# Optimal Vibration Control of Flexible Systems

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## Abstract

A control scheme for active vibration control of a flexible beam is proposed. The controller consists of a Linear Quadratic Regulator and of a spline based state reconstructor. The weighting matrices of the LQR are selected so as to modify only a limited number of system's poles, specifically the ones acting in the frequency range of the disturbances, so limiting the spillover phenomenon. On the other hand the spline reconstructor gives an "instantaneous" reconstruction of the state variables and spatially filters the beam deformation measurements without introducing phase-lag, and this contributes to increase the robustness of the control scheme.
Computer simulations show that the proposed scheme allows to reduce the vibration of a single-span beam subject to a persistent, multifrequency disturbance, reduces the interaction between controlled and uncontrolled dynamics hence increases the robustness against spillover phenomenon.

## 1 Introduction

Flexible structure are characterized by low damping factor, hence an active control scheme is necessary to increase the damping factor. The design of these active vibration controllers calls for a detailed dynamic model of the flexible structure. Unfortunately the motions of flexible systems are described by partial differential equations and, except for some simple cases, no close solution of these equations can be expected. To overcome this difficulty the partial differential equations are usually replaced, via spatial discretization, by a finite set of simultaneous ordinary differential ones. Implicit in this approach is a system truncation: a system of infinite order is replaced by a finite order one [1],[2].
It has been shown ([3],[4]) that a controller, designed on a finite dimensional model approximation, can destabilize the real system. This phenomenon was firstly investigated by Balas in [5] and was termed "spillover". The spillover phenomenon is due to the interaction between modelled and unmodelled dynamics. The *input spillover*, i.e. the excitation of the unmodelled dynamics by mean of the inputs, may deteriorate the performance of the control scheme, but it does not affect the stability property of the system. On the contrary, in the presence of a feedback controller, as the deformation measurements contain both modelled and un-modelled dynamics (*observer spillover*), the closed loop system may be unstable These phenomenon is dramatically emphasized in presence of dynamic observers needed for state feedback controllers.
The spillover phenomenon may be suppressed by avoiding the interaction between modelled and unmodelled dynamics. In order to reduce this interaction various approaches have been proposed. In [1] the use of a large number of sensors is suggested, in [6] an observer is designed on the basis of a system model of order greater than that used for the controller design, in [5] the sensor data are prefiltered with a comb filter, in [7] and in [8] it is shown that a control system based on distributed actuators and distributed sensors is not affected by spillover phenomenon (for further approaches see [9] and the references therein).

In this work we develop a technique for designing an optimal state feedback controller which reduces the interaction between modelled and un-modelled dynamics without using distributed sensors. This goal is obtained through a combined effect of a state observer which reconstructs only some system modes, and of a controller which modifies the dynamics of the reconstructed modes only.

It is important to note that the energy stored in vibrating structures is usually concentrated in the first modes of the system. Then it is possible to obtain good performance by using a controller which increase the damping factor of only the first modes without modifying the high-frequency behaviour where the model is less accurate. Such a controller can be designed through a proper choice of the weighting matrices of the performances index of an LQ regulator.

As far as the observer is concerned we propose to use a state reconstructor based on spline shape functions which interpolate the available deformation measurements by taking into account the boundary conditions. In particular a spline shape function for each class of physical homogeneous state variables (e.g. displacement, velocity of deformation, etc.) is defined. This reconstructor is not based on a mathematical model of the structure neither it use information about system inputs, then it is intrinsically robust with respect to parameter uncertainties and it may work correctly even in presence of persistent disturbances. However the most interesting property is that it operates as a spatial filter, i.e. represents the spatial deformation by means of only the low frequency modes.

In Section 2 the model of the vibrating structure, a flexible beam, used for the controller design is described. In Section 3 a procedure for the design of an optimal controller which modifies the dynamic of only a limited number of modes is developed. In Section 4 the spline based reconstructor is presented. In Section 5 the robustness analysis of the control scheme is developed. Finally in Section 6 some conclusive remarks are presented.

# 2 Model of a Flexible Beam

In this paper we consider the one-dimensional cantilever beam shown in Fig. 1. By considering only bending deformations in the vertical plane, the free motions are described by the well known equation:

$$m(x)\,\frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[E\,I_z\frac{\partial^2 v(x,t)}{\partial x^2}\right] = 0 \qquad (1)$$

where $m(x)$ is the mass per unit of length at abscissa $x$, $v(x,t)$ is the vertical displacement, $EI_z$ is the flexural rigidity. The boundary conditions are

$$\left.\begin{aligned} v(x,t) &= 0 \\ EI_z\frac{\partial v(x,t)}{\partial x} &= 0 \end{aligned}\right\} \quad \text{at } x = 0 \quad \forall\, t \qquad (2)$$

and

$$\left.\begin{aligned} EI_z\frac{\partial^2 v(x,t)}{\partial x^2} &= 0 \\ EI_z\frac{\partial^3 v(x,t)}{\partial x^3} &= 0 \end{aligned}\right\} \quad \text{at } x = L \quad \forall\, t \qquad (3)$$

where $L$ is the length of the beam.

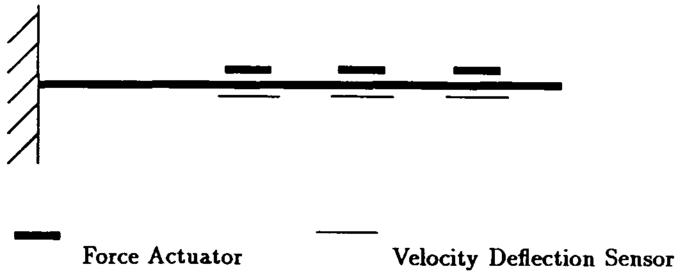


Figure 1: A cantilever beam.

In order to obtain a finite dimensional model, we divide the flexible beam in $n$ elements, and assume that the shape of each of them is described, at each instant, by the corresponding elastic strain. This hypothesis is realistic if the inertia forces acting on the $i$-th element and due to the distributed mass of the element, are negligible with respect to all the forces exerted by the remaining part of the structure on the extreme of the element itself. This can be considered as a straightforward extension of the De Saint Venant principle.

This approach was developed by the authors in an earlier paper [2]. The resulting dynamic model of the beam in the presence of both control and disturbance discrete forces is given by:

$$M\ddot{q} + F\dot{q} + Kq = u \qquad (4)$$

where $q = (v_2, \alpha_2, \dots, v_{n+1}, \alpha_{n+1})^T$ is the vector of lagrangian coordinates, i.e. displacements $v_i$ and slopes $\alpha_i$ at abscissae $x_i$, $i = 1, \dots, n+1$ of the spatial discretization ($x_1 = 0$, $x_{n+1} = L$), $M$ and $K$ are mass and stiffness matrix respectively, $F$ is the proportional damping matrix and $u$ is the vector of external generalized forces (for further details see [2]).
Equation (4) may be rewritten in the equivalent state-space form

$$\dot{z} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}F \end{bmatrix} z + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} u = Az + Bu \tag{5}$$

where

$$z = (v_2, \alpha_2, \dots, v_{n+1}, \alpha_{n+1}, \dot{v}_2, \dot{\alpha}_2, \dots, \dot{v}_{n+1}, \dot{\alpha}_{n+1})^T \in \Re^N \tag{6}$$

Model (5), unlike the models based on the modal approach, is characterized by the fact that the lagrangian coordinates and the generalized forces represent actual displacements and actual inputs, respectively.

# 3 Feedback Controller

The main aim of an active vibration controller is to subtract elastic energy to the flexible structure. The sources of this energy may be considered the disturbances acting on the system: earthquakes, wind, pressure waves at the fuselage in turbofan, asymmetries of the engine in cars, and so on.
All these disturbances present a limited frequency spectrum. Hence the controller must be able to reduce the resonance peaks of the system at the frequencies within the disturbance bandwidth.
It is important to note that, even if in presence of discontinuous disturbances the high frequency modes may be excited, because of the exponential stability property of system (4) and of the low amplitude of their residues, they "naturally" goes to zero provided that the control signal doesn't excite them. As a consequence the controller should be able to lower the magnitude frequency response in correspondence of the first $n$-resonance peaks of the system, where $n$ essentially depends upon the frequency spectrum of the disturbances.
Although the frequency shaping technique [10] could be used to allow this aim, unfortunately this technique doesn't guarantee robustness properties. Therefore we propose to use an LQ regulator with a proper choice of the weighting matrices. This choice is based on the following

**Theorem 1** *Let's consider system (5), and, for any integer $\nu < N/2$, let $\Gamma = (\Gamma_1^T \ \Gamma_2^T)$ be the eigenvector matrix of $A$, ordered with increasing values of the associated eigenvalues, where $\Gamma_1 \in \Re^{2\nu \times 2\nu}$. Then, for any given positive number $q$, the LQ regulator minimizing the performance index*

$$J = \int_0^\infty \left( z^T Q z + u^T R u \right) dt \tag{7}$$

*with*

$$Q = \Gamma^{-T} \hat{Q} \Gamma^{-1} \qquad \hat{Q} = \begin{bmatrix} qI_{2\nu} & 0 \\ 0 & 0 \end{bmatrix} \tag{8}$$

*modifies only the first $2\nu$ eigenvalues of the system. Moreover only the dynamics associated with these eigenvalues affect the control signal.*

*Proof.* The LQ problem given by system (5) and the performance index (7), (8), can be rewritten in the eigenvector basis, as

$$\min \ J = \int_0^\infty \left( \eta^T \begin{bmatrix} qI_{2\nu} & 0 \\ 0 & 0 \end{bmatrix} \eta + u^T R u \right) dt \tag{9}$$

s.t.

$$\dot{\eta} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \eta + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u \tag{10}$$

where $\eta = \Gamma^{-1} z$ are the modal coordinates, $\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} = \Gamma^{-1} A \Gamma$, $\Lambda_1 \in \Re^{2\nu \times 2\nu}$, and $\begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 \end{bmatrix}^T = \Gamma^{-1} B$. Hence, by Lemma 1 given in Appendix, Theorem 1 follows. ∎

It is important to note that, even if in the control signal there are no contribution of the modelled high-frequency dynamics, the direct feedback of actual measurements may introduce spillover. Indeed the measurements contain all the frequencies of the system, hence we must introduce a filtering system which screens out the unwanted frequency.

# 4 State Reconstruction

In order to implement the proposed state feedback controller, taking into account that only a limited number of measurements are available, a state observer is

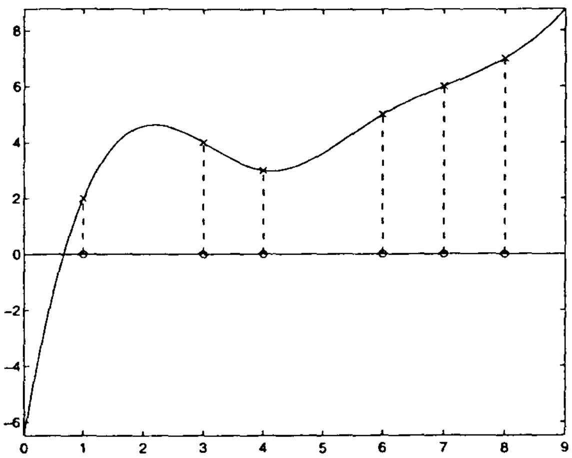

Figure 2: Spline interpolation function fitting the dates (×) in the knots (∘).

needed. Classical observers, besides to increase the order of the controller, considerably deteriorate the performance of the whole system especially in the presence of model uncertainties and of unknown persistent external disturbances. Moreover their presence emphasizes the spillover phenomenon.

In this work we propose a technique that allows to obtain an "instantaneous" reconstruction of state variables which works correctly even in presence of persistent disturbances or parameter variations and reduces the spillover effect [11]. The technique is based on the spline approach.

*The spline approach*

Let $\tilde{y} = (y_1, \ldots, y_m)$ be $m$ samples of an unknown function $y(x)$ at $m$ knots $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_m)$ (see Fig. 2). A cubic spline interpolating these samples is composed by $r = m - 2$ polynomials which coefficients $\beta_{i1}, \ldots, \beta_{i4}$, $i = 1, \ldots, r$ can be arranged in a vector $\beta$ given by:

$$\beta = D(\overline{x})\tilde{y} \tag{11}$$

where $D$ is a matrix depending upon the abscissae $\overline{x}_i$ of the knots and on boundary conditions imposed at the extreme of the $x$-interval in terms of values of the interpolated function and/or its derivative [12].

The spline reconstruction of the function $y(x)$ at the generic abscissa $x$ can be written as

$$\hat{y}(x) = s(x, \overline{x})D(\overline{x})\tilde{y} \tag{12}$$

where $s(x, \overline{x})$ is a vector function which selects the piece of cubic spline at abscissa $x$.

As shown in [13], the error between $\hat{y}(x)$ and the actual value $y(x)$ and the error between the $x$-derivatives of $\hat{y}(x)$ and of $y(x)$ are bounded and depends on the maximum interval $\Delta x$ between two consecutive measurement points. From equation (12) it follows that, once the boundary conditions have been specified, the values that the function $\hat{y}$ assumes in a finite number of specified points $x_1, \ldots, x_n$ can be expressed in the compact form

$$\hat{Y} = S(x_1, \ldots, x_n, \overline{x})D(\overline{x})\tilde{y} = T(x_1, \ldots, x_n, \overline{x})\tilde{y} \tag{13}$$

where

$$S(x_1, \ldots, x_n, \overline{x}) = \begin{pmatrix} s(x_1, \overline{x}) \\ \vdots \\ s(x_n, \overline{x}) \end{pmatrix} \tag{14}$$

*The state reconstruction problem*

Let's return now to the single-span beam described in Section 2. The state vector of the spatial discretized model at instant $t$ is composed by the following variables:

$$\Big[v(x_2,t),\ldots,v(x_{n+1},t) \vdots \alpha(x_2,t),\ldots,\alpha(x_{n+1},t) \vdots \dot{v}(x_2,t), \ldots,\dot{v}(x_{n+1},t) \vdots \dot{\alpha}(x_2,t),\ldots,\dot{\alpha}(x_{n+1},t)\Big]^T$$

Let's assume that a limited number $k$ of vertical velocity measurements $\tilde{\dot{v}}(t) = (\tilde{\dot{v}}_1, \ldots, \tilde{\dot{v}}_k)$ at abscissae $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_k)$ are available at each time instant. Moreover, in virtue of (2) and (3), at the extreme $x = 0$ and $x = L$ of the beam we can assume that, $\forall\, t$,

$$\left.\begin{array}{l} \dot{v}(x,t) = 0 \\ \dfrac{\partial \dot{v}(x,t)}{\partial x} = 0 \end{array}\right\} \text{ at } x = 0 \qquad \left.\begin{array}{l} \dfrac{\partial^2 \dot{v}(x,t)}{\partial x^2} = 0 \\ \dfrac{\partial^3 \dot{v}(x,t)}{\partial x^3} = 0 \end{array}\right\} \text{ at } x = L \tag{15}$$

Then, using the spline approach, we can reconstruct the vertical velocity at the generic abscissa $x$ and at the points $x_2, \ldots, x_{n+1}$ of the spatial discretization of the beam. In particular by equation (12) and (13) we obtain:

$$\hat{\dot{v}}(x,t) = s(x, \overline{x})D(\overline{x})\tilde{\dot{v}}(t) \tag{16}$$

$$\left(\hat{\dot{v}}(x_2,t), \ldots, \hat{\dot{v}}(x_{n+1},t)\right)^T = T(x_2, \ldots, x_{n+1}, \overline{x})\tilde{\dot{v}}(t) \tag{17}$$

As far as the remaining state variables are concerned, namely the vertical displacements, the slopes and slope velocities, they can be reconstructed by the vertical velocities measurements in the following way:

$$\left(\hat{v}(x_2,t),\ldots,\hat{v}(x_{n+1},t)\right)^T = T(x_2,\ldots,x_{n+1},\overline{x}) \int_0^t \tilde{\dot{v}}(\tau)d\tau \tag{18}$$

where it has been implicitly assumed $v(x,0)=0$.
Moreover

$$\hat{\dot{\alpha}}(x,t) = \frac{d}{dx}s(x,\overline{x})D(\overline{x})\tilde{\dot{v}}(t) \tag{19}$$

then

$$\left(\hat{\dot{\alpha}}(x_2,t),\ldots,\hat{\dot{\alpha}}(x_{n+1},t)\right)^T = S_\alpha(x_2,\ldots,x_{n+1},\overline{x})D(\overline{x})\tilde{\dot{v}}(t) = T_\alpha(x_2,\ldots,x_{n+1},\overline{x})\tilde{\dot{v}}(t) \tag{20}$$

$$(\hat{\alpha}(x_2,t),\ldots,\hat{\alpha}(x_{n+1},t))^T = T_\alpha(x_2,\ldots,x_{n+1},\overline{x}) \int_0^t \tilde{\dot{v}}(\tau)d\tau \tag{21}$$

where

$$S_\alpha(x_2,\ldots,x_{n+1},\overline{x}) = \begin{pmatrix} \left.\frac{d}{dx}s(x,\overline{x})\right|_{x=x_2} \\ \vdots \\ \left.\frac{d}{dx}s(x,\overline{x})\right|_{x=x_{n+1}} \end{pmatrix} \tag{22}$$

The state reconstructor scheme is shown in Fig. 3, the *re-ordination* block is a combinatorial one which places the reconstructed variables in the same order of the $z$ vector. It is interesting to note that the proposed reconstructor is not based on the mathematical model of the beam and it does not use any information about inputs. Then it is intrinsically robust against parametric variations and works correctly also in the presence of external persistent disturbances, provided that the disturbance frequencies don't exceed the frequencies associated with the reconstructed modes.
An other interesting feature of the proposed state reconstructor is its intrinsic filtering property. As a matter of facts, the beam deformation is approximated, at each instant, by means of a cubic spline. In [13] it is shown that the spline interpolation function has the property of minimizing the overall curvature among all functions fitting the measured values. Then, at each instant, the deformation is approximated by means of the lowest spatial frequency modes. This spatial filtering, unlike the classical time-filtering, does not introduce phase-lag, hence it does not deteriorate the stability properties. A practical consequence of this filtering action is that, even thought the disturbance inputs excite the high-frequency modes of the system, they tend to be screen out in the reconstructed state variables. In the next section we will show, via computer simulations, that the spline observer can dramatically reduce the spillover phenomenon.

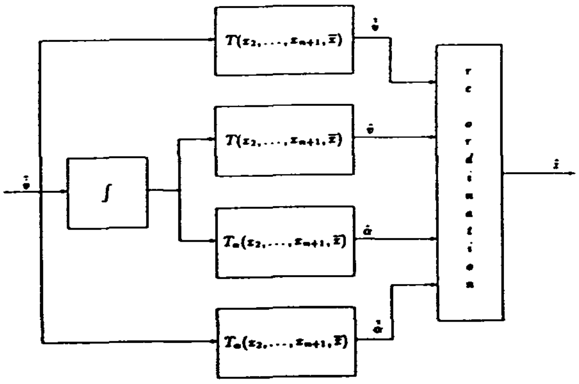


Figure 3: Reconstructor schema.

# 5 Simulation Results and Robustness Analysis

In order to show the effectiveness of the proposed controller, we consider a flexible system shown in Fig. 1 with the following parameters: $EI_z = 2.55 \cdot 10^{13}\,[Kg \cdot mm^3/sec^2]$, $m = 0.28 \cdot 10^{-5}\,[Kg/mm]$, $L = 2500\,[mm]$. Moreover three force actuators at $x_1 = 1100$ [mm], $x_2 = 1650$ [mm], and $x_3 = 2200$ [mm], and three collocated vertical velocity sensors have been considered. In order to obtain a finite dimensional model, the beam was divided into 9 parts of the same length. Moreover we consider the presence of the following force disturbance acting vertically at the tip of the beam

$$d(t) = \sum_{k=1}^{3} a_k\,[1+\sin(\omega_a t)]\sin\left(k\omega t + d\sin(\omega_p t)\right) \tag{23}$$

where $\omega = 250$ rad/s, $\omega_a = 25$ rad/s, $\omega_p = 0.8$ rad/s, $d = 0.1$, $a_1 = 5$ mN, $a_2 = 3$ mN, $a_3 = 2$ mN (see Fig. 4). On the basis of the disturbance spectrum, of the natural frequencies of the system and of the sensor locations, the matrices $Q$ and $R$ in the performance in-

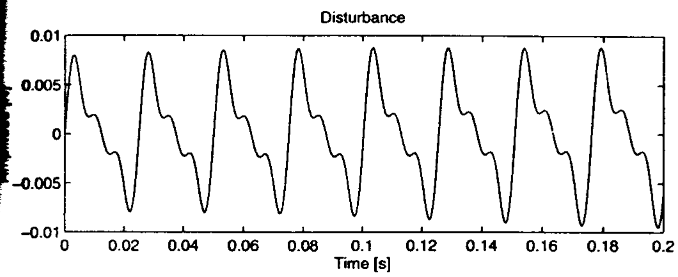


Figure 4: Disturbance time history.

dex (7) were chosen so as to penalize the first three modes:

$$Q = 10^8 \Gamma^{-T} \begin{bmatrix} I_6 & 0 \\ 0 & 0 \end{bmatrix} \Gamma^{-1}; \quad R = \mathrm{diag}[1, 1, 1] \quad (24)$$

In Fig. 5 the bode diagrams of the open loop system and of the closed one are shown. It is interesting to note that the the high-frequency peaks are not modified. The performance of the proposed controller (op-

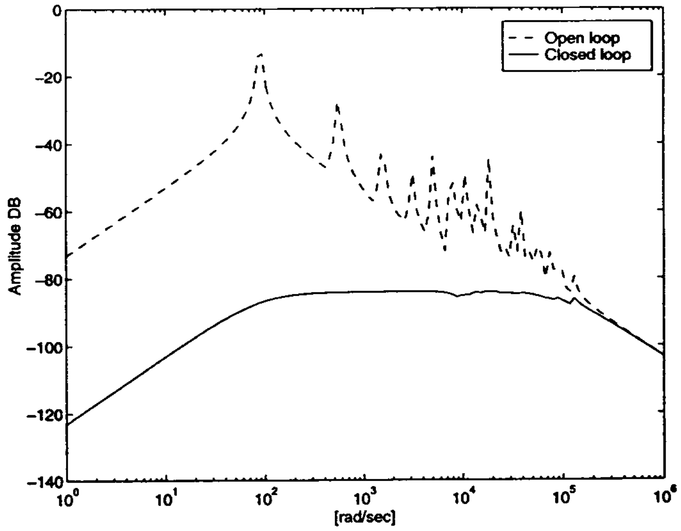


Figure 5: Maximum singular value $\overline{\sigma}$ plot of the beam model and of the closed loop system.

timal feedback of the reconstructed state) is shown by comparing the open loop behaviour (Fig. 6) and closed loop responses (Fig. 7).

In order to evaluate the property of reducing the spillover effect of the proposed control scheme, we considered a model obtained by dividing the beam in 18 parts. It has been noted that, by using the direct feedback of the state variables associated with the "9 part-model", the closed loop system becomes unstable. This

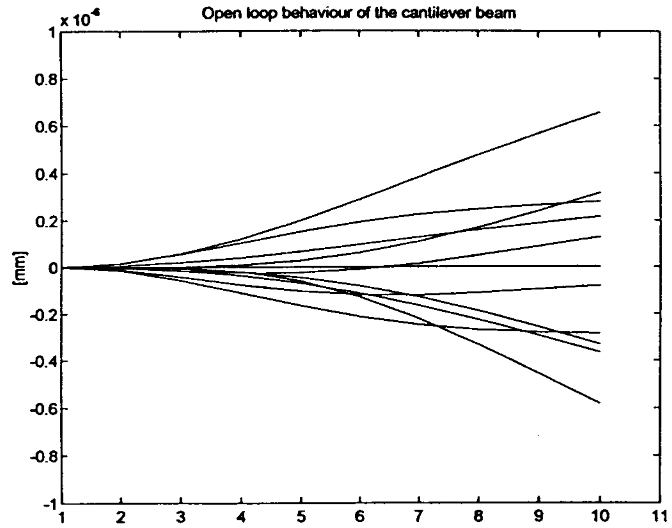


Figure 6: Open loop behaviour.

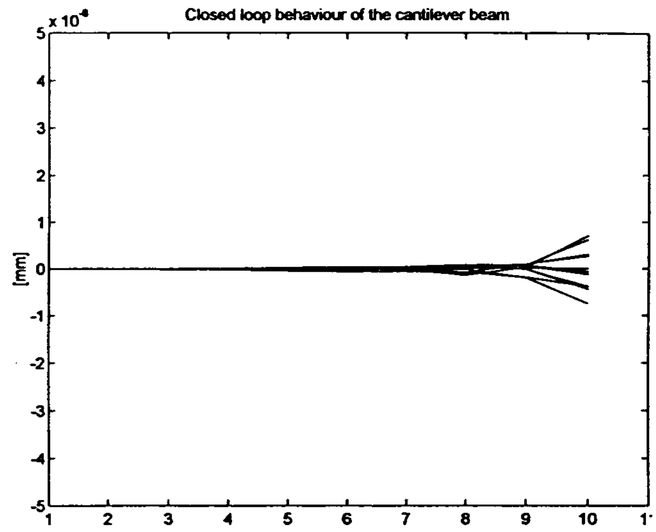


Figure 7: Closed loop behaviour.

is not the case when the reconstructed state variables are used.
To have a confirmation of the stability robustness property of the proposed control scheme with respect to un-modelled high frequency dynamics, we have schematized these dynamics as multiplicative unstructured perturbation. In other terms we assumed that the actual infinite-dimensional transfer function of the beam $P_t(s)$ can be written as:

$$P_t(s) = (I + \Delta(s))\,P(s) \tag{25}$$

where $P(s)$ is the transfer function of the "9 part-model", and $\Delta(s)$ represents the un-modelled high frequency dynamics. Note that equation (25) represents the multiplicative uncertainty at the plant output (see Fig. 8) which is the critical point for the spillover phenomenon. A measure of the robustness of the controller is given by the so called "generalized stability margin function" given by:

$$l(\omega) = \frac{1}{\overline{\sigma}\left(K(j\omega)(I - K(j\omega)P(j\omega))^{-1}\right)} \tag{26}$$

In [14] it is shown that if the transfer function $P(s)$ is proper and has no poles on the imaginary axis and the perturbation $\Delta(s)$ is proper and is such that $\Delta(s)$ and $(I+\Delta(s))P(s)$ have equal numbers of poles in the right half plane then the compensator $K(s)$ which stabilizes $P(s)$ will also stabilize all perturbated system such that

$$\overline{\sigma}(\Delta(j\omega)) < l(\omega) \tag{27}$$

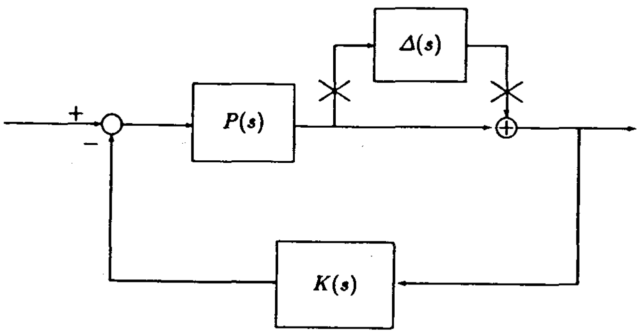


Figure 8: Output multiplicative uncertainty.

In Fig. 9 the robust margin functions $l(\omega)$ for various control schemes are shown. In particular both an LQ controller with the state weighting matrix $Q = \alpha I$, and with $Q$ chosen as suggested in Theorem 1 are shown. Finally, for the latter case, also the presence of a spline reconstructor is considered. We can see that a suitable choice of the control law and the use of a spatial filtering as the spline reconstructor allows to dramatically increase the robustness margin of the control scheme.

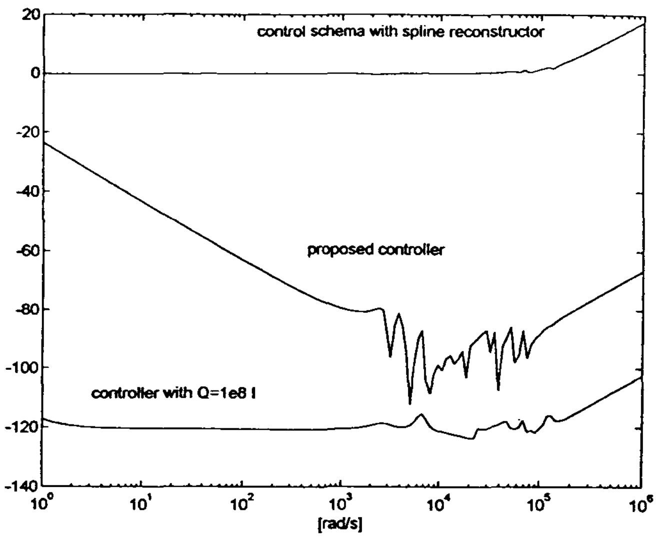


Figure 9: Diagram of the robust margin $1/\overline{\sigma}\left(K(I - KP)^{-1}\right)$ for different control schemes.

# 6 Conclusion

In this paper a control scheme for flexible system was presented. The proposed control is based on a proper LQ regulator and on a spline reconstructor. The Q and R matrices in the LQ index have been chosen so as to weight a limited number of modes only. The spline reconstructor, unlike classical dynamic observer, is an "instantaneous" estimator of the state variables and is not based on a copy of the system, hence it is intrinsically robust. Moreover the reconstructor introduces a spatial filtering on the high-frequency modes which allows to screen out the un-modelled dynamics without introducing any phase-lag in the control loop.
It has been shown that this controller is robust against the spillover phenomenon.

# Appendix

**Lemma 1** *Let us consider the system*

$$\dot{x} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{28}$$

*and the following performances index*

$$J = \int_0^\infty (x^T \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} x + u^T Ru)dt \tag{29}$$

*where $Q_1 > 0$ has the same dimensions of the $A_1$ block. If the $A_2$ matrix is Hurwitz then the optimal feedback control modifies the eigenvalues of $A_1$ matrix only.*

*Proof.* The only semi-definite positive solution of the Riccati equation is

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{30}$$

where $P_1$ is the solution of the Riccati equation:

$$A_1^T P_1 + P_1 A_1 - P_1 B_1 R^{-1} B_1^T P_1 + Q_1 = 0 \tag{31}$$

Hence the closed loop matrix is given by

$$A_c = \begin{bmatrix} A_1 - B_1 R^{-1} B_1^T P_1 & 0 \\ -B_2 R^{-1} B_1^T P_1 & A_2 \end{bmatrix} \tag{32}$$

i.e. only the eigenvalues of $A_1$ are modified. ■

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