ORDER REDUCTION OF LIGHTLY DAMPED SYSTEMS BY APPROXIMATED BALANCED REALIZATION

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ABSTRACT

A method of approximating the controllability gramian, observability gramian and the balancing transformation for lightly damped mechanical systems is presented. the approximation uses the special structure of the system i.e. the positive definitness of the inertia, damping and stiffness matrices, and the fact that the damping is small to reduce the amount of computation considerably. Furthermore, in one variation of the method, one can avoid the calculation of the entire balancing transformation matrix and calculate only the parts that are required for order reduction. In cases where the reduced order is much smaller than the original that leads to another substantial reduction of computation effort.

INTRODUCTION

The problem of approximating a high-order, linear, time invariant dynamic system by a lower order model is one of the fundamental problems of system theory and has received renewed interest in the last decade. Several approaches to the problem were suggested. The method that represents the beginning of the new era in model order reduction is the truncated balanced realization method (Moore, 1982). In this method a state transformation is used to obtain a realization with controllability and observability gramians which are diagonal and equal. This identifies the strong modes of the system which are retained while other modes are truncated. The method is heuristic and formally does not incorporate any explicit criterion. However, it can be shown that it is closely related to L_2 minimization, i.e. quadratic error criterion. This criterion is optimized in the optimal projection method (Hyland and Bernstein, 1985) and its extensions, e.g. (Halevi, 1992). It should be noted that in many cases the truncated balanced realization method results in reduced order models that are very close to the optimal ones. Another method that uses a similar criterion is the component cost analysis (Skelton and Yousouff, 1983).

In undamped or lightly damped structures the most common method of order reduction is modal truncation. Mathematically this is a special case of the method of partial fraction expansion. In general, i.e. in systems that may be overdamped or with large damping factor, this method does not yield good approximations. However, for undamped systems with disjoint natural frequencies it gives the optimal approximation, and for lightly damped systems good approximation where the accuracy depends on the level of damping and the distance between the natural frequencies.

There is a considerable volume of works dealing with the properties of the truncated balanced realization method, e.g. (Enns, 1984, Kabamba, 1985, Moore, 1982, Pernebo and Silverman, 1983) and its usefulness seems to be evident. The computational aspects have also been considered and efficient and reliable numerical algorithms for a general system were presented (Laub et al, 1987, Safonov and Chiang, 1988). Nevertheless the main problem in the application of the method to structures seems to be the computational burden. The steps that are involved in this method are

(i) Calculating the controllability and observability gramians.
(ii) Calculating the balancing transformation and the balanced realization.
(iii) Truncation of the balanced realization.

The orders of models of structures can be as large as hundreds of thousands and steps (i) and (ii) in such cases may require unacceptably long computation time. Another source of difficulty is the fact that due to the small damping, the system dynamics contains terms which may be several orders of magnitude apart.

Several works deal with the application of the truncated balanced realization to structures (Mottershead and Friswell. 1993, Williams, 1990 and 1994) and the algorithms there exploit some properties particular to those systems. The main difference between those methods and the method that is proposed in this paper is that they consider the exact solution of the balancing problem while we look for an *approximate* one.

The main idea in the suggested method is to express the lightly damped system as a small perturbation from the undamped. The analysis that follows attempts to calculate only the nonnegligible terms of the gramians and the balancing transformation and thus a substantial reduction in the computation is achieved.

The material is organized as follows. Section 2 contains the mathematical statement of the problem and some preliminary results. Section 3 deals with the calculation of the gramians and section 4 with the balancing transformation. An example that demonstrates the use of the approximated method is given in section 5. Section 6 summarizes the results of the paper.

2. PROBLEM STATEMENT AND PRELIMINARIES

We consider the system

$$M\ddot{z} + C\dot{z} + Kz = Fu \tag{1}$$

$$y = H_v \dot{z} \tag{2}$$

where $z \in R^N$ is a vector of generalized coordinates and $u \in R^m$ is a vector of generalized forces, i.e. forces and displacements that enter the system via stiffness elements. The output $y \in R^r$ is defined as some linear combinations of the generalized velocities. The inertia matrix M is symmetric and positive definite and for the sake of simplicity we assume that the symmetric stiffness matrix K is positive definite as well. That means that there are no rigid body degrees of freedom, or, if the original system has them, our system is defined as the deviation from the rigid body motion. The damping matrix C is nonnegative definite and may be singular. However it is assumed that the system is asymptotically stable, i.e. all the solutions of

$$\det\left(Ms^2 + Cs + K\right) = 0 \tag{3}$$

have strictly negative real part. A complete analysis of the conditions that M, K and C should satisfy for that may be found in (Bernstein and Bhat, 1994). We just mention that these conditions are mild and are practically satisfied for most structures with $C \geq 0$. Since we are interested in systems with light damping we consider the case where $\|C\| << (\|M\| \|K\|)^{1/2}$, and use the parametrization

$$C = \alpha C_0 \tag{4}$$

where $1 >> \alpha > 0$ is a scalar and $\|C_0\|$ is (roughly) of the order of magnitude $(\|M\| \|K\|)^{1/2}$. We will show later that the exact choice of α does not change the results as long as it is sufficiently small and so is $\|C\|$.

Defining the state vector $x = [z^T \ \ \dot{z}^T]^T$, the state space realization of the system which has an order $n = 2N$ is given as

$$\dot{x} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -\alpha M^{-1}C_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix} u \tag{5}$$

$$y = [0 \quad H_v] x \tag{6}$$

Or, with obvious notation

$$\dot{x} = Ax + Bu \tag{7}$$

$$y = Hx \tag{8}$$

It is assumed that (A,B,H) is minimal, i.e. controllable and observable. The controllability gramian of the system is defined as

$$Q \stackrel{\Delta}{=} \int_0^\infty e^{At} B B^T e^{A^T t} dt \tag{9}$$

and it satisfies the Lyapunov equation

$$AQ + QA^T + BB^T = 0 \tag{10}$$

Similarly the observability gramian is given by

$$P \stackrel{\Delta}{=} \int_0^\infty e^{A^T t} H^T H e^{At} dt \tag{11}$$

or

$$PA + A^T P + H^T H = 0 \tag{12}$$

Suppose one uses a state transformation $x = Vx'$ then $(A,B,H) \rightarrow (V^{-1}AV,\ V^{-1}B,\ HV)$ and the new gramians are given by

$$Q' = V^{-1} Q (V^{-1})^T \quad ; \quad P' = V^T P V \tag{13}$$

It was shown by Moore (1982) that there exists a transformation Vb such that the resulting gramians are diagonal and equal

$$Q_b = P_b = \mathrm{diag}\{G_i\} \qquad \sigma_1 \geq \sigma_2 \quad \geq \ldots\ldots \geq \sigma_n > 0 \tag{14}$$

Such realizations are called balanced. Let the new realization (A_b, B_b, H_b) be

$$A_b = \begin{bmatrix} A_{b_{11}} & A_{b_{12}} \\ A_{b_{21}} & A_{b_{22}} \end{bmatrix} \;;\quad B_b = \begin{bmatrix} B_{b_1} \\ B_{b_2} \end{bmatrix} \;;$$

$$H_b = [H_{b1} \quad H_{b2}] \tag{15}$$

where the partitioning is conformal, i.e. the dimensions of A_{b11}, B_{b1} and H_{b1} are $n_r \times n_r$, $n_r \times m$ and $r \times n_r$ respectively for some $n_r < n$. Then it is well known (Moore, 1982) that $(A_{b11}, B_{b1}, H_{b1})$ is a good (yet not optimal in any sense) n_rth order approximation of (A, B, H).

The first step in the derivation is the calculation of the modal form of the undamped system (M, K). To simplify the analysis we assume that the natural frequencies of the system are distinct. Let Ω be the diagonal matrix of the natural frequencies and T the mass normalized modal matrix. Then we have

$$T^T M T = I_N \;,\quad T^T K T = \Omega^2 \tag{16}$$

Using the transformation $z = T\xi$ and premultiplying eq. (1) by T^T we have

$$\ddot{\xi} + T^T C T \dot{\xi} + \Omega^2 \xi = T^T F u \tag{17}$$

and the state space realization

$$\dot{x} = \begin{bmatrix} 0 & I \\ -\Omega^2 & -\alpha T^T C_o T \end{bmatrix} x + \begin{bmatrix} 0 \\ T^T F \end{bmatrix} u \tag{18}$$

$$y = [0 \;\; HT] x \tag{19}$$

This realization can be obtained from (5) - (6) by the state transformation

$$V = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \tag{20}$$

and using the relationships (16). The common assumption of proportional damping will make $T^T C T$ diagonal as well however we do not make it and $T^T C_o T$ is considered to be a full symmetric matrix. An important observation which is a consequence of the stability of the system is that all the diagonal elements of $T^T C_o T$ are strictly positive. In the sequel we will develop approximate expressions for the gramians and the balancing transformation that take into account the special structure of the system and in particular the fact that α is small.

As the last preliminary result, we present the following two lemmas. These results are very simple (and probably might be found, one way or another, in linear algebra texts) but are used several times in the derivation.

Lemma 2.1: Let L_1 and L_2 be two symmetric matrices and let D be a diagonal matrix with distinct nonzero diagonal elements. Then $L_1 = DL_2$ (or $L_1 = L_2 D$) only if L_1 and L_2 are both diagonal

Proof: the (i,j) and the (j,i) elements of L_1 are given by

$$L_1(i,j) = D_i L_2(i,j)$$
$$L_1(j,i) = D_j L_2(j,i)$$

But $L_1(i,j) = L_1(j,i)$, $L_2(i,j) = L_2(j,i)$ and $D_i \quad D_j$. Hence $L_1(i,j) = 0 \quad i \quad j$ and the same holds for L_2.

Lemma 2.2: Let D be a diagonal matrix with distinct nonzero diagonal elements. Then the only solution of

$$L + L^T = 0$$
$$DL + L^T D = 0$$

is $L = 0$.

Proof: The (i,j) elements of the equations are

$$L(i,j) + L(j,i) = 0$$
$$D_i L(i,j) + D_j L(j,i) = 0$$

Since $D_i \quad D_j$ the two equations are independent and the trivial solution is the only solution. The (i,i) elements are zero because of the skew symmetry of L.

3. CALCULATION OF THE GRAMIANS.

3.1 Controllability Gramian.

We start with the realization (18) - (19) which we write as

$$\dot{x} = (A_o + \alpha A_1) x + B u \tag{21}$$

$$y = H x \tag{22}$$

where

$$A_o = \begin{bmatrix} 0 & I \\ -\Omega^2 & 0 \end{bmatrix},\; A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\alpha \overline{C}_o \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ \overline{F} \end{bmatrix},\; H = [0 \;\; \overline{H}] \tag{23}$$

and

$$\overline{C}_o = T^T C_o T\ ,\quad \overline{F} = T^T F\ ,\quad \overline{H} = H_v T \tag{24}$$

The Lyapunov equation which determines the controllability gramian Q is

$$(A_o + \alpha A_1)Q + Q(A_o + \alpha A_1)^T + BB^T = 0 \tag{25}$$

Notice that the gramian exists only for asymptotically stable systems. A_o is the matrix of the undamped system and all of its eigenvalues are on the imaginary axis, i.e. it is not asymptotically stable. Therefore the Lyapunov equation for $\alpha = 0$ does not have a solution. Since we assume that the damped system is stable, a solution exists for all $\alpha>0$. To approximate the solution for small α we use the following structure

$$Q = \alpha^{-1} Q_1 + Q_0 \tag{26}$$

Substituting it into eq. (25) and equating like powers of α we get for α^{-1} and α^0

$$A_o Q_1 + Q_1 A_o^T = 0 \tag{27}$$

$$A_o Q_o + A_1 Q_1 + Q_o A_o^T + Q_1 A_1^T + BB^T = 0 \tag{28}$$

We partition Q_1 and Q_o as

$$Q_1 = \begin{bmatrix} q_1 & q_{12} \\ q_{12}^T & q_2 \end{bmatrix};\quad Q_o = \begin{bmatrix} \overline{q}_1 & \overline{q}_{12} \\ \overline{q}_{12}^T & \overline{q}_2 \end{bmatrix}$$

Then eq. (27) can be written as

$$\begin{bmatrix} q_{12} + q_{12}^T & q_2 - q_1\Omega^2 \\ q_2 - \Omega^2 q_1 & -\Omega^2 q_{12} - q_{12}^T\Omega^2 \end{bmatrix} = 0$$

From Lemmas 2.1 and 2.2 it follows immediately that $q_{12} = 0$ and that q_1 and q_2 are diagonal. Using the fact that $q_{12} = 0$ eq. (28) can now be written as

$$\begin{bmatrix} \overline{q}_{12} + \overline{q}_{12}^T & \overline{q}_2 - \overline{q}_1\Omega^2 \\ \overline{q}_2 - \Omega^2\overline{q}_1 & -\Omega^2\overline{q}_{12} - \overline{q}_{12}^T\Omega^2 - \overline{C}_o q_2 - q_2\overline{C}_o + \overline{F}\overline{F}^T \end{bmatrix} = 0 \tag{29}$$

The (1,1) block implies that $\overline{q}_{12}$ is skew symmetric, which in turn implies that the diagonal elements of $\Omega^2\overline{q}_{12} + \overline{q}_{12}^T\Omega^2$ are zero. So by looking at the diagonal entries of the (2,2) block we find that

$$(q_2)_{ii} = \left(\overline{F}\overline{F}^T\right)_{ii} / 2(\overline{C}_o)_{ii} \tag{30}$$

This completely determines q_2. q_1, which is diagonal as well, is given by

$$(q_1)_{ii} = (q_2)_{ii} / \omega_i^2 \tag{31}$$

To simplify the notation in the forthcoming derivation we will use

$$f_i \triangleq (\overline{F}\overline{F}^T)_{ii},\quad c_i = (\overline{C}_o)_{ii} \tag{32}$$

Substituting $\overline{q}_{12}^T = -\overline{q}_{12}$ into the (2,2) block we obtain

$$\overline{q}_{12}\Omega^2 - \Omega^2\overline{q}_{12} = \overline{C}_o q_2 + q_2\overline{C}_o - \overline{F}\overline{F}^T \tag{33}$$

or

$$(\overline{q}_{12})_{ij} = (\overline{C}_o q_2 + q_2\overline{C}_o - \overline{F}\overline{F}^T)_{ij} / (\omega_j^2 - \omega_i^2) \tag{34}$$

A more explicit formula is

$$(\overline{q}_{12})_{ij} = \left[(\overline{C}_o)_{ij}((q_2)_{ii} + (q_2)_{jj}) - (\overline{F}\overline{F}^T)_{ij}\right] / (\omega_j^2 - \omega_i^2) \tag{35}$$

Since $\overline{q}_{12}$ is skew symmetric only the upper triangular part ($j>i$) needs to be calculated. Eq. (29) does not fully define $\overline{q}_1$ and $\overline{q}_2$ and there are infinitely many diagonal matrices satisfying this equation. However, to minimize the error of our approximated solution we set $\overline{q}_1 = \overline{q}_2 = 0$. This point will be discussed later.

To summarize, the solution algorithm is as follows

1) Calculate the diagonal q_2 from eq. (30) (N elements)
2) Calculate the diagonal q_1 from eq. (31) (N elements)
3) Calculate the skew symmetric $\overline{q}_{12}$ from eq. (35). $(N \cdot (N-1)/2$ elements$)$.

Q is then given by

$$Q \cong \begin{bmatrix} \alpha^{-1} q_2 & \overline{q}_{12} \\ \overline{q}_{12}^T & \alpha^{-1} q_2 \end{bmatrix} \tag{36}$$

If one is interested in the gramian of the original system (5)-(6) then the transformation is $q_1 \to Tq_1T^T$, $q_2 \to Tq_2T^T$ $\overline{q}_{12} \to T\overline{q}_{12}T^T$ and all the zero submatrices remain unchanged. Also the skew symmetry of $\overline{q}_{12}$ does not change by the congruent transformation.

3.2 Error Analysis

We can analyze the error associated with the approximation (36) from two points of view: equation error and solution error. In general the equation error is

$$E = \alpha(A_1Q_o + Q_oA_1^T) \tag{37}$$

Assuming that $\overline{q}_1$ and $\overline{q}_2$ are nonzero we get

$$E = \alpha\begin{bmatrix} 0 & \overline{q}_{12}\overline{C}_o \\ \overline{C}_o\overline{q}_{12}^T & \overline{C}_o\overline{q}_2 + \overline{q}_2\overline{C}_o \end{bmatrix} \tag{38}$$

The choice $\overline{q}_2 = 0$ (which implies $\overline{q}_1 = 0$) minimizes the norm of the error. A more meaningful approach is to look at the error of the solution. For analysis purposes we add one more term to the series representation of Q.

$$Q \cong \alpha^{-1}Q_1 + Q_o + \alpha Q_\alpha \tag{39}$$

The coefficient matrices of α^{-1} and α^{0} are made zero by the choice of Q_0 and Q_1. Partitioning

$$Q_\alpha = \begin{bmatrix} \hat{q}_1 & \hat{q}_{12} \\ \hat{q}_{12}^T & \hat{q}_2 \end{bmatrix} \tag{40}$$

and equating the α term to zero we get (assuming for the moment general $\overline{q}_1$ and $\overline{q}_2$).

$$\begin{bmatrix} \hat{q}_{12} + \hat{q}_{12}^T & \hat{q}_{12} - \hat{q}_1\Omega^2 - \overline{q}_{12}\overline{C}_o \\ \hat{q}_2 - \Omega^2\hat{q}_1 - \overline{C}_o\overline{q}_{12}^T & -\Omega^2\hat{q}_{12} - \hat{q}_{12}^T\Omega^2 - \overline{C}_o\overline{q}_2 - \overline{q}_2\overline{C}_o \end{bmatrix} = 0 \tag{41}$$

$\hat{q}_{12}$ is skew symmetric hence the diagonal of $-\Omega^2\hat{q}_{12} - \hat{q}_{12}^T\Omega^2$ is zero. The diagonal terms of $\overline{C}_o$ are non zero, therefore $\overline{q}_2$ must be zero. This leads to two conclusions. First, that the choice $\overline{q}_1$ and $\overline{q}_2$ both equal to zero which was made earlier is justified. Secondly, $\overline{q}_2 = 0$ makes $\hat{q}_{12}$ zero as well so the nonzero terms of order α reside only in the (1,1) and (2,2) submatrices which according to our calculation have order of α^{-1}. The error is therefore two order of α magnitude smaller. Actually $\hat{q}_1$ and $\hat{q}_2$ can be calculated except the diagonal elements which are less important since they are added directly to terms of order α^{-1}. Hence one can find an even more accurate approximation where the error is of order α^2.

3.3 Observability Gramian.

Similarly to what we did in the previous subsection we write P as

$$P \cong \alpha^{-1}P_1 + P_o \tag{42}$$

and the Lyapunov equation becomes

$$\left(\alpha^{-1}P_1 + P_o\right)\left(A_o + \alpha A_1\right) + \left(A_o + \alpha A_1\right)^T\left(\alpha^{-1}P_1 + P_o\right) + H^TH = 0 \tag{43}$$

Equating like powers of α we get for α^{-1} and α^{0}

$$P_1A_o + A_o^TP_1 = 0 \tag{44}$$

$$P_oA_o + P_1A_1 + A_o^TP_o + A_1^TP_1 + H^TH = 0 \tag{45}$$

As before P_1 and P_o are partitioned as

$$P_1 = \begin{bmatrix} p_1 & p_{12} \\ p_{12}^T & p_2 \end{bmatrix} \; ; \; P_o = \begin{bmatrix} \overline{p}_1 & \overline{p}_{12} \\ \overline{p}_{12}^T & p_2 \end{bmatrix}$$

Eq. (44) becomes

$$\begin{bmatrix} -p_{12}\Omega^2 - \Omega^2 p_{12}^T & \overline{p}_1 - \Omega^2\overline{p}_2 \\ \overline{p}_1 - \overline{p}_2\Omega^2 & \overline{p}_{12}^T + \overline{p}_{12} - p_2\overline{C}_o - \overline{C}_o p_2 + \overline{H}^T\overline{H} \end{bmatrix} = 0 \tag{46}$$

and as for Q_1 we get that $p_{12} = 0$ and that p_1 and p_2 are diagonal. The partitioned form of eq. (45) is

$$\begin{bmatrix} -p_{12}\Omega^2 - \Omega^2 p_{12}^T & \overline{p}_1 - \Omega^2\overline{p}_2 \\ \overline{p}_1 - \overline{p}_2\Omega^2 & \overline{p}_{12}^T + \overline{p}_{12} - p_2\overline{C}_o - \overline{C}_o p_2 + \overline{H}^T\overline{H} \end{bmatrix} = 0 \tag{47}$$

From the (1,1) block it follows that $\overline{p}_{12}\Omega^2$ is skew symmetric so that the diagonal of $\overline{p}_{12}$ is zero. Hence

$$(p_2)_{ii} = h_i/2c_i \tag{48}$$

where h_i is the (i,i) element of $\overline{H}^T\overline{H}$. p_1 is obtained immediately from

$$(p_1)_{ii} = \omega_i^2 h_i / 2c_i \tag{49}$$

Premultiplying the (2,2) sub block of (47) by Ω^2 and noting that $\Omega^2\overline{p}_{12}^T = -\overline{p}_{12}\Omega^2$ we get

$$\Omega^2\overline{p}_{12} - \overline{p}_{12}\Omega^2 = \Omega^2\left(p_2\overline{C}_o + \overline{C}_o p_2 - \overline{H}^T H\right) \tag{50}$$

Hence for $i \neq j$ $((\overline{p}_{12})_{ii} = 0)$

$$(\overline{p}_{12})_{ij} = \omega_i^2\left[\left(\overline{C}_o\right)_{ij}\left((p_2)_{ii} + (p_2)_{jj}\right) - \left(H^TH\right)_{ij}\right] / (\omega_i^2 - \omega_j^2) \tag{51}$$

$\overline{p}_{12}$ is not skew symmetric however the (i,j) and (j,i) terms differ only in the ω_i^2 term that multiplies the rest of the expression so the amount of calculation is not far from that of the skew symmetric $\bar{q}_{12}$. Using arguments similar to the controllability gramian case we set $\bar{p}_1 = \bar{p}_2 = 0$. The solution algorithm for P is similar to that of Q and involves eqs (48), (49) and (51), and it is given by

$$P \cong \begin{bmatrix} \alpha^{-1}p_1 & \bar{p}_{12} \\ \bar{p}_{12}^T & \alpha^{-1}p_2 \end{bmatrix}$$

4. BALANCING

In section 3 approximate expressions were derived for the controllability and observability gramians for the case of light damping, i.e. $\alpha << 1$. At this point one can balance those matrices exactly using any standard method to get the balanced realization of the system. However, the motivation for the approximations was to reduce the amount of computational and to continue that we present now an approximation of the balancing transformation.

As a first step we employ the state transformation

$$V_{b1} = \begin{bmatrix} \Omega^{-1}T_{b1} & 0 \\ 0 & T_{b1} \end{bmatrix} \tag{52}$$

where

$$T_{b1} = \text{diag}\left\{ \left(\frac{f_i}{h_i} \right)^{1/4} \right\} \tag{53}$$

Then, from the expressions for q_1, q_2, p_1 and p_2 we get

$$T_{b1} = \text{diag}\left\{ \left(\frac{f_i}{h_i} \right)^{1/4} \right\} \tag{54}$$

$$P_{b1} = \begin{bmatrix} \alpha^{-1}\Sigma & \Omega^{-1}T_{b1}\bar{P}_{12}T_{b1} \\ T_{b1}P_{12}^T\Omega^{-1} & \alpha^{-1}\Sigma \end{bmatrix} \tag{55}$$

where

$$\Sigma = \text{diag}\left\{ \frac{(f_i h_i)^{1/2}}{2c_i} \right\} \tag{56}$$

At this stage we already have a crude approximation for the balanced system, since if we neglect the α^0 terms as compared to α^{-1}, Q_{b1} and P_{b1} are diagonal and equal. Order reduction based on this transformation will be exactly the same as modal truncation where the elements of Σ_1 indicate which modes should be retained, and the 'new' $\bar{C}_o$ consists of the columns and rows of those modes.

The second step is to consider the transformation

$$V_{b2} = \begin{bmatrix} I + \frac{1}{2}\alpha^2 L_1 L_2 & \alpha L_1 \\ \alpha L_2 & I + \frac{1}{2}\alpha^2 L_2 L_1 \end{bmatrix} \tag{57}$$

For $\alpha << 1$, V_{b2}^{-1} is given by

$$V_{b2}^{-1} = \begin{bmatrix} I + \frac{1}{2}\alpha^2 L_1 L_2 & -\alpha L_1 \\ -\alpha L_2 & I + \frac{1}{2}\alpha^2 L_2 L_1 \end{bmatrix} \tag{58}$$

where the deviation from the identity matrix in the product is of order α^4. Applying this transformation we get

$$Q_{b2} = \begin{bmatrix} \alpha^{-1}\Sigma + \alpha E_1 & \tilde{q}_{12} - L_1\Sigma - \Sigma L_2^T + \alpha^2 E_{12} \\ \tilde{q}_{12}^T - \Sigma L_1^T - L_2\Sigma + \alpha^2 E_{12}^T & \alpha^{-1}\Sigma + \alpha E_2 \end{bmatrix} \tag{59}$$

$$P_{b2} = \begin{bmatrix} \alpha^{-1}\Sigma + \alpha \hat{E}_1 & \tilde{p}_{12} + \Sigma L_1 + L_2^T\Sigma + \alpha^2 E_{12} \\ \tilde{p}_{12}^T + L_1^T\Sigma + \Sigma L_2 + \alpha^2 \hat{E}_{12}^T & \alpha^{-1}\Sigma + \alpha \hat{E}_2 \end{bmatrix} \tag{60}$$

where

$$\tilde{q}_{12} = \Omega T_{b1}^{-1}\bar{q}_{12}T_{b1}^{-1}, \quad \tilde{p}_{12} = \Omega^{-1}T_{b1}\bar{p}_{12}T_{b1} \tag{61}$$

and E_1, E_2, E_{12}, $\hat{E}_1$, $\hat{E}_2$ and $\hat{E}_{12}$ are unknown matrices. These matrices cannot be calculated since earlier in the deviation we neglected terms of order α in the diagonal sub blocks of the gramians. Our goal now is to nullify the terms of order α^0. That leads to

$$L_1\Sigma + \Sigma L_2^T = \tilde{q}_{12} \tag{62}$$

$$\Sigma L_1 + L_2^T\Sigma = -\tilde{p}_{12} \tag{63}$$

Subtracting (63) premultiplied by Σ from (62) postmultiplied by Σ we get

$$L_1\Sigma^2 - \Sigma^2 L_1 = \tilde{q}_{12}\Sigma + \Sigma\tilde{p}_{12} \tag{64}$$

The diagonals of both sides are identically zero and the off diagonal terms of L_1 are given by

$$(L_1)_{ij} = \left[\Sigma_i(\tilde{p}_{12})_{ij} + \Sigma_j(\tilde{q}_{12})_{ij}\right]/(\Sigma_j^2 - \Sigma_i^2) \tag{65}$$

Similarly we obtain

$$(L_2)_{ij} = \left[\Sigma_i(\tilde{p}_{12})_{ji} + \Sigma_j(\tilde{q}_{12})_{ji}\right]/(\Sigma_j^2 - \Sigma_i^2) \tag{66}$$

With this choice Q_{b2} and Q_{b2} have identical diagonal of order α^{-1} and the rest of their elements have order α or α^2. This is a good approximation of the exact balanced form. The diagonal elements of L_1 and L_2 provide more degrees of freedom however it is not clear how to use them.

If one is willing to trade accuracy for simplicity, a simpler transformation is

$$\hat{V}_{b2} = \begin{bmatrix} I & \alpha L_1 \\ \alpha L_2 & I \end{bmatrix} \tag{67}$$

$$\hat{V}_{b2}^{-1} = \begin{bmatrix} I & -\alpha L_1 \\ -\alpha L_2 & I \end{bmatrix} \tag{68}$$

where the deviation from identity in the product $\hat{V}_{b2}\hat{V}_{b2}^{-1}$ is of order α^2 instead of α^4. The derivation in equations (59) - (66) remains unchanged so there is no loss of accuracy in that respect. The advantage of this transformation becomes clear from the following argument. Let V_b be the exact balancing transformation and partition it and its inverse as

$$V_{b1} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \quad , \quad V_b^{-1} = \begin{bmatrix} \hat{V}_1 \\ \hat{V}_2 \end{bmatrix} \tag{69}$$

where V_1 consists of the first n_r columns of V_b and $\hat{V}_1$ of the first n_r rows of V_b^{-1}. Then the reduced order model which is obtained by truncating the balanced realization, as explained in section 2, is given by

$$A_{b11} = \hat{V}_1 A V_1 \;;\quad B_{b1} = \hat{V}_1 B \;;\quad H_{b1} = H V_1 \tag{70}$$

So only $\hat{V}_1$ and V_1 are required for the order reduction process. Considering $\hat{V}_{b2}$ and $\hat{V}_{b2}^{-1}$ that means that only N_r columns and rows of L_1 and L_2 have to be calculated. That requires the calculation of the same columns and rows of $\bar{q}_{12}$ and $\bar{p}_{12}$. In cases where $N_r << N$ such approximation leads to a sharp reduction in the required computation.

5. EXAMPLE

We consider the (synthetic) system in figure 1 with $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, equal spring constants $k = 1$ and equal damping elements $C = \alpha$ the input is the force f and the output is defined as the velocity of m_2. In matrix form

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} , \quad K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

$$C_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \; f = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad , \quad h = [1 \;\; 0 \;\; 0]$$

Notice that the input and the output are noncollocated and that the damping is not proportional. The errors in the controllability gramians was calculated as follows:

$$Eq = \frac{\| Q_{exact} - Q_{app} \|_2}{\| Q_{exact} \|_2}$$

$$Ep = \frac{\| P_{exact} - P_{app} \|_2}{\| P_{exact} \|_2}$$

and are depicted in figure 2 for $0 \le \alpha \le 0.2$ where for $\alpha = 0.2$ the damping ratios were about 4 percent. In this upper value the assumption $\alpha << 1$ barely holds but the approximation is still very accurate with relative errors in the vicinity of 2 percent.

6. SUMMARY

A method for approximating the controllability gramian, observability gramian and the balancing transformation for a lightly damped structure was presented. The actual structure was presented as a small perturbation from an undamped system by means of the small parameter α. Then in the analysis that was carried out, high powers of α were neglected and as a result the calculation was drastically simplified. For example, for each of the gramians only $N + N(N - 1)/2$ terms need to be calculated, and an explicit formula for each one of them exists. This should be compared to a number of operation of order $(2N)^3$ in the general solution.

The entire derivation hinges on the modal form of the system and 'fluctuate' about it. As a result the mathematical steps do not obscure the physical insight into the system. As a matter of fact there is a clear physical interpretation, that was not given as a result of space limitation, to the structure of the gramians and to their dependence on α (α^{-1}, α^0 etc.).

For the simplicity of presentation and because of space limitation only the case of velocity output and distinct natural frequencies was considered. More general results were obtained using this method and will be reported in future publications.

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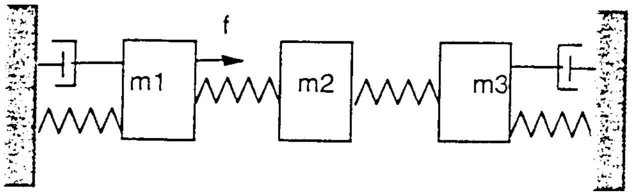

Figure 1 : The system in the example.

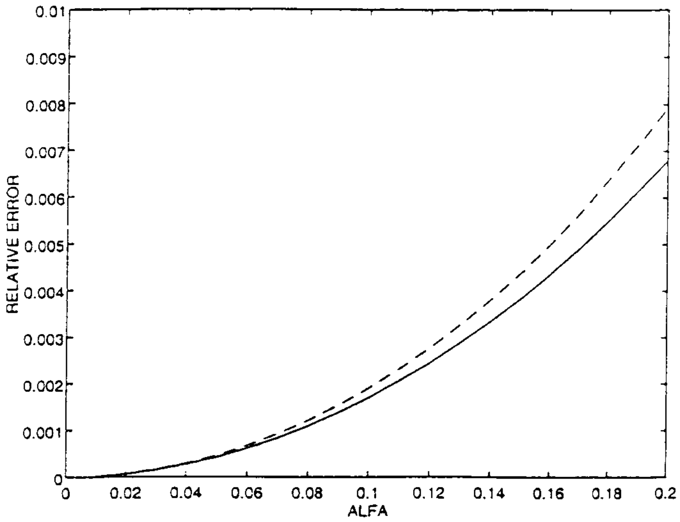

Figure 2 : The relative errors Eq (solid) and Ep (dashed) vs α.