0}, \quad \forall h \in \mathbb{Z}, \qquad (21b)$$

so that (20) holds if and only if the following conditions are satisfied for some $\bar{t}_0 \in \mathbb{Z}$:

$$\mathcal{R}_\tau^S = \mathcal{R}_\tau^M, \quad \tau = \bar{t}_0, \bar{t}_0+1, \ldots, \bar{t}_0+\omega-1. \tag{22}$$

As already mentioned, it is of particular interest to consider the special case of a linear ω-periodic system of the form (13) that is system equivalent to model (1) at some time t_0, and to check whether such a sistem is also a state-space representation of the input-output relation $\mathcal{R}^M$ of model (1); such a problem will be called Problem 2. Also Problem 1 will be studied under the same Assumption 2 that was considered in Lemma 2. The following proposition and lemma will be useful.

Proposition 7 *[2] For the ω-periodic model* (1) *and its corresponding ω-stacked form* (4) *at time t_0, if Assumption 2 holds, its ω-stacked transfer matrix $G_{t_0}^M(z)$ satisfies conditions (i) and (ii) of Lemma 2, and its order $\overline{n}$ is less than or equal to $m\omega$, possibly after a preliminary operation of the type (a) has been carried out on* (4), *then $S_{t_0}^M(\Delta)$ is strictly system equivalent to the following matrix:*

$$\left[\begin{array}{cc|c} -I_{m\omega-\overline{n}} & 0 & 0 \\ 0 & E_{t_0} - \Delta I_{\overline{n}} & J_{t_0} \\ \hline 0 & L_{t_0} & P_{t_0} \end{array}\right] =: S_{t_0}^{Ma}(\Delta) \tag{23}$$

where $E_{t_0}, J_{t_0}, L_{t_0}$, and P_{t_0} are constant matrices, and E_{t_0} is nonsingular.

Lemma 4 *Under Assumption 1, if there exists a state-space representation of the input-output relation $\mathcal{R}^M$ of model* (1), *then its ω-stacked transfer matrix at any initial time t_0 coincides with the ω-stacked transfer matrix $G_{t_0}^M(z)$ of model* (1) *at the same initial time t_0.*

Proof. In view of the procedure for the computation of the solutions of equation (4a) given in [21], which was based on multiplying both sides of (4a) by a unimodular polynomial matrix $L(\Delta)$ such that $L(\Delta)\mathcal{T}_{t_0}(\Delta)$ is in the Hermite lower triangular form, and denoting by μ the maximum degree in Δ among all the entries of $\mathcal{T}_{t_0}(\Delta)$, $L(\Delta)\mathcal{T}_{t_0}(\Delta)$, and $\mathcal{V}_{t_0}(\Delta)$, and by ν the maximum degree in Δ among all the entries of $\mathcal{U}_{t_0}(\Delta)$, $L(\Delta)\mathcal{U}_{t_0}(\Delta)$, and $\mathcal{W}_{t_0}(\Delta)$, it is possible to check that (6) holds if all the initial conditions that make unique the solution of (4a) in $\xi_{t_0}(h)$ are chosen equal to zero and if the values of $u_{t_0}(h)$ are chosen to be zero for each $h = 0, 1, \ldots, \nu + m\mu$.

Let S be a state-space representation of $\mathcal{R}^M$ of the form (13). The ω-stacked form (4) at any initial time t_0 for system S can be written as follows:

$$\mathcal{R}_n(\Delta)x_{t_0}(h) = \mathcal{A}_{t_0}x_{t_0}(h) + \mathcal{B}_{t_0}u_{t_0}(h), \tag{24a}$$
$$y_{t_0}(h) = \mathcal{C}_{t_0}x_{t_0}(h) + \mathcal{D}_{t_0}u_{t_0}(h), \tag{24b}$$

where $\mathcal{A}_{t_0}$, $\mathcal{B}_{t_0}$, $\mathcal{C}_{t_0}$, $\mathcal{D}_{t_0}$ are the ω-stacked forms at time t_0 of matrices $A(t)$, $B(t)$, $C(t)$, $D(t)$, respectively.

The z-transform of the ω-stacked form at time t_0 of the output response of S, from the initial state $x(t_0) = x_0$, has the following expression:

$$y_{t_0}(z) = G_{t_0}^S(z)u_{t_0}(z) + \Psi_{t_0}(z)x_0, \tag{25}$$

where $G_{t_0}^S(z) := \mathcal{C}_{t_0}(\mathcal{R}_n(z) - \mathcal{A}_{t_0})^{-1}\mathcal{B}_{t_0} + \mathcal{D}_{t_0}$ and $\Psi_{t_0}(z) := z\mathcal{C}_{t_0}(\mathcal{R}_n(z) - \mathcal{A}_{t_0})^{-1}\Gamma$, $\Gamma := [0\ 0\ \ldots\ 0\ I_n]^T$.

Now, rewrite $G_{t_0}^S(z)$ as $G_{t_0}^S(z) = G_{t_0}^M(z) + \tilde{G}_{t_0}(z)$, then equation (25) becomes:

$$y_{t_0}(z) = G_{t_0}^M(z)u_{t_0}(z) + \tilde{G}_{t_0}(z)u_{t_0}(z) + \Psi_{t_0}(z)x_0. \tag{26}$$

Assume, by contradiction, that one of the entries of matrix $\tilde{G}_{t_0}(z)$ is nonzero, say the entry $\tilde{g}_{i,j,t_0}(z)$ in the i-th row and j-th column, and denote by $\ell_0, \ell_0 \in \mathbf{Z}$, the multiplicity of the null zero of $\tilde{g}_{i,j,t_0}(z)$ (where $\ell_0 < 0$ means that $\tilde{g}_{i,j,t_0}(z)$ has a null pole of multiplicity ℓ_0), and by $\ell_p \geq 0, \ell_p \in \mathbf{Z}$, the maximum multiplicity of the null pole in the entries of the i-th row $[\Psi_{t_0}(z)]_i$ of matrix $\Psi_{t_0}(z)$. Choose an input function $\bar{u}(\cdot) \in U_{t_0}^*$ such that only the j-th entry of the z-transform $\bar{u}_{t_0}(z)$ of its ω-stacked form at time t_0 $\bar{u}_{t_0}(h)$ is nonzero, and, in particular, equal to $1/z^\ell$, with $\ell \in \mathbf{Z}, \ell > \max(\nu + m\mu, \ell_p + \ell_0)$. In view of (20), there exists an initial state x_0 of system S such that the ω-stacked form at time t_0 of the output response of system S from the initial state x_0 at time t_0 under the input function $\bar{u}(\cdot)$ coincides with the ω-stacked form at time t_0 of the output response of model (1) to the input function $\bar{u}(\cdot)$, corresponding to zero initial conditions of $\xi_{t_0}(h)$.

In view of (6) and (26), this implies that the following equality must be satisfied for some x_0:

$$0 = [\Psi_{t_0}(z)]_i x_0 + \tilde{g}_{i,j,t_0}(z)\frac{1}{z^\ell}. \tag{27}$$

Now, rewrite $\tilde{g}_{i,j,t_0}(z)$ as $\tilde{g}_{i,j,t_0}(z) = \hat{g}_{i,j,t_0}(z)z^{\ell_0}$. By partial fraction expansion of the rational right-hand side of (27), the hypothesis $\ell - \ell_0 > \ell_p$ yields $\hat{g}_{i,j,t_0}(0) = 0$, that is a contradiction. □

The following theorem gives the solution of Probelm 1, under the same Assumption 2 that was considered in Lemma 2.

Theorem 1 *Under Assumption 2, Problem 1 admits a solution if and only if conditions (i) and (ii) of Lemma 2 hold.*

Proof. *(Necessity)* If Problem 1 admits a solution, by Lemma 4 any state-space representation of $\mathcal{R}^M$ has the same ω-stacked transfer matrix as model (1), at any initial time t_0. Since a state-space representation of $\mathcal{R}^M$ has the form (13), it is well known that its ω-stacked transfer matrix satisfies conditions (i) and (ii) of Lemma 2 (see, e.g., [23]).

(Sufficiency) It is recalled that, by Proposition 5 and Lemma 2, if Assumption 2 and conditions (i) and (ii) of Lemma 2 hold for a $t_0 = \bar{t}_0 \in \mathbf{Z}$, then they hold for all $t_0 \in \mathbf{Z}$. Now, for an arbitrary $t_0 \in \mathbf{Z}$, by Lemma 2 it is possible to find a linear system S of the form (13) that is system equivalent to model (1) at time t_0. Hence, taking into account Proposition 4, the sets of input-output pairs $\mathcal{R}_{t_0}^S$ and $\mathcal{R}_{t_0}^M$ satisfy the following equality:

$$\mathcal{R}_{t_0}^S = \mathcal{R}_{t_0}^M. \tag{28}$$

On the basis of (28), condition (22) (and therefore condition (20)), can be proved by showing that:

$$\mathcal{R}_{t_0+i}^S = \mathcal{R}_{t_0+i}^M, \quad i = 1, 2, \ldots, \omega - 1. \tag{29}$$

Equality (29), in turn, is implied by (28) and by the following ones:

$$\mathcal{R}^S_{t_0}|_{T_{t_1}} = \mathcal{R}^S_{t_1}, \quad t_1 = t_0+1,\ldots,t_0+\omega-1, \quad (30a)$$

$$\mathcal{R}^M_{t_0}|_{T_{t_1}} = \mathcal{R}^M_{t_1}, \quad t_1 = t_0+1,\ldots,t_0+\omega-1, \quad (30b)$$

where $\mathcal{R}^S_{t_0}|_{T_{t_1}}$ and $\mathcal{R}^M_{t_0}|_{T_{t_1}}$ denote the set of all the input-output pairs obtained by restricting to the interval T_{t_1} the input-output pairs in the set $\mathcal{R}^S_{t_0}$, and $\mathcal{R}^M_{t_0}$, respectively. In view of the closure property (17), and of the similar property of system S that consists of (17) rewritten with $\mathcal{R}^S_{t_0}$ and $\mathcal{R}^S_{t_1}$ instead of $\mathcal{R}^M_{t_0}$ and $\mathcal{R}^M_{t_1}$, respectively, conditions (30) are implied by the following ones:

$$\mathcal{R}^S_{t_0}|_{T_{t_1}} \supset \mathcal{R}^S_{t_1}, \quad t_1 = t_0+1,\ldots,t_0+\omega-1, \quad (31a)$$

$$\mathcal{R}^M_{t_0}|_{T_{t_1}} \supset \mathcal{R}^M_{t_1}, \quad t_1 = t_0+1,\ldots,t_0+\omega-1. \quad (31b)$$

As for condition (31a), for a given initial time $t_1 \in \mathbb{Z}, t_1 > t_0$, and a given pair $(u_1(\cdot), y_1(\cdot)) \in \mathcal{R}^S_{t_1}$, there exists an initial state $x(t_1) = x_1$ of system S such that the output response under the input signal $u(t) = u_1(t)$ for all $t \geq t_1$, coincides with the function $y_1(t)$ for all $t \geq t_1$. In addition, system S is time reversible by Assumption 2 [2]; hence, there exists an initial state $x(t_0) = x_0$ such that the free response of S from x_0 at the initial time t_0 coincides with x_1 at time t_1. This implies that, for any pair $(u_1(\cdot), y_1(\cdot)) \in \mathcal{R}^S_{t_1}$, there exists a pair $(u_0(\cdot), y_0(\cdot)) \in \mathcal{R}^S_{t_0}$ such that $(u_0(\cdot)|_{T_{t_1}}, y_0(\cdot)|_{T_{t_1}}) = (u_1(\cdot), y_1(\cdot))$, i.e., condition (31a) holds.

Condition (31b) can be proved similarly. By virtue of Proposition 7, rewritten with t_1 instead of t_0, and of a proposition similar to Proposition 4 that can be stated for strict system equivalence, for a given initial time $t_1 \in \mathbb{Z}, t_0 < t_1 < t_0+\omega$, the output responses $y(t)$ of model (1) from the initial time t_1 can be obtained in ω-stacked form from a linear time-invariant system S_a characterized by a quadruplet of constant matrices $(E_{t_1}, J_{t_1}, L_{t_1}, P_{t_1})$, with E_{t_1} being nonsingular (see (23)). Therefore, for the given time t_1 and for any pair $(u_1(\cdot), y_1(\cdot)) \in \mathcal{R}^M_{t_1}$, there exists an initial state x^a_1 of system S_a such that the ω-stacked form of $y_1(t)$ can be obtained from the initial state x^a_1 of S_a under the ω-stacked form of $u_1(t)$ as input signal. In view of the nonsingularity of E_{t_1} and of the periodicity of model (1) (which, by (21b), implies that also $\mathcal{R}^M_{t_1-\omega}$ can be obtained in ω-stacked form through S_a, as well as $\mathcal{R}^M_{t_1}$), there exists an initial state $x^a_{-\omega} := E^{-1}_{t_1} x^a_1$ of system S_a such that, defining as follows the input function $u_{-\omega}(\cdot) \in U^*_{t_1-\omega}$:

$$u_{-\omega}(t) = 0, \; t = t_1-\omega, t_1-\omega+1, \ldots, t_1-1,$$

$$u_{-\omega}(t) = u_1(t), \; \forall t \in T_{t_1},$$

and denoting by $y_{-\omega}(\cdot) \in Y^*_{t_1-\omega}$ the output response of model (1) that is obtained in ω-stacked form through system S_a as its output response from $x^a_{-\omega}$ to the ω-stacked form of $u_{-\omega}(\cdot)$, the following relations hold:

$$(u_{-\omega}(\cdot), y_{-\omega}(\cdot)) \in \mathcal{R}^M_{t_1-\omega} \quad (32a)$$

$$u_{-\omega}(\cdot)|_{T_{t_1}} = u_1(\cdot), \qquad y_{-\omega}(\cdot)|_{T_{t_1}} = y_1(\cdot). \quad (32b)$$

Since $t_0 > t_1 - \omega$, relation (32a) yields:

$$(u_{-\omega}(\cdot)|_{T_{t_0}}, y_{-\omega}(\cdot)|_{T_{t_0}}) \in \mathcal{R}^M_{t_0}. \quad (33)$$

Since $t_1 > t_0$, (32b) and (33) imply $(u_1(\cdot), y_1(\cdot)) \in \mathcal{R}^M_{t_0}|_{T_{t_1}}$. □

Corollary 1 *If model* (1) *satisfies Assumption 2, then an ω-periodic system of the form* (13) *that is system equivalent to model* (1) *at some initial time t_0, is a state-space representation of the input-output relation $\mathcal{R}^M$ of model* (1).

Corollary 1 gives the solution to Problem 2, while Theorem 1 gives the solution to Problem 1 under Assumption 2, i.e., under the same assumption under which the conditions for the existence of a system of the form (13) that is system equivalent to model (1) were given in [2].

At the best author's knowledge, Problem 1 in the general case (i.e., without the assumption that $\det \mathcal{T}_{t_0}(0) \neq 0$) is still open, as well as the problem similar to Problem 2 that can be defined for large system equivalence instead of system equivalence. However, the results in [3] allow to study the causality of the input-output relation $\mathcal{R}^M$ of model (1). This is the object of the following discussion.

It was shown in [21] that, for each initial time $t_0 \in \mathbb{Z}$, there exists a set X_{t_0}, with each element of it being the "initial condition" of the function $\xi(\cdot) \in \Xi^*_{t_0}$ (consisting of the $\nu_j \geq 0$ values of the j-th compenent of $\xi_{t_0}(h)$ for $h = 0, 1, \ldots, \nu_j - 1$, for each $j = 1, 2, \ldots, m\omega$), such that, for each $u(\cdot) \in U^*_{t_0}$, equation (1a) admits a unique solution $\xi(\cdot) \in \Xi^*_{t_0}$ (hence also equation (1b) admits a unique solution $y(\cdot) \in Y^*_{t_0}$) for each element of X_{t_0}, i.e., for each "initial condition". Therefore, according to the following definition, the input-output relation $\mathcal{R}^M$ of model (1) admits at least one parametrization.

Definition 1 If, for each initial time $t_0 \in \mathbb{Z}$, there exist a set X_{t_0} and a family of functions $\mathbf{F}_{t_0} = \{f_{t_0,\alpha} : T_{t_0} \times U^*_{t_0} \to \mathbb{R}^q, \alpha \in X_{t_0}\}$ such that

(i) for each $(u(\cdot), y(\cdot)) \in \mathcal{R}^M_{t_0}$, there exists an element $\alpha \in X_{t_0}$ such that

$$y(t) = f_{t_0,\alpha}(t, u(\cdot)), \quad \forall t \in T_{t_0}; \quad (34)$$

(ii) for each $\alpha \in X_{t_0}$, the following relation holds:

$$(u(\cdot), f_{t_0,\alpha}(\cdot, u(\cdot))) \in \mathcal{R}^M_{t_0}, \quad \forall u(\cdot) \in U^*_{t_0}, \quad (35)$$

then the pair $(X_{t_0}, \mathbf{F}_{t_0})$ is called a *parametrization of* $\mathcal{R}^M_{t_0}$, and the family of such parametrizations of $\mathcal{R}^M_{t_0}$, $t_0 \in \mathbb{Z}$, is called a *parametrization of the input-output relation $\mathcal{R}^M$ of model* (1).

On this basis, the following definition of causality of the input-output relation $\mathcal{R}^M$ of model (1) is standard (see [25] for a similar one).

Definition 2 If, for each initial time $t_0 \in \mathbb{Z}$, there exists a parametrization $(X_{t_0}, \{f_{t_0,\alpha} : T_{t_0} \times U^*_{t_0} \to \mathbb{R}^q, \alpha \in X_{t_0}\})$ of $\mathcal{R}^M_{t_0}$ that satisfies the following property:

$$u_1(\cdot)|_{[t_0,t]} = u_2(\cdot)|_{[t_0,t]}, \; u_1(\cdot), u_2(\cdot) \in U^*_{t_0}, t \in T_{t_0},$$

$$\Rightarrow f_{t_0,\alpha}(t, u_1(\cdot)) = f_{t_0,\alpha}(t, u_2(\cdot)), \; \forall \alpha \in X_{t_0}, \quad (36)$$

then the input-output relation $\mathcal{R}^M$ of model (1) is said to be *causal*. Model (1) is said to be *causal* if its input-output relation $\mathcal{R}^M$ is causal.

The causality of linear ω-periodic input-ouput maps was studied in [20], where it was shown that conditions (i) and (ii) of Lemma 2 are necessary and sufficient for the causality of such a map. In this paper, the more general case of the input-output relation $\mathcal{R}^M$ is considered. In [2, 21] it was shown that conditions (i) and (ii) of Lemma 2 are necessary for the causality of model (1), on the basis of a stronger notion of causality. By the following theorem, such conditions are sufficient for model (1) to be causal, for the above given definition of causality.

Theorem 2 *If conditions (i) and (ii) of Lemma 2 are satisfied, then the linear ω-periodic model* (1) *is causal.*

Proof. If conditions (i) and (ii) of Lemma 2 hold for some $t_0 = \bar{t}_0$, by Lemma 2 they hold for all $t_0 \in \mathbb{Z}$. By Lemma 3, for each $t_0 \in \mathbb{Z}$ there exists an ω-periodic system S_{t_0} of the form (13) that is large system equivalent at time t_0 to model (1). By Proposition 6, for each $t_0 \in \mathbb{Z}$, S_{t_0} trivially provides a parametrization of $\mathcal{R}^M_{t_0}$. In addition, for each $t_0 \in \mathbb{Z}$ such a parametrization satisfies property (36), thus proving the causality of $\mathcal{R}^M$, hence the causality of model (1). □

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