

# OPTIMAL DESIGN OF ROBUST CONTROLLERS FOR UNCERTAIN DISCRETE-TIME SYSTEMS

Ioannis K. Konstantopoulos and Panos J. Antsaklis

Department of Electrical Engineering

University of Notre Dame

Notre Dame, IN 46556, U.S.A.

antsakli@saturn.ee.nd.edu

ikonstan@elgreco.helios.nd.edu

## Abstract

This paper presents an algorithm for the design of robust output feedback controllers for linear uncertain discrete-time systems. The algorithm utilizes a version of the Broyden-Fletcher-Goldfarb-Shanno (BFGS) optimization method of conjugate directions and minimizes a performance index that includes an LQR term to optimize performance and a robustness term which is based on recently developed bounds. The minimization of only the robustness term which corresponds to the maximization of the uncertainty bound is also studied. The case of unstructured perturbations in  $A$  has been the only one studied in the literature; the present algorithm not only introduces a unified approach to both unstructured and structured perturbations in  $A$  but also is shown to improve considerably the existing unstructured uncertainty bound. Several other cases involving unstructured/structured perturbations in all the state-space matrices are also presented and examples, including an aircraft control system, are used to illustrate the results.

## 1. Introduction

The problem of determining a linear feedback control law for uncertain linear systems has drawn considerable attention. Several criteria have been used to characterize the system uncertainties, so that the stability (asymptotic, quadratic, exponential) of the uncertain systems is guaranteed if these criteria are satisfied, and several robust controller design methods have been developed. These methods are based on  $H_\infty$  techniques (as in [3], [5], [24]), the LQR/LQG formulation (as in [4], [16], [22]), the Guaranteed Cost Control approach (as in [14], [17]), optimization techniques (as in [6], [7]) and the theory of Lyapunov stability/Riccati equations (as in [20], [21]).

All the above controller design approaches share the same general objective, which is to find a stabilizing controller

that satisfies some stability conditions or is robust in some sense, without considering the maximization of any of the robust stability bounds existing in literature. This has been done in [23] for continuous systems with structured uncertainties in the system matrix  $A$  and in [11] for discrete-time systems with unstructured uncertainties in  $A$ . The design in the first paper relies on the selection of a weighting matrix not directly associated with the structured uncertainties and in the latter on the bound developed in [10]. In both papers, the information about the uncertainty bound is part of the minimizing quantity, which also includes the classical LQR cost. Therefore, the controller design objective is twofold, that is to minimize the LQR cost and maximize the perturbation bounds. A similar approach was used earlier in [19] for continuous systems with structured uncertainties in all the state-space matrices, under some quite restrictive assumptions imposed on the perturbation matrices. Although the maximization of some stability bound is not considered in the design process, the information about the structured uncertainty is directly included in the minimizing quantity.

Here, we present a unified output feedback controller design approach for both cases of unstructured and structured perturbations in  $A$ . Note that from the above discussion it is clear that for discrete-time systems, only the case of unstructured perturbations in  $A$  has been studied in the literature. Our approach is based on new results for both the structured and unstructured cases, which were recently developed in [12]; these theorems have shown to provide bounds that improve the ones obtained via the methodology suggested in [10] and used in [11]. Our design not only provides a stabilizing static output feedback controller that improves the unstructured bound for  $A$  derived in [11], but is also capable of finding another such controller that maximizes the bound for the case that  $A$  is perturbed by known uncertainty matrices. In addition, several other interesting cases are studied. In all these cases, the minimizing quantity consists of two terms; one is the robustness term, which is associated with the specific unstructured/structured bound we wish to maximize

and the other is the LQR term, which is associated with the specific control performance we wish to maintain. Our approach is also applied to a minimizing quantity consisted of only the robustness term, in order to find the controller that maximizes the stability bounds, without considering any control specifications. Note that only the case of static output feedback is studied, since it can easily be shown that the case of dynamic output feedback can be reduced to that of static feedback as well. Finally note that our minimization algorithm utilizes a version of the Broyden family method of conjugate directions, which is based on the BFGS rule, [2], and that the case of state feedback can be easily considered as a special case of the output feedback case for  $C = I$ .

The paper is organized as follows. In section 2, we present without proofs the new theorems of [12] for the cases of unstructured and structured perturbations in discrete-time systems. In section 3, we study the case of unstructured/structured perturbations in the system matrix  $A$ , and present an algorithm based on the BFGS rule that solves the minimization problem. In section 4, we consider unstructured/structured perturbations in either the input matrix  $B$  or the output matrix  $C$  and in section 5, unstructured/structured perturbations in either  $(A, B)$  or  $(A, C)$ . In section 6, we study the case of unstructured perturbations in all state-space matrices. In section 7, we provide illustrative examples for some of the cases studied above and finally in section 8, concluding remarks are included.

## 2. Preliminaries

We consider the linear discrete-time system

$$x(k+1) = Ax(k) \quad (1)$$

where  $x \in \mathbb{R}^n$  is the state vector and  $A$  an asymptotically stable matrix. Then, for every symmetric positive definite matrix  $Q$ , we can find a symmetric positive definite matrix  $P$ , which is the unique solution of the Lyapunov equation

$$A^T P A - P + Q = 0 \quad (2)$$

When  $A$  is perturbed by the matrix  $\Delta A$ , then for the perturbed system

$$y(k+1) = (A + \Delta A) y(k) \quad (3)$$

the following theorem has been proven in [12]. First define

$$\Omega_1 = A^T P Z^{-1} P A \quad (4)$$

**Theorem 2.1** Consider the linear discrete-time system (1) where  $A$  is an asymptotically stable matrix that satisfies (2). Suppose that  $A \rightarrow A + \Delta A$ , then the perturbed system (3) remains asymptotically stable, if  $(\Delta A)^T (\alpha Z + P) (\Delta A) + \frac{1}{\alpha} \Omega_1 < Q$  or

$$\sigma_{\max}(\Delta A) < \sqrt{\frac{\sigma_{\min}(Q) - \sigma_{\max}(\frac{1}{\alpha} \Omega_1)}{\sigma_{\max}(\alpha Z + P)}} \quad (5)$$

where  $P, Q$  are defined in (2),  $\Omega_1$  in (4),  $Z$  can be any positive definite matrix of appropriate dimensions, and  $\alpha$  any positive number that satisfies

$$\alpha > \frac{\sigma_{\max}(\Omega_1)}{\sigma_{\min}(Q)} \quad (6)$$

When the perturbation matrix  $\Delta A$  is described by

$$\Delta A = \sum_{i=1}^m \kappa_i A_i = (\tilde{K} \otimes I_n)^T \tilde{A} \quad (7)$$

where  $\kappa_i, i = 1, \dots, m$  denote real, uncertain parameters and  $A_i, i = 1, \dots, m$  are constant, known matrices, the following theorem has been proven in [12]. Obviously, we have used the definitions  $\tilde{K} = [\kappa_1 \ \kappa_2 \ \dots \ \kappa_m]^T$ , and  $\tilde{A} = [A_1^T \ A_2^T \ \dots \ A_m^T]^T$ .

**Theorem 2.2** The linear discrete-time system (3) with structured perturbations of the form of (7) remains asymptotically stable, if the uncertainty parameters satisfy

$$\sum_{i=1}^m \kappa_i^2 < \frac{\sigma_{\min}(Q) - \sigma_{\max}(\frac{1}{\alpha} \Omega_1)}{\sigma_{\max}^2(\tilde{A}) \sigma_{\max}(\alpha Z + P)} \quad (8)$$

where  $\Omega_1, \kappa_i, \tilde{A}$  are defined in (4), (7), (7) respectively,  $Z$  can be any positive definite matrix of appropriate dimensions, and  $\alpha$  any positive number that satisfies (6).

The above theorems provide bounds that improve the ones obtained via the methodology of [10]; this is demonstrated next. The main point of the approach used for the theorems above is the appropriate selection of a positive definite matrix  $Z$  and a positive number  $\alpha$  that maximize the stability region within which the uncertain parameters vary.

**Example 2.1** Consider the following uncertain discrete-time system (3) from [10] with

$$A = \begin{pmatrix} 0.20 & 0.30 \\ 0.10 & -0.15 \end{pmatrix} \quad (9)$$

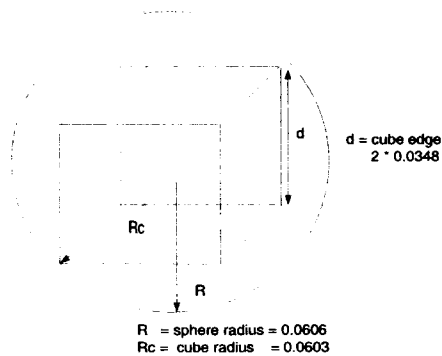


Figure 1: Example 2.2

We choose  $Q = I$ ,  $\alpha = 0.2702$  and  $Z = \begin{pmatrix} 2.0399 & -0.2037 \\ -0.2037 & 1.4586 \end{pmatrix}$ . Using (5) of Theorem 2.1, we obtain  $\sigma_{\max}(\Delta A) < 0.6787$ , which compares favorably to the result of [10], which is  $\sigma_{\max}(\Delta A) < 0.6373$ .

**Example 2.2** Consider the same nominal system, but now with structured perturbations of (7), that is

$$A_1 = \begin{pmatrix} 10 & 0.1 \\ -1 & 5 \end{pmatrix}, A_2 = \begin{pmatrix} -0.5 & 9 \\ 0 & -3 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0.6 \\ 1 & 0.3 \end{pmatrix} \quad (10)$$

We choose  $Q = I$ ,  $\alpha = 0.40$ ,  $Z = \begin{pmatrix} 1.3462 & -0.1184 \\ -0.1184 & 0.8786 \end{pmatrix}$ .

Using (8) of Theorem 2.2, we obtain  $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 < (0.0606)^2$ , that is a sphere with radius  $R = 0.0606$ , whereas the method of [10] gives  $|\kappa_i| < 0.0348$  for  $i = 1, 2, 3$ . As we can see in Fig. 1, the defined cube is completely included in the sphere found above, which shows that our bound is less conservative than the one of [10].

### 3. Perturbations in A

We consider the linear discrete-time system

$$x(k+1) = Ax(k) + B_0 u(k), \quad y(k) = C_0 x(k) \quad (11)$$

where  $x \in \mathbb{R}^n$  is the state vector,  $u \in \mathbb{R}^r$  is the input vector and  $y \in \mathbb{R}^q$  is the output vector. Both unstructured and structured perturbations for the system matrix  $A$  are of interest here, that is  $A = A_0 + \Delta A$  and  $A = A_0 + \sum_{i=1}^m \kappa_i A_i$ . We apply the static output feedback law  $u(k) = Ky(k)$ . Defining  $\bar{A}_0 = A_0 + B_0 K C_0$ , the closed-loop systems are described respectively by  $x(k+1) = (\bar{A}_0 + \Delta A)x(k)$ , and  $x(k+1) = (\bar{A}_0 + \sum_{i=1}^m \kappa_i A_i)x(k)$ .

#### 3.1. Design without performance specifications

Our objective is to find a stabilizing output feedback gain  $K$  that maximizes the bounds of (5) and (8). Due to the similarity between these two relations, we present here a unified approach for both the unstructured and structured cases. For the closed-loop systems above, relations (2) and (4) can be translated into

$$\bar{A}_0^T P \bar{A}_0 - P + Q = 0, \quad \Omega_1 = \bar{A}_0^T P Z^{-1} P \bar{A}_0 \quad (12)$$

Since  $Q$  in (12) is selected beforehand, in order to maximize the bounds of (5), (8), we need to

- (A.1) : minimize  $\sigma_{\max}(\alpha Z + P)$
- (A.2) : minimize  $\sigma_{\max}(\frac{1}{\alpha} \Omega_1)$

For (A.1), we choose to minimize the quantity -see [13]-  $J_{11} = \text{Tr}(\alpha^2 Z^2 + 2\alpha ZP + P^2)$ . For (A.2), we have  $\sigma_{\max}(\frac{1}{\alpha} \Omega_1) \leq \frac{1}{\alpha} \sigma_{\max}^2(\bar{A}_0) \sigma_{\max}^2(P) \sigma_{\max}(Z^{-1})$ . Since  $Z$  is selected beforehand and an upper bound of  $\sigma_{\max}(P)$ , that is  $\text{Tr}(P^2)$ , is already minimized in  $J_{11}$ , for (A.2) we simply choose to minimize  $J_{12} = \frac{1}{\alpha} \text{Tr}(\bar{A}_0^T \bar{A}_0)$ , which is an upper bound of  $\frac{1}{\alpha} \sigma_{\max}^2(\bar{A}_0)$ . Note that the minimization of the sum of  $\text{Tr}(\frac{1}{\alpha} \bar{A}_0^T \bar{A}_0)$  and  $\text{Tr}(P^2)$  is an indirect and harder way to minimize their product; in other words, we impose a more demanding task on the minimizing process. Note also that  $\alpha$  is included in the minimizing quantity, because we need to satisfy the positiveness of the numerator, as indicated in (6). Therefore, the minimizing quantity is given as  $J'_1 = J_{11} + J_{12}$  under the condition that (12) holds. This is clearly a constrained minimization problem. By including (12) in  $J'_1$ , we finally reduce the problem to an unconstrained minimization one, with the minimizing quantity finally given by

$$J_1 = \text{Tr}[\alpha^2 Z^2 + 2\alpha PZ + P^2 + \frac{1}{\alpha} \bar{A}_0^T \bar{A}_0 + L_1 (\bar{A}_0^T P \bar{A}_0 - P + Q)] \quad (13)$$

where  $L_1 \in \mathbb{R}^{n \times n}$  is the Lagrange multiplier matrix. Next, we need the following properties from [1]

$$\frac{\partial}{\partial X} \text{Tr}(X^2) = 2X^T \quad (14)$$

$$\frac{\partial}{\partial Y} \text{Tr}(A_1 Y B_1) = A_1^T B_1^T \quad (15)$$

$$\frac{\partial}{\partial Y} \text{Tr}(A_2 Y^T B_2) = B_2 A_2 \quad (16)$$

$$\frac{\partial}{\partial Y} \text{Tr}(A_3 Y B_3 Y^T) = A_3 Y B_3 + A_3^T Y B_3^T \quad (17)$$

for any  $X \in \mathbb{R}^{n \times n}$ ,  $Y \in \mathbb{R}^{n \times m}$ ,  $A_1 \in \mathbb{R}^{l \times n}$ ,  $B_1 \in \mathbb{R}^{m \times l}$ ,  $A_2 \in \mathbb{R}^{l \times m}$ ,  $B_2 \in \mathbb{R}^{n \times l}$ ,  $A_3 \in \mathbb{R}^{n \times n}$ ,  $B_3 \in \mathbb{R}^{m \times m}$ . With these properties, we have

$$\frac{\partial J_1}{\partial L_1} = \Delta_{L_1}^1 = \bar{A}_0^T P \bar{A}_0 - P + Q \quad (18)$$

$$\frac{\partial J_1}{\partial \alpha} = \Delta_\alpha^1 = 2\alpha \text{Tr}(Z^2) + 2 \text{Tr}(PZ) - \frac{1}{\alpha^2} \text{Tr}(\bar{A}_0^T \bar{A}_0) \quad (19)$$

$$\frac{\partial J_1}{\partial P} = \Delta_P^1 = 2P + 2\alpha Z + \bar{A}_0 L_1^T \bar{A}_0^T - L_1^T \quad (20)$$

$$\frac{\partial J_1}{\partial K} = \Delta_K^1 = \frac{2}{\alpha} B_0^T B_0 K C_0 C_0^T + \frac{2}{\alpha} B_0^T A_0 C_0^T + B_0^T P B_0 K C_0 (L_1 + L_1^T) C_0^T + B_0^T P A_0 (L_1 + L_1^T) C_0^T \quad (21)$$

To minimize (13), we use a version of the Broyden family method of conjugate directions, which is based on the Broyden-Fletcher-Goldfarb-Shanno (BFGS) update rule; details in [2]. The proposed algorithm is presented next.

**Initialization Step** Let  $\epsilon > 0$  be the termination scalar. Choose an initial stabilizing gain

$$K_1 = \begin{pmatrix} (\tau_1^1)^T \\ \vdots \\ (\tau_r^1)^T \end{pmatrix} \quad (22)$$

where  $(\tau_l^1)^T, l = 1, \dots, r$  are the  $1 \times q$  rows of  $K_1$ , which stabilizes  $(A_0, B_0, C_0)$ , that is  $\bar{A}_0$  stable. Also, choose an initial symmetric positive definite matrix  $D_1 \in \mathbb{R}^{r \times r}$ . Let

$$y_1 = x_1 = ((\tau_1^1)^T \dots (\tau_r^1)^T)^T \quad (23)$$

be a column vector consisting of the transposes of the rows of  $K_1$ . Also let  $k = j = 1$  and go to the *Main Step*.

### Main Step

M1. Substitute the gain matrix  $K_j$  in the gradients of (18)-(20), set them to zero, that is  $\Delta_{L_1}^1 = 0$ ,  $\Delta_\alpha^1 = 0$ ,  $\Delta_P^1 = 0$ , and solve for  $P$ ,  $\alpha$ ,  $L_1$  respectively, in that order.

M2. Substitute these parameters in (21) and compute

$$\Delta_{K_j}^1 = \begin{pmatrix} (\sigma_1^j)^T \\ \vdots \\ (\sigma_r^j)^T \end{pmatrix} \quad (24)$$

where  $(\sigma_l^j)^T, l = 1, \dots, r$  are the  $1 \times q$  rows of  $\Delta_{K_j}^1$ .

M3. Define  $\nabla J_1(y_j) = ((\sigma_1^j)^T \dots (\sigma_r^j)^T)^T$ . If  $\|\nabla J_1(y_j)\| < \epsilon$ , STOP. The optimal gain is  $K_j$ . Otherwise, go to M4.

M4. If  $j > 1$ , update the positive definite matrix  $D_j$

$$D_j = D_{j-1} + \frac{p_{j-1} p_{j-1}^T}{p_{j-1}^T q_{j-1}} \left[ 1 + \frac{q_{j-1}^T D_{j-1} q_{j-1}}{p_{j-1}^T q_{j-1}} \right] - \frac{[D_{j-1} q_{j-1} p_{j-1}^T + p_{j-1} q_{j-1}^T D_{j-1}]}{p_{j-1}^T q_{j-1}} \quad (25)$$

where  $p_{j-1} = \lambda_{j-1} d_{j-1} = y_j - y_{j-1}$ , and  $q_{j-1} = \nabla J_1(y_j) - \nabla J_1(y_{j-1})$ .

M5. Define  $d_j = -D_j \nabla J_1(y_j)$ , and let  $\lambda_j$  be an optimal solution to the problem of minimizing  $J_1(y_j + \lambda d_j)$  subject to  $\lambda \geq 0$ . Let  $y_{j+1} = y_j + \lambda_j d_j = ((\tau_1^{j+1})^T \dots (\tau_r^{j+1})^T)^T$ , which implies that

$$K_{j+1} = \begin{pmatrix} (\tau_1^{j+1})^T \\ \vdots \\ (\tau_r^{j+1})^T \end{pmatrix} \quad (26)$$

where obviously  $(\tau_l^{j+1}), l = 1, \dots, r$  are  $q \times 1$  column vectors.

M6. If  $j < r q$ , replace  $j$  by  $j + 1$  and repeat the *Main Step*. Otherwise, if  $j = r q$ , let  $y_1 = x_{k+1} = y_{r q+1}$ , replace  $k$  by  $k + 1$ , let  $j = 1$  and repeat the *Main Step*.  $\square$

Several issues need to be discussed here. First, note that the line search in (M5) is restricted to stabilizing gain matrices. Therefore, the selected new gain matrix needs first to stabilize the closed-loop system and then minimize  $J_1$ . Since our algorithm is an indirect version of the BFGS algorithm, as an alternative to the stopping criterion of (M3), we could use another quite practical criterion. Specifically, we may consider monitoring  $J_1$  and stop when we see that  $J_1$  is sufficiently small and the derived bound derived is satisfactorily large. From (19), we can easily see that there is at least one real positive solution for  $\alpha$ . For our algorithm, we choose to keep the largest value of  $\alpha$ , since we also need to satisfy the positiveness of the numerator of (5) and (8), as discussed before. Finally, note that for optimization problems similar to the one we study here, alternative methods based on gradient-type and nongradient-type algorithms have been proposed in [8] and [18] respectively.

### 3.2. Design with performance specifications

In the previous subsection, we focused on finding an output feedback gain  $K$  that maximizes the bounds of (5) and (8). If, in addition to this objective, we also wish to attain a specific control performance, then we need to include in our minimizing quantity a term that evaluates this control performance. Therefore, we consider the familiar LQR cost  $J_2 = \sum_{k=0}^{\infty} x^T(k) Q_1 x(k) + u^T(k) R_1 u(k)$ ,

where  $Q_1, R_1$  are positive definite matrices of appropriate dimensions. For the nominal system  $(A_0, B_0, C_0)$  with the output feedback law  $u(k) = Ky(k)$ , we finally choose, see [13], the following modified cost  $J_2 = \text{Tr}(P_2 X_0)$ , where  $P_2$  is the solution of the Lyapunov equation

$$\bar{A}_0^T P_2 \bar{A}_0 - P_2 + \bar{Q} = 0 \quad (27)$$

We have defined  $\bar{Q} = Q_1 + C_0^T K^T R_1 K C_0$ ,  $X_0 = E[x(0)x^T(0)] > 0$ . Usually, [15], we consider  $x(0)$  uniformly distributed on a sphere of radius  $\sigma$ , that is  $X_0 = \sigma I_n$ , with  $\sigma = 1$  the obvious choice. Therefore, the overall minimizing quantity, which is associated with both the robustness of  $\bar{A}_0$  and the control performance of the closed-loop system is given by

$$J_A = \text{Tr}[\alpha^2 Z^2 + 2\alpha PZ + P^2 + \frac{1}{\alpha} \bar{A}_0^T \bar{A}_0 + L_1(\bar{A}_0^T P \bar{A}_0 - P + Q) + P_2 X_0 + L_2(\bar{A}_0^T P_2 \bar{A}_0 - P_2 + \bar{Q})] \quad (28)$$

where, similarly to (13), we have reduced the problem to an unconstrained minimization one by including (27). Due to the introduction of  $P_2$  and  $L_2$  in the new cost  $J_A$ , we need to consider its partial derivatives with respect to these new matrix variables as well. For the same reason, we have some additional terms in  $\Delta_K^1$  of (21).

$$\frac{\partial J_A}{\partial L_1} = \Delta_{L_1}^A = \Delta_{L_1}^1 \quad (29)$$

$$\frac{\partial J_A}{\partial L_2} = \Delta_{L_2}^A = \bar{A}_0^T P_2 \bar{A}_0 - P_2 + \bar{Q} \quad (30)$$

$$\frac{\partial J_A}{\partial \alpha} = \Delta_{\alpha}^A = \Delta_{\alpha}^1 \quad (31)$$

$$\frac{\partial J_A}{\partial P} = \Delta_P^A = \Delta_P^1 \quad (32)$$

$$\frac{\partial J_A}{\partial P_2} = \Delta_{P_2}^A = X_0^T + \bar{A}_0 L_2^T \bar{A}_0^T - L_2^T \quad (33)$$

$$\begin{aligned} \frac{\partial J_A}{\partial K} = \Delta_K^A = & \Delta_K^1 + R_1 K C_0 (L_2 + L_2^T) C_0^T \\ & + B_0^T P_2 B_0 K C_0 (L_2 + L_2^T) C_0^T \\ & + B_0^T P_2 A_0 (L_2 + L_2^T) C_0^T \end{aligned} \quad (34)$$

To minimize  $J_A$ , the algorithm of the previous subsection can be used again, the only difference being that steps (M1), (M2) have to be replaced by the following

M1a. Substitute the gain matrix  $K_j$  in the gradients of (29)-(33), set them to zero, that is  $\Delta_{L_1}^A = 0$ ,  $\Delta_{L_2}^A = 0$ ,  $\Delta_{\alpha}^A = 0$ ,  $\Delta_P^A = 0$ ,  $\Delta_{P_2}^A = 0$  and solve for  $P, P_2, \alpha, L_1, L_2$  respectively, in that order.

M2a. Substitute these parameters in (34) and compute

$$\Delta_{K_j}^A = \begin{pmatrix} (\sigma_1^j)^T \\ \vdots \\ (\sigma_r^j)^T \end{pmatrix} \quad (35)$$

where  $(\sigma_l^j)^T, l = 1, \dots, m$  are the  $1 \times q$  rows of  $\Delta_{K_j}^A$ .

#### 4. Perturbations in B or C

We consider perturbations in either the input matrix  $B$  or the output matrix  $C$ . Since both cases are similar, we study the case of perturbations in  $B$ . Therefore, we consider the linear discrete-time system

$$x(k+1) = A_0 x(k) + Bu(k), \quad y(k) = C_0 x(k) \quad (36)$$

with static output feedback for both unstructured and structured perturbations in the input matrix  $B$ , that is  $B = B_0 + \Delta B$ ,  $B = B_0 + \sum_{i=1}^m \lambda_i B_i$  respectively.

##### 4.1. Unstructured perturbations

It can easily be shown, [13], that the stability of the closed-loop system is maintained, if  $\sigma_{\max}(\Delta B) < \frac{\sqrt{\frac{\sigma_{\min}(Q) - \sigma_{\max}(\frac{1}{\alpha} \Omega_1)}{\sigma_{\max}(\alpha Z + P)}}}{\sigma_{\max}(K C_0)}$ . Hence, in addition to the minimization objectives of (A.1) and (A.2), we also need to

- (A.3) : minimize  $\sigma_{\max}(K C_0)$

Instead of (A.3), we choose to minimize its upper bound, that is  $J_3 = \text{Tr}[(K C_0)^T (K C_0)]$ . The minimizing quantity is  $J_{B_u}^r = J_1 + J_3$  when no performance specifications are considered, where  $J_1$  is defined in (13). When performance specifications are considered, then  $J_{B_u}^{r,p} = J_A + J_3$ , where  $J_A$  is defined in (28). The algorithm of the previous section can be used here for  $J_{B_u}^r$  and  $J_{B_u}^{r,p}$  as well, the only difference being that the term  $\frac{\partial J_3}{\partial K} = 2 K C_0 C_0^T$  needs to be added to (21) and (34).

##### 4.2. Structured perturbations

It can easily be shown, [13], that the stability of the closed-loop system is maintained, if  $\sum_{i=1}^m \lambda_i^2 < \frac{\sigma_{\min}(Q) - \sigma_{\max}(\frac{1}{\alpha} \Omega_1)}{\sigma_{\max}(B^* K C_0) \sigma_{\max}(\alpha Z + P)}$ , where  $B^* = (B_1^T \dots B_m^T)^T$ . Now, in addition to (A.1) and (A.2), we also need to

- (A.4) : minimize  $\sigma_{\max}(B^* K C_0)$

We choose to minimize  $J_4 = \text{Tr}[(B^* K C_0)^T (B^* K C_0)]$  so that the minimizing quantities are now  $J_{B_s}^r = J_1 + J_4$ ,  $J_{B_s}^{r,p} = J_A + J_4$ . The algorithm of the previous section

applies here for  $J_{B_s}^r$  and  $J_{B_s}^{r,p}$  as well, the only difference now being that the term  $\frac{\partial J_A}{\partial K} = 2 (B^*)^T B^* K C_0 C_0^T$  needs to be added to ( 21 ) and ( 34 ).

## 5. Perturbations in (A, B) or (A, C)

We consider perturbations in the system matrix  $A$  and the input matrix  $B$  or in  $A$  and the output matrix  $C$ . Since both cases are similar, we study the case of perturbations in  $(A, B)$ .

### 5.1. Unstructured perturbations

We consider the linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = C_0 x(k) \quad (37)$$

where  $A = A_0 + \Delta A$ ,  $B = B_0 + \Delta B$  and static output feedback is applied. Here, the stability of the closed-loop system is maintained, if  $\sigma_{\max}(\Delta A) + \sigma_{\max}(\Delta B) \sigma_{\max}(K C_0) < \sqrt{\frac{\sigma_{\min}(Q) - \sigma_{\max}(\frac{1}{\alpha} \Omega_1)}{\sigma_{\max}(\alpha Z + P)}}$ . It can be shown, [13], that in order to maximize the stability region defined above, we need to satisfy the objectives (A.1), (A.2), (A.3). We see that the present case has the same objectives with the case of unstructured perturbations in  $B$  or  $C$  we studied before.

### 5.2. Structured perturbations

We consider again the system of ( 37 ) with  $A = A_0 + \sum_{i=1}^{m_A} \kappa_i A_i$ ,  $B = B_0 + \sum_{j=1}^{m_B} \lambda_j B_j$ . Here, the closed-loop system remains asymptotically stable, if  $\sum_{i=1}^{m_A} \hat{\theta}_i^2 < \frac{\sigma_{\min}(Q) - \sigma_{\max}(\frac{1}{\alpha} \Omega_1)}{\sigma_{\max}(\hat{\Pi}) \sigma_{\max}(\alpha Z + P)}$ , where  $\hat{\theta}_i = \kappa_i$ ,  $i = 1, \dots, m_A$  and  $\hat{\theta}_i = \lambda_{i-m_A}$ ,  $i = m_A + 1, \dots, m_A + m_B$ , and  $\hat{\Pi} = (A_1^T \dots A_{m_A}^T (B_1 K C_0)^T \dots (B_{m_B} K C_0)^T)^T$ .

Therefore, in addition to (A.1), (A.2), we also need to

- (A.5) : minimize  $\sigma_{\max}(\hat{\Pi})$

Similarly to before, we choose to minimize  $J_5 = \text{Tr}[\hat{\Pi}^T \hat{\Pi}]$ , so that the minimizing quantities are  $J_{AB}^r = J_1 + J_5$ ,  $J_{AB}^{r,p} = J_A + J_5$ . Hence, our algorithm can be used again, the only difference being that the term  $\frac{\partial J_5}{\partial K} = \sum_{j=1}^{m_B} 2 B_j^T B_j K C_0 C_0^T$  needs to be added to ( 21 ) and ( 34 ).

## 6. Perturbations in (A, B, C)

We consider unstructured perturbations in all system matrices. Specifically, we consider the linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) \quad (38)$$

where  $A = A_0 + \Delta A$ ,  $B = B_0 + \Delta B$ ,  $C = C_0 + \Delta C$ . Here, the stability of the closed-loop system is maintained, if

$$\sigma_{\max}(\Delta_{ABC}) < \sqrt{\frac{\sigma_{\min}(Q) - \sigma_{\max}(\frac{1}{\alpha} \Omega_1)}{\sigma_{\max}(\alpha Z + P)}} \quad (39)$$

where

$$\begin{aligned} \sigma_{\max}(\Delta_{ABC}) &= \sigma_{\max}(\Delta A) \\ &+ \sigma_{\max}(\Delta B) \sigma_{\max}(K) \sigma_{\max}(C_0) \\ &+ \sigma_{\max}(B_0) \sigma_{\max}(K) \sigma_{\max}(\Delta C) \\ &+ \sigma_{\max}(\Delta B) \sigma_{\max}(K) \sigma_{\max}(\Delta C) \end{aligned} \quad (40)$$

This inequality defines a region in  $\mathbb{R}^3$  for  $\sigma_{\max}(\Delta A)$ ,  $\sigma_{\max}(\Delta B)$  and  $\sigma_{\max}(\Delta C)$ . It can be shown, see [13], that in order to maximize the volume of this region, we need to satisfy objectives (A.1), (A.2) above, and also

- (A.6) : minimize  $\sigma_{\max}(K)$

which corresponds to the minimization of  $J_6 = \text{Tr}(K^T K)$ . Therefore, the minimizing quantities are  $J_{ABC}^r = J_1 + J_6$ ,  $J_{ABC}^{r,p} = J_A + J_6$ . Our algorithm can be used again, with the addition of the term  $\frac{\partial J_6}{\partial K} = 2K$  to ( 21 ) and ( 34 ).

## 7. Illustrative examples

**Example 7.1** Consider the scalar system

$$x(k+1) = 0.5 x(k) + u(k), \quad x(0) = 1.0 \quad (41)$$

with state feedback  $u(k) = Kx(k)$ . This system was studied in [11], where the LQR cost  $J'_2 = \sum_{k=0}^{\infty} x^2(k) + u^2(k)$  was used, that is  $Q_1 = R_1 = 1$ . The derived bound for unstructured perturbations in the system matrix  $A$  was  $\sigma_{\max}(\Delta A) < 0.8436$ , for a gain of  $K = -0.3436$ . We apply our method for the same LQR term. Choosing  $Q = 1.30$ ,  $Z = 0.60$ , initial stabilizing gain  $K_1 = 0.1$  and positive definite matrix  $D_1 = 0.001$ , we obtain a stabilizing gain of  $K = -0.49998$ , which corresponds to  $\sigma_{\max}(\Delta A) < 0.99998$ , which compares favorably to the result of [11] given above. The components of the minimizing quantity ( 28 ) that are associated with the robustness and the performance objectives are  $J_1 = 1.69$  and  $J_2 = 1.25$  respectively.

Neglecting the performance specifications, as indicated by the LQR cost, and focusing on just the maximization of the robustness bound, that is the minimization of (13), we obtain a stabilizing gain of  $K = -0.49999$ , which corresponds to  $\sigma_{\max}(\Delta A) < 0.99999$  and  $J_1 = 1.6900$ . Note that the same  $Q$ ,  $Z$ ,  $K_1$  and  $Q_1$  have been used. As we see, in this scalar case, we obtain almost the same results for the final stabilizing gain  $K$ , the uncertainty bound and the robustness component  $J_1$  of the minimizing quantity, no matter whether the LQR term is included or not in the minimizing quantity. Note, however, that this is not the case, in general, for MIMO systems, as we can see in the examples of [13] and the example that follows.

**Example 7.2** Consider an aircraft longitudinal control system from [9], whose the linearized continuous dynamic model is given by

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -0.0582 & 0.0651 & 0 & -0.171 \\ -0.303 & -0.685 & 1.109 & 0 \\ -0.0715 & -0.658 & -0.947 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x(t) \\ &+ \begin{pmatrix} 0 & 1 \\ -0.0541 & 0 \\ -1.11 & 0 \\ 0 & 0 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x(t) \end{aligned} \quad (42)$$

where  $x(t) = (\alpha(t) \ \beta(t) \ \psi(t) \ \theta(t))^T$ ,  $u(t) = (\eta(t) \ \tau(t))^T$ ,  $\alpha(t)$  and  $\beta(t)$  are the forward and vertical speeds,  $\psi(t)$  is the pitch rate and  $\theta(t)$  is the pitch angle. The control inputs  $\eta(t)$  and  $\tau(t)$  are the elevator angle and throttle position respectively. Note that all states are assumed available for measurement. We consider the discrete-time model for  $T = 0.5$  sec. The state-space matrices are given by

$$\begin{aligned} A_d &= \begin{pmatrix} 0.9692 & 0.0283 & -0.0112 & -0.0842 \\ -0.1302 & 0.6469 & 0.3584 & 0.0059 \\ -0.0086 & -0.2126 & 0.5644 & 0.0007 \\ -0.0041 & -0.0621 & 0.3873 & 1.0001 \end{pmatrix} \\ B_d &= \begin{pmatrix} 0.0017 & 0.4924 \\ -0.1385 & -0.0344 \\ -0.4266 & -0.0041 \\ -0.1170 & -0.0009 \end{pmatrix}, \quad C_d = I_4 \end{aligned} \quad (43)$$

We study the case of structured perturbations in the system matrix  $A_d$ ; specifically we assume that

$$\begin{aligned} \Delta A_d &= \kappa_1 \begin{pmatrix} 0.1 & 0.15 & 0 & 0 \\ 0.05 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0.05 \end{pmatrix} \\ &+ \kappa_2 \begin{pmatrix} 0 & 0 & 0 & 0.05 \\ 0 & -0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.05 \\ 0 & 0 & 0.05 & 0 \end{pmatrix} \end{aligned} \quad (44)$$

First we need to find the static output feedback matrix that maximizes the stability bound (8), without considering performance specifications. We choose  $Q = 10^{-2}I_4$ ,  $Z = \begin{pmatrix} 0.0110 & 0 & 0 & 0 \\ 0 & 0.0148 & -0.0007 & -0.0110 \\ 0 & -0.0007 & 0.0199 & -0.0028 \\ 0 & -0.0110 & -0.0028 & 0.0047 \end{pmatrix}$ , initial gain  $K_1 = \begin{pmatrix} -0.0264 & -0.1722 & 3.0531 & 10.2700 \\ -1.6068 & 0.2706 & 0.0224 & -0.0742 \end{pmatrix}$ , that places the closed-loop poles at  $(0.20, 0.70, -0.50 \pm 0.25j)$ , and  $D_1 = I_8$ . Our algorithm converges to the stabilizing gain  $K = \begin{pmatrix} -0.1089 & -0.5016 & 2.2162 & 1.5402 \\ -1.8981 & 0.0748 & 0.0317 & -0.1342 \end{pmatrix}$  which corresponds to the  $J_1 = 0.4468$ . With this gain, the stability region for the uncertain parameters is defined by the circle  $\kappa_1^2 + \kappa_2^2 < (0.6161)^2$ .

Next, we include the LQR cost in the minimizing quantity (28) for  $Q_1 = 0.1I_4$ ,  $R_1 = 0.01I_2$ ,  $X_0 = 0.1I_4$ . We consider the same  $Q$ ,  $Z$ ,  $D_1$ ,  $K_1$  as before and obtain the output gain  $K = \begin{pmatrix} -0.0672 & -0.1988 & 1.4512 & 0.9723 \\ -1.9253 & 0.0895 & 0.0590 & 0.0641 \end{pmatrix}$ , which corresponds to  $J_1 = 0.3985$  and  $J_2 = 0.0932$ . This gain defines the circle  $\kappa_1^2 + \kappa_2^2 < (0.5269)^2$ . Note that the uncertainty radius here is more conservative compared to the case of only the robustness specifications studied above.

Note that for the examples presented above and the numerous examples of [13], our algorithm proved to be quite fast. The algorithm, written in MATLAB code, converged in just several iterations of the *Main Step*; this took approximately a minute on a Sun SPARCstation 10.

## 8. Conclusions

A fast optimization algorithm for the design of robust output feedback controllers for linear uncertain discrete-time systems based on the BFGS rule has been presented. The minimizing quantity reflects the twofold optimization objective, which is the simultaneous maximization of established uncertainty bounds and the minimization of an LQR performance criterion. The first objective is based on recently established improved bounds that were developed in [12]. Note that the algorithm has also been applied to the case where the LQR term is not included in the minimizing

quantity, so that the only objective is the **design of a stabilizing controller that maximizes the uncertainty bounds**. In that case, the derived stability bounds are, in general, larger than the ones derived in the case of the robustness/LQR minimizing quantity. This was expected, since the inclusion of the LQR term in the minimizing quantity added an additional requirement to the optimization task.

Previous work was restricted to the case of unstructured perturbations in  $A$ . Here, a unified approach to both unstructured and structured perturbations in  $A$  has been presented. It has been shown that the present design process improves significantly the unstructured bound of [11]. Additionally, the cases of unstructured/structured perturbations in  $B$  or  $C$ , in  $(A, B)$  or  $(A, C)$ , together with the case of unstructured perturbations in  $(A, B, C)$  have also been studied. Note that the case of structured perturbations in all the state-space matrices remains to be addressed.

## References

- [1] M. Athans, "The matrix minimum principle," *Informat. and Contr.*, vol. 11, no. 5/6, pp. 592-606, 1967.
- [2] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty, *Nonlinear Programming: Theory and Applications*. New York: John Wiley & Sons, Inc., 1993.
- [3] C. E. de Souza, M. Fu, L. Xie, " $H_\infty$  analysis and synthesis of discrete-time systems with time-varying uncertainty," *IEEE Trans. Automat. Contr.*, vol. 38, no. 3, pp. 459-462, 1993.
- [4] J. Douglas and M. Athans, "Robust linear quadratic designs with real parameter uncertainty," *IEEE Trans. Automat. Contr.*, vol. 39, no. 1, pp. 107-111, 1994.
- [5] M. Fu, L. Xie, and C. E. de Souza, " $H_\infty$  control for linear systems with time-varying parameter uncertainty," in *Control of Uncertain Dynamic Systems*, S. P. Bhattacharyya and L. H. Keel, Eds., pp. 63-75, Boca Raton, FL: CRC Press, 1991.
- [6] K. Gu, Y. H. Chen, M. A. Zohdy, and N. K. Loh, "Quadratic stabilizability of uncertain systems: a two level optimization setup," *Automatica*, vol. 27, no. 1, pp. 161-165, 1991.
- [7] K. Gu, "Designing stabilizing control of uncertain systems by quasiconvex optimization," *IEEE Trans. Automat. Contr.*, vol. 39, no. 1, pp. 127-131, 1994.
- [8] H. P. Horisberger and P. R. Belanger, "Solution of the optimal constant output feedback problem by conjugate gradients," *IEEE Trans. Automat. Contr.*, vol. 19, pp. 434-435, 1974.
- [9] J. Jiang, "Design of reconfigurable control systems using eigenstructure assignments," *Int. J. Contr.*, vol. 59, no. 2, pp. 395-410, 1994.
- [10] S. R. Kolla, R. K. Yedavalli, and J. B. Farison, "Robust stability bounds on time-varying perturbations for state-space models of linear discrete-time systems," *Int. J. Contr.*, vol. 50, no. 1, pp. 151-159, 1989.
- [11] S. R. Kolla and J. B. Farison, "Reduced-order dynamic compensator design for stability robustness of linear discrete-time systems," *IEEE Trans. Automat. Contr.*, vol. 36, no. 9, pp. 1077-1081, 1991.
- [12] I. K. Konstantopoulos and P. J. Antsaklis, "Robust stability of linear continuous and discrete-time systems under parametric uncertainty," *Technical Report of the ISIS Group at the University of Notre Dame, ISIS-94-006*, March 1994.
- [13] I. K. Konstantopoulos and P. J. Antsaklis, "Design of output feedback controllers for robust stability and optimal performance of discrete-time systems," *Technical Report of the ISIS Group at the University of Notre Dame, ISIS-94-009*, September 1994.
- [14] O. I. Kosmidou, "Robust stability and performance of systems with structured and bounded uncertainties: an extension of the guaranteed cost control approach," *Int. J. Contr.*, vol. 52, no. 3, pp. 627-640, 1990.
- [15] W. S. Levine and M. Athans, "On the determination of optimal constant output feedback gains for linear multi-variable systems," *IEEE Trans. Automat. Contr.*, vol. 15, no. 1, pp. 44-48, 1970.
- [16] J. S. Luo and A. Johnson, "Stability robustness of the discrete-time LQG system under plant perturbation and noise uncertainty," *Int. J. Contr.*, vol. 55, no. 6, pp. 1491-1502, 1992.
- [17] J. S. Luo, A. Johnson, and P. P. J. Van Den Bosch, "Minimax guaranteed cost control for linear continuous-time systems with large parameter uncertainty," *Automatica*, vol. 30, no. 4, pp. 719-722, 1994.
- [18] J. M. Mendel and J. Feather, "On the design of optimal time-invariant compensators for linear stochastic time-invariant systems," *IEEE Trans. Automat. Contr.*, vol. 20, no. 5, pp. 653-657, 1975.
- [19] G. Menga and P. Dorato, "Observer-feedback design for linear systems with large parameter uncertainties," in *Proc. 1974 IEEE Conf. Decision Contr.*, Phoenix, AZ, pp. 872-878, 1974.
- [20] M.-L. Ni and H.-X. Wu, "A Riccati equation approach to the design of linear robust controllers," *Automatica*, vol. 29, no. 6, pp. 1603-1605, 1993.
- [21] S.-C. Tsay, I.-K. Fong, and T.-S. Kuo, "Robust linear quadratic optimal control for systems with linear uncertainties," *Int. J. Contr.*, vol. 53, no. 1, pp. 81-96, 1991.
- [22] S.-D. Wang, T.-S. Kuo, Y.-H. Lin, C.-F. Hsu, and Y.-T. Juang, "Robust control design for linear systems with uncertain parameters," *Int. J. Contr.*, vol. 46, no. 5, pp. 1557-1567, 1987.
- [23] R. K. Yedavalli, "Dynamic compensator design for robust stability of linear uncertain systems," in *Proc. 25th IEEE Conf. Decision Contr.*, Athens, Greece, pp. 34-36, 1986.
- [24] K. Zhou, P. P. Khargonekar, J. Stoustrup, and H. H. Niemann, "Robust Stability and performance of uncertain systems in state space," in *Proc. 31st IEEE Conf. Decision Contr.*, Tucson, AZ, pp. 662-667, 1992.