Convergence and Approximation of Nonlinear Algorithms, Operators and Semigroups

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1. Introduction

In this paper we intend to present and discuss several new and recent results on the convergence and approximation of nonlinear algorithms, operators and semigroups, with particular emphasis on the case when they act on different Banach spaces. Such approximation and convergence problems arise naturally in many situations, including, for example, parameter estimation and identification theory and the numerical solution of partial differential equations. Thus the introduction of different spaces as well as the corresponding projection and embedding operators is motivated, inter alia, by the approximation of differential equations via difference equations, since the difference operators act on spaces different from the one on which the differential operator acts. A similar situation occurs in identification problems (see, for example, the book [2] and the paper [4]).

2. Main Results

We begin by considering families of approximations to Banach space. Let Z and X_n be (real) Banach spaces with norms $|\cdot|$ and $|\cdot|_n$, $n = 1, 2, \ldots$, respectively, and let X be a closed linear subspace of Z. We make the following assumptions.

For each $n = 1, 2, \ldots$ there exist mappings $P_n\colon Z \mapsto X_n$ and $E_n\colon X_n \mapsto Z$ satisfying

(i) $|P_n x - P_n y|_n \le M|x-y|$ for all x and y in Z,

and

(ii)
$|E_n x_n - E_n y_n| \le M|x_n - y_n|_n$ for all x_n and y_n in X_n,

where M is independent of n;

(iii) $\lim_{n\to\infty} |E_n P_n x - x| = 0$ for all x in X;

(iv) $P_n E_n x_n = x_n$ for all x_n in X_n.

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Note that we do not assume that the spaces X_n are subspaces of X, or that the mappings P_n and E_n are bounded linear operators. See [8], [15] and [16] for related settings.

The sequence $\{E_n\}$ is said to be asymptotically linear if it satisfies

(v)
If $\lim_{n\to\infty} E_n x_n = x$, $\lim_{n\to\infty} E_n y_n = y$ and $x, y \in X$,

then

$\lim_{n\to\infty} E_n(\alpha x_n + \beta y_n) = \alpha x + \beta y$ for all scalars α and β.

For positive r we denote the resolvent $(I + rA)^{-1}$ of an operator A by J_r^A. The domain of A will be denoted by $\mathrm{D}(A)$ and its closure by $\mathrm{cl}(\mathrm{D}(A))$. The range of A will be denoted by $\mathrm{R}(A)$.

Theorem 1. *Let $A + \omega I$ be an* m*-accretive operator in X and let S be the semigroup generated by $-A$. Let $A_n + \omega I$ be an* m*-accretive operator in X_n, $n = 1, 2, \ldots$, and let S_n be the semigroup generated by $-A_n$. Assume that $\{E_n\}$ is asymptotically linear. If*

$$\lim_{n\to\infty} E_n J_{r_0}^{A^n} x_n = J_{r_0}^A x \textit{ for some } r_0 > 0 \textit{ with } r_0\omega < 1$$

whenever $x \in X$, $x_n \in X_n$ and $\lim_{n\to\infty} E_n x_n = x$,

then

$$\lim_{n\to\infty} E_n S_n(t) x_n = S(t)x$$

whenever $x \in \mathrm{cl}(\mathrm{D}(A))$, $x_n \in \mathrm{cl}(\mathrm{D}(A_n))$ and

$\lim_{n\to\infty} E_n x_n = x$, and the convergence is uniform on bounded t-intervals.

To prove this theorem we first show that its hypotheses imply resolvent convergence for all $r > 0$ with $r\omega < 1$. Then we modify an idea due to J. Kisynski [9] (see also [7]) according to which a convergent sequence of semigroups is viewed as a single semigroup on an appropriate space of convergent sequences. A different, more direct, proof yields a companion theorem where the asymptotic linearity of the sequence $\{E_n\}$ is not assumed, but the resolvent convergence of the operators for all positive r with $r\omega < 1$ is one of the hypotheses. Complete proofs of these two theorems, as well as several corollaries, examples and applications, can be found in [10]. These new versions of the nonlinear Trotter-Kato-Neveu theorem are nonlinear analogs of the recent results for semigroups of linear operators presented in [8]. They also include a result in an unpublished manuscript of Ph. Benilan, M. G. Crandall and A. Pazy, as well as the one space results in [3] and [7].

Turning our attention to nonlinear algorithms, we continue with the following nonlinear version of Chernoff's theorem.

Theorem 2. *Let $A+\omega I$ be an accretive operator in X satisfying $R(I+rA) \supset \mathrm{cl}(\mathrm{D}(A))$ for all $0 < r < r_0$, and let S be the semigroup generated by $-A$. Let $\{\rho_n\}$ be a sequence of positive numbers converging to 0, and for each n let $F(\rho_n)$ be a mapping from a closed convex subset C_n of X_n into itself. Suppose that*
(i) $|F(\rho_n)x_n - F(\rho_n)y_n|_n \leq \alpha_n|x_n - y_n|_n$ for all x_n and y_n in C_n, where $\alpha_n = 1+\omega\rho_n + \mathrm{o}(\rho_n)$;
(ii) $P_n\left(\mathrm{cl}(\mathrm{D}(A))\right) \subset C_n$ for each n;
(iii) $\lim_{n\to\infty} E_n\left(I + r(I - F(\rho_n))/\rho_n\right)^{-1} P_n x = J_r^A x$ for all x in $\mathrm{cl}(\mathrm{D}(A))$ and $0 < r < r_0$.

If $\{k_n\}$ is a sequence of integers such that $\lim_{n\to\infty} k_n\rho_n = t$, then $\lim_{n\to\infty} E_n F(\rho_n)^{k_n} P_n x = S(t)x$ for all x in $\mathrm{cl}(\mathrm{D}(A))$, and the convergence is uniform on bounded t-intervals.

This theorem is a nonlinear analog of the linear result presented in [13]. It includes the one space linear [5] and nonlinear [3] theorems. In its proof we use the companion theorem to Theorem 1 mentioned above and a lemma in [12]. A complete proof of Theorem 2, as well as an example of its applicability, can be found in [11].

Next we consider the converse of our results, namely the question of convergence versus resolvent consistency. In general Banach spaces they are not equivalent because there are semigroups which do not have unique generators [6]. But with some restrictions on X and X_n we can show that the convergence of semigroups does imply the convergence of resolvents by using the geometry of sufficiently nice Banach spaces. This shows that the hypotheses of our theorems are not only sufficient, but also necessary. We illustrate this with the following converse of Theorem 1.

Theorem 3. *Let X^* and X_n^* be uniformly convex dual Banach spaces with moduli of convexity $\delta_X(\varepsilon)$ and $\delta_n(\varepsilon)$ respectively. Let $A+\omega I$ be an accretive operator in X such that $R(I+rA) \supset \mathrm{cl}(\mathrm{D}(A))$ for $r > 0$ with $r\omega < 1$, and let S be the semigroup generated by $-A$. For each n, let $A_n + \omega I$ be an accretive operator in X_n such that $R(I + rA_n) \supset \mathrm{cl}(\mathrm{D}(A_n))$ for $r > 0$ with $r\omega < 1$, and let S_n be the semigroup generated by $-A_n$. Suppose that*
(i) the sequence $\{E_n\}$ is asymptotically linear;
(ii) $\delta(\varepsilon) = \inf\{\delta_X(\varepsilon), \delta_n(\varepsilon) : n \geq 1\}$ is positive;
(iii) $P_n\left(\mathrm{cl}(\mathrm{D}(A))\right) \subset \mathrm{cl}(\mathrm{D}(A_n))$ for each $n \geq 1$;
(iv) $\mathrm{cl}(\mathrm{D}(A))$ is convex;
(v) $\lim_{n\to\infty} E_n S_n(t) P_n x = S(t)x$ for $x \in \mathrm{cl}(\mathrm{D}(A))$ and the convergence is uniform on bounded t-intervals.

Then $\lim_{n\to\infty} E_n J_r^{A_n} P_n x = J_r^A x$ for $r > 0$ with $r\omega < 1$ and $x \in \mathrm{cl}(\mathrm{D}(A))$.

In the linear case (that is, when A, A_n, P_n and E_n are all linear), such a result is a consequence of Lebesgue's Dominated Convergence Theorem because $(I + rA)^{-1}x = \int_0^\infty e^{-rt} S(t)x\,dt$ for all x in X and positive r. In the nonlinear case the proof is more difficult. We use Banach limits and the "optimization method" for establishing strong convergence in infinite-dimensional Banach spaces. This method also yields a converse of Theorem 2. Complete proofs can be found in [10] and in [11]. For examples of other problems to which variants of this "optimization method" were applied see [14].

3. Conclusion

Finally, we mention that our ideas are useful in other settings too. We refer, in particular, to nonlinear Volterra integral equations and to functional differential equations, and to more general product formulas. In this way we can obtain extensions of the results in [1] and in [14]. It would also be of interest to quantify our results by obtaining rates of convergence.

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