Finite Dimensional Nonlinear Output Feedback Disturbance Attenuation Control Problems

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Abstract. In general, nonlinear output feedback control problems are infinite dimensional. In this paper we present classes of nonlinear output feedback $\mathcal{H}^\infty$ control problems for which the optimal observer dynamics are linear, and the optimal control law is linear feedback. These solutions are obtained by incorporating an information state approach to a dynamic game, where the controller plays against the initial and final states, and the disturbances entering the unobservable and observable variables. The game is solved by first solving explicitly the first order partial differential information state equation, to recover an equivalent full information game subject to linear optimal observer dynamics. The solution of the full information game yields the optimal control law, which is linear in the observer state. The results are applied to parameter identification problems with drift terms depending affinely on the unknown parameters. The optimal control law is shown to be linear feedback.

Key words: Nonlinear dynamic games, output feedback, information state, exact optimal controls, parameter estimation.

1. Introduction

Since the pioneer work of Zames [1] on the $\mathcal{H}^\infty$, or robust, control problem of linear systems, there has been an increasing interest in having a complete theory to cope with the disturbance attenuation problem (see [2]). Originally, this problem was posed and solved using frequency domain methods (which are restricted to linear systems) by introducing the $\mathcal{H}^\infty$ norm. The frequency formulation approach of the disturbance attenuation problem has been linked to the theory of dynamic games using state space models (see [3]), as zero-sum dynamic games with two opposing players; the disturbance input and the control input. For feedback control systems, the theory of dynamic games has been a powerful tool for solving both linear and nonlinear disturbance attenuation problems. This is due to the fact that the $\mathcal{H}^\infty$ operator norm can be interpreted in terns of the $L^2 - gain$ of the system which makes the theory of dissipated systems applicable, (see [4]).

However, for analogous output feedback control systems, several issues associated with the controller design are not so well developed and understood. This is due in part, to the fundamental difficulty that for general nonlinear output feedback control problems one has to introduce an observer state (summarizing the observation history) on which the control action should be based, and in part, because the information state is infinite dimensional.

Our main objectives in this paper are the following:

1. Formulate the nonlinear disturbance attenuation problem as a dynamic game using an information state approach;

2. Show that large classes of nonlinear systems admit finite dimensional solutions of the information state equation, and that the optimal observer dynamics are linear, reminiscent of that associated with linear $\mathcal{H}^\infty$ control problems;

3. Show that the same classes yield optimal control laws which are linear feedback, reminiscent of that associated with linear $\mathcal{H}^\infty$ control problems;

4. Solve parameter identification problems when the unknown parameters enter bilinearly in the unobservable dynamics and compute explicitly the optimal control law.

Recently, using similar ideas, it is has been shown in [5, 6, 7, 8, 9, 10, 11], that the information state associated with partially observable exponential of integral control problems can be solved explicitly when nonlinearities appear in the dynamics of the unobservable state. In particular, in [5, 7] it is shown that the finite dimenionallity feature of the information state is preserve when the measurements are quadratic in the unobservable state. In [9, 10], the results on finite dimensional information state are generalized considerably; it is shown that finite dimensional controllers exist whenever the sensor measurements are quadratic in the unobservable state and the dynamics of the unobservable states are nonlinear. Furthermore, it is shown that large classes of nonlinear stochastic control problems yield optimal feedback control laws, reminiscent of linear-quadratic-Gaussian (LQG) and linear-exponential-quadratic-Gaussian (LEQG) tracking problems.

It is important to note that this paper presents for the first time, examples of nonlinear $\mathcal{H}^\infty$ control systems with optimal control laws expressed explicitly as a function of the observer state.

2. Problem Statement

2.1 Dynamics

The nonlinear system to be considered in this section consists of an $\Re^n$−valued unobservable variable $x(\cdot)$ and an $\Re^d$−valued observable variable $y(\cdot)$ described by the equations

$$\begin{aligned} \dot{\tilde{x}}_t &= F_t\tilde{x}_t + g_t(\tilde{x}) + f_t \\ &+ B_tu(t,y) + G_tw_t, \quad \tilde{x}(0), \qquad (1) \\ y_t &= H_t\tilde{x}_t + h_t + N_t^{\frac{1}{2}}v_t, \quad y(0) = 0, \qquad (2) \end{aligned}$$

where $t \in [0,T]$, $u(\cdot)$ is the control input, $w(\cdot), v(\cdot)$ are deterministic disturbances, and $x(0)$ is an unknown initial state.

We define the observation history by $Y_t \doteq \{y(s); 0 \leq s \leq T\}$ and introduce the following assumptions:

A1. $F_t \in \Re^{n\times n}, f_t \in \Re^n, H_t \in \Re^{d\times n}, h_t \in \Re^d$;
A2. $N_t \in \Re^{d\times d}, N = N^*, N > 0$;
A3. $G_t \in \Re^{n\times n}, G > 0$;
A4. $g : [0,T] \times \Re^n \to \Re^n$, remains to be determined;
A5. $u(\cdot)$ takes values in $U = \Re^m$ and the set of admissible controls is defined by

$$\begin{aligned} \mathcal{U} &\doteq \{u : [0,T] \times C([0,T];\Re^d) \to U, \\ &\quad u(\cdot,y) \in L^2([0,T];\Re^m)\}; \end{aligned}$$

A6. $(w(\cdot), v(\cdot)) \in L^2([0,T];\Re^{n+d})$, $(x(0), x(T)) \in \Re^{2n}$ are unknown;
A7. $p : \Re^n \to \Re, \ell : [0,T] \times \Re^n \times \mathcal{U} \to \Re, \varphi : \Re^n \to \Re$;
A8. $p(x(0)) = -\frac{1}{2\theta}|P(0)^{-\frac{1}{2}}(\tilde{x}(0) - \xi)|^2 + \frac{1}{\theta}\phi(\tilde{x}(0), 0)$.

Definition 0.1 *(Disturbance Attenuation). Given $\theta > 0$ and $t \in [0,T]$, the disturbance attenuation problem consists of determining a control $u \in \mathcal{U}$ such that for all $w(\cdot) \in L^2([0,T];\Re^n), v(\cdot) \in L^2([0,T];\Re^d)$, $(\tilde{x}(0); \tilde{x}(T)) \in \Re^{2n}$ the following inequality holds:*

$$\begin{aligned} I^\theta(u) &\doteq \frac{\int_0^T \ell(t,\tilde{x}_t,u(t,y))dt}{-p(x(0)) + \int_0^T \frac{1}{2}\left(|w_t|^2 + |N_t^{-\frac{1}{2}}v_t|^2\right)dt - \varphi(\tilde{x}_T)} \\ &\leq \frac{1}{\theta}. \end{aligned}$$

Whenever the upper value of this disturbance attenuation is bounded, the optimal disturbance attenuation level specified by θ^* (i.e., $\frac{1}{\theta^*}$) is obtained from $\frac{1}{\theta^*} \doteq \inf_{u\in\mathcal{U}} \sup_{(v,\tilde{x}_T)} \sup_{(w,\tilde{x}_0)} I^\theta(u)$.

Recall that in the general formulation of the disturbance attenuation tracking problem one has to specify the signals to be controlled. In our formulation these signals are represented by the function $\ell(\cdot)$. Also, since we are interested in the tracking problem we introduce the assumption

A9. $2\ell(t,\tilde{x},u) \doteq \zeta_1 + \zeta_2$, where

$$\begin{aligned} \zeta_1 &= Q_t\tilde{x}.\tilde{x} + R_tu.u + 2m_tx + 2n_tu, \\ Q &= Q^* \geq 0, \quad R = R^* > 0, \\ \zeta_2 &= \frac{1}{\theta}|G_t^{-1}.g(t,\tilde{x})|^2 + \tilde{\ell}(t,\tilde{x},u). \end{aligned}$$

Remark 0.2 *Notice that part of ζ_2 is given by $|G_t^{-\frac{1}{2}}(\dot{x} - F_tx - f_t - B_tu_t - G_tw_t)|^2$, therefore, the controller is to be designed so that tracking is achieved while deviations of the model from being linear are penalized.*

2.2 Dynamic Game

The disturbance attenuation problem will be solved using a game theoretic information state approach. That is, we identify an information state which summarizes all the information available to the controller and, thus, carries the information $\{Y_s, u_s; s \in [0,T]\}$. To this end we introduce the functional

$$\begin{aligned} J^\theta(u(\cdot)) &\doteq \sup_{(v,\tilde{x}_T)} \sup_{(w,\tilde{x}_0)} \left\{ \frac{1}{\theta}p(\tilde{x}_0) + \int_0^T (\ell(t,\tilde{x}_t,u_t)dt \right. \\ &\left. - \frac{1}{2\theta}\left[|w_t|^2 + |v_t|^2\right]\right) dt + \frac{\varphi(\tilde{x}_T)}{\theta} \bigg\}, \qquad (3) \end{aligned}$$

and we define the sup pairing "$< \cdot, \cdot >$" by $< p, q > \doteq \sup_{x\in\Re^n} \{p(x) + q(x)\}$. From the definition of $J^\theta(u(\cdot))$ we deduce the alternative definition of θ^* given by $\theta^* \doteq \sup\{\theta; \ \inf_{u\in\mathcal{U}} J^\theta(u(\cdot)) < 0\}$.

Recall that for feedback controls (i.e., $u \equiv u(t,x)$) (see also [3]) the disturbance attenuation problem can be cast in terms of minimizing over $u \in \mathcal{U}$, the functional $J^\theta(u(\cdot))$, subject to the evolution in time of the state $x(\cdot)$. Since in the present context the control is output feedback (i.e., $u \equiv u(t,y)$), we shall derive an evolution equation for the "information state" on which the control action should be based. The following theorem derived in [12], by applying large deviations techniques to the partially observable exponential of integral control problem is now introduced. The results of the theorem has been previously incorporated independently in [13], as a new means for solving nonlinear disturbance attenuation control problems.

Theorem 0.3 *For a given output path $y(\cdot) \in L^2([0,T];\Re^d)$, starting state $\tilde{x}_t = \tilde{x}$, and for a fixed strategy $u \in \mathcal{U}$ define the information state $q^\theta(\cdot)$ by*

$$\begin{aligned} q^\theta(\tilde{x},t) &\doteq \sup_{w\in\Re^n}\sup_{\tilde{x}_0\in\Re^n}\left\{\frac{1}{\theta}p(\tilde{x}_0)+\int_0^t(\ell(s,\tilde{x}_s,u_s)\right. \\ &-\left.\frac{1}{2\theta}\left[|w_s|^2+|N^{-\frac{1}{2}}(y_s-H_s\tilde{x}_s-h_s)|^2\right]\Big)\,ds\right\}, \quad (4)\end{aligned}$$

where the unobservable state $\tilde{x}^u(\cdot)$ satisfies the backward equation

$$\dot{\tilde{x}}_s = F_s\tilde{x}_s + g_s(\tilde{x}_s) + f_s + B_s u_s + G_s w_s, \quad \tilde{x}_t = \tilde{x},$$

where $s \le t \le T$. Then, $q_t^\theta \equiv q^\theta(\tilde{x},t)$ satisfies the 1st order partial differential equation (PDE) evolving forward in time

$$\begin{aligned} \frac{\partial}{\partial t}q_t^\theta &= -D_{\tilde{x}}q_t^\theta.(F_t\tilde{x}+f_t+g_s(\tilde{x})+B_s u(s,y)) \\ &+ \ell(t,\tilde{x},u_t)-\frac{1}{2\theta}|N^{-\frac{1}{2}}(y_t-H_t\tilde{x}_t-h_t))|^2 \\ &+ \sup_w\left\{-D_{\tilde{x}}q_t^\theta.G_t w_t-\frac{1}{2\theta}|w_t|^2\right\}, \quad (5) \\ q^\theta(\tilde{x},0) &= \frac{1}{\theta}p(\tilde{x}_0), \quad (6)\end{aligned}$$

where $D_{\tilde{x}} \doteq \left(\frac{\partial}{\partial\tilde{x}_1},\frac{\partial}{\partial\tilde{x}_2},\ldots,\frac{\partial}{\partial\tilde{x}_n}\right)$. The optimal disturbance attenuation problem is cast in terms of minimizing over $u \in \mathcal{U}$, the functional

$$J^\theta(u(\cdot)) = \sup_{y\in L^2([0,T];\mathcal{R}^d)}\left\{<q_T^\theta,\frac{\varphi}{\theta}>\right\}. \quad (7)$$

3. Finite Dimensional Problem

3.1 Solution of Information State Equation

The control problem stated in Theorem 0.3 by (5), (6), (7), although fully observable, is infinite dimensional. In this section we shall determine classes of nonlinear functions $g(\cdot)$ entering the unobservable dynamics (1) that yield explicit solutions of (5), (6). Equivalently, the optimal observer dynamics will be determined in terms of a finite number of ODE's which form the sufficient statistics of the estimation problem. First, we give a precise definition of the system to be considered.

Control System Σ_1: Suppose A1-A9 hold and the dynamics and observations are given by (1), (2), respectively.

Theorem 0.4 *Suppose there exist functions $\phi \in C_{t,x}^{1,1}(\Re^n\times[0,T])$ satisfying*

$$\begin{aligned}\frac{\partial\phi_t}{\partial t} &+ (F_t\tilde{x}+f_t).D_{\tilde{x}}\phi_t = \frac{1}{2}\tilde{x}.\tilde{\Lambda}_t\tilde{x}+\tilde{x}.\tilde{\sigma}_t+\delta_t \\ &+\frac{\theta}{2}\tilde{\ell}(t,\tilde{x},u)-\frac{1}{2}|G_t.D_{\tilde{x}}\phi_t|^2-B_t u.D_{\tilde{x}}\phi_t, \quad (8)\end{aligned}$$

where the functions $\tilde{\ell}(\cdot),\tilde{\Lambda}(\cdot),\tilde{\sigma}(\cdot),\delta(\cdot)$ are to be chosen so that (8) yields explicit solutions, and there exists $\theta \le \theta^*$ such that

$$H_t^*N_t^{-1}H_t+\tilde{\Lambda}_t-\theta Q_t \ge 0, \quad \forall t\in[0,T].$$

Then, the information state of system Σ_1 with nonlinear drift term $g(\cdot)$ given by

$$g_t(\tilde{x}) = G_tG_t^*D_{\tilde{x}}\phi(\tilde{x},t) \quad (9)$$

admits explicit solutions given by

$$q^\theta(\tilde{x},t) = \frac{1}{\theta}\phi(\tilde{x},t)-\frac{1}{2\theta}|P_t^{-\frac{1}{2}}(\tilde{x}-r_t)|^2+\mathcal{I}_{0,t}^\theta(u), \quad (10)$$

where

$$\begin{aligned}\mathcal{I}_{0,t}^\theta(u) &\doteq \int_0^t\frac{1}{2}\left\{[Q_s-\frac{\tilde{\Lambda}_s}{\theta}]r_s.r_s+R_su_s.u_s\right. \\ &+ 2r_s.[m_s^*-\frac{\tilde{\sigma}_s}{\theta}]+2n_su_s-2\frac{\delta_s}{\theta} \\ &- \left.\frac{1}{\theta}|N_s^{-\frac{1}{2}}(y_s-H_sr_s-h_s)|^2\right\}ds.\end{aligned}$$

Further, the disturbance attenuation problem of system Σ_1 is now equivalent to the finite dimensional completely observable minmax game of determining the optimal $u \in \mathcal{U}$ that minimizes the functional

$$\begin{aligned} J^\theta(u(\cdot)) &= \sup_y\left\{<\frac{1}{\theta}\phi-\frac{1}{2\theta}|P_T^{-\frac{1}{2}}(\tilde{x}-r_T)|^2,\frac{\varphi}{\theta}>\right. \\ &+ \left.\mathcal{I}_{0,T}^\theta(u)\right\}, \quad (11)\end{aligned}$$

subject to optimal observer dynamics described by only two statistics $r(\cdot), P(\cdot)$ satisfying the ODE's:

$$\begin{aligned}\dot{r}_t &= \left\{F_t-P_t\left(\tilde{\Lambda}_t-\theta Q_t\right)\right\}r_t+f_t \\ &- P_t\tilde{\sigma}_t+B_tu_t+\theta P_tm_t^* \\ &+ P_tH_t^*N_t^{-1}\hat{y}_t,\ \hat{y}_t \doteq y_t-H_tr_t-h_t,\ r(0), \quad (12)\end{aligned}$$

$$\begin{aligned}\dot{P}_t &= F_tP_t+P_tF_t^*-P_t\left(H_t^*N_t^{-1}H_t\right. \\ &+ \left.\tilde{\Lambda}_t-\theta Q_t\right)P_t+G_tG_t^*, \quad P(0). \quad (13)\end{aligned}$$

Proof. Using the hypothesis of the theorem verify that (10) satisfies (5), (6). □

According to our earlier investigations in [9, 10] associated with stochastic control problems, there is a fundamental difficulty in solving (8) because of the presence of the term $B_tu_t.D_{\tilde{x}}\phi_t$; the functions $g(\cdot)$ resulting from these solutions will not yield interesting nonlinear systems Σ_1 which can be encountered in real applications. Of course, one way to remove this terms from (8) is to use $\tilde{\ell}$ to cancel $B_t u_t . D_{\tilde{x}}\phi_t$. Alternatively, we introduce the following control system:

Suppose the dynamics and observations are described by

$$\begin{aligned}
\begin{bmatrix} \dot{x}_t \\ \dot{z}_t \end{bmatrix} &= \begin{bmatrix} F_{11}(t) & F_{12}(t) \\ F_{21}(t) & F_{22}(t) \end{bmatrix} \begin{bmatrix} x_t \\ z_t \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \\
&+ \begin{bmatrix} g_t(x_t) \\ 0 \end{bmatrix} + \begin{bmatrix} B_t^1 u(t,y) \\ B_t^2 u(t,y) \end{bmatrix} \\
&+ \begin{bmatrix} G_t^1 & 0 \\ 0 & G_t^2 \end{bmatrix} \begin{bmatrix} w_t^1 \\ w_t^2 \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ z(0) \end{bmatrix}, \qquad (14) \\
&\equiv F_t \tilde{x}_t + g(t,\tilde{x}_t) + f_t + B_t u(t,y) + G_t w_t, \\
y_t &= H_1(t)x_t + H_2(t)z_t + h_t + N_t^{\frac{1}{2}} dv_t, \qquad (15) \\
&\equiv H_t \tilde{x}_t + h_t + N_t^{\frac{1}{2}} v_t, \quad y(0) = 0,
\end{aligned}$$

respectively, where $\tilde{x}^* \equiv (x^*, z^*)$, and $x(\cdot), z(\cdot)$ are, respectively, $\Re^{n_1}, \Re^{n-n_1}$—valued unobservable states. In addition, we introduce the assumption

A10: $G_t^1 \in \Re^{n_1 \times n_1}, G^1 > 0$, $\tilde{\ell}(t,\tilde{x},u) \doteq \tilde{\ell}(t,x,u)$, $F_{12} = 0, B^1 = 0$.

Control System Σ_2: Suppose A1, A2, A4-A10 hold and the dynamics and observations are given by (14), (15), respectively.

The equivalent of Theorem 0.4 applied to system Σ_2 is stated next.

Theorem 0.5 *The results of Theorem 0.4 remain valid for the nonlinear control system Σ_2 provided (8), (9), (10) are, respectively, replaced by the following equations:*

$$\begin{aligned}
\frac{\partial \phi_t}{\partial t} &+ \frac{1}{2}|G_t^1 . D_x \phi_t|^2 + (F_{11}(t)x + f_1(t)) . D_x \phi_t \\
&= \frac{1}{2} x.\Lambda_t x + x.\sigma_t + \delta_t + \frac{\theta}{2}\tilde{\ell}(t,x,u), \qquad (16)
\end{aligned}$$

$$g_t(x) = G_t^1 G_t^{1,*} D_x \phi(x,t), \qquad (17)$$

$$\begin{aligned}
q^\theta(\tilde{x},t) &= \frac{1}{\theta}\phi(x,t) - \frac{1}{2\theta}|P_t^{-\frac{1}{2}}(\tilde{x} - r_t)|^2 \\
&+ \mathcal{I}_{0,t}^\theta(u), \qquad (18)
\end{aligned}$$

where

$$\tilde{\Lambda}_t = \begin{bmatrix} \Lambda_t & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{\sigma}_t = \begin{bmatrix} \sigma_t \\ 0 \end{bmatrix}.$$

Lemma 0.6 *Consider the nonlinear system Σ_2 and suppose $F_{11} = 0, G^1 = I_{n_1}, f_1 = 0$, and $g \in C_x^1(\Re^{n_1})$ is the gradient of a potential, that is,*

$$g(x) = D_x \phi(x),$$

If the function $\tilde{\ell}(\cdot)$ is given by

$$\tilde{\ell}(t,x,u) = \frac{1}{\theta}|g(t,x)|^2,$$

then (16) is satisfied. Therefore, the optimal observer dynamics are obtained from (12), (13) by setting $\tilde{\Lambda} = \tilde{\sigma} = F_{11} = f_1 = 0$.

Proof. If the hypothesis of the theorem are satisfied by Theorem 0.5 we know that any time independent function $\phi : \Re^{n_1} \to \Re$ will satisfy (16), simply by setting $\Lambda = 0, \sigma = 0, \delta = 0$, thus $g(\cdot)$ is the gradient of some potential function and hence, the optimal observer dynamics are obtained from (12), (13). □

3.2 Examples of Finite Dimensional Systems

In this section we shall present certain classes of nonlinear functions $g(\cdot)$ that admit finite dimensional solutions for the information state equation.

Theorem 0.7 *Suppose $u \in \hat{\mathcal{U}}$ and there exists $\theta \leq \theta^*$ such that*

$$H_t^* N_t^{-1} H_t + \tilde{\Lambda}_t - \theta Q_t \geq 0, \qquad \forall t \in [0,T].$$

Let $\Delta_t \in \Re^{n_1 \times n_1}$, $\zeta_t \in \Re^{n_1}$, $n_t \in \Re$, *and define*

$$\Gamma_2(t,x) \doteq \frac{1}{2}\Delta_t x.x + x.\zeta_t + \eta_t,$$

The information state $q^\theta(\cdot)$ associated with system Σ_2 given in Theorem 0.5 admits explicit representations, at least for the following two classes:

Class 1 (Rational Nonlinearities). *A solution of (16) is*

$$\phi_{R_2}(x,t) = \log \Gamma_2(x,t)$$

provided the function $\tilde{\ell}$ is defined by

$$\tilde{\ell}(t,x,u) \doteq \frac{1}{\theta}|G_t^1 . D_x \phi_{R_2}(x,t)|^2.$$

This implies that the nonlinear drift term $g(\cdot)$ is

$$g_t(x) = \frac{G_t^1 G_t^{1,*}}{\frac{1}{2}\Delta_t x.x + x.\zeta_t + \eta_t}\left(\Delta_t x + \zeta_t\right),$$

where

$$\begin{aligned}
\dot{\Delta}_t &+ F_{11}(t)^* \Delta_t + \Delta_t F_{11}(t) = \delta_t \Delta_t, \\
\dot{\zeta}_t &+ F_{11}(t)^* \zeta_t + \Delta_t f_1(t) = \delta_t \zeta_t, \\
\dot{\eta}_t &+ f_1(t).\zeta_t = \delta_t \eta_t, \\
\Lambda_t &= 0, \quad \sigma_t = 0, \quad \delta_t = \textit{arbitrary}.
\end{aligned}$$

Class 2 (Exponential Nonlinearities). *Suppose* $\gamma^1, \gamma^2 : [0,T] \to \Re$. *A solution of (16) is*

$$\phi_{E_2}(x,t) = \log\left\{\gamma_t^1 \exp\left(\Gamma_2(x,t)\right) + \gamma_t^2 \exp\left(-\Gamma_2(x,t)\right)\right\}$$

provided the function $\tilde{\ell}$ *is defined by*

$$\tilde{\ell} \doteq \frac{1}{\theta}|G_t^1.D_x\phi_{E_2}(x,t)|^2.$$

This implies that the nonlinear drift term $g(\cdot)$ *is*

$$\begin{aligned} g_t(x) &= \frac{\gamma_t^1 \exp\left(\Gamma_2(x,t)\right) - \gamma_t^2 \exp\left(-\Gamma_2(x,t)\right)}{\gamma_t^1 \exp\left(\Gamma_2(x,t)\right) + \gamma_t^2 \exp\left(-\Gamma_2(x,t)\right)} \\ &\times \; G_1^1 G_t^{1,*}\left(\Delta_t x + \zeta_t\right), \end{aligned}$$

where

$$\begin{aligned} \dot{\Delta}_t &+ F_{11}(t)^*\Delta_t + \Delta_t F_{11}(t) = 0, \\ \dot{\zeta}_t &+ F_{11}(t)^*\zeta_t + \Delta_t f_1(t) = 0, \\ \dot{\eta}_t &+ f_1(t).\zeta_t = \frac{1}{2}\frac{d}{dt}\left(\log\frac{\gamma_t^1}{\gamma_t^2}\right), \\ \Lambda_t &= 0, \quad \sigma_t = 0, \quad \delta_t = \frac{1}{2}\frac{d}{dt}\left(\log \gamma_t^1\gamma_t^2\right). \end{aligned}$$

Proof. Substitute the solutions into the evolution equation of $\phi(\cdot)$. □

One can obtain similar results by considering the control system Σ_1. Additional classes with finite dimensional information state equations are found in [9, 10]. An example with polynomial nonlinearities is presented next.

Example 0.8 *Suppose* $x : [0,T]\times\Omega \to \Re$, $z : [0,T]\times\Omega \to \Re$, $y : [0,T]\times\Omega \to \Re$ *and consider*

$$\begin{aligned} \dot{x}_t &= -\Pi x_t^3 + w_t^1, \quad x(0), \;\; \Pi \geq 0, \\ \dot{z}_t &= x_t + z_t + u(t,y) + w_t^2, \quad z(0), \\ y_t &= x_t + z_t + v_t. \end{aligned}$$

If we set $\tilde{\ell}(t,x,u) = \frac{1}{\theta}|x^3\Pi|^2$, *the information equation is finite dimensional and is obtained from Theorem 0.5 by setting* $\phi(x,t) = -\frac{\Pi}{4}x^4$.

3.3 Generalizations

We have thus far presented general classes of nonlinear systems that yield finite dimensional optimal observer dynamics. Next, we shall obtain additional generalizations by simply modifying the signal to be controlled introduced in A9. Although we shall consider only system Σ_1, the reader should be convinced that similar generalizations will also hold for system Σ_2. To this end we introduce the next assumption.

A11: $2\ell(t,\tilde{x},u) \doteq \zeta_1 + \tilde{\zeta}_2$, where ζ_1 is given in A9 while $\tilde{\zeta}_2$ is given by

$$\begin{aligned} \tilde{\zeta}_2 &= \frac{1}{\theta}|G_t^{-1}.\left[F_t\tilde{x} + f_t + g(t,\tilde{x}) + B_t u(t,y)\right]|^2 \\ &+ \frac{\partial}{\partial t}\left((G_tG_t^*)^{-1}D_{\tilde{x}}g(t,\tilde{x})\right). \end{aligned}$$

Control System Σ_3: Suppose A1-A8, A11 hold and the dynamics and observations are given by (1), (2).

Define

$$\begin{aligned} Q_t^\theta &\doteq Q_t + \frac{1}{\theta}F_t^*(G_tG_t^*)^{-1}F_t, \quad t \in [0,T), \\ R_t^\theta &\doteq R_t + \frac{1}{\theta}B_t^*(G_tG_t^*)^{-1}F_t, \quad t \in [0,T), \\ m_t^\theta &\doteq m_t + \frac{1}{\theta}f_t^*(G_tG_t^*)^{-1}F_t, \quad t \in [0,T), \\ n_t^\theta &\doteq n_t + \frac{1}{\theta}f_t^*(G_tG_t^*)^{-1}B_t, \quad t \in [0,T). \end{aligned}$$

Theorem 0.9 *Suppose there exists a* $\theta \leq \theta^*$ *such that*

$$H_t^*N_t^{-1}H_t - \theta Q_t^\theta \geq 0, \qquad \forall t \in [0,T].$$

Assume (without loss of generality) $F_t.B_t = 0$. *Then, the information state of system* Σ_3 *with nonlinear drift term* $g(\cdot)$ *given by*

$$g_t(\tilde{x}) = G_tG_t^*D_{\tilde{x}}\phi(\tilde{x},t) \tag{19}$$

admits explicit solutions given by

$$q^\theta(\tilde{x},t) = \frac{1}{\theta}\phi(\tilde{x},t) - \frac{1}{2\theta}|P_t^{-\frac{1}{2}}(\tilde{x}-r_t)|^2 + \mathcal{I}_{0,t}^\theta(u), \tag{20}$$

where

$$\begin{aligned} \mathcal{I}_{0,t}^\theta(u) &\doteq \frac{1}{2}\int_0^t \Big\{Q_s^\theta r_s.r_s + R_s^\theta u_s.u_s \\ &+ 2r_s.m_s^{\theta,*} + 2n_s^\theta u_s + \frac{1}{\theta}|G_s^{-1}.f_s|^2 \\ &- \frac{1}{\theta}|N_s^{-\frac{1}{2}}(y_s - H_sr_s - h_s)|^2\Big\}\,ds. \end{aligned}$$

The disturbance attenuation problem of system Σ_3 *is equivalent to the minmax problem of determining the optimal control* $u \in \mathcal{U}$ *that minimizes the functional*

$$\begin{aligned} J^\theta(u(\cdot)) &= \sup_y\Big\{<\frac{1}{\theta}\phi - \frac{1}{2\theta}|P_T^{-\frac{1}{2}}(\tilde{x}-r_T)|^2, \frac{\varphi}{\theta}> \\ &+ \mathcal{I}_{0,T}^\theta(u)\Big\}, \end{aligned} \tag{21}$$

subject to optimal observer dynamics described the ODE's:

$$\begin{aligned} \dot{r}_t &= \left\{F_t + \theta P_tQ_t^\theta\right\}r_t + f_t + B_tu_t + \theta P_tm_t^{\theta,*} \\ &+ P_tH_t^*N_t^{-1}\hat{y}_t, \; \hat{y}_t \doteq y_t - H_tr_t - h_t, \; r(0), \end{aligned} \tag{22}$$

$$\begin{aligned} \dot{P}_t &= F_tP_t + P_tF_t^* - P_t\left(H_t^*N_t^{-1}H_t\right. \\ &+ \left.-\theta Q_t^\theta\right)P_t + G_tG_t^*, \quad P(0). \end{aligned} \tag{23}$$

Proof. Follow the derivation of Theorem 0.4. □

4. Exact Optimal Control Laws

In Section 3 we have presented nonlinear systems for which the information state equations admit finite dimensional solutions. In this section we shall present sufficient conditions for systems $\Sigma_1, \Sigma_2, \Sigma_3$ to yield optimal control laws reminiscent of that associated with linear $\mathcal{H}^\infty$ tracking control problems.

Without loss of generality, the developments of this section will be carried out based on the results of Section 3.3 (Theorem 0.9), that is, by considering system Σ_3. From Theorem 0.9 we know that in order to compute the optimal control law we need to minimize over $u \in \mathcal{U}$ the functional (21) subject to observer dynamics (22), (23). To this end we define

$$\begin{aligned}\tilde{x}_* &\doteq \arg\sup_{\tilde{x}\in\Re^n}\left\{\frac{1}{\theta}\phi(\tilde{x},t) - \frac{1}{2\theta}|P_T^{-\frac{1}{2}}(\tilde{x}-r_T)|^2\right. \\ &+ \left.\frac{\varphi(\tilde{x})}{\theta}\right\} = \arg\sup_{\tilde{x}\in\Re^n}\left\{q^\theta(\tilde{x},T) + \frac{\varphi(\tilde{x})}{\theta}\right\}, \qquad (24)\end{aligned}$$

and we assume $\tilde{x}_*$ is unique. If uniqueness fails we restrict the state space to $X \subset \Re^n$. If we now define

$$\hat{\varphi}(r,T) \doteq q^\theta(\tilde{x}_*,T) + \frac{\varphi(\tilde{x}_*)}{\theta} \qquad (25)$$

we have the following full information minmax game:

Full Information Game: Minimize over $u \in \mathcal{U}$ the functional

$$J^\theta(u(\cdot)) = \sup_{y\in L^2([0,T];\Re^d)}\{\hat{\varphi}(r,T) + \mathcal{I}^\theta_{0,T}(u)\} \qquad (26)$$

subject to (22), (23), where $\mathcal{I}_{0,t}(u)$ is given in Theorem 0.9. Towards solving this game, for $\tilde{x} \in \Re^n, p \in \Re^n$ we introduce the Hamiltonian

$$\begin{aligned}\mathcal{H}(x,p) &= \inf_{u\in\mathcal{U}}\left\{p.B_t u + \frac{1}{2}\left(R^\theta_t u.u + 2n^\theta_t u\right)\right\} \\ &+ \frac{1}{2}\left(Q^\theta_t\tilde{x}.\tilde{x} + 2m^\theta_t\tilde{x}\right),\end{aligned}$$

and we define the operator

$$\tilde{A} \doteq \sum_{i=1}^n\left(\sum_{j=1}^n\left(\tilde{F}_{i,j}\tilde{x}_j + \theta P_{i,j}m^{\theta,*}_j\right) + f_i\right)\frac{\partial}{\partial\tilde{x}_i},$$

where

$$\tilde{F}_t = F_t + \theta P_t Q^\theta_t, \quad \tilde{\alpha}(t) = P_t H^*_t N_t^{-1} H_t P_t.$$

If we now define the cost-to-go function

$$\begin{aligned}S(r,t) &\doteq \inf_{u\in\mathcal{U}}\sup_{y\in L^2([0,T];\Re^d)}\{\hat{\varphi}(r,T) \\ &+ \mathcal{I}_{t,T}(u);\ r_t = r\}, \qquad (27)\end{aligned}$$

by using dynamic programming arguments (see [11]) we deduce the following Hamilton-Jacobi (HJ) equation

$$\begin{aligned}\frac{\partial}{\partial t}S(r,t) &+ \tilde{A}(t)S(r,t) + \frac{\theta}{2}D_r S(r,t).\tilde{\alpha}_t D_r S(r,t) \\ &+ \mathcal{H}(r, D_r S(r,t)) = 0, \qquad (28)\\ S(r,T) &= \hat{\varphi}(r,T). \qquad (29)\end{aligned}$$

Note that we arrive at (28) by performing the maximization over y. In addition, it should be clear from the context of (28), (29) that Isaac's saddle point inequalities are satisfied. Consequently, we have the following verification theorem.

Theorem 0.10 *Consider the control system Σ_3 and assume the admissible controls are of separated form $u(t) \equiv u(t,r)$. Denoting by $S(\cdot)$ the solution of the HJ equation (28), (29) we have*

$$S(r(0),0) \leq J^\theta(u(\cdot)), \qquad \forall u \in \mathcal{U}.$$

Further, letting $u^*(t) = u^*(t,r)$, where u^* is a Borel measurable function minimizing $\mathcal{H}$ given by

$$u^*(t) = -R^{\theta,-1}_t B^*_t D_r S(r,t) - R^{\theta,-1}_t n^{\theta,*}_t,$$

and $r(\cdot) \equiv r^{u^*}(\cdot)$ is the corresponding solution of

$$\begin{aligned}\dot{r}_t &= \left(F_t + \theta P_t Q^\theta_t\right)r_t + f_t + B_t u^*(t) + \theta P_t m^{\theta,*}_t \\ &+ P_t H^*_t N_t^{-1}\hat{v}_t, \quad r(0), \quad \hat{v}_t \doteq y_t - H_t r_t - h_t,\end{aligned}$$

we have

$$S(r(0),0) = J^\theta(u^*(\cdot)) = \inf_{u\in\mathcal{U}}\sup_{y\in L^2([0,T];\Re^d)} J^\theta(u(\cdot)).$$

Proof. See [11]. □

Clearly, the results of Theorem 0.10 imply that for any nonlinear function $\phi(\cdot)$ obtained from Theorem 0.9 (hence, $g_t(\tilde{x}) = G^1_t G^{1*}_t D_x\phi(\tilde{x},t)$), the optimal observer dynamics are reminiscent of that associated with linear $\mathcal{H}^\infty$ or, robust tracking problems. Similar results are obtained for the optimal control law in the next theorem.

Theorem 0.11 *Consider the control system Σ_3. The optimal control law $u^* \in \mathcal{U}$ minimizing the total cost function $J^\theta(u(\cdot))$ is linear feedback, reminiscent of that associated with the linear $\mathcal{H}^\infty$ or, robust tracking problems, if the following hold:*

1. *The function $g(\cdot)$ is defined by*

$$g(t,\tilde{x}) \doteq G_t G^*_t D_{\tilde{x}}\phi(\tilde{x},t), \qquad (30)$$

2. *The function $\varphi(\cdot)$ is defined by*

$$\varphi(\tilde{x},T) \doteq -\phi(\tilde{x},T) + \frac{\theta}{2}\left(Q_T\tilde{x}.\tilde{x} + 2m_T\tilde{x}\right). \qquad (31)$$

Proof. If the conditions 1, 2 of the theorem are satisfied, we know from (24) that

$$\tilde{x}_* = (I - \theta P_T Q_T)^{-1}(r_T + \theta P_T m_T^*).$$

Also, from (25) we have

$$\begin{aligned}\hat{\varphi}(r,T) &= \frac{1}{2} r_T . Q_T (I - \theta P_T Q_T)^{-1} r_T \\ &+ m_T (I - \theta P_T Q_T)^{-1} + \frac{\theta}{2} m_T (I - \theta P_T Q_T)^{-1} P_T m_T^* .\end{aligned}$$

Therefore, the terminal condition of the HJ equation (see (29) is a quadratic function of r. Hence, the result follows. □

For completeness, we present next the solution of system Σ_3 when conditions 1, 2 of Theorem 0.11 hold.

Theorem 0.12 *(Exact Optimal Control Laws). Consider the control system Σ_3 and assume conditions 1, 2 of Theorem 0.11 hold. Denote by $\tilde{\rho}(AB)$ the spectral radius of AB, and define*

$$\theta^* \doteq \left\{ \sup \theta;\ P \geq 0,\ \tilde{S} \geq 0,\ \ \tilde{\rho}(P\tilde{S}) < \frac{1}{\theta},\ \ \forall t \in [0,T] \right\},$$

where $P(\cdot)$ is given in Theorem 0.9 and $\tilde{S}(\cdot)$ is the solution of the Riccati differential equation

$$\begin{aligned}\dot{\tilde{S}}_t &+ F_t^* \tilde{S}_t + \tilde{S}_t F_t - \tilde{S}_t \left(B_t R_t^{\theta,-1} B_t^* \right. \\ &- \left. \theta G_t G_t^* \right) \tilde{S}_t + Q_t^\theta,\ S_T = Q_T .\end{aligned}$$

Then, for $\theta \leq \theta^*$ the optimal control law corresponding to the class of control systems Σ_3 is given by

$$\begin{aligned}u^*(t) &= -R_t^{\theta,-1} B_t^* \left(\Sigma_t r_t + k_t^* \right) - R_t^{\theta,-1} n_t^{\theta,*} \\ &= -R_t^{\theta,-1} n_t^{\theta,*} - R_t^{\theta,-1} B_t^* \left((I - \theta \tilde{S}_t P_t)^{-1} \tilde{S}_t r_t + k_t^* \right),\end{aligned}$$

where $r(\cdot) \equiv r^{u^*}(\cdot)$, $P(\cdot)$ satisfy (22), (23), respectively, while the control gains are

$$\begin{aligned}\dot{\Sigma}_t &+ \Sigma_t \left(F_t + \theta P_t Q_t^\theta \right) + \left(F_t^* + \theta Q_t^\theta P_t \right) \Sigma_t + Q_t^\theta \\ &- \Sigma_t \left\{ B_t R_t^{\theta,-1} B_t^* - \theta P_t H_t^* N_t^{-1} H_t P_t \right\} \Sigma_t = 0, \\ \Sigma_T &= \frac{1}{4} \left\{ (I - \theta Q_T P_T)^{-1} Q_T + Q_T (I - \theta P_T Q_T)^{-1} \right\}, \\ \dot{k}_t &+ k_t \left(F_t + \theta P_t H_t^* N_t^{-1} H_t P_t \Sigma_t \right. \\ &+ \left. \theta P_t Q_t^\theta - B_t R_t^{\theta,-1} B_t^* \Sigma_t \right) \\ &+ m_t^\theta + \left(f_t^* + \theta m_t^\theta P_t - n_t^\theta R_t^{\theta,-1} B_t^* \right) \Sigma_t = 0, \\ k_T &= m_T (I - \theta P_T Q_T)^{-1} .\end{aligned}$$

Furthermore, the optimal total cost associated with system Σ_3 is given by

$$\begin{aligned}J^\theta(u^*(\cdot)) &= \hat{\mathcal{I}}_{0,T} + \frac{1}{2} \left(\Sigma(0) r(0) . r(0) \right. \\ &+ \left. 2k(0) r(0) + \rho(0) \right),\end{aligned}$$

where the functions $\hat{\mathcal{I}}(\cdot), \rho(\cdot)$ are given by

$$\begin{aligned}\dot{\rho}_t &+ \theta k_t P_t H_t^* N_t^{-1} H_t P_t k_t^* + 2k_t \left(f_t + \theta P_t m_t^{\theta,*} \right) \\ &- |R_t^{\theta,-\frac{1}{2}} \left(B_t^* k_t^* + n_t^{\theta,*} \right)|^2 = 0, \quad \rho_T = 0, \\ \hat{\mathcal{I}}_{0,T} &= \frac{\theta}{2} m_T \left(I - \theta P_T Q_T \right)^{-1} P_T m_T^* \\ &+ \frac{1}{2\theta} \int_0^T |G_t^{-1} . f_t|^2 dt .\end{aligned}$$

Proof. See [11]. □

5. Parameter Estimation and Control

In this section we suppose that systems Σ_i, $i = 1,2,3$ contain unknown constant parameters $\Theta_1, \Theta_2, ..., \Theta_{n-n_1}$ which we desire to estimate as well. For simplicity, we consider the case when the $\Re^{n_1}$–valued unobservable state $x(\cdot)$ and observation $y(\cdot)$ are described by

$$\begin{aligned}\dot{x}_t &= F_{11}(t) x_t + f_1(t) + g(x_t, \Theta) \\ &+ B_t^1 u(t,y) + w_t^1,\ \ x(0), \qquad (32) \\ y_t &= H_1(t) x_t + H_2(t)\Theta + h_t + N_t^{\frac{1}{2}} v_t, \qquad (33) \\ &\equiv H_t \tilde{x}_t + h_t + N_t^{\frac{1}{2}} v_t,\ \ y(0) = 0,\end{aligned}$$

where $\tilde{x} = \begin{pmatrix} x \\ \Theta \end{pmatrix}$. The nonlinear function $g(\cdot)$ is given by

$$g(x, \Theta) = \sum_{i=1}^{n-n_1} \Theta_i A_i x .$$

Here Θ is an $n - n_1$–dimensional vector (i.e., $\Theta \subset \Re^{n-n_1 \times n-n_1}$). Since Θ is a constant vector we can take this as part of the dynamics by introducing the equation

$$\dot{\Theta} = 0, \qquad \Theta(0). \qquad (34)$$

The parameter estimation and control problem of interest is defined as follows:

Parameter/Control System Σ_Θ: Suppose A1-A2, A4-A8 hold (with $\tilde{x}^* = (x^*, \Theta^*)$), the dynamics are given by (32), (34), the observations are given by (33), and the signal to be controlled $\ell(\cdot)$ is given by A11 with $\tilde{\zeta}_2$ defined by

$$\tilde{\zeta}_2 = \frac{1}{\theta} |F_{11}(t) x + f_1(t) + g(x,\Theta) + B_t^1 u(t,y)|^2 .$$

Although system Σ_Θ is slightly different from systems $\Sigma_i, 1 \leq i \leq 3$, the analysis of the previous sections goes through, by taking $g(x,\Theta) = D_x \phi(x,\Theta)$, where $\phi(x,\Theta) = \frac{1}{2} x . \sum_{i=1}^{n-n_1} \Theta_i A_i x$. The information state equation for system Σ_Θ is explicitly solvable and is obtained exactly as in Theorem 0.9. In addition, if we employ Theorem 0.11, that is, by setting $\varphi(\tilde{x}) = -\phi(x, \Theta) + \frac{\theta}{2}(Q_T\tilde{x}.\tilde{x} + 2m_T\tilde{x})$, the optimal control law is obtained exactly as in Theorem 0.12; thus, it is linear feedback and equivalent to that associated with the linear $\mathcal{H}^\infty$, or robust tracking problems. We note that additional generalization of system Σ_Θ are possible and should appear elsewhere.

Finally, we point out that the analogous risk-sensitive stochastic control problem of the current paper is treated in [14].

6. Conclusion

In this paper we have considered nonlinear output feedback $\mathcal{H}^\infty$, or robust control tracking problems, and we have presented explicit solutions of the information state equations in terms of a finite number of sufficient statistics forming the observer dynamics. We have then derived sufficient conditions for constructing nonlinear output feedback tracking problems which are equivalent to linear $\mathcal{H}^\infty$ control tracking problems. In addition, we have shown that the class of parameter identification problems with the unknown parameters entering affinely in the drift of the unobservable state, lead to optimal control laws that can be implemented in real-time.

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