CONTROLLER DESIGN TO MINIMIZE A COMPOSITE MEASURE OF THE ℓ_1 AND THE $\mathcal{H}_2$ NORMS

M.V. Salapaka, M. Dahleh, P.G. Voulgaris and M.A. Dahleh

ABSTRACT

In this paper we consider the problem of minimizing a given positive linear combination of the ℓ_1 norm and the square of the $\mathcal{H}_2$ norm of the closed loop over all internally stabilizing controllers. The problem is analysed for the discrete-time, SISO, linear time invariant case. It is shown that a unique optimal solution always exists which can be obtained by solving a finite dimensional convex optimization problem with an *a priori* determined dimension. It is also established that the solution is continuous with respect to changes in the coefficients of the linear combination.

1. Notation

The following notation is employed in this paper:

$|x|_1$ — The 1-norm of the the vector $x \in R^n$.

$|x|_2$ — The 2-norm of the the vector $x \in R^n$.

$\hat{x}(\lambda)$ — The λ transform of a right sided real sequence $x = (x(k))_{k=0}^{\infty}$ defined as $\hat{x}(\lambda) := \sum_{k=0}^{k=\infty} x(k)\lambda^k$.

ℓ_1 — The Banach space of right sided absolutely summable real sequences with the norm given by $\| x \|_1 := \sum_{k=0}^{\infty} |x(k)|$.

ℓ_∞ — The Banach space of right sided, bounded sequences with the norm given by $\| x \|_\infty := sup_k |x(k)|$.

c_0 — The subspace of ℓ_∞ with elements x that satisfy $\lim_{k\to\infty} x(k) = 0$.

ℓ_2 — The Banach space of right sided square summable sequences with the norm given by $\| x \|_2 := \left[\sum_{k=0}^{k=\infty} x(k)^2\right]^{\frac{1}{2}}$

$\mathcal{H}_2$ — The isometric isomorphic space of ℓ_2 under the λ transform $\hat{x}(\lambda)$ wih the norm given by $\| \hat{x}(\lambda) \|_2 = \| x \|_2$.

X^* — The dual space of the Banach space X. $< x, x^* >$ denotes the value of the bounded linear functional x^* at $x \in X$.

$W(X^*, X)$ — The weak star topology on X^* induced by X.

T^* — The adjoint operator of $T : X \to Y$ which maps Y^* to X^*.

We have from functional analysis that $(\ell_1)^* = \ell_\infty$, $(c_0)^* = \ell_1$, $(\ell_2)^* = \ell_2$.

2. Introduction

Consider the standard feedback configuration of Figure 1 and let ϕ_{zw} be the closed loop transfer function which maps the exogenous input w to the regulated output z.

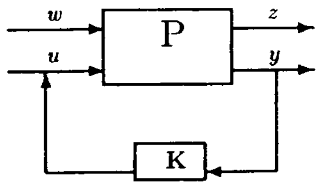

Figure 1: Plant Controller Configuration

Many important control problems can be reduced to this setup where the objective is to minimize a suitably defined measure of ϕ_{zw}. In the standard ℓ_1 problem the design of an internally stabilizing controller such that the ℓ_∞ norm of the regulated output z due to the worst case magnitude bounded disturbance w, is addressed. It is shown in [3] that for the 1-block case, the problem reduces to solving a finite dimensional linear program. The analogous problem with the signal measures being the ℓ_2 norm is the standard $\mathcal{H}_\infty$ problem. The standard $\mathcal{H}_2$ problem is concerned with the minimization of the energy contained in the

The first two authors are with University of California at Santa Barbara, the third is with University of Illinois at Urbana-Champaign and the fourth is with the Massachusetts Institute of Technology

pulse response of the closed loop, ϕ_{zw}. This can be viewed as minimizing the variance of the regulated output z due to a white noise input w. Both problems have been extensively analyzed in the past and solutions have been provided (e.g., [7]).

All of the previous design problems refer to a single performance measure. It is well known (see for example [2]) that optimization with respect to a particular norm may not necessarily yield good performance with respect to another. Thus, if enhanced performance is required with respect to multiple measures then it is necessary to include all these measures directly into the design process. In the recent years such considerations have led researchers to focus on the design of controllers to satisfy mixed performance criteria. One of the main problems in this class is the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design. For this problem the interest is on the interplay between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance measures of the closed loop. Several state-space results associated with this problem and variants are available. The interested reader may consult [6,13,14,15] to mention only a few.

In [5] it is shown that a wide variety of control problems reduce to convex optimization problems and it is argued that the present technology makes it possible to deem the problem solved if it can be reduced to a convex problem. In this light it is appropriate to exploit as much structure in the problem as possible so that the standard software available becomes computationally more efficient. Within this context several results on multiobjective functions involving the ℓ_1 norm are becoming available. In [4] the problem of minimizing the ℓ_1 norm of the closed loop under linear inequality constraints is addressed. Every such problem is equivalent to a linear programming problem which has a canonical dual problem associated with it. Contrary to the finite dimensional case it is not true that every infinite dimensional linear program has the same optimal value as its dual. However it was shown by the authors that under some conditions this "duality gap" does not exist between the primal and the dual which is advantageous from a computational point of view. The problem of minimizing the ℓ_1 norm of the closed loop while keeping the $\mathcal{H}_\infty$ norm under a prescribed level falls under the above category.

In [9] the problem of minimizing the ℓ_1 norm of a single input single output transfer function while keeping the $\mathcal{H}_\infty$ norm of the closed loop system under a specified value is reduced to solving a sequence of finite dimensional convex optimization problems and an unconstrained $\mathcal{H}_\infty$ problem. In [10] a similar problem in continuous time of minimizing the maximum amplitude due to a specified input while keeping the $\mathcal{H}_\infty$ norm below a given level is reduced to solving a finite dimensional convex constrained optimization problem and a standard unconstrained $\mathcal{H}_\infty$ problem.

In [8] it is shown that the problem of minimizing the $\mathcal{H}_2$ norm of ϕ_{zw} while keeping its ℓ_1 norm below a specified level reduces to a finite dimensional quadratic programming problem. Optimal solution is shown to be unique. In [11] it is shown that the problem of minimizing the ℓ_1 norm of ϕ_{zw} while keeping its $\mathcal{H}_2$ norm below a prescribed level reduces to a finite dimensional convex optimization problem with an *a priori* determined dimension. It is also shown that the optimal is unique whenever the $\mathcal{H}_2$ norm constraint is active.

In this paper we consider the problem of minimizing a given positive linear combination of the ℓ_1 norm and the square of the $\mathcal{H}_2$ norm of ϕ_{zw} over all stabilizing controllers. This cost function has strong relations with the notion of Pareto optimal solutions with respect to the ℓ_1 and $\mathcal{H}_2$ norms and can be used to generate all such solutions. The underlying optimization principle that allows us to transform the problem into a tractable finite dimensional convex problem is the Lagrange duality theorem [1].

The paper is organized as follows. In section 3 the problem statement is made precise and the relation of the problem to Pareto optimality is established. In section 4 it is shown that the problem has a unique solution and in section 5 the problem is reduced to a finite dimensional convex optimization problem. In section 6 an example is given to illustrate the theory developed and in section 7 conclusions are given.

3. Problem Formulation

Consider the standard feedback problem represented in Figure 1 where P and K are the plant and the controller respectively. Let w represent the exogenous input, z represent the output of interest, y is the measured output and u is the control input where z and w are assumed scalar. Let ϕ be the closed loop map which maps $w \to z$. From Youla parametrization [16] it is known that all achievable closed loop maps under stabilizing controllers are given by $\phi = h - u * q$ ($*$ denotes convolution), where $h, u, q \in \ell_1$; h, u depend only on the plant P and q is a free parameter in ℓ_1. Throughout the paper we make the following assumption:

Assumption 1 *All the zeros of $\hat{u}$ (the λ transform of u) inside the unit disc are real and distinct. Also, $\hat{u}$ has no zeros on the unit circle.*

The assumption that all zeros of $\hat{u}$ which are inside the open unit disc are real and distinct is not restrictive and is made to streamline the presentation of the paper. Let the zeros of u which are inside the unit disc be given by $z_1, z_2, \ldots, z_n$. Let

$$\Phi := \{\phi : \textit{there exists } q \in \ell_1 \textit{ with } \phi = h - u * q\}.$$

Φ is the set of all achievable closed loop maps under stabilizing controllers. Let $A : \ell_1 \rightarrow R^n$ be given by

$$A = \begin{pmatrix} 1 & z_1 & z_1^2 & z_1^3 & \dots \\ 1 & z_2 & z_2^2 & z_2^3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z_n & z_n^2 & z_n^3 & \dots \end{pmatrix},$$

and $b \in R^n$ be given by

$$b = \begin{pmatrix} h(z_1) \\ h(z_2) \\ \vdots \\ h(z_n) \end{pmatrix}.$$

Theorem 1 *The following is true:*

$$\begin{aligned} \Phi &= \{\phi \in \ell_1 : \hat{\phi}(z_i) = \hat{h}(z_i) \text{ for all } i = 1, \dots, n\} \\ &= \{\phi \in \ell_1 : A\phi = b\}. \end{aligned}$$

Proof: Given in [2]. ∎

The problem of interest is :

Given $c_1 > 0, c_2 > 0$ obtain a solution to the following mixed objective problem:

$$\begin{aligned} \nu & \qquad (1) \\ := \; & inf\{c_1 \| \phi \|_1 + c_2 \| \phi \|_2^{\,2} : \phi \in \ell_1, \; A\phi = b\} \qquad (2) \end{aligned}$$

In the following sections we will study the existence, structure and computation of the optimal solution. Before we initiate our study towards these goals it is worthwhile to point out certain connections between the cost under consideration and the notion of Pareto optimality.

3.1 Relation to Pareto Optimality

The notion of Pareto optimality can be stated as follows (see for example, [5]). Given a set of m nonnegative functionals f_i, $i = 1, \dots, m$ on a normed linear space X, a point $x_0 \in X$ is Pareto optimal with respect to the vector valued criterion $f := (f_1, \dots, f_m)$ if there does not exist any $x \in X$ such that

$$f_i(x) \le f_i(x_0) \; \forall i \in \{1, \dots, m\} \text{ and}$$

$$f_i(x) < f_i(x_0) \text{ for some } i \in \{1, \dots, m\}.$$

Under certain conditions the set of all Pareto optimal solutions can be generated by solving a minimization of weighted sum of the functionals as the following theorem indicates.

Theorem 2 *[12] Let X be a normed linear space and each nonnegative functional f_i be convex. Also let*

$$S_m := \{c \in R^m : c_i \ge 0, \; \sum_{i=1}^{m} c_i = 1\}$$

and for each $c \in R^m$ consider the following scalar valued optimization:

$$\inf_{x \in X} \sum_{i=1}^{m} c_i f_i(x).$$

If $x_0 \in X$ is Pareto optimal with respect to the vector valued criterion $f(x)$, then there exists some $c \in S_m$ such that x_0 solves the above minimization. Conversely, given $c \in S_m$, if the above minimization has at most one solution x_0 then x_0 is Pareto optimal with respect to $f(x)$. ∎

In the next section we show that there is a unique solution ϕ_0 to Problem (2) for all c_1 and c_2. Furthermore, since u is assumed to be a scalar, there is a unique optimal $q \in \ell_1$. Hence, in view of the aforementioned theorem we have that if we restrict our attention to parameters c_1 and c_2 such that $(c_1, c_2) \in S_2 := \{(c_1, c_2) : c_1 + c_2 = 1, \; c_1, c_2 \ge 0\}$, we will recover the set of all Pareto optimal solutions with respect to the vector valued function

$$f(q) = (\| h - u * q \|_1, \| h - u * q \|_2^{\,2})$$

where $q \in \ell_1$. Indeed, note that all the assumptions of convexity and linearity are satisfied. Also note in the case where $c_2 = 0$ we have the standard ℓ_1 optimization which can have many optimal solutions [2]; nontheless there is a unique solution among them which minimizes the $\mathcal{H}_2$-norm and this is in fact a Pareto optimal solution. Moreover, in the case where $c_1 = 0$ we have a standard $\mathcal{H}_2$ problem which is known to have a unique solution [2]. For all other values of $(c_1, c_2) \in S_2$ we have a unique solution (to be shown in the sequel).

To summarize, if ϕ_0 is the optimal solution for Problem (2) with $(c_1, c_2) \in S_2$, then there does not exist a preferable alternative ϕ with $\phi = h - u * q$ for some $q \in \ell_1$ such that

$$\| \phi \|_1 \le \| \phi_0 \|_1 \text{ and } \| \phi \|_2 < \| \phi_0 \|_2$$

or,

$$\| \phi \|_1 < \| \phi_0 \|_1 \text{ and } \| \phi \|_2 \le \| \phi_0 \|_2.$$

Conversely, all possible ϕ's that enjoy Pareto optimality can be recovered by the solution to Problem 2 with $(c_1, c_2) \in S_2$.

As a final note we mention that if (c_1, c_2) do not satisfy $c_1 + c_2 = 1$ then we can generate a new set of parameters $\bar{c}_1$, $\bar{c}_2$ as $\bar{c}_1 = \frac{c_1}{c_1 + c_2}$, $\bar{c}_2 = \frac{c_2}{c_1 + c_2}$ with $\bar{c}_1 + \bar{c}_2 = 1$. These new parameters would yield the same optimal solution as with (c_1, c_2).

4. Existence, Uniqueness and Properties of the Optimal Solution

In the first part of this section we show that Problem (2) always has a solution. In the second part we show that any solution to Problem (2) is a finite impulse response sequence in the third we give an *a priori* bound on the length.

4.1 Existence of a Solution

Here we show that a solution to (2) always exists. We use the following lemma to prove the main result of this subsection.

Lemma 1 *(Banach Alaoglu) Let X be a Banach space with X^* as its dual then the set $\{x^* : x^* \in X^*, \| x^* \| \leq M\}$ is $W(X^*, X)$ compact for any $M \in R$.*

Theorem 3 *There exists $\phi_0 \in \Phi$ such that*

$$c_1\| \phi_0 \|_1 + c_2\| \phi_0 \|_2^{\,2} = \inf_{\phi\in\Phi}\{c_1\| \phi \|_1 + c_2\| \phi \|_2^{\,2}\},$$

where $\Phi{:=}\{\phi \in \ell_1 : A\phi = b\}$. Therefore the infimum in (2) is a minimum.

Proof: We denote the feasible set of our problem by $\Phi{:=}\{\phi \in \ell_1 : A\phi = b\}$. Let $B := \{\phi \in \ell_1 : c_1\| \phi \|_1 + c_2\| \phi \|_2^{\,2} \leq \nu + 1\}$. It is clear that

$$\nu = \inf_{\phi\in\Phi\cap B}\{c_1\| \phi \|_1 + c_2\| \phi \|_2^{\,2}\}.$$

Therefore given $i > 0$ there exists $\phi_i \in \Phi \cap B$ such that $c_1\| \phi_i \|_1 + c_2\| \phi_i \|_2^{\,2} \leq \nu + \frac{1}{i}$. B is a bounded set in $\ell_1 = c_0^*$. It follows from the Banach Alaoglu lemma that B is $W(c_0^*, c_0)$ compact. Using the fact that c_0 is separable we know that there exists a subsequence $\{\phi_{i_k}\}$ of $\{\phi_i\}$ and $\phi_0 \in \Phi \cap B$ such that $\phi_{i_k} \to \phi_0$ in the $W(c_0^*, c_0)$ sense, that is for all v in c_0

$$< v, \phi_{i_k} > \to < v, \phi_0 > \; as \; k \to \infty. \tag{3}$$

Let the j^{th} row of A be denoted by a_j and the j^{th} element of b be given by b_j . Then as $a_j \in c_0$ we have as $k \to \infty$,

$$< a_j, \phi_{i_k} > \to < a_j, \phi_0 > \; for\ all\ j = 1, 2, \ldots, n. \tag{4}$$

As $A(\phi_{i_k}) = b$ we have $< a_j, \phi_{i_k} >= b_j$ for all k and for all j which implies $< a_j, \phi_0 >= b_j$ for all j. Therefore we have $A(\phi_0) = b$ from which it follows that $\phi_0 \in \Phi$. This gives us $c_1\| \phi_0 \|_1 + c_2\| \phi_0 \|_2^{\,2} \geq \nu$.

From (3) we have for all N,

$$\sum_{j=0}^{N}\{c_1|\phi_{i_k}(j)| + c_2(\phi_{i_k}(j))^2\} \to \tag{5}$$

$$\sum_{j=0}^{N}\{c_1|\phi_0(j)| + c_2(\phi_0(j))^2\} \; as \; k \to \infty. \tag{6}$$

As $c_1\| \phi_{i_k} \|_1 + c_2\| \phi_{i_k} \|_2^{\,2} \leq \nu + \frac{1}{i_k}$ we have that

$$\sum_{j=0}^{N}\{c_1|\phi_{i_k}(j)| + c_2(\phi_{i_k}(j))^2\} \leq \nu + \frac{1}{i_k}. \tag{7}$$

Letting $k \to \infty$ in (7) and using (6) we have that for all N

$$\sum_{j=0}^{N}\{c_1|\phi_0(j)| + c_2(\phi_0(j))^2\} \leq \nu.$$

By letting $N \to \infty$ in the above inequality we conclude that $c_1\| \phi_0 \|_1 + c_2\| \phi_0 \|_2^{\,2} \leq \nu$. Therefore, it follows that $c_1\| \phi_0 \|_1 + c_2\| \phi_0 \|_2^{\,2} = \nu$. This proves the theorem. ■

4.2 Structure of Optimal Solutions

In this subsection we use a Lagrange duality result to show that every optimal solution is of finite length. First we give the following definitions, where we denote the interior of a set by *int*.

Definition 1 *Let P be a convex cone in a vector space X. We write $x \geq y$ if $x - y \in P$. We write $x > 0$ if $x \in int(P)$. Similarly $x \leq y$ if $x - y \in -P := N$ and $x < 0$ if $x \in int(N)$.*

Definition 2 *Let X be a vector space and Z be a vector space with positive cone P. A mapping $G : X \to Z$ is convex if $G(tx + (1-t)y) \leq tG(x) + (1-t)G(y)$ for all x, y in X and t with $0 \leq t \leq 1$ and is strictly convex if $G(tx + (1-t)y) < tG(x) + (1-t)G(y)$ for all x, y in X and t with $0 < t < 1$.*

The following is a Lagrange duality theorem.

Theorem 4 *[11,1] Let X be a Banach space, Ω be a convex subset of X, Y be a finite dimensional space, Z be a normed space with positive cone P. Let $f : \Omega \to R$ be a real valued convex functional, $g : X \to Z$ be a convex mapping, $H : X \to Y$ be an affine linear map and $0 \in int[range(H)]$. Define*

$$\mu_0 := inf\{f(x) : \; g(x) \leq 0, \; H(x) = 0, \; x \in \Omega\}. \tag{8}$$

Suppose there exists $x_1 \in \Omega$ such that $g(x_1) < 0$ and $H(x_1) = 0$ and suppose μ_0 is finite. Then,

$$\mu_0 = max\{\varphi(z^*, y) : z^* \geq 0, \; z^* \in Z^*, \; y \in Y\}, \tag{9}$$

where $\varphi(z^*, \; y){:=} \; inf \; \{ \; f(x) + < g(x), z^* > + < H(x), y >: x \in \Omega \; \}$ and the maximum is achieved for some $z_0^* \geq 0$, $z_0^* \in Z^*$, $y_0 \in Y$.

Furthermore if infimum in (8) is achieved by some $x_0 \in \Omega$ then

$$< g(x_0), z_0^* > + < H(x_0), y_0 >= 0, \tag{10}$$

and

x_0 *minimizes* $f(x) + < g(x), z_0^* > + < H(x), y_0 >$, (11)

over all $x \in \Omega$. (12)

We refer to (8) as the **Primal** problem and to (9) as the **Dual** problem.

Lemma 2

$$\nu = \max_{y \in R^n} \inf_{\phi \in \ell_1} \{c_1 \| \phi \|_1 + c_2 \| \phi \|_2{}^2 + < b - A\phi, y >\}. \quad (13)$$

Proof: We will apply Theorem 4 to get the result. Let X, Ω, Y, Z in Theorem 4 correspond to ℓ_1, ℓ_1, R^n, R respectively. Let $\gamma := \nu + 1$, $g(\phi) := c_1 \| \phi \|_1 + c_2 \| \phi \|_2{}^2 - \gamma$ and $H(\phi) := b - A\phi$. With this notation we have $Z^* = R$.

A has full range which implies $0 \in int[range(H)]$. From Theorem 3 we know that there exists ϕ_0 such that $g(\phi_0) = c_1 \| \phi_0 \|_1 + c_2 \| \phi_0 \|_2{}^2 - \gamma = -1 < 0$ and $H(\phi_0) = 0$. Therefore all the conditions of Theorem 4 are satisfied. From Theorem 4 we have

$$\nu = \max_{z \geq 0, y \in R^n} \inf_{\phi \in \ell_1} \{c_1 \| \phi \|_1 + c_2 \| \phi \|_2{}^2 + < g(\phi), z > + < b - A\phi, y >\}.$$

Let $z_0 \in R$, $y_0 \in R^n$ be a maximizing solution to the right hand side of the above equation. ϕ_0 being the solution of the primal we have from (10) that $< g(\phi_0), z_0 > + < H(\phi_0), y_0 >= 0$ which implies that $< g(\phi_0), z_0 >= 0$. As $g(\phi_0) \neq 0$ we conclude that $z_0 = 0$. This proves the theorem. ■

Lemma 3 *The following is true:*

$$\nu = \max_{y \in R^n} \inf_{\phi \in \ell_1, \phi(i)v(i) \geq 0} \{c_1 \| \phi \|_1 + c_2 \| \phi \|_2{}^2 \quad (14)$$

$$- < \phi, v > + < b, y >\}, \quad (15)$$

where $v := A^* y$.

Proof: From (13) it easily follows that

$$\nu = \max_{y \in R^n} \inf_{\phi \in \ell_1} \{c_1 \| \phi \|_1 + c_2 \| \phi \|_2{}^2 - < \phi, v > + < b, y >\},$$

where $v := A^* y$. Suppose $\phi \in \ell_1$ is such that $\phi(i)v(i) < 0$ for some i. Then choose $\phi_1 \in \ell_1$ such that $\phi_1(j) = \phi(j)$ for all $j \neq i$ and $\phi_1(i) = 0$. It follows that $c_1 \| \phi \|_1 + c_2 \| \phi \|_2{}^2 - < \phi, v > > c_1 \| \phi_1 \|_1 + c_2 \| \phi_1 \|_2{}^2 - < \phi_1, v >$.

Therefore, in the infimization above we can restrict ϕ to satisfy $\phi(i)v(i) \geq 0$ for all i. This proves the lemma. ■

The following theorem shows that the solution to (2) is unique and that it is a finite impulse response sequence.

Theorem 5 *Define* $\mathcal{T} := \{\phi \in \ell_1$ *: there exists* L^* *with* $\phi(i) = 0$ *if* $i \geq L^*\}$. *The following is true:*

$$\nu = \max_{y \in R^n} \inf_{\phi \in \mathcal{T}, \phi(i)v(i) \geq 0} \{c_1 \| \phi \|_1 + c_2 \| \phi \|_2{}^2 \quad (16)$$

$$- < \phi, v > + < b, y >\}, \quad (17)$$

where $v(i) = (A^* y)(i)$. *Also, the solution to the primal (2) is unique and the solution belongs to* $\mathcal{T}$.

Proof: Let $y_0 \in R^n$ be the solution to the right hand side of (17). Define $v_0 := (A^* y_0)(i)$ and let

$$L(\phi) := c_1 \| \phi \|_1 + c_2 \| \phi \|_2{}^2 - < \phi, v_0 > + < b, y_0 > .$$

It is clear from Lemma (3) that

$$\nu = \inf_{\phi \in \ell_1} L(\phi) = \inf_{\phi(i)v_0(i) \geq 0} L(\phi).$$

As v_0 is in ℓ_1 we know that there exists L^* such that $v_0(i)$ satisfies $|v_0(i)| < c_1$ if $i \geq L^*$. Suppose ϕ is such that $\phi(i)v(i) \geq 0$. Then it follows that

$$L(\phi) = \sum_{i=0}^{\infty} \{\phi(i)(c_1 sgn(v_0(i)) - v_0(i)) + c_2(\phi(i))^2\} + < y_0, b > .$$

If it is also true that $|v_0(i)| < c_1$ then it follows that

$$\phi(i)(c_1 sgn(v_0(i)) - v_0(i)) + c_2(\phi(i))^2 \geq 0,$$

with the equality holding only when $\phi(i) = 0$. Therefore, in the infimization of

$$\inf_{\phi(i)v_0(i) \geq 0} L(\phi)$$

we can restrict ϕ to satisfy $\phi(i) = 0$ whenever $|v_0(i)| < c_1$. It follows that we can restrict ϕ to $\mathcal{T}$ in the infimization because if $i \geq L^*$ then $|v_0(i)| < c_1$.

In Theorem 3 we showed that there exists a solution ϕ_0 to the primal (2). ¿From Theorem 4 we know that ϕ_0 is a solution to $\inf_{\phi \in \ell_1} L(\phi)$. As $L(\phi)$ is strictly convex in ϕ we conclude that the solution to the primal (2) is unique. ¿From the previous discussion it follows that $\phi_0 \in \mathcal{T}$ and that ϕ_0 is a solution to the problem

$$\inf_{\phi(i)v_0(i) \geq 0} \sum_{i=0}^{L^*} \{c_1 |\phi(i)| + c_2(\phi(i))^2 - \phi(i)v_0(i)\} + < y_0, b >$$

This proves the theorem. ■

4.3 *A priori* Bound on the Length of the Optimal Solution

In this section we give an *a priori* bound on the length of the solution to (2). First we establish the following three lemmas.

Lemma 4 *Let ϕ_0 be a solution of the primal (2). Let y_0 represent the corresponding dual solution as obtained in (17). Let $v_0 := A^* y_0$ then,*

$$\begin{aligned} c_2\phi_0(i) &= \tfrac{v_0(i)-c_1}{2} \quad \textit{if } v_0(i) > c_1 \\ &= \tfrac{v_0(i)+c_1}{2} \quad \textit{if } v_0(i) < -c_1 \\ &= 0 \quad \textit{if } |v_0(i)| \le c_1. \end{aligned}$$

Also, $\| v_0 \|_\infty \le \alpha$ where $\alpha = 2c_2(\| h \|_1 + \frac{c_2}{c_1}\| h \|_2{}^2) + c_1$.

Proof:Let $L(\phi)$ be defined as in the proof of Theorem (5). We have shown that

$$\nu = \inf_{\phi \in \ell_1} L(\phi) = \inf_{\phi(i)v_0(i) \ge 0} L(\phi).$$

Suppose $|v_0(i)| = c_1$. As ϕ_0 minimizes $L(\phi)$ we have $\phi_0(i) = 0$. In the proof of Theorem (5) we have shown that if $|v_0(i)| < c_1$ then $\phi_0(i) = 0$. Therefore, $c_2\phi_0(i) = 0$ if $|v_0(i)| \le c_1$.

Suppose $v_0(i) > c_1$ then it is easy to show that there exists $\phi(i)$ such that $\phi(i) \ge 0$ and

$$\phi(i)(c_1\ sgn(v_0(i)) - v_0(i)) + c_2(\phi(i))^2 < 0.$$

As any optimal minimizes $L(\phi)$ we know that

$$\phi_0(i)(c_1 sgn(v_0(i)) - v_0(i)) + c_2(\phi_0(i))^2 < 0,$$

which implies $\phi_0(i) > 0$. Solving for the optimal by putting the derivative equal to zero we have $c_1 - v_0(i) + 2c_2\phi_0(i) = 0$ (Differentiation is valid because $\phi_0(i) > 0$). This implies that $c_2\phi_0(i) = \frac{v_0(i)-c_1}{2}$. Similarly, it follows that $c_2\phi_0(i) = \frac{v_0(i)+c_1}{2}$ when $v_0(i) < -c_1$. It also follows that

$$\| v_0 \|_\infty \le\ 2c_2\| \phi_0 \|_\infty + c_1 \le 2c_2\| \phi_0 \|_1 + c_1 \le$$

$$2c_2(\| h \|_1 + \frac{c_2}{c_1}\| h \|_2{}^2) + c_1.$$

The last inequality follows from the fact that h is an achievable closed loop map. This implies that $\alpha := 2c_2(\| h \|_1 + \frac{c_2}{c_1}\| h \|_2{}^2) + c_1$ is an *a priori* upper bound on $\| v_0 \|_\infty$. This proves the lemma. ■

Lemma 5 *[2] If $y \in R^n$ is such that $\| A^*y \|_\infty \le \alpha$ then there exists a positive integer L^* independent of y such that $|(A^*y)(i)| < c_1$ for all $i \ge L^*$.*

Proof: Define

$$A_L^* = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ z_1 & z_2 & z_3 & \dots & z_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_1^L & z_2^L & z_3^L & \dots & z_n^L \end{pmatrix},$$

$A_L^* : R^n \to R^{L+1}$. With this definition we have $A_\infty^* = A^*$. Let $y \in R^n$ be such that $\| A^*y \|_\infty \le \alpha$. Choose any L such that $L \ge (n-1)$. As $z_i, i = 1, \dots, n$ are distinct A_L^* has full column rank. A_L^* can be regarded as a linear map taking $(R^n, \| \cdot \|_1) \to (R^{L+1}, \| \cdot \|_\infty)$. As A_L^* has full column rank we can define the left inverse of A_L^*, $(A_L^*)^{-l}$ which takes $(R^{L+1}, \| \cdot \|_\infty) \to (R^n, \| \cdot \|_1)$. Let the induced norm of $(A_L^*)^{-l}$ be given by $\| (A_L^*)^{-l} \|_{\infty,1}$. $y \in R^n$ is such that $\| A^*y \|_\infty \le \alpha$ and therefore $\| A_L^*y \|_\infty \le \alpha$. It follows that,

$$\| y \|_1 \le \| (A_L^*)^{-l} \|_{\infty,1} \| A_L^*y \|_\infty \le \tag{18}$$

$$\| (A_L^*)^{-l} \|_{\infty,1}\ \alpha. \tag{19}$$

Choose L^* such that

$$\max_{k=1,\dots,n} |z_k|^{L^*}\ \| (A_L^*)^{-l} \|_{\infty,1}\ \alpha < c_1. \tag{20}$$

There always exists such an L^* because $|z_k| < 1$ for all $k = 1, \dots, n$. Note that L^* does not depend on y. For any $i \ge L^*$ we have

$$\begin{aligned} |(A^*y)(i)| &= |\sum_{k=1}^{k=n} z_k^i y(k)| \le \max_{k=1,\dots,n} |z_k|^i \| y \|_1 \\ &\le \max_{k=1,\dots,n} |z_k|^i\ \| (A_L^*)^{-l} \|_{\infty,1}\ \alpha \\ &\le \max_{k=1,\dots,n} |z_k|^{L^*}\ \| (A_L^*)^{-l} \|_{\infty,1}\ \alpha. \end{aligned}$$

The second inequality follows from 19. From 20 we have $|(A^*y)(i)| < c_1$ if $i \ge L^*$. This proves the lemma. ■

Theorem 6 *Every solution ϕ_0 of the primal (2) is such that $\phi(i) = 0$ if $i \ge L^*$ where L^* given in Lemma (5) can be determined a priori.*

Proof: Let y_0 be the dual solution to (2) and let $v_0 := A^*y_0$. From Lemma 4 we know that $\| v_0 \|_\infty \le \alpha$ where $\alpha = 2c_2(\| h \|_1 + \frac{c_2}{c_1}\| h \|_2{}^2) + c_1$. Applying Lemma 5 we conclude that there exists L^* (which can be determined apriori) such that $|v_0(i)| < c_1$ if $i \ge L^*$. We have shown in the proof of Theorem (5) that $\phi_0(i) = 0$ if $|v_0(i)| < c_1$. We conclude that $\phi_0 = 0$ if $i \ge L^*$. This proves the theorem. ■

The above theorem shows that the Problem (2)is a finite dimensional convex minimization problem. In fact, it is a quadratic programming problem which can be solved numerically with very efficient methods (see for example [1]).

5. Continuity of the Optimal Solution

In this section we show that the optimal is continuous with respect to changes in the parameters c_1 and c_2. First, we prove the following lemma:

Lemma 6 *Let $\{f_k\}$ be a sequence of functions which map R^m to R. If f_k converges uniformly to a function f on a set $S \subset R^m$ then*

$$\lim_{k\to\infty} \min_{x\in S} f_k(x) = \min_{x \in S} f(x),$$

provided that the minima exist.

Proof : Let $\min_{x \in S} f(x) = f(x_0)$ for some $x_0 \in S$. Given $\epsilon > 0$ we know from convergence of the sequence $\{f_k\}$ to f that there exists an integer K such that if $k > K$ then

$$|f_k(x_0) - f(x_0)| < \epsilon$$
$$\Rightarrow f_k(x_0) < \epsilon + f(x_0)$$
$$\Rightarrow \min_{x \in S} f_k(x) < \epsilon + f(x_0)$$
$$\Rightarrow \lim_{k \to \infty} \min_{x \in S} f_k(x) < \epsilon + f(x_0).$$

As ϵ is arbitrary we have $\lim_{k \to \infty} \min_{x \in S} f_k(x) \leq f(x_0)$. Now we prove the other inequality. Given $\epsilon > 0$ we know that there exists an integer K such that if $k > K$ then

$$|f_k(x) - f(x)| < \epsilon \text{ for any } x \in S$$
$$\Rightarrow f_k(x) > f(x) - \epsilon \geq f(x_0) - \epsilon \text{ for any } x \in S$$
$$\Rightarrow \min_{x \in S} f_k(x) > f(x_0) - \epsilon$$
$$\Rightarrow \lim_{k \to \infty} \min_{x \in S} f_k(x) > f(x_0) - \epsilon.$$

As ϵ is arbitrary we have $\lim_{k \to \infty} \min_{x \in S} f_k(x) \geq f(x_0)$. This proves the lemma ∎

Theorem 7 *Let $c_1^k \in [a_1, b_1]$ and $c_2^k \in [a_2, b_2]$ where $a_1 > 0$, $a_2 > 0$. Let ϕ_k be the unique solution to the problem*

$$\nu_k := \min_{A\phi = b} c_1^k \| \phi \|_1 + c_2^k \| \phi \|_2^2, \qquad (21)$$

and let ϕ_0 be the solution to the problem

$$\nu := \min_{A\phi = b} c_1 \| \phi \|_1 + c_2 \| \phi \|_2^2, \qquad (22)$$

where $c_1 \in [a_1, b_1]$ and $c_2 \in [a_2, b_2]$. If $c_1^k \to c_1$ and $c_2^k \to c_2$ then $\phi_k \to \phi_0$.

Proof : We prove this theorem in three parts; first we show that we can restrict the proof to a finite dimensional space, second we show that $\nu_k \to \nu$ and finally we show that $\phi_k \to \phi_0$. Let y_k represent the dual solution of (21) and let $v_k := A^* y_k$. Let α_k the upper bound on $\| v_k \|_\infty$ be as given by Lemma 4. Therefore, $\alpha_k = 2c_2^k(\| h \|_1 + \frac{c_2^k}{c_1^k} \| h \|_2^2) + c_1^k \leq 2b_2(\| h \|_1 + \frac{b_2}{a_1} \| h \|_2^2) + b_1$. Let this bound be denoted by d. Choose L^* such that

$$\max_{k=1,\dots,n} |z_k|^{L^*} \;\| (A_L^*)^{-l} \|_{\infty,1}\, d < a_1.$$

where L is such that $L \geq (n-1)$. Therfore, it follows that

$$\max_{k=1,\dots,n} |z_k|^{L^*} \;\| (A_L^*)^{-l} \|_{\infty,1}\, \alpha_k < c_1^k.$$

for all k. From arguments similar to that of Lemma 5 and Theorem 6 it follows that $\phi_k(i) = 0$ if $i \geq L^*$ for all k. Therefore we can assume that $\phi_k \in R^{L^*}$.

Now, we prove that $\nu_k \to \nu$. Let ϕ_1 be the solution of the problem

$$\nu_1 := \min_{A\phi = b} b_1 \| \phi \|_1 + b_2 \| \phi \|_2^2.$$

As $c_1^k \leq b_1$ and $c_2^k \leq b_2$ we have that $\nu_k \leq \nu_1$ for all k. Therefore, for any k we have $c_1^k \| \phi_k \|_1 + c_2^k \| \phi_k \|_2^2 \leq \nu_1$ which implies $\| \phi_k \|_1 \leq \frac{\nu_1}{c_1^k} \leq \frac{\nu_1}{a_1}$ and $\| \phi_k \|_2^2 \leq \frac{\nu_1}{c_2^k} \leq \frac{\nu_1}{a_2}$.

Let $f_k(\phi) := c_1^k \| \phi \|_1 + c_2^k \| \phi \|_2^2$ and $f(\phi) := c_1 \| \phi \|_1 + c_2 \| \phi \|_2^2$. Let $S := \{\phi \in R^{L*} : A\phi = b, \| \phi \|_1 \leq \frac{\nu_1}{a_1}, \| \phi \|_2^2 \leq \frac{\nu_1}{a_2}\}$. Then it is clear that

$$\nu_k := \min_{\phi \in S} c_1^k \| \phi \|_1 + c_2^k \| \phi \|_2^2.$$

We prove now that f_k converges to f uniformly on S. Given $\epsilon > 0$ choose K such that if $k > K$ then $|c_1^k - c_1| < \frac{\epsilon a_1}{2\nu_1}$ and $|c_2^k - c_2| < \frac{\epsilon a_2}{2\nu_1}$. Then for any $\phi \in S$ we have

$|f_k(\phi) - f(\phi)| = |(c_1^k - c_1) \| \phi \|_1 + (c_2^k - c_2) \| \phi \|_2^2| \leq |c_1^k - c_1| \frac{\nu_1}{a_1} + |c_2^k - c_2| \frac{\nu_1}{a_2} < \epsilon$.

Therefore, it follows that f_k converges uniformly to f on S. From Lemma 6 it follows that $\nu_k \to \nu$.

We now prove that $\phi_k \to \phi_0$. Let $B := \{\phi \in R^{L*} : \| \phi \|_1 \leq \frac{\nu_1}{a_1}\}$ then we know that $\phi_k \in B$ which is compact in $(R^{L*}, \| \,.\, \|_1)$. Therefore there exists a subsequence ϕ_{k_i} of ϕ_k and $\overline{\phi} \in R^{L^*}$ such that $\phi_{k_i} \to \overline{\phi}$.

As $c_1^k \to c_1$, $c_2^k \to c_2$ and $\phi_{k_i} \to \overline{\phi}$ we have that $f_{k_i}(\phi_{k_i}) \to f(\overline{\phi})$. As ν_k converges to ν it follows that $f_{k_i}(\phi_{k_i}) \to f(\phi_0)$ (note that $\nu_{k_i} = f_{k_i}(\phi_{k_i})$ and $\nu = f(\phi_0)$) and therefore $f(\overline{\phi}) = f(\phi_0)$. As $A\phi_{k_i} = b$ for all i we have that $A\overline{\phi} = b$. From uniqueness of the solution of (22) it follows that $\overline{\phi} = \phi_0$. Therefore we have established that $\phi_{k_i} \to \phi_0$. From uniqueness of the solution of (22) it also follows that $\phi_k \to \phi_0$. This proves the theorem. ∎

6. Conclusions

In this paper we solved the problem of minimizing a linear positive combination of the ℓ_1 norm and the square of the $\mathcal{H}_2$ norm of the closed loop transfer function for discrete-time SISO feedback systems. This way all Pareto optimal solutions with respect to ℓ_1 and $\mathcal{H}_2$ norms can be generated and trade-off studies can be performed. The unique solution to the problem is readily obtained by a quadratic programming problem of *a priori* determined demension. Also, continuity of the solution with respect to the linear combination coefficients of the problem was established.

The main tool for the development of the paper is the Lagrange duality theory. Such a tool can also be used for MIMO problems where the above SISO results seem to have natural extentions. This remains the subject of current research.

7. References

[1] D. G. Luenberger, *Optimization by Vector Space Methods.* NY: John Wiley and Sons, Inc., 1969.

[2] M.A. Dahleh and I.J. Diaz-Bobillo, *Control of Uncertain Systems: A Linear Programming approach,* Prentice Hall (in print).

[3] M.A. Dahleh and J.B. Pearson, "ℓ_1 optimal feedback controllers for MIMO discrete-time systems", *IEEE Trans. Automat. Control,* Vol AC-32, no 4, pp. 314-322, April 1987.

[4] N.Elia, M.A. Dahleh and I.J. Diaz-Bobillo, "Controller design via infinite dimensional linear programming", *IEEE Proceedings Automat. Control,* Vol AC-3, pp. 2165-2169, San Fransisco, California, June 1993.

[5] S.P. Boyd and C.H. Barratt, *Linear Controller Design: Limits Of Performance,* Prentice Hall, Englewood Cliffs, New Jersey, 1991.

[6] P.P. Khargonekar and M.A. Rotea, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control; A convex optimization approach", *IEEE Trans. Automat. Control,* Vol AC-36, no 7, pp.824-837, 1991.

[7] J.C.Doyle, K. Glover, P. Khargonekar and B.A. Francis, "State space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems", *IEEE Trans. Automat. Control .,* Vol 34, no 8, pp. 831-847, August 1989.

[8] Petros Voulgaris, "Optimal $\mathcal{H}_2/\ell_1$ control via duality theory" (Preprint).

[9] M. Sznaier, "Mixed $\ell_1/\mathcal{H}_\infty$ Controllers for SISO Discrete Time Systems", *IEEE Proc. Decision and Control,* Vol 1, pp 34-38, Dec. 1993.

[10] M. Sznaier and F. Blanchini, "Mixed $\mathcal{L}_\infty/\mathcal{H}_\infty$ Suboptimal Controllers for SISO Continuous-Time Systems", *IEEE Proc. Decision and Control,* Vol 3, pp 34-38, Dec. 1993.

[11] Murti V. Salapaka, M. Dahleh, Petros Voulgaris, " Mixed Objective Control Synthesis: Optimal $\ell_1/\mathcal{H}_2$ Control", (Preprint)

[12] N.O. Da Cuhna and E. Polak, "Constrained minimization under vector valued criteria in finite dimensional spaces", *J. Math. Amal. and Appl.,* vol 19, pp 103-124, 1967.

[13] J.C. Doyle, K. Zhou, and B. Bodenheimer, "Optimal control with mixed H_2 and H_∞ performance objectives," *Proceedings of ACC,* Pittsburg, PA, 1989.

[14] D. Mustafa, K. Glover, and D.J.N. Limebeer, "Solutions to the H_∞ general distance problem which minimize an entropy integral," *Automatica,* vol.77, pp. 193-199,1991.

[15] D. S. Bernstein and W. M. Haddad, "LQG control with an H_∞ performance bound: A Riccati equation approach," *IEEE Trans. Automat. Control,* vol. 34, pp. 293–305, March 1989.

[16] D. C. Youla, H. A. Jabr and J. J. Bongiorno, "Modern wiener-hopf design of optimal controllers - part 1: The multivariable case", *IEEE Trans. Automat. Control,* Vol AC-21, June 1976.