DELAY-DEPENDENT STABILITY OF LINEAR TIME-DELAY SYSTEMS: AN LMI APPROACH

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ABSTRACT

This paper considers the problem of stability analysis of linear systems with a single, possibly varying, time-delay. We develop a method based on linear matrix inequalities to determine a bound for the time-delay which ensures global uniform asymptotic stability. The proposed method has the advantage that can be tested numerically very efficiently and was shown, via a simulation example, to give less conservative results.

Keywords: Time-delay systems; stability; linear matrix inequalities.

1. INTRODUCTION

The stability of dynamical systems involving delayed states is a problem of theoretical and practical interest as time-delays are frequently encountered in many processes and very often is the cause for instability. A number of techniques for stability analysis of linear systems with time-delay in the state variable have been reported in the literature over the past decades. Criteria for global uniform stability which are independent of the size of the time-delay have been proposed by a number of investigators; see, e.g. [1], [4] and [13]. Since for these stability criteria the time-delay is allowed to be arbitrary large, these stability results are, in general, conservative for many important applications. Recently, increasing attention has been devoted to the development of methods for delay-dependent stability analysis, i.e. stability criteria which depend on the size of the time-delay (e.g. [5], [6], [8], [9], [11] and [12]). Over the past few years, delay-dependent stability criteria, which are given in terms of the solution of either a Lyapunov or Riccati equation have been proposed in [8], [11] and [12]. A common feature of the latter results is that they involve the tuning of a scalar and/or a symmetric positive definite matrix. However, to the best of our knowledge, no tuning procedure for such scalar and matrix is available, which makes the use of these methods somehow difficult, specially when it is required to find the largest possible bound for the time-delay which ensures global uniform asymptotic stability.

This paper is concerned with the problem of stability analysis of linear systems with delayed state. We consider the case of a single, possibly varying, time-delay and attention will be focused on developing delay-dependent stability criteria based on *linear matrix inequalities*. The stability analysis problem treated here is to determine an upper-bound $\bar{\tau}$ for the time-delay $\tau(t)$ such that the system is globally uniformly asymptotically stable for any $\tau(t)$ satisfying $0 \leq \tau(t) \leq \bar{\tau}$. The linear matrix inequality (LMI) approach developed in this paper has few advantages over the existing methods for delay-dependent stability analysis such as those in [8], [11] and [12]. Firstly, the LMI approach is computationally very efficient as it can be solved numerically using interior point methods; see, e.g. [2] and [7]. Secondly, it does not involve any tuning of a scaling parameter and/or a positive definite matrix as is the case with the methods of [8], [11] and [12]. Thirdly, the problem of finding the largest possible bound for the time-delay which ensures global uniform asymptotic stability can be easily solved using the proposed LMI approach.

Notation. $\Re^+ = [0, \infty)$, $\Re^n$ denotes the n dimensional Euclidean space, $\Re^{n \times m}$ is the set of all $n \times m$ real matrices, $\| \cdot \|$ denotes the Euclidean vector norm, and the notation $X > Y$ (respectively, $X \geq Y$), where X and Y are symmetric matrices, means that the matrix $X - Y$ is positive definite (respectively, positive semi-definite).

2. MAIN RESULT

Consider the following linear time-delay system

$$\dot{x}(t) = Ax(t) + A_d x(t - \tau(t)) \quad (1)$$
$$x(\theta) = \phi(\theta), \quad \forall\,\theta \in [-s, 0], \quad s > 0 \quad (2)$$

where $\tau(t)$ is a varying time-delay which satisfies

$$0 \le \tau(t) \le s, \; \forall\, t \ge 0,$$

$x(t) \in \Re^n$ is the state, A and A_d are $n \times n$ real matrices, and $\phi(\cdot)$ is the initial condition.

In this paper we shall develop delay-dependent conditions for global uniform asymptotic stability of the system (1)-(2). More specifically, our objective is to determine a bound $\bar{\tau}$ for the time-delay such that the system (1)-(2) is globally uniformly asymptotically stable for any $\tau(t)$ satisfying $0 \le \tau(t) \le \bar{\tau}$. A linear matrix inequality approach will be developed in this paper for solving the latter stability analysis problem.

We shall adopted the following assumption for the system of (1)-(2).

Assumption 1 *The matrix $A + A_d$ has all its eigenvalues in the open left-half plane.*

We observe that the above assumption, which corresponds to the asymptotic stability of system (1)-(2) without time-delay, is indeed necessary for the global uniform asymptotic stability of system (1)-(2) in the presence of time-delay.

The next theorem provide an LMI method for obtaining a bound on the time-delay guaranteeing the global uniform asymptotic stability of system (1)-(2).

Theorem 1 *Consider the system (1)-(2) satisfying Assumption 1. Then given a scalar $\bar{\tau} > 0$, this system is globally uniformly asymptotically stable for any time-delay $\tau(t)$ satisfying $0 \le \tau(t) \le \bar{\tau}$ if any of the following equivalent conditions holds:*

(i) There exist symmetric matrices $P > 0$, $P_1 > 0$ and $P_2 > 0$ solving the following LMIs:

$$\begin{bmatrix} (A+A_d)^T P + P(A+A_d) + 2\bar{\tau}P & \bar{\tau}PA_dA & \bar{\tau}PA_d^2 \\ \bar{\tau}A^TA_d^TP & -\bar{\tau}P_1 & 0 \\ \bar{\tau}(A_d^2)^TP & 0 & -\bar{\tau}P_2 \end{bmatrix} < 0. \quad (3)$$

$$P_1 - P \le 0, \qquad P_2 - P \le 0. \quad (4)$$

(ii) There exist symmetric matrices $X > 0$, $X_1 > 0$ and $X_2 > 0$ solving the following LMIs:

$$X(A+A_d)^T + (A+A_d)X + 2\bar{\tau}X + \bar{\tau}A_dAX_1A^TA_d^T + \bar{\tau}A_d^2X_2(A_d^2)^T < 0 \quad (5)$$

$$X - X_1 \le 0, \qquad X - X_2 \le 0. \quad (6)$$

Before proceeding to the proof of Theorem 1 we recall the following inequality.

Proposition *For any z, $y \in \Re^n$ and for any symmetric positive definite matrix $X \in \Re^{n\times n}$,*

$$-2z^Ty \le z^TX^{-1}z + y^TXy.$$

For the sake of simplicity of notation, we shall omit in the sequel the dependence on t in the time-delay.

Proof of Theorem 1

(i) Consider the following time-delay system:

$$\dot{\xi}(t) = (A + A_d)\xi(t) - A_d\int_{-\tau}^{0}[A\xi(t+\theta) + A_d\xi(t-\tau+\theta)]d\theta \quad (7)$$

$$\xi(\theta) = \psi(\theta), \; \forall\,\theta \in [-2s, 0] \quad (8)$$

where $\psi(\cdot)$ is the initial condition and τ is a varying time-delay which satisfies $0 \le \tau(t) \le s$ for all $t \ge 0$.

Let $x(t)$, $t \ge 0$, be a trajectory of the system (1)-(2). Since

$$x(t-\tau) = x(t) - \int_{-\tau}^{0}\dot{x}(t+\theta)d\theta = x(t) - \int_{-\tau}^{0}[Ax(t+\theta) + A_dx(t-\tau+\theta)]d\theta$$

it follows that any solution of (1)-(2) is also a solution of (7)-(8). This implies that the uniform asymptotic stability of system (7)-(8) will ensure the uniform asymptotic stability of (1)-(2); see, e.g. [3]. In the sequel we will study the stability of system of (7)-(8) in order to ascertain the uniform asymptotic stability of system (1)-(2).

Introduce the Lyapunov function candidate for the system of (7)-(8)

$$V(\xi) = \xi^T(t)P\xi(t) \quad (9)$$

where P is a symmetric positive definite matrix. Then the time-derivative of $V(\xi)$ along the solution of (7)-(8) is given by

$$\dot{V}(\xi) = \xi^T(t)\left[(A+A_d)^TP + P(A+A_d)\right]\xi(t) + \eta_1(\xi,t) + \eta_2(\xi,t) \quad (10)$$

where

$$\eta_1(\xi,t) = -2\int_{-\tau}^{0}\xi^T(t)PA_dA\xi(t+\theta)d\theta$$

$$\eta_2(\xi,t) = -2\int_{-\tau}^{0}\xi^T(t)PA_d^2\xi(t-\tau+\theta)d\theta.$$

Note that in view of the Proposition, we have that for any $n \times n$ symmetric matrices $P_1 > 0$ and $P_2 > 0$

$$\eta_1(\xi,t) \leq \int_{-\tau}^{0} \left[\xi^T(t) P A_d A P_1^{-1} A^T A_d^T P \xi(t) + \xi^T(t+\theta) P_1 \xi(t+\theta)\right] d\theta$$
$$= \tau \xi^T(t) P A_d A P_1^{-1} A^T A_d^T P \xi(t) + \int_{-\tau}^{0} \xi^T(t+\theta) P_1 \xi(t+\theta) d\theta \qquad (11)$$

and

$$\eta_2(\xi,t) \leq \int_{-\tau}^{0} \left[\xi^T(t) P A_d^2 P_2^{-1} (A_d^2)^T P \xi(t) + \xi^T(t-\tau+\theta) P_2 \xi(t-\tau+\theta)\right] d\theta$$
$$= \tau \xi^T(t) P A_d^2 P_2^{-1} (A_d^2)^T P \xi(t) + \int_{-\tau}^{0} \xi^T(t-\tau+\theta) P_2 \xi(t-\tau+\theta) d\theta. \qquad (12)$$

In the light of the Razumikhin-type stability theorem (see the Appendix), we assume that for some real number $\delta > 1$, the following inequality holds:

$$V[\xi(\theta)] < \delta V[\xi(t)], \ \forall \theta \in [t-2s,\ t]. \qquad (13)$$

Hence, using (11)-(12) in (10) and considering (13) we obtain

$$\dot{V}(\xi) < \xi^T(t) \left[(A+A_d)^T P + P(A+A_d) + 2\tau\delta P + \tau P A_d A P_1^{-1} A^T A_d^T P + \tau P A_d^2 P_2^{-1} (A_d^2)^T P\right] \xi(t) \qquad (14)$$

where P_1 and P_2 are any $n \times n$ real symmetric positive definite matrices such that $P_1 \leq P$ and $P_2 \leq P$.

Now, we define the the matrix function

$$S(P, P_1, P_2, \alpha) \triangleq (A+A_d)^T P + P(A+A_d) + \alpha P A_d A P_1^{-1} A^T A_d^T P + \alpha P A_d^2 P_2^{-1} (A_d^2)^T P + 2\alpha P \qquad (15)$$

where P, P_1 and P_2 are real symmetric positive definite matrices and α is a real number.

Since $S(P, P_1, P_2, \alpha)$ is monotone increasing with respect to α (in the sense of positive definiteness), we have that if for some scalar $\bar{\tau} > 0$ there exist symmetric positive definite matrices P, P_1 and P_2 satisfying the inequalities

$$S(P, P_1, P_2, \bar{\tau}) < 0, \quad P_1 \leq P. \quad P_2 \leq P, \qquad (16)$$

then there exists a sufficiently small $\delta > 1$ such that for any $\tau(t) \leq \bar{\tau}$

$$W \triangleq -\left[(A+A_d)^T P + P(A+A_d) + 2\tau\delta P + \tau P A_d A P_1^{-1} A^T A_d^T P + \tau P A_d^2 P_2^{-1} (A_d^2)^T P\right] > 0.$$

This implies that for any $\tau(t) \leq \bar{\tau}$

$$\dot{V}(\xi) < -\lambda_{min}(W) \|\xi(t)\|^2$$

where $\lambda_{min}(W)$ denotes the minimum eigenvalue of W. Hence, it follows from the Razumikhin-type theorem that the system (7)-(8) is globally uniformly asymptotically stable for any time-delay $\tau(t) \leq \bar{\tau}$.

Finally, using Schur complements we obtain that the inequalities of (16) are equivalent to the those of (3)-(4).

(ii) Using the new variables $X \triangleq P^{-1}$, $X_1 \triangleq P_1^{-1}$ and $X_2 \triangleq P_2^{-1}$, it can be easily obtained that the conditions of (16) are equivalent to the following inequalities

$$X(A+A_d)^T + (A+A_d)X + 2\bar{\tau}X + \bar{\tau} A_d A X_1 A^T A_d^T + \bar{\tau} A_d^2 X_2 (A_d^2)^T < 0$$

$$X - X_1 \leq 0, \qquad X - X_2 \leq 0$$

which concludes the proof.

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Remark Theorem 1 provides delay-dependent conditions for global uniform asymptotic stability of linear time-delay systems in terms of the solvability of linear matrix inequalities. This is in contrast with the results of [8], [11] and [12] which are given in terms of the solution of either a Lyapunov or Riccati equation. We note that a common feature of the methods of [8], [11] and [12] is that they involve the tuning of a scalar and/or a symmetric positive definite matrix. However, it happens that both the scalar and the matrix to be tuned enter the bound on the time-delay nonlinearly and, to the best of our knowledge, no tuning procedure for such scalar and matrix is available. This makes the use of these methods somehow difficult, specially when one wants to find the largest possible bound for the time-delay which ensures global uniform asymptotic stability.

The stability criteria of Theorem 1 have the advantage that they are given in terms of the solution of linear matrix inequalities, and thus do not involve any parameter tuning, as is the case with the methods of [8], [11] and [12]. Indeed, the stability criteria proposed in this paper can be tested numerically very efficiently using interior point algorithms, which have been recently developed for solving linear matrix inequalities; see, e.g. [2] and [7]. Observe that the LMI (3) of criterion (i) is of dimensions $3n \times 3n$ whereas the LMI (5) of criterion (ii) is $n \times n$. This makes the criterion (ii) numerically more attractive.

Another advantage of the LMI approach is that the problem of finding the largest $\bar{\tau}$ can also be easily solved without the need of tuning any parameter. For instance, using the stability criterion (ii) of Theorem 1, the largest value of $\bar{\tau}$ can be computed by solving the following quasi-convex optimization problem in X, X_1, X_2 and $\bar{\tau}$:

$$\begin{aligned}&\text{maximize} \quad \bar{\tau}\\ &\text{subject to } X > 0,\ X_1 \geq X,\ X_2 \geq X,\ \bar{\tau} > 0 \text{ and } (5).\end{aligned}$$

Note that the above optimization problem has the form of a generalized eigenvalue problem, which can be solved numerically very efficiently; see, e.g. [2] and [7]. Similar optimization procedure also applies to the stability criterion (i). □

3. AN EXAMPLE

Consider the following linear time-delay system which has been analysed in [11] and [12]:

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - \tau(t)) \tag{17}$$

where $0 \leq \tau(t) \leq s$ for all $t \geq 0$, where $s > 0$, and with the initial condition

$$x(\theta) = \phi(\theta), \quad \forall\, \theta \in [-s, 0]. \tag{18}$$

With either of the stability results of Theorem 1, it has been obtained using the software package LMI Lab that the system (17)-(18) is globally uniformly asymptotically stable for any time-delay $\tau(t)$ satisfying $0 \leq \tau(t) \leq 0.8571$. On the other hand, the criterion of [11] guarantees the global uniform asymptotic stability of (17)-(18) if $0 \leq \tau(t) < 0.4629$, whereas the method of [12] (with the correction as suggested in [14]) gives a bound of 0.2189. Note that the time-delay bound provided by our method is 85% larger than that obtained in [11], which is a significant improvement. Hence, for this example, Theorem 1 gives a less conservative bound for the time-delay which ensures global uniform asymptotic stability than those obtained via the methods of [11] and [12].

4. CONCLUSIONS

The problem of stability analysis of linear time-delay systems has been addressed. A linear matrix inequality approach to delay-dependent stability analysis has been proposed. The LMI approach developed in this paper has two significant advantages. First, it is computationally very efficient as it can be solved numerically using interior point algorithms. Another advantage is that the problem of finding the largest bound for the time-delay to ensure global uniform asymptotic stability can be easily solved using the proposed LMI approach and does not involve the tuning of a scaling parameter nor a positive definite matrix, as is the case with some of the existing delay-dependent stability criteria.

APPENDIX

Razumikhin-Type Stability Theorem

Let $\mathcal{C}_n = C([-s, 0], \Re^n)$ for a given $s > 0$ denotes the Banach space of continuous functions mapping the interval $[-s, 0]$ into $\Re^n$ with the topology of uniform convergence. For $\phi \in \mathcal{C}_n$, define $\|\phi\|_c = \sup_{-s \leq \theta \leq 0} \|\phi(\theta)\|$ and $\mathcal{C}_n^a$, where $a > 0$, denotes the set $\{\phi \in \mathcal{C}_n : \|\phi\|_c < a\}$. Moreover, let $x_t \in \mathcal{C}_n$ for a given t, be defined by $x_t(\theta) = x(t + \theta)$, $\forall\, \theta \in [-s, 0]$.

Consider the functional differential equation of retarded type

$$\dot{x}(t) = f(t, x_t) \tag{A.1}$$

where $f : \Re^+ \times \mathcal{C}_n^a \rightarrow \Re^n$ is continuous and $f(t, 0) = 0$ for all $t \in \Re$. It is assumed that for any $\phi \in \mathcal{C}_n^a$ and for any $t_0 \in \Re$, (A.1) with the initial condition

$$x_{t_0}(\theta) = \phi(\theta), \quad \forall \theta \in [-s, 0] \tag{A.2}$$

possesses a unique solution.

Theorem A.1 ([3]) *Consider the retarded functional differential equation (A.1) and let u, v, w : $\Re^+ \rightarrow \Re^+$ be continuous, positive definite functions, u, w non-decreasing, and v strictly increasing. Suppose $p : \Re^+ \rightarrow \Re^+$ is a continuous non-decreasing function satisfying $p(h) > h$ for $h > 0$. If there exists a continuous function $V : \Re \times \Re^n \rightarrow \Re^+$ such that:*

(a) $u(\|x\|) \leq V(t, x) \leq v(\|x\|), \quad \forall t \in \Re, \quad x \in \Re^n$

(b) For any $t_0 \in \Re$, the derivative of $V(t, x)$ along the solution of (A.1)-(A.2), defined as

$$\dot{V}(t, x(t)) = \lim_{h \rightarrow 0^+} \sup \frac{1}{h}[V(t + h, x(t + h)) - V(t, x(t))]$$

satisfies $\dot{V}(t, x(t)) \leq -w(\|x\|)$, if $V(t + \theta, x(t + \theta)) \leq p(V(t, x(t)))$ for all $\theta \in [-s, 0]$, $t \geq t_0$, then the solution $x = 0$ of (A.1) is uniformly asymptotically stable. Moreover, if $u(h) \rightarrow \infty$ as $h \rightarrow \infty$, then the solution $x = 0$ is globally uniformly asymptotically stable.

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