Solving Hidden Markov Problem By Spectral Approach

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May 22, 1995

Abstract The objective of this paper is to develop a stochastic spectral approach to solve hidden Markov problems based on a Wiener Chaos expansion. We prove that the set of Fourier coefficients in the Cameron-Martin development of the unnormalized posterior probability is a solution of a simple recursive system of infinite order deterministic equations. This decomposition separates observations and parameters which suggests a natural numerical algorithm for solving this problem.

1 Introduction

The objective of this paper is to develop a stochastic spectral approach to solve hidden Markov problems. Specifically we estimate the distribution of a continuous time Markov chain given noisy observations of its path. Similar approach was developed by Mikulevicius-Rozovskii [4] for nonlinear filtering of diffusion Markov processes (see also [3], [5]).

We consider the filtering scheme where the signal process $x(t)$ is a homogeneous jump Markov process. The observation process $y(t)$ is of the form

$$y(t) = y_0 + \int_0^t h(x(s))ds + W(t),$$

where $W(t)$ is a Brownian motion. A fundamental objective of the filtering theory is to compute the conditional distribution of the signal process $x(t)$ given the observations $y(s), s \leq t$. This problem reduces to solving an infinite system of stochastic differential equations (Rozovskii-Shiryaev [5])

$$\begin{aligned} dq_i(t) &= \sum_j \lambda_{ji} q_j(t)dt + h_i q_i(t)dy(t) & (1.1)\\ q_i(0) &= p_0^i & (1.2) \end{aligned}$$

*This work was partially supported by ARO Grant DAAH 04-95-1-0164

where $\{p_0^i, i = 1, 2, \ldots\}$ is the law of $x(0)$ and $h_i = h(i)$. A solution to this equation is usually referred to as unnormalized posterior probability.

In this paper we present a spectral decomposition of a solution of (1.1), (1.2) based on the Cameron-Martin orthogonal development of L_2-functionals of a Gaussian process (Cameron-Martin [1], Hida [2]). Specifically we prove that this solution can be written in the form (see Theorem 2.1)

$$q_i(t) = \sum_{\alpha \in J} \frac{1}{\sqrt{\alpha!}} \varphi_\alpha^i(t)\, \xi_\alpha(y) \quad a.s. \tag{1.3}$$

where J is the set of multi-indices, $\xi_\alpha(y)$ are products of Hermite polynomials of Wiener integrals $\int_0^t m_j(s)dy(s)$, where $\{m_k\}$ is a complete orthonormal system in $L_2(0,t)$. We also prove that the Fourier coefficients $\{\varphi_\alpha^i(t), i = 1, 2, \ldots, \alpha \in J\}$ satisfy the recursive system of infinite order deterministic equations

$$\begin{aligned} \frac{d\varphi_\alpha^i(s)}{ds} &= \sum_j \lambda_{ji}\varphi_\alpha^j(s) + \sum_k \alpha_k m_k(s) h_i \varphi_{\alpha(k)}^i(s)\\ \varphi_\alpha^i(0) &= p_0^i 1_{\{|\alpha|=0\}}. \end{aligned}$$

It is important that decomposition (1.3) separates observations and parameters which allows the time consuming computation of deterministic Fourier coefficients $\varphi_\alpha^i(t)$ to be performed off-line and leaves only the computationally simple stochastic part to be performed on-line. Expansion (1.3) is a double infinite sum. We study the error results from truncating the expansion (see Theorem 2.2). Many well known time discretization schemes can be derived from (1.3), for example the splitting-up approximation.

2 Main Results

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{x(t), 0 \leq t \leq T\}$ be a homogeneous continuous time Markov process with transition probability density $[\lambda_{ij};\ i,j = 1,2,\ldots]$. Let the observation process be of the form

$$y(t) = y_0 + \int_0^t h(x(s))ds + W(t) \tag{2.1}$$

where $\sup_i |h_i| < \infty$ and $W(t)$ is a one dimensional Brownian motion independent of $x(t)$. It is known that the posterior probability $P(x(t) = i|\mathcal{F}_t^y)$ can be written as $q_i(t)/\sum_i q_i(t)$, where $\{q_i(t),\ 0 \le t \le T\}$ is the unique solution of the infinite system of stochastic differential equations (Rozovskii-Shiryaev [5])

$$dq_i(t) = \sum_j \lambda_{ji} q_j(t)dt + h_i q_i(t)dy(t) \tag{2.2}$$

$$q_i(0) = p_0^i \tag{2.3}$$

where $\{p_0^i, i = 1,2,\ldots\}$ is the law of $x(0)$.

Let

$$\rho(T) = \exp\{-\int_0^T h(x(s))dw(s) - \frac{1}{2}\int_0^T |h(x(s))|^2 ds\}.$$

It is a standard fact that the measure $\tilde{P}$ defined by $d\tilde{P} = \rho(T)dP$ is a probability measure on $(\Omega, \mathcal{F})$ with respect to which y is a Brownian motion. Let J be the set of multi-indices. For $\alpha \in J$, define the following notations: $|\alpha| = \sum_i \alpha_i$, $\alpha! = \prod_k \alpha_k!$ and $\alpha(j) = (\alpha_1, \ldots, \max(\alpha_j - 1, 0), \alpha_{j+1}, \ldots)$. Let $\{m_k(s)\}$ be a complete orthonormal system (CONS) in $L_2(0,t)$ and $\xi_k = \int_0^t m_k(s)dy_s$. Further, for $\alpha \in J$, let

$$\xi_\alpha(y) = \prod_k \frac{H_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}},$$

where

$$H_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{-\frac{x^2}{2}}$$

is the jth Hermite polynomial. Then $\{\xi_\alpha(y), \alpha \in J\}$ is a CONS in $L_2(\Omega, \mathcal{F}_t^y, \tilde{P})$ (Cameron-Martin [1], Hida [2]). Let $l_2 = \{(a_1, a_2, \ldots) : \sum_i a_i^2 < \infty\}$. Denote $|\cdot|_2$ the norm in l_2.

Theorem 2.1 *If $\{q_i(t), i = 1,2,\ldots\}$ is the solution of (2.2), (2.3), then*

$$q_i(t) = \sum_{\alpha \in J} \frac{1}{\sqrt{\alpha!}} \varphi_\alpha^i(t) \xi_\alpha(y) \quad P-a.s. \tag{2.4}$$

where $\varphi_\alpha(s) = \{\varphi_\alpha^i(s), i = 1,2,\ldots\}$ satisfies the following recursive system of deterministic equations

$$d\varphi_\alpha^i(s)/ds = \sum_j \lambda_{ji}\varphi_\alpha^j(s) + \sum_k \alpha_k m_k(s) h_i \varphi^i_{\alpha(k)}(s) \tag{2.5}$$

$$\varphi_\alpha^i(0) = p_0^i 1_{\{|\alpha|=0\}}, \tag{2.6}$$

for $s \le t$. Moreover, the series (2.4) converges in $L_2(\Omega, \tilde{P})$.

For $\alpha \in J$, denote $d(\alpha) = max\{k \ge 1 : \alpha_k > 0\}$ and set $J_N^n = \{\alpha \in J : |\alpha| < N, d(\alpha) \le n\}$. Expansion (2.4) is a double infinite sum, denote by $q_{N,n}(t)$ the truncated expansion which includes only terms corresponding to α in J_N^n.

Theorem 2.2 *The error bound of $q_{N,\infty}$ is given by*

$$\tilde{E}|q_{N,\infty}(t) - q(t)|_2^2 \le e^{|h|_\infty^2 t} \frac{(|h|_\infty^2 t)^{N+1}}{(N+1)!}. \tag{2.7}$$

If $m_0(s) = \frac{1}{\sqrt{t}}, m_k(s) = \sqrt{\frac{2}{t}}\cos(\frac{\pi k s}{t})$, then

$$\tilde{E}|q_{N,n}(t) - q_t|_2^2 \le e^{|h|_\infty^2 t}\left[\frac{(|h|_\infty^2 t)^{N+1}}{(N+1)!} + c\frac{t^3}{n}\right]. \tag{2.8}$$

For a diffusion type state process $x(t)$, similar result was proved in [3]. It is clear that for $N \ge 2$ the error is of order $O(t^3)$. First we will demonstrate how the splitting-up approximation can be derived from (2.4) - (2.6). Let us take $N = \infty, n = 1$ and for $|\alpha| = k$, write $\varphi_k(s)$ for $\varphi_\alpha(s)$. Solution of (2.5), (2.6) can be approximated by

$$\varphi_k^i(s) \approx \left(\frac{s\, h_i}{\sqrt{t}}\right)^k p_i(s), \tag{2.9}$$

where $p_i(s) = P(x(s) = i)$. This can be shown by induction. If $k = 0$, this is the exact solution by Kolmogorov forward equation. Suppose (2.9) holds up to $k - 1 (\ge 0)$. From Lemma 3.1 (below), (2.5), (2.6) follows that for $s \le t$,

$$\begin{aligned}
&\varphi_k^i(s)\\
&= \frac{k}{\sqrt{t}}\int_0^s \sum_j h_j \varphi_{k-1}^j(r) p_{ji}(s-r)dr\\
&\approx \frac{k}{\sqrt{t}}\int_0^s \left(\frac{r}{\sqrt{t}}\right)^{k-1} \sum_j p_j(r)p_{ji}(s-r)h_j^k dr,
\end{aligned}$$

where p_{ij} is the transition probability for x. Approximate $p_j(r)p_{ji}(s-r)$ by $p_j(s)p_{ji}(0)$, then the summation gives $h_i^k p_i(s)$, and we arrive at (2.9). When $s = t$,

$$\begin{aligned}
q_{\infty,1}^i(t) &\approx \sum_{k=0}^\infty \frac{1}{k!}\left(\sqrt{t}\, h_i\right)^k H_k(\xi_1)p_i(t)\\
&= \exp\{h_i y(t) - \frac{1}{2}h_i^2 t\}p_i(t), \qquad (2.10)
\end{aligned}$$

the last equality holds by the fact that $\exp\{\theta z - z^2/2\} = \sum_{k=0}^\infty z^k H_k(\theta)/k!$. Formula (2.10) is the splitting-up approximation.

Corollary 2.1 *The error of splitting-up method is*

$$\sup_{0\le t\le T} \tilde{E}|q_{\infty,1}(t)-q_t|_2^2 \le c\, e^{|h|_\infty^2 T}\, T^3.$$

3 Proofs

Proof of Theorem 2.1 The theorem is proved if solution of (2.5), (2.6) exists and is unique (see [4]). This follows by induction and Lemma 3.1 below.

Lemma 3.1 *Let $l_1 = \{(a_1,a_2,\ldots) : \sum_i |a_i| < \infty\}$. If $u_0 \in l_1$ and $g(t) = (g_1(t), g_2(t),\ldots)$ satisfying $\sup_{0\le t\le T}\sum_i |g_i(t)| < \infty$, then there exists a unique solution to*

$$\frac{du_i(t)}{dt} = \sum_j \lambda_{ji}u_j(t) + g_i(t) \tag{3.1}$$

$$u_i(0) = u_0^i. \tag{3.2}$$

The solution satisfies $\sup_{0\le t\le T}\sum_i |u_i(t)| < \infty$ and can be written as

$$u_i(t) = \sum_k u_0^k p_{ki}(t) + \int_0^t \sum_k g_k(t)p_{ki}(t-s)ds. \tag{3.3}$$

In addition, if $u_0 \in l_1 \bigcap l_2$ and $\sup_{0\le t\le T}\sum_i g_i^2(t) < \infty$, then $\sup_{0\le t\le T}\sum_i u_i^2(t) < \infty$.

This lemma is well understood and we ommit the proof.

Proof of Theorem 2.2 (cf. Theorem 2.2 in [3]) Since $\{\xi_\alpha\}$ is a CONS in $L_2(\Omega,\tilde{P})$,

$$\tilde{E}|q_{N,\infty}(t)-q(t)|_2^2 = \sum_{k>N}\sum_{|\alpha|=k}\frac{|\varphi_\alpha(t)|_2^2}{\alpha!}, \tag{3.4}$$

where (see [4])

$$\sum_{|\alpha|=k}\frac{|\varphi_\alpha(t)|_2^2}{\alpha!} \tag{3.5}$$

$$= \int_0^t\int_0^{s_k}\cdots\int_0^{s_2} |T_{t-s_k}h\cdots T_{s_2-s_1}hT_{s_1}p_0|_2^2\, ds^k,$$

where $T_{r-s}f(i) = \sum_j f(j)p_{ji}(r-s)$ and $ds^k = ds_1\cdots ds_k$. Direct computation yields

$$|T_{t-s_k}hT_{s_k-s_{k-1}}h\cdots hT_{s_1}p_0|_2^2 \le |h|_\infty^{2k}.$$

Now we substitute the latter into (3.5), integrate and sum over k in (3.4). To prove (2.8) it is sufficient to determine $\tilde{E}|q_{N,n}(t)-q_{N,\infty}(t)|_2^2$ which is given by

$$\sum_{l>n}\sum_{k=1}^N\sum_{|\alpha|=k,d(\alpha)=l}\frac{|\varphi_\alpha(t)|_2^2}{\alpha!}. \tag{3.6}$$

For $|\alpha| = k$, let $i_1^\alpha \le i_2^\alpha \le \cdots \le i_k^\alpha$ where $\alpha_{i_j^\alpha} \ne 0$. It is known that (see [3])

$$\varphi_\alpha^i = \int_0^t\int_0^{s_k}\cdots\int_0^{s_2} F_i(t;s^k)\, E_\alpha(s^k)\, ds^k$$

where

$$F_i(t;s^k) = (T_{t-s_k}h\cdots T_{s_2-s_1}hT_{s_1}p_0)_i,$$

$$E_\alpha(s^k) = \sum_{\sigma\in\mathcal{P}_k} m_{i_1^\alpha}(s_{\sigma(1)})\cdots m_{i_k^\alpha}(s_{\sigma(k)})$$

and $\mathcal{P}_k$ is the permutation group of $\{i_1^\alpha,\ldots,i_k^\alpha\}$. Denote $s_j^k = (s_1,\ldots,s_{j-1},s_{j+1},\ldots,s_k)$ and note that

$$E_\alpha(s^k) = \sum_{j=1}^k m_l(s_j)\, E_{\alpha(l)}(s_j^k).$$

Changing the order of integration we arrive at

$$\varphi_\alpha^i = \sum_{j=1}^k \int_0^t\int_0^{s_k}\cdots\int_0^{s_{j+1}}\int_0^{s_{j-1}}\cdots \tag{3.7}$$

$$\cdots\int_0^{s_2}\left(\int_{s_{j-1}}^{s_{j+1}} F_i(t;s^k)m_l(s_j)ds_j\right) E_{\alpha(l)}(s_j^k)ds_j^{k-1}$$

where $s_0 = 0, s_{k+1} = t$. For $m_0(s) = \frac{1}{\sqrt{t}}$ and $m_k(s) = \sqrt{\frac{2}{t}}\, cos(\frac{\pi k s}{t})$, integration by parts gives

$$\int_{s_{j-1}}^{s_{j+1}} F_i(t;s^k)\, m_l(s_j)ds_j$$

$$= F_i(t;s_1,\ldots,s_{j-1},s_{j+1},s_{j+1},\ldots,s_k)M_l(s_{j+1})$$

$$- F_i(t;s_1,\ldots,s_{j-1},s_{j-1},s_{j+1},\ldots,s_k)\, M_l(s_{j-1})$$

$$- \int_{s_{j-1}}^{s_{j+1}} \frac{\partial F_i(t;s^k)}{\partial s_j} M_l(s_j)ds_j \tag{3.8}$$

where $M_l(s) = \frac{\sqrt{2t}}{\pi l} sin\frac{\pi l s}{t}$. The summation over j of the first two terms of (3.8) cancels out all the terms except $-F_i(t;s_2,\ldots,s_k)M_l(0)E_{\alpha(l)}(s_1^k)$ and $F_i(t;s_1,\ldots,s_{k-1})M_l(t)E_{\alpha(l)}(s_k^k)$, which are both zero. So, we have

$$\varphi_\alpha^i(t) = \sum_{j=1}^k\int_0^t\int_0^{s_k}\cdots\int_0^{s_{j+1}}\int_0^{s_{j-1}}\cdots$$

$$\cdots\int_0^{s_2} f_{i,l}(t;s_j^k)\, E_{\alpha(l)}(s_j^k)\, ds_j^k,$$

where

$$f_{i,l}(t;s_j^k) = -\int_{s_{j-1}}^{s_{j+1}}\frac{\partial F_i(t;s^k)}{\partial s_j} M_l(s_j)\, ds_j. \tag{3.9}$$

Since $\{\alpha(l) : |\alpha| = k, d(\alpha) = l\} \subset \{\alpha : |\alpha| = k-1\}$,

$$\sum_{|\alpha|=k,d(\alpha)=l} \frac{|\varphi_\alpha(t)|_2^2}{\alpha!} \leq$$

$$\sum_{|\alpha|=k-1} \sum_i \sum_{j=1}^{k} \frac{k}{\alpha!} \left(\int_0^t \int_0^{s_k} \cdots \int_0^{s_{j+1}} \int_0^{s_{j-1}} \cdots \right.$$

$$\left. \cdots \int_0^{s_2} f_{i,l}(t; s_j^k)\, E_\alpha(s_j^k)\, ds_j^k \right)^2 .$$

Note that the summation over α in the RHS of the latter inequality gives

$$\int_0^t \int_0^{s_k} \cdots \int_0^{s_{j+1}} \int_0^{s_{j-1}} \cdots \int_0^{s_2} f_{i,l}^2(t; s_j^k)\, ds_j^k,$$

(see [3]). So

$$\sum_{|\alpha|=k,d(\alpha)=l} \frac{|\varphi_\alpha(t)|_2^2}{\alpha!} \leq k \sum_{j=1}^{k} \int_0^t \int_0^{s_k} \cdots$$

$$\cdots \int_0^{s_{j+1}} \int_0^{s_{j-1}} \cdots \int_0^{s_2} |f_{\cdot,l}(t; s_j^k)|_2^2 ds_j^k . (3.10)$$

Applying Holder's inequality in (3.9) we see that $|f_{i,l}(t; s_j^k)|^2$ is bounded by

$$\frac{c}{l^2} t(s_{j+1} - s_{j-1}) \int_{s_{j-1}}^{s_{j+1}} \left(\frac{\partial F_i(t; s^k)}{\partial s_j}\right)^2 ds_j .$$

Note that $\partial F_i(t; s^k)/\partial s_j =$

$$\left(T_{t-s_k} h \cdots h L T_{s_j - s_{j-1}} h T_{s_{j-1} - s_{j-2}} \cdots T_{s_1} p_0\right)_i$$
$$- \left(T_{t-s_k} h \cdots h L T_{s_{j+1} - s_j} h \cdots T_{s_1} p_0\right)_i , (3.11)$$

where L is the operator $(Lu)_i = \sum_j \lambda_{ji} u_j$. It is readily checked that L is a bounded linear map from l_1 to $l_1 \bigcap l_2$. Let $|Lu|_1, |Lu|_2 \leq \tilde{c}|u|_1$. Applying the inequality $|T_s v|_1 \leq |v|_1$ repeatedly in (3.11), we obtain

$$\sum_i \left| \frac{\partial F_i(t; s^k)}{\partial s_j} \right|^2 \leq 4\tilde{c}^2 |h|_\infty^{2k} .$$

Therefore

$$|f_{\cdot,l}(t; s_j^k)|_2^2 \leq \frac{c}{l^2}\, |h|_\infty^{2k}\, t^2\, (s_{j+1} - s_{j-1}).$$

Substituting into (3.10) and integrate, we arrive at

$$\sum_{|\alpha|=k,d(\alpha)=l} \frac{|\varphi_\alpha(t)|_2^2}{\alpha!} \leq \frac{c}{l^2} \frac{k(|h|_\infty^2 t)^{k-1}}{(k-1)!}\, t^3 .$$

This together with (3.6) and (2.7) gives the error bound in (2.8). □

4 References

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