A Polynomial Approach to the Solution of the Invariant Polynomial Assignment Problem for Periodic Systems *

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Abstract

This paper considers the problem of assigning the closed loop invariant polynomials of a unity feedback control system, where the plant is a linear, discrete-time, periodic system. By a matrix algebraic approach, the solution is provided in terms of a linear, discrete-time periodic controller whose associate transfer matrix originates from the solution of a suitable diophantine matrix equation. The particular form chosen for this solution guarantees the causality of the periodic realization of the associate transfer matrix.

1 Introduction

Various classes of processes, such as periodically time-varying networks and filters (for example switched-capacitors circuits and multirate digital filters), chemical processes, multirate sampled-data systems, can be modeled through a linear periodic system (see, e.g., [1], [2] and references therein). Moreover, the study of linear periodic systems can be helpful even for the stabilization and control of time-invariant linear systems through a periodic controller, that has been recently investigated [3], [4], [5], [6], [7], [8], and for the stabilization and control of a class of bilinear systems [9], [10], [11].

In the discrete-time case, a control theory is developing with the help of algebraic and geometric techniques and contributions on several control problem have been given, including eigenvalues assignment, state and output dead-beat control, disturbance decoupling, model matching, adaptive control, robust control and optimal H_2/H_∞ control (see, e.g., [2], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]).

The aim of this paper is to analyze the invariant polynomial assignment problem for the class of discrete-time linear periodic systems. This problem generalizes the characteristic polynomial assignment, which, for the same class of systems, was solved by a geometric approach in [17], [21], [23]. For time-invariant plants, the invariant polynomial assignment was considered in [4], [7], [25], [26].

The paper is organized in the following way. In Section 2 preliminary definitions and results are given. The problem considered in this paper is formally stated in Subsection 3.1, and conditions for its solvability are constructively established in Subsection 3.2.

2 Preliminary results

Consider the ω-periodic discrete-time system Σ described by

$$x(k+1) = A(k)x(k) + B(k)u(k) \quad (2.1)$$

$$y(k) = C(k)x(t) + D(k)u(t) \quad (2.2)$$

where $k \in \mathbb{Z}$, $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^p$ is the input, $y(k) \in \mathbb{R}^q$ is the output and $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are periodic matrices of period ω (briefly, ω-periodic). Denote also by $\Phi(k, k_0)$, $k \geq k_0$, the transition matrix associated with $A(\cdot)$.

It is well-known that, for any initial time $k_0 \in \mathbb{Z}$, the output response of system Σ for $k \geq k_0$, to given

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initial state $x(k_0)$ and control function $u(\cdot)$, can be obtained through the time-invariant *associated system of* Σ *at time* k_0, denoted by $\Sigma^a(k_0)$ [27]. $\Sigma^a(k)$ is represented by

$$\begin{aligned} x_k(h+1) &= E_k x_k(h) + J_k u_k(h) \quad (2.3) \\ y_k(h) &= L_k x_k(h) + M_k u_k(h) \quad (2.4) \end{aligned}$$

where $E_k := \Phi(\omega+k,k)$, $J_k := [(J_k)_1 \cdots (J_k)_\omega]$, $(J_k)_i := \Phi(\omega+k,i+k)B(i-1+k)$, $i = 1,\cdots,\omega$, $L_k := [(L_k)'_1 \cdots (L_k)'_\omega]'$, $(L_k)_i := C(i-1+k)\Phi(i-1+k,k)$, $i = 1,\cdots,\omega$, $M_k := [(M_k)_{ij} \in \mathbb{R}^{q\times p}$, $i,j = 1,\cdots,\omega]$, with $(M_k^L)_{ij} := C(i-1+k)\Phi(i-1+k,j+h)B(j-1+k)$, if $i > j$, $(M_k^L)_{ij} := D(i-1+h)$, if $i = j$ and $(M_k^L)_{ij} := 0$, if $i < j$.

In fact, if $x_k(0) = x(k)$ and $u_k(h) := [u'(h\omega + k) \;\; u'(h\omega+k+1) \cdots u'(h\omega+k+\omega-1)]'$ for all $h \in \mathbb{Z}^+$, then $x_k(h) = x(k+h\omega)$ and $y_k(h) = [y'(h\omega+k) \;\; y'(h\omega+k+1) \cdots y'(h\omega+k+\omega-1)]'$ for all $h \in \mathbb{Z}^+$. The notion of associated system at time k allows one to analyze structural and stability properties and pole-zero-structures of periodic systems [1], [28], [29], [30], [31]. For example, the subspace of reachable (unobservable) states of system Σ at time k is readily seen to coincide with that of system $\Sigma^a(k)$ if it is expressed in terms of matrices E_k, J_k, L_k and M_k [29]. Obviously, $\Sigma^a(k+\omega) = \Sigma^a(k)$ for all integer k.

The notions of invariant zero, transmission zero and pole of the ω-periodic system Σ at time k are defined with reference to the following $\omega q \times \omega p$ matrix

$$W_k(d) = L_k d(I_n - dE_k)^{-1}J_k + M_k, \qquad (2.5)$$

where I_n denotes the identity matrix of dimension n and $d := z^{-1}$ is the backward shift operator. The rational matrix $W_k(d)$ is the transfer matrix of the associated system of Σ at time k and is called the *associated transfer matrix of* Σ *at time* k. A complete analysis of pole-zero structure of system Σ is reported in [29] and [32] making use of the associated transfer matrix characterized with the forward shift operator z. The following result, that follows from Lemma 2.1 in [29], shows the dependence of $W_k(d)$ with respect to the initial time k.

Lemma 2.1 *For any integer k it holds that:*

$$W_{k+1}(d) = \begin{bmatrix} 0 & I_{q(\omega-1)} \\ d^{-1}I_q & 0 \end{bmatrix} W_k(d) \begin{bmatrix} 0 & dI_p \\ I_{p(\omega-1)} & 0 \end{bmatrix}. \qquad (2.6)$$

As a consequence of this result the rank r of $W_k(d)$ is independent of time k (see, e.g., [29] for a similar result with the forward shift operator z).

The transfer matrix $W_k(d)$ can be factored as

$$W_k(d) = A_k^{-1}(d)\,B_k(d) = \bar{B}_k(d)\,\bar{A}_k^{-1}(d), \qquad (2.7)$$

where $A_k(d)$ and $B_k(d)$ are relatively left prime *(rlp)* polynomial matrices and $\bar{A}_k(d)$ and $\bar{B}_k(d)$ are relatively right prime *(rrp)* polynomial matrices. Moreover, denoting with $T_k(d)$ the Smith-McMillan form of $W_k(d)$ the following relation holds

$$W_k(d) = U_k^{W,L}(d)\,T_k(d)\,U_k^{W,R}(d), \qquad (2.8)$$

with

$$T_k(d) = \begin{bmatrix} \mathrm{diag}\{\epsilon_1^k(d)/\psi_1^k(d),\cdots,\epsilon_r^k(d)/\psi_r^k(d)\} & 0_{r,\omega p-r} \\ 0_{\omega q-r,r} & 0_{\omega q-r,\omega p-r} \end{bmatrix},$$

where $\epsilon_i^k(d)$ and $\psi_i^k(d)$ $(i = 1,\ldots,r)$ are coprime polynomials such that $\epsilon_i^k(d)$ $(\psi_{i+1}^k(d))$ divides $\epsilon_{i+1}^k(d)$ $(\psi_i^k(d))$, and $U_k^{W,L}(d)$ and $U_k^{W,R}(d)$ are unimodular matrices. The $\epsilon_i^k(d)$'s and $\psi_i^k(d)$'s $(i = 1,\ldots,r)$ are uniquely determined by $W_k(d)$ up to arbitrary real scalars. Analogously to the time-invariant case [25], the polynomials $\psi_i^k(d)$ $(i = 1,\ldots,r)$ are called the *invariant polynomials of* Σ *at time* k. As shown in [29], [32], under the hypothesis of reachability and observability of Σ at all times, the product of these polynomials characterizes the stability properties of Σ.

By (2.8), the *rlp* polynomial matrices $A_k(d)$ and $B_k(d)$) and the *rrp* polynomial matrices $\bar{A}_k(d)$ and $\bar{B}_k(d)$ satisfying (2.7) are given by

$$A_k(d) = \left[\mathrm{diag}\{\psi_1^k(d),\cdots,\psi_r^k(d)\} \;\; I_{q-r}\right](U_k^{W,L})^{-1}, \qquad (2.9)$$

$$B_k(d) = \left[\mathrm{diag}\{\epsilon_1^k(d),\cdots,\epsilon_r^k(d)\} \;\; 0_{q-r,p-r}\right]U_k^{W,R}, \qquad (2.10)$$

$$\bar{A}_k(d) = (U_k^{W,R})^{-1}\left[\mathrm{diag}\{\psi_1^k(d),\cdots,\psi_r^k(d)\} \;\; I_{q-r}\right], \qquad (2.11)$$

$$\bar{B}_k(d) = U_k^{W,L}\left[\mathrm{diag}\{\epsilon_1^k(d),\cdots,\epsilon_r^k(d)\} \;\; 0_{q-r,p-r}\right]. \qquad (2.12)$$

The above definition of invariant polynomials of Σ at time k and equations (2.7), (2.8), (2.9), (2.10), (2.11) and (2.12) yield the following result. (Note that, two polynomials are called associate if their ratio is a scalar [25].)

Lemma 2.2 *For any integer k, the invariant polynomials of* Σ *at time k are associate of the invariant polynomials of the Smith forms of $A_k(d)$ and $\bar{A}_k(d)$.*

Denote by $\chi(p,q,\omega)$ the class of $\omega q \times \omega p$ rational matrices

$$W(d)=\begin{bmatrix} W_{11}(d) & W_{12}(d) & \cdots & W_{1\omega}(d) \\ W_{21}(d) & W_{22}(d) & \cdots & W_{2\omega}(d) \\ \vdots & \vdots & \ddots & \vdots \\ W_{\omega 1}(d) & W_{\omega 2}(d) & \cdots & W_{\omega\omega}(d) \end{bmatrix},$$
$$W_{ij}(d) \in \mathbb{C}^{q\times p}\, i,j=1,\cdots,\omega, \quad (2.13)$$

with $W_{ij}(0)=0$, $i<j$, $i,j=1,\ldots,\omega$. The class $\chi(q,p,\omega)$ characterizes the transfer matrices of ω-periodic systems. In fact, the causality of ω-periodic system Σ implies that the associated transfer matrix of Σ at time k belongs to the class $\chi(p,q,\omega)$ for all $k \in \mathbb{Z}$ [33]. Then, the causality of Σ implies that the roots of invariant polynomials of Σ at time k are different to zero for all integer k. This in turn implies that matrices $A_k(0)$ and $\bar{A}_k(0)$ are nonsingular. Foregoing considerations and Lemma 2.1 allow us to prove the following result.

Lemma 2.3 *The invariant polynomials of Σ at time k are independent of k.*

Moreover, $\chi(p,q,\omega)$ characterizes also the class of rational matrices that can be realized by an ω-periodic system of the form (2.1), (2.2). The solution of the minimal realization problem for the periodic case is described by a system reachable and observable at any time whose matrices have generally time-varying dimensions. In general, the subspaces of reachable states and/or observable states may have time-varying dimensions. Therefore, it is natural, in order to consistently solve the problem of minimal realization, to allow for state-space description having time-varying dimensions. The possibility of computing a "quasi" minimal (reachable and observable at lest in one time) uniform (fixed-dimension) realization is also available. Efficient algorithms for the computation of minimal or quasi minimal realization of a given transfer matrix are introduced in [33] and [34].

Moreover, the associated system at a given time k of a composite system obtained connecting ω-periodic subsystems coincides with the same connection of the associated systems at time k of the composing subsystems [32].

3 Main result

3.1 Problem statement

Consider an ω-periodic controller Σ_G for system Σ acting in an unity feedback error actuated servo system and described by

$$x_G(k+1) = A_G(k)x_G(k)+B_G(k)e(k), \quad (3.1)$$
$$u(k) = C_G(k)x_G(t)+D_G(k)e(t), \quad (3.2)$$

where $e(k):=r(k)-y(k)$, $r(\cdot)$ being the external reference to be tracked, and $x_G(k) \in \mathbb{R}^{n_G}$ is the state.

The problem considered in this paper is formally stated as follows.

Problem 3.1 Given an ω-periodic system Σ reachable and observable at all times, and m causal polynomials $s_1(d)$, $s_2(d)$, $\ldots$, $s_m(d)$ such that $s_{i+1}(d)$ divides $s_i(d)$, find an ω-periodic controller Σ_G described by (3.1), (3.2), such that the closed loop system be minimally realized and its invariant polynomials be associated of $s_i(d)$, $i=1,2,\ldots,m$.

Note that, by Lemma 2.3, Problem 3.1 has been stated independently of the time instant k.

The $\omega p \times \omega q$ associated transfer matrix of Σ_G at time k is expressed by

$$G_k(d) = L_k^G d(I_{n_G} - dE_k^G)^{-1}J_k^G + M_k^G, \quad (3.3)$$

where matrices $L_k^G \in \mathbb{R}^{\omega p\times n_G}$, $E_k^G \in \mathbb{R}^{n_G\times n_G}$, $J_k^G \in \mathbb{R}^{n_G\times \omega q}$ and $M_k^G \in \mathbb{R}^{\omega p\times\omega q}$ are defined as matrices L_k, E_k, J_k and M_k with matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ substituted respectively by matrices $A_G(\cdot)$, $B_G(\cdot)$, $C_G(\cdot)$ and $D_G(\cdot)$.

The causality of system Σ_G implies that $G_k(d)$ belongs to the class $\chi(p,q,\omega)$.

Let $G_k(d)$be factored as

$$G_k(d) = P_k^{-1}(d)\,Q_k(d) \quad (3.4)$$

where $P_k(d)$ and $Q_k(d))$ are *rlp* polynomial matrices.

3.2 Problem solution

The solvability condition of the problem considered is stated in the following theorem.

Theorem 3.1 *Problem 3.1 has a solution if and only if $m \le min(\omega q,\omega p)$.*

Proof (Necessity) Let $W_k^C(d)$ be the $\omega q\times\omega q$ associated transfer matrix at time k of the ω-periodic closed loop system Σ_C described by (2.1), (2.2), (3.1), (3.2), and let it be factored as

$$W_k^C(d) = C_k^{-1}(d)\,D_k(d) = \bar{D}_k(d)\,\bar{C}_k^{-1}(d). \quad (3.5)$$

where $C_k(d)$ and $D_k(d)$ are *rlp* polynomial matrices and $\bar{D}_k(d)$ and $\bar{C}_k(d)$ are *rrp* polynomial matrices.

By Lemma 2.2 applied to the closed loop system, the invariant polynomials of Σ_C at time k are associated to the invariant polynomials of the Smith forms of $C_k(d)$ and $\bar{C}_k(d)$. Moreover, taking into account that the closed loop system Σ_C is required to be free of hidden modes, it follows that the invariant polynomials of Σ_C are also associate to the invariant polynomials of the $\omega p \times \omega p$ polynomial matrix $X_k(d)$ such that [25]

$$P_k(d)\bar{A}_k(d) + Q_k(d)\bar{B}_k(d) = X_k(d). \qquad (3.6)$$

As the invariant polynomials of the $\omega p \times \omega p$ polynomial matrix $X_k(d)$ are associate to the invariant polynomials of the $\omega q \times \omega q$ polynomial matrices $C_k(d)$ and $\bar{C}_k(d)$, it follows that the number m of invariant polynomials that can be assigned can not be larger than $min(\omega p, \omega q)$.

(Sufficiency) By the primeness of the pair $(\bar{B}_k(d), \bar{A}_k(d))$, equation (3.6), can be solved with respect to $P_k(d)$, $Q_k(d)$, for any choice of $X_k(d)$. The matrix $Q_k(d)$ will be sought of the form $Q_k(d) = dQ_k^a$ because, as it will be shown later, this guarantees the causality of the ω-periodic controller Σ_G . With this choice of $Q_k(d)$, equation (3.6), can be rewritten as

$$P_k(d)\bar{A}_k(d) + dQ_k^a(d)\bar{B}_k(d) = X_k(d). \qquad (3.7)$$

By the causality of Σ, one has that $det\bar{A}_k(0) \neq 0$, so that right primeness of $\bar{A}_k(d)$ and $\bar{B}_k(d)$ implies right primeness of $\bar{A}_k(d)$ and $d\bar{B}_k(d)$; this in turn implies that also equation (3.7) can be solved with respect to $P_k(d)$ and $Q_k^a(d)$ for each $X_k(d)$. The general solution of (3.7) is of the form

$$[\,P_k(d) \quad Q_k^a(d)\,] = [\,X_k(d) \quad T(d)\,]\begin{bmatrix} L_2(d) & R_2(d) \\ M_2(d) & S_2(d) \end{bmatrix}, \qquad (3.8)$$

where $T(d)$ is an arbitrary polynomial matrix and where the matrix

$$U(d) := \begin{bmatrix} L_2(d) & R_2(d) \\ M_2(d) & S_2(d) \end{bmatrix}$$

is a unimodular matrix whose elements are polynomial matrices such that

$$\begin{aligned} L_2(d)\bar{A}_k(d) + M_2(d)d\bar{B}_k(d) &= I_p, \\ R_2(d)\bar{A}_k(d) + S_2(d)d\bar{B}_k(d) &= 0. \end{aligned}$$

Unimodularity of $U(d)$ implies that the matrix $[P_k(d) \quad Q_k^a(d)]$ is full row rank for all $d \in \mathbb{C}$, if and only if the same property holds for $[X_k(d) \quad T(d)]$. As $T(d)$ is arbitrary, this last requirement can always be satisfied, so that $P_k(d)$ and $Q_k^a(d)$ solution of (3.7) are *rlp* polynomial matrices. Nonsingularity of $X_k(0)$ and $\bar{A}_k(0)$ implies nonsingularity of $P_k(0)$, so that also $P_k(d)$ and $Q_k(d) = dQ_k^a(d)$, solutions of (3.6), are *rlp* polynomial matrices.

The transfer matrix $G_k(k) = P_k(d)^{-1}Q_k(d) = P_k(d)^{-1}Q_k^a(d)d$ belongs to the class $\chi(p,q,\omega)$, so that it can be realized in terms of ω-periodic state-space realization of the kind (3.1), (3.2). This last step can be accomplished using the minimal realization procedures given in [33]. $\triangle$

4 Concluding remarks

The invariant polynomial assignment problem has been introduced and solved for linear periodic discrete-time systems. A matrix algebraic approach has been considered and the solution has been provided in terms of a linear periodic discrete-time controller whose associate transfer matrix originates from the solution of a suitable diophantine matrix equation. The synthesis procedure of the controller solving the problem is given in the sufficiency proof of the main theorem.

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