Sub-Optimum $\mathcal{H}_\infty$ Problem for 2-D MIMO Systems

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Abstract: Robust $\mathcal{H}_\infty$ control problem for multi-input multi-output 2-D systems is attacked by polynomial techniques. The design procedure is shown to consist of the same major steps as its 1-D counterpart: one linear 2-D matrix polynomial equation and two quadratic 2-D matrix equations called 2-D J-spectral factorization. As usual for the 2-D case, every particular step is more involved both theoretically and numerically.

1 Introduction

Polynomial techniques have become very popular in the last decade. They were applied to solve standard tasks in communication and control. In addition, they have been employed recently to cope with some new problems of robustness as $\mathcal{H}_\infty$ optimization for standard (1-D) linear systems. On the other hand, polynomial techniques have been generalized to cover standard problems for some non-standard classes of systems. This namely comprehends two-dimensional (2-D) linear systems and filters, which are now of rising importance in many areas of engineering and science. Influenced by both the trends mentioned above, a generalization of polynomial techniques to cope with $\mathcal{H}_\infty$ problems in control of scalar 2-D systems has been published recently [1]. The present paper is the first attempt to study $\mathcal{H}_\infty$ problems for multi-input multi-output 2-D systems. Namely, the sub-optimum $\mathcal{H}_\infty$ problem is tackled as a prerequisite to approach the optimum solution.

It is shown that, as expected, all the basic steps of the design procedure remain the same as in 1-D case. However, their content is different and, namely, completely new numerical algorithms are required to perform every particular step of the design. In fact, similar situation happened when other 1-D control problems have being generalized to 2-D [2, 3].

A causal sequential matrix $S(z_1,z_2)=\sum_{i,j\geq 0} S_{ij} z_1^i z_2^j$ is called *absolutely summable* if $\sum \|S_{ij}\| < \infty$ and *minimum phase* if, moreover, its inverse S^{-1} is absolutely summable as well. A *conjugate* to S is denoted by S^* and defined by $S^*(z_1,z_2) = S^T(z_1^{-1},z_2^{-1})$.

A finite-extent causal 2-D sequential matrix p is called (2-D) *polynomial* matrix. (in the indeterminates z_1 and z_2). A square 2-D polynomial matrix $P(z_1,z_2)$ is *stable* if $\det P(z_1,z_2) \neq 0$ on the *closed unit bidisc*

$$\{(z_1,z_2) \in \mathcal{C} \times \mathcal{C} \mid |z_1| \leq 1, |z_2| \leq 1\}. \tag{1}$$

A 2-D sequential matrix $R(z_1,z_2)$ is called *recurrent* if it is described by a 2-D matrix polynomial fraction[1] $R(z_1,z_2) = N(z_1,z_2)D^{-1}(z_1,z_2) = \bar{D}^{-1}\bar{N}(z_1,z_2)(z_1,z_2)$.

The key concept of the paper is the ∞-norm, which is a useful tool in many areas of control and filtering. For a recurrent absolutely summable sequential matrix R is defined via

$$\begin{aligned} \|R(z_1,z_2)\|_\infty &= \\ &= \sup_{|z_1|=1,|z_2|=1} \lambda_{\max}^{1/2}\left(R^*(z_1,z_2)R(z_1,z_2)\right) \\ &= \sup_{\omega_1,\omega_2} \lambda_{\max}^{1/2}\left(R^T(e^{-j\omega_1},e^{-j\omega_2})R(e^{j\omega_1},e^{j\omega_2})\right) \end{aligned} \tag{2}$$

where $\lambda_{\max}(A)$ denotes the largest eigenvalue of the (constant) matrix A. The number $\lambda_{\max}^{1/2}(A^{\mathrm{H}}A)$ is the largest of the *singular values*[2]

$$\sigma_i(A) = \lambda_i^{1/2}(A^{\mathrm{H}}A), \quad i = 1,\ 2,\ \cdots, n, \tag{3}$$

of the $m \times n$ matrix A.

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[1] Throughout the paper, all polynomial fraction will possess causal denominator and, hence, they will be well defined. For more details on polynomial matrix inversions see [4, 5].

[2] Here the superscript H denotes the Hermitian.

In particular, for a 1-D sequence this reads

$$\begin{aligned} \|R(z_1, z_2)\|_\infty &= \sup_{|z_1|=1} \lambda_{\max}^{1/2} \left(R^*(z_1)R(z_1)\right) \\ &= \sup_{\omega_1} \lambda_{\max}^{1/2} \left(R^T(e^{-j\omega_1})R(e^{j\omega_1})\right) \end{aligned} \tag{4}$$

2 1-D Standard Solution

As a starting point, let us consider the standard 1-D problem which is defined by the well-known configuration of Fig 1. The block marked "G" is the "plant," that is, the system to be controlled. The signal w represents external, uncontrollable inputs. The signal u is the control input. The output z has the meaning of control error, which ideally should be zero. The signal y, finally, is the measured output, available for feedback via the controller, which is the block marked "K." Such a structure has been attacked for 1-D systems via polynomial techniques by Kwakernaak [6].

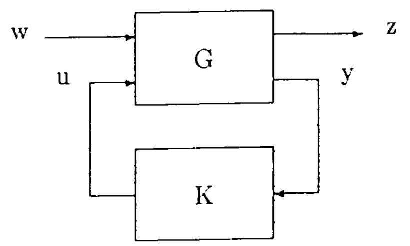

Figure 1: The standard $\mathcal{H}_\infty$ problem

The plant is represented by the rational transfer matrix G such that

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}. \tag{5}$$

The dimensions of the subblocks of G follows from

$$\begin{array}{cc} & \begin{array}{cc} k_1 & k_2 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \end{array}. \tag{6}$$

Without loss of generality, it is assumed [6] that

1. The dimension of the external input w is at least as great as that of the observed output y, that is, $k_1 \geq m_2$, and G_{21} has full normal row rank.

2. The dimension of the control error z is at least as great as that of the control input u, that is, $m_1 \geq k_2$, and G_{12} has full normal column rank.

Using a proper transformation, the assumptions can be always met.

In the configuration of Fig. 1, the closed-loop transfer function H from the external input w to the control error z is easily found to be given by

$$H = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}. \tag{7}$$

In the polynomial approach, we assume that the plant is represented in left coprime matrix fraction form

$$G = D^{-1}N. \tag{8}$$

Similarly, we represent the compensator K in *right* matrix fraction form as

$$K = YX^{-1}. \tag{9}$$

Closed-loop stability is determined by the roots of the closed-loop characteristic polynomial which, when partitioning

$$D = \begin{array}{c} \begin{array}{cc} m_1 & m_2 \end{array} \\ [D_1 \quad D_2] \end{array}, \qquad N = \begin{array}{c} \begin{array}{cc} k_1 & k_2 \end{array} \\ [N_1 \quad N_2] \end{array}, \tag{10}$$

reads

$$\det\left([D_1 \quad D_2X - N_2Y]\right). \tag{11}$$

These roots are precisely the closed-roots poles. From them, however, only the roots of

$$\det(D_\circ X - N_\circ Y) \tag{12}$$

are assignable with $G_{22} = D_\circ^{-1}N_\circ$ a left coprime polynomial matrix fraction representation of G_{22}. Following [6], we shall call the controller (9) stabilizing if and only if it makes all the assignable poles stable.

For the above structure, the typical problem is [6] as follows:

Definition - Standard $\mathcal{H}_\infty$-optimal problem. Determine the controller K such that

- the closed-loop system is stable,
- $\|H\|_\infty$ is minimal. $(\star)$

It is well known that the solution of $\mathcal{H}_\infty$-optimal problem consists of a chain of successive solutions of $\mathcal{H}_\infty$-sub-optimal problems with λ approaching the optimal value $\lambda_{\min}$. In every step of iteration, the following problem solution is exercised:

Definition - Sub-optimal problem. For a given $\lambda \geq 0$ (and, in fact, $\lambda \geq \lambda_{\min}$) determine the controller k_λ such that

- the closed-loop system is stable,
- $\|H_\lambda\|_\infty \leq \lambda$ $(\star\star)$

Besides its repeated appearance in the $\mathcal{H}_\infty$-optimum search, the sub-optimal solution itself is known to guarantee certain level of robustness [6]. This problem has recently been solved both via state space methods [7] and polynomial techniques [6]. We now briefly review the polynomial solution of this problem developed by Kwakernaak [6]. Although Kwakernaak derived his polynomial solution for continuous-time systems, it applies, *mutatis mutandis* for discrete-time systems as well.

The underlaying philosophy is to transform the original problem to the solution of the rational matrix inequality

$$\begin{bmatrix} X^* & Y^* \end{bmatrix} \Pi_\lambda \begin{bmatrix} X \\ Y \end{bmatrix} \geq 0 \text{ on the imaginary axis,} \tag{13}$$

which parameterizes the class of all controllers (9) that warrant ($\star\star$). Here, for λ nonnegative, Π_λ is the rational matrix

$$\Pi_\lambda = \begin{bmatrix} D_2^* \\ -N_2^* \end{bmatrix} (N_1 N_1^* - \lambda^2 D_1 D_1^*)^{-1} [D_2 \quad -N_2]. \tag{14}$$

The next step is to find its *rational* symmetric factorization of

$$\Pi_\lambda = Z_\lambda^* J Z_\lambda, \tag{15}$$

where Z_λ is a square rational matrix such that Z_λ is both stable and minimum-phase. This rational factorization can be reduced to two *polynomial* matrix J-spectral factorizations, one for the denominator, the other for the numerator. The former is the polynomial J-spectral cofactorization

$$N_1 N_1^* - \lambda^2 D_1 D_1^* = Q_\lambda J' Q_\lambda^*, \tag{16}$$

with Q_λ square such that its determinant is Hurwitz.

Once Q_λ has been determined, we may obtain polynomial matrices Δ_λ and Λ_λ by the left-to-right fraction conversion

$$Q_\lambda^{-1}[D_2 \quad -N_2] = \Delta_\lambda \Lambda_\lambda^{-1}. \tag{17}$$

By the second polynomial J-spectral factorization

$$\Delta_\lambda^* J' \Delta_\lambda = \Gamma_\lambda^* J \Gamma_\lambda, \tag{18}$$

with Γ_λ square such that its determinant is Hurwitz, we obtain the rational J-spectral factor Z_λ as

$$Z_\lambda = \Gamma_\lambda \Lambda_\lambda^{-1}. \tag{19}$$

This problem of computation of (16)-(18), which is called *J-spectral factorization*, has received much attention recently and workable algorithms have been already derived [8].

Then, according to Kwakernaak [6], the class of all controllers (9) that guarantee ($\star\star$) is given by

$$K = \tilde{Y}\tilde{X}^{-1}, \tag{20}$$

where

$$\begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix} = Z_\lambda^{-1} \begin{bmatrix} A \\ B \end{bmatrix}, \tag{21}$$

with A and B rational stable and A square such that

$$\|BA^{-1}\|_\infty \leq 1. \tag{22}$$

There are two obvious special choices: $A = I$ and $B = 0$ (*central* solution) or $A = B = I$ (*equalizing* solution).

To result in a stabilizing controller, a stable A should be chosen in (21).

3 2-D Case

Let us consider the standard structure again but now all the signals u, w and z, y are vector 2-D causal sequences and R is a 2-D bicausal recurrent sequential matrix. Consequently, D, N and and all their parts are 2-D polynomial matrices with D bicausal.

The controller (9), in general, is expected with X, Y ranging 2-D causal (not necessarily recurrent) sequential matrices with X bicausal. It is stabilizing provided that

$$\det(D_o X - N_o Y) \tag{23}$$

is a minimum phase 2-D sequence (or, in particular, a stable 2-D polynomial).

It can be shown that the solution of the 2-D problem follows the lines of the 1-D solution: At first, the matrix polynomial J-spectral factorization (16) should be performed to find a bicausal minimum phase 2-D sequential matrix $Q_\lambda(z_1, z_2)$ such that

$$N_1 N_1^* - \lambda^2 D_1 D_1^* = Q_\lambda J' Q_\lambda^* \tag{24}$$

This operation is a difficult and not yet quite clear numerical problem even with a positive definite J because of its 2-D nature.

Then, the left-to-right conversion (17) is to be performed. The linear operations for 2-D polynomial matrices are now quite well understood and can be performed relatively easily (see, e.g., [9]).

As the last step, J-spectral factorization (18) must be computed to get a 2-D sequential matrix $\Gamma_\lambda(z_1, z_2)$ such that det Γ_λ is minimum phase and

$$\Delta_\lambda^* J' \Delta_\lambda = \Gamma_\lambda^* J \Gamma_\lambda \tag{25}$$

This is probably the most difficult step as explained in the next section.

4 2-D Spectral Factorization

The main difference between 1-D and 2-D solutions emerges in the operation of spectral factorization. So the numerator factorization (24)

$$N_1 N_1^* - \lambda^2 D_1 D_1^* = Q_\lambda J' Q_\lambda^* \tag{26}$$

exists whenever $\Pi(z_1, z_2)$ is nonzero on the *unit torus*

$$\{(z_1, z_2) \in C \times C \mid |z_1| = 1, |z_2| = 1\} \tag{27}$$

but, in general, it results in a factor $Q_\lambda(z_1, z_2)$ with an infinite number of terms. Hence, $Q_\lambda(z_1, z_2)$ is an infinite 2-D sequential matrix rather than a 2-D polynomial [10]. As an example, for the symmetric scalar left hand side polynomial

$$z_1^{-1} + z_2^{-1} + 5 + z_1 + z_2$$

it is not possible to find a 2-D polynomial $Q_\lambda(z_1, z_2)$ to satisfy (24).

An algorithm to compute $Q_\lambda(z_1, z_2)$ in the scalar case has been published (see [10]) but in practice, it always ends in a truncated (i.e., polynomial) version $\bar{Q}_\lambda(z_1, z_2)$ of the sequence $Q_\lambda(z_1, z_2)$. Of course, the truncation must retain the stability of $\bar{Q}_\lambda(z_1, z_2)$ which is yet another problem. The matrix case encountered her, in addition, combines both indefiniteness and two-dimensionality and remains unsolved thus far. It will be a subject of further research.

The solution of (25) faces the same problems, of course.

5 Conclusion

The problem of $\mathcal{H}_\infty$ robust control has been discussed for 2-D systems with scalar standard structure. It was shown that the 1-D polynomial design procedure [6] fits but its basic steps (spectral and J-spectral factorizations) are to be replaced by their 2-D versions. While the solution of the linear 2-D matrix polynomial equation (17) is known [9], the problem of 2-D matrix J-spectral factorization ((24) and (25)) is a subject of further research.

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