Feedback Stabilization of Homogeneous Systems of Odd Degree

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Abstract: The purpose of this paper is to derive a simple necessary and sufficient condition of stabilization at the origin for a homogeneous vector field by means of homogeneous feedbacks of the same degree, which are explicity given.

Keywords: Stabilizability, Nonlinear systems, Homogeneous vector fields, Homogeneous feedbacks, Lyapunov function.

1 Introduction

The stabilization of nonlinear control systems by means of smooth feedbacks on $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$, is one of the most important problem in control theory. It has attracted the interest of increasing number of authors in the last decay, see e.g. [2, 3, 6, 7, 8, 9]. In this paper, we address such a problem for systems of the form:

$$\begin{cases} \dot{x} = f(x) + Bu \\ x \in \mathbb{R}^n, u \in \mathbb{R}^m \end{cases} \tag{1}$$

where f is a homogeneous vector field (i.e, a vector field whose components $f_i, i = 1, ..., n$ are all homogeneous functions of the same degree) and B is a $n \times m$ matrix. We shall give a necessary and sufficient condition for the existence of a feedback u mapping $\mathbb{R}^n$ into $\mathbb{R}^m$ which is also a homogeneous function with the same homogeneity degree as f, and which makes the origin globally asymptotically stable equilibrium point for the system (1) in closed-loop. Furthermore, we construct explicitly the stabilizing feedback u.

For the system considered in this work, the matrix B has not necessarily rank $n-1$ as assumed in [5], and the Lyapunov function used for giving the stabilizing feedback is not necessarily a homogeneous quadratic one as supposed in [1] where, in addition, the given condition is sufficient but not necessary. So, our result appears as a generalization of [1, 5]. An example which does satisfies neither the conditions of [1] nor those of [5], and which can be stabilized by our propesed feedback, is given.

2 Main result

Throughout this paper, $||x||$ will denote the usual Euclidean norm of a point $x \in \mathbb{R}^n$ and ${}^tA$ will denote the transposed matrix of A. Let $f(x) = {}^t(f_1(x), ..., f_n(x))$ be a vector field of $\mathbb{R}^n$ such that all its components $f_i(x)$ $(i = 1, ..., n)$ are homogeneous functions of the same odd degree $k \geq 1$. In the sequel, we will say that f is a homogeneous vector field and that k is its degree of homogeneity. Let B be an $n \times m$ matrix; we will say that the system

$$\dot{x} = f(x) + Bu \tag{2}$$

admits a homogeneous stabilizing feedback if there exists a function $u(x) = {}^t(u_1(x), ..., u_m(x))$ such that each component $u_j(x)$ $(j = 1, ..., m)$ is a homogeneous function of degree k and the origin is a globally asymptotically stable equilibrium point for the closed-loop system

$$\dot{x} = f(x) + Bu(x) \tag{3}$$

We remark that the right hand side of (3) is still a homogeneous vector field. We recall that for ordinary differential equations with homogeneous right-hand side local and global asymptotic stability are equivalent [4].

The following theorem provides a necessary and sufficient condition for the existence of a homogeneous stabilizing feedback. Its proof is based on an application of the so-called Lyapunov second method. Indeed, we shall make use of a Lyapunov function for testing the stability of the closed-loop system.

In a suitable basis of $\mathbb{R}^n$, the system (2) can be written:

$$\begin{cases} \dot{y}_1 = g_1(y) \\ \vdots \\ \dot{y}_q = g_q(y) \\ \dot{y}_{q+1} = g_{q+1}(y) + \tilde{u}_{q+1} \\ \vdots \\ \dot{y}_n = g_n(y) + \tilde{u}_n \end{cases} \tag{4}$$

where $n-q$ is the rank of B and $g = {}^t(g_1, ..., g_n)$ is such that $g_i(y)(i = 1, ..., n)$ are homogeneous functions of the same odd degree $k \geq 1$. Notice that such a change of coordinates does not affect the properties of stabilizability, hence the stabilizability of system (2) is equivalent to those of system (4), so throughout this section, we consider the system as in the form (4).

Let us introduce:

Assumption (H) . There exists a smooth, proper, definite positive and homogeneous function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that:

$$\sum_{i=1}^{q} \frac{\partial V}{\partial y_i}(y) g_i(y) < 0, \quad \forall y \in E^*$$

where:

$$E = \left\{ y \in \mathbb{R}^n : \frac{\partial V}{\partial y_{q+1}}(y) = ... = \frac{\partial V}{\partial y_n}(y) = 0 \right\}$$

and

$$E^* = E \setminus \{0\}.$$

Theorem *The control system (4), where g is as stated above, is globally asymptotically stable by a homogeneous feedback of the same odd degree as g, if and only if, the assumption (H) holds.*

Proof: First, since V is supposed to be definite, positive and homogeneous, then its degree of homogeneity is even noted d. The condition (H) is sufficient, to prove this we establishe the following result

Proposition *If (H) holds, then it implies the existence of $\lambda \in \mathbb{R}$, $\lambda > 0$, such that, the control $\tilde{u} = {}^t(\tilde{u}_{q+1}, ..., \tilde{u}_n)$ defined by*

$$\begin{cases} \tilde{u}_i(y) = -\lambda \|y\|^{k-d+1} \dfrac{\partial V}{\partial y_i}(y) & \text{if } y \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

is a homogeneous stabilizing feedback for the system (4).

Remark 1 Since V is smooth on $\mathbb{R}^n$, it is easy to see that $\tilde{u}$ is at least C^{k-1} function on $\mathbb{R}^n$ and smooth on $\mathbb{R}^n \setminus \{0\}$

Proof of the proposition : Let, C^+ be the closed cone:

$$C^+ = \{ y \in \mathbb{R}^n : \ \nabla V(y) g(y) \geq 0 \}$$

and C^- be its complementary cone:

$$C^- = \{ y \in \mathbb{R}^n : \ \nabla V(y) g(y) < 0 \}$$

and let S be the unit sphere of $\mathbb{R}^n$

$$S = \{ y \in \mathbb{R}^n : \|y\| = 1 \}$$

Now, the derivative of V along trajectories of (4-5) is

$$\dot{V}(y) = \nabla V(y) g(y) - \lambda \|y\|^{k-d+1} \sum_{i=q+1}^{n} \left(\frac{\partial V}{\partial y_i}(y) \right)^2$$

$\dot{V}$ is a homogeneous function of even degree $d+k-1$; hence, its sign doesn't change along any ray issuing from the origin. This sign can be evaluated on the sphere S. According to the Lyapunov theorem [4], in order to complete the proof, we have to show that $\dot{V}$ is negative on the whole sphere S.
For any $y \in S$ we have:

$$\dot{V}(y) = \nabla V(y) g(y) - \lambda \sum_{i=q+1}^{n} \left(\frac{\partial V}{\partial y_i}(y) \right)^2$$

Let δ_1 and δ_2 be the numbers defined as follow:

$$\delta_1 = \max_{y \in S} \nabla V(y) g(y), \quad \delta_2 = \min_{y \in S \cap C^+} \sum_{i=q+1}^{n} \left(\frac{\partial V}{\partial y_i}(y) \right)^2$$

Since S and $S \cap C^+$ are compact, δ_1 and δ_2 exist. Furthermore, under assumption (H), we remark that $E \subset C^- \cup \{0\}$, hence $E \cap C^+ = \{0\}$, which implies

$$\sum_{i=q+1}^{n}\left(\frac{\partial V}{\partial y_i}(y)\right)^2 > 0, \quad \forall y \in S \cap C^+$$

so, $\delta_2 > 0$. Now, we choose $\lambda > \frac{\delta_1}{\delta_2}$. If $y \in S \cap C^-$ then obviously $\dot{V}(y) < 0$, and if $y \in S \cap C^+$ then $\dot{V}(y) \leq \delta_1 - \lambda\delta_2 < 0$ because of the choice $\lambda > \frac{\delta_1}{\delta_2}$. Thus, the proposition is proved.

Let us prove that (H) is a necessary condition for the existence of a stabilizing continuous and homogeneous feedback of the same degree as g.
Suppose that such a feedback exists, which makes system (4) globally asymptotically stable, then, since the right-hand side of (4) is an autonomous and homogeneous vector field, there exists a smooth, proper, definite positive and homogeneous function V [4, 10] such that for all y in $\mathbb{R}^n \setminus \{0\}$, we have,

$$\dot{V}(y) = \nabla V(y)g(y) + \sum_{i=q+1}^{n}\frac{\partial V}{\partial y_i}(y)\tilde{u}_i(y) < 0.$$

hence, if $y \in E^*$ we have:

$$\dot{V}(y) = \sum_{i=1}^{q}\frac{\partial V}{\partial y_i}(y)g_i(y) < 0.$$

the proof of our theorem is now completed.

Remark 2 Notice that (H), without the homogeneity assumption on Lyapunov function V, is still a necessary condition of stabilizability of system (2) by a continuous feedback in closed-loop, see [10].

Example Let us consider the following system on $\mathbb{R}^3$:

$$\left\{\begin{array}{ll} \dot{x}_1 & = -2x_1^3 - 5x_1^2x_2 + 4x_1x_2^2 - x_2^3 \\ & \quad +x_1x_3^2 = f_1(x_1,x_2,x_3) \\ \dot{x}_2 & = x_1^3 + 4x_1^2x_2 + 5x_1x_2^2 - 2x_2^3 \\ & \quad +x_2x_3^2 = f_2(x_1,x_2,x_3) \\ \dot{x}_3 & = x_1^2x_3 + 2x_1x_2x_3 + 5x_2^3 \\ & \quad +u = f_3(x_1,x_2,x_3) + u \end{array}\right. \tag{6}$$

We remark that this system does not satisfy the hypotheses of [5]; indeed, rank of B is $1 \neq 3-1 = 2$.

We remark also that we can not find a 3×3 symetric and definite positive matrix P which satisfies the hypotheses of [1]:

$$Ker^tBP \subset \{x \in \mathbb{R}^3 :^t xPf(x) < 0\} \cup \{0\}$$

Indeed; suppose that such a matrix exists,

$$P = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & h \end{pmatrix}$$

it follows

$$Ker^tBP = \{x \in \mathbb{R}^3 : cx_1 + ex_2 + hx_3 = 0\}$$

then for $x \in Ker^tBP$, we have:

$$\begin{aligned} {}^txPf(x) &= \left(a - \frac{c^2}{h}\right)x_1f_1(x) + \left(b - \frac{c.e}{h}\right)x_1f_2(x) \\ &\quad + \left(b - \frac{c.e}{h}\right)x_2f_1(x) + \left(d - \frac{e^2}{h}\right)x_2f_2(x) \\ &= (x_1,x_2)A\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \end{aligned}$$

where

$$A = \begin{pmatrix} a - \frac{c^2}{h} & b - \frac{c.e}{h} \\ b - \frac{c.e}{h} & d - \frac{e^2}{h} \end{pmatrix}$$

which is a 2×2 symetric and definite positive matrix as P. hence,

$${}^txPf(x) < 0, \quad \forall x \in Ker^tBP \setminus \{0\} \subset \mathbb{R}^3$$

is equivalent to

$$\left\{\begin{array}{l} (y_1,y_2)A\begin{pmatrix} g_1(y_1,y_2) \\ g_2(y_1,y_2) \end{pmatrix} < 0 \\ (y_1,y_2) \in \mathbb{R}^2 \setminus \{0\} \end{array}\right. \tag{7}$$

where

$$\left\{\begin{array}{ll} g_1(y_1,y_2) = & -2y_1^3 - 5y_1^2y_2 + 4y_1y_2^2 - y_2^3 \\ & +y_1\left(-\frac{c}{h}y_1 - \frac{e}{h}y_2\right)^2 \\ g_2(y_1,y_2) = & y_1^3 + 4y_1^2y_2 + 5y_1y_2^2 - 2y_2^3 \\ & +y_2\left(-\frac{c}{h}y_1 - \frac{e}{h}y_2\right)^2 \end{array}\right.$$

this last inequality is not possible, because for all 2×2 symetric and definite positive matrix D,

$$D = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$$

and for any $y = {}^t(y_1, y_2) \in \mathbb{R}^2$ we have

$$yD\begin{pmatrix} g_1(y) \\ g_2(y) \end{pmatrix} = \chi(y_1, y_2) + \xi(y_1, y_2) \qquad (8)$$

where

$$\chi(y_1, y_2) = (\alpha y_1^2 + 2\beta y_1 y_2 + \gamma y_2^2)\Big(-\frac{c}{h}y_1 - \frac{e}{h}y_2\Big)^2$$

and

$$\begin{aligned}\xi(y_1, y_2) = \quad & (-2\alpha + \beta)y_1^4 \\ & +(-5\alpha + 2\beta + \gamma)y_1^3 y_2 + (4\alpha + 4\gamma)y_1^2 y_2^2 \\ & +(-\alpha + 2\beta + 5\gamma)y_1 y_2^3 + (-\beta - 2\gamma)y_2^4\end{aligned}$$

we remark that:

$$\chi(y_1, y_2) \geq 0, \quad \forall y \in \mathbb{R}^2$$

so, in order to have (8)< 0, $\forall(y_1, y_2) \in \mathbb{R}^2 \setminus \{0\}$, we must have,

$$\xi(y_1, y_2) < 0 \qquad (9)$$

Now, if we take in the one hand $y_1 = y_2$ and in the other hand $y_1 = -y_2$, and taking in to account (9), we obtain:

$$-4\alpha + 4\beta + 8\gamma < 0 \quad \text{and} \quad 8\alpha - 4\beta - 4\gamma < 0$$

which implies, $4\alpha + 4\gamma < 0$. Since D is a symetric and definite positive matrix, this last inequality is not possible. Thus, condition of [1] is not met.

Now, we can take, for the above system, the following definite positive, proper and homogeneous function defined for any $y = {}^t(y_1, y_2, y_3)$ in $\mathbb{R}^3$ by

$$\begin{aligned}V(y_1, y_2, y_3) &= 4y_1^2 y_2^2 + 2y_1 y_2(y_1^2 + 2y_1 y_2 - y_2^2) \\ &+(y_1^2 + 2y_1 y_2 - y_2^2)^2 + y_3^4 = \tilde{V}(y_1, y_2) + y_3^4\end{aligned}$$

which satisfies assumption (H). Indeed, in this case

$$E^* = \Big\{y \in \mathbb{R}^3 \setminus \{0\} : \frac{\partial V}{\partial y_3}(y) = 0; i.e, y_3 = 0\Big\}$$

so, we can easily verify that for all (y_1, y_2, y_3) in E^* we have

$$\frac{\partial V}{\partial y_1}(y) f_1(y) + \frac{\partial V}{\partial y_2}(y) f_2(y) = -2(y_1^2 + y_2^2)\tilde{V}(y_1, y_2) < 0$$

According to our theorem,

$$u(y_1, y_2, y_3) = -4\lambda y_3^3$$

is a stabilizing feedback for a suitable value of λ. In order to give an estimation for λ, one has to compute δ_1 and δ_2.

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