Robust Output Stabilizing Control For Discrete Uncertain MIMO Systems

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Abstract
Given a discrete uncertain system with multiple inputs and multiple outputs (MIMO), matched uncertainties, and a measured output, a necessary and sufficient condition is given for the existence of a min-max output feedback stabilizing controller. The min-max controller, defined in [1]-[2], is in general *state* dependent. It is shown that the condition for realizing the min-max control-law via *output* feedback is that the transfer matrix of an auxiliary system, precisely defined in the paper, can be made a discrete strictly positive real transfer matrix by premultiplication by a constant matrix F. The existence problem of an output stabilizing controller for uncertain systems was treated extensively and solved for the continuous case. Only recently, the problem was solved for single input single output (SISO) discrete systems, [20]. In this paper the results of [20] are generalized to MIMO systems. Using the results developed in the paper, the complete min-max control may be designed entirely in the frequency domain.
The paper contains also some new results on discrete positive real matrices which are essential for deriving the existence conditions.

I. Introduction
In recent years a considerable amount of work has been devoted to the problem of stabilizing discrete uncertain dynamic systems with bounded uncertainties (e.g. [3]÷[10]). All the mentioned works dealt with controllers which are state dependent.
Consider an uncertain discrete system:

$$x(k+1)=[A+\Delta A(k)]x(k)+[B+\Delta B(k)]u(k)+v(k) \quad (1.1)$$

where $x\in\Re^n$, $u\in\Re^m$ and A, B are of appropriate dimensions. $v(k)\in\Re^n$ represents the external disturbances. Under the assumption that the uncertainties satisfy the matching conditions, system (1.1) can be represented as:

$$x(k+1)=Ax(k)+B\{u(k)+\eta(x(k), k)\} \quad (1.1a)$$

where $\eta(x, k)$ represents the matched uncertainties of the system, and is unknown but is assumed to be cone bounded, i.e.:

$$\eta(x(k),k)\le\rho_o+\rho_1\|x(k)\| \quad (1.2)$$

In (1.2), ρ_1 is a function of $\Delta A(k)$ and $\Delta B(k)$ and ρ_0 is a function of v(k).
In [1], [2] the following state feedback control law for system (1.1a) was derived:

$$u(k)=-\left(B^TPB\right)^{-1}B^TPAx(k) \quad (1.3)$$

where P solves the discrete Lyapunov equation:

$$A^TPA-P=-LL^T \quad (1.4)$$

for a positive definite (p.d.) LL^T. In [1], [2] it is shown that for $\Delta B=0$, (1.3) is a min-max controller for (1.1a) since it minimizes the Lyapunov function's difference, ΔV, for the worst case of the uncertainties (minimum on u of the *bound* on $\max_\eta\{\Delta V\}$). The properties of the discrete min-max controller are as follows:
Define:

$$\xi=\frac{\underline{\lambda}\left(LL^T\right)}{\overline{\lambda}\left(B^TPB\right)} \quad (1.5)$$

where P, $LL^T>0$ satisfy the Lyapunov equation (1.4) and $\underline{\lambda}(Q)$, $\overline{\lambda}(Q)$ denote the smallest and largest eigenvalue of a square matrix Q, respectively. If

$$\rho_1<\sqrt{\xi} \quad (1.6)$$

then
1. In the absence of external disturbances (i.e. v(k)=0), (1.3) guarantees asymptotic stability of system (1.1a) even when $\Delta B\neq0$.
2. In the presence of external disturbances ($v(k)\neq0$) and for $\Delta B=0$, (1.3) assures uniform boundedness.

The min-max controller (1.3) requires, for its realization, all the states of the system. In practice, however, usually not all state variables are available- i.e. either some of them are not accessible or it is very costly to measure them all. The question therefore arises: When can the discrete min-max state control-law be realized via *output* feedback?

In continuous systems [11]÷[17] a stabilizing controller for an uncertain system is also based on state feedback. Steinberg and Corless [18] have shown that for *continuous* systems, a sufficient condition for realizing an output feedback stabilizing controller for an uncertain system with matched uncertainty is that the open loop is Strictly Positive Real (s.p.r.). Following Steinberg and Corless, Magaña and Zak [19] tried to define conditions for realizing an output min-max controller for a *discrete* system with the following output:

$$y(k)=Cx(k) \tag{1.7}$$

They pointed out that realization of such a controller requires a Positive Definite (p.d) matrix P which solves the discrete Lyapunov equation (1.4) for a p.d. LL^T, and satisfies:

$$B^TPA=C \tag{1.8}$$

Since searching for such a P requires the use of numerical decision methods, it is noted in [19] that "one would be tempted to extend Steinberg and Corless's results [regarding continuous systems] to the discrete-time case". However, as it is clear from corollary 2.3 of section II, a system with the output (1.7) can never be Discrete Strictly Positive Real (d.s.p.r.) even if there exists P that solves the Lyapunov equation (1.4) and satisfies (1.8). Magaña and Zak's final conclusion was that the question of a system theoretic interpretation of the existence of the matrices P and L that satisfy (1.4), (1.8) has not yet been resolved, and remains an open problem.
In [20] the conditions for realizing the min-max control via output feedback, (i.e. the condition for the existence of P that solves (1.4) and satisfies (1.8)) were derived for *SISO* systems. It was also shown that those conditions are not analogous to the conditions for the continuous case. In this paper the results of [20] are generalized to the *MIMO* case.
The rest of the paper is organized as follows. In section II some new results concerning Discrete Positive Real (d.p.r) and Discrete Strictly Positive Real (d.s.p.r.) systems, needed for the development in the paper, are presented. Section III presents the conditions for realizing the min-max controller via output feedback in multiple outputs and multiple inputs systems. Conclusions are presented in section IV.

II Preliminaries

The purpose of this section is to present some well known definitions and results as well as new ones, on discrete positive realness (d.p.r.), discrete strictly positive realness (d.s.p.r.) and on projection matrices. These results will be utilized in the subsequent development.

Definition 2.1 (Discrete Positive Real-d.p.r.) [21]:
A square matrix G(z) of real-rational functions is discrete positive real (d.p.r.) if it has the following properties:

G(z) has elements analytic in $|z|>1$ and

$G^*(z)+G(z)\geq 0$ in $|z|>1$

where $G^*(z)$ denotes the transpose conjugate of G(Z).

Lemma 2.2 (Discrete Positive Real (d.p.r.) Lemma) [21], [22]:
Let G(z) be a square (m×m) transfer function matrix with all poles inside the unit disc (u.d.) (or simple poles only on the u.d.) then G(z) is d.p.r. iff there exist $L\in R^{n\times v}$, $W\in R^{v\times m}$, $P\in R^{n\times n}$, P>0 and symmetric, such that:

$$A^TPA-P=-LL^T \tag{2.1}$$
$$B^TPA=C-W^TL^T \tag{2.2}$$
$$B^TPB=D+D^T-W^TW \tag{2.3}$$

where {A, B, C, D} is a minimal realization of G(z).

Corollary 2.3:
A necessary condition for d.p.r. is that the system is non strictly proper and $D+D^T>0$.

Proof: The proof is readily obtained using equation (2.3).

In the results to follow, projection matrices will be used. Hence, in the following, we present a basic definition and a lemma of such matrices.

Definition 2.4 [23]:
A projection matrix Ω is a matrix that satisfies:

$$\Omega^2=\Omega \tag{2.4}$$

If a projection matrix Ω is symmetric, then it is called an orthogonal projection matrix.

Lemma 2.5 [23]:
a symmetric matrix Ω is an orthogonal projection matrix iff all its eigenvalues are either 1 or 0.

The next lemma is the key for proving theorem 2.7.

Lemma 2.6:
Given $X \in R^{\nu \times m}$, and the following equation:

$$W^T W = W^T X \tag{2.5}$$

all solutions (for W) of (2.5) are given by:

$$W = \Omega X \tag{2.6}$$

where Ω is any orthogonal projection matrix.

Proof: the proof is given in the appendix.

the next theorem is crucial for proving the main result of the paper.

Theorem 2.7 :
Let {A, B, C, D} be a minimal realization of a stable and a square transfer matrix G(z), where A is nonsingular. The system is d.p.r. and D is symmetric and equal to:

$$D = \tfrac{1}{2} C A^{-1} B \tag{2.7}$$

iff there exists $\tilde{P} > 0$ which solves the discrete Lyapunov equation (2.1) and satisfies:

$$A^T \tilde{P} A - \tilde{P} = -\tilde{L}\tilde{L}^T \tag{2.1a}$$
$$B^T \tilde{P} A = C \tag{2.2a}$$
$$B^T \tilde{P} B = 2D \tag{2.3a}$$

(i.e. the d.p.r. equations hold for W=0 and a symmetric D)

Proof:
Sufficiency: Suppose there exists $\tilde{P} > 0$ such that equations (2.1a), (2.2a), (2.3a) hold, then according to lemma 2.2, the system is d.p.r. with W=0. Multiplying (2.2a) by $A^{-1}B$, yields:

$$B^T \tilde{P} B = C A^{-1} B \tag{2.8}$$

which together with (2.3a) implies (2.7). And since $\tilde{P}$ is symmetric then (2.8) implies that D in (2.7) is also symmetric.
Necessity: Suppose that D is given as in (2.7) and it is symmetric. Suppose also that G(z) is d.p.r. Then, according to lemma 2.2, there exist matrices L, W, and P>0 such that equations (2.1÷2.3) hold. Substitution of C from (2.2) into (2.7) and then substitution of D (which is symmetric) into (2.3) yields:

$$W^T W = W^T L^T A^{-1} B \tag{2.9}$$

According to lemma 2.6, *all* the solutions of (2.9), for W, are given by:

$$W = \Omega L^T A^{-1} B \tag{2.10}$$

where Ω is any orthogonal projection matrix. Substitution of W from (2.10) into (2.2), yields:

$$\begin{aligned} C &= B^T P A + B^T A^{-T} L \Omega L^T = \\ &= B^T \left(P + A^{-T} L \Omega L^T A^{-1}\right) A \end{aligned} \tag{2.11}$$

or:

$$C = B^T \tilde{P} A \tag{2.12}$$

where

$$\tilde{P} = P + A^{-T} L \Omega L^T A^{-1}. \tag{2.13}$$

Since Ω is symmetric so does $\tilde{P}$. It is left to show that if P satisfies the Lyapunov equation (2.1) then there exists $\tilde{L}$ such that $\tilde{P}$ in (2.13) solves (2.1a). Substitution of P from (2.13) into the Lyapunov equation (2.1) yields:

$$\begin{aligned} &A^T\left(\tilde{P} - A^{-T} L \Omega L^T A^{-1}\right) A - \\ &-\left(\tilde{P} - A^{-T} L \Omega L^T A^{-1}\right) = -L L^T \end{aligned} \tag{2.14}$$

or:

$$A^T \tilde{P} A - \tilde{P} = -L(I - \Omega) L^T - A^{-T} L \Omega L^T A^{-1} \tag{2.15}$$

Since all eigenvalues of Ω are equal to 1 or 0, then $I-\Omega \geq 0$, and hence, the right hand side of (2.15) is nonpositive definite, which implies that there exists $\tilde{L}$ such that $\tilde{P}$ and $\tilde{L}$ satisfy (2.1a).

Remark: It is worthwhile noting that (2.7) is equivalent to the property that the products of all poles and zeros of G(z), are identical except for a sign. This can be shown using the fact that if (2.7) holds, then $G(0) = -G(\infty)$.

In the subsequent development use is made of the *strictly* positive real property of discrete-time systems. Hence, the following definition and lemma are in order.

Definition 2.8 (Discrete Strictly Positive Real- d.s.p.r.):
A transfer matrix G(z) is d.s.p.r. iff there exists $\varepsilon > 0$ such that $G\left(z \cdot e^{-\varepsilon}\right)$ is d.p.r.

Lemma 2.9:

Consider the transfer function $G(z)=C(zI-A)^{-1}B+D$ where D is symmetric and given by (2.7). If G(z) is d.s.p.r., then equations (2.2a), (2.3a) hold with P that solves the Lyapunov equation (2.1) for a *positive definite* matrix LL^T $\left(LL^T>0\right)$.

Proof:

The proof for the SISO case is given in [20]. The proof for the MIMO case follows along the same lines using theorem 2.7 instead of the counterpart theorem for SISO systems.

III. Output min-max control-law

In this section we treat MIMO systems with *given* outputs and seek conditions which guarantee that the min-max control-law can be applied via the given outputs. Furthermore, when the conditions are satisfied, an explicit expression for the output control law is given both in terms of the state space representation and the transfer matrix of the system.

Consider the following nominal system:

$$\begin{aligned} x(k+1)&=Ax(k)+Bu(k) \\ y(k)&=Cx(k) \end{aligned} \tag{3.1}$$

where $x\in\Re^n$, $u\in\Re^m$, $y\in\Re^p$

Consider the following assumptions:.

(i) C and B are of full ranks.

(ii) The number of outputs is greater or equal to the number of inputs, $p\geq m$

(iii) {A, B, C} is minimal, i.e. the system is controllable and observable.

(iv) A is nonsingular.

(v) The nominal system is asymptotically stable, or it can be stabilized via output feedback. (Without loss of generality, we shall assume that the system is asymptotically stable).

The next theorem presents the conditions under which the min-max controller (1.3) can be applied to system (3.1) which satisfies (i)÷(v) via the given output. The theorem deals with strictly proper systems. Later it will be extended also to proper systems.

Theorem 3.1:

Consider the system {A, B, C} defined in (3.1) with the assumptions (i)÷(v). Let the corresponding transfer matrix of (3.1) be $G(z)=C(zI-A)^{-1}B$.

Define an auxiliary system:

$$G_a(z)=G(z)+D_a \tag{3.2}$$

where

$$D_a=\tfrac{1}{2}CA^{-1}B=-\tfrac{1}{2}G(0) \tag{3.3}$$

The min-max control law can be applied to the original system (3.1) via the output iff there exists a matrix $F\in\Re^{m\times p}$ such that:

1) $D_F \triangleq FD_a$ is symmetric

2) $G_F(z)\triangleq FG_a(z)$ is d.s.p.r.

The output controller is then given by:

$$u(k)=-(2D_F)^{-1}Fy(k)=\{FG(0)\}^{-1}Fy(k) \tag{3.4}$$

Proof:

The realization of $G_F(z)$ is given by $\{A,B,C_F,D_F\}$ where $C_F\triangleq FC$ and $D_F\triangleq\tfrac{1}{2}C_FA^{-1}B$. Hence, according to theorem 2.7 and lemma 2.9, there exists P that solves the Lyapunov equation (2.1) for $LL^T>0$ and satisfies:

$$B^TPA=C_F=FC \tag{3.5}$$

$$B^TPB=2D_F=2FD_a \tag{3.6}$$

Iff D_F is symmetric and $G_F(z)$ is d.s.p.r.

According to corollary 2.3, the positive realness of $G_F(z)$ guarantees that D_F is nonsingular. Now substitute (3.5) and (3.6) into the control law (3.4) and use y=Cx to obtain the equivalency of the output control law (3.4) and the min-max control law (1.3). Using (3.3), the right hand side of (3.4) follows immediately

Corollary 3.2:

Necessary conditions for the existence of an output min-max controller for system (3.1) with the assumptions (i)÷(v) are:

(a) $CA^{-1}B$ is of full column rank.

(b) $G_a(z)$ which is defined in (3.2), (3.3), is minimum-phase.

Proof:

According to theorem 3.1, a necessary (and a sufficient) condition for the existence of an output min-max controller, is that $G_F(z)$ is d.s.p.r. According to corollary 2.3, the positive realness of $G_F(z)$ implies the nonsingularity of D_F, which implies the necessity of (a), and also that F is of full row rank. The strictly positive realness of $G_F(z)$ implies also that $G_F(z)$ is minimum-phase. Since F is of full row rank, then the zeros of $G_a(z)$ are also zeros of $G_F(z)$. This implies the necessity of (b). Note that (a) is the reason for assumption (ii).

__Remark 1__: It is clear from (3.4), that when the conditions of the theorem are satisfied, the design of the output min-max controller is based upon either the transfer matrix or the state space representation of the system, and there is no need to compute the matrices P and L.

__Remark 2__: Note that unlike in the continuous case, the conditions are applied to the auxiliary system. Consequently, FG(z) need not be d.s.p.r. and may be even nonminimum phase and still may satisfy the conditions of the theorem. This has been demonstrated in the SISO case [20].

__Remark 3__: It should be emphasized that if the conditions of the above theorem are satisfied, then every control law of the form $u(k) = -K\left(B^T PA\right)x(k)$, where K is a m×p gain matrix, can be realized via the given output. However the specific gain in (3.4) leads to the min-max control defined in [2]. This remark applies to the next corollary as well.

The following corollary extends theorem 3.1 to proper systems.

__Corollary 3.3:__

Consider now the case where a proper system is given by:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{3.7}$$

Let $G(z) = C(zI - A)^{-1}B + D$ be the corresponding transfer function.

Similar to the strictly proper case, define the auxiliary system $\left\{A, B, C, D_a \triangleq \frac{1}{2}CA^{-1}B\right\}$, with the corresponding transfer function:

$$G_a(z) = G(z) - \tfrac{1}{2}\{G(0) + G(\infty)\} \tag{3.8}$$

if there exists F such that:

1) $FD_a = \frac{1}{2}F[G(\infty) - G(0)]$ is symmetric
2) $FG_a(z)$ is d.s.p.r.
3) $F(2D_a - D) = -FG(0)$ is nonsingular

then the min-max control-law (1.3) can be applied to the proper system (3.7) via the ***given*** output, using the following output control-law:

$$u(k) = -\{F(2D_a - D)\}^{-1}Fy(k) \tag{3.9}$$

or in terms of the transfer function:

$$u(k) = \{FG(0)\}^{-1}Fy(k) \tag{3.10}$$

__Proof:__

The control law (3.9) can be written as follows:

$$\{F(2D_a - D)\}u(k) = -Fy(k) = -FCx(k) - FDu(k) \tag{3.11}$$

or

$$u(k) = -\tfrac{1}{2}(FD_a)^{-1}FCx(k) \tag{3.12}$$

note that according to corollary 2.3, FD_a is nonsingular.

It is clear from the proof of theorem 3.1, that if $G_F(z)$ is d.s.p.r., then there exists P that solves (2.1) for $LL^T>0$ and satisfies:

$$FC=B^TPA \tag{3.13}$$

Using the definition of D_a, (3.13) implies that:

$$FD_a = \tfrac{1}{2}B^TPB \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12) leads immediately to the equivalence of the output control (3.9) and the min-max control law. Noting that $D = G(\infty)$ and that $D_a = -\frac{1}{2}[G(0) - G(\infty)]$ the equivalency of (3.9) and (3.10) follows immediately.

__Remark 4__: If D of the original system (3.7) already satisfies $D = \frac{1}{2}CA^{-1}B$, then $G_a(z) = G(z)$. In such a circumstance, the theorem is applied directly to the system.

The question of the existence of F that satisfies the conditions of theorem 3.1 or corollary 3.3, remains an open question. In the case where there exists such a matrix F, then it is not unique. It was mentioned in the introduction that the bound on the allowable uncertainties is a function of the matrices P and L (equation (1.5)), and according to (3.5), P depends on F. The question of how to select F among all the matrices F that satisfy the

conditions of the theorem, in order to achieve the best robustness properties is currently under investigation.
In the case where the number of inputs is equal to the number of outputs (p=m), F if a *square* nonsingular matrix. In this case, the control laws (3.4) and (3.10) are reduce to:

$$u(k)=\{G(0)\}^{-1}y(k) \tag{3.15}$$

and the matrix F, which is essential for checking the existence condition, does not appear in the control law and consequently does not affect the robustness properties of the controller, though, it affects the allowable bounds on the uncertainties.

IV. Conclusions

The problem of realizing the discrete min-max controller for discrete uncertain MIMO systems with matched and bounded uncertainties via a given output was considered. Necessary and sufficient conditions were derived for the existence of an *output* min-max control-law. Using the results of this paper, the checking of the existence conditions, as well as the complete design of the output controller, may be performed entirely in the frequency domain.

Appendix: proof of lemma 2.6

It is clear from (2.5) that Rank(W)≤Rank(X) and that a linear combination of the rows of W is equal to a linear combination of the rows of X. Hence all solutions of (2.5) are given by:

$$W=\Omega X \tag{A.1}$$

where Ω is square but otherwise a completely general matrix. Substitution of (A.1) into (2.5) yields:

$$X^T\Omega^T\Omega X = X^T\Omega^T X \tag{A.2}$$

If X has full row rank then (A.2) implies $\Omega^T\Omega=\Omega^T$, which implies that Ω is symmetric and therefore is an orthogonal projection matrix.
If Rank(X)=r<v then there exists an orthogonal matrix U $\left(U^TU=I_v\right)$ such that:

$$UX=\begin{bmatrix} X_1 \\ \hline 0 \end{bmatrix}\begin{matrix} r \\ v-r \end{matrix} \tag{A.3}$$

where $X_1\in\Re^{r\times m}$ has a full row rank. Rewriting equation (2.5) in the following form:

$$W^TU^TUW = W^TU^TUX \tag{A.4}$$

yields:

$$\tilde{W}^T\tilde{W} = \tilde{W}^T\begin{bmatrix} X_1 \\ \hline 0 \end{bmatrix} \tag{A.5}$$

where:

$$\tilde{W}\triangleq UW \tag{A.6}$$

According to (A.1), the solutions of (A.5) are given by:

$$\tilde{W}=\tilde{\Omega}\begin{bmatrix} X_1 \\ \hline 0 \end{bmatrix}=\left[\begin{array}{c|c} \tilde{\Omega}_1 & \tilde{\Omega}_{12} \\ \hline \tilde{\Omega}_{21} & \tilde{\Omega}_2 \end{array}\right]\begin{bmatrix} X_1 \\ \hline 0 \end{bmatrix}=\begin{bmatrix} \tilde{\Omega}_1X_1 \\ \hline \tilde{\Omega}_{21}X_1 \end{bmatrix} \tag{A.7}$$

From (A.7) it follows:

$$\tilde{W}^T\tilde{W} = X_1^T\left(\tilde{\Omega}_1^T\tilde{\Omega}_1+\tilde{\Omega}_{21}^T\tilde{\Omega}_{21}\right)X_1 \tag{A.8}$$

and

$$\tilde{W}^T\begin{bmatrix} X_1 \\ \hline 0 \end{bmatrix} = X_1^T\tilde{\Omega}_1^TX_1 \tag{A.9}$$

Equating the right hand sides of (A.8) and (A.9) (according to (A.5)) and using the fact that X_1 has a full row rank, yields:

$$\tilde{\Omega}_1^T\tilde{\Omega}_1+\tilde{\Omega}_{21}^T\tilde{\Omega}_{21}=\tilde{\Omega}_1^T \tag{A.10}$$

which implies that $\tilde{\Omega}_1$ is *symmetric*, hence (A.10) can be rewritten as:

$$\tilde{\Omega}_1-\tilde{\Omega}_1^2=\tilde{\Omega}_{21}^T\tilde{\Omega}_{21} \tag{A.11}$$

From (A.7) it is seen that $\tilde{W}$ depends upon $\tilde{\Omega}_1$ and $\tilde{\Omega}_{21}$ only. Therefore, all solutions for $\tilde{W}$ which is given in (A.7) can be expressed by a symmetric $\tilde{\Omega}$ of the following form:

$$\tilde{\Omega}=\left[\begin{array}{c|c} \tilde{\Omega}_1 & \tilde{\Omega}_{21}^T \\ \hline \tilde{\Omega}_{21} & \tilde{\Omega}_2 \end{array}\right] \tag{A.12}$$

with $\tilde{\Omega}_2$ being *any* symmetric matrix.

Using (A.6) it can easily be shown that:

$$W = U^T\tilde{\Omega}UX \triangleq \Omega X \qquad \text{(A.13)}$$

and since $\tilde{\Omega}$ is symmetric so does Ω. Now it remains to show that $\tilde{\Omega}_2$ can be chosen such that $\tilde{\Omega}$ in (A.12) is an projection matrix (which immediately implies that Ω is also an projection matrix), and since it is symmetric, it is an orthogonal projection.

Suppose $\tilde{\Omega}_1$ has ρ eigenvalues that are equal to 1 (note that ρ may be also equal to zero). In the following it is shown that those ρ eigenvalues are also the eigenvalues of $\tilde{\Omega}$. Since $\tilde{\Omega}_1$ is a symmetric matrix, then there exists an orthogonal matrix T such that:

$$T\tilde{\Omega}_1T^T = \left[\begin{array}{c|c} I_\rho & 0 \\ \hline 0 & \Lambda \end{array}\right] \qquad \text{(A.14)}$$

where Λ is a diagonal matrix which contains all eigenvalues of $\tilde{\Omega}_1$ which are different from 1.

Premultiplying and postmultiplying both sides of (A.11) by T and T^T respectively, yields:

$$T\left(\tilde{\Omega}_1 - \tilde{\Omega}_1^2\right)T^T = \left[\begin{array}{ccc|c} 0 & & & \\ & \text{o} & & 0 \\ & & 0 & \\ \hline & 0 & & \Lambda - \Lambda^2 \end{array}\right] = T\tilde{\Omega}_{21}^T\tilde{\Omega}_{21}T^T \qquad \text{(A.15)}$$

hence, the first ρ rows of $T\tilde{\Omega}_{21}^T$ must vanish, and it can be written as: $T\Omega_{21}^T = \left[\begin{array}{c} 0 \\ \hline \hat{\Omega}_{21}^T \end{array}\right]$. Define now the following orthogonal matrix:

$$\tilde{T} = \left[\begin{array}{c|c} T & 0 \\ \hline 0 & I_{v-r} \end{array}\right] \qquad \text{(A.16)}$$

and use the following similarity transformation on $\tilde{\Omega}$:

$$\tilde{T}\tilde{\Omega}\tilde{T}^T = \left[\begin{array}{c|c} T\tilde{\Omega}_1T & T\tilde{\Omega}_{21}^T \\ \hline \tilde{\Omega}_{21}T^T & \tilde{\Omega}_2 \end{array}\right] = \left[\begin{array}{c|c|c} I_\rho & 0 & 0 \\ \hline 0 & \Lambda & \hat{\Omega}_{21}^T \\ \hline 0 & \hat{\Omega}_{21} & \tilde{\Omega}_2 \end{array}\right] \qquad \text{(A.17)}$$

that yields to a block diagonal matrix, which means that the eigenvalues of $\tilde{\Omega}$ are ρ times 1 and the eigenvalues of:

$$\tilde{\Omega}_{22} \triangleq \left[\begin{array}{c|c} \Lambda & \hat{\Omega}_{21}^T \\ \hline \hat{\Omega}_{21} & \tilde{\Omega}_2 \end{array}\right] \qquad \text{(A.18)}$$

Note that $\tilde{\Omega}_2$ is *any* symmetric matrix. Choosing:

$$\tilde{\Omega}_2 = I_{v-r} - \hat{\Omega}_{21}\left(I_{r-\rho} - \Lambda\right)^{-1}\hat{\Omega}_{21}^T \qquad \text{(A.19)}$$

and using the equality $\Lambda - \Lambda^2 = \hat{\Omega}_{21}^T\hat{\Omega}_{21}$,which is derived from (A.15), and the definition of $\hat{\Omega}_{21}^T$, some algebra will show that $\tilde{\Omega}_{22}^2 = \tilde{\Omega}_{22}$. Hence, it is shown that all eigenvalues of $\tilde{\Omega}$, which is symmetric, are either 1 or 0 and therefore it is an orthogonal projection matrix.

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References:

[1] M. Corless, "Stabilization of uncertain discrete-time systems," Proceeding of the IFAC workshop on model error concepts and compensation, Boston, 1985.

[2] J. Manela, "Deterministic control of uncertain linear discrete and sampled-data systems," Ph. D. dissertation, Univ. Calif., Berkeley, 1985.

[3] M. Corless and J. Manela, "Control of uncertain discrete-time systems," in Proc. Amer. Contr. Conf., Seattle, WA, June 1986, pp. 515-520.

[4] K.H. Keinitz, "Guaranteed cost control approach to the stabilization of uncertain linear discrete-time systems," INT. J. Systems SCI., vol. 21, No. 4, pp. 643-652, 1990.

[5] M.E. Magaña and S.H. Zak, "Robust state feedback stabilization of discrete-time uncertain dynamical systems," IEEE Trans. Automat. Cont., vol.33, No. 9, pp.887-891, Sep. 1988.

[6] W.C. Yang and M. Tomizuka, "Discrete robust control via state feedback for single input systems," IEEE Trans. Automat. Cont., vol. 35, pp.590-598, May. 1990.

[7] M.S. Mahmoud and A.A. Bahnasawi, "Asymptotic stability for a class of linear discrete systems with bounded uncertainties," IEEE Trans. Automat. Cont., vol.33, No. 6, pp.572-575, Jun. 1988.

[8] J. B. Farison and S.R. Kolla, "Comments on "Asymptotic stability for a class of linear discrete

systems with bounded uncertainties,"" IEEE Trans. Automat. Cont., vol.35, No. 3, pp.382-384, Mar. 1990.
[9] M. Corless, "Comments on "Asymptotic stability for a class of linear discrete systems with bounded uncertainties,"" IEEE Trans. Automat. Cont., vol.35, No. 3, pp.384-384, Mar. 1990.
[10] M.S. Mahmoud and A.A. Bahnasawi, "Authors' reply," IEEE Trans. Automat. Cont., vol.35, No. 3, pp.384-384, Mar. 1990.
[11] S. Gutman, "Uncertain dynamical systems-A Lyapunov min-max approach," IEEE Trans. Automat. Cont., vol.AC-24, No. 3, pp.437-443, Jun. 1979.
[12] S. Gutman and Z. Palmor, "Properties of min-max controllers in uncertain dynamical systems," SIAM J. contr. Optimiz., vol. 20, no. 6, pp. 850-861, Nov. 1982.
[13] E. Cheres, Z.J. Palmor and S. Gutman, "Design of input matrices of stabilizing controllers for uncertain systems," Int. J. Sys. SCI, vol 22, No. 4, pp. 735-742, 1991.
[14] B. R. Barmish, M. Corless, and G. Leitman, "a new class of stabilizing controllers for uncertain dynamical systems," SIAM J. contr. Optimiz., vol. 21, no. 2, pp. 264-255, Mar. 1983.
[15] G. Leitman, "Deterministic control of uncertain systems, "Acta Astronautica, vol. 7, p. 1457, 1980.
[16] G. Leitmann, "Guaranteed ultimate boundedness for class of uncertain linear dynamical systems," IEEE Trans. Automat. Cont., vol. AC-23, p.1109, 1978.
[17] G. Leitmann, "On one approach to the control of uncertain systems," Trans. ASME J. Dyn. Sys., Meas. and Cont., vol 115, No. 2(B), pp. 373-380, Jun. 1993.
[18] A Steinberg and M. Corless, "Output feedback stabilization of uncertain dynamical systems," IEEE Trans. Automat. Cont., vol.AC-30, pp.1025-1027, Oct. 1985.
[19] M.E. Magaña and S.H. Zak, "Robust output feedback stabilization of discrete time uncertain dynamical systems," IEEE Trans. Automat. Cont., vol.33, pp.1082-1085, Nov. 1988.
[20] N. Sharav-Schapiro, Z.P. Palmor and A. Steinberg, "Min-max output control for discrete uncertain SISO systems," in Proc. workshop on robust control via variable structure and Lyapunov techniques, Benevento, Italy, pp. 42-48, 1994.
[21] L. Hitz and B.D.O. Anderson, "Discrete positive real functions and their application to system stability," Proc. IEE. vol. 116, pp. 153-155, Jan 1969.
[22] G. C. Goodwin and K.S. Sin, "Adaptive filtering prediction and control," Prentice-Hall, 1984.
[23] C.R. Rao and S.K. Mitra, Generalized inverse of matrices and its applications. New York: Wiley, 1971.