LEARNING CONTROL OF CHAOTIC DYNAMICAL SYSTEMS

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Abstract

This paper presents an equivalence relation between chaotic dynamical systems and stochastic systems under an invariant measure. Built on this relationship, an equivalent stochastic system model for certain chaotic dynamics can be constructed. State prediction and control of chaotic systems can be achieved by learning using time series, based on the concept of the equivalent stochastic system model.

Keywords: Chaotic dynamics
Stochastic system
Learning control

1 Introduction

Study of chaotic dynamics has been of interest to many physicists, chemists and mathematicians and it has received attention from researchers working in other fields of science and engineering. Besides exploring the precise nature of chaotic behavior of many dynamical systems, researchers desire to control chaos. Chaotic dynamics are always found in many engineering systems, where there exist complex, hierarchical and distributed structures, incomplete system information, uncertainties and immeasurable states. For example, in a weakly interconnected transmission network of an electric power system, a gas supply system, large-scale integrated electronic circuits, aerodynamics, industrial robots and nonlinear optics, chaotic behavior has been observed for a long time. However, there is an obvious lack of methodologies to analyze and control the chaos in such chaotic systems. In recent years, Ott, Grebogi and York (OGY) [1] have suggested a method of controlling a chaotic dynamical system by stabilizing one of the many unstable periodic orbits embedded in a chaotic attractor, using small time-dependent perturbations in the form of feedback to an accessible system parameter. OGY stressed that all values needed to achieve control can be obtained from an experimental signal starting with the well-known embedding technique. Recently the OGY method has been successfully applied to some experimental systems [2, 3, 4]. OGY's idea has broadened view of many researchers working on problems of controlling chaos. Based on this idea, Pyragas [5] has proposed two methods of permanent control in the form of feedback. Both the methods are concerned with construction of a special form of a time-continuous perturbation to stabilize unstable periodic orbits.

However, existing work on controlling chaos has only been achieved based on some specific dynamical systems and it is restricted within low-dimensional chaotic systems. Control of chaos has only been treated in a simple manner as a conventional dynamic process control problem. As a natural extension of nonlinear system analysis and control theory, research on control of chaotic dynamical systems has received great attention in recent years. Most existing work concerns known dynamical systems with complete system information or artificial systems. Theoretical studies of controlling chaos in unknown dynamical systems with local information and measurable states have so far not been attempted.

Chaotically behaved deterministic dynamical systems appear to behave in a pseudo-random manner. This apparently random behavior of chaotic systems stems from their inherent properties determined by sensitivity of system initial states to system nonlinearities. It has been noted that chaotic systems behave as similar as that of a certain class of stochastic systems whose randomness is caused by stochastic properties of system parameters, states and external disturbances. Therefore, the theory of statistical properties of dynamical systems and some concepts and techniques used in stochastic systems have been employed to study chaotic dynamics [6, 7]. Nevertheless the relationship between chaotic and stochastic systems has not been identified due to a lack of understanding exact similarities of behavior of the two systems and due to a failure to establish a relevant methodology for the study.

This paper investigates an equivalence relation between chaotic and stochastic systems under an invariant measure. Based on this relationship, system reconstruction and control can be achieved using learning techniques. Since motions of chaotic dynamical systems, after some transients, settle down to strange attractors and sustain for a long time, this provides a great opportunity in terms of time period for a computer to learn the properties of the chaotic systems through space-time patterns. As long as learning proceeds, prediction and control of future states of the chaotic systems can be achieved in theory. In this paper, a feedback control strategy using the concept of the stochastic system model is proposed to control chaos in dynamical systems. Two typical chaotic systems are employed to evaluate the effectiveness of the proposed strategy.

2 Relation between chaotic and stochastic systems

Consider the following nonlinear dynamical system:

$$x_{t+1} = f(x_t). \tag{2.1}$$

The system is chaotic, if the ω-limit set of system 2.1 is a strange attractor. This is equivalent to that the Lyapunov exponent σ of system 2.1 is larger than 0, described as:

$$\sigma = \lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t-1} log\,|f'(x_i)| > 0 \tag{2.2}$$

If the ω-limit set of system 2.1 is a strange attractor with a certain probability measure, from the view of probability, there exists a stochastic system with noise perturbation which is equivalent to the chaotic dynamical system.

Definition 1. If the ω-limit set of system 2.1, $\omega(x)$, has a probability measure $\mu(x)$, $x \in M$, M is a Borel set, and

$$\mu(x) = \lim_{t\to\infty} \pi(x,t), \qquad x \in M \tag{2.3}$$

where $\pi(x,t)$ is the probability distribution function of a stochastic process $x(t)$, then system 2.1 is said to be equivalent to the stochastic process.

Theorem 1. If the chaotic system 2.1 is equivalent to the Itô stochastic differential equation:

$$\frac{dx}{dt} = g(x) + \mathcal{N}(x,t), \qquad x \in M \tag{2.4}$$

where $\mathcal{N}(x,t)$ is the Gaussian noise with a zero mean value and a covariance function $\Omega(x)$ given as:

$$\Omega(x) = |f(x)| \tag{2.5}$$

then:

$$g(x) = \frac{1}{2p(x)}\{c + \frac{\partial}{\partial x}[|f(x)|p(x)]\} \tag{2.6}$$

where $p(x)$ is the probability density function, c is an arbitrary constant.

Proof: Suppose $p(x,t)$ is the probability density function of stochastic differential equation 2.4, then it satisfies the Fokker–Planck equations:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}[g(x)p(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\Omega(x)p(x,t)]. \tag{2.7}$$

As the steady state probability density function is independent of time, it can be seen that:

$$\begin{aligned} \lim_{t\to\infty} p(x,t) &= p(x), \\ \lim_{t\to\infty} \frac{\partial p(x,t)}{\partial t} &= 0. \end{aligned} \tag{2.8}$$

Noting the conditions for $\mathcal{N}(x,t)$, with a few mathematical operations, we have:

$$\frac{\partial}{\partial x}\{[2g(x)p(x)] - \frac{\partial}{\partial x}[|f(x)|p(x)]\} = 0. \tag{2.9}$$

This leads to:

$$2g(x)p(x) - \frac{\partial}{\partial x}[|f(x)|p(x)] = c, \tag{2.10}$$

and

$$g(x) = \frac{1}{2p(x)}\{c + \frac{\partial}{\partial x}[|f(x)|p(x)]\}. \qquad \square$$

The above theorem shows that the behavior of chaotic system 2.1 can be described as a stochastic process. Thus its dynamics is characterized by the initial distribution $p(x_0)$ and the transition probability density function $p(x_{k+1}|x_k), k = 1, 2, ..., n$. $p(x_k)$ and $p(x_{k+1}|x_k)$ can be obtained using the time series of the chaotic dynamics conveniently. In other words, if the probability density function of a chaotic system has been obtained, the chaotic dynamics can be represented by an equivalent stochastic system model which is constructed based on the probability density function. An example is presented as follows to show the equivalence relation between the chaotic and stochastic systems. Let us consider a typical chaotic system, the Logistic map, $x_{t+1} = 4.0x_t(1-x_t)$. Its probability density function is [6]:

$$p(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \qquad x \in (0,1).$$

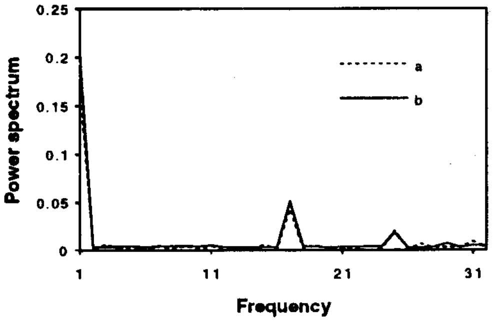
According to equation 2.6, the function in the equivalent stochastic system is:

$$g(x) = (1-2x) + \frac{c\pi\sqrt{x(1-x)}}{2}.$$

where c is an arbitrary constant, chosen to be 0. Considering $\Omega(x) = \sigma^2(x)$ and $\mathcal{N}(x,t) = \sigma(x)\mathcal{N}_o(t)$, where $\mathcal{N}_o(t)$ is the standard Gaussian random process, the equivalent stochastic differential equation can be obtained as follows:

$$\frac{dx}{dt} = (1-2x) + 2\sqrt{x(1-x)}\mathcal{N}_o(t) \tag{2.11}$$

The power spectra of the chaotic map and the equivalent stochastic system 2.11 have been obtained based on the time series of the two systems. The comparison is given in **Figure 1** which shows that the two systems have the similar power spectra.



a. Power spectrum of the equivalent stochastic system

b. Power spectrum of the chaotic map

Figure 1: Comparison of power spectra of the chaotic and stochastic systems

Inversely, an equivalent chaotic system can be constructed if a steady state probability density function of a stochastic system is given. This is discussed as follows.

Definition 2. If there exists a probability measure function $\mu(A)$, for arbitrary subset A in the Borel set M, it satisfies:

$$\begin{aligned} \mu(A) &= \mu[f^{-1}(A)] \\ f^{-1}(A) &= \{x \in M | f(x) \in A\}, \end{aligned} \quad (2.12)$$

then μ is a f-invariant measure of the mapping f: $[M, \mu] \rightarrow M$.

Theorem 2. If a steady state probability density function of a stochastic process is $p(x)$, $x \in M \subset R$, and the stochastic process is yielded from the following self mapping:

$$f : [M, \mu] \longrightarrow M,$$

then

$$p(y) = \sum_{i=1}^{n} \frac{p(x_i)}{|f'(x_i)|} \quad (2.13)$$

where $y \in \{f(x_i) | i = 1, 2, \ldots, n\}$.

Proof: For an arbitrary subset $A = [\alpha_0, \beta_0]$, we have:

$$f^{-1}(A) = \bigcup_{i=1}^{n} [a_i, b_i]$$

where $\alpha_0 = \min f(a_i), \beta_0 = \max f(b_i), i = 1, 2, ..., n$. As $f(x)$ has a f-invariant measure μ, then

$$\int_{\alpha_0}^{\beta_0} p(y)dy = \sum_{i=1}^{n} \int_{a_i}^{b_i} p(x_i)dx_i. \quad (2.14)$$

By integral transformation, $x = f^{-1}(y)$, the solution of the above equation can be obtained as follows:

$$p(y) = \sum_{i=1}^{n} \frac{p(x_i)}{|f'(x_i)|}. \quad \square$$

If $p(x)$ is an unimodal and symmetric function, then the map $f(x)$ is also unimodal and symmetric. In this case, equation 2.13 becomes:

$$p(y) = \frac{2p(x_1)}{|f'(x_1)|}; x_1 \in [a_0, \frac{b_0 + a_0}{2}] \quad (2.15)$$

$$f(x_2) = f(b - x_1); x_2 \in [\frac{b_0 + a_0}{2}, b_0] \quad (2.16)$$

where $b = b_0 + a_0$.

An example is given to interpret the above theorem. Suppose that a stochastic process has a probability density function, $p(x) = 6x(1 - x)$, $x \in [0, 1]$, then based on equations 2.15 and 2.16, the self mapping of the stochastic system can be formed as follows:

$$6(1 - y)ydy = 12x(1 - x), \qquad y = f(x). \quad (2.17)$$

Integrating the above equation gives:

$$\begin{aligned} y^2 - \frac{2}{3}y^3 &= 2x^2 - \frac{4}{3}x^3; & x \in [0, 0.5) \\ y^2 - \frac{2}{3}y^3 &= 2(1 - x)^2 - \frac{4}{3}(1 - x)^3; & x \in [0.5, 1] \end{aligned} \quad (2.18)$$

The chaotic map is constructed based on the above self mapping, $f : x \mapsto y$, formulated as $x_{k+1} = f(x_k)$. The chaotic map is shown in **Figure 2**. Its probability density function obtained from a large number of samples (simulated for 10,000 iterations) and the given probability density function are plotted in **Figure 3** which shows the similarity of the two functions.

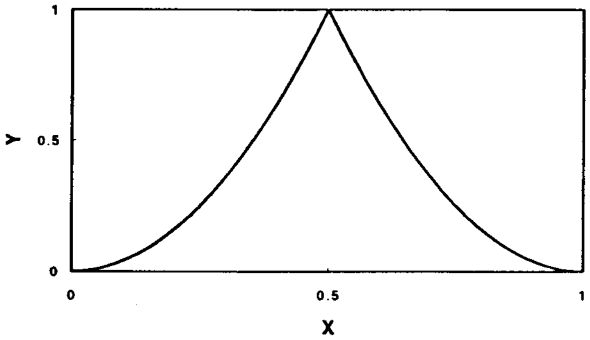


Figure 2: A chaotic map constructed from a stochastic process

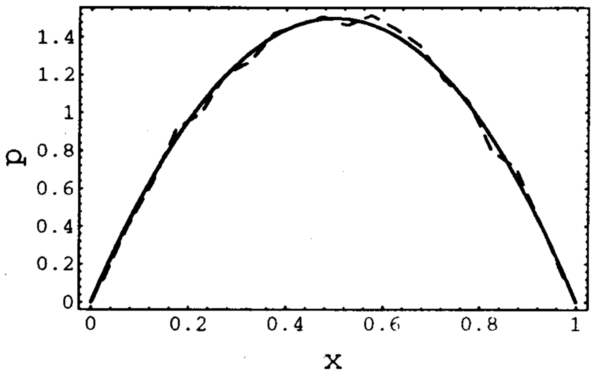


Figure 3: Probability density obtained from the chaotic mapping

3 Construction of an equivalent stochastic model

An equivalent stochastic model can be used to reconstruct a unknown chaotic system as mentioned in the previous section. The reconstruction can be achieved based on the probability density function which may be obtained using learning approaches to the time series of the chaotic system. A simple learning process of the chaotic system is introduced as follows.

Suppose that the chaotic system 2.1 is equivalent to the stochastic system 2.4 under the definitions 1 and 2. First, the invariant interval of the chaotic dynamics is divided into n subintervals, $I^i = [x^i_{min}, \ x^i_{max}], i = 1, 2, ...n$ and $x^1_{min} = x_{min}$, $x^n_{max} = x_{max}$, $x^j_{max} = x^{j+1}_{min}$, $(j = 1, 2, ...n-1)$, where x_{min} and x_{max} correspond to the minimum and maximum of x respectively. The points which are allocated in the subinterval I^i are assigned a same value denoted as x^i. Secondly, using the time average to replace the set average, the equivalent stochastic model for the chaotic system can be obtained through a learning process as:

$$P_k(x^i) = N_k(x^i)/k; \quad x^i \in I^i, \quad i = 1, 2, ..., n \tag{3.1}$$

where P, N and k are the probability, frequency and number of samples respectively. As a matter of fact, the above model can be realized using the following recurrence formula:

$$P_{k+1}(x^i) = \begin{cases} \frac{k}{k+1} P_k(x^i) + \frac{1}{k+1}, & x \in x^i \\ \frac{k}{k+1} P_k(x^i), & x \notin x^i \end{cases} \tag{3.2}$$

where $0 \leq P_k(x^i) \leq 1, i = 1, 2, ..., n$, and $\sum_{i=1}^{n} P_k(x^i) = 1$.

The transition probability $P(x_{k+1}|x_k)$ can be obtained analogously. Based on these statistic quantities, if an analytical representation of the probability density function can be approximated, an explicit equivalent stochastic model will be constructed. Otherwise, the statistic properties, obtained through the learning and related to the nature of the chaotic system, can be used for prediction and control of future states of the chaotic system, which is discussed in the following section.

4 State prediction and feedback control

Based on the equivalence relation between the chaotic and stochastic systems which is characterized by $P(x_k)$ and $P(x_{k+1}|x_k)$, it is possible to predict the future state x_{k+1} from state x_k. The prediction can be achieved according to the minimum variance method:

$$\hat{x}_{k+1} = \{x^i | \min_{x^i} Var(\tilde{x}^i)\}, \tag{4.1}$$

where $\tilde{x}$ is the estimate error and $\tilde{x}^i = x^i - \hat{x}_{k+1}$. This method is based on the following performance index:

$$\min_{x^i} E\{(x^i - \hat{x}_{k+1})^2\}. \tag{4.2}$$

According to the index 4.2, $\hat{x}_{k+1}$ can be obtained as a conditional mean:

$$\hat{x}_{k+1} = \sum_{i=1}^{n} x^i P(x^i|x_k). \tag{4.3}$$

The equation 4.3 may be regarded as a weighted mean of states. In its special case, x_{k+1} can be predicted using the maximum transition probability, which is given as:

$$\hat{x}_{k+1} = \{x^i | \max_{x^i} P(x^i|x_k)\} \tag{4.4}$$

To control a chaotic system, a control input u_k can be applied to system 2.1 as follows:

$$x_{k+1} = f(x_k) + u_k. \tag{4.5}$$

The feedback control is designed as:

$$u_k = \phi(x_k). \tag{4.6}$$

The closed loop system becomes:

$$x_{k+1} = f(x_k) + \phi(x_k), \tag{4.7}$$

where ϕ is the control law to be designed.

To control a unknown chaotic system, the estimated model or the predicted state:

$$\hat{x}_{k+1} = \hat{f}(x_k) \tag{4.8}$$

can be used to obtain the control signal. From equation 4.5, it has been noted that:

$$u_k = x_{k+1} - f(x_k). \tag{4.9}$$

If the state x_{k+1} is expected to follow a desired trajectory x_s, the feedback control using the predicted state can be obtained as:

$$u_k = x_s - \hat{x}_{k+1} = x_s - \hat{f}(x_k). \tag{4.10}$$

The controlled system becomes:

$$\begin{aligned} x_{k+1} &= f(x_k) + x_s - \hat{f}(x_k) \\ &= x_s + [f(x_k) - \hat{f}(x_k)] \\ &= x_s + \tilde{x}_{k+1}. \end{aligned} \tag{4.11}$$

If the prediction is accurate, then $\tilde{x}_{k+1} \to 0$, and $x_{k+1} \to x_s$. Otherwise, a track error defined as:

$$e_{k+1} = x_s - x_{k+1} \tag{4.12}$$

is always found. In order to compensate this error, an appropriate integration is introduced, which is given as follows:

$$u_k = x_s - \hat{x}_{k+1} + \frac{1}{T}\sum_{j=1}^{k} e_j \tag{4.13}$$

where T is the integration time constant. Equation 4.11 is modified as:

$$x_{k+1} = x_s + \tilde{x}_{k+1} + \frac{1}{T}\sum_{j=1}^{k} e_j \tag{4.14}$$

and the track error can be then expressed as:

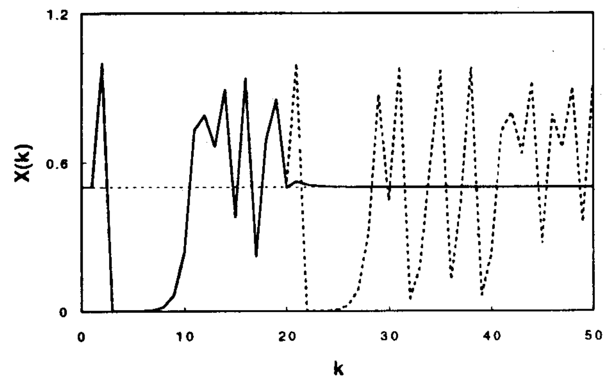
$$e_{k+1} = -(\tilde{x}_{k+1} + \frac{1}{T}\sum_{j=1}^{k} e_j) \approx \frac{1}{T}\sum_{j=1}^{k} \tilde{x}_j - \tilde{x}_{k+1}. \tag{4.15}$$

where $\tilde{x}_j = x_j - \hat{x}_j$. It can be seen that if $\frac{1}{T}\sum_{j=1}^{k}\tilde{x}_j$ approximately equals to $\tilde{x}_{k+1}$, then $e_{k+1} \rightarrow 0$ and $x_{k+1} \rightarrow x_s$.

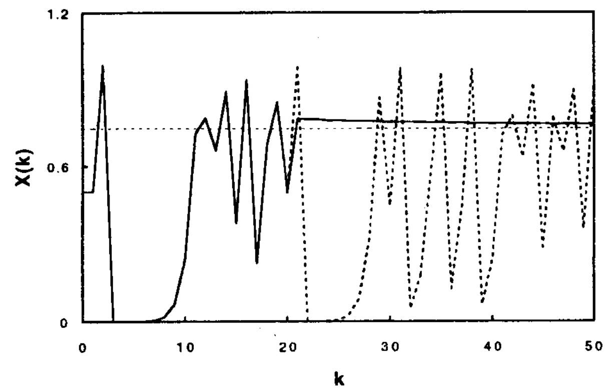
5 Examples

In order to evaluate the proposed control strategy, the Logistic map and the Henon map are employed in the simulation study. For the Logistic map $x_{t+1} = 4.0x_t(1.0 - x_t)$, its invariant interval is $[0, 1]$ which is divided into 20 subintervals. The system state is expected to be controlled to two desired values, $x_s = 0.5$ and $x_s = 0.75$, respectively. The system responses are presented in **Figure 4** (a) and (b) respectively. From the figures it can be seen that after the control is applied at iteration 20, the chaotic system can be controlled to the desired point immediately. However, the trajectory has a slow convergence rate during the time evolution when the setpoint is close to the fixed point of the map, which is shown in the **Figure 4** (b).

For the Henon map, $x_{t+1} = 1 - ax_t^2 + by_t$, $y_{t+1} = x_t$, its invariant interval is approximately $[-1.3, 1.3]$ which is divided into 20 subintervals. Two desired values, $x_s = 0.5$ and $x_s = -0.1$, are expected. **Figure 5** (a) and (b) illustrate the system responses to the two setpoints respectively. The simulation results show that the prediction and control perform satisfactorily, based on the concept of the equivalent stochastic system model.



(a) $x_s = 0.5$

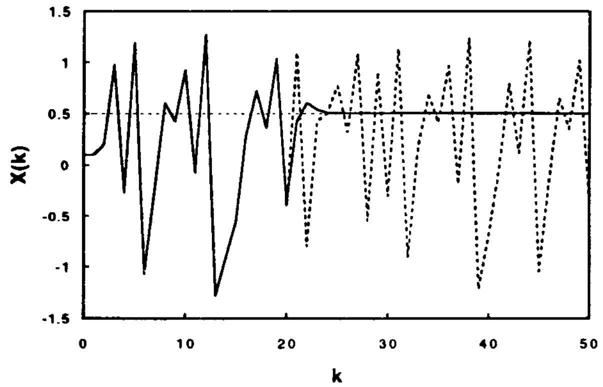


(b) $x_s = 0.75$

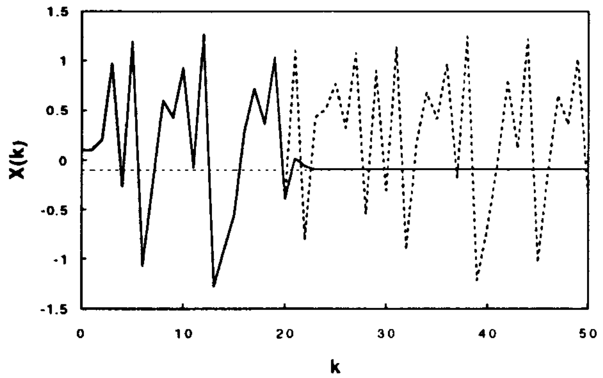
Figure 4: Dynamic response of the chaotic system

6 Conclusion

An equivalence relation between chaotic and stochastic systems has been investigated. Based on the relationship, an equivalent stochastic system model of the chaotic system can be constructed by learning using time series. The equivalent model can be used to predict states of the chaotic system and to develop a feedback controller.



(a) $x_s = 0.5$



(b) $x_s = -0.1$

Figure 5: Dynamic response of the chaotic system

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