COMPUTATION OF THE IMPULSIVE BEHAVIOR OF MULTIVARIABLE LINEAR SYSTEMS USING A DIVISION ALGORITHM

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ABSTRACT

A new method is proposed which allows the evaluation of the impulsive behavior of a linear multivariable system of arbitrary order. The proposed method is in closed matrix form, that is only constant (real) matrix arithmetic(i.e. multiplication and addition) is involved among the polynomial matrices and therefore the algorithm can be easily programmed in a digital computer.

1 INTRODUCTION

Let us consider a linear multivariable system $(\sum)$ whose dynamics are described by a polynomial matrix model (PMD) $\sum$ [Callier and Desoer (1982)]:

$$A(\rho)\beta(t) = B(\rho)u(t) \qquad (1)$$

$$y(t) = C(\rho)\beta(t) + D(\rho)u(t) \qquad (2)$$

where $\rho \doteq \frac{d}{dt}$ the differential operator , $A(\rho) = \sum_{i=0}^{n} A_i \in \Re[\rho]^{r\times r}$, $n \geq 1$, with $\text{rank}_{\Re} A_n \leq r$, $B(\rho) = \sum_{i=0}^{d} B_i \in \Re[\rho]^{r\times m}$, $d \geq 0, C(\rho) = \sum_{i=0}^{d_1} C_i \in \Re[\rho]^{p\times r}$, $d_1 \geq 0$, $D(\rho) = \sum_{i=0}^{d_2} D_i \in \Re[\rho]^{p\times m}$, $d_2 \geq 0$ and $\beta(t) : [0^-,\infty) \rightarrow \Re^r$ the pseudo state , $u(t) : [0^-,\infty) \rightarrow \Re^m$ the input vector and $y(t) : [0^-,\infty) \rightarrow \Re^p$ the output vector of $(\sum)$.

The purpose of the present paper is to find analytical expressions of the impulsive part of the pseudostate $\beta(t)$. . The impulsive behavior in the pseudostate $\beta(t)$ arise naturally in many control systems of the form (1)-(2) either due to the operation of inconsistent initial conditions or due to the structure at infinity of the coefficient matrices of the PMD (Σ). In order to understand better the impulsive behavior and analyze ways to eliminate it we have to develop certain methods to describe these impulsive terms. Our method evaluates the impulsive terms of the PMD (1)-(2) computationally directly in terms of the coefficient matrices of $A(\rho), B(\rho)$ using a division algorithm. In the literature, the impulsive behavior of generalized state space systems (which represent a special case of PMDs of the form (1)-(2)) have been examined extensively in the last few years [see Verghese (1978), Campell (1982), Cobb (1984), Ozcaldiran (1985) Lewis (1986)]. Here we propose a method which gives *the most general solution* to the problem of computation the impulsive terms of multivariable systems of arbitrary order as these described by (1)-(2).

2 MAIN RESULTS

Consider the PMD (Σ) as in (1)-(2). The complete solution (evaluation of the pseudostate $\beta(t)$) has generally the form:

$$\beta_c(t) = \beta_{\text{hom}}(t) + \beta_n(t) \tag{3}$$

where $\beta_c(t)$ is the complete (whole) solution, whereas $\beta_{\text{hom}}(t)$ describes the part of the solution which corresponds to the homogeneous part of the differential equation (1) (zero input response):

$$A(\rho)\beta(t) = 0 \tag{4}$$

and $\beta_n(t)$ corresponds to the non homogeneous part of the differential equation (1):

$$A(\rho)\beta(t) = B(\rho)u(t) \tag{5}$$

assuming zero initial conditions of the pseudostate $\beta(t)$ and of the input $u(t)$ (zero state response).

To determine the impulsive terms we make use of a recently developed algorithm [Fragulis et al. (1991)] which finds the inverse of an arbitrary polynomial matrix using only multiplication and addition of constant matrices. The recursive formulas which are obtained from that algorithm can be used also for the evaluation of the Laurent expansion of the inverse of a polynomial matrix. Specifically that algorithm computes H_j where $j = \widehat{q}_r, \widehat{q}_{r-1}, \ldots, 1, 0, -1, \ldots$

$$\begin{gathered} A^{-1}(s) = H_{pol}(s) + H_{spr(s)} \\ = \left[H_{\widehat{q}_r} s^{\widehat{q}_r} + H_{\widehat{q}_r - 1} s^{\widehat{q}_r - 1} + \ldots + H_1 s + H_0\right] \\ \left[+H_{-1} s^{-1} + \ldots\right] \end{gathered} \tag{6}$$

where $\widehat{q}_r$ is the zero at infinity of $A(s)$ with maximum degree ($\widehat{q}_r \equiv 0$ if $A(s)$ has no infinite zeros). $H_{pol}(s), H_{spr(s)}$ are the polynomial and the strictly proper part of $A^{-1}(s)$ respectively.

First we consider the computation of $\beta_n(t)$ which corresponds to the non homogeneous part of the differential equation (1). Using the Laplace transformation in (1) we obtain:

$$\begin{gathered} A(s)\beta(s) = B(s)u(s) \Rightarrow \\ \beta(s) = A^{-1}(s)B(s)u(s) \end{gathered} \tag{7}$$

Easily we can see that we have to find a more convenient expression for $A^{-1}(s)B(s)$. The previous product can be considered also as a left division :

$$B(s) = A(s)Q(s) + R(s) \tag{8}$$

where $Q(s), R(s)$ are the quotient and the remainder respectively of the division (8). The above division (i.e. the computation of $Q(s), R(s)$ in (8)) is a problem arising in various fields of control systems' analysis and synthesis. Numerous methods have been developed which are based on the manipulations of constant matrices only [see e.g. Zhang (1986)]. From (8) we obtain

$$A^{-1}(s)B(s) = Q(s) + A^{-1}(s)R(s) \tag{9}$$

with $A^{-1}(s)R(s)$ strictly proper. Since our aim is to find the impulsive terms of the PMD (Σ), from (7) and (9) we need only the quotient $Q(s)$ which is a polynomial matrix. Hence using equation (6) we obtain:

$$\begin{gathered} A^{-1}(s)B(s) = \\ H_{pol}(s)B(s) + H_{spr}(s)B(s) \Rightarrow \end{gathered}$$

$$\begin{gathered} A^{-1}(s)B(s) = \\ H_{pol}(s)B(s) + G_{pol}(s) + G_{spr}(s) \end{gathered} \tag{10}$$

where $G_{pol}(s), G_{spr}(s)$ are the polynomial and the strictly proper part of the product $H_{spr}(s)B(s)$ respectively(take in mind that $B(s)$ is a polynomial matrix and is multiplied by the strictly proper matrix $H_{spr}(s)$).

Now combining (9),(10) it is clear that the quotient $Q(s)$ (which is a polynomial matrix) is equal to :

$$Q(s) = H_{pol}(s)B(s) + G_{pol}(s) \tag{11}$$

Now we shall find a closed matrix formula for the evaluation of $Q(s)$ in terms of the coefficient matrices of $B(s), A(s)$ such that only constant matrix multiplication and addition is involved. From the form of $B(s)$ in(1) and equation (6) we obtain :

$$H_{pol}(s)B(s) = \left[H_0 + H_1 s + \ldots + H_{\widehat{q}_r} s^{\widehat{q}_r}\right] \times \left[B_0 + B_1 s + \ldots + B_d s^d\right]$$

which after some matrix manipulations takes the following form:

$$H_{pol}(s)B(s) = \left[H_0, H_1, \ldots, H_{\widehat{q}_r}\right] \times \begin{bmatrix} B_0 & B_1 & \cdots & B_d & 0 & \cdots & \cdots \\ 0 & B_0 & \cdots & B_{d-1} & B_d & \cdots & \cdots \\ 0 & 0 & \cdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & & & & \\ 0 & 0 & \cdots & B_{d-\widehat{q}_r} & B_{d-\widehat{q}_r+1} & \cdots & B_d \end{bmatrix} \times \begin{bmatrix} I_r \\ sI_r \\ \vdots \\ s^d I_r \\ -- \\ s^{d+1} I_r \\ \vdots \\ s^{\widehat{q}_r+d} I_r \end{bmatrix} \tag{12}$$

In the same manner we obtain that the form of $G_{pol}(s) \doteq$ polynomial part of $[H_{spr}(s)B(s)]$

$$G_{pol}(s) = \text{pol. part} \left\{ \begin{array}{c} \left[H_{-1}s^{-1} + H_{-2}s^{-2} + \ldots H_{-d}s^{-d}\right] \\ \times \left[B_0 + B_1 s + \ldots + B_d s^d\right] \end{array} \right\}$$

which after some matrix manipulations can be written as:

$$G_{pol}(s) = \left[H_{-d}, H_{-(d-1)}, \ldots, H_{-2}, H_{-1}\right] \times \begin{bmatrix} B_d & 0 & \cdots & 0 & 0 & 0 \\ B_{d-1} & B_d & \cdots & 0 & 0 & 0 \\ B_{d-2} & B_{d-1} & & \vdots & \vdots & \\ \vdots & \vdots & \ddots & B_d & 0 & 0 \\ B_1 & B_2 & \cdots & B_{d-1} & B_d & 0 \end{bmatrix} \times \begin{bmatrix} I_r \\ sI_r \\ \vdots \\ s^d I_r \end{bmatrix} \tag{13}$$

Combining (12)-(13) the quotient $Q(s)$ of the division (9) is :

$$Q(s) = \left[H_{-d}, H_{-(d-1)}, \ldots, H_{-2}, H_{-1} \mid H_0, H_1, \ldots, H_{\widehat{q}_r}\right] \times$$

$$\left[\begin{array}{cccccc} B_d & 0 & \cdots & 0 & 0 & 0 \\ B_{d-1} & B_d & \cdots & 0 & 0 & 0 \\ B_{d-2} & B_{d-1} & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & B_d & 0 & 0 \\ B_1 & B_2 & \cdots & B_{d-1} & B_d & 0 \\ -- & -- & -- & -- & -- & -- \\ B_0 & B_1 & \cdots & & B_{d-1} & B_d \\ 0 & B_0 & B_1 & \cdots & & B_{d-1} \\ \vdots & \vdots & & \ddots & & \\ 0 & 0 & \cdots & & & B_{d-\widehat{q}_r} \end{array}\right. \cdots \left.\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 \\ -- & -- & & -- \\ 0 & 0 & \cdots & 0 \\ B_d & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ B_{d-\widehat{q}_r+1} & \cdots & & B_d \end{array}\right]$$

$$\times \begin{bmatrix} I_r \\ sI_r \\ \vdots \\ s^d I_r \\ --- \\ s^d I_r \\ \vdots \\ s^{\widehat{q}_r+d} I_r \end{bmatrix}$$

$$= Q_0 + Q_1 s + \cdots + Q_{\widehat{q}_r+d} s^{\widehat{q}_r+d} \tag{14}$$

where $Q_i \; i = 0, 1, \ldots, \widehat{q}_r+d$ are the matrices which are obtained after the multiplication of the involved coefficient matrices.

To evaluate the impulsive solutions $\beta_n(t)$ due to the non homogeneous part of the PMD ($\sum$) (see(5)) we take the inverse Laplace transformation of (14) and we obtain:

$$\beta_n(t) = L_-^{-1}\left[\text{pol. part of} \left\{A^{-1}(s)B(s)u(s)\right\}\right] = L_-^{-1}\left[Q(s)\right] \Rightarrow$$

$$\beta_n(t) = L_-^{-1}\left[\begin{array}{c} Q_0u(s) + Q_1su(s) + \\ \cdots + Q_{\widehat{q}_r+d}s^{\widehat{q}_r+d}u(s) \end{array}\right] = \\ = Q_0u(t) + Q_1u^{(1)}(t) + \cdots \\ + Q_{\widehat{q}_r+d}u^{(\widehat{q}_r+d)}(t) \tag{15}$$

where $u^{(i)}(t)$ $i = 0, 1, \ldots, \widehat{q}_r + d$ denotes the i-th derivative of u(t) in the distributional sense [Kailath (1980), Campell (1982), Cobb (1984)]. Let us now denote $u^{[i]}(t)$ the $i-th$ (ordinary) derivative of $u(t)$. The two types of derivatives are related by the formula (see Campell (1982) p.52):

$$u^{(i)}(t) = u^{[i]}(t) + \delta(t)u^{[i-1]}(0) + \\ \ldots + \delta^{(i-1)}(t)u(0) \qquad i = 1, 2, \ldots \tag{16}$$

where $\delta^{(i)}(t)$ $i = 0, 1, \ldots,$ denotes the unit impulse and its derivatives. From (15) and (16) we obtain :

$$\beta_n(t) = Q_0u(t) + Q_1\left[u^{[1]}(t) + \delta(t)u(0^-)\right] + \\ \ldots + \\ Q_{\widehat{q}_r+d}\left[\begin{array}{c} u^{[\widehat{q}_r+d]}(t) + \delta(t)u^{[\widehat{q}_r+d-1]}(0^-) + \\ \ldots + \delta^{(\widehat{q}_r+d-1)}(t)u(0^-) \end{array}\right] \Rightarrow$$

$$\beta_n(t) = \left[\begin{array}{c} Q_0u(t) + Q_1u^{[1]}(t) + \\ \ldots + Q_{\widehat{q}_r+d}u^{[\widehat{q}_r+d]}(t) \end{array}\right] + \\ \left[\begin{array}{c} Q_1u(0^-) + Q_2u^{[1]}(0^-) + \\ \ldots + Q_{\widehat{q}_r+d}u^{[\widehat{q}_r+d-1]}(0^-) \end{array}\right]\delta(t) + \\ \ldots + \left[Q_{\widehat{q}_r+d}u(0^-)\right]\delta^{(\widehat{q}_r+d-1)}(t)$$

which
after some rearrangements among the matrices we obtain:

$$\beta_n(t) = \beta_n^f(t) + \beta_n^{imp}(t) \tag{17}$$

where $\beta_n^f(t)$:

$$\beta_n^f(t) = \sum_{i=0}^{\widehat{q}_r+d} Q_iu^{[i]}(t) \tag{18}$$

denotes the part of $\beta_n(t)$ that is impulse-free and $\beta_n^{imp}(t)$

$$\beta_n^{imp}(t) = \\ \sum_{i=0}^{\widehat{q}_r+d-1}\delta^{(i)}(t)\left[\sum_{j=i}^{\widehat{q}_r+d-1}Q_{j+1}u^{[j-i]}(0^-)\right] \tag{19}$$

denotes the impulsive part of $\beta_n(t)$.

In the sequel we shall consider the impulsive terms which are obtained from the homogeneous part of the PMD (Σ) (see equation (4)) . Taking the Laplace transformed equation (4) we obtain :

$$\beta(s) = A^{-1}(s)a(s) \in \Re^{r\times 1} \tag{20}$$

where $a(s) \in \Re^{r\times 1}[s]$ is the initial condition vector associated with the initial values of $\beta(t)$ and its $(n-1)$-derivatives at $t = 0^-$ *i.e.* $\beta(0^-)$, $\beta^{(1)}(0^-)$, ... , $\beta^{(n-1)}(0^-)$ given by [Callier and Desoer (1982)] :

$$a(s) = [s^{n-1}I_r, s^{n-2}I_r, \ldots, sI_r, I_r] \times F \tag{21}$$

$$F = \begin{bmatrix} A_n & 0 & \cdots & 0 \\ A_{n-1} & A_n & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_1 & A_2 & \cdots & A_n \end{bmatrix}\begin{bmatrix} \beta(0^-) \\ \beta^{(1)}(0^-) \\ \vdots \\ \beta^{(n-1)}(0^-) \end{bmatrix} \tag{22}$$

Applying the same method as before (evaluation of the quotient $Q_1(s)$ of the division between the $A^{-1}(s)$ and $a(s)$) after some matrix manipulations we finally arrive at the following special form for $\beta(s)$ which appeared originally in [Fragulis (1993)] :

$$\beta_{\text{hom}}(t) = L\ \Delta(t)F \tag{23}$$

where :

$$L = [H_{\widehat{q}_r-1}, \ldots, H_1, H_0 \mid \\ \ldots \mid H_{-1}, H_{-2}, \ldots, H_{-(n-1)}] \in \Re^{r\times[r(\widehat{q}_r+n-1)]} \tag{24}$$

$$\Delta(t) = \begin{bmatrix} 0 & 0 \dots & \dots & 0 & \delta(t)^{(\widehat{q}_r-1)} I_r \\ 0 & 0 \dots & \dots & \delta(t)^{(\widehat{q}_r-1)} I_r & \delta(t)^{(\widehat{q}_r-2)} I_r \\ \vdots & \vdots & & \vdots & \vdots \\ \delta(t)^{(q_1-1)} I_r & \delta(t)^{(q_1-2)} I_r \dots & \dots & \delta(t)^{(1)} I_r & \delta(t) I_r \\ \delta(t)^{(q_1-2)} I_r & \delta(t)^{(q_1-3)} I_r \dots & \dots & \delta(t) I_r & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \delta(t)^{(1)} I_r & \delta(t) I_r \dots & \dots & 0 & 0 \\ \delta(t) I_r & 0 \dots & \dots & 0 & 0 \end{bmatrix} \tag{25}$$

Thus we have found :

$$\beta_{\text{hom}}(t) = L\ \Delta(t) F = \beta_0 \delta(t) + \beta_1 \delta^{(1)}(t) + \dots + \beta_{\widehat{q}_r-1} \delta^{(\widehat{q}_r-1)}(t) \tag{26}$$

where β_i $i = 0, 1, \dots, \hat{q}_{r-1}$ are $r \times 1$ vectors obtained after some manipulations in the terms of (26) . From equation (26) and the definition of F in (22) it follows that if the initial conditions $\beta(0^-)$, ..., $\beta^{(\widehat{q}_r-1)}(0^-)$ are appropriate then $\beta(t)$ has an "impulsive behavior" at $t = 0^-$ which consists of a Dirac impulse $\delta(t)$ and its $(\hat{q}_r - 1)$ distributional derivatives . In other words when the initial conditions are imposed on $\beta(t)$ and its $(\hat{q}_{r-1})$ derivatives at $t = 0^-$, $\beta(t)$ may exhibit an impulsive behavior at $t = 0^-$ which is a consequence of the fact that (4) forces $\beta(t)$ and $\beta^{(i)}(t)$, $i = 1, 2, \dots, \hat{q}_{r-1}$ to satisfy certain constraints at $t = 0^-$. The exact derivation of these constraints and their relation to the structure at $s = \infty$ of $A(s)$ are examined in [Vardulakis and Fragulis (1989)] explicitly.

Now taking in mind that the impulsive part of the complete solution (pseudostate) $\beta(t)$ of (1) has the form:

$$\beta_c^{imp}(t) = \beta_{\text{hom}}(t) + \beta_n^{imp}(t)$$

and from the forms of $\beta_{\text{hom}}(t)$ in (26) and $\beta_n^{imp}(t)$ in (19) we finally obtain:

$$\beta_c^{imp}(t) = \sum_{i=0}^{\widehat{q}_r-1} \beta_i \delta^{(i)}(t) + \sum_{i=0}^{\widehat{q}_r+d-1} \delta^{(i)}(t) \left[\sum_{j=i}^{\widehat{q}_r+d-1} Q_{j+1} u^{[j-i]}(0^-) \right] \tag{27}$$

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