Approximation of Stochastic Evolution Equations in Hilbert Spaces *

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Abstract In this paper we consider the Ito's stochastic differential equation in Hilbert spaces. We discuss and analyze several difference approximations in time. Applications to the the Zakai equation and the Kushner equation in nonlinear filtering theory are presented.

1 Introduction

Consider an abstract Ito's stochastic differentiable equation

(1.1) $dx(t)+Ax(t)\,dt = Bx(t)\,dw(t),\ x(0)=x\in H,$

where H is a Hilbert space and A is a single-valued maximal monotone operator on H and $w(t)$ is a Wiener process in a separable Hilbert space E with nuclear covariance operator Q. Let $(\Omega,\mathcal{F},P)$ be the probability space with an increasing family of sub σ algebra $\{\mathcal{F}_t\}$ of $\mathcal{F}$ that is right continuous and complete with respect to the probability P. Assume that $w(t)$ is $\mathcal{F}_t$-adapted and the initial condition an H-valued, $\mathcal{F}_0$ measurable random variable. Assume for $x\in$dom(A) $Bx\in\mathcal{L}(E,H)$. In Section 2 we consider the time discretization schemes of (1.1)

(1.2) $x_k-x_{k-1}+\Delta t\,Ax_k = Bx_{k-1}\,(w(t_k)-w(t_{k-1})),$

where $\Delta t>0$, $t_0=0$ and $t_k=k\Delta t$ and summarize the results in [It1] concerning the convergence of the discrete-time solution $\{x_k\}$ to a unique strong solutions to (1.1) under appropriate conditions on the operators A and B.

The stochastic partial differential equation (1.1) arises in the nonlinear filtering problem as the so-called Zakai equation and Kushner equation. The Zakai equation is linear and the abstract formulation of Section 2 is applied to the Zakai equation. However, the Kushner equation can not be treated under the abstract formulation because of its indefinite cubic nonlinearity. We will sketch a construction of nonnegative solutions to the Kushner equation by a difference approximation.

The other objective is to develop the higher order difference approximation for (1.1) based on the operator splitting methods when A, B are linear. In order to develop higher order (3rd and 4th order) methods it is necessary to use the following Winner integral

$$\Delta z_k = \int_{t_{k-1}}^{t_k}(s-\frac{t_{k-1}+t_k}{2})\,dw(s)$$

besides the increment $\Delta w_k = w(t_k)-w(t_{k-1})$ of the Winner process w. Note that 1, $(s-\frac{t_{k-1}+t_k}{2})$ are the first two Taylor elements at the midpoint $t_{k-\frac{1}{2}}=\frac{t_{k-1}+t_k}{2}$. A brief discussion of such constructions is given in Section 4.

2 Stochastic Evolution Equations

In this section we summarize the results in [It1] for the stochastic evolution equation (1.1) and the difference approximation (1.2). Let $S=((0,T)\times\Omega,\mathcal{B}\times\mathcal{F},dt\times dP)$. For the initial condition $x\in$dom(A) we define the solution of (1.1) as follows.

Definition A continuous $\mathcal{F}_t$-adapted H-valued process $x(t)$ such that $Ax(t)\in L^2(S;H)$ is a solution of (1.1) if

$$(2.1)\qquad x(t) = x-\int_0^t Ax(s)\,ds+\int_0^t Bx(s)\,dw(s)$$

for $t\in[0,T]$, where the stochastic integral is defined in the Ito's sense (e.g., see [KR]).

This definition corresponds to the strong solution for the deterministic case. Compare it with the weak or variational solution of [KR] (also, see Section 2.1).

Assume the monotonicity condition: for $x_1,\ x_2\in$dom(A)

$$(\mathrm{C}_1)\qquad \begin{aligned}&(Ax_1-Ax_2,\,x_1-x_2)-\frac{1}{2}\,|Bx_1-Bx_2|_Q^2\\ &+\rho\,|x_1-x_2|_H^2\ \geq\ 0\quad\text{for some }\rho\geq 0.\end{aligned}$$

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*Research supported in part by the Air Force Office of Scientific Research under grant AFOSR F49620-93-1-0198.

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Theorem 2.1 Assume C_1 holds. Then, the strong solution to (1.1) is unique.

Next, we assume the boundedness condition: for $x \in \mathrm{dom}(A)$

$$(C_2) \qquad 2\,(Ax,\, x) - |Bx|_Q^2 + \rho\,|x|_H^2 + \alpha \geq 0$$

for some $\rho,\ \alpha \geq 0$.

Then, the discrete-time solution $\{x_k\}$ to (1.2) is defined and satisfies

$$(2.2) \qquad \begin{aligned} & E\,(|x_k|^2 + \Delta t\,|Bx_k|_Q^2) + \sum_{k=1}^{m} 2\Delta t\,(Ax_k,\, x_k\,) \\ & \leq e^{2\rho T}\,(E\,(|x_0|^2 + \Delta t\,|Bx_0|_Q^2) + \alpha) \end{aligned}$$

for $1 \leq k \leq m$, independent of $\Delta t > 0$, where $\Delta t = \frac{T}{m}$. But, it is not sufficient to prove the convergence of $\{x_k\}$. We need *a priori* estimate for the sequence $\{x_k\}$ generated by (1.2); given $x \in \mathrm{dom}(A)$ there exist a constant $\delta > 0$ such that

$$(C_3) \quad E\sum_{k=1}^{m} \Delta t\,|Ax_k|^2 \leq \delta \quad \text{uniformly in } \Delta t > 0.$$

Define a function $x_\lambda(t)$ by

$$x_\lambda(t) = x_k \quad \text{if } t \in [k\lambda, (k+1)\lambda),\ k \geq 0.$$

where $\lambda = \Delta t$. The following theorem is proved in [It1].

Theorem 2.2 Assume that the conditions (C_1)-(C_3) hold. Then the equation (1.1) has the unique solution $x(t)$. Moreover, the sequence generated by (1.2) converges to $x(t)$ in the sense that

$$Ax_\lambda(t) \to Ax(t) \text{ weakly in } L^2(S; H)$$

$$x_\lambda(t) \to x(t) \text{ weakly star in } L^\infty(0, T; L^2(\Omega; H)),$$

as $\lambda = \Delta t \to 0$ and we have

$$(2.3) \qquad \begin{aligned} & E\,|x(t)|^2 - E\,|x|^2 \\ & +E\int_0^t (2\,(Ax(s),\, x(s)) - |Bx(s)|_Q^2)\,ds = 0. \end{aligned}$$

We refer to [It1] for examples for which Theorem 2.2 are applied.

2.1 Stochastic Evolution Equations under Gelfand Triple

In order to prove the convergence of the sequence $\{x_k\}$ generated by (1.2) without condition C_3 we can use the Gelfand triple formulation as in [KR]. Assume that $A = A_1 + A_2$ where A_i is hemicontinuous, monotone, bounded operator on Banach space V_i for each $i = 1,\ 2$, where $V_i,\ 1 =, 2$ are real, separable, reflexive Banach spaces and densely and continuously embedded into H. Let $V = V_1 \cap V_2$. Then V is a Banach space with norm $|x|_V = |x|_{V_1} + |x|_{V_2}$. Assume A satisfies

(A_1) Hemicontinuity of A:

the function $\langle A(x_1 + \lambda\,x_2),\, x\rangle$ is

continuous in λ on R,

(A_2) Monotonicity of (A, B):

$$\langle Ax_1 - Ax_2,\, x_1 - x_2\rangle - \tfrac{1}{2}\,|Bx_1 - Bx_2|_Q^2$$

$$+\rho\,|x_1 - x_2|_H^2 \geq 0,$$

(A_3) Coercivity of (A, B):

$$\langle Ax,\, x\rangle - \tfrac{1}{2}\,|Bx|_Q^2 + \rho\,(1 + |x|_H^2)$$

$$\geq \alpha\,(|x|_{V_1}^{p_1} + |x|_{V_2}^{p_2}),$$

(A_4) Boundedness of A:

$$|A_i x|_{V_i^*} \leq M\,(\tfrac{1}{2} + |x|_{V_i}^{p_i - 1}) \text{ foe each } i.$$

Then, the following theorem is proved in [It1].

Theorem 2.3 Assume that $A = A_1 + A_2$ with A_i hemicontinous, monotone, bounded operator on V_i for $i = 1,\ 2$ and that $(A_1) - (A_4)$ hold. Then equation (1.1) has the unique solution $x(t) \in L^{p_1}(S; V_1) \cap L^{p_2}(S; V_2) \cap L^\infty(0, T; L^2(\Omega; H))$. Moreover the sequence generated by (1.2) converges to $x(t)$ in the sense that

$$Ax_\lambda(t) \to Ax(t) \text{ weakly in } L^q(S; V^*) \quad \text{and}$$

$$x_\lambda(t) \to x(t) \text{ weakly in } L^p(S; V)$$

$$\text{and weakly star in } L^\infty(0, T; L^2(\Omega; H)),$$

as $\lambda = \Delta t \to 0$ and we have (2.3).

Theorem 2.3 can be applied to the following example. Let $A_1 = -\Delta \in \mathcal{L}(H_0^1(D), H^{-1}(D))$ and $A_2\phi = |\phi|^{p-2}\phi$ on $L^p(D)$ where D is a bounded open set in R^d with sufficiently smooth boundary and Δ denotes the Laplacian. The operator B is, for example, defined by $B\phi = \sigma\,(\nabla\phi, |\phi|^{r-1}\phi) \in R^{d+1}$ where $r = p/2$ and σ is an appropriately chosen constant . Setting $V_1 = H_0^1(D)$, $V_2 = L^p(D)$, $H = L^2(D)$, $E = R^{d+1}$ and $p_1 = 2$, $p_2 = p$ one can show that the conditions in Theorem 2.3 are satisfied. Compare this result with Theorem 2.2 that defines the strong solution.

3 Nonlinear Filtering Equations

Applications of the stochastic dynamics (1.1) include the Zakai equation and the Kushner equation in the nonlinear filtering problem [Ro] as follows. A signal process $x(t) \in R^d$ satisfies the Ito stochastic differential equation

(3.1) $dx(t) = g(x(t))\,dt + \sigma(x(t))\,dw_1(t), \quad x(0) = x,$

and the observation process $y(t) \in R^p$ is given by

(3.2) $\qquad dy(t) = h(x(t))\,dt + dw_2(t), \quad y(0) = 0.$

Assume that $w_1(t)$, $w_2(t)$ are $\mathcal{F}_t$-adapted independent Wiener process with covariance I and R, respectively. The initial condition x is a R^d-valued, $\mathcal{F}_0$ measurable random variable with probability density $\pi_0(x)$. Assume that the functions g, σ and h are bounded and that g, σ are Lipschitz. Then the unnormalized conditional probability density function $p(t) = p(t,x)$:

$$E\left[\phi(x(t)) \mid y(s),\ 0 \le s \le t\right] = \frac{\int \phi(x)p(t,x)dx}{\int p(t,x)dx}.$$

satisfies the Zakai equation

(3.3) $\quad dp(t) + Ap(t)\,dt = Bp(t)\,dy(t), \quad p(0) = \pi_0,$

where

(3.4) $$-A\phi = \frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial}{\partial x_j}\phi) - \frac{\partial}{\partial x_i}(a_i\phi)$$

and

(3.5) $$B\phi = hR^{-1}\phi$$

with

$$a = \frac{1}{2}\sigma\sigma^* \text{ and } a_i = g_i - \frac{\partial}{\partial x_j}a_{i,j}.$$

The Zakai theory is based on the change of probability measure [Ro]. Let $\eta(t)$ be a stochastic process defined by

$$\eta(t) = \exp\left(-\int_0^t h^*(x)R^{-1}dw_2(s) - \frac{1}{2}\int_0^t h^*(x)R^{-1}h(x)\,ds\right).$$

Define the probability measure $\tilde{P}$ on $(\Omega, \mathcal{F})$ by

$$\left.\frac{d\tilde{P}}{dP}\right|_{\mathcal{F}_t} = \eta(t).$$

Then the observation process $y(t)$ becomes an $\mathcal{F}_t-$ adapted Wiener process with covariance R on $(\Omega, \mathcal{F}, \tilde{P})$.

The normalized probability density

$$\pi(t,x) = p(t,x)/\int_{R^d} p(t,x)\,dx$$

satisfies the Kushner equation

(3.6)
$$d\pi(t) + A\pi(t)\,dt = (h\pi(t) - \pi(t)\int_{R^d} h\pi\,dx)\,(dy(t) - \int_{R^d} h\pi\,dx),$$

with $\pi(0) = \pi_0$.

In [It2] we analyze the convergence of numerical approximations of the Zakai equation (3.3) by choosing

$$H = L^2(R^d) \text{ and } V = H^1(R^d).$$

Assume that there exist positive constants ρ, β such that

$$\langle A\phi,\,\phi\rangle - \frac{1}{2}\,|B\phi|_R^2 + \rho\,|\phi|_H^2 \ \ge\ \frac{1}{2}\beta\,|\phi|_V^2 \quad \text{for all } \phi \in V.$$

Let Δp_k, $1 \le k \le m$ be the approximation error

$$\Delta p_k = p_k - p(t_k) \text{ at } t_k = k\,\Delta t$$

with $\Delta t = \frac{T}{m}$. In [It2] we obtain the following convergence rate of the Euler scheme (1.2) of the Zakai equation:

(3.7) $$p_k - p_{k-1} + \Delta t\,Ap_k = B\Delta y_k.$$

Theorem 3.1 Assume that the solution $p(t)$ of the Zakai equation (3.3) satisfies the following regularity:

$$\tilde{E}\,|Bp(t)|_V^2 \le M_1 \text{ and } \int_0^T \tilde{E}\,|Ap(t)|_V^2\,dt \le M_1$$

for some $M_1 > 0$, independent of $t \in [0,T]$. Then the Euler scheme (3.7) is of first order in the sense that

$$\tilde{E}\,|\Delta p_k|_H^2 + \frac{\beta}{2}\,\tilde{E}\sum_{j=1}^k \Delta t e^{2\rho(t_k - t_j)}\,|\Delta p_j|_V^2 \le M_2\,\frac{e^{2\rho T}-1}{2\rho}\Delta t,$$

for $1 \le k \le m$ and some $M_2 > 0$.

In [It2] we also consider the time discretization of the Zakai equation (3.3) based on the Milshtein approximation of the Ito stochastic integral:

$$\begin{aligned} &p_k - p_{k-1} + \Delta t\, A p_k \\ (3.8) \quad &= (B \Delta y_k + \frac{1}{2} BB(\Delta y_k^2 - \Delta t))\, p_{k-1}. \end{aligned}$$

and obtain the convergence rate estimate:

Theorem 3.2 Assume that the solution $p(t)$ of the Zakai equation (3.3) satisfies the regularity

$$\tilde{E}\, |Bp(t)|_V^2 \le M_3 \text{ and } \tilde{E}\, |Ap(t)|_V^2 \le M_3.$$

Then, the Milshtein scheme (3.8) is of second order in the sense that

$$\tilde{E}\, |\Delta p_k|_H^2 + \tfrac{\beta}{4}\, \tilde{E}\, \textstyle\sum_{j=1}^k \Delta t e^{2\rho(t_k - t_j)} |\Delta p_k|_V^2$$

$$\le M_4 \frac{e^{2\rho T} - 1}{2\rho} \Delta t^2$$

for $1 \le k \le m$ and some $M_4 > 0$.

Remark: The regularity assumptions on solutions to the Zakai (3.3) can be verified under certain smoothness assumptions on the function g, σ, h and b and the initial condition π_0 (e.g., see [Ro]). In [It1] the case when $w_1(t)$, $w_2(t)$ are correlated and spatial approximations of the Zakai equation are also investigated.

3.1 Operator Splitting Methods

Consider the approximation scheme based on the Trotter product formula (e.g., [BGR],[FL],[Pi]):

$$(3.9) \quad p_k - p_{k-1} + \Delta t\, A p_k = (e^{(B\, \Delta y_k - \frac{\Delta t}{2} BB)} - I)\, p_{k-1},$$

where

$$\hat{p}(t) = T^B(\Delta t)\, p_{k-1} = e^{(B\, \Delta y_k - \frac{\Delta t}{2} BB)}\, p_{k-1}$$

is the semigroup generated by the Winner process $y(\cdot)$ and satisfies

$$\hat{p}(t) - p_{k-1} = \int_{t_{k-1}}^t B \hat{p}(s) dy(s), \quad t \ge t_{k-1}.$$

Thus, (3.9) is equivalently written as

$$p_k = (I + \Delta t A)^{-1} T^B(\Delta t) p_{k-1}$$

where $(I + \Delta t A)^{-1}$ is the Hile approximation of the semigroup $S(t)$, $t \ge 0$ generated by $-A$ on H. The Milshtein scheme is closely related to (3.9) in the sense that

$$T^B(\Delta t) = \sum_{k=1}^{\infty} \frac{1}{k!} B^k \sqrt{\Delta t}^k\, H_k(\frac{\Delta y}{\sqrt{\Delta t}}).$$

where $H_k(\cdot)$ is the k-th Hermite polynomial on R, and that the term

$$p_{k-1} + (B \Delta y_k + \frac{1}{2} BB(\Delta y_k^2 - \Delta t))\, p_{k-1}$$

appears in the Milshtein scheme (3.8) is the second order approximation of $T^B(\Delta t)\, p_{k-1}$. In fact, it can be proved that the splitting scheme (3.9) is of second order and Theorem 3.2 also holds for the sequence generated by (3.9).

3.2 Approximation of the Kushner equation

In this section we give a difference method that overcomes the difficulty of the indefinite cubic nonlinearity of the Kushner equation. Consider the difference scheme:

(3.10)

$$\pi_k + \Delta t\, (A + \pi_{k-1}[h]\, (h - \pi_{k-1}[h]))\, \pi_k = \xi_k$$

$$\xi_k(x) = e^{(h(x) - \pi_{k-1}[h]) \Delta y_k - \frac{1}{2} (h(x) - \pi_{k-1}[h])^2}$$

where $\Delta t = \frac{T}{m}$, $\Delta y_k = y_{t_k} - y_{t_{k-1}}$ and

$$\pi_{k-1}[h] = \frac{\int h(x) \pi_{k-1}\, dx}{\int \pi_{k-1}\, dx}$$

The scheme (3.10) is again splitting the deterministic part and the stochastic part.

It can be shown that the sequence $\{\pi_k\}$ generated by (3.10) satisfies

$$\pi_k \ge 0 \text{ a.e. in } S$$

and uniformly integrable on S. Based on this fact, it can be proved that the sequence $\{\pi_k\}$ converges to the unique nonnegative solution to the Kushner equation in the sense of Theorem 2.3.

4 Higher Order Difference Methods

In this section we discuss higher order approximations to the Zakai equation. Both the Milshtein scheme (3.8) and operator splitting scheme (3.9) are based on the increment $\Delta y_k = y_{t_k} - y_{t_{k-1}}$ of the Winner process $y(t)$ and of second order. In order to develop a higher order scheme we need to use the additional Winner integral

$$\Delta z_k = \int_{t_{k-1}}^{t^k} (s - \frac{t_{k-1} + t_k}{2})\, dy(s).$$

This point is clearly seen from the following discussion. Given $f \in L^2(0,T)$ (deterministic), consider the Winner integral

$$X = \int_0^T f(s)\, dy(s).$$

If we approximate this just by the increment of the Winner process y, i.e.,

$$\hat{X}_1 = \sum_{k=1}^{m} f(t_{k-\frac{1}{2}})\Delta y_k$$

then we have the estimate

$$\tilde{E}\,|X - \hat{X}_1|^2 \le \frac{T}{12}(\Delta t)^2\,|f'|_\infty.$$

On the other hand, if approximate X by

$$\hat{X}_2 = \sum_{k=1}^{m} f(t_{k-\frac{1}{2}})\Delta y_k + f'(t_{k-\frac{1}{2}})\Delta z_k$$

then we have the estimate

$$\tilde{E}\,|X - \hat{X}_2|^2 \le \frac{T}{80}(\Delta t)^4\,|f''|_\infty.$$

Note that $\hat{X}_2$ is the Wiener integral on $[0,T]$ when the function f is replaced by the first Taylor polynomial of f at $t_{k-\frac{1}{2}}$

$$f(t_{k-\frac{1}{2}}) + f'(t_{k-\frac{1}{2}})\,(s - t_{k-\frac{1}{2}})$$

on each subinterval (t_{k-1}, t_k).

Now, we extend this idea to the Zakai equation. First, note that

$$\tilde{E}\,|\Delta y_k|^2 = O(\Delta t) \text{ and } \tilde{E}\,|\Delta z_k|^2 = O(\Delta t)^3.$$

We construct the third order method by substituting the Milshtein approximation into the Picard iterate, i.e.,

$$\begin{aligned} p_k^{3rd} &= S(\Delta t)p_{k-1} \\ &+ \int_0^{\Delta t} S(t-s)BS(s)(I + B(y(s+t_{k-1}) - y(t_{k-1})) \\ &+ \frac{1}{2}BB((y(s+t_{k-1}) - y(t_{k-1}))^2 - s))p_{k-1}\, dy(s+t_{k-1}) \end{aligned}$$

where $S(t) = S^{-A}(t)$, $t \ge 0$ is the semigroup generated by $-A$ on H. Then, if we approximate Winner integral in the order $\sqrt{\Delta t}^3$ we obtain the formula

(4.1)
$$p_k = \left[S(\Delta t)\sum_{k=0}^{3} \frac{1}{k!} B^k \sqrt{\Delta t}^k H_k\left(\frac{\Delta y_k}{\sqrt{\Delta t}}\right) + C\Delta z_k \right] p_{k-1}$$

where we use the fact that

$$\begin{aligned} C &= \frac{d}{ds}\left(S(t-s)BS(s)\right)(t_{k-\frac{1}{2}}) \\ &= AS(\frac{\Delta t}{2})BS(\frac{\Delta t}{2}) - S(\frac{\Delta t}{2})BS(\frac{\Delta t}{2})A. \end{aligned} \tag{4.2}$$

We can repeat this procedure again to obtain the forth order method, i.e.,

$$\begin{aligned} p_k^{4th} &= S(\Delta t)p_{k-1} \\ &+ \int_0^{\Delta t} S(t-s)BS(s) \\ &\times \left(\sum_{k=0}^{3} \frac{1}{k!} B^k \sqrt{s}^k H_k\left(\frac{y(s+t_{k-1}) - y(t_{k-1})}{\sqrt{s}}\right)\right. \\ &+ \left[AS(\frac{s}{2})BS(\frac{s}{2}) - S(\frac{s}{2})BS(\frac{s}{2})A\right] \\ &\times \int_0^s (\sigma - \frac{\Delta t}{2})\, dy(\sigma + t_{k-1})\, p_{k-1})\, dy(s + t_{k-1}). \end{aligned}$$

Then, if we approximate Winner integral in the order $\sqrt{\Delta t}^4$ we obtain the formula

(4.3)
$$\begin{aligned} p_k &= \left[S(\Delta t)\sum_{k=0}^{4} \frac{1}{k!} B^k \sqrt{\Delta t}^k H_k\left(\frac{\Delta y_k}{\sqrt{\Delta t}}\right) + C\Delta z_k \right] p_{k-1} \\ &+ \frac{1}{2}(CB + BC)\Delta y_k \Delta z_k\, p_{k-1} \end{aligned}$$

where we used the fact that

$$\begin{aligned} &\int_0^{\Delta t} (s - \frac{\Delta t}{2}) \int_0^s dw(\sigma)\, dw(s) \\ &= \int_0^{\Delta t}\int_0^s (\sigma - \frac{\Delta t}{2})\, dw(\sigma)\, dw(s) \\ &= \frac{1}{2}w(t)\int_0^{\Delta t} (s - \frac{\Delta t}{2})\, dw(s). \end{aligned}$$

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