AN APPROXIMATE FOURIER EXPANSION WITH UNCORRELATED COEFFICIENTS

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ABSTRACT

In any transform coding scheme the central operation is the reduction of correlation and thereby, with appropriate coding of the transform coefficients, allows data compression to be achieved. The transform efficiency and ease of implementation are to a large extent mutually incompatible. There are various orthogonal transforms such as Karhunen-Loeve, discrete cosine, Haar, discrete Fourier etc., but the application depends upon the criteria applicable in any particular case. In this paper, we present a discrete approximate Fourier expansion (AFE) of stationary and non-periodic signals with (theoretically) uncorrelated coefficients. Some mathematical properties are derived and preliminary simulation results compare the performance of such an expansion with that of discrete cosine transform.

1. INTRODUCTION

Transform coding is one of the well known approaches to efficient waveform representation at medium to low bit rates. The first step in transform coding, the transform, represents a method of converting a signal into a form that is more amenable to quantization. This step is typically considered a "decorrelation step" to allow each of the coefficients to be quantized independently of the others. The efficiency of a compression algorithm is measured by it's data compressing ability, the resulting distortion, as well as by the implementation complexity. The optimal decorrelation transformation is the Karhunen-Loeve transform (KLT) [1]. The KLT has the property that for any integer $M \leq N$ where M is the size of the transform and N is the size of data vector, it packs the maximum average energy into some M coefficients [2]. Unfortunately, no efficient computation of the KLT exists, and also it does not have desirable properties of trigonometric series. Another complication in applying KLT is that its basis functions are not fixed but are data dependent. For stationary random sequences there are other unitary transforms which approach to energy packing efficiency of the KLT. Examples are discrete cosine, Fourier and sine transforms. These transforms are members of a large family of sinusoidal transforms all of which have a performance equivalent to the KLT as the size N of the data vector approaches infinity [3]. For continuous-time non-periodic signals, an approximate trigonometric series expansion with uncorrelated coefficients is briefly discussed in reference [4]. The approximation can be made arbitrarily close by making a transform parameter small enough. In this paper we extend this expansion to sampled signals and we explore the capability of coding 1-D signals and images.

The paper is organized as follows: In section 2, we present the approximate Fourier expansion for sampled stationary and non-periodic signals and we show that the resulting AFE coefficients are uncorrelated. In section 3, some mathematical properties of this expansion will be presented, and section 4 presents an error analysis of the expansion. In section 5, we determine the transform efficiency by computing the decorrelation efficiency of the expansion. Finally, section 6 presents experimental results followed by conclusion in section 7.

2. DISCRETE APPROXIMATE FOURIER EXPANSION

For continuous-time non-periodic signals, an approximate Fourier expansion can be expressed as [4]:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_o t} \tag{1}$$

where c_k are random variables given by:

$$c_k = \int_{-\infty}^{\infty} x(t) \frac{\sin\left(\frac{\omega_o t}{2}\right)}{\pi t} e^{-jk\omega_o t} dt \tag{2}$$

We extend the approximate Fourier expansion to sampled signals as:

$$\hat{x}(n) = \sum_{k=-M}^{M} c_k e^{jk\omega_o n} \tag{3}$$

where:

$$c_k = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin\left(\frac{\omega_o n}{2}\right)}{\pi n} e^{-jk\omega_o n}, \quad k = 0, \pm 1, .., \pm M \tag{4}$$

We will verify the validity of such an expansion later. It will be shown later that depending on the value of ω_o, the coefficients c_k could be periodic or pseudo-periodic. Since ω_o is a user defined parameter, we can write $\omega_o = \frac{2\pi}{L}$ where

L is a positive real number. The AFE for discrete signals can now be written as:

$$\hat{x}(n) = \sum_{k=0}^{M} c_k e^{j\frac{2\pi kn}{L}} \tag{5}$$

$$c_k = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin\left(\frac{\pi n}{L}\right)}{\pi n} e^{-j\frac{2\pi kn}{L}} \tag{6}$$

where the value of M is yet to be determined.

For two-dimensional signals we can consider a separable extension of (5) and (6). If k_1 and k_2 are integers and assuming the same $\omega_o = \frac{2\pi}{L}$ along both frequency axes, then in 2-dimensions, the AFE pair can be extended as follows:

$$\hat{x}(m,n) = \sum_{k_1=0}^{M} \sum_{k_2=0}^{M} c_{k_1,k_2} e^{j\frac{2\pi}{L}(k_1 m + k_2 n)} \tag{7}$$

$$c_{k_1,k_2} = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} x(m,n) \frac{\sin\left(\frac{\pi m}{L}\right)\sin\left(\frac{\pi n}{L}\right)}{\pi^2 mn} e^{-j\frac{2\pi}{L}(k_1 m + k_2 n)} \tag{8}$$

3. PROPERTIES OF THE EXPANSION

In this section, we will examine some important properties of this expansion. We will consider only the case of 1-D sampled signals.

a) Mean value of coefficients:

Property:

$$E\{c_k\} = \begin{cases} E\{x(n)\} & for \quad k = 0 \\ 0 & otherwise \end{cases} \tag{9}$$

Proof:

$$E\{c_k\} = E\{x(n)\} \sum_{k=-\infty}^{\infty} \frac{\sin(\omega_o n/2)}{\pi n} e^{-jk\omega_o n} \tag{10}$$

The summation term in (10) represents the Fourier transform of $\frac{\sin\left(\frac{\omega_o n}{2}\right)}{\pi n}$ evaluated at $\omega = k\omega_o$. Since this is zero except when $k = 0$ where it has the value of 1, the property is proved.

b) Correlation of coefficients:

Property: The coefficients of the AFE are uncorrelated.

Proof: Consider the ideal low-pass filters of center frequency $k\omega_o$ and bandwidth ω_o:

$$H_k(w) = \begin{cases} 1 & \left(k - \frac{1}{2}\right)\omega_o < w < \left(k + \frac{1}{2}\right)\omega_o \\ 0 & otherwise \end{cases} \tag{11}$$

where $-\pi \le \omega \le \pi$ is a discrete frequency variable. The corresponding impulse response is given by:

$$h_k(n) = e^{jk\omega_o n} \frac{\sin\left(\frac{\omega_o n}{2}\right)}{\pi n} \tag{12}$$

If $x(n)$ is the input to this filter, the output of k will be:

$$y_k(n) = \sum_{\tau=-\infty}^{\infty} x(\tau) e^{jk\omega_o(n-\tau)} \frac{\sin\left[\frac{\omega_o(n-\tau)}{2}\right]}{\pi(n-\tau)} \tag{13}$$

At $n = 0$ we have:

$$y_k(0) = \sum_{\tau=-\infty}^{\infty} x(\tau) e^{-jk\omega_o \tau} \frac{\sin\left(\frac{\omega_o \tau}{2}\right)}{\pi\tau} = c_k \tag{14}$$

Since individual filters are non-overlapping, their outputs c_k and c_m are orthogonal i.e.,

$$E\{y_k(0)y_m^*(0)\} = E\{c_k c_m^*\} = 0, \quad for \quad k \ne m \tag{15}$$

Since from property (a), $E\{c_k\} = 0$ except at k=0, we conclude that the coefficients are also uncorrelated.

(c) Periodicity:

Property: If $\omega_o = \frac{2\pi}{L}$, and L is a rational number, then the coefficients c_k and the reconstructed signal $\hat{x}(n)$ will be periodic.

Proof: From (6) it is evident that if the complex exponential is periodic, then the coefficients will be periodic. If $L = \frac{P}{Q}$ where P, Q are integers, then $e^{-j2\pi\frac{Q}{P}kn}$ will be periodic with period P. From (5) it can also be noticed that the reconstructed signal will be periodic with the same period. Under these conditions, the upper limit of the summation in (5) is $M = P$. In the special case where L is an integer ($Q = 1$), then the period will be L. If the signal x(n) is of finite duration L, then the pair of (5), (6) can be viewed as a DFT pair of the signal $x(n)\frac{\sin\left(\frac{\pi n}{L}\right)}{\pi n}$, i.e. the signal x(n) windowed by the main lobe of the sinc function. It should also be noted that when L is not an integer, then there will be a "pseudo-periodicity" in the coefficients with period $P = int\{L\}$, i.e. the coefficients within a pseudo-period P will not be exactly equal, but similar. From this it can be concluded that an integer value of L will always result in a computation of less number of coefficients.

4. ERROR ANALYSIS

In this section, we will compute the mean square error between original and reconstructed signal. The main objective here is to evaluate an upper bound on ω_o so that error does not exceed certain percentage of the average power of $x(n)$. We define the mean square error as:

$$e = E\{|x(n) - \hat{x}(n)|^2\}$$

$$e = 2E\{|\hat{x}(n)|^2\} - 2E\{|x(n)\hat{x}(n)^*|\}$$

We have used the fact that $E\{|x(n)|^2\} = E\{|\hat{x}(n)|^2\}$ (the proof is omitted for now). Using c_k=$y_k(0)$ (output of k filter) as proved in section (2), we get

$$E\{x(n)c_k^*\} = E\{x(n)y_k^*(0)\} = R_{xy_k}(n)$$

But cross-power-spectrum of $x(n)$ and $y_k(n)$ equals $S_x(w)\,H^*_k(w)$; therefore

$$E\{x(n)c_k^*\} = \frac{1}{2\pi}\int_{\pi}^{\pi} S_x(w)H_k^*(w)e^{jwn}dw = \frac{1}{2\pi}\int_{(k-1/2)\omega_o}^{(k+1/2)\omega_o} S_x(w)e^{jwn}dw$$

Summing along all k's, we get:

$$\sum_{k=-M}^{M} E\{x(n)c_k^*\}e^{-jk\omega_o n} = \frac{1}{2\pi}\sum_{k=-M}^{M}\int_{(k-1/2)\omega_o}^{(k+1/2)\omega_o} S_x(w)e^{j(w-k\omega_o)n}dw$$

$$E\{x(n)\hat{x}^*(n)\} = \frac{1}{2\pi}\sum_{k=-M}^{M}\int_{(k-1/2)\omega_o}^{(k+1/2)\omega_o} S_x(w)e^{j(w-k\omega_o)n}dw$$

Using this equation and the fact that: $E\{|x(n)|^2\} = E\{|\hat{x}(n)|^2\}$, we get after some manipulations that:

$$e = \frac{1}{\pi}\sum_{k=-M}^{M}\int_{(k-1/2)\omega_o}^{(k+1/2)\omega_o} S_x(w)[1-\cos(w-k\omega_o)n]\;dw \tag{16}$$

The second part of this equation depends on n and is generated by $E\{|x(n)\hat{x}^*(n)|\}$. This shows that the $x(n)$ and $\hat{x}(n)$ are individually but not jointly wide-sense stationary. If they were, the mean-square error e would be independent of time.

Upper bound on error: In equation (4) if $\omega_o n < \pi$ for every n in (0,N-1) where N is the length of the signal, then from equation (16) we have

$$1-\cos(w-k\omega_o)n \;=\; 2\sin^2\frac{(w-k\omega_o)n}{2} \;<\; 2\sin^2\left(\frac{\omega_o n}{4}\right)$$

for every w such that $|w-k\omega_o| < \frac{\omega_o}{2}$.

Therefore,

$$e \le \frac{2}{\pi}\sin^2\left(\frac{\omega_o n}{4}\right)\int_{-\pi}^{\pi} S_x(w)dw$$

Rearranging the right hand side of the above equation, we get

$$e \le 4\sin^2\left(\frac{\omega_o n}{4}\right)\;\frac{1}{2\pi}\int_{-\pi}^{\pi} S_x(w)dw$$

$$e \le 4\sin^2\left(\frac{\omega_o n}{4}\right)\;E\{|x(n)|^2\} \tag{17}$$

This means that in order for mean square error not to exceed certain percentage of the average power of $x(n)$, ω_o should be chosen such that

$$4\sin^2(\omega_o N/4) \ll 1 \tag{18}$$

The above is a worst case estimate assuming that $S_x(w)$ is concentrated at the end points of each integration interval i.e. from $(k\text{-}1/2)\omega_o$ to $(k\text{+}1/2)\omega_o$. If $S_x(w)$ does not vary appreciably in these intervals, then equation (16) can be modified by replacing $S_x(w)$ by a constant in each integration interval as

$$e = \frac{1}{\pi}\sum_{k=-M}^{M} S_x(w)\left[\omega_o - \frac{\sin(\omega_o n/2)}{n} - \frac{\sin(\omega_o n/2)}{n}\right] = \frac{1}{\pi}\sum_{k=-M}^{M} S_x(w)\;\;\omega_o\left[1-\frac{\sin(\omega_o n/2)}{\omega_o n/2}\right]$$

$$e = 2\left[1-\frac{\sin(\omega_o n/2)}{\omega_o n/2}\right]\;\frac{1}{2\pi}\sum_{k=-M}^{M}\omega_o S_x(w)$$

This can be simplified as:

$$e \;=\; 2\;\; E\{|x(n)|^2\}\;\left[1-\frac{\sin\left(\frac{\omega_o n}{2}\right)}{\frac{\omega_o n}{2}}\right] \tag{19}$$

This means that in the limit as ω_o tends to zero, mean square error approaches zero in mean-square sense.

5. DETERMINATION OF TRANSFORM EFFICIENCY

In this section, we concentrate exclusively upon the ability of the approximate Fourier expansion to remove correlation from the data source. In order to compare its performance with other transforms, we will demonstrate that the basis set of the Approximate Fourier Expansion provides a good approximation to the eigen vectors of the class of Toeplitz matrices as:

$$\psi = \begin{bmatrix} 1 & \rho & \rho^2 & . & . & \rho^{N-1} \\ \rho & 1 & \rho & . & . & \rho^{N-2} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \rho^{N-1} & \rho^{N-2} & . & . & . & 1 \end{bmatrix}, \quad 0<\rho<1.$$

For image processing applications, ψ above provides a useful model for the data covariance matrix corresponding to the rows and columns of an image matrix. The covariance matrix in the transform domain is denoted by Ψ and is given by[5]

$$\Psi = \Lambda\psi\Lambda^{*T} \tag{20}$$

where Λ is the 2-D matrix representation of an orthogonal transform, Λ^* is its complex conjugate. We will determine transform decorrelation efficiency by calculating the decrease in inter-element correlation in the transform domain covariance matrix compared with that in the data domain equivalent. To demonstrate the approach, we take the image block size of N=8 and evaluate equation (20) for various values of ρ. The decorrelation efficiency is then given by[6]

$$\eta = 1-\left(\frac{\sum_{j,k=1}^{N} Y_{j,k}}{\sum_{j,k=1}^{N} X_{j,k}}\right),\quad j \ne k \tag{21}$$

where $\sum X_{j,k}$ is total sum of data covariance entries and $\sum Y_{j,k}$ is total sum of transform covariances. This decorrelation efficiency was computed for the approximate Fourier expansion, discrete Fourier and discrete cosine transforms and the result is shown in Table 1. From the results, it is clear that for relatively less correlated data, the performance of the approximate Fourier expansion is better than discrete Fourier and discrete cosine transforms, where as for very highly correlated data, its performance is better than discrete Fourier and comparable to discrete cosine transform.

6. EXPERIMENTAL RESULTS

In this section we present preliminary experimental results. The performance of this transform will be evaluated using objective fidelity criterion. An objective fidelity criterion is the mean square signal to noise ratio (SNR_{ms}). Here we define mean square signal to noise ratio as follows:

$$SNR_{ms} = \frac{\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}(\hat{x}_{ij})^2}{\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}(x_{i,j}-\hat{x}_{i,j})^2} \tag{22}$$

Since the performance of this transform depends on the choice of $\omega_o = \frac{2\pi}{L}$, this transform was applied to a length of a speech signal and the "Lena" image, and the SNR_{ms} was computed for various values of L. Results are shown in Figures 1 through 4. Figures 1.a and 2.a display the original speech signal and "Lena" image, whereas Figures 1.b, 1.c and 2.b, 2.c show the reconstructed speech signal and "Lena" image for different values of L. It is obvious from Figures 1.b, 1.c and 2.b and 2.c that smaller values of ω_o produce closer approximations. This is expected since smaller values of ω_o result in finer frequency resolution, whereas larger values of ω_o produce a coarse resolution resulting in a large error. This is in contrast to discrete cosine and Fourier transforms where the frequency resolution is determined by the size of the data vector. Figures 3 and 4 show SNR_{ms} of the reconstructed speech signal and "Lena" image respectively when ω_o varies from high to low. It is clear from these Figures that small values of ω_o (and hence more coefficients to be computed) yield higher SNR_{ms}. Moreover, the computation of a relatively large no. of coefficients compacts the energy in first few coefficients. This was experimentally verified by computation of 32 and 102 A.F.E coefficients of the speech signal. The variance of first 16 coefficients in both cases is shown in Fig. 5. It is clear that the area under the variance curve of first 16 coefficients of 102-point expansion is smaller than that of 32-point expansion. Figure 6 displays the autocorrelation of the AFE coefficients of the speech signal. For comparison purposes, the autocorrelation of discrete cosine transform and DFT coefficients is also shown in Fig. 6. It is clear that the autocorrelation of AFE coefficients is similar to that of DCT, where as the performance of AFE is better than that of discrete Fourier transform.

7. CONCLUSIONS AND FUTURE WORK

We have presented a preliminary analysis of an approximate Fourier expansion of stationary non-periodic sampled signals along with some simulation results. It was shown that the frequency resolution of uncorrelated coefficients is user-defined which is in contrast to the discrete cosine and Fourier transforms where the maximum resolution depends on the size of data vector. Preliminary simulations showed that the decorrelation efficiency of the approximate Fourier expansion matches closely to that of DCT and, therefore, it can be used for transform coding of multidimensional signals.

More work needs to be done to determine the potential of this AFE in signal and image encoding for the purpose of compression. In addition, the relation to other expansions such as wavelet expansions can be determined. All these task will be explored in the immediate future.

7. REFERENCES

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	0.75	0.8	0.85	0.9	0.95	0.97
A.F.E	97.15	97.59	97.73	97.83	97.9	97.92
D.C.T	92.71	95.39	96.64	97.82	98.94	99.37
D.F.T	69.76	78.26	83.12	88.38	94.02	96.37

Table 1. Percentage decorrelation efficiency of transforms for various values of inter-element correlation

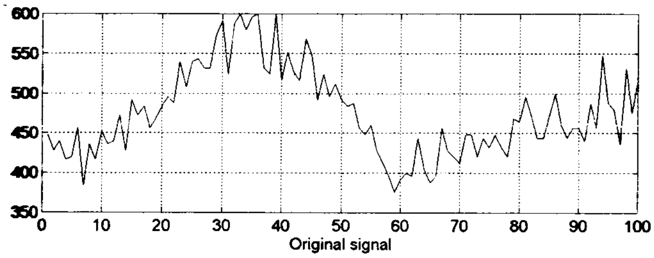

Figure 1.a Original speech signal

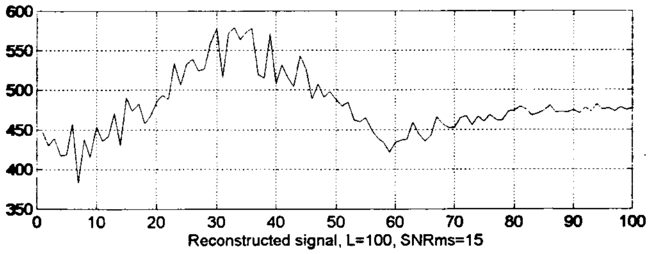

Figure 1.b Speech signal reconstructed with L=100

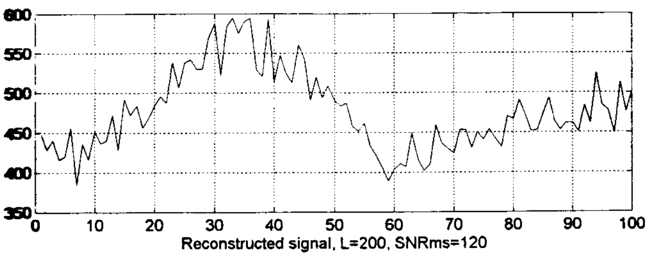

Figure 1.c Speech signal reconstructed with L=200

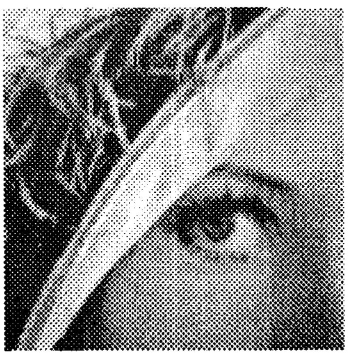

Figure 2.a Original "Lena" image

Figure 2.b Image reconstructed with L=140

Figure 2.b **Image reconstructed** with L=256

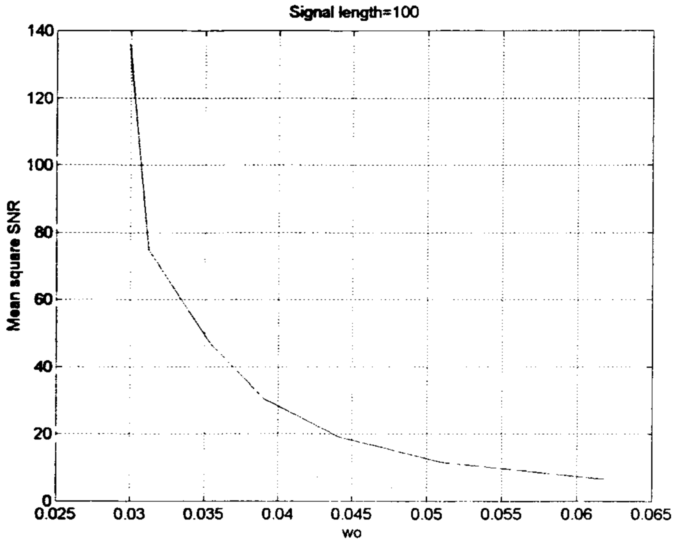

Figure 3. SNR_{ms} for the speech signal

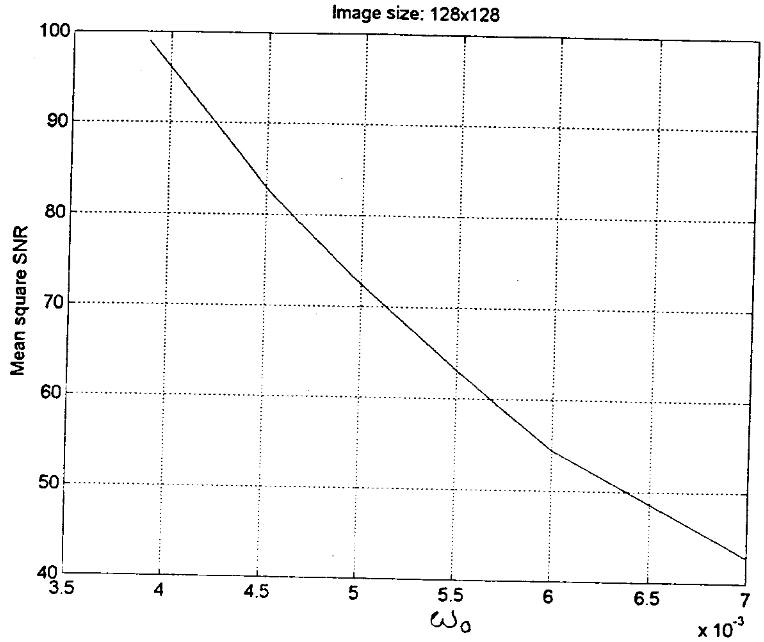

Figure 4. SNR_{ms} for the "Lena" image

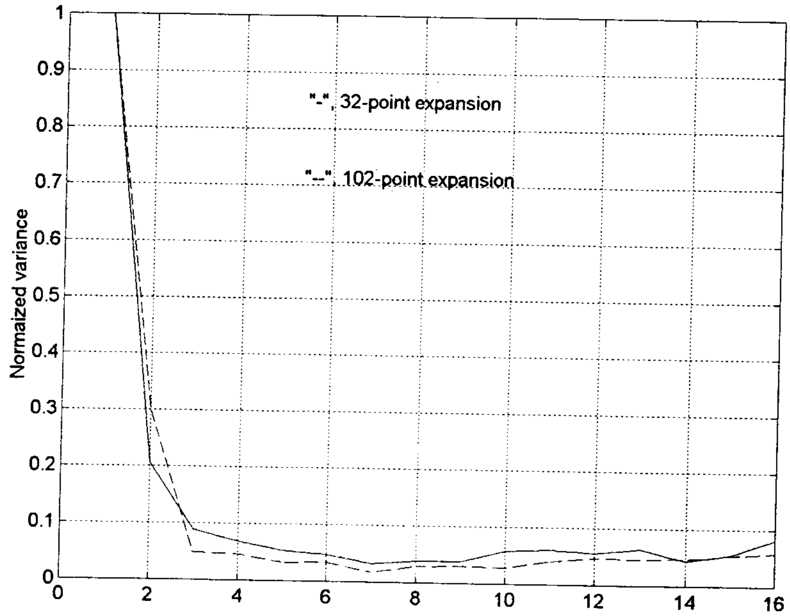

Figure 5. Variance of first 16 coefficients of 32-point and 102-point approximate Fourier expansion

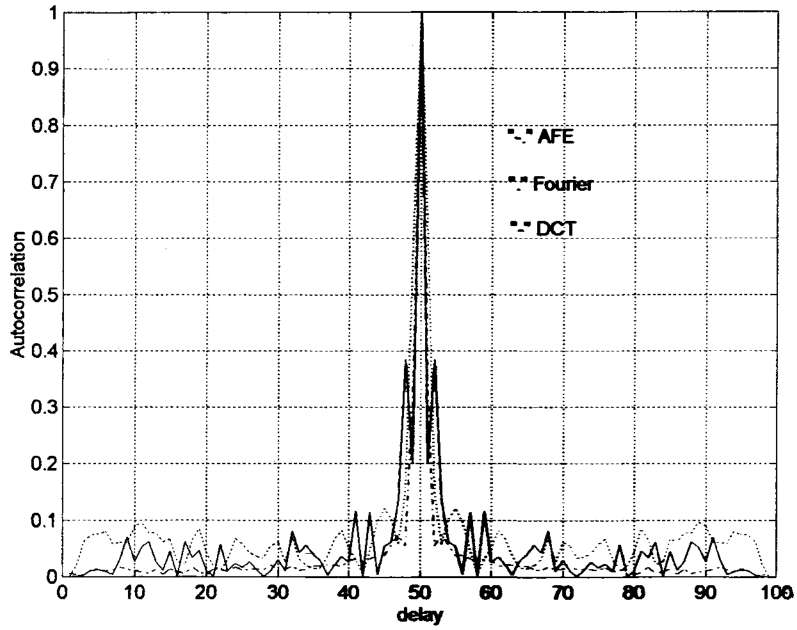

Figure 6. Autocorrelation of the A.F.E, D.C.T, and D.F.T coefficients.