STABILITY OF FUZZY CONTROL SYSTEMS VIA INTERVAL MATRIX METHOD

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ABSTRACT

Interest in the stability criterion for fuzzy control systems has grown in recent years. One of the most important difficulties with the creation of new stability criteria for any fuzzy control system has been the analytical interpretation of the linguistic part of the fuzzy controllers' IF-THEN rules. Often fuzzy control systems are being designed with very modest or no prior knowledge of a solid mathematical model which, in turn, makes it difficult to tap on relatively many tools for the stability of conventional control systems. In this paper with the help of Takagi-Sugeno fuzzy IF-THEN rules in which their consequents are analytical, a sufficiency condition is proposed to check the stability of fuzzy control systems. The scheme is based on the stability theory of interval matrices and is independent, but comparable to the Lyapunov approach. The scheme is numerically easy to check.

1. Introduction

One of the most fundamental issues in any control system-fuzzy or others is the stability. Briefly, a system is said to be stable if it would come to its equilibrium state after any external inputs, initial conditions and/or disturbances have impressed the system. The issue of stability is of even greater relevance when questions of safety, lives, and environment are at state like in such systems as nuclear reactors, traffic systems and airplanes autopilots, etc. The stability test of fuzzy control systems or lack of it has been a subject of criticism by many control engineers in some control engineering literature (IEEE, 1993).

Almost any linear or nonlinear system under the influence of a closed-loop conventional controller has one type of stability test or other. For example, the stability of a linear time-invariant system can be tested by a wide variety of methods such as Routh-Hurwitz, Root Locus, Bode Plots, Nyquist Criterion, and even through traditionally nonlinear systems methods of Lyapunov, Popov and Circle Criterion. The common requirement in all these tests is the availability of a mathematical model-be it in time or frequency domain. A reliable mathematical model for a very complex system, for example, may, in practice, be unavailable or unfeasible. In such cases, a fuzzy controller may be designed based on expert knowledge or experimental practice. However, the issue of the stability of a fuzzy control system still remains and must be addressed. The aim of this paper is to present a sufficiency condition to test for fuzzy control systems stability. Fuzzy controllers represent static nonlinearities (Jamshidi, 1996) and as such the stability problems belongs to nonlinear control systems. Next section, a brief survey of fuzzy control systems stability will be given. For a comprehensive survey consult the upcoming book by Jamshidi (1996).

2. Fuzzy control systems stability classes

From the viewpoint of stability, a fuzzy controller can be either acting as a conventional (low-level) controller or as a supervisory (high-level) controller (Jamshidi, 1996). Depending on the existence and nature of a system's mathematical model and the level in which fuzzy rules are being utilized for control and robustness, four classes or fuzzy control suitability problems can be distinguished. These four classes are (See Figure 2.1):

Class 1: Process model is crisp and linear and fuzzy controller is low level.

Class 2: Process model is crisp and nonlinear and the fuzzy controller is low level.

Class 3: Process model (linear or nonlinear) is crisp and a fuzzy tuner or an adaptive fuzzy controller is present at high level.

Class 4: Process model is fuzzy and fuzzy controller is low level.

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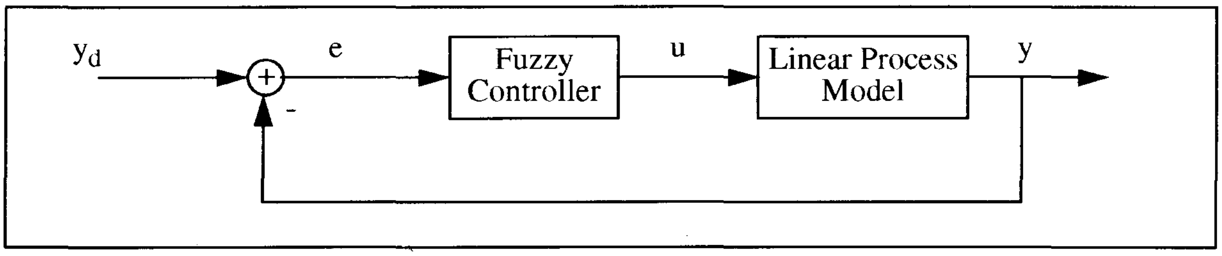
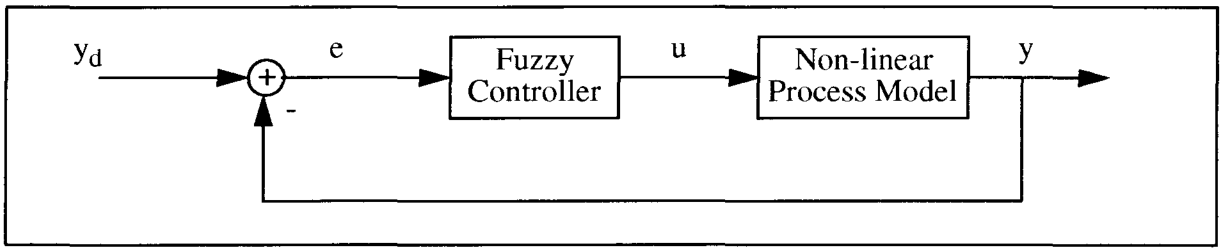
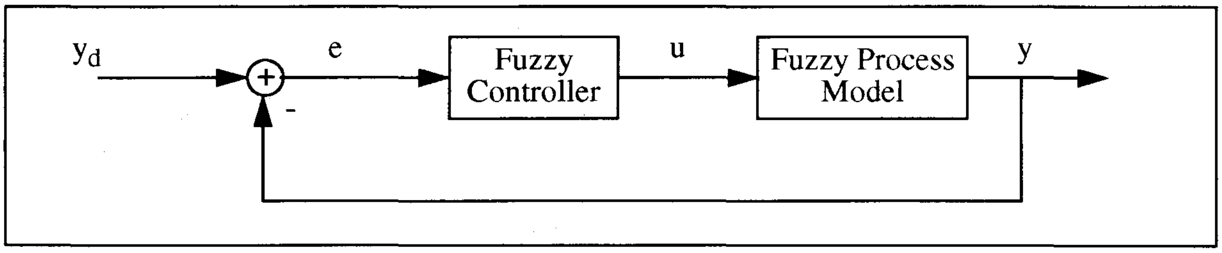


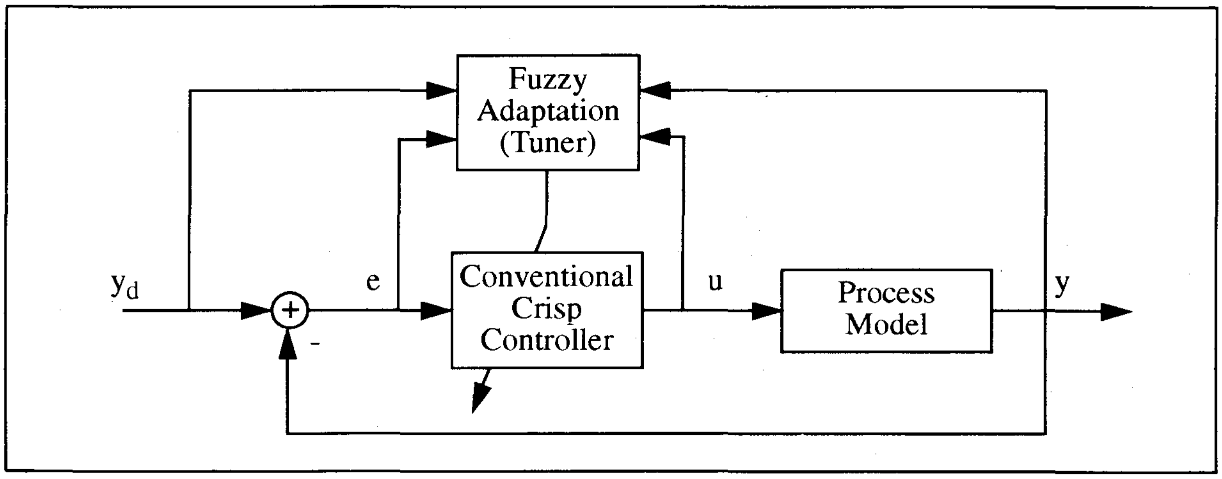
Class 1 of fuzzy control system stability problem.



Class 2 of fuzzy control system stability problem.



Class 3 of fuzzy control system stability problem.



Class 4 of fuzzy control system stability problem.

Figure 2.1 Stability Classes of Fuzzy Control Systems

In this paper we are concerned mainly with the first three classes. For the last class, traditional nonlinear control theory would fail and is beyond the scope of this paper. The techniques for testing the stability of the first two classes of systems are shown in Table 1. As shown, the methods are divided into two main groups-time and frequency.

Time Domain Methods

The time-domain methods are primarily based on the state-space approach. The basic approach here is to subdivide the state space into a finite number of cells based on the definitions of the membership functions. Now, if a separate rule is defined for every cell, a cell-to-cell trajectory can be constructed from the system's output induced by the new outputs of the fuzzy controller. If every cell of the modified state space is checked, one can identify all the equilibrium points including the system's stable region. This method should be used with some care since the inaccuracies in the modified description could cause oscillatory phenomenon around the equilibrium points.

Stability Analysis Methods	
Time Domain	Frequency Domain
State-space	Harmonic Balance
Lyapunov Theory	
Hyperstability Theory	Circle Criterion
Bifurcation Theory	
Graph Theory	Popov Criterion

TABLE 1 Stability Analysis Methods for Fuzzy Control Systems with Known Model

The second class of methods is based on the Lyapunov's method (Tanaka and Sugeno, 1992, Jamshidi, 1996). The approach is along the same lines as in classical approach of Lyapunov Stability, i.e. show that the time derivative of the Lyapunov function at the equilibrium point is negative semi definite. Many approaches have been proposed. One approach is to define a Lyapunov function and then derive the fuzzy controller's architecture out of the stability conditions. Another approach uses Aiserman's method (Jamshidi, 1996) to find an adopted Lyapunov function, while representing the fuzzy controller by a nonlinear algebraic function $u = f(y)$, when y is the system's output. A third method calls for the use of so-called facet functions, where the fuzzy controller is realized by boxwise multilinear facet functions with the system being described by a state space model. To test stability, a numerical parameter optimization scheme is needed.

Hyperstability approach has been used to check stability of first class of systems. The basic approach her is to restrict the
input-output behavior of the nonlinear fuzzy controller by inequality and to derive conditions for the linear part of the closed-loop system to be satisfied for stability.

Bifurcation theory (Jamshidi, 1996) can be used to check stability of fuzzy control systems of second class of systems This approach represents a tool in deriving stability conditions and robustness indices for stability from small gain theory. The fuzzy controller, in this case, is described by a nonlinear vector function. The stability, in this scheme, could only be lost if one of the following conditions become true: (i) the origin becomes unstable if a pole crosses the imaginary axis into the right-half plane-static bifurcation, (ii) the origin becomes unstable if a pain of poles would cross over the imaginary axis and assumes positive real parts-Hopf bifurcation, or (iii) new additional equilibrium points are produced.

Frequency Domain Methods

There are three primary groups of methods which have been considered her (see Table 1). The Harmonic Balance approach (Jamshidi, 1996) has been used to check the stability of the first two classes of fuzzy control systems. The main idea is to check if permanent oscillations occur in the system and whether these oscillations with known amplitude or frequency are stable. The nonlinearity (fuzzy controller) is described by a complex-valued describing function and the condition of Harmonic balance is tested. If this condition is satisfied, then a permanent oscillation exists. This approach is equally applicable to MIMO systems.

Circle criterion has been used to check stability of the first class of systems. In both criterion, certain conditions on the linear process model and static nonlinearity (controller) must be satisfied. It is assumed that the characteristic value of the nonlinearity remains within certain bounds, and the linear process model must be open-loop stable with proper transfer function. Both criteria can be graphically evaluated in simple manners.

The stability of adaptive fuzzy control systems has been treated by Jamshidi (1996) and is best used when it is augmented with the design process.

3. Fuzzy System Stability via Interval Matrix Method

Recent results on the stability of time-varying discrete interval matrices by Han and Lee (1994) can lead us to some more conservative, but computationally more convenient, stability criteria for fuzzy systems of the Takagi-Sugeno type shown below

Pi: IF x(k) is A1and . . . and x(k - n + 1) is An THEN
$x^i(k+1)=a1ix(k)+ . . . +anix(k-n+1)$ (3.1)

with i = 1, . . . ,p. Before we can state the new criteria some preliminary discussions will be necessary.

Consider a linear discrete-time system described by a difference equation in state form:

$$x(k + 1) = (A + G(k))\, x(k)\,,\ x(0) = xo \qquad (3.2)$$

where A is n x n constant asymptotically stable matrix, x is the n x 1 state vector, and G(k) is an unknown n x n time-varying on the perturbation matrix's maximum modulus, i.e.

$$|G(k)| < Gm \quad \text{for all } k \qquad (3.3)$$

where and the inequality holds elementwise. Now, consider the following theorem:

Theorem 3.1 The time-varying discrete-time system (3.2) is asymptotically stable if

$$\rho(\, |A| + Gm\,) < 1 \qquad (3.4)$$

where $\rho(\bullet)$ stands for maximum of the eigenvalues. The proof of this theorem is straight forward, based on the evaluation of the spectral norm $\|x(k)\|$ of x(k) and showing the if condition (3.4) holds, then lim $\|x(k)\| = 0$. The entire Proof can be found in Han and Lee (1994).

Definition 3.1 An interval matrix AI(k) is an n x n matrix whose elements consist of intervals [bij, cij] for i, j = 1, . . ., n, i.e.

$$AI(k) = \{\, [bij, cij] \,\}. \qquad (3.5)$$

Definition 3.2 The center matrix, Ac and the maximum difference matrix, Am of AI(k) in (3.5) are defined by

$$Ac = (B + C)/2 \qquad Am = (C - B)/2 \qquad (3.6)$$

where B = { bij} and C = { cij } . Thus, the interval matrix AI(k) in (3.5) can also be rewritten as

$$AI(k)= [Ac - Am\,,\, Ac + Am\,] = Ac + \Delta A(k) \qquad (3.7)$$

with $|\Delta A(k)| <= Am$.

Lemma 3.1 The interval matrix AI(k) is asymptotically stable if matrix Ac is stable and

$$\rho(\, |Ac| + Am\,) < 1 \qquad (3.8)$$

or in canonical form,

$$\rho\,(\, |T^{-1}|\, Ac\, |T| + |T^{-1}|\, Am\, |T|\,) < 1 \qquad (3.9)$$

The proof of this Lemma can also be found in Han and Lee (1994). The above Lemma can be used to check the sufficient condition for the stability of fuzzy systems of Takagi-Sugeno type given by

IF x(k) is Al and . . . and x(k - n + l) is An THEN
$x^{i}(k + 1) = Ai\, x(k)$
(3.10)

Consider a set of m fuzzy rules like (3.10), i.e.

IF x(k) is A11 and . . . and x(k - n + l) is An1 THEN
$x^{1}(k + 1) = A1\, x(k)$

.
. (3.11)
.

IF x(k) is A1m and . . . and x(k - n + l) is Anm THEN
$x^{m}(k + 1) = Am\, x(k)$

where Ai matrices for i = 1, . . ., m are defined by

$$Ai = \begin{bmatrix} a1i & a2i & \dots & ani \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots 1 & 0 \end{bmatrix} \qquad (3.12)$$

One can now formulate all the m matrices Ai , i = 1,..., m as an interval matrix of the form (3.5) by simply finding the minimum and maximum of all the elements at the top row of all the Ai matrices. In other words, we have

$$AI(k) = \begin{bmatrix} [a1\ a1] & \dots & & [an\ an] \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & \cdots & & \\ 0 & 0 & \dots 1 & 0 \end{bmatrix} \qquad (3.13)$$

where aiand ai for i = 1, . . ., m are the minimum and maximum of the respective elements of the first rows of Ai in (3.12). Using the above definitions and observations, the fuzzy system (3.11) can be rewritten in compact form as

$$\text{IF } x(k) \text{ is A THEN } x(k + 1) = AI(k)\, x(k) \qquad (3.14)$$

where the antecedent of the above rule corresponds to all the m rules of (3.11), i.e. x(k) corresponds to all m x(k)'s and A corresponds to all the m "and"- ed antecedents. Now, we consider the following lemma:

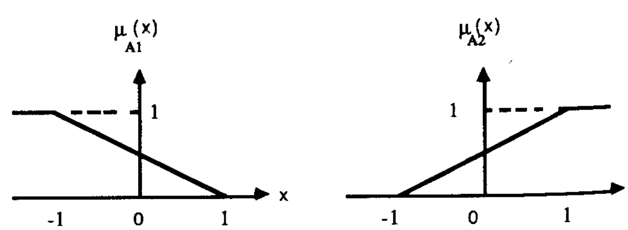


Figure 3.1 Fuzzy sets for the antecedents of Example 3.1

Lemma 3.2 The fuzzy system (3.14) is asymptotically stable if the interval matrix AI(k) in (3.13) is asymptotically stable, i.e., the conditions of Lemma 3.1 are satisfied.

Example 3.1 Consider the fuzzy system

p1: IF x(k - l) is A1 THEN $x^{1}(k + 1) = -\, x(k) - 0.2x(k - 1)$

p2: IF x(k - 1) is A2 THEN $x^{2}(k + 1) = x(k) - 0.2x(k - 1)$

where the fuzzy sets Ai, i = 1, 2 are shown in Figure 3.1 . It is desired to check the stability of this system.

SOLUTION: The system's two canonical matrices, written in the form of an interval matrix (3.13) is given by

$$AI(k) = \begin{bmatrix} [-1,1] & -0.2 \\ 1 & 0 \end{bmatrix}$$

The center and maximum difference matrices (see Equation (3.6) are

$$Ac = \begin{bmatrix} 0 & -0.2 \\ 1 & 0 \end{bmatrix}$$

and

$$Am = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Then, condition (3.8) would become,

$$\rho\, (\, |Ac| + Am\,) \; = \; 1.17 \; > 1 \qquad (3.8)$$

Thus, the stability of fuzzy system under consideration is inconclusive. In fact, it can be shown that while the individual rules do represent stable discrete-time systems, the combined two-rule fuzzy system is unstable (Jamshidi, 1996).

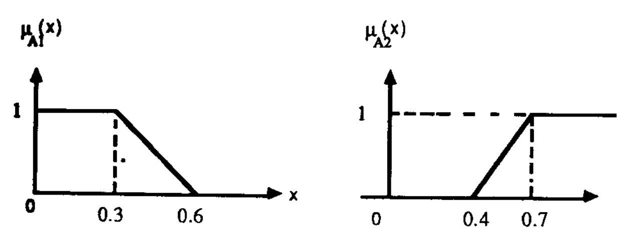


Figure 3.2 Fuzzy sets for the antecedents of Example 3.2

Example 3.2 Consider a two-rule fuzzy system

p1: IF x(k) is A1 THEN $x^1(k + 1) = 0.3x(k) + 0.5x(k - 1)$

p2: IF x(k) is A2 THEN $x^2(k + 1) = 0.2x(k) + 0.2x(k - 1)$

where Ai, i = 1, 2, are piecewise continuous fuzzy sets in Figure 3.2.

It is desired to check if this system is stable by interval matrix method.

SOLUTION: The Ai matrices are

$$A1 = \begin{bmatrix} 0.3 & 0.5 \\ 1 & 0 \end{bmatrix} \quad A2 = \begin{bmatrix} 0.2 & 0.2 \\ 1 & 0 \end{bmatrix}$$

Consider the center and the maximum difference matrices for this system

$$Ac = \begin{bmatrix} 0.25 & 0.35 \\ 1 & 0 \end{bmatrix}$$

and

$$Am = \begin{bmatrix} 0.05 & 0.15 \\ 0 & 0 \end{bmatrix}$$

and matrix

$$|Ac| + Am = \begin{bmatrix} 0.3 & 0.5 \\ 1 & 0 \end{bmatrix}$$

and $\rho(\, |Ac| + Am\,) = 0.873 < 1$. Hence, in view of the above inequality and the fact that Ac is stable, by Lemma 3.2 the interval matrix

$$AI(k) = \begin{bmatrix} [0.2,0.3] & [0.2,0.5] \\ 1 & 0 \end{bmatrix}$$

is asymptotically stable and the fuzzy system is stable.

4. Conclusions

In this brief paper a sufficiency condition has been presented to check for the asymptotic stability of fuzzy control systems with Takagi-Sugeno type rules, i.e. Equation (3.1). This criterion presented here is somewhat conservative. It is noted that if condition (3.8) or (3.9) is not satisfied it does not mean that the system is necessarily unstable. On the other hand, if the condition is true, then the system is, in fact, stable. This paper hopes to serve as a starting point for many new results to come toward a solid stability theory for fuzzy control systems. This challenge still exists for both control engineers and mathematicians.

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