Switching strategies for stabilization of a class of nonlinear systems. *

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Abstract

In this paper we present a very simple, yet effective approach to the construction of piecewise constant feedback stabilization strategies for a class of nonlinear systems without drift in which the difference between the number of state and control variables is equal to one or two. The approach exploits a set of guiding functions whose number is equal to the number of the controls in the system. The guiding functions are chosen in such a way as to permit a sequence of controls which result in a monotonic decrease in one of the guiding functions while the remaining ones are varying in an oscillatory way. The strategy is formulated in such a way that no chattering occurs; the oscillations in the values of the guiding functions can be big. It is shown that in the case of the class of system considered, the choice of such guiding functions is particularly straightforward. The proposed feedback control is global in that the origin is globally attractive, and the trajectories of the controlled system converge to the origin exponentially.

Keywords: feedback stabilization, drift-free systems, discontinuous control.

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1 Introduction

We consider systems of the type

$$\dot{\xi} = \sum_{i=1}^{m} f_i(\xi) v_i, \tag{1}$$

where $f_1, ..., f_m$ are linearly independent, smooth vector fields in $I\!R^n$, $m = n-1$, or else $m = n-2$, and v_i are Lebesgue integrable control functions on the interval $[0, \infty)$. A trajectory of system (1) is an absolutely continuous function $t \to x(t) \in I\!R^n$ which satisfies (1) almost everywhere and corresponds to an admissible control $v \stackrel{def}{=} (v_1, ..., v_m)$.

Further, we will assume that the vector fields $f_i, i = 1, ..., m$ are real analytic and that the system is completely controllable, i.e. that for every pair of points ξ_1 and ξ_2 there exists an admissible control which steers the system from ξ_1 to ξ_2.

For real analytic systems, complete controllability is equivalent to the well known LARC (Lie algebraic rank condition): if $L(f_1, ..., f_m)$ denotes the Lie algebra of vector fields generated by $f_1, ..., f_m$, and $L(f_1, ..., f_m)(\xi) \stackrel{def}{=} \{f(\xi) : f \in L(f_1, ..., f_m)\}$, then $L(f_1, ..., f_m)(\xi)$ must span $I\!R^n$ for all $\xi \in I\!R^n$.

Although the LARC guarantees the existence of an admissible control which steers the system from any point ξ_1 to any point ξ_2, it is not obvious how to construct such a control explicitly.

Our interest in this paper is to propose a feedback strategy which steers system (1) from any given intial point ξ_1 to the origin with uniformly bounded, piecewise constant controls $u(\xi) \in [-1, 1]$, $\xi \in I\!R^n$. As will become clear later, the discontinuities of the feedback control $u(\xi)$ will only occur at isolated points, hence no problems will arize in the context of existence of solutions to (1).

The problem of nonlinear stabilization is typically approached by imposing a suitable 'control' Lyapunov function $V(\xi)$ and deriving a control function $u(\xi)$ which renders $\frac{d}{dt}V(\xi) < 0$ along the trajectories of the system , see [1, 2, 3]. For systems of type (1) where there is no drift vector field on the right hand side of the system equation, this approach is not possible. An alternative is to find a *time varying* Lyapunov function $V(\xi, t)$, and a corresponding *time varying* feedback control $u(\xi, t)$ which satisfy $\frac{d}{dt}V(\xi, t) < 0$ in some sense. Several papers have been devoted to this approach, [7, 5, 6, 8]. Although there are many advantages of this approach (such as the fact that it leads to continuous feeback controls), there is one disadvantage: finding a suitable Lyapunov function is sometimes difficult.

In this paper we take a different route. We attempt to find m functions $V_i(\xi)$ $i \in \{1, ..., m\}$, henceforth called 'guiding functions' for system (1), whose behaviour along the trajectories of the controlled system will not be limited to $\frac{d}{dt}V_i(\xi) < 0$. We will therefore allow some guiding functions to increase and aim at designing the control laws $v_i(\xi)$ $i \in \underline{m}$ in such a way that the sum $V(\xi) \stackrel{def}{=} \sum_{i=1}^{m} V_i(\xi)$ decreases *on average*.

It is shown that such a design is particularly simple if we allow discontinuous controls v_i, $i \in \underline{m}$. The principal idea of the guiding function strategy is the following.

Suppose that we can find functions $V_i(\xi)$, $i \in \underline{m}$ such that:

(a) each V_i, $i \in \underline{m}$ is semi-positive definite on $\mathbb{R}^n$ while their sum V is stricly positive definite, decrescent and proper on $\mathbb{R}^n$,

(b) the value of each of the functions V_i, $i \in \underline{m-1}$ can be manipulated independently of the value of V_m in that: at any point p, if $V_i(p) \neq 0$ for some $i \in \underline{m-1}$ then there exist controls $v_i, i \in \underline{m}$ such that V_i $i \in \underline{m-1}$ are 'steered' to zero in finite time while V_m maintains its value at p.

(c) the value of V_m can always be decreased over a finite interval of time if the remaining V_i, $i \in \underline{m-1}$ are allowed to vary freely.

Under the above assumptions, it is clear that a feedback strategy based on the guiding functions V_i, $i \in \underline{m}$ can be focused on the decrease on V_m alone. To start with, the strategy attempts to employ controls which provide for

$$\frac{d}{dt}V(\xi) = \sum_{i=1}^{m} \frac{d}{dt}V_i(\xi) < 0 \qquad (2)$$

If this becomes impossible, due to the fact

$$\frac{d}{dt}V_i(p) = 0 \quad \text{for all } i \in \underline{m} \qquad (3)$$

regardless to the values of the controls $v_i, i \in \underline{m}$, then the controls are changed to achieve a decrease of V_m while the remaining V_i are permitted to increase (see assumption (c)). After achieving a decrease in V_m, another set of controls is employed which maintains the previous value of V_m and restores the previous values of V_i, $i \in \underline{m}$.

Eventually, when V_m is sufficiently small, the controls guaranteed by assumption (b) are employed to decrease V_i to zero.

Repeating the above procedure results in asymptotic convergence of V to zero.

It is shown here that the above strategy is indeed feasible, in that the guiding functions are easy to define and satisfy the desired properties (a)-(c), in the case when the vector fields $f_1, ..., f_{n-2}$ or else the vector fields $f_1, ..., f_{n-3}$ are simultaneously rectifiable.

Without the loss of generality, the strategy employs uniformly bounded, piecewise constant controls v_i. This makes it attractive for applications to problems with control constraints imposed a priori on the system.

The strategy is first tested on the model of Reeds Shepp car where, it is shown to achieve what is the 'intuitively best' type of control. In the absence of disturbances, the control is dead-beat and is accomplished in three steps. At the end of the first step the car assumes a position which is regarded as bad : namely the position when it is sideways to its goal - the origin. This position of the car 'requires' the car to displace sideways, (in the direction of the Lie bracket of the vector fields corresponding to the rotation and rolling movements of the car), in order to decrease its distance from zero. In its second stage the strategy makes the car to rotate 90 degrees and drives it straight to the origin.

2 Notation and assumptions

We use the symbol $\xi(t;\xi_0,t_0)$ (or shortly $\xi(t)$) to denote the trajectory of the controlled system (1) passing through the point (ξ_0,t_0).

Discontinuous stabilization problem

DSP: Find a feedback control strategy in terms of uniformly bounded, piecewise constant controls

$$v_i(\xi) \in [-1,1] \quad i \in \underline{m} \tag{4}$$

such that:

for any initial point $\xi(0)=\xi_1 \in I\!R^n$.

$$\xi(t;\xi_1,0) \to 0 \tag{5}$$

as $t \to \infty$.

We need the following assumptions:

Controllability :

A1. For systems for which $m=n-1$:

$$span\{f_i(\xi),[f_i,f_j](\xi),\ i,j \in \underline{n-1}\} = I\!R^n \tag{6}$$

for all $\xi \in I\!R^n$;

For systems for which $m=n-2$:

$$span\{f_i(\xi),[f_i,f_j](\xi),[f_i,[f_j,f_k]](\xi), \\ i,j,k \in \underline{n-2}\} = I\!R^n \tag{7}$$

for all $\xi \in I\!R^n$.

Rectifiability :

A2. There exist diffeomorphic state feedback transformations $\xi = T_{n-1}(x)$, $v = U_{n-1}(\xi,u)$, and $\xi = T_{n-2}(x)$, $v=U_{n-2}(\xi,u)$ such that in the new coordinates x and in terms of the new control v :

the system with $m=n-1 \geq 2$ assumes the form :

$$\dot{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ . \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ . \\ 0 \end{bmatrix} u_2 + \dots + \begin{bmatrix} 0 \\ . \\ 1 \\ 0 \\ 0 \end{bmatrix} u_{n-2} + \begin{bmatrix} 0 \\ . \\ 0 \\ h_1(x) \\ h_2(x) \end{bmatrix} u_{n-1}$$
$$\stackrel{def}{=} g_1(x)u_1 + g_2(x)u_2 + \dots \\ \dots + g_{n-2}(x)u_{n-2} + g_{n-1}(x)u_{n-1} \tag{8}$$

the system with $m=n-2 \geq 3$ assumes the form :

$$\dot{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ . \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ . \\ 0 \end{bmatrix} u_2 + \dots + \begin{bmatrix} 0 \\ . \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_{n-3} + \begin{bmatrix} 0 \\ . \\ 0 \\ 1 \\ h_1(x) \\ h_2(x) \end{bmatrix} u_{n-2}$$
$$\stackrel{def}{=} g_1u_1 + g_2u_2 + \dots \\ \dots + g_{n-3}(x)u_{n-3} + g_{n-2}(x)u_{n-2} \tag{9}$$

where h_1, h_2, are some smooth functions of the new state variable x.

Sufficient conditions for A2 to hold at least in some neighbourhood of the origin are known, see e.g. [4] :

A necessary and sufficient condition for system (1) to be feedback equivalent with (8) is that $L(f_1,\dots,f_{n-2})$ has constant dimension equal to $n-2$ in a neighbourhood of the origin.

3 The feedback strategy and its properties

For a system in a rectified form (8) we introduce the following semi-positive definite guiding functions:

$$V_i(x) \stackrel{def}{=} \frac{1}{2}x_i^2, \quad i \in \underline{n-2} \tag{10}$$

$$V_{n-1}(x) \stackrel{def}{=} \frac{1}{2}[x_{n-1}^2 + x_n^2] \tag{11}$$

Clearly,

$$V(x) \stackrel{def}{=} \sum_{i=1}^{n-1} V_i(x) = (1/2)x^T x \tag{12}$$

and hence V is positive definite, proper and decrescent, as required. The first stage of the control strategy employs the standard feedback control in order to decrease V :

$$u_i(x) \stackrel{def}{=} -sign(x^T g_i(x)), \text{ for } x \in I\!R^n \setminus S \tag{13}$$

$i \in \underline{n-1}$. Since

$$\frac{d}{dt}V_i(x) = x^T g_i(x) u_i, \quad i \in \underline{n-1} \tag{14}$$

this yields

$$\frac{d}{dt}V(x) = \sum_{i=1}^{n-1} |x^T g_i(x)| < 0 \tag{15}$$

along the controlled trajectory until, at some time instant t, $x(t) \stackrel{def}{=} p \in S$, where the set S consist of points at which all the terms in (15) are identically zero :

$$S \stackrel{def}{=} \{x \in I\!R^n : x^T g_i(x) = 0, i \in \underline{n-1}\} \tag{16}$$

Clearly $p \in S$ implies $V_i(p) = 0, i \in \underline{n-2}$, so that $V(p) = V_{n-1}(p)$ on S. Also, $0 \in S$, so the trajectory of the system which satisfies (15) must approach S in finite or infinite time. In fact we can easily show:

Proposition 1 *The controls defined in (13) steer the system (8) to a point $x \neq 0$, $x \in S$ in finite time, or else the controlled trajectory converges asymptotically to the origin.*

Once $x(t) = p \in S$, the above strategy fails in that $\frac{d}{dt}V(p) = 0$ regardless of the controls.

At this point, the proposed strategy enters its second phase . Since the first derivative of V can no longer be influenced by the choice of controls u_i, we calculate the second derivative of V with respect to time. The second derivatives of its component functions $V_i, i \in \underline{n-1}$, (*while assuming that the controls can only take constant values*) take the form:

$$\begin{aligned} \tfrac{d^2}{dt^2}V_i(x) &= (g_i^T(x) + x^T \nabla g_i(x))\dot{x}u_i \\ &= (g_i^T(x) + x^T \nabla g_i(x)) \textstyle\sum_{j=1}^{n-1} g_j(x) u_j u_i \\ &= ||g_i(x)||^2 u_i^2 + \textstyle\sum_{j=1}^{n-1} x^T \nabla g_i(x) g_j(x) u_j u_i \end{aligned} \tag{17}$$

for $i \in \underline{n-1}$, since $g_i^T g_j = 0$ for $i \neq j$. Next, since

$$||g_i||^2 = 1, \quad \nabla g_i(x) = 0 \quad \text{for } i \in \underline{n-2} \tag{18}$$

we get

$$\frac{d^2}{dt^2}V_i(x) = u_i^2, \quad i \in \underline{n-2} \tag{19}$$

and

$$\begin{aligned} &\tfrac{d^2}{dt^2}V_{n-1}(x) = \\ &= (||g_{n-1}(x)||^2 + x^T \nabla g_{n-1}(x) g_{n-1}(x)) u_{n-1}^2 \\ &\quad + \textstyle\sum_{j=1}^{n-2} x^T \nabla g_{n-1}(x) g_j(x) u_j u_{n-1} \end{aligned} \tag{20}$$

Due to (18) we also have

$$[g_j, g_{n-1}](x) = \nabla g_{n-1}(x) g_j(x), \text{ for all } x \in I\!R^n \tag{21}$$

for all $j \neq n-1$, so that (20) becomes

$$\begin{aligned} &\tfrac{d^2}{dt^2}V_{n-1}(x) = \\ &= (||g_{n-1}(x)||^2 + x^T \nabla g_{n-1}(x) g_{n-1}(x)) u_{n-1}^2 \\ &\quad + \textstyle\sum_{j=1}^{n-2} x^T [g_j, g_{n-1}](x) u_j u_{n-1} \end{aligned} \tag{22}$$

From (19) and (22) it also follows that

$$\frac{d}{dt}x^T g_i(x) = u_i, \quad i \in \underline{n-2} \tag{23}$$

and

$$\begin{aligned} &\tfrac{d}{dt}x^T g_{n-1}(x) = \\ &= (||g_{n-1}(x)||^2 + x^T \nabla g_{n-1}(x) g_{n-1}(x)) u_{n-1} \\ &\quad + \textstyle\sum_{j=1}^{n-2} x^T [g_j, g_{n-1}](x) u_j \end{aligned} \tag{24}$$

By virtue of the controllability assumption, we can always choose an index $i \in \underline{n-2}$ such that

$$|x^T g_i(x)| = max\{|x^T[g_j, g_{n-1}](x)|, j \in \underline{n-2}\} > 0 \tag{25}$$

Setting,

$$\begin{aligned} u_i(x) &= 1 \qquad \text{and} \\ u_j(x) &= 0 \quad \text{for all } j \neq i \end{aligned} \tag{26}$$

results in the increase of V_i along the controlled trajectory, while V_j for $j \neq i$ stays constant. Most importantly, in the process of the above, the 'coefficient' $x^T g_{n-1}(x)$ in V_{n-1} changes from zero to a nonzero value since, by virtue of (24) and (26),

$$\frac{d}{dt} x^T g_{n-1}(x) = x^T[g_i, g_{n-1}](x) \neq 0 \tag{27}$$

along the controlled trajectory with controls as in (26).

It is logical to assume that controls (26) are employed until $x^T g_{n-1}(x)$ reaches its maximal value, and hence until the projection of the current value of the state onto the Lie bracket , $x^T[g_j, g_{n-1}](x)$ changes sign, or else until the value of $V_i(x)$ becomes comparable with the value of $V(p)$ at a point p at which S was last traversed, i.e. until

$$x^T[g_i, g_{n-1}](x) = 0 \quad \text{or} \tag{28}$$

$$V_i(x) \geq \alpha V(p) \tag{29}$$

where $\alpha > 0$ is a given constant. At this point, the fact that $x^T g_{n-1}(x) \neq 0$ can be taken advantage of by resetting the controls to:

$$\begin{aligned} u_{n-1}(x) &= -sign(x^T g_{n-1}(x)) \qquad \text{and} \\ u_j(x) &= 0 \quad \text{for all } j \neq n-1 \end{aligned} \tag{30}$$

This causes a decrease in the guiding function V_{n-1} while the values of the other guiding functions stay unchanged. After $x^T g_{n-1}(x)$ reaches zero again, V_i is restored to its previous value (zero) by

$$\begin{aligned} u_i(x) &= -1 \qquad \text{and} \\ u_j(x) &= 0 \quad \text{for all } j \neq i \end{aligned} \tag{31}$$

and the next cycle is started by choosing a possibly new index value i which satisfies (25).

It should be noted that the "oscillations" in the x_i , $i \neq n-1$ components of the state (as caused by controls (26) and (31)) can be big. No chattering or excessive switching takes place. The evolution of of V_{n-1} consists of intervals in which V_{n-1} stays constant, alternated by intervals in which V_{n-1} is strictly decreasing. In the meantime, the remaining guiding functions are oscillating freely. Quantitatively, the decrease in V_{n-1} can be assessed as follows

Proposition 2 *For every bounded region $G \subset I\!R^n$ there exists a constant $\gamma \in (0,1)$ such that if $x(t_0) = p \in G \cap S$ then the control sequence given by (26) and (30) results in the following decrease in the guiding function V_{n-1}*

$$\begin{aligned} V_{n-1}(x(t_1)) - V_{n-1}(p) &= -\textstyle\int_{t_0}^{t_1} |x^T(t) g_{n-1}(x(t))| dt \\ &\leq -\gamma |x^T(p)[g_i, g_{n-1}](p)| \leq -\tfrac{\gamma}{n} V(p) \end{aligned} \tag{32}$$

over the time interval $[t_0, t_1]$. Here, t_1 corresponds to the time instant at which $x^T(t_1) g_{n-1}(x(t_1)) = 0$ under the action of controls given by (30) .

The above can be formalized into the following algorithmic feedback strategy:

Stabilization feedback strategy for systems with $m = n-1$

Data: $\alpha > 0$.

•1 If $x \in I\!R^n \setminus S$, apply the controls

$$u_i(x) = -sign(x^T g_i(x)), \; i \in \underline{n-1} \tag{33}$$

•2 If $x \in S$,

•2a Select index $i \in n-2$ satisfying

$$|x^T g_i(x)| = max\{|x^T[g_j, g_{n-1}](x)|, \; j \in \underline{n-2}\} \tag{34}$$

•2b Employ the controls

$$\begin{aligned} u_i(x) &= 1 \qquad \text{and} \\ u_j(x) &= 0 \quad \text{for all } j \neq i \end{aligned} \tag{35}$$

until

$$\begin{aligned} x^T[g_i, g_{n-1}](x) &= 0 \quad \text{or} \\ V_i(x) &\geq \alpha V(p) \end{aligned} \tag{36}$$

where p is the value of the state of the controlled system for which the set S is last traversed.

•2c Employ the controls

$$u_{n-1}(x) = -sign(x^T g_{n-1}(x)) \quad \text{and}$$
$$u_j(x) = 0 \quad \text{for all } j \neq n-1 \qquad (37)$$

until $x^T g_{n-1}(x) = 0$.

•2d Employ the controls

$$u_i(x) = -1 \quad \text{and}$$
$$u_j(x) = 0 \quad \text{for all } j \neq i \qquad (38)$$

until $x^T g_i(x) = 0$ and reapeat Steps 2a-2d.

Propositions 1-2 provide a basis for proving the following

Theorem 1 *The stabilization feedback strategy is well defined. Every trajectory of the controlled system converges to the origin exponentially. The origin is globally attractive.*

For a system in a rectified form (9) we introduce the guiding functions similarly as before:

$$V_i(x) \stackrel{def}{=} \frac{1}{2}x_i^2, \quad i \in \underline{n-3} \qquad (39)$$
$$V_{n-2}(x) \stackrel{def}{=} \frac{1}{2}[x_{n-2}^2 + x_{n-1}^2 + x_n^2] \qquad (40)$$

A similar control procedure, as outlined above can be applied here until the control system traverses a set $S_1 \subset S$, where

$$S_1 \stackrel{def}{=} \{x \in I\!R^n : x^T g_i(x) = 0,$$
$$x^T[g_i, g_j](x) = 0, i, j \in \underline{n-2}\} \qquad (41)$$

As $0 \in S_1$, Proposition 1 holds with the set S replaced by S_1.

However, the controls given in (26) can no longer influence the 'coefficient' $x^T g_{n-1}(x)$ of $\frac{d}{dt}V_{n-1}(x)$ at a point $x = p \in S_1$ (for the reason that (27) is no longer true).

At this point it is necessary to calculate third derivatives of $V_i, i \in \underline{n-2}$ in order to see the way in which to change V_{n-2}.

For brevity of exposition, let us discuss here only the case of a system with state dimension $n = 4$ i.e. a system whose 'rectified' form is:

$$\dot{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ h_1(x) \\ h_2(x) \end{bmatrix} u_2$$
$$\stackrel{def}{=} g_1(x)u_1 + g_2(x)u_2 \qquad (42)$$

In this case $V_1(x) \stackrel{def}{=} (1/2)x_1^2$ and $V_2(x) \stackrel{def}{=} (1/2)[x_2^2 + x_3^2 + x_4^2]$. After some calculation, it is possible to show that

$$\frac{d^3}{dt^3}V_2(x) = s_1(x)g_2(x)u_2^3 + s_1(x)g_1(x)u_2^2u_1$$
$$+s_2(x)g_2(x)u_2^2u_1 + s_2(x)g_1(x)u_2u_1^2 \qquad (43)$$

where

$$s_1(x) \stackrel{def}{=} [2g_2^T\nabla g_2 + g_2^T\nabla\{\nabla g_2^T x\}$$
$$+x^T\nabla^2 g_2](x) \qquad (44)$$
$$s_2(x) \stackrel{def}{=} [[g_1, g_2]^T + x^T\nabla[g_1, g_2]](x) \qquad (45)$$

It can further be seen that

$$\frac{d^3}{dt^3}V_2(x) = k_1(x)u_2^3 + k_2(x)u_2^2u_1$$
$$+(k_3(x) + x^T[g_2, [g_1, g_2]](x))u_2^2u_1$$
$$+(k_4(x) + x^T[g_1, [g_1, g_2]](x))u_2u_1^2 \qquad (46)$$

where $k_i(x), i = 1, 2, 3, 4$ are some smooth functions of x which vanish for $x \in S_1$. Since, for $x \in S_1$ either

$$x^T[g_2, [g_1, g_2]](x) \neq 0, \text{ or} \qquad (47)$$
$$x^T[g_1, [g_1, g_2]](x) \neq 0 \qquad (48)$$
$$(49)$$

then it is possible to make $\frac{d^3}{dt^3}V_2(x) < 0$ by choosing a suitable sequence of controls $u_1(x)$, $u_2(x)$. As a consequence, it is possible to show that

Proposition 3 *For every bounded region $G \subset I\!R^n$ there exists a constant $\gamma \in (0, 1)$ such that if $x(t_0) = p \in G \cap S_1$ then there exists a sequence of controls $u_1(x)$ and $u_2(x)$ with values in $[-1, 1]$ which yield*

$$V_2(x(t_1)) - V_2(p) \leq -\frac{\gamma}{n}V(p) \qquad (50)$$

for some $t_1 > t_0$.

The detailed control strategy relevant to this case will be presented elsewhere.

4 Conclusion

A simple piecewise constant feedback strategy was proposed for stabilization of a class of systems without drift. The approach is not really limited to systems with $m = n - 1$ or $m = m - 2$. Systems for which the controllability algebra is spanned by higher order Lie brackets can , in principle, be considered. However, the calculation of the higher order derivatives of the last guiding function V_m, required in this case, are more complex.

Furthermore, the guiding function approach is not limited to systems which are feedback equivalent with their corresponding 'rectified' form. Finding a suitable set of guiding functions can often be easy even if the system does not comply with any of the rectified forms. To see this, consider for example the well known example

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= xv - yu \end{aligned} \tag{51}$$

the two guiding functions needed in this case can, for example, be introduced as follows:

$$\begin{aligned} V_1 &\stackrel{def}{=} \tfrac{1}{2}y^2 \\ V_2 &\stackrel{def}{=} \tfrac{1}{2}x^2 + \tfrac{1}{4}(z - xy)^2 \end{aligned} \tag{52}$$

An easy calculation shows that $S = \{x \in \mathbb{R}^3 : x = y = 0\}$ as

$$\frac{d}{dt}V(x, y, z) = (x - zy)u + (y + zx)v \tag{53}$$

and the controls $u = -sign(x - zy)$ and $v = -sign(y + zx)$ can be applied until the system trajectory traverses S.

For constant controls u, v:

$$\begin{aligned} \tfrac{d^2}{dt^2}V_1(x, y, z) &= v^2 \\ \tfrac{d^2}{dt^2}V_2(x, y, z) &= (1 + 2y^2)u^2 - (z - xy)uv \end{aligned} \tag{54}$$

It follows that, whenever $p \in S$ then setting, for example, $u = 0$ and $v = 1$ produces a change in V_1 while V_2 stays unchanged. Also, the time derivative of the 'coefficient' associated with u is

$$\frac{d}{dt}(x - zy) = (1 + y^2)u - (z - xy)v \tag{55}$$

and hence $|x - zy|$ grows away from zero, as required. The guiding functions, as introduced by (52), satisfy our assumptions (a)-(b) and the control strategy can be applied as specified above.

5 Example

The above feedback strategy was applied to the Reeds-Shepp car model :

$$\dot{x}(t) = f_1(x(t))u_1 + f_2(x(t))u_2 \tag{56}$$

where

$$\begin{aligned} x(t) &\stackrel{def}{=} [x_1(t), x_2(t), x_3(t)]^T \in \mathbb{R}^n \\ f_1(x) &= [1, 0, 0]^T \\ f_2(x) &= [0, sin(x_1), cos(x_1)]^T \end{aligned} \tag{57}$$

The initial condition is $[x_1,\ x_2, x_3](0) = [1., 3., 3.]$ and the constant α is 10. Figure 2. shows the state variables versus time. It is visible that the surface $S = \{x \in \mathbb{R}^n : x_1 = 0\} \cap \{x \in \mathbb{R}^n : x_2 sin(x_1) + x_3 cos(x_1) = 0\} = \{x \in \mathbb{R}^n : x_2 = 0\}$ is reached at about $t = 3.2$. The strategy then enters its second phase . The desired (but inaccessible) direction of motion is $[f_1, f_2]$ at any point $(0, x_2, 0)$, $x_2 > 0$. (or else $-[f_1, f_2]$ when $x_2 < 0$).

Figure 1 shows the actual trajectory of the car's centre of mass. At the end of the first phase of the control strategy the car finds itself in a position sideways to its goal - the origin. Any further decrease of the global guiding function V is impossible at this point since an instantaneous sideways motion of the car is impossible. In Step 2b of the strategy the car is rotated in place by $(\pi/2)$ which is the point at which $x^T[f_1, f_2](x) = 0$, and at which $x^T f_2(x)$ achieves its maximum. The application of Step 2c now results in a straight line motion of the car to the origin. In this case the controller achieves its goal in a finite number of steps, which demonstrates its effectiveness.

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Figure 1. Trajectory of the car model : plot of $x_2(t)$ vs. $x_3(t)$.

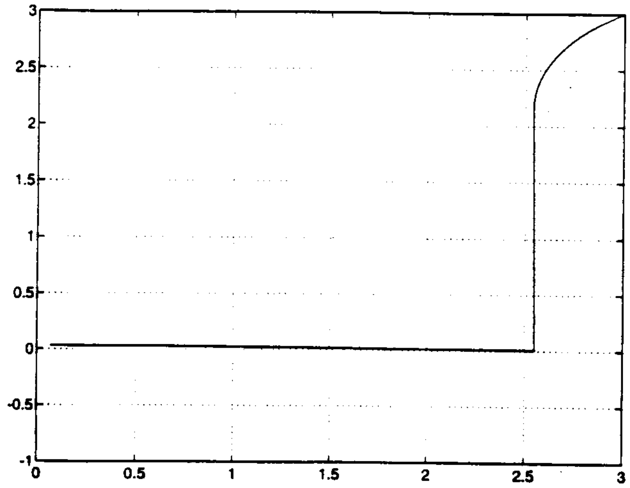

Figure 2. Plot of state variables of the car model $x_1(t), x_2(t), x_3(t)$.

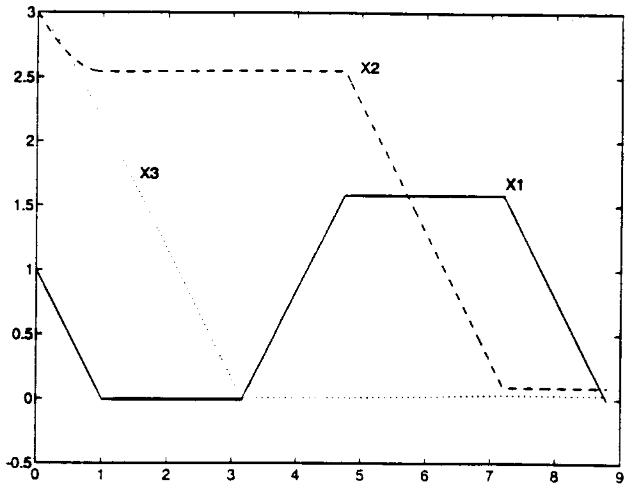

Figure 3. Plot of the guiding functions $V_1(x(t)), V_2(x(t)), V(x(t))$.