Optimum Damping Design for an Abstract Wave Equation

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1. Introduction

In this paper, we investigate different spatial damping distributions for an abstract wave equation which would result in exponential stability of the solutions while satisfying other appropriately defined performance criteria. While eventual decay of solutions is of paramount importance both in theory and applications, different physical systems modeled by the wave equation may require additional specific responses from the system. In applications to large flexible structures, for example, damping design addresses not only the dissipation of vibrational energy over a long period of time, but also reduction and control of displacement and vibrations of the structure so that a desired type of motion such as rapid slewing over large angles is achieved in a specified time interval. Other design specifications such as constraints on the amount of mass of the structure, or whether a particular response is required during the motion, or at the end of the time interval would result in different active or passive damping designs. In this study, our attempt is to address the question of "optimal" damping design in an abstract setting and precisely define and analyze various design criteria which are of importance in applications. We will further illustrate our results in application to a one-dimensional damped wave equation, and will present numerical results for different damping designs for this example.

2. Mathematical Model

The linear model for many problems related to study of flexible structures and acoustics can be given by the following second order abstract wave equation :

$$M\ddot{u} + D_a\dot{u} + A_0 u = 0 \qquad (1)$$

in a Hilbert space H where A_0 is an elliptic operator and M and D_a represent the mass and damping operators respectively. To cast the problem in the weak form we will follow the theoretical framework as outlined in [1], and assume a Hilbert space $V \subset H$ that is densely and continuously embedded in H. Define a bounded, symmetric sesquilinear form $\sigma_1(u, \hat{u}) : V \times V \to \mathcal{C}$ which is continuous and coercive on V. This sesquilinear form defines a densely defined operator, the stiffness operator A_0, in H where

$$\sigma_1(u, \hat{u}) = \langle A_0 u, \hat{u}\rangle_H$$

for $u \in$ dom (A_0), and $\hat{u} \in V$. Similarly, we can define the following bounded symmetric sesquilinear forms on H, $\rho(u, \hat{u}) = \langle Mu, \hat{u}\rangle_H$ and $\mu_a(u, \hat{u}) = \langle D_a u, \hat{u}\rangle_H$. In order to write equation (1) in the first order weak form we define the following product spaces, $\mathcal{H} = V \times H$ with product norm $\|.\|_{\mathcal{H}}$ and $\mathcal{V} = V \times V$, and a sesquilinear $\sigma : \mathcal{V} \to \mathcal{C}$ in the following way:

$$\sigma((u, \hat{u}), (\phi, \hat{\phi})) = -\langle \hat{u}, \phi\rangle_V + \sigma_1(u, \hat{\phi}) + \mu_a(\hat{u}, \hat{\phi}).$$

Define $w = (u, \hat{u}), \chi = (\phi, \hat{\phi}) \in \mathcal{V}$, and write equation (1) in the weak form as

$$\langle \dot{w}(t), \chi\rangle_{\mathcal{H}} + \sigma(w(t), \chi) = 0.$$

The above weak form gives rise to the following first order state equation in $\mathcal{H}$

$$\dot{w}(t) = \mathcal{A}w(t)$$

where

$$\mathcal{A}_a = \begin{bmatrix} 0 & I \\ -M^{-1}A_0 & -M^{-1}D_a \end{bmatrix}$$

with domain defined as

$$\text{dom}\,(\mathcal{A}_a) = \{(\phi, \psi) \in \mathcal{H} | \psi \in V \text{ and } A_0\phi + D_a\psi \in H\}.$$

By Lumer-Phillips theorem, one can show that $\mathcal{A}_a$ generates a C_0 semigroup, $S_a(t)$, in the state space $\mathcal{H} = V \times H$ if D_a is a bounded self-adjoint, and nonnegative operator on V. Our design goal is to model D_a which is dependent on the design parameter(s) $a \in (\mathcal{Q}_a =$ the Design Space) in such a way so that the norm of the semigroup solution of the equation above decays to zero in a desired manner.

3. Design Criteria

To formulate a performance index that is based on the dynamical behavior of the solutions one can consider the three following possibilities.

The first one is based on minimizing $\|S_a(\tau)\|_{\mathcal{H}}$, given $\tau > 0$. While this criterion is useful in many applications where the performance measure is based on a decay factor for a desired time interval, mathematical characterization of this problem does not yield formulation of an easily implementable performance index.

A second frequently used criterion in the engineering literature is minimization of the largest eigenvalue of the operator $\mathcal{A}_a$, i.e.,

$$\min_{a\in\mathcal{Q}_a} \quad \sup \mathrm{Re}\, \sigma(\mathcal{A}_a)$$

where $\sigma(\mathcal{A}_a)$ is the spectrum of the operator $\mathcal{A}_a$. While this criterion is widely used in the finite dimensional models, its use in conjunction with infinite dimensional wave equation presents us with several problems: The first problem is related to characterization of $\sigma(\mathcal{A}_a)$, which is difficult to do in many cases. But even in cases where $\sigma(\mathcal{A}_a)$ is easily defined, we still need to have the spectrum determined growth assumption satisfied, (see [2]):

$$\inf \omega = \{\|S_a(t)\| \leq Me^{\omega t} \quad \omega \in \mathcal{R}\} = \sup \mathrm{Re}\, \sigma(\mathcal{A}_a).$$

It has been shown (see [1]) that if the damping operator μ_a is uniformly coercive then $S_a(t)$ is an analytic semigroup and $\sigma(\mathcal{A}_a)$ is sectorial and the spectrum growth determinant condition is satisfied. But in general the vertical asymptote of $\sigma(\mathcal{A}_a)$ is difficult to examine. Even in cases where the first two problems are circumvented, minimization of the slowest decay rate which the criterion amounts to does not result in overall reduction of the energy in a finite amount of time.

The third criterion which is based on physical considerations of minimizing the total energy of the system over a long time interval is more easily characterized and realized in actual physical systems than the other two criteria. This criterion for our problem can be defined as

$$\min_{a\in\mathcal{Q}_a} \int_0^\infty \|\mathcal{R}^{1/2} S_a(t)u\|_{\mathcal{H}}^2 dt.$$

Minimization of the total energy is realized by the characterization of the Datko Lemma which basically states that if $\mathcal{A}_a$ is exponentially stable on $\mathcal{H}$ then the minimum of the total energy is given in terms of the solution to a Lyapunov equation, in other words the following are equivalent:

- $\mathcal{A}_a$ is exponentially stable on $\mathcal{H}$.
- $\int_0^\infty \|S_a(t)u\|^2 dt$ is finite for all $u \in \mathcal{H}$.
- There exists a bounded nonnegative, and self-adjoint operator Π_a on $\mathcal{H}$ that satisfies the following Lyapunov equation

$$(\mathcal{A}_a^*\Pi_a + \Pi_a\mathcal{A}_a + \mathcal{R})u = 0 \qquad (2)$$

for all $u \in$ dom $(\mathcal{A}_a)$.

Then we have

$$\int_0^\infty \|R^{1/2}S_a(t)u\|_{\mathcal{H}}^2 dt = \langle \Pi_a u, u\rangle_{\mathcal{H}}$$

where R is a coercive, self-adjoint operator on $\mathcal{H}$. In order to develop a criterion that is independent of the state vector u we consider the following performance measures that are based on minimizing the total energy.

$$\min_{a\in\mathcal{Q}_a} \quad \|\Pi_a\| = \sup_{|u|=1} \langle \Pi_a u, u\rangle_{\mathcal{H}} \qquad (3)$$

and

$$\min_{a\in\mathcal{Q}_a} \quad \mathrm{tr}\, \Pi_a Q = E(\Pi_a u, u). \qquad (4)$$

Here, we assume that the initial data u is a random vector with normal distribution of zero mean and covariance Q, a nuclear operator, and E denotes the expectation over the initial condition. In general Π is not compact in $\mathcal{H}$, therefore it is not always possible to define the trace norm of Π. In this sense, the second criterion is the weighted trace norm of Π with respect to Q which in engineering applications is chosen to be the subspace spanned by the dominant eigenfunctions for the nominal plant. If Π_a is compact then the first criterion amounts to minimizing zing the L^∞ norm of Π_a, and the second criterion is equivalent to minimizing L^1 norm of Π_a.

Our goal is to solve the optimization problems based on (3) and (4) subject to some constraints on the parameter a. In the following section we consider a specific example and will present numerical experiments to illustrate our theoretical results.

4. A One Dimensional Damped Wave Equation

In this section, we consider the following one dimensional wave equation on interval $(-1,1)$

$$u_{tt} = u_{xx} + a(x)u_t \quad \text{with } u(-1,t) = u(1,t) = 0.$$

For this problem we are interested in finding the optimal spatial distribution of damping subject to some constraints on the total amount of damping material available. We choose the set of admissible parameters $\mathcal{Q}_a$ as the set of functions a of bounded variation on $(-1,1)$, $BV(-1,1)$, satisfying

$$0 < \underline{a} \le a(x) \le \bar{a} \tag{5}$$

and

$$\int_{-1}^{1} a(x)\,dx = a_{tot} = \text{ Total Mass.} \tag{6}$$

Here, $BV(-1,1)$ is chosen as a compact set of $L_1(-1,1)$ which allows jump discontinuities. We can also use the finite dimensional parameterization by the piecewise constant functions; i.e, if one models $a(x)$ as a piecewise constant function

$$a(x) = \sum_{i=1}^{ns} a_i \chi_{[x_i - x_{i-1}]}(x)$$

$$-1 = x_0 < x_1 < \dots < x_{ns} = 1$$

where a_i represents the amount of damping distribution over ns subintervals (x_{i-1}, x_i), then the goal is to find the optimal values for a_i with respect to criteria (3) or (4) subject to the constraints (5) and (6).

5. Necessary Optimality Conditions

In order to show that the optimization problem discussed in the previous section is well-posed, we need to show for each criterion the existence of an optimal parameter and discuss the necessary optimality conditions that characterize the optimal solution. Here, we will only mention the main results and refer the reader to [3] for detailed statement and proofs. For the first criterion

$$\min_{a\in\mathcal{Q}_a} \quad f(a) = \min_{a\in\mathcal{Q}_a} \quad \|\Pi_a\| = \sup_{|u|=1} \langle \Pi_a u, u\rangle_{\mathcal{H}}$$

if we show lower-semi-continuity of $f(a)$, from compactness of the set $\mathcal{Q}_a$ we can obtain existence of a minimizing parameter a. We can show that $f(a) = \|\Pi_a\|$ is lower-semicontinous provided the following condition is satisfied for all $u \in \mathcal{H}$

$$\Pi_a u \to \Pi_{\hat{a}} u \text{ as } a \to \hat{a} \text{ in } \mathcal{Q}_a.$$

In [3], we have shown that Π_a does satisfy the above continuity condition with respect to parameter a. In fact, we can prove the following stronger result:

Theorem 1 *Suppose* dom $(\mathcal{A}_a)$ *and* dom $(\mathcal{A}_a^*)$ *are independent of $a \in \mathcal{Q}_a$, a compact set , and also $\delta\mathcal{A}_{\hat{a}} = \mathcal{A}_a - \mathcal{A}_{\hat{a}}$ satisfies*

$$\|\delta\mathcal{A}_{\hat{a}}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \to 0 \text{ as } \|a - \hat{a}\| \to 0,$$

then

$$\|\Pi_{\hat{a}} - \Pi_{\hat{a}}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \to 0 \text{ as } \|a - \hat{a}\| \to 0,$$

where Π_a is the solution of the Lyapunov equation

$$\mathcal{A}_a^*\Pi_a + \Pi_a\mathcal{A}_a + R = 0.$$

We can similarly argue existence of an optimizer for the second criterion

$$\min_{a\in\mathcal{Q}_a} \quad J(a) = \min_{a\in\mathcal{Q}_a} \quad \text{tr } \Pi_a Q$$

by using the fact that the operator Q is nuclear, therefore its range is relatively compact and also by observation above that Π_a is continuous in norm with respect to parameter a. In this case the sensitivity operator $\frac{\partial\mathcal{A}_a}{\partial a}$ is characterized by the following equations:

$$\mathcal{A}_a^*\Pi + \Pi\mathcal{A}_a + R = 0,$$

$$\mathcal{A}_a\Lambda_a + \Lambda\mathcal{A}_a^T + Q^T = 0,$$

$$\left((\frac{\partial\mathcal{A}_a}{\partial a})^T\Pi + \Pi\frac{\partial\mathcal{A}_a}{\partial a}, \Lambda \right)_{\mathcal{R}^{m\times m}} = 0$$

where m is the dimension of the range of Q.

6. Finite Dimensional Approximations

For performing the numerical approximations, we employ a Legendre-Tau method which is a variation of the well-know Galerkin technique. In this method the approximate solution is expanded in terms of the Legendre polynomials, $L_n(x)$, which are orthogonal with respect to the $L^2(-1,1)$ norm. These basis elements do not individually satisfy the boundary conditions as in Galerkin method. The boundary conditions are imposed on the approximate solution by use of a non-orthogonal projection operator. For more details on implementations of Legendre-tau method to the wave equation see [3,4].

For a second order wave equation we seek an approximate solution in the form

$$u_n(t,x) = \sum_{j=0}^{n} \xi_j(t) L_j(x).$$

The vector $\xi(t) = (\xi_0, \xi_1, ..., \xi_{n-2})$ satisfies

$$M^n \ddot{\xi}(t) + D^n \dot{\xi}(t) + K^n \xi(t) = 0 \qquad (7)$$

and a_{n-1}, and $, a_n$ are determined as linear combinations of $a_0, a_1, ..., a_{n-2}$ by applying the boundary conditions on the solution u_n. The mass matrix M^n, damping matrix D^n, and the stiffness matrix K^n, and are given by

$$\begin{aligned}(M^n)_{i,j} &= \langle L_i, L_j \rangle_{L_2(-1,1)},\\ (D^n)_{i,j} &= \sum_{k=1}^{ns} \int_{x_{k-1}}^{x_k} a_k L_i L_j dx,\\ (K^n)_{i,j} &= (S^2 P_n)_{i,j}.\end{aligned}$$

In the expression for K^n, S^2 is the matrix representation of the second order differential operator with respect to the Legendre polynomials and P_n is the projection operator that imposes the Dirichlet boundary conditions at the two ends on the approximate solution.

The first order form of (7) for $\eta = [\xi^T, \dot{\xi}^T]$ is

$$\dot{\eta} = \mathcal{A}^n \eta$$

where

$$\mathcal{A}^n = \begin{bmatrix} 0_{n-1\times n-1} & I_{n-1\times n-1} \\ -M^{-n}K^n & -M^{-n}D^n \end{bmatrix}.$$

In the above, M^{-n} denotes the inverse of the mass matrix M^n. For approximating the total energy we take R in (2) to be the identity, and we write its matrix representation as

$$\mathcal{R}^n = \begin{bmatrix} K^n & 0_{n-1\times n-1} \\ 0_{n-1\times n-1} & M^n \end{bmatrix}.$$

Assuming $(\mathcal{A}^n, \mathcal{R}^n)$ is detectable, then the total energy in the finite dimensional space is given by

$$E^n(u) = \int_0^\infty \eta^T \mathcal{R}^n \eta dt = \eta_0^T \Pi^n \eta_0 \qquad \eta(0) = \eta_0$$

where Π^n is the matrix representation of the finite-dimensional approximation to Π and is equal to $\mathcal{R}^{-n}\tilde{\Pi}^n$ where $\tilde{\Pi}^n$ satisfies the following Lyapunov equation

$$(\mathcal{A}^n)^T \tilde{\Pi}^n + \tilde{\Pi}^n \mathcal{A}^n + \mathcal{R}^n = 0.$$

The finite dimensional approximation of the first performance index (3) can be written as

$$\min_{a\in\mathcal{Q}_a} \quad \max \operatorname{eig} (\mathcal{R}^{-n}\tilde{\Pi}^n) \qquad (8)$$

To calculate the approximate performance index (4), we consider operator Q to be the projection onto a space spanned by the m dominant undamped eigenfunctions of the equation. If Φ_{mn} denotes the matrix representation of the orthogonal projection that projects the finite-dimensional solution space to the m-dimensional space of range of Q, then the matrix representation of the finite-dimensional performance index becomes:

$$\min_{a\in\mathcal{Q}_a} \quad \operatorname{tr} \left((\Phi_{mn}\mathcal{R}^{-n}\Phi_{mn}^T)(\Phi_{mn}\tilde{\Pi}^n\Phi_{mn}^T) \right) \qquad (9)$$

7. Numerical Results

To perform numerical experiments for various damping designs, we took the number of Legendre polynomials in our approximations to be 20, and the number of subdivisions for distribution of the damping material to be 40. Also, to calculate the second performance criterion, we took m, the number of undamped dominant modes to be 7. We first experimented with a few damping designs and calculated the value of performance indices (8) and (9) in each case. The following figures demonstrate the distribution of a_i's over the $(-1, 1)$ interval for these examples:

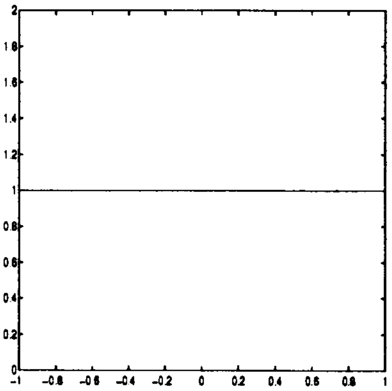

Figure 1: Uniform distribution

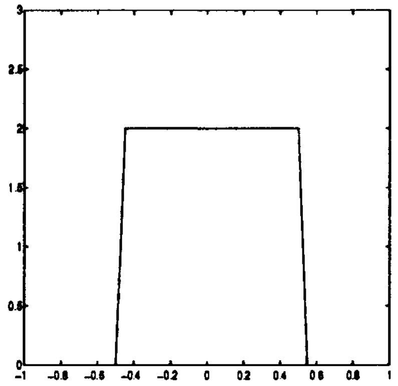

Figure 2: Center distribution

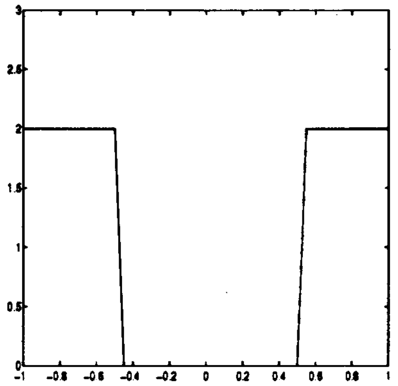

Figure 3: Corner distribution.

Next, we performed optimization of damping distributions with respect to the criteria (8), and (9) and found the optimal values of a_i's in each case. The graphs of optimal values of a_i for each performance index over the interval $(-1, 1)$ are shown below.

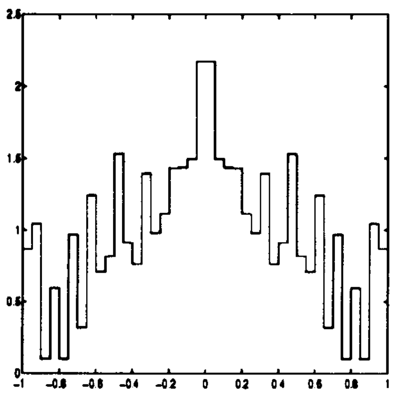

Figure 4: First optimal distribution with $\min \max \mathrm{eig}\,(\Pi^n) = 1.2959$

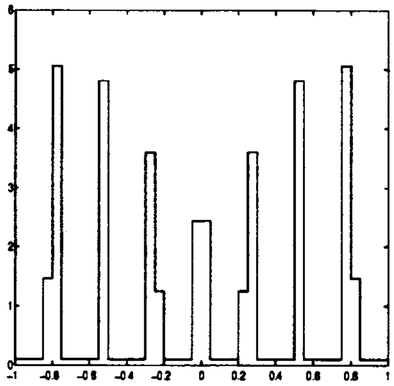

Figure 5: Second optimal distribution with min tr = 12.6311

The following table compares the different designs and the corresponding values of the two performance indices.

Damping designs	Min(max(eig))	Min (tr)
Uniform Distribution	1.4354	14.3064
Center Distribution	499.1038	15.3697
Corner Distribution	3.2690	18.6927
Optimal Distribution 1	1.2959	14.6744
Optimal Distribution 2	12.6311	9.3611

Table 1: Comparison of different designs

¿From this table, one can see that different performance criteria yield different optimal damping designs, and a design that performs well with respect to one criterion, may perform poorly with respect to the others,(compare the results for the center and corner distributions). But overall, the uniform damping design seems to perform quite well with respect to either criterion. The results also indicate that much is to be gained by performing the optimization. The key point in optimizing damping designs is to carefully choose the performance criterion that is most suited to the problem in hand. Practical and theoretical considerations should both be taken into account in choosing the proper criterion. For example, depending on the amount of information on the physical modeling of the initial state vector or the number of dominant vibrational modes that need to be suppressed one may choose either criterion that fits the requirements of the problem. The numerical results we have presented here are only preliminary efforts in optimizing damping designs and our future efforts will address numerous issues concerning the numerical and theoretical optimization of these designs.

8. References

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