Two-parameter Compensation Scheme : Application to the Minimum Number of Unstable Zeros Decoupling Problem

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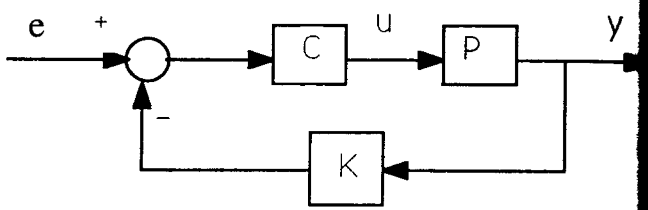
Abstract *: In this paper we examine the problem of internally stabilizing and simultaneously diagonally decoupling a linear multivariable system with a "minimum number of unstable zeros" (in some sens) in the decoupled system by two-parameter compensation. Based on our results in [4] where it was proved that any given plant (P) of full row rank can be decoupled with internal stability by the considered configuration (computation of the decoupling compensators is easy). The existence of some solution to the decoupling problem with a "minimum number of unstable zeros" in the decoupled system is derived.*

Keywords *: linear multivariable systems, decoupling, two-parameter compensation, stability, precompensator, unstable zeros.*

I. INTRODUCTION

The linear time-invariant system decoupling problem (or noninteracting control) was extensively investigated in the last decades by geometric approach as well as a transfer approach. For authoritative references representing important steps in the development of this theory, see [4], [6], [7], [10], [11] and [15]

In the paper [4], by using the transfer approach, we have presented an algebraic design method for diagonal input-output maps which can achieved by a stabilizing two-parameter compensation. This problem is also referred to a two-degree of freedom design in [7].
More precisely, we have considered the following multi-input multi-output (M.I.M.O) system $\sum(P,K,C)$:



where P is a given proper transfer of a plant ; and K, C are proper controllers to design such that the system $\sum(P,K,C)$ shown in the figure is internally stable and the input-output map $e \rightarrow y$ is diagonal and nonsingular with "a minimum number of unstable zeros". This problem is refered to a decoupling problem by two-parameter compensation with internal stability.
Based on the results given in [3], the following are proved in [4] :

a) Any plant P of full row rank can be decoupled by the closed-loop system shown in the scheme above with internal stability ;

b) For the "generic" case of plants P (the genericity is in the sens that det.D and the biggest invariant factor of N are coprime, where (N, D) is any right coprime factorization of P) we have exhibited a solution to the decoupling problem which garanties a minimum number of unstable zeros in the decoupled system.

The aim of the present note is to investigate the stable decoupling problem with a minimum number of unstable zeros in the general case. We show that the solution to the problem above is not unique, and some solutions can be derived from the particular solution to the decoupling problem with internal stability, presented in [4]. Our approach will largely use some results concerning the "skew-prime" proper and stable matrices and the "skew-complements" of these matrices.

The paper is organized as follows. A short mathematical background is given in section II. Section III gives the formulation of the considered problem and some preliminary results. The section IV is devoted to the existence of solutions to the "minimum number of unstable zeros" decoupling problem. Some concluding remarks end the paper.

II. NOTATIONS AND PRELIMINARIES

We denote by $\mathcal{R}_{ps}$ the ring of proper and stable rational functions, and by ∂° the degree function defined from $\mathcal{R}_{ps}$ to $\mathbf{Z}_+$ by :

- ∂° f(s)= ordre of infinite zero + number of finite unstable zeros of f(s) (counted with their multiplicity), if $f(s) \neq 0$
- $\partial^\circ f(s) = \infty$, if $f(s) = 0$.

Note that $\mathcal{R}_{ps}$ is an Euclidean Domain (with this degree function).

$\mathcal{R}_p$ denotes the ring of proper rational functions. $\mathcal{M}_p$ and $\mathcal{M}_{ps}$ denote the set of matrices over $\mathcal{R}_p$ and $\mathcal{R}_{ps}$ respectively. $\mathcal{U}_p$ and $\mathcal{U}_{ps}$ are the groups of units of $\mathcal{M}_p$ and $\mathcal{M}_{ps}$ respectively. We denote by Λ the ring of $\mathcal{M}_{ps}$ - matrices which are diagonal and nonsingular.

Let $P \in \mathcal{M}_p$. We say that $(N, D) \in \mathcal{M}_{ps} \times \mathcal{M}_{ps}$ is a right coprime factorization (r.c.f) of P if :

i) $P = N D^{-1}$

ii) N, D are right coprime (r.c) ; i.e. : $XN + YD = I$; with X and Y in $\mathcal{M}_{ps}$.

Let us recall that i f (N, D) is a r.c.f of $P \in \mathcal{M}_p$, then all the r.c.f of P are given by (NU, DU) where $U \in \mathcal{U}_{ps}$.

DEFINITION II.1 : Let A, B $\in \mathcal{M}_{ps}$. We say that (A, B) are skew-prime if there exists X and Y in $\mathcal{M}_{ps}$ such that : $XA + BY = I$.

The order in which A and B are cited is taken in consideration. This avoids the definition of internal and external skew-primness as in [16].

THEOREM II.2 : Let A, B $\in \mathcal{M}_{ps}$, of dimensions qxp and pxm, respectively. Then the following conditions are equivalent :

1) the pair of matrices (A,B) is skew-prime

2)there exists a pair of $\mathcal{M}_{ps}$-matrices, $\bar{A}$ and $\bar{B}$, of sizes rxm and qxr, respectively such that :

$AB = \bar{A}\bar{B}$ and (A, $\bar{B}$) are left coprime, and ($\bar{A}$, B) are right coprime.

Furthermore if A is square and det.A≠0 (resp. B is square and det.B≠0), then :

i) $B = U \bar{B} V$ (resp. $A = U \bar{A} V$) where U and V are in $\mathcal{U}_{ps}$

ii) det B = u. det $\bar{B}$ (resp. det A = u. det $\bar{A}$) where u is a unit of $\mathcal{R}_{ps}$.

The pair ($\bar{B}$, $\bar{A}$) is said to be a skew-complement of the pair (A, B).

This theorem is proved by Wolovich ([16]) for the polynomial functions case, the proof remains valid for any principal ideal domain, in particularly for $\mathcal{R}_{ps}$.

III. PROBLEM DESCRIPTION AND PRELIMINARY RESULTS

We consider the M.I.M.O linear, time invariant system $\sum(P,K,C)$ described by the configuration below :

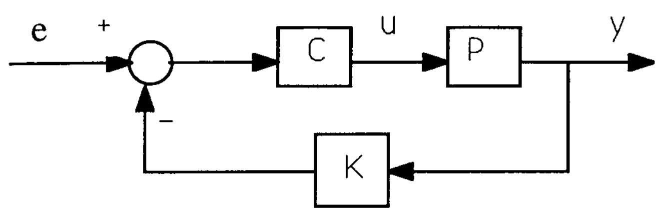


Fig.1. The system $\sum(P,K,C)$

Let us consider the following problem : given a plant $P \in \mathcal{M}_p$ of size ℓxm, rank$P = \ell$, (the rank of the matrices is defined as the number of columns or rows which are independent over $\mathcal{R}_p$), we wish to design two proper controllers K and C such that the resulting feedback system $\sum(P,K,C)$ shown in Fig.1 is internally stable and the input-output map : $e \rightarrow y$ is nonsingular, stable and decoupled and have a "minimum number of unstable zeros " (i.e : $PC.(I + KPC)^{-1} \in \Lambda$ and $\partial^{\circ}PC.(I + KPC)^{-1}$ is minimal).

To give the full solution of this problem we need some preliminary results concerning the decoupling problem. This is done hereafter.

DEFINITION III.1 : let $P = N D^{-1} \in \mathcal{M}_p$ where (N, D) is any r.c.f of P. We say that $Q \in \mathcal{M}_{ps}$, such that $PQ \in \mathcal{M}_{ps}$, is admissible relatively to P iff:

i) $Q = D.R.\Delta$; with $R \in \mathcal{M}_{ps}$ and $\Delta \in \mathcal{M}_{ps} \cap \mathcal{U}_p$

ii) (D,R) are skew-prime and (NR, Δ) are skew-prime.

THEOREM III.2 ([3]) : Let $P \in \mathcal{M}_p$. The two following statements are equivalent:

i) there exists controllers K and C in $\mathcal{M}_p$ such that the system $\sum(P,K,C)$ is internally stable

ii) there exists a stable precompensator Q (i.e: $Q \in \mathcal{M}_{ps}$) which is admissible relatively to P and the transfer matrix from e to y (Fig.1) is equal to the product $PQ \in \mathcal{M}_{ps}$.

It appears, from the theorem above, that the decoupling problem with internal stability using the configuration shown in the Fig.1 is equivalent to the following one. Given a plant $P \in \mathcal{M}_p$ of size ℓxm, rank$P = \ell$, we have to prove the existence of a precompensator $Q \in \mathcal{M}_{ps}$ of seize mxℓ, such that :

i) $PQ \in \mathcal{M}_{ps}$, diagonal and non singular ie. $PQ \in \Lambda$;

ii) Q is admissible relatively to P ;

iii) PQ have a "minimum number of unstable zeros".

The solution of the decoupling problem when relaxing the condition iii) above is contained in the following theorem :

THEOREM III.3 Let $P \in \mathcal{M}_p$ of size ℓxm, rank$P = \ell$. Then there always exists $Q \in \mathcal{M}_{ps}$ of size mxℓ such that :

i) $PQ \in \Lambda$

ii) Q is admissible relatively to P.

Proof : The complete proof of the theorem can be found in [4].

Let $P \in \mathcal{M}_p$ of size ℓxm, rank $P = \ell$. An r.c.f (N, D) of P can be derived from the Smith MacMillan form of P (see [14]) :
$P = N D^{-1}$ with :

$$N = V[\overset{\ell}{\mathbf{n}} \,.......\, \overset{m-\ell}{\mathbf{0}}] \tag{3.1}$$

$V \in \mathcal{U}_{ps}$ and $\mathbf{n}$ = diag.$\{n_1, ..., n_\ell\}$ with n_i / n_{i+1}, for i = 1,..., ℓ-1, and $n_i \in \mathcal{R}_{ps}$ for i = 1,..., ℓ .

$$D = W.\begin{array}{c} \begin{array}{cc} \ell & m-\ell \end{array} \\ \left[\begin{array}{c|c} \mathbf{d} & 0 \\ \hline 0 & I_{m-\ell} \end{array}\right] \end{array} \tag{3.2}$$

$W \in \mathcal{U}_{ps}$ and $\mathbf{d} = \text{diag.}\{d_1, ..., d_\ell\}$ with d_{i+1} / d_i, for $i = 1,..., \ell-1$, and $d_i \in \mathcal{R}_{ps}$ for $i = 1,..., \ell$.
Furthermore the pairs (n_i, d_i) are coprime.

Let : $Q = D.Q_0$ where

$$Q_0 = \left[\begin{array}{c} \bar{\mathbf{n}} \\ \hline 0 \end{array}\right] V^{-1} \in \mathcal{M}_{ps} \tag{3.3}$$

with $\bar{\mathbf{n}} = \text{diag}\{\bar{n}_1, ..., \bar{n}_\ell\}$; where $\bar{n}_i = n_\ell . n_i^{-1} \in \mathcal{R}_{ps}$, for $i=1, .., \ell$.

$$P.Q = N.\ Q_0 = \begin{bmatrix} n_\ell & & 0 \\ & \ddots & \\ 0 & & n_\ell \end{bmatrix} = n_\ell . I_\ell \tag{3.4}$$

where I_ℓ is the $(\ell x \ell)$-identity matrix. In this way we can see that PQ is decoupled.

It is shown in [4] that $Q = D.Q_0$, where D and Q_0 are defined as in (3.2) and (3.3) is admissible relatively to P.

IV. DECOUPLING WITH MINIMUM NUMBER OF UNSTABLE ZEROS

It is clear that the solution $Q=D.Q_0$ given in the last section may introduce a large number of unstable zeros in the system. Then it makes sens to characterize a stable and admissible precompensator Q ($\in \mathcal{M}_{ps}$) which garanties a "minimum number of unstable zeros" (in some sens) in the transfer matrix of a decoupled system given by $PQ \in \Lambda$.

DEFINITION IV.1 : let $P \in \mathcal{M}_p$ and $Q \in \mathcal{M}_{ps}$ such that $PQ \in \Lambda$ and Q is admissible relatively to P. We say that Q is minimal if for any admissible $Q' \in \mathcal{M}_{ps}$ such that $PQ' \in \Lambda$, we have $\partial^\circ \det.(PQ) \leq \partial^\circ \det.(PQ')$.

This definition implies that the number of unstable zeros in the decoupled system PQ is smallest than any one other decoupled system PQ'.
The notion of minimum which will be considered in the next is always in the sens of the definition IV.1.

Every $Q \in \mathcal{M}_{ps}$ which satisfies the definition IV.1 is said to be an admissible minimum decoupling precompensator (a.m.d.p). It is clear, from theorem III.2, that any a.m.d.p Q gives at least a pair of proper controllers (K,C) solution to the decoupling problem defined in section III with a minimum number of unstable zeros in the decoupled system.

Relaxing the admissibility condition for the precompensator $Q \in \mathcal{M}_{ps}$ we firstly give the parametrization of all precompensators $Q \in \mathcal{M}_{ps}$ such that, for a given plant $P \in \mathcal{M}_p$, $PQ \in \Lambda$. And secondly, we derive some solution to the a.m.d.p problem.

A. *Parametrization of the set of decoupling and stabilizing precompensators*

For any given plant $P \in \mathcal{M}_p$, of size ℓxm, rank $P = \ell$. We denote by $\mathbf{C} = \{Q \in \mathcal{M}_{ps}$, of size m$x\ell$, such that $PQ \in \Lambda\}$ the set of all stable precompensators Q which ensure, for given plant $P \in \mathcal{M}_p$, that the product PQ is stable, diagonal and nonsingular.

We are now able to characterize the elements of the set $\mathbf{C}$. Before doing this, let us recall the following :
for any given plant $P \in \mathcal{M}_p$, of size ℓxm, rank $P = \ell$, w e consider the r.c.f (N, D) given by (3.1) and (3.2). We know that a decoupling solution is given by the precompensator Q defined as :

$Q = D.Q_0 \in \mathcal{M}ps$, of seize $m \times \ell$, with

$$Q_0 = \begin{bmatrix} \bar{n} \\ ---- \\ 0 \end{bmatrix} V^{-1} \in \mathcal{M}ps$$

with $\bar{n} = \text{diag}\{\bar{n}_1, \ldots, \bar{n}_\ell\}$; where $\bar{n}_i = n_\ell . n_i^{-1} \in \mathcal{R}ps$, for $i=1, .., \ell$ and satisfies :

$$PQ = V\,\mathbf{n}\,.\bar{n}\,V^{-1} = n_\ell .I_\ell .$$

In this way, we see that PQ is decoupled.

Now define :

- β_i = g.c.d. $Col_i\{\bar{n}\,V^{-1}\}$ - the greatest common divisor of the elements of the column number i of the matrix $\bar{n}\,V^{-1}$ for $i=1,..., \ell$. (4.1)
- $\beta = \text{diag}\{\beta_i\} \in \Lambda$ for $i=1, .., \ell$. (4.2)
- $R_0 = \bar{n}\,.\,V^{-1}.\,\beta^{-1} \in \mathcal{M}ps$ (4.3)

Note that it is easy to see that :
$PQ = DQ = V\mathbf{n}\,.\,\bar{n}\,V^{-1} = V\mathbf{n}\,R_0\,\beta = \beta_0\beta \in \Lambda$
where $\beta_0 = V\mathbf{n}\,R_0 \in \Lambda$

PROPOSITION IV.2 : Every $Q \in \mathbf{C}$ is given by the following form :

$$Q_0 = D.\begin{bmatrix} R_0\,\gamma \\ ---- \\ X \end{bmatrix}$$

with $\gamma \in \Lambda$ and $X \in \mathcal{M}ps$.

Proof : For a complete proof of this proposition, the reader is refered to [1, 4].

This proposition gives a characterization of the set $\mathbf{C}$ of all the stable precompensators $Q \in \mathcal{M}ps$, which are not necessarily admissible, such that the product $PQ \in \Lambda$, for a given plant $P \in \mathcal{M}p$ of size $\ell \times m$.

B. *Problem solution*

In this part we are going to give a solution to the decoupling problem with a minimum number of unstable zeros. Before doing this , we will precise the notion of minimum given in definition IV.1.

PROPOSITION VI.3 : The precompensator

$$Q = D.\begin{bmatrix} R_0\,\gamma \\ ---- \\ X \end{bmatrix} \in \mathbf{C}$$

is minimal iff for all $\bar{Q} = D.\begin{bmatrix} R_0\,\bar{\gamma} \\ ---- \\ X \end{bmatrix}$ *which is in $\mathbf{C}$ and is admissible, we have $\partial^\circ(\det \gamma) \leq \partial^\circ(\det \bar{\gamma})$, where γ and $\bar{\gamma}$ are in Λ.*

Before stating the main theorem, we give an intermediary result which derive a subset of admissible decoupling precompensators. This subset will be used in the main theorem.

LEMMA IV.4 : For every $Q = D.\begin{bmatrix} R_0\,\gamma \\ ---- \\ X \end{bmatrix}$ *which is in $\mathbf{C}$ and is admissible, it is true that* $Q_1 = D.\begin{bmatrix} R_0 \\ ---- \\ 0 \end{bmatrix}.\gamma$ *is in $\mathbf{C}$ and admissible.*

Proof: the proof is given in [1,4].

Let us now give the main result solving the problem in consideration :

MAIN THEOREM : Let $Q_1 = D.\begin{bmatrix} R_0 \\ ---- \\ 0 \end{bmatrix}.\beta_1$ *be in* $\mathbf{C}$ *and admissible, with* β_1 *in* Λ*. If* β_1 *and* β *(defined in (4.1) and (4.2)) are coprime,*

then $Q_{min} = D.\begin{bmatrix} R_0 \\ ---- \\ 0 \end{bmatrix}$ *is admissible relatively to P.*

The following corollary allows us to completely characterize a solution of the problem.

COROLLARY IV.5 : For every

$Q = D.\begin{bmatrix} R_0 \\ ---- \\ 0 \end{bmatrix}.\gamma$ *which is in* $\mathbf{C}$ *and is admissible with* γ *in* Λ*, there exists* β_1 *and* γ_1 *in* Λ *such that :*

i) $\gamma = \beta_1 . \gamma_1$ *and* β_1 *divides* β *(* $\beta = \alpha . \beta_1$ *with* α *in* Λ *);*

ii) Let $Q_1 = D.\begin{bmatrix} R_0 \\ ---- \\ 0 \end{bmatrix}.\beta_1$ *be in* $\mathbf{C}$ *is admissible.*

Since the set of the divisors, which are not units, of β (this set is denoted $\mathfrak{D}(\beta)$) is finite, there exists at least one β_j in $\mathfrak{D}(\beta)$ such that ∂° (det β_j) is minimal and $Q_j = Q_{min}\ \beta_j$ in $\mathbf{C}$ is admissible; this means that Q_j is a solution of the a.m.d.p problem.
The fact that the set $\mathfrak{D}(\beta)$ is finite is due to the finiteness of the non-units elementary divisors of β.

The proves of the main theorem and the corollary are given in [1].

V. CONCLUSION

In this paper we considered the problem of decoupling with a minimum number of unstable zeros under internal stability condition. It appeared from the results that the solution is not unique. But we shown that a particular solution can be derived using the results obtained in [4] concerning the decoupling problem with internal stability.
One can note that the notion of skew-prime matrices plays a key role in solving the problem.
We do think that a challenging research is to derive a constructive algorithm to get an explicit solution of the problem.

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