A SUFFICIENT CONDITION FOR STABILITY OF ANALOG NEURAL NETWORKS WITH DELAYS

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Abstract

This paper presents a generalized sufficient condition which guarantees stability of analog neural networks with delays. The condition derived using a Lyapunov functional describes the relation between the neuronal gains and the network weights. It can be used to assess stability of the neural networks with multiple time delays handily. The condition is an extension of the results presented in [1].

Keywords: Neural networks, Time delay
Lyapunov functional, Stability

1 Introduction

Theoretical study of neural dynamics and hardware implementation of artificial neural networks has advanced rapidly in the recent years [2]-[7]. In particular, with advances in very-large-scale integration (VLSI) technology, electronic implementations of analog neural networks have created a path leading to neural computers. However, many problems such as switching delays, integration and communication delays in the hardware implementation have arisen, which deteriorate dynamic performance and lead to instability of hardware neural networks. Study of neural dynamics with consideration of these problems becomes more important in order to manufacture high quality microelectronic neural networks. The time delay is a key issue which causes instability of hardware neural networks [1, 4, 8]. This problem is introduced as follows.

The dynamic equations for analog neural networks without delay can be described as follows [1, 4, 8]:

$$C_i\frac{du_i}{dt} = -\frac{u_i}{R_i} + \sum_{j=1}^{N} T_{ij} f_j(u_j(t)), \quad i = 1, 2, \dots, N \tag{1.1}$$

where the variable $u_i(t)$ represents the voltage on the input of the ith neuron. Each neuron is characterized by an input capacitance C_i, a delay τ_j, and a transfer function f_j. The element of the connection matrix, T_{ij}, has a value $1/R_{ij}$ when the noninverting output of jth neuron is connected to the input of ith neuron through a resistance R_{ij}, and a value $-1/R_{ij}$ when the inverting output of jth neuron is connected to the input of ith neuron through a resistance R_{ij}. The parallel resistance at the input of each neuron is defined as $R_i = (\sum_j |T_{ij}|)^{-1}$. The circuit equations for a network of N neurons with delayed outputs coupled via a resistive interconnection matrix are the same as equation 1.1, with addition of delays τ_j, which are described as follows:

$$C_i\frac{du_i}{dt} = -\frac{u_i}{R_i} + \sum_{j=1}^{N} T_{ij} f_j(u_j(t-\tau_j)), i = 1, \dots, N. \tag{1.2}$$

When the connection matrix $T = \{T_{ij}\}$ is symmetric, it is well known that system 1.1 is always a dynamics in gradient convergence. This has been the basis for applications of the model to the associative memory and optimization problems. However, when T is symmetric and provided the neuronal gains (defined as a slope of $f_i(u)$ at $u = 0$) are sufficiently high, system 1.2 is not convergent necessarily. Divergence may happen even if the delays τ_j are very small. In particular, even if the delays are all the same ($\tau_j = \tau$) across the network, the dynamics of system 1.2 is still not convergent as shown by Marcus and Westervelt [1]. An extensive analysis of the effect of one common delay and the stability of system 1.2 has been conducted in [1], especially considering network architectures. In [4] different possible delay values are allowed across the network and the network dynamics, in particular oscillations, are considered instead of fixed point stability. The time delay affecting the networks' learning has also been studied in [4].

Marcus and Westervelt have investigated the case in which $C_i = C$, $\tau_j = \tau$ and T_{ij} are symmetric in system 1.2. By rescaling time, delay and T_{ij} the new variables, $t' = \frac{t}{RC}$, $\tau'_j = \frac{\tau_j}{RC}$ and $J_{ij} = RT_{ij}$, are obtained. Neglecting $'$ without losing the generality, linearizing $f_j(u_j(t-\tau_j)$ around the equilibrium gives:

$$\frac{du_i}{dt} = -u_i + \sum_{j=1}^{N} \beta_i J_{ij} u_j(t-\tau) \qquad (1.3)$$

where β_i is the gain. It is convenient to represent the linearized form of the N delay equations as amplitudes x_i ($i = 1, 2, \ldots, N$) along the N eigenvectors of the connection matrix J_{ij}:

$$\frac{dx_i}{dt} = -x_i + \sum_{j=1}^{N} \beta_i \lambda_i x_i(t-\tau) \qquad (1.4)$$

where λ_i is the ith eigenvalue of matrix J. Denoting λ_{min} and λ_{max} as the minimal and maximal eigenvalues of matrix J respectively, the following results can be obtained using the characteristic equations.

(1) If $\beta_i < \frac{1}{\lambda_{max}}$, $\frac{1}{\beta_i} > -\lambda_{min}(\omega^2+1)^{\frac{1}{2}}$ and $\omega = -tg(\omega\tau)$, $\frac{\pi}{2} < \omega\tau < \pi$, the origin is stable.

(2) If $\beta_i > \frac{1}{\lambda_{max}}$, or $\beta_i < \frac{1}{\lambda_{max}}$ and $\frac{1}{\beta_i} < -\lambda_{min}(\omega^2+1)^{\frac{1}{2}}$, $\omega = -tg(\omega\tau)$, $\frac{\pi}{2} < \omega\tau < \pi$, the origin is unstable. $\beta_i = \frac{1}{\lambda_{max}}$ is a pitchfork bifurcation, whereas $\beta = -\lambda_{min}(\omega^2+1)^{\frac{1}{2}}$, $\omega = -tg(\omega\tau)$, $\frac{\pi}{2} < \omega\tau < \pi$ is a Hopf bifurcation.

In this paper, a Lyapunov functional [9] is employed to investigate the stability of the continuous Hopfield neural network with time delays. A generalized sufficient condition that guarantees stability of analog neural networks with delay is presented. The condition can be described as: the equilibrium of analog neural networks with delay is globally asymptotically stable as long as the product of the two-norm of connection matrix and the neuronal gain is less than 1. This condition is an extension of the results presented in [1].

2 Condition for stability of neural networks with delay

Consider the following autonomous time delay equation:

$$\dot{x}(t) = f(x_t), \qquad (2.1)$$

where $f : C \to R^n$ is completely continuous and solutions of equation 2.1 depend continuously on the initial function. We denote by $x(\phi)$ the solution through $(0, \phi)$, $\phi \in C$ and C denotes $C([-\tau, 0], R^n)$.

If $V : C \to R$ is a continuous functional assumed as a Lyapunov functional, we define the derivative of V along the solution of equation 2.1 as follows:

$$\frac{dV(\phi)}{dt}|_{(5)} = \lim_{h\to 0^+} \sup \frac{1}{h}[V(x_h(\phi)) - V(\phi)]. \quad (2.2)$$

Lemma 1: Suppose $V : C \to R$ is continuous and there exist nonnegative functions $a(r)$ and $b(r)$ such that $a(r) \to \infty$ as $r \to \infty$

$$a(|\phi(0)|) \le V(\phi), \qquad \frac{dV(\phi)}{dt} < -b(|\phi(0)|), \tag{2.3}$$

then the solution $x = 0$ of equation 2.1 is stable and every solution is bounded. If, in addition, $b(r)$ is positive definite, then every solution of equation 2.1 approaches to zero as $t \to \infty$.

Based on the above lemma, let us consider the continuous Hopfield neural networks with delays. Assuming $C_i = C$ and $R_i = R$, equation system 1.2 can be rewritten as:

$$RC\frac{du_i(t)}{dt} = -u_i(t) + R\sum_{j=1}^{N} T_{ij} f_j(u_j(t-\tau_j)). \tag{2.4}$$

Using the transformations $t' = \frac{t}{RC}$, $\tau'_j = \frac{\tau_j}{RC}$, $J_{ij} = RT_{ij}$; $\sum_j T_{ij} = 1$ and neglecting $'$, equation system 1.2 becomes:

$$\frac{du_i(t)}{dt} = -u_i(t) + \sum_{j=1}^{N} J_{ij} f_j(u_j(t-\tau_j)). \tag{2.5}$$

Denoting a two-norm of matrix $T = \{T_{ij}\}$ by $||T||_2$, (defined as $[\lambda_{max}(T^T T)]^{\frac{1}{2}}$) and $\beta = \max\{\beta_1, \beta_2, \ldots, \beta_N\}$, the main result is as follows.
Theorem: If $\beta||J||_2 < 1$, then the equilibrium of system 2.5 is unique and asymptotically stable.

The proof is given as follows:
(i) The equilibrium of system 2.5 is unique:

Suppose system 2.5 has a non-zero equilibrium $X^0 = (X_1^0, X_2^0, \ldots, X_N^0)^T$, then $X_i^0 = \sum_{j=1}^{N} J_{ij} f_j(X_j^0)$, $i = 1, 2, \ldots, N$. Let $Y^0 = (Y_1^0, Y_2^0, \ldots, Y_N^0)^T$, $Y_i^0 = f_i(X_i^0), i = 1, 2, \ldots, N$, then $X^0 = JY^0$, $X^{0T}X^0 = X^{0T}JY^0$, $|X^0|^2 =\le |X^0|||J||_2|Y^0|$ and $|Y_j^0| = |f_j(X_j^0)| \le \beta_j|X_j^0| \le \beta|X_j^0|$, therefore

$$|Y^0| \le \beta|X^0|$$

and

$$|X^0|^2 \le \beta||J||_2|X^0|^2$$

which results in $\beta||J||_2 \ge 1$, contradicting with the assumption. Thus $|X^0| = 0$ and the origin is the unique equilibrium of system 2.5.
(ii) The equilibrium of system 2.5 is asymptotically stable:

Let $\phi = (\phi_1, \phi_2, \ldots, \phi_N)^T$, $a(r) = r^2$ and V functional be:

$$V(\phi) = \sum_{i=1}^{N} \phi_i^2(0) + \sum_{i=1}^{N} \int_{-\tau_i}^{0} \phi_i^2(\theta) d\theta, \tag{2.6}$$

$a(r)$ tends to $+\infty$ as $t \to \infty$, and obviously $a(|\phi(0)|) \le V(\phi)$. Differentiating the V functional with respect to equation 2.5, we have:

$$\begin{aligned}
\frac{dV(u_t)}{dt} &= 2\sum_{i=1}^{N} u_i(t)\dot{u}_i(t) + \sum_{i=1}^{N} u_i^2(t) \\
&\quad - \sum_{i=1}^{N} u_i^2(t-\tau_i) \\
&= -\Big(|u(t)|^2 + \sum_{i=1}^{N} u_i^2(t-\tau_i)\Big) \\
&\quad + 2\sum_{i=1}^{N}\sum_{j=1}^{N} u_i(t) J_{ij} f_j(u_j(t-\tau_j)).
\end{aligned} \tag{2.7}$$

In equation 2.7, let $\eta = (f_1(u_1(t-\tau_1)), f_2(u_2(t-\tau_2)), \ldots, f_N(u_N(t-\tau_N))^T$ and $U(t) = (u_1(t), u_2(t), \ldots, u_N(t))^T$, then the second term in the right-hand side of equation 2.7 becomes $U^T(t)J\eta$. Suppose it is positive, based on the *Cauchy* inequality, we have:

$$|\eta| \le \beta\Big(\sum_{i=1}^{N} u_i^2(t-\tau_i)\Big)^{\frac{1}{2}}$$

and

$$U^T(t)J\eta \le \beta|u(t)|||J||_2\Big(\sum_{i=1}^{N} u_i^2(t-\tau_i)\Big)^{\frac{1}{2}}, \tag{2.8}$$

then

$$\frac{dV(u_t)}{dt} \le -\Big(|u(t)|^2 + \sum_{i=1}^{N} u_i^2(t-\tau_i)\Big)$$

$$
\begin{aligned}
& +2\beta|u(t)|||J||_2\Big(\sum_{i=1}^{N} u_i^2(t-\tau_i)\Big)^{\frac{1}{2}} \\
\leq\ & -(1-\beta||J||_2)|u(t)|^2 \\
& +(\beta||J||_2 - 1)\Big(\sum_{i=1}^{N} u_i^2(t-\tau_i)\Big) \\
\leq\ & -(1-\beta||J||_2)|u(t)|^2. \qquad (2.9)
\end{aligned}
$$

Define $b(r)$ as $(1-\beta||J||_2)|r|^2$, according to the lemma, the equilibrium of system 2.5 is asymptotically stable.

Corollary: If the matrix J is symmetric, λ_{max} denotes the m..ximum of absolute eigenvalue of matrix J and $\beta|\lambda_{max}| < 1$, then the equilibrium of system 2.5 is unique and asymptotically stable.

An example is given as follows to show the abo.e corollary:

$$
J = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}
$$

The eigenvalues of matrix J is $\lambda_{1,2} = \frac{1}{2}$, and $\lambda_3 = -1$. According to the theorem, if $\beta < 1$, then the equilibrium of system 2.5 is unique and asymptotically stable with respect to arbitrary delays.

3 Conclusion

This paper presents a generalized sufficient condition which guarantees stability of analog neural networks with delays. The condition can be described as follows: the equilibrium of analog neural networks with delay is globally asymptotically stable as long as the product of the two-norm of connection matrix and neuronal gain is less than 1. The condition can be used to design stable analog neural networks with delays in practical applications.

References

[1] Marcus, C. M., Westervelt, R. M., 'Stability of analog neural networks with delay', *Physical Review A*, **39**, (2), 1989, pp.347-359.

[2] Denker, J. S., 'Neural network for computing (Snowbird UT 1986)', *Proceedings of the conference on neural networks for computing*, AIP Conf. Proc.151, New York, 1986.

[3] Marcus, C. M., Waugh, F. R. and Westervelt, R. M., 'Nonlinear dynamics and stability of analog neural networks', *Physica D*, **51**, 1991, pp.234-247.

[4] Baldi, P. and Atiya, A. F. 'How delays affect neural dynamics and learning', *IEEE Trans. on Neural Networks*, **5**, (4), July 1994, pp.612-621.

[5] Wang, L., Pichler, E. E. and Ross, J., 'Oscillations and chaos in neural networks: an exactly solvable model', *Proc. Natl. Acad. Sci. USA*, **87**, December 1990, pp.9467-9471.

[6] Hopfield, J. J., 'Neural networks and physics systems with emergent collective computation abilities', *Proc. Natl. Acad. Sci. USA*, **79**, (2), 1982, pp.2554-2558.

[7] Hopfield, J. J. and Tank, D. W., 'Neural computation of decisions optimization problems', *Biological Cybernetics*, **52**, 1985, pp.141-152.

[8] Hopfield, J. J., 'Neurons with graded response have collective computational properties like those of two–state neurons', *Proc. Natl. Acad. Sci. USA*, **81**, (10), 1984, pp.3088-3092.

[9] Hale, J. K., 'Introduction to functional differential equations', New York, Springer Verlag, 1993.