OPTIMAL ROBUST FILTERING FOR UNCERTAIN LINEAR SYSTEMS WITH MEASURABLE INPUTS

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ABSTRACT

This paper deals with the robust minimum variance filtering problem for time-varying systems subject to a measurable input and to norm-bounded time-varying parameter uncertainty, in both the state and the output matrices of the state-space model. The problem addressed is the design of linear filters having an error variance with a guaranteed upper bound for any allowed uncertainty and any input of bounded energy. Three types of input signals are considered: a signal that is *a priori* known for the whole time interval, an unknown signal of very large bandwidth that is perfectly measured on-line, and a large bandwidth signal that is measured ahead of time in a fixed preview time interval.

Keywords: Robust filtering; $\mathcal{L}_2$ filtering; uncertain systems.

1. INTRODUCTION

One of the reasons for the recent development of $\mathcal{H}_\infty$ filtering is the fact that these filters are less sensitive than the $\mathcal{L}_2$ estimators to the exact knowledge of the dynamic model of the system under consideration, see, e.g. [1]. It has, in fact, been noticed that Kalman filters may fail to provide a guaranteed error variance in presence of parameters uncertainties [2]-[4]. This is the reason why a considerable interest has been paid to the design of robust estimators that achieve a prescribed upper-bound to the estimation error variance, for any admissible modelling uncertainty [6]-[9].

The design of robust filters, on finite horizon, for systems with an ellipsoidal-type parameter uncertainty in the state and the input noise matrices has been studied in [5]. Robust "quadratically stable" filters for linear systems with norm-bounded parameter uncertainty which provide an optimal bound for the error variance, in the stationary regime, have been developed in [6] and [7]. We note that the filter design in [7] allows for uncertainty in the state matrix only, and can be viewed as a particular case of [6], which treated the design of robust, reduced-order, filters in the presence of parameter uncertainty in both the state and the output matrices. An alternative technique of designing robust filters for systems with norm-bounded uncertainty has been developed in [8]. However, the proposed filter is sub-optimal, in the sense of the error variance upper-bound. Very recently a general treatment of the robust $\mathcal{L}_2$ estimation problem has been presented in [9] for both the time-varying and the stationary cases.

One of the main handicaps of the above methods is that they are confined to cases where the exogenous signals are all white noises of zero mean, and that they cannot easily treat cases where a part of these signals is either *a priori* known, or is measured on-line or with a preview. While such a part imposes no difficulty in the nominal filter design, where no uncertainty is present, it causes a problem in the uncertain case. In the nominal design case, the known signal is added to the estimator in order to perfectly cancel its effect on the estimation error [10]. Unfortunately, this cancellation is no longer possible to achieve in the case where the dynamic model of the process to be estimated is not perfectly known.

The present paper addresses the robust $\mathcal{L}_2$ estimation problem for signal processes with both parameter uncertainties and a known input signal. Since the estimation error will be affected by both the process and measurement noise signals and the known input, one of the main issues of this paper is how to incorporate

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the information on the input to optimally reduce a measure of the estimation error variance.

In this paper we generalize the robust $\mathcal{L}_2$ estimation approach of [9] to cope with norm bounded uncertainty in the case where the process is driven by a partially known input signal. An estimator is looked for that provides the smallest possible upper-bound to the estimation error for all the admissible process parameters. We treat the problem in the finite-horizon setting. Our estimation is based on both the measurement of the process output and the known signal. We constrain our filter to be linear, and show in Section 3 that the part of the filter that is based on the process measurement is identical to the one used in [9]. The other part of the filter, the one that is driven by the known input, is calculated in Section 4. It is obtained by applying the recent results of [11] that have been obtained for robust tracking control.

Three types of the 'known' input signal are considered. The first type is a signal that is known *a priori* for the whole filtering horizon. The other two types relate to an output of a large bandwidth linear system driven by a white noise. The second type assumes that this signal is measured on-line, while the third type assumes that the signal is measured with a fixed preview. Three different filters are obtained in Section 4 for the three types of inputs.

The time-varying results are extended in Section 5 to the stationary infinite horizon case. Conditions are given there that guarantee the convergence of the results of Sections 3 and 4, in the time-invariant case, in the limit where the horizon extends to infinity.

The results of this paper are demonstrated by an example in Section 6. This example demonstrates the significant improvement that can be achieved in signal estimation with the new technique. It also shows the advantage of preview in reducing the estimation error.

2. PROBLEM STATEMENT

We consider the following system:

$$(\Sigma):\ \dot{x}(t) = [A(t)+\Delta A(t)]x(t) + B(t)w(t) + B_r(t)r(t), \quad x(0)=x_0 \tag{1}$$

$$y(t) = [C(t)+\Delta C(t)]x(t) + D(t)w(t) \tag{2}$$

$$z(t) = L(t)x(t) \tag{3}$$

where $x \in \Re^n$ is the state, x_0 is a zero-mean random vector, $y \in \Re^m$ is the measurement, $w \in \Re^p$ is a zero-mean, white noise signal with identity power spectrum density matrix which is uncorrelated with x_0 for all $t \geq 0$, $r \in \Re^r$ is either a known input signal, or a stochastic signal that is uncorrelated with w and x_0, $z \in \Re^q$ is a linear combination of the state variables to be estimated over the time-horizon $[0,T]$, $A(t)$, $B(t)$, $B_r(t)$, $C(t)$ and $L(t)$ are known bounded time-varying matrices that describe the nominal system of (1)-(3), and $\Delta A(t)$ and $\Delta C(t)$ are unknown matrices representing parameter uncertainties. The admissible uncertainties are assumed to be of the form:

$$\Delta A = H_1(t)F(t)E(t), \quad \Delta C = H_2(t)F(t)E(t) \tag{4}$$

where $F(t) \in \Re^{i \times j}$ is an unknown matrix satisfying

$$\| F(t) \| \leq 1, \ \forall t \tag{5}$$

and $H_1(t)$, $H_2(t)$ and $E(t)$ are known bounded time-varying matrices of appropriate dimensions that specify how the uncertain parameters in $F(t)$ enter the nominal matrices $A(t)$ and $C(t)$. Throughout this paper, $\| X \|$, for $X \in \Re^{m \times n}$, denotes its largest singular value. For the sake of notation simplification, we shall omit in the sequel the dependence on t in the matrices when there is no possibility of confusion.

We observe that the case where the input and measurement noise signals are uncorrelated zero-mean white signals, say $v_1(t)$ and $v_2(t)$, respectively, with identity power spectrum density matrices, is a particular case of (1)-(3) where $w = [v_1^T \ \ v_2^T]^T$ and where the matrices B and D are replaced by $[B \ \ 0]$ and $[0 \ \ D]$, respectively.

It is assumed that the the covariance matrix of the initial state, x_0, is unknown but is such that

$$\mathbf{E}\left\{x_0 x_0^T\right\} \leq \bar{X}_0$$

where $\bar{X}_0 > 0$ is a known symmetric matrix, $\mathbf{E}\{\cdot\}$ denotes the expectation operator, and the notations $X > 0$ and $X \geq 0$, for a symmetric matrix X, means that X is positive definite and positive semidefinite, respectively.

Three types of information patterns for the input signal r will be considered: (i) r is *a priori* known over the entire time horizon $[0,T]$; (ii) r is an unknown signal of very large bandwidth that is perfectly measured on-line; (iii) r is a large bandwidth signal that is measured ahead of time in a fixed preview time interval.

Since in this paper we consider cases where r is not necessarily known *a priori*, the optimization criterion should be based on an average over the statistics of the unknown part of r. We thus assume, throughout the paper, that this part of r is an output of a linear, strictly proper, system that is driven by a standard white noise. the parameters of the latter system are unknown, and its bandwidth is assumed to be very large.

We define the history of $r(\cdot)$ at time l by

$$R_l = \{r(\tau),\ 0 \leq \tau \leq l\}$$

and we denote by $\bar{R}_l$ the future information on $r(\cdot)$ at time l, namely

$$\bar{R}_l = \{r(\tau),\ l < \tau \leq T\}.$$

In this paper we are concerned with the design of robust linear filters for estimating z over the time-horizon $[0, T]$ with a guaranteed estimation error variance, irrespective of the uncertainty. More specifically, we look for a linear estimate of the signal z over the time-horizon $[0, T]$ of the form

$$\hat{z} = \mathcal{G}_y y + \mathcal{G}_r r \tag{6}$$

where $\mathcal{G}_y$ and $\mathcal{G}_r$ are linear operators. The operator $\mathcal{G}_y$ is assumed to be causal and independent of r, whereas $\mathcal{G}_r$ can be either causal or noncausal depending on whether the exogenous signal $r(\cdot)$ is, respectively, measured on-line or known *a priori*. The operator $\mathcal{G}_y$ and $\mathcal{G}_r$ are to be determined in order to ensure that the worst-case quadratic performance cost

$$J = \frac{1}{T} \sup_{\|F\| \leq 1} \int_0^T \mathbf{E}_{w,x_0,r}\left\{e^T(t)e(t)\right\} dt \tag{7}$$

will satisfy a certain upper bound for all admissible uncertainties and for all r of energy less than 1, where e is the estimation error defined by

$$e \triangleq z - \hat{z}$$

and where the operator $\mathbf{E}_{w,x_0,r}\{\cdot\}$ denotes the expectation over the exogenous signal w, over the initial condition x_0, and over $\bar{R}_{t+h}$, where h is determined by the information structure on r. Moreover, this upper bound is required to be as small as possible.

We conclude this section by introducing the following assumption for the system (1)-(3):

Assumption 2.1

$[D(t) \quad H_2(t)]$ *is of full row rank for all* $t \in [0, T]$.

The above assumption means that the robust filtering problem is non-singular. Observe that if the parameter uncertainty in the output matrix disappears, i.e. $H_2 = 0$, Assumption 2.1 reduces to $DD^T > 0$, which is a standard assumption in the Kalman filtering problem for the nominal system of (1)-(3).

3. THE ROBUST FILTER

Due to the linearity of the system (1)-(3) and the filter (6), it can be easily found that the estimation error is given by

$$e = \mathcal{G}_{ew} w + \mathcal{G}_{ex_0} x_0 + \mathcal{G}_{er} r \tag{8}$$

where $\mathcal{G}_{ew}$, $\mathcal{G}_{ex_0}$ and $\mathcal{G}_{er}$ are linear operators defined by:

$$\begin{aligned} \mathcal{G}_{ew} &= \mathcal{G}_{zw} - \mathcal{G}_y \mathcal{G}_{yw} && (9) \\ \mathcal{G}_{ex_0} &= \mathcal{G}_{zx_0} - \mathcal{G}_y \mathcal{G}_{yx_0} && (10) \\ \mathcal{G}_{er} &= \mathcal{G}_{zr} - \mathcal{G}_y \mathcal{G}_{yr} - \mathcal{G}_r && (11) \end{aligned}$$

where $\mathcal{G}_{yw}$ and $\mathcal{G}_{zw}$ are the operators from w to y and z, respectively, $\mathcal{G}_{yx_0}$ and $\mathcal{G}_{zx_0}$ are the operators from x_0 to y and z, respectively, and $\mathcal{G}_{yr}$ and $\mathcal{G}_{zr}$ are the operators from r to y and z, respectively. State space realizations of these operators can be easily obtained from the state space model of (1)-(3). We also denote

$$e = e_{w,x_0} + e_r \tag{12}$$

where

$$e_{w,x_0} \triangleq \mathcal{G}_{ew} w + \mathcal{G}_{ex_0} x_0, \qquad e_r \triangleq \mathcal{G}_{er} r. \tag{13}$$

We note that that in view of (10), the part of the estimation error that stems from w and x_0, namely e_{w,x_0}, depends only on the operator $\mathcal{G}_y$ and thus, it is not affected by the choice of the operator $\mathcal{G}_r$. This fact will be of importance in the derivation of the robust filter in the sequel.

Since r is independent of w and x_0 and the operator $\mathcal{G}_y$ is independent of r, we find that

$$\int_0^T \mathbf{E}_{w,x_0,r}\{e^T(t)e(t)\}dt = \int_0^T \mathbf{E}_{w,x_0}\left\{e^T_{w,x_0}(t)e_{w,x_0}(t)\right\} dt + \int_0^T \mathbf{E}_{\bar{R}_t}\left\{e_r^T(t)e_r(t)\right\} dt \tag{14}$$

where $\mathbf{E}_{w,x_0}\{\cdot\}$ is the expectation over the unmeasurable signals w and x_0, and $\mathbf{E}_{\bar{R}_t}\{\cdot\}$ is the expectation over $\bar{R}_{t+h}$. Note that the first term in the right-hand side of (14) is independent of r.

Without loss of generality, we consider the following state space realization for the operator $\mathcal{G}_y$ of the estimator of (6)

$$\begin{aligned} \dot{\hat{x}}_y(t) &= A_y(t)\hat{x}_y(t) + B_y(t)y(t), \quad \hat{x}_y(0) = 0 && (15) \\ \hat{z}_y(t) &= L_y(t)\hat{x}_y(t) && (16) \end{aligned}$$

where A_y, B_y and L_y are bounded time-varying matrices. Note that in view of (1)-(3) and (15)-(16), the operator from (w, x_0) to e_{w,x_0} can be described by the following state space model:

$$\begin{aligned} \dot{\xi} &= [\bar{A} + \bar{H}F\bar{E}]\xi + \bar{B}w, \quad \xi(0) = [x_0^T \quad 0]^T && (17) \\ e_{w,x_0} &= \bar{L}\xi && (18) \end{aligned}$$

where

$$\bar{A} = \begin{bmatrix} A & 0 \\ B_yC & Ay \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ B_yD \end{bmatrix},$$

$$\bar{H} = \begin{bmatrix} H_1 \\ B_yH_2 \end{bmatrix}, \quad \bar{E} = [E \quad 0], \quad \bar{L} = [L \quad -L_y].$$

Since the first term in the right side of (14) does not depend on the operator $\mathcal{G}_r$, the results of [9] can be applied to find the optimal $\mathcal{G}_y$, in the sense that for for all $t \in [0,T]$, it minimizes an upper bound on the worst-case error variance

$$\sup_{\|F\|\leq 1} \mathbf{E}_{w,x_0} \left\{ e^T_{w,x_0}(t) e_{w,x_0}(t) \right\}.$$

More specifically, it has been established in [9] that, if for some scalar $\varepsilon > 0$, there exists a bounded solution $\bar{P}(t)$ over $[0,T]$ to the Riccati differential equation (RDE)

$$\dot{\bar{P}} = \bar{A}\bar{P} + \bar{P}\bar{A}^T + \varepsilon\bar{P}\bar{E}^T\bar{E}\bar{P} + \bar{B}\bar{B}^T + \varepsilon^{-1}\bar{H}\bar{H}^T;$$
$$\bar{P}(0) = \mathrm{diag}\left\{\bar{X}_0, 0\right\} \tag{19}$$

then the variance of the estimation error, e_{w,x_0}, satisfies the bound

$$\mathbf{E}\left\{e^T_{w,x_0}(t)e_{w,x_0}(t)\right\} \leq \mathrm{tr}\left\{\bar{L}\bar{P}(t)\bar{L}^T\right\}, \; \forall t \in [0,T] \tag{20}$$

for all admissible uncertainties, where $\mathrm{tr}\{\cdot\}$ stands for the matrix trace and $\mathrm{diag}\{\cdots\}$ denotes a block diagonal matrix. The robust filter of [9] minimizes the upper bound on the error variance in (20).

In order to present the method for determining the operator $\mathcal{G}_y$, we begin by introducing the Riccati differential equations:

$$\dot{Y} = AY + YA^T + \varepsilon YE^TEY + BB^T + \varepsilon^{-1}H_1H_1^T;$$
$$Y(0) = \bar{X}_0 \tag{21}$$

and

$$\begin{aligned} \dot{X} = & \left(A - \hat{B}\hat{D}^T\hat{V}^{-1}C\right)X + X\left(A - \hat{B}\hat{D}^T\hat{V}^{-1}C\right)^T \\ & + X\left(\varepsilon E^TE - C^T\hat{V}^{-1}C\right)X \\ & + \hat{B}\left(I - \hat{D}^T\hat{V}^{-1}\hat{D}\right)\hat{B}^T; \quad X(0) = \bar{X}_0 \end{aligned} \tag{22}$$

where

$$\hat{B} = \left[B \quad \frac{1}{\sqrt{\varepsilon}}H_1\right], \quad \hat{D} = \left[D \quad \frac{1}{\sqrt{\varepsilon}}H_2\right], \quad \hat{V} = \hat{D}\hat{D}^T$$

and where ε is a positive scalar to be chosen.

The next theorem presents the optimal filter $\mathcal{G}_y$, in the sense of minimizing the upper bound on the error variance in (20).

Theorem 3.1 ([9]) *Consider the system (1)-(3) satisfying Assumption 2.1. Then, there exists a filter $\mathcal{G}_y$ of the form of (15)-(16) that minimizes the bound on the error variance of (20) over $[0,T]$ if and only if the RDE (21) has a bounded solution over $[0,T]$. Under this condition, the RDE (22) has a solution $X(t)$ over $[0,T]$ and the optimal filter $\mathcal{G}_y$ is described by the state space realization*

$$\dot{\hat{x}}_y = A_y\hat{x}_y + B_yy, \quad \hat{x}_y(0) = 0 \tag{23}$$
$$\hat{z}_y = L\hat{x}_y \tag{24}$$

where

$$A_y = A + \varepsilon XE^TE - B_yC \tag{25}$$
$$B_y = \left(XC^T + BD^T + \varepsilon^{-1}H_1H_2^T\right) \cdot \left(DD^T + \varepsilon^{-1}H_2H_2^T\right)^{-1}. \tag{26}$$

Moreover, this filter guarantees that for all $t \in [0,T]$

$$\sup_{\|F\|\leq 1} \mathbf{E}_{w,x_0} \left\{e^T_{w,x_0}(t)e_{w,x_0}(t)\right\} \leq \mathrm{tr}\left\{L(t)X(t)L^T(t)\right\}. \tag{27}$$

Remark 3.1 It should be noted that although the filter of (23)-(24) does not depend on the solution of (21), in order for this filter to provide a bound on the error variance it does not suffice to find a solution to (22) over $[0,T]$ for a suitable $\varepsilon > 0$. Observe that it is also required to verify if for this ε the RDE (21) has a bounded solution over $[0,T]$. It may happen that there exist values of ε for which (22) has a bounded solution over $[0,T]$ while (21) has an escape-point in $[0,T]$. For such values of ε, the resulting filter cannot guarantee the bound on the error variance in (27). □

Once $\mathcal{G}_y$ is obtained, the problem remains one of finding $\mathcal{G}_r$ that minimizes the bound on

$$\frac{1}{T}\int_0^T \mathbf{E}_{\bar{R}_t}\{e_r^T(t)e_r(t)\}dt. \tag{28}$$

The latter problem will be solved in the next section.

4. THE OPTIMAL $\mathcal{G}_R$

First, in view of (11) and (13), we denote

$$e_r = z_r - \hat{z}_r \tag{29}$$
$$z_r \triangleq \mathcal{G}_{z_r r}r, \quad \hat{z}_r \triangleq \mathcal{G}_r r \tag{30}$$

where $\mathcal{G}_{z_r r}$ is the operator defined by

$$\mathcal{G}_{z_r r} = \mathcal{G}_{zr} - \mathcal{G}_y\mathcal{G}_{yr}. \tag{31}$$

Given the filter $\mathcal{G}_y$ of (23)-(24), it can be easily derived that a state space realization of the operator $\mathcal{G}_{z_r r} : r \to z_r$ is as follows

$$\begin{aligned} \dot{\eta} &= [A_\eta + H_\eta F E_\eta]\eta + B_\eta r, \quad \eta(0) = 0 \quad (32) \\ z_r &= L_\eta \eta \quad (33) \end{aligned}$$

where

$$A_\eta = \begin{bmatrix} A & 0 \\ B_y C & Ay \end{bmatrix}, \; B_\eta = \begin{bmatrix} B_r \\ 0 \end{bmatrix} \quad (34)$$

$$H_\eta = \begin{bmatrix} H_1 \\ B_y H_2 \end{bmatrix}, \; E_\eta = [E \quad 0], \; L_\eta = [L \quad -L] \quad (35)$$

and where A_y and B_y are given by (25) and (26), respectively.

Hence, it follows from the above that the problem of finding the filter $\mathcal{G}_r$ coincides with the problem of finding an estimate $\hat{z}_r = \mathcal{G}_r r$ for the signal z_r of (32)-(33) which provides a guaranteed bound for

$$J_r \triangleq \frac{1}{T}\int_0^T \mathbf{E}_{\bar{R}_t}\left\{[z_r(t) - \hat{z}_r(t)]^T[z_r(t) - \hat{z}_r(t)]\right\} dt \quad (36)$$

for all r of energy less than one, and for all $\| F \| \leq 1$.

Note that the problem of finding such an $\mathcal{G}_r$ is similar to a special case of the robust $\mathcal{H}_\infty$ tracking problem that has been treated in [11]. Since the system of (32)-(33) involves no measurement noise, a modification of the results of [11] should be used which allows for zero measurement noise.

This modification can be simply carried out by using the method of [12]. This method replaces the standard $\mathcal{H}_\infty$ observer RDE, which is also used in [11], with a similar equation whose solution guarantees the required performance of the observer for a zero measurement noise. Hence, applying the modified results of [11] to determine the operator $\mathcal{G}_r$, we obtain the following results.

Theorem 4.1 *Consider the following Riccati differential equation:*

$$-\dot{P} = A_\eta^T P + P A_\eta + P H_\eta H_\eta^T P + E_\eta^T E_\eta; \quad P(T) = 0. \quad (37)$$

Then if there exists a bounded solution $P(t)$ to (37) over $[0,T]$, the following filters yield the minimizing $\mathcal{G}_r$ for the three different information patterns on r:

Filtering with r *a priori* known over $[0,T]$

$$\begin{aligned} \dot{\hat{x}}_r &= A_r \hat{x}_r + B_\eta r + B_\theta \theta \quad \hat{x}_r(0) = 0 \\ \hat{z}_r &= L_\eta \hat{x}_r \end{aligned}$$

where

$$A_r = A_\eta + H_\eta H_\eta^T P, \qquad B_\theta = H_\eta H_\eta^T$$

and θ is given by

$$\dot{\theta} = -A_r^T \theta - P B_\eta r; \quad \theta(T) = 0.$$

Filtering with r measured on-line

$$\begin{aligned} \dot{\hat{x}}_r &= A_r \hat{x}_r + B_\eta r \; ; \quad \hat{x}_r(0) = 0 \\ \hat{z}_r &= L_\eta \hat{x}_r . \end{aligned}$$

Filtering with r measured with a preview h

$$\begin{aligned} \dot{\hat{x}}_r &= A_r \hat{x}_r + B_\eta r + B_\theta \theta_p \; ; \quad \hat{x}_r(0) = 0 \\ \hat{z}_r &= L_\eta \hat{x}_r \end{aligned}$$

where $\theta_p = \mathbf{E}_{\bar{R}_t}\{\theta\}$ for the given h.

Remark 4.1 It can be easily shown using the strict bounded real lemma for time-varying-systems (see, e.g. [14]) that the existence of a bounded solution $Y(t)$ to the RDE (21) over $[0,T]$ guarantees that the RDE (37) possesses a bounded solution $P(t)$ over $[0,T]$ as well. In view of this fact, the existence of a bounded solution $Y(t)$ to (21) over $[0,T]$ suffices to ensure the existence of the optimal filters $\mathcal{G}_y$ and $\mathcal{G}_r$ of Theorems 3.1 and 4.1. □

5. THE STATIONARY CASE

This section addresses the robust filter design in the stationary case. To this end we assume that T tends to infinity, the noise w in system Σ is stationary and the matrices A, B, B_r, C, L, E, H_1, and H_2 are constant. The uncertainty matrix F is still allowed to be time-varying. Attention is focused on guaranteeing that the asymptotic value of (7) is within a certain bound for all r of average power less than one, irrespective of the uncertainties. Note that now the filter is also required to be asymptotically stable.

We make the following assumption for the system (1)-(3):

Assumption 5.1

(i) The system (1) is quadratically stable.

(ii) $[D \quad H_2]$ is of full row rank.

Remark 5.1 The quadratic stability assumption implies that the system (1) is exponentially stable for all admissible uncertainties. It should be noted that due to the presence of time-varying parameter uncertainty, Assumption 5.1 (i) is required in order to guarantee the uniform asymptotic stability of the estimation error dynamics. The reason is that the estimation error dynamics is driven by the state of the system Σ and thus the quadratic stability of the latter system ensures the boundedness of the estimation error. Note that the asymptotic stability of the nominal state matrix A is a necessary condition for Assumption 5.1 to hold.

Similarly to Assumption 2.1, Assumption 5.1 (ii) means that the robust filtering problem is nonsingular. □

We observe, from results in [13], that if there exist bounded solutions $Y(t)$, $X(t)$, and $P(t)$, $\forall\, t \in [0, \infty)$, to (22), (21) and (37), respectively, these solutions will converge, as $t \to \infty$, to the stabilizing solutions of the associated algebraic Riccati equations (AREs), namely

$$AY + YA^T + \varepsilon YE^TEY + BB^T + \varepsilon^{-1}H_1H_1^T = 0 \quad (38)$$

$$\begin{aligned} \left(A - \hat{B}\hat{D}^T\hat{V}^{-1}C\right)X + X\left(A - \hat{B}\hat{D}^T\hat{V}^{-1}C\right)^T \\ + X\left(\varepsilon E^TE - C^T\hat{V}^{-1}C\right)X \\ + \hat{B}\left(I - \hat{D}^T\hat{V}^{-1}\hat{D}\right)\hat{B}^T = 0 \end{aligned} \quad (39)$$

and

$$A_\eta^TP + PA_\eta + PH_\eta H_\eta^TP + E_\eta^TE_\eta = 0\,. \quad (40)$$

In view of the above, it can be easily shown that the results of Sections 3 and 4 remain valid asymptotically, when the solutions $Y(t)$, $X(t)$ and $P(t)$ of the RDEs (21), (22) and (37) are replaced by the stabilizing solutions $Y = Y^T \geq 0$, $X = X^T \geq 0$ and $P = P^T \geq 0$ of the AREs (38), (39) and (40), respectively.

We denote the asymptotic values of the error e_{w,x_0} in (13) and of the cost function J_r in (36) by e_w and J_{r_s}, respectively. Also, observe that now $\mathrm{tr}\left\{\bar{L}\bar{X}\bar{L}^T\right\}$ is a guaranteed bound on the variance of e_w where $\bar{X}$ is the positive semidefinite stabilizing solution of the ARE corresponding to (19). Hence, we have the following result, which is the stationary counterpart of Theorems 3.1 and 4.1.

Theorem 5.1 *Consider the system (1)-(3) satisfying Assumption 5.1. Then there exists a time-invariant, asymptotically stable filter of the form (6) that minimizes the bound* $\mathrm{tr}\left\{\bar{L}\bar{X}\bar{L}^T\right\}$ *on the variance of* e_w *and provides an optimized guaranteed bounded for* J_{r_s} *if and only if the AREs (38) and (40) have stabilizing solutions* $Y = Y^T \geq 0$ *and* $P = P^T \geq 0$*, respectively. Under this condition, the optimal filter is given by (23)-(24) and the estimators of Theorem 5.1, where* $X = X^T \geq 0$ *is the stabilizing solution of the ARE (39), which is guaranteed to exist.*

Remark 5.2 In parallel with the Remark 4.1, it can be easily shown using the strict bounded real lemma (see, e.g. [14]) that the existence of a stabilizing solution $Y = Y^T \geq 0$ to the ARE (38) guarantees that there exists a stabilizing solution $P = P^T \geq 0$ to the ARE (40) as well. Furthermore, in view of Theorem 2.3 of [15], it can be also established that the existence of a symmetric positive semidefinite stabilizing solution to (38) is equivalent to the quadratic stability of the system (1). This implies that Assumption 5.1 (i) ensures the existence of the optimal filters $\mathcal{G}_y$ and $\mathcal{G}_r$ of Theorem 5.1. □

Remark 5.3 Similarly to the finite-horizon case, although the stabilizing solution of the ARE (38) plays no role in the calculation of the filter $\mathcal{G}_y$ of (23)-(24), in order for this filter to provide a bound on the error variance of e_w, it does not suffice to find a positive semidefinite stabilizing solution to (39) for a suitable $\varepsilon > 0$. Observe that it is also required to verify if for this ε the ARE (38) has a positive semidefinite stabilizing solution. It may happen that there exist values of ε for which (38) has a positive semidefinite stabilizing solution but not (38). For such values of ε, the resulting filter $\mathcal{G}_y$ cannot guarantee that $\mathrm{tr}\{LXL^T\}$ is a bound on the error variance of e_w for all admissible uncertainties. □

Remark 5.4 The filter $\mathcal{G}_y$ of Theorem 5.1 minimizes the bound on the variance of e_w for a fixed ε. However, since different values of ε give rise to different values for the bound on the variance of e_w, we can still minimize this bound with respect to the parameter ε.

It can be shown, using monotonicity results on algebraic Riccati equations, that if the ARE (38) has a positive semidefinite stabilizing solution for $\varepsilon = \bar{\varepsilon} > 0$, then (38) also has a positive semidefinite stabilizing solution for any $\varepsilon \in (0, \bar{\varepsilon}]$. Therefore, it follows that if the robust filter of Theorem 5.1 can be found for a given $\varepsilon > 0$, then there exists an $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*]$ the robust filter is guaranteed to exists. Observe that ε^* is the largest positive ε such that the ARE (38) admits a stabilizing solution $Y = Y^T \geq 0$. This allows us to carry out the minimization of the upper bound on the estimation error variance with respect to ε, namely

$$\min_{\varepsilon \in (0, \varepsilon^*]} \left\{\mathrm{tr}\left[LX(\varepsilon)L^T\right];\; X(\varepsilon) = X^T(\varepsilon) \geq 0\right\}.$$

where X is the stabilizing solution of (39). □

6. EXAMPLE

In this section we treat the simple example of [9] with an additional known input. We consider the system whose state-space description is given by:

$$\dot{x} = \begin{bmatrix} 0 & -1+\delta \\ 1 & -0.5 \end{bmatrix} x + \begin{bmatrix} -2 \\ 1 \end{bmatrix} w + \begin{bmatrix} 1 \\ -1.1 \end{bmatrix} r \quad (41)$$

$$y = [-100 \quad 100]x + v \quad (42)$$

$$z = [1 \quad 0]x \quad (43)$$

where w and v are uncorrelated, zero-mean, white noise signals with unit power spectrum density, and δ is an unknown parameter satisfying $|\delta| \leq 0.3$.

Note that the above system is of the form of Σ with

$$H_1 = [1 \quad 0]^T, \quad H_2 = 0, \quad E = [0 \quad 0.3].$$

The stationary robust filtering method of Section 5 is applied to system (41)-(43). First, we determined the optimal $\mathcal{G}_y$ in the sense of minimizing the bound $\mathrm{tr}\{LXL^T\}$ on the asymptotic variance of the part of estimation error that stems from w. In view of Remark 5.4 it was found that the optimal $\mathcal{G}_y$ is obtained for $\varepsilon = 0.3286$ and that the optimal bound is 2.312. The corresponding optimal filter is given by (23)-(24) with

$$A_y = \begin{bmatrix} -161.48 & 160.55 \\ 186.35 & -185.79 \end{bmatrix}, \quad B_y = \begin{bmatrix} -1.6148 \\ 1.8536 \end{bmatrix}.$$

The optimal $\mathcal{G}_r$ is obtained by solving (40) first. We obtained $P = \mathrm{diag}\{P_1, 0\}$, where

$$P_1 = \begin{bmatrix} 0.1027 & -0.0053 \\ -0.0053 & 0.1006 \end{bmatrix}.$$

The results of Theorem 5.1 are then derived for the case of the case of r measured on-line and for the case where it is measured with a preview.

We summarize our results in two tables. Table 1 describes the performance of the filter $\mathcal{G}_y$ by simulating it with the system of (41)-(43) for various values of the uncertain parameter δ. We compare our results with those achieved for the Kalman filter that is designed for the nominal system of (41)-(43), namely for $\delta = 0$.

Filter	Actual Error Variance ($r(t) \equiv 0$)		
	$\delta = 0$	$\delta = 0.3$	$\delta = -0.3$
Kalman Filter	0.0266	12.990	3.321
$\mathcal{G}_y$	0.618	1.288	0.670

Table 1. Comparison between $\mathcal{G}_y$ and the Kalman Filter.

The results for the filter $\mathcal{G}_r$ of Section 5 are described in Table 2. This table shows simulation results for $r(t) = 10\sin(0.4t)$ for the same values of δ that were used in Table 1. In this table we bring the asymptotic values achieved for the cost function of (28). We compare our results with those that are obtained by the intuitive approach for choosing $\mathcal{G}_r$, namely taking A_η instead of A_r in the dynamics of the filter for r that is measured on-line. We also compare our results with those achieved for the Kalman filter with the input r that is designed for the nominal system of (41)-(43).

Filter	Cost Function		
	$\delta = 0$	$\delta = 0.3$	$\delta = -0.3$
Kalman Filter	0	11.32	2.803
$\mathcal{G}_r$: on-line	2.02	10.24	10.36
$\mathcal{G}_r$: preview, h=8	0.70	9.95	7.91
Intuitive Design	0	15.76	3.90

Table 2. Comparison between $\mathcal{G}_r$, Kalman Filter and the Intuitive Design.

Table 2 shows that the two new designs of $\mathcal{G}_r$ reduces the error that is obtained in the worst case, $\delta = 0.3$, for the intuitive design by more than 35%. It also shows the improvement gained by a preview of $h = 8$ secs. in relation to the case where r is measured on-line. This improvement is bigger for $\delta = -0.3$ than for $\delta = 0.3$. It should be noted that, although the worst case performance of the the Kalman filter is only slightly inferior to that of our filters $\mathcal{G}_r$, the former filter is very sluggish. Indeed, the dominant time constant of the Kalman filter is 303 secs., whereas the dominant time constant of the filter $\mathcal{G}_r$ for both the cases of r measured on-line and with a preview is only 4.17 secs.

7. CONCLUSIONS

In the present paper we have introduced a robust estimation method that copes with both unmeasurable stochastic noise inputs and measurable inputs (or disturbances) that are either measured or known *a priori*. We have treated the time-varying finite-horizon case and discussed the convergence of the results in the stationary infinite-horizon case.

In the case where the process considered has no measured or known input, the vector θ will be identically zero. In this case it is easy to see that our filter recovers the robust $\mathcal{H}_2$ filter of [9] and provides, in the stationary case, the upper bound of $\mathrm{tr}\{LXL^T\}$ for our performance index. We also observe that in the case where there is no parameter uncertainty the matrix P is identically zero which leads to an identically zero θ. Our filter recovers then the standard Kalman filter with a known input signal.

In our problem we have not assumed any *a priori* knowledge of a model that produces the input signal r. We observe that when the latter is known, it may be incorporated into the filter design by augmenting the system Σ to include this model. However, this *a priori* knowledge is in many cases inaccurate and hardly available.

One *ad hoc* way of treating the known signal is to cancel its effect for the nominal system, namely to take $F(t) \equiv 0$ in (32). The results of Theorems 4.1 and 5.1 show that this intuitive method is not optimal, and instead of trying to minimize the effect of r on the system (A_η, B_η, C_η) one should aim at canceling it for the system (A_r, B_η, C_η).

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