On Observability of Linear Delay Systems with Unknown Input

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Abstract

For linear systems with delays we consider the problem of the observability with unknown input and initial function. Abstract condition is given using output nulling subspace described using frequency domain approach. In the state space framework one gives geometric conditions for perfect observability, in particular, for a finite time intervalle. We use the description of the delay systems by systems without delay which represent the system in a finite time interval. One notes also the connection with other control problems (invertibility, disturbance decoupling).

Keywords: Delay systems, Observability, Invertibility, Decoupling.

1 Preliminaries

The present paper is concerned with linear delay systems, i.e. systems described by equations

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A_0x(t) + A_1x(t-h) + B_0u(t) \\ y(t) & = & C_0x(t) \end{array} \right. \qquad (1)$$

The state space X is $\mathbb{R}^n$, the input space U is $\mathbb{R}^p$ and the output space Y is $\mathbb{R}^m$, h stands for the delay. In this paper we assume that matrices B_0 and C_0 are of full rank. The control function is assumed to be Laplace-transformable, i.e. the integral

$$\int_0^\infty e^{-st}u(t)dt$$

converge absolutely for $\operatorname{Re} s > \alpha$.

One can also consider more general systems with several commensurate delays, with delays in input and output. The approach developped in this paper may be extended for those cases.

The initial conditions are given by

$$x(0) = x_0, \quad x(t) = \varphi(t) \quad \text{for } t \in [-h, 0[\qquad (2)$$

The purpose of this study is to determine conditions when from the observation of the output $y(t)$ on a finite interval of time, one may determine the state of the system, the input $u(t)$ and the initial conditions being, in general unknown. This notion, called perfect observability, for a system without delay ($A_1 = 0$), was considered first by Basile and Marro [1]. The well known criteria of observability for the case $A_1 = 0$ is that the supremal (A_0, B_0)-invariant subspace contained in $\operatorname{Ker} C_0$, noted $\mathcal{V}^*$, is the trivial subspace $\{0\}$. There is other equivalent conditions, for example the rank condition:

$$\operatorname{rank} \begin{bmatrix} sI - A_0 & -B_0 \\ C_0 & 0 \end{bmatrix} = n + m, \quad \forall s \in \mathbb{C}.$$

Miniuk [5] shows that an analogous condition holds for delay systems like (1): the perfect observability of the delay system is equivalent to the condition

$$\operatorname{rank} \begin{bmatrix} sI - A_0 - A_1e^{-hs} & -B_0 \\ C_0 & 0 \end{bmatrix} = n + m, \qquad (3)$$

for all $s \in \mathbb{C}$. In this paper geometric conditions are given for perfect observability. We introduce and consider also the notion of the perfect observability in finite time which is, for delay systems, different from perfect observability. The subspaces introduced for observability are used for other control problems.

*Supported in part by ESPRIT Basic Research Project No 8924 (SESDIP)

1.1 System description

We use a step-by-step description of the delay system. This description was introduced by Olbrot [7] and Zmood [14]. This approach is based on the idea that the system (1) may be described by the equations

$$\left\{\begin{array}{lcl} \dot{z}_k(t) & = & F_k z_k(t) + G_k v_k(t) \\ w_k(t) & = & H_k z_k(t) \end{array}\right. \tag{4}$$

where the matrices F_k, G_k and H_k are given by

$$F_k = \begin{bmatrix} A_0 & 0 & 0 & \dots & 0 \\ A_1 & A_0 & 0 & \dots & 0 \\ 0 & A_1 & A_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_0 \end{bmatrix},$$

$$G_k = \begin{bmatrix} B_0 & 0 & 0 & \dots & 0 \\ 0 & B_0 & 0 & \dots & 0 \\ 0 & 0 & B_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_0 \end{bmatrix},$$

$$H_k = \begin{bmatrix} C_0 & 0 & 0 & \dots & 0 \\ 0 & C_0 & 0 & \dots & 0 \\ 0 & 0 & C_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & C_0 \end{bmatrix}.$$

Let Z_k be the matrices:

$$Z_0 = I, \quad Z_k = \begin{bmatrix} I \\ e^{F_{k-1}h} Z_{k-1} \end{bmatrix}.$$

It is well known that the solution of the system (1) with the input $u(t) = 0$ and the initial function $\varphi = 0$ may be written as

$$\begin{array}{rcl} x(t) & = & \Phi(t)x_0 \\ & = & [0, \dots, 0, I] e^{F_k(t-kh)} Z_k x_0, \quad t \in [k, k+1]. \end{array}$$

For arbitrary initial function and control function one has

$$\begin{array}{rcl} x(t) & = & \Phi(t)x_0 + \int_0^h \Phi(t-\tau)A_1\varphi(\tau)d\tau \\ & + & \int_0^t \Phi(t-\tau)Bu(\tau)d\tau. \end{array}$$

Another expression of the fundamental solution may be given using the so called "determining equations" introduced by Gabasov and Kirillova (see for instance [3]). Let the matrices $Q_i(j)$ be defined by

$$Q_i(j) = A_0 Q_{-1}(j) + A_1 Q_{i-1}(j-1),$$

$$Q_0(0) = I, \quad Q_i(j) = 0 \quad \text{if} \quad ij < 0.$$

Then the fundamental solution $\Phi(t)$ may be written as follows

$$\Phi(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{k} Q_i(j) \frac{(t-j)^i}{i!} \quad \text{for} \quad t \in [kh, (k+1)h].$$

The relation between the two different expressions for the fundamental solution $\Phi(t)$ may be explained by the equality

$$F_k^i = \begin{bmatrix} Q_i(0) & 0 & \dots & 0 \\ Q_i(1) & Q_i(0) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Q_i(k) & Q_i(k-1) & \dots & Q_i(0) \end{bmatrix}.$$

In [6, 10, 11] the expression given here was used to describe structure at infinity and to solve some control problems as disturbance and row-by-row decoupling.

1.2 Geometrics tools

As in [6] we consider the ouput nulling subspace in frequency domain. First we define strictly proper function.

Definition 1.1 *A function $f(s)$ defined and and analytical for $Res > s_0$ is said strictly proper if $sf(s) \longrightarrow f_0$ as $s \longrightarrow +\infty$, s being real and f_0 a constant.*

Note that if $f(s)$ is strictly proper then $f(s) \longrightarrow 0$ as $s \longrightarrow \infty$. According to this definition the transfer function matrix $T(s) = C_0(sI - A_0 - A_1e^{-s})^{-1}B_0$ is a strictly proper function matrix.

Definition 1.2 *The subspace $\mathcal{V}_\Sigma$ which elements are $x \in \mathbb{R}^n$ such that*

$$x = (sI - A_0 - A_1e^{-s})\xi(s) - B\omega(s), \quad s > s_0,$$

where s_0 is a real number and ξ, ω are strictly proper functions, such that $C_0\xi(s) \equiv 0$, is called the output nulling subspace in frequency domain.

Note that the subspace $\mathcal{V}_\Sigma$ is in $\mathbb{R}^n$. In [15], in a infinite dimensional setting, was considered the *supremal frequency invariant subspace*. If we consider the infinite dimensional representation of the delay system (1) in the subspace $\mathbb{R}^n \times L_2(-h, 0; \mathbb{R}^n)$ then the subspace $\mathcal{V}_\Sigma$ is the projection on $\mathbb{R}^n$ of the intersection of the *supremal frequency invariant subspace* and the subspaces $\mathcal{X}_0 \subset \mathbb{R}^n \times L_2(-h, 0; \mathbb{R}^n)$ which elements are $(x, 0), x \in \mathbb{R}^n$. In a similar way one can define the ouput nulling subspace in the time domain.

Definition 1.3 *The output nulling subspace in the time domain, noted $\mathcal{V}_{ol}$, is the subspace of those $x \in \mathbb{R}^n$ such that there exist an initial function φ and a control $u(t)$ such that the corresponding output $y(t)$ vanishes for all $t \geq 0$. In the particular case when the initial function is 0, the corresponding subspace is noted $\mathcal{V}_{ol}^0$.*

Let us remark that if $A_1 = 0$ then the subspaces $\mathcal{V}_\Sigma$ and $\mathcal{V}_{ol}$ concide with the subspace $\mathcal{V}^*$ which is given by the algorithm:

$$\mathcal{V}_0 = \mathbb{R}^n,$$

$$\mathcal{V}_i = \operatorname{Ker} C_0 \cap A_0^{-1}(\mathcal{V}_{i-1} + \operatorname{Im} B),$$

$\mathcal{V}^* = \cap_{i \in \mathbb{N}} \mathcal{V}_i$. And there exists a matrix L_0 such that $\mathcal{V}^*$ is invariant under the transformation $A_0 + B_0 L_0$ (see [1, 13]).

2 Observability

Let us give the precise definition of perfect observability.

Definition 2.1 *The system (1) is said perfectly observable if $\mathcal{V}_{ol} = \{0\}$.*

The perfect observability means that given an output $y(t) \equiv 0$ the corresponding initial conditions which generate this output is $(x_0, \varphi) = (0, 0)$ for all possible control $u(t)$. One can also consider a weaker notion when the initial function is known to be 0. In this case we say that the system is weakly perfectly observable, this notion correspond to the condition $\mathcal{V}_{ol}^0 = 0$.
In the sequel we say "observability" instead of "perfect observability" if no confusion occurs.

First let us show the following Lemma.

Lemma 2.2 $\mathcal{V}_{ol}^0 \subset \mathcal{V}_{ol} \subset \mathcal{V}_\Sigma$.

Proof. The first inclusion is obvious. Suppose that $x_0 \in \mathcal{V}_{ol}$ then there exists an initial condition φ and a control $u(t)$ such that the corresponding solution $x(t)$ verifies

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B_0 u(t), \quad C_0 x(t) = 0, \quad t \geq 0.$$

The Laplace transform gives

$$x_0 = (sI - A_0 - A_1 e^{-hs})\hat{x}(s) - B\hat{u}(s)$$

with $\hat{x}$ and $\hat{u}$ strictly proper functions and $\hat{x}(s) \in \operatorname{Ker} C_0$ for all s. This means that x_0 is in $\mathcal{V}_\Sigma$. ∎

The main result is given by the following theorem.

Theorem 2.3 *The system (1) is observable if and only if the following equivalent conditions holds*

$$\text{i) rank} \begin{bmatrix} sI - A_0 - A_1 e^{-hs} & -B_0 \\ C_0 & 0 \end{bmatrix} = n + m$$

ii) $\mathcal{V}_\Sigma = \{0\}$.

In the sequel the matrix

$$\begin{bmatrix} sI - A_0 - A_1 e^{-hs} & -B_0 \\ C_0 & 0 \end{bmatrix} = n + m$$

will be noted by $K(s)$.

Proof. The condition *i)* was in fact proved in [5]. One gives here another proof following the scheme $i) \Rightarrow, ii) \Rightarrow \mathcal{V}_{ol} = \{0\} \Rightarrow \operatorname{rank} K(s) = n + m$.

1. Assume that rank $K(s) = n+m$ and $x_0 \in \mathcal{V}_\Sigma$. Then $x_0 = (sI - A_0 - A_1 e^{-hs})\xi(s) - B\omega(s)$, with $\xi(s)$ and $\omega(s)$ strictly proper functions and $C_0 \xi(s) = 0$. This may be written as

$$\begin{bmatrix} x_0 \\ 0 \end{bmatrix} = K(s) \begin{bmatrix} \xi(s) \\ \omega(s) \end{bmatrix}.$$

Let $K_l^{-1}(s)$ be the left inverse of the matrix $K(s)$. Then we have

$$K_l^{-1} \begin{bmatrix} x_0 \\ 0 \end{bmatrix} = \begin{bmatrix} \xi(s) \\ \omega(s) \end{bmatrix}.$$

Let us note by K_{ij} the blocks of the matrix K_l^{-1}:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} sI - A_0 - A_1 e^{-hs} & -B_0 \\ C_0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

We have then $-K_{21} B_0 = I$ which means that $-K_{21}$ is a constant matrix. In the other hands, $\omega(s) = -K_{21} x_0$. Then ω cannot be a strictly proper function if $x_0 \neq 0$. This implies that $\mathcal{V}_\Sigma = \{0\}$.

2. If $\mathcal{V}_\Sigma = \{0\}$ then the Lemma 2 gives $\mathcal{V}_{ol} = \{0\}$.

3. Suppose now that $\mathcal{V}_{ol} = \{0\}$ but rank $K(s) < n+m$, i.e. for some s_1 $\operatorname{Ker} K(s_1) \neq \{0\}$:

$$(s_1 I - A_0 - A_1 e^{-hs_1})\xi_1 - B\omega_1 = 0$$

and $C_0 \xi_1 \equiv 0$. Let F be the matrix defined such that $F\xi_1 = \omega_1$. If $x(t) = e^{s_1 ht}\xi_1$ and $u(t) = -Fe^{s_1 ht}\xi_1$, then a simple calculation gives

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B_0 u(t)$$

and $C_0 x(t) \equiv 0$. this means that $\xi_1 \in \mathcal{V}_{ol}$. But $(\xi_1, \omega_1) \neq (0, 0)$, and $\xi_1 = 0$ implies $\omega_1 = 0$, because of the assumption: B_0 of full rank. Finally $\mathcal{V}_{ol} \neq \{0\}$. This ends the proof. ∎

Remark 2.4 The assumption of B_0 being of full rank is not essential: in the general case one replaces in the proof B_0 by $\underline{B}_0$ which is composed by the $\underline{m}$ independent columns of B_0 and m by $\underline{m}$. On the other hand, the Theorem 2.3 yields a necessary condition of perfect observability: $\operatorname{rank} C_0 \geq \operatorname{rank} B_0$.

The conditions given by the Theorem 2.3 is not easy to verify. In the following section one obtains a more simple but only sufficient condition of perfect observability as a consequence of a condition of perfect observability in finite time.

3 Observability in finite time

For time invariant systems without delay perfect observability does not depend on the time of observation. If a system is observable on $[0, T]$, then it is observable in $[0, T_1]$ for all T_1. For delay systems, it is not the case. The following example may illustrate this situation.

Example. Let consider the system (1) with

$$A_0 = 0,\ A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_0 = [1 \quad 1].$$

A simple computation shows that the initial state $(x_0, 0)$ with $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not observable on $[0, h]$ but all states are observable on $[0, 2h]$.

In the other hand the condition of perfect observability does not insure the existence of a finite time interval on which the system may be observable. As in many applications one considers a systems in a finite time interval, it is interesting to know conditions of observability in a finite time interval. This section is devoted to this problem.

The approach developped here make use of tools introduced in [7] for controllabilty and extended in [8] for delay systems of neutral type. Consider the systems (4). Let $\mathcal{V}_k^*$ be the supremal (F_k, G_k)-invariant contained in $\operatorname{Ker} H_k$. It is well known that this subspace contains the elements z_k^0 such that there exists a function v_k such that the output w_k corresponding to the initial condition z_k^0 and the control v_k is equal to zero. In the other hand, there exists a feedbak R_k such that $\mathcal{V}_k^*$ is $(F_k, +G_k R_k)$-invariant. Let now $\mathcal{K}_k^*$ be the (F_k, G_k)-controllability subspace in $\operatorname{Ker} H_k$, i.e. the subpace of all reachable states from the origin with a trajectory lying in $\mathcal{V}_k^*$. Note that if R_k is as defined here, the subspace $\mathcal{K}_k^*$ is also $(F_k, +G_k R_k)$-invariant. Just in order to simplify, we consider the delay system with a zero initial function φ Later will be considered the more general case.

Lemma 3.1 *An initial condition x_0 is such that the corresponding output of the system (1) verifies*

$$y(t, x_0, 0, u(t)) = 0, \quad \text{for} \quad t \in [0, (k+1)h]$$

for some control $u(t)$ if and only if

$$\begin{bmatrix} x_0 \\ 0 \end{bmatrix} \in \left(I_k - J_k e^{(F_k + G_k R_k)h} \right) \mathcal{V}_k^* + J_k \mathcal{K}_k^*,$$

where I_k is the identity in $\mathbb{R}^{n \times (k+1)}$ and J_k is given by

$$J_k = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ I & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

and I is the identity in $\mathbb{R}^n$.

Proof. Suppose that x_0 is such that there exists a control $u(t)$ and

$$y(t, x_0, 0, u(t)) = 0, \quad \text{for} \quad t \in [0, k].$$

Then the corresponding solution of the system (4), say $z_k(t)$ is in $\mathcal{V}_k^*$ for all $t \in [0, h]$. The corresponding initial condition $z_k(0)$ is given by

$$z_k(0) = \begin{bmatrix} x_0 \\ z_{k-1}(h) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} + J_k z_k(h).$$

The value $z_k(h)$ is in $\mathcal{V}_k^*$ and may be written as

$$z_k(h) = e^{(F_k + G_k R_k)h} z_k(0) + \zeta_k,$$

where $\zeta_k \in \mathcal{K}_k^*$. This gives

$$\begin{bmatrix} x_0 \\ 0 \end{bmatrix} = (I_k - J_k e^{(F_k + G_k R_k)h}) z_k(0) - J_k \zeta_k.$$

As $z_k(0) \in \mathcal{V}_k^*$, one obtains the desired result.

Conversely, suppose that

$$\begin{bmatrix} x_0 \\ 0 \end{bmatrix} = \left(I_k - J_k e^{(F_k + G_k R_k)h} \right) \mathcal{V}_k^* + J_k \mathcal{K}_k^*.$$

Then

$$\begin{bmatrix} x_0 \\ 0 \end{bmatrix} = \left(I_k - J_k e^{(F_k + G_k R_k)h} \right) z_k + J_k \zeta_k,$$

with

$$\zeta_k = \int_0^h e^{(F_k + G_k R_k)(h - \tau)} G_k v_k(\tau) d\tau.$$

This yields to

$$\begin{bmatrix} x_0 \\ 0 \end{bmatrix} = z_k + J_k z_k(h),$$

where

$$z_k(h) = e^{(F_k+G_kR_k)h} z_k + \int_0^h e^{(F_k+G_kR_k)(h-\tau)} G_k v_k(\tau) d\tau.$$

It is easy to see that, by construction, $z_k(t) \in \mathcal{V}_k^*$ and

$$z_k(0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} + J_k z_k(h) = \begin{bmatrix} x_0 \\ z_{k-1}(h) \end{bmatrix}.$$

This means that $H_k z_k(t) = 0$ for $t \in [0,h]$. The corresponding solution of the system (1) with the initial condition $x(0) = x_0$ and $\varphi = 0$ verifies: $C_0 x(t) = 0$ for $t \in [0,(k+1)h]$, i.e. $y(t,x_0,0,u(t)) = 0$ in the same interval for some control $u(t)$. ∎

Now we can formulate a condition of perfect observability one a finite time interval. By this we mean that the initial condition x_0 may be determinated using the observation of $y(t)$ on $[0,T]$, the control $u(t)$ being unknown. As it was assumed, the initial function is here $\varphi = 0$.

Theorem 3.2 *The system (1) is perfectly observable on* $[0,T]$, *for* $0 < T \leq (k+1)h$, *if and only if*

$$\begin{bmatrix} x_0 \\ 0 \end{bmatrix} \in \left(I_k - J_k e^{(F_k+G_kR_k)h}\right) \mathcal{V}_k^* + J_k \mathcal{K}_k^* \qquad (5)$$

implies $x_0 = 0$.

Proof. Suppose that the system is observable on $[0,T]$ with $0 < T \leq (k+1)h$ and x_0 verifies (5). Then there , by the Lemma, we can find a control $u(t)$ such that $y(t,x_0,0,u(t) = 0 = y(t,0,0,0)$ on $[0,(k+1)h$. Our assumption gives $x_0 = 0$. The converse result may be obtained in a similar way. ∎

In order to consider the same results for the observability with unknown initial function φ we have to replace the matrix G_k by the matrix

$$\Gamma_k = \begin{bmatrix} A_1 & B_0 & 0 & 0 & \dots & 0 \\ 0 & 0 & B_0 & 0 & \dots & 0 \\ 0 & 0 & 0 & B_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \dots & B_0 \end{bmatrix}.$$

Then all the results of this section hold with an initial function φ and Γ_K instead of G_k . In fact the initial function has the same role as a control function.

Remark 3.3 The Lemma allows in fact to descrine all the initial conditions x_0 which are not observable when the control (and the initial functions) may be choosen. It is easy to see that for each k the initial conditions such that $y(t,x_0,0,u(t)) = 0$ on $[0,(k+1)h]$ define a linear subspace. We denote this subspace by $\mathcal{N}_k^*$. The Lemma gives

$$\mathcal{N}_k^* = \left\{ x | \begin{bmatrix} x \\ 0 \end{bmatrix} \in \left(I_k - J_k e^{(F_k+G_kR_k)h}\right) \mathcal{V}_k^* + J_k \mathcal{K}_k^* \right\},$$

for all k and $\mathcal{N}_k^*$ does not depend on R_k.

4 Other applications

The subspaces $\mathcal{V}_{ol}, \mathcal{V}_\Sigma$, was used to charecterize observability. For systems without delays $\mathcal{V}_{ol}$, and $\mathcal{V}_\Sigma$ are invariant under feedback. They coincide with the supremal (A_0,B_0)-invariant subspace contained in $\operatorname{Ker} C_0$. Then in other control problems those subspaces may be used: disturbance decoupling, row-by-row decoupling, model matching ...etc (see for example [1], [2]). For delay systems the situation is quite different because of the lack of feedback invariance. In [6] and [9] the subpace $\mathcal{V}_\Sigma$ was used to formulate conditions of invertibility, disturbance rejection, model matching and row-by-row decoupling essentially in frequency domain. The compensators used there in order to solve the given problems are not, in general, realizable by static state feedback without prediction. In the other hand it is still unknown how to calculate the subspace $\mathcal{V}_\Sigma$. However the considerations used for observability in finite time may be extended for other control problems. The subspace stands for a geometric tool to describe control problems on a finite time interval. Let us give one particular application. For other appilcation the approach is similar.

Consider the system

$$\begin{cases} \dot{x}(t) &= A_0 x(t) + A_1 x(t-h) + B_0 u(t) + Dq(t) \\ y(t) &= C_0 x(t), \end{cases} \qquad (6)$$

where $q(t)$ is unknown disturbance. The problem of disturbance rejection consists on finding a compensator such that the closed loop system is such that the output does not depend on $q(t)$. In the frequency domain it means that there exists a compensator $C(s)$ such that

$$T(s)C(s) + T_D(s) = 0,$$

where the matrix $C(s)$ is strictly proper. The necessary and sufficient condition is

$$\operatorname{Im} D \subset \mathcal{V}_\Sigma.$$

But it is not clear how to design the feedback control law. For the case of a static state feedback without prediction [6] contains a result on partial disturbance rejection, i.e. rejection in a finite time interval. Let P_k be the matrix

$$P_k = \begin{bmatrix} D & 0 & 0 & \dots & 0 \\ 0 & D & 0 & \dots & 0 \\ 0 & 0 & D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & D \end{bmatrix}.$$

Proposition 4.1 [6] *There exists a non anticipative feedback law $u(t) = L_0x(t) + L_1x(t-h) + \dots + L_kx(t-kh)$ such that the output $y(t)$ of the compensed system is not affected by the disturbance $q(t)$ over the interval $[0,(k+1)h]$ if and only if*

$$\operatorname{Im} P_k \subset \mathcal{V}_k^*.$$

This simple result was not extended for other control problems like row-by-row decoupling. Our purpose is to give an approach wich may be used for other control problems. The following result shows as the problem of disturbance rejection may be formulated in terms of the subspaces $\mathcal{N}_k^*$, wich is in $\mathbb{R}^n$. This subspace is, as we think, the good geometric tool for control problems of delay systems in a finite time interval.

Proposition 4.2 *There exists a non anticipative feedback law $u(t) = L_0x(t) + L_1x(t-h) + \dots + L_kx(t-kh)$ such that the output $y(t)$ of the compensed system is not affected by the disturbance $q(t)$ over the interval $[0,(k+1)h]$ if and only if*

$$\operatorname{Im} D \subset \mathcal{N}_k^*.$$

Proof. Suppose that there exists a control law as indicated such that the output does not depend on the disturbance on the interval $[0,(k+1)h]$. Let $\overline{\Phi}(t)$ be the fundamental matrix of the closed loop system. Then it is easy to see that for all q we have $C_0\overline{\Phi}(t)q = 0$ for $t \in [0,(k+1)h]$. This gives

$$C_0\Phi(t)Dq + C_0\int_0^t \Phi(t-\tau)B_0u(\tau)d\tau = 0.$$

This means that $y(t, Dq, 0, u(t)) = 0$ on $[0,(k+1)h]$. By the Lemma one obtains $Dd \in \mathcal{N}_k^*$.
Conversely, if $Dq \in \mathcal{N}_k^*$ for all q, one has

$$\begin{bmatrix} Dq \\ 0 \end{bmatrix} \in \left(I_k - J_k e^{(F_k+G_kR_k)h}\right)\mathcal{V}_k^* + J_k\mathcal{K}_k^*.$$

This gives

$$\begin{aligned} \begin{bmatrix} Dq \\ 0 \end{bmatrix} &= I_k z_k - J_k e^{(F_k+G_kR_k)h} z_k \\ &+ J_k \int_0^h e^{(F_k+G_kR_k)(h-\tau)} G_k v_k(\tau)d\tau \end{aligned}$$

for some control $v_k(t)$. As

$$z_k(h) = e^{(F_k+G_kR_k)h} z_k + \int_0^h e^{(F_k+G_kR_k)(h-\tau)} G_k v_k(\tau)d\tau$$

is in $\mathcal{V}_k^*$, it may be written as $z_k(h) = e^{(F_k+G_k\overline{R}_k)h} z_k$ for some matrix $\overline{R}_k$. Moreover, $\overline{R}_k$ can be choosen in the following form:

$$\overline{R}_k = \begin{bmatrix} L_0 & 0 & 0 & \dots & 0 \\ L_1 & L_0 & 0 & \dots & 0 \\ L_2 & L_1 & L_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ L_k & L_{k-1} & L_{k-2} & \dots & L_0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} Dq \\ 0 \end{bmatrix} = I_k z_k - J_k e^{(F_k+G_k\overline{R}_k)h} z_k$$

and

$$z_k(0) = z_k = \begin{bmatrix} Dq \\ 0 \end{bmatrix} + J_k e^{(F_k+G_k\overline{R}_k)h} z_k = \begin{bmatrix} Dq \\ z_{k-1}(h) \end{bmatrix}$$

is in $\mathcal{V}_k^*$. Hence $z_k(t) = e^{(F_k+G_k\overline{R}_k)h} z_k(0)$ is in $\operatorname{Ker} H_k$ and the fundamental matrix $\overline{\Phi}(t)$ corresponding to the closed loop system (i.e. defined as in the Section 1 with $F_k + G_k\overline{R}_k$ instead of F_k)is such that

$$C_0\overline{\Phi}(t)Dq = 0, \quad t \in [0,(k+1)h].$$

for all q. This means that the disturbance does not affect the output in the interval $[0,(k+1)h]$. ∎

This Proposition shows that the approach developped here may be extended for other control problem, for example for the problem of row-by-row decoupling (see [10] for an approach based on the structure at infinity of the transfer matrix function). It is also interesting to find some simple algorithm for the computation of the subspaces $\mathcal{N}_k^*$.

5 Conclusion

The conditions of perfect observability given here, in particular condition in the Section 3, are more efficient and may be verified. The criterion using the subspace $\mathcal{V}_\Sigma$ shows that this subspace plays an essential role in several control problems. It may be interesting to further investigate the properties if this susbspace in order to precise when it is of feedback type. The introduction of observability in finite time allows to give a good tool in order to investigate control problems for delay systems in finite time interval. The interesting problem of row-by-row decoupling may be studied via the approach developped here.

References

[1] G. Basile, G. Marro, *On observability of linear time-invariant systems with unknown inputs,* J. Optimization Theory and Applications, vol. 3, No 6, pp.410-415, 1969.

[2] C. Commault, J.M. Dion, J. Descusse, J.F. Lafay, M. Malabre, *Influence de la structure à l'infini des systèmes linéaires sur la résolution de problèmes de commande*, A.P.I.I., **20**, No 3, pp. 207–252, 1986.

[3] R. Gabasov, F. M. Kirillova, *Kachestvenaya teoria optimalnykh protsessov,*, Nauka, Moskow, 1971. (In russian).

[4] H. Górecki, S. Fuksa, P. Grabowski, A Korytowski, *Analysis and synthesis of time delay systems,* J. Wiley Ed., 1989.

[5] S. A. Miniuk, *On the theory of ideal observability for linear systems with delay,* Differentsialnie Uravnenia, vol. 15, No 12, 1978.

[6] M. Malabre, R. Rabah, *Structure at infinity, model matching and disturbance rejection for linear systems with delays.* Kybernetika, **29**, No 5, pp. 485–498, 1993.

[7] A.W. Olbrot, *Algebraic criteria of controllability to zero function for linear constant time-lag systems,* Control and Cybernetics, vol.2, pp 59-77, 1973.

[8] R. Rabah, *Structural properties and controllability for delay systems of neutral type,* Proccedings of the IFAC Conference on System Structure and Control, Nantes, 5-7 July 1995.

[9] R. Rabah, M. Malabre, *Structure at infinity for infinite dimensional systems,* Internal report No 94.6, Laboratoire d'Automatique de Nantes, Ecole Centrale de Nantes, 1994.

[10] O. Sename, R. Rabah, J.F. Lafay, *Decoupling without prediction of linear systems with delays: a structural approach,* Systems and Control Letters, to appear.

[11] O. Sename, J.F. Lafay, R. Rabah, *Controllability indices of linear systems with delays,* Proceeding of the 2nd IEEE Mediterranean Symposium on New Directions in Control and Automation, June 19-22, Chania, Greece, 1994.

[12] A.C. Tsoi, *Recent advances in the algebraic system theory of delay differential equations,* in the book: Recent theoretical developments in control, M.J. Gregson Ed., Academic Press, 1978, pp.67-127.

[13] W. M. Wonham, *Linear multivariable control: a geometric approach, Springer-Verlag,* New York, 1979.

[14] R.B. Zmood, *The euclidean space controllability of control systems with delays,* SIAM J. of Control, vol. 12. No 4, 1974, pp.609-623.

[15] H. Zwart, *Geometric theory for infinite dimensional systems,* Lecture Notes in Control and Information Sciences, **115**, 1989.