ESTIMATION OF TEMPORALLY DISCONTINUOUS PARAMETERS IN GENERAL PARABOLIC EVOLUTION SYSTEMS

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Abstract. In this paper we present a unified convergence theory for estimating temporally discontinuous parameters in a general class of linear parabolic systems. We apply this theory to estimate parameters in the Euler-Bernoulli beam equation.

1. Introduction. Inverse or parameter identification problems arise in several contexts, including bioremediation of contaminated groundwater [13], in population biology problems [3], and in physical models for flexible sturctures [4, 5, 8, 9]. The inverse problems then consist of estimating these parameters, using data obtained from experimental observations. The goal of this paper is to present a general convergence and stability theory for approximation methods for the treatment of temporally discontinuous parameter identification problems involving distributed parameter systems.

General theory for parameter estimation in an abstract setting can be found in [7]. In that work (and its many references) one finds that key components in inverse problem analyses are continuity of the system state with respect to the parameter, compactness of parameter spaces, and convergence of numerical approximations that is uniform with respect to the parameters and consistent with the topology of the observation space. For general autonomous linear parabolic problems, the paper [6] contains the relevant analysis. The sesquilinear form approach contained therein provides a unified way to handle a wide variety of problems, with conditions that can be verified in a straightforward manner. In the paper [1] results which extend the framework of [6] to nonautonomous parabolic problems were established in order to allow general coverage of many problems, together with verifiable conditions on the sesqulinear form that determines the dynamics. In this paper we weaken one of the conditions imposed on the sesquilinear form in order to extend the results in [1] to allow the estimation of temporally discontiuous parameters.

Estimation of discontinuous parameters is crucial in certain applications. For example, the reproduction function of an individual in a population model depends is usually represented in terms of a discontinuous function

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of the form

$$\beta(t,y) = \begin{cases} 0 & y_B \le y < y_A \\ \beta(t,y) & y_A < y \end{cases}$$

where y_B is the birth size or age and y_A is the adult size or age (see for example, [12]). The stiffness of a beam in a flexible structure may decrease over a long period of time and could form a discontinuity at a certain time, due to aging. Fluctuations in water tables due to precipitation cause changes in groundwater velocity fields which in some cases must be determined from tracer movements. To analyze these situations, it is essential to generalize the existing theory to include time dependent parameter problems.

Our theory is based on the weak version of the system in terms of sesquilinear forms used in [6] and [1]. The theory depends on the following properties of the time and parameter ($q \in Q$) dependent sesquilinear form $\sigma(t,q)(\cdot,\cdot)$ describing the system: continuity with respect to the parameter, uniform boundedness (both in time and the parameter), and uniform coercivity in time and the parameter.

The paper is organized as follows. In Section 2, we present a theoretical framework for the approximation and applications of this theory to the Euler-Bernoulli beam equation is discussed in Section 3.

2. Approximation theory for identification problems. Let H be a Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and corresponding norm $|\cdot|$. Let V be a Hilbert space that is densely and continuously imbedded in H, with norm $\|\cdot\|$ and imbedding constant K : for each $\phi \in V$, we have $|\phi| \le K\|\phi\|$. We use these spaces to form a Gelfand triple structure $V \hookrightarrow H = H^* \hookrightarrow V^*$. We consider the following abstract differential equation on H

$$\begin{cases} \dot{u}(t,q) = A(t,q)u(t,q) + f(t,x,q) \\ \\ u(0,q) = u_0(q) \end{cases} \tag{2.1}$$

with parameter q belonging to the set

$$\begin{gathered} BV_\infty = \{f \in L^1([0,T],\tilde{Q}) : \\ f \in BV([0,T],Q) \cap L^\infty([0,T],Q), \\ \|f\|_{L^\infty([0,T],Q)} \le M_1 \text{ and } TV(f) \le M_2\} \end{gathered}$$

Where

$$TV(f) = sup \sum_{i=1}^{M} \|f(t_{i+1}) - f(t_i)\|_Q$$

the supremum taken over all finite partitions $0 \le t_1 < \cdots < t_M \le T$, with Q being compactly embedded in the normed linear space $\tilde{Q}$. It is well known that $BV([0,T],Q)$ is compactly embedded in the space $L^1([0,T],\tilde{Q})$ (see, [10]) hence since BV_∞ is a closed subset of $BV([0,T],Q)$ then it is compact in $L^1([0,T],\tilde{Q})$. The operator A is assumed to be determined by a time and parameter dependent sesquilinear form on V; i.e, $\sigma(\cdot,\cdot)(\cdot,\cdot) : [0,\infty) \times Q \times V \times V \to C$, where $\sigma(t,q)(\cdot,\cdot)$ is sesquilinear for each $t \in [0,\infty)$ and $q \in Q$. Concerning σ, we make the following assumptions

(Σ0) The function $\sigma(\cdot,q)(\phi,\psi)$ is measurable on $[0,\infty)$, for fixed $\phi,\psi \in V$ and $q \in BV_\infty$.

(Σ1)
There exists $K_0 > 0 \ni |\sigma(t,q)(\phi,\psi)| \le K_0\|\phi\| \cdot \|\psi\| \; \forall \phi,\ \psi \in V,\ q \in BV_\infty$ uniformly in t on each interval $[0,T]$.

(Σ2) There exists $c_0 > 0,\ \lambda_0 \in R \ni \sigma(t,q)(\phi,\phi) + \lambda_0|\phi|^2 \ge c_0\|\phi\|^2,\ \forall \phi \in V,\ q \in BV_\infty$ uniformly in t on each interval $[0,T]$.

(Σ3) For $q, \hat{q} \in BV_\infty$, a.e. $t \geq 0$ and all $\phi, \psi \in V$, we have that

$$|\sigma(t,q)(\phi,\psi) - \sigma(t,\hat{q})(\phi,\psi)| \\ \leq \|q(t) - \hat{q}(t)\|_{\widetilde{Q}}\|\phi\| \cdot \|\psi\|.$$

Under these assumptions there exists a family of uniquely determined linear operators $A(t,q) : \mathrm{dom}(A(t,q)) \to H$, with dense domains, satisfying $\sigma(t,q)(\phi,\psi) = \langle -A(t,q)\phi, \psi\rangle$, $\forall\phi \in \mathrm{dom}\ A(t,q)$, $\psi \in V$.

Our main goal in this paper is a convergence theory for least squares based parameter estimation. Toward that end, we next consider an approximation method based on a sequence of Hilbert spaces H^N, $N = 1, 2, \ldots$, with orthogonal projections $P^N : H \to H^N$. The following assumption about these approximations will be needed for our convergence results.

(A1) The subspaces H^N are subsets of V, and $\forall v \in V$, we have that $\|P^N v - v\| \to 0$.

This assumption is satisfied by many finite element and spectral schemes (see [7, 11, 14]). The Galerkin approach to approximation involves restricting $\sigma(t,q)$ to $H^N \times H^N$, yielding bounded linear operators $A^N(t,q)$ satisfying

$$\sigma(t,q)(\phi^N, \psi^N) = -\langle A^N(t,q)\phi^N, \psi^N\rangle.$$

Using the above assumptions the following theorem has been proved in [2].

Theorem 2.3. Suppose that (Σ0) – (Σ3), and **(A1)** hold, and that $q^N \to q$ in $L^1([0,T],\widetilde{Q})$. Then we have that $u^N(t,q^N) \to u(t,q)$, in H, uniformly on [0,T].

We have thus obtained, based on the assumptions given above, that $u^N(t;q^N) \to u(t;q)$ in H, when $q^N \to q$ in $L^1([0,T],\widetilde{Q})$. To put this result into the context of least squares estimation, we consider a continuous map $\mathcal{C}: H \to Z$, where Z is a normed linear space. Given $z \in Z$, one determines an appropriate parameter value for the system by minimizing

$$J(q) = \|\mathcal{C}u(q) - z\|^2.$$

The continuous dependence results above indicate that a minimizer exists within the compact set BV_∞.

In order to compute minimizers, we must make some approximations. Approximation u^N of the state variable u, as discussed above, lead to a cost functional

$$J^N(q) = \|\mathcal{C}u^N(q) - z\|_Z^2$$

to be minimized. The convergence results of the previous sections guarantee that if $q^N \to q$ then $J^N(q^N) \to J(q)$, which will give us (as in [7]) subsequential convergence of minimizers.

Below we discuss an example, illustrating the application of this general theory.

3. Euler-Bernoulli beam equation with Kelvin-Voigt damping

The following example indicates the straightforward manner in which the assumptions above may be verified.

We consider the following equation:

$$\begin{gathered} y_{tt} + (EIy_{xx} + c_D I y_{xxt})_{xx} = f(t,x) \\ y(t,0) = y_x(t,0) = 0 \\ EIy_{xx} + c_D I y_{xxt}(t,l) = 0, \\ (EIy_{xx}(t,l) + c_D I y_{xxt}(t,l))_x = 0 \\ y(0,x) = y_0(x), \qquad y_t(0,x) = y_1(x) \end{gathered} \tag{3.4}$$

This equation describes the transverse vibrations of a cantilevered Euler-Bernoulli beam with Kelvin-Voigt internal damping. The state variable $u(t,x)$ is the displacement along the beam at time t at position x. The parameter $EI(t,x)$ is the stiffness coefficient, and $c_D I(t,x)$ is the Kelvin-Voigt damping coefficient which reflects the assumption that the bending moment depends not only on the strain, but also on the strain rate as well. The function f represents external distributed forces applied to the beam. For further details on this model, see [5].

The equation above may be written as a first-order system in the usual way: Define $w(t,x) = [y(t,x), \dfrac{\partial y}{\partial t}(t,x)]^T$ and $\hat{F} = [0, f(t,x)]^T$. Then denoting $\frac{\partial}{\partial x^2}$ by D^2, we see that (3.4) is equivalent to

$$w_t = \mathcal{A}(t,q)w(t,x) + \hat{F}(t,x), \qquad (3.5)$$

where (formally)

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -D^2(EID^2) & D^2(c_D I D^2) \end{bmatrix}$$

with the initial condition $w(0) = w_0 = (y_0, y_1)$.

To write equation (3.5) in a weak formulation, we define the following spaces

$$H_L^2(0,l) = \{u \in H^2(0,l) | u(0) = u_x(0) = 0\}$$
$$H = H_L^2(0,l) \times L^2(0,l)$$
$$V = H_L^2(0,l) \times H_L^2(0,l).$$

The inner product on the $L^2(0,l)$ will be denoted by $(\cdot,\cdot)$ and for the space $H_L^2(0,l)$ we use the inner product

$$\langle\langle \phi, \psi \rangle\rangle = (\phi_{xx}, \psi_{xx})$$

and the associated norm $|||\cdot|||$ (which is equivalent to the usual $H_L^2(0,l)$ norm by Poincare's inequality). Finally, the inner products for the spaces H, V will be taken to be the usual product space inner products, and will be denoted as in the abstract formulation $\langle\cdot,\cdot\rangle$ and $|\cdot|$, and $\langle\cdot,\cdot\rangle_V$ and $\|\cdot\|$ respectively. Then as in [6] the weak form of the beam can be written in terms of the following sesquilinear form: with $w = (u,v)$ and $\mathcal{X} = (\phi,\psi)$ elements of V, define

$$\sigma(t,q)(w,\mathcal{X}) = -\langle\langle v, \phi \rangle\rangle + \sigma_1(t,q)(u,\psi) + \sigma_2(v,\psi)$$

Where

$$\sigma_1(t,q)(u,\psi) = (EIu_{xx}, \psi_{xx}),$$
$$\sigma_2(t,q)(v,\psi) = (c_D I v_{xx}, \psi_{xx}).$$

Then with $w = (y, \dot{y})$, the weak form of equation (3.4) can be written

$$\langle \dot{w}(t), \mathcal{X}\rangle + \sigma(t,q)(w(t),\mathcal{X}) = \langle \hat{F}(t), \mathcal{X}\rangle,$$

$$\text{for } \mathcal{X} \in V$$

In terms of the abstract formulation developed in the previous section we will set $q = (EI, c_D I)$, $\Omega = (0,l)$,

$$\tilde{Q} = L^\infty(\Omega) \times L^\infty(\Omega)$$

and,

$$\begin{aligned} Q \;=\; \{ &(EI, c_D I) \in (C^{0,1}(\overline{\Omega}))^2 : \\ & c_0 \le EI \le c_1, \\ & c_2 \le c_D I \le c_3 \\ & |EI|_{C^{0,1}} \le c_4,\ |c_D I|_{C^{0,1}} \le c_5 \\ & c_0, c_1, c_2 \text{ and } c_3 > 0 \} \end{aligned}$$

Here the space $C^{0,1}(\overline{\Omega})$ is the space of Lipschitz continuous functions with norm

$$\|f\|_{C^{0,1}} = \sup_{\Omega} |f(x)| + \sup_{x,y \in \Omega\, x \neq y} \left| \frac{f(x) - f(y)}{x - y} \right|.$$

Using same arguments as in former example one can establish the compactness of the set Q in $\tilde{Q}$. Standard arguments and the fact that $EI(t,x) \geq c_0$ and $c_D I(t,x) \geq c_2$ easily verify the assumptions $(\Sigma 0) - (\Sigma 2)$. For verification of $(\Sigma 3)$ we let $w = (u,v), \mathcal{X} = (\phi, \psi) \in V$, and suppose that $q^N \to q$ in $L^1([0,T], \tilde{Q})$. Then we have that

$$\begin{aligned}
&|\sigma(t,q)(w,\mathcal{X}) - \sigma(t,q^N)(w,\mathcal{X})| \\
&\quad \leq |\sigma_1(t,q)(u,\psi) - \sigma_1(t,q^N)(u,\psi)| \\
&\quad + |\sigma_2(t,q)(v,\psi) - \sigma_1(t,q^N)(v,\psi)| \\
&\quad \leq \|EI(t) - EI^N(t)\|_\infty \left(\int_0^l |u_{xx}|^2 dx \right) \\
&\qquad \left(\int_0^l |\psi_{xx}|^2 dx \right)^{\frac{1}{2}} + \\
&\quad \|c_D I(t) - c_D I^N(t)\|_\infty \left(\int_0^l |v_{xx}|^2 dx \right)^{\frac{1}{2}} \\
&\qquad \left(\int_0^l |\psi_{xx}|^2 dx \right)^{\frac{1}{2}} \\
&\quad \leq \|q^N(t) - q(t)\|_{\tilde{Q}} \|w\| \|\mathcal{X}\|
\end{aligned}$$

We will point out that the above choice of $\tilde{Q}$ and Q is not an optimal one in the sense that if one considers $\tilde{Q} = L^1(0,l)$ and $Q = \{f \in L^1(0,l) : \|f\|_{L^\infty(0,l)} \leq C_1 \text{ and } TV(f) \leq C_1\}$ and a modification of $(\Sigma 3)$ then it can be shown that a covergence proof holds for parameters that are discontinuous both in time and space (see [2]).

To illustrate the computational methods analyzed herein, we estimated the stiffness parameter EI as a function of time from computationally generated data. Our FORTRAN program uses cubic B-splines to approximate the solution of the Euler Bernoulli differential equation, and we used piewise constant functions to estimate the parameter. In the computations given below, we use a beam of length 1, with $\rho = 1, \gamma = .01, c_D I = .01$. We used 15 cubic B-splines for the computations. For the generated data, we used for EI the function

$$EI(t) = \begin{cases} 12 & t < 0.35 \\ 10 & t \geq 0.35 \end{cases}$$

which is constant with respect to the spatial variable. The initial displacement and velocity are taken to be 0, and the forcing function is given by

$$f(t,x) = 100 \sin(5\pi t) * \frac{1}{.01} \chi_{[.495,.505]}(x),$$

which approximates a δ function in the spatial variable.

For data, we sampled the displacement $u(t_i, x = 1)$ at 200 uniformly spaced time points t_i in the time interval $[0,1]$, as generated with the above model. In order to examine the behavior of the least squares identification procedure, we used as data the actual model generated signal, as well as the signal modified by Gaussian noise: $z_i = u(t_i, 1) * (1 + \epsilon_i)$, was used for data, with ϵ_i a random sample from a zero mean Gaussian random number generator. We used $\sigma = .01$ and $\sigma = .1$ for standard deviations for the noise.

Our identification algorithm was given the known values of $\rho, \gamma, c_D I$ and f, and was used to estimate $EI(t)$ using using 10 piecewise constants functions. In order to implement the above mentioned compactness constraints, we used a penalized (or regularized) least squares cost of the form

$$\begin{aligned}
J(EI) &= \sum_{i=1}^{200} |z_i - u(t_i, 1; EI)|^2 \\
&\quad + \beta \int_0^1 \sqrt{|\dot{EI}(t)|^2 + \alpha}\ dt
\end{aligned}$$

with α small positive constant and two different choices of β, depending on the noise level. Note that the integral term is (at least for smooth EI) a differentiable approximation to the total variation of EI.

In Figure 1, we see the results of minimizing J_β with 10 (dotted-line) piecewise constant functions. The true EI function is the solid line. We used the constant function $EI = 12$ as our initial guess in the optimization, which was carried out using the package `lmdif1` from `netlib`. The regularization parameter β used was 10^{-4} and $\alpha = 10^{-5}$.

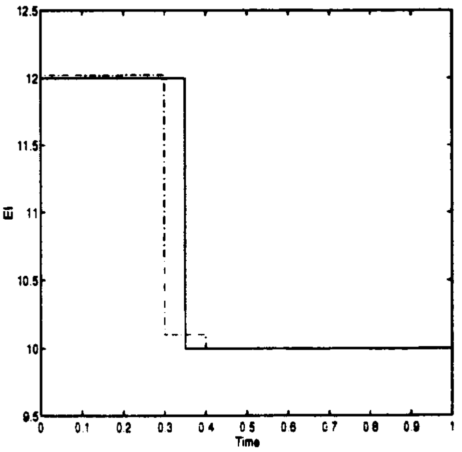

Fig1. EI estimates using 10 step fuctions, no noise in the data.

Figure two represents the same procedure as Figure 1 when the data was corrupted by the above described noise. The solid line is again the true EI; the dashed line is the estimate with $\sigma = .01$; and the dot-dash line, $\sigma = .05$. The regularization parameter values used for the two estimation runs were $\alpha = 10^{-5}$, $\beta = 10^{-4}$ and $\beta = 10^{-3}$, respectively.

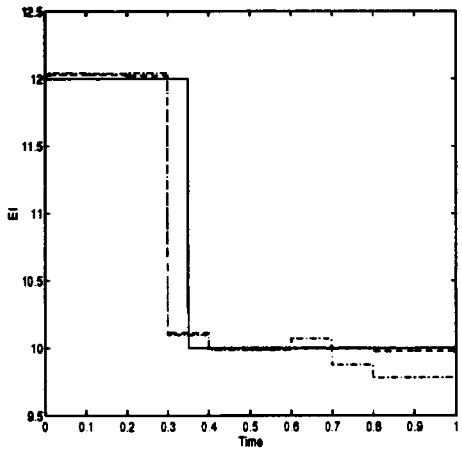

Fig 2. EI estimates using 10 step functions with two levels of noise using standard deviation $\sigma = 0.01, 0.05$

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