ROBUST FEEDBACK SYNTHESIS FOR DISTURBANCE REJECTION WITH PREFERRED CONTROL EFFORTS

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Abstract

A linear time invariant MIMO plant with more inputs than outputs which is known to belong to a set $\mathcal{P}$ is considered. In order to decrease the plant output due to disturbances, it is embedded in a feedback structure. A design method is given to find a controller to meet the following closed-loop specifications: (i) closed loop robust stability and disturbance rejection performances; and (ii) the control efforts are members of a preferred optimal set for one plant from the set $\mathcal{P}$ and n disturbances, or for n pairs of plants and disturbances (n is the number of plant outputs). The Quantitative Feedback Theory (QFT) is the framework for the proposed design technique.

1 Introduction

A system is embedded in a feedback structure in order to reduce its output sensitivity to plant uncertainty for tracking commands, and to decrease output which is due to unknown disturbances. This has given rise to the question of how to design the feedback parameters. The latest and most famous design techniques which include robustness are H_∞, μ synthesis and QFT. The main advantages of the QFT technique are that (i) it offers a design for the exact amount of plant uncertainty, which naturally saves controller bandwidth compared to techniques that suit only special structures of uncertainty; (ii) almost no iterations are needed during the design process; and (iii) the uncertain plant may be non-minimum phase including delay, and/or can be given in a table form for dense enough frequencies. Another property of QFT that should be considered an advantage is that the closed-loop performances to be achieved are given at each frequency, rather than by norms of matrix transfer functions.

A very important property of a design is that the control efforts should be optimal for a given optimal criterion. Clearly an optimal criterion on each plant is superior to a global one, as for example the H_∞ criterion (Doyle *et al.*, 1989). Most likely there exist no controllers and/or prefilters that will force optimal control efforts for each plant in the uncertain set for one or a set of tracking and/or disturbance inputs. Here, the framework of the QFT (Yaniv and Horowitz, 1986; Yaniv, 1992) is used to extend the general QFT specs, such as disturbance rejection (Yaniv and Horowitz 1986) and margins (Yaniv 1992), to force the control efforts to lie in a preferred set. This means that the control efforts are close to optimal (within given tolerances) either for one preferred plant from the uncertain set $\mathcal{P}$ and n disturbances, or for n pairs of plants and disturbances, where n is the number of plant outputs.

The paper is set out as follows. The problem is stated in Section 2. In Section 3 the design algorithm is developed; Section 4 is devoted to a design example and the conclusions are given in Section 5.

2 Motivation and Problem Statement

Notation 2.1 *Bold capital letters denote a matrix and the same indexed lower case letter denotes its elements. Lower case bold letters denote a vector, with the same indexed lower case non-bold letter as its elements.*

Consider the feedback system shown in Fig. 1, where the uncertain plant $\mathbf{P} = [p_{ij}]$ belongs to the family $\mathcal{P}$. The controller $\mathbf{WG}$ is a product of the pre-controller $\mathbf{W}$ and the diagonal controller $\mathbf{G}$. If the plant has more inputs than output then the same closed loop output can be achieved by different control efforts. Moreover it is natural that the disturbance $\mathbf{d}$ has a special structure, for example side wind on a four wheel vehicle will force correlated side force and moment around the vehicle center of gravity. This example raises the question of how to design the feedback of such system to achieve good closed loop performance with minimal control efforts. The fact that most practical system have structured uncertainty, as in flight control and vehicle control (Ackerman 1993) motivates to use a structured robust framework. Based on all these, the following problem is formulated.

Let the following notations and definitions hold:

- n, m - number of rows (outputs) and columns (inputs) of $\mathbf{P}$; $n < m$.
- $\mathbf{G} = diag(g_i)$, the nxn diagonal component of the controller.
- $\mathbf{W}$ - the mxn non-diagonal component of the controller.
- $\mathbf{P}_i$, $\mathbf{u}_i$ and $\mathbf{d}_i$ - n chosen plants in $\mathcal{P}$ for which $\mathbf{u}_i$ is the optimal control efforts that minimizes the plant output $\mathbf{y}_i$

  $$\mathbf{y}_i = \mathbf{P}_i \mathbf{u}_i + \mathbf{d}_i, \quad i = 1, ..., n, \qquad (1)$$

  for a certain criterion (see Remark 2.1).
- $\tilde{\mathbf{u}}_i$ - the closed-loop input to $\mathbf{P}_i$ for the disturbance $\mathbf{d}_i$.
- $\mathbf{a}_i(\omega)$ - a vector of m elements that represents the control effort tolerances on the input of the plant $\mathbf{P}_i$ for the disturbance $\mathbf{d}_i$; its elements are positive.

Remark 2.1 *The optimal control efforts $\mathbf{u}_i$ can satisfy any given criterion, such as LQ, H_∞ or any engineering criterion.*

Problem 2.1 *Show a method of designing* $\mathbf{G}$ *and* $\mathbf{W}$ *such that for all* $\mathbf{P} \in \mathcal{P}$, *the system defined in Fig. 1 is internally stable, and satisfies the specs*

$$|\tilde{\mathbf{u}}_i(i\omega) - \mathbf{u}_i(i\omega)| \le \mathbf{a}_i(\omega), \quad i = 1, .., n, \qquad (2)$$

where the last vector inequality means element by element inequality.

Remark 2.2 *The motivation for this kind of specs on the control efforts are: (i) Inequality 2 bounds the maximum allowed deviation at each frequency of the true control efforts from the optimal control efforts for the pair* $\mathbf{P}_i$, $\mathbf{d}_i$; *and (ii) Time domain specs on the average on any finite interval, of the form*

$$\int_0^t |\tilde{\mathbf{u}}_i(\tau) - \mathbf{u}_i(\tau)|^2 d\tau \le \int_0^t \mathbf{a}_i^2(\tau) d\tau \qquad (3)$$

can be replaced by specs of the form defined by inequality 2, see Krishnan and Cruickshanks (1977). They also argued that in practice, inequality (3) force almost the same as inequality (2) but in time domain, that is

$$|\tilde{\mathbf{u}}_i(t) - \mathbf{u}_i(t)| \le \mathbf{a}_i(t), \quad i = 1, .., n, \qquad (4)$$

which is a highly recommended spec for the common saturation phenomenon.

3 Development of the Design Procedure

In Fig. 1 the input to $\mathbf{P}_i$ owing to the disturbance $\mathbf{d}_i$ is

$$\tilde{\mathbf{u}}_i = -[\mathbf{I} + \mathbf{WGP}_i]^{-1} \mathbf{WGd}_i. \qquad (5)$$

The plant input $\tilde{\mathbf{u}}_i$ for a given plant output is not unique, because the plant has more inputs than outputs. Conditions on the structure of $\mathbf{W}$ such that the plant input will be the optimal $\mathbf{u}_i$ if its output is the optimal output given by equation 1 are now developed.

Lemma 3.1 *Let* $\mathbf{P}_i$ *and* $\mathbf{d}_i$ *be* n *pairs* $(i = 1, .., n)$ *of plants and disturbances such that for* $\mathbf{d} = \mathbf{d}_i$ *and* $\mathbf{P} = \mathbf{P}_i$ *in Fig. 1 the plant input is* $\mathbf{u}_i$ *and* $[\mathbf{u}_1 ... \mathbf{u}_n]$ *has full rank except in a finite number of points in the complex plane. Then*

$$\mathbf{W} = [\mathbf{u}_1 ... \mathbf{u}_n]\mathbf{H} \stackrel{\text{def}}{=} \mathbf{UH}, \qquad (6)$$

where $\mathbf{H}$ *is an* nxn *non-singular matrix transfer function, i.e., det(*$\mathbf{H}$*) has a finite number of zeros.*

Proof: Let $\mathbf{x}_i$ be the input to $\mathbf{W}$ for the pair $\mathbf{P}_i$, $\mathbf{d}_i$. Then

$$[\mathbf{u}_1...\mathbf{u}_n] = \mathbf{W}[\mathbf{x}_1...\mathbf{x}_n] \stackrel{\text{def}}{=} \mathbf{WX}, \qquad (7)$$

whose left side has full rank except in a finite number of points in the complex plane, thus $\mathbf{X}^{-1}$ exists because $m > n$ (m is the number of rows in equation 7). This in turn implies equation 6, where $\mathbf{H} = \mathbf{X}^{-1}$.

Lemma 3.2 *Suppose equation 6 is true, the output of* $\mathbf{P}_i$ *is* $\mathbf{P}_i\mathbf{u}_i$, *and* $\mathbf{P}_i\mathbf{U}$ *and* $\mathbf{H}$ *are invertible. Then the input to* $\mathbf{P}_i$ *is* $\mathbf{u}_i$[1].

Let $\tilde{\mathbf{u}}_i$ denote the input to $\mathbf{P}_i$. Since $\mathbf{P}_i\mathbf{U}$ and $\mathbf{H}$ are invertible, then from Fig. 1

$$\begin{aligned} \tilde{\mathbf{u}}_i &= \mathbf{UH}(\mathbf{P}_i\mathbf{UH})^{-1}\mathbf{P}_i\mathbf{u}_i \\ &= \mathbf{U}(\mathbf{P}_i\mathbf{U})^{-1}\mathbf{P}_i\mathbf{u}_i \\ &= i \textit{ column of } \mathbf{U}(\mathbf{P}_i\mathbf{U})^{-1}\mathbf{P}_i\mathbf{U} = \mathbf{u}_i. \;\square(8) \end{aligned}$$

A natural consequence from Lemma 3.1-3.2 is to split $\mathbf{W}$ in Fig. 1 into

$$\mathbf{W} = [\mathbf{u}_1...\mathbf{u}_n]\mathbf{H} = \mathbf{UH}, \qquad (9)$$

where

$$\mathbf{U} = [\mathbf{u}_1...\mathbf{u}_n], \qquad (10)$$

as shown in Fig. 2. A natural design process will then run as follows: First design the square controller $\mathbf{H}$, and then design a controller $\mathbf{G}$ that solves Problem 2.1.

Remark 3.1 *Note that a solution is not guaranteed because even if the plant and* $\mathbf{U}$ *are minimum phase,* $\mathbf{W} = \mathbf{UH}$ *might change its high-frequency gain sign, as in the following example. The two* $1x2$ *plant has two cases,* $\mathbf{P}_1 = [1 \quad -2]/s$ *and* $\mathbf{P}_2 = [1 \;\; 1]/s$, *and the optimal control effort for both plants is* $\mathbf{U} = [1 \; 1]$, *which gives* $\mathbf{P}_1\mathbf{U} = -1/s$ *and* $\mathbf{P}_2\mathbf{U} = 1/s$. *Clearly an LTI controller cannot even stabilize these two plants simultaneously.*

A design method for the controller $\mathbf{G}$ to satisfy the closed-loop specs given in Problem 2.1 which is based on the sequential QFT concept is now developed. Recall that the QFT design technique for nxn MIMO system turns the design process into an n step sequence of n SISO problems with the same uncertain plant and controller for each step. The solution of the ith set of n SISO problems is the controller g_i, giving a combined solution $\mathbf{G} = diag(g_i)$. Solving these SISO problems is a routine task in QFT. The main task during the design of each SISO problem is to find *Bounds* on an open-loop transfer function. Each *Bound* is a closed curve on the complex plane, dividing it into two regions. One of these is called $\sigma(\omega)$, so that if the open loop transfer function is shaped such that at each frequency ω it belongs to $\sigma(\omega)$ and satisfies the Nyquist stability criterion, the synthesis procedure must succeed. If several specs are imposed, the *Bound* used is the *Bound* of all intersections of the regions $\sigma(\omega)$ for all specs. In order to establish an algorithm to find *Bounds* on the open loop of each design step that force the control efforts to satisfy the specs of Problem 2.1, the following equations are provided. Substituting equation 1 in equation 5 gives

$$\begin{aligned} \tilde{\mathbf{u}}_i - \mathbf{u}_i &= [\mathbf{I} + \mathbf{WGP}_i]^{-1}\mathbf{WGP}_i\mathbf{u}_i - \mathbf{u}_i \\ &= -[\mathbf{I} + \mathbf{WGP}_i]^{-1}\mathbf{u}_i. \qquad (11) \end{aligned}$$

Using the notations

$$\begin{aligned} \mathbf{G}_k &= diag(0,...,0,g_k,0,...,0), \\ \mathbf{C}_{ki} &= \mathbf{I} + \mathbf{W}[\mathbf{G} - \mathbf{G}_k]\mathbf{P}_i, \\ \mathbf{w}_k &= (w_{1k}, w_{2k}, ..., w_{mk})' \textit{ and} \\ \mathbf{p}_k^i &= \textit{the } k \textit{ row of } \mathbf{P}_i, \qquad (12) \end{aligned}$$

then (the proof is in the Appendix)

$$[\mathbf{I} + \mathbf{WGP}_i]^{-1} = \mathbf{C}_{ki}^{-1} - \frac{g_k \; \mathbf{C}_{ki}^{-1}\mathbf{w}_k\mathbf{p}_k^i\mathbf{C}_{ki}^{-1}}{1 + g_k\mathbf{p}_k^i\mathbf{C}_{ki}^{-1}\mathbf{w}_k}. \quad (13)$$

Substituting equation 13 in equation 11 gives

$$\tilde{\mathbf{u}}_i - \mathbf{u}_i = -\mathbf{C}_{ki}^{-1}\mathbf{u}_i + \frac{g_k \; \mathbf{C}_{ki}^{-1}\mathbf{w}_k\mathbf{p}_k^i\mathbf{C}_{ki}^{-1}}{1 + g_k\mathbf{p}_k^i\mathbf{C}_k^{-1}\mathbf{w}_k}\mathbf{u}_i; \quad (14)$$

and the specs 2 will be

$$|\mathbf{C}_{ki}^{-1}\mathbf{u}_i - \frac{g_k \; \mathbf{C}_{ki}^{-1}\mathbf{w}_k\mathbf{p}_k^i\mathbf{C}_{ki}^{-1}}{1 + g_k\mathbf{p}_k^i\mathbf{C}_{ki}^{-1}\mathbf{w}_k}\mathbf{u}_i| \leq \mathbf{a}_i; i = 1,..,n. \qquad (15)$$

Remark 3.2 *The solution of inequality 15 is a domain* $\sigma_k(\omega)$ *in the complex plane such that if* $g_k(i\omega) \in \sigma_k(\omega)$ *then inequality 15 is valid. For a given* k *it includes* $m*n$ *inequalities, i.e,* $i = 1, ..., n$ *and* m *is the length of the vector* $\mathbf{a}_i$.

The proposed sequential process to find $\sigma_k(\omega)$ will then be as follows: In the first step, $k = 1$, solve inequality 15 for each one of the plants $\mathbf{P}_i$ when the unknown g_i, $i = 2, .., n$, are assumed to be infinity. The intersection of all these solutions is $\sigma_1(\omega)$. Now

[1]The plant input for a given plant output is not unique because the plant has more inputs than outputs

shape g_1 to satisfy the *Bounds* of $\sigma_1(\omega)$. In the kth step, solve inequality 15 for each one of the plants $\mathbf{P}_i$ where g_i for $i = 1,..,k-1$ are known and the unknown g_i, $i = k+1,..,n$, are assumed to be infinity. The intersection of all these solutions is $\sigma_k(\omega)$. Finally, shape g_k to satisfy the *Bounds* of $\sigma_k(\omega)$.

There are computing difficulties in calculating $\mathbf{C}_{ki}^{-1}$ because some of its elements are ∞ and/or ≈ 0. The following is a simple formula to calculate it when some of the g_i's are 0 and/or ∞. Let $\mathbf{G}^{\alpha} = diag(g_k)$ be a diagonal controller whose diagonal elements $\alpha = \alpha_1, ..., \alpha_l$ are zero, $\mathbf{G}_{\alpha}^{-1}$ a diagonal matrix whose diagonal elements α are 1 and the elements α_i, $i > l$ are $1/g_{\alpha_i}$ (for $g_{\alpha_i} = \infty$ it is 0), $\mathbf{P}_{\alpha}$ the same as $\mathbf{P}$ except that the α rows are zero, and $\mathbf{W}_{\alpha}$ the same as $\mathbf{W}$ except that the α columns are zero. Then (for proof see Appendix)

$$[\mathbf{I} + \mathbf{W}\mathbf{G}^{\alpha}\mathbf{P}]^{-1} = \mathbf{I} - \mathbf{W}_{\alpha}[\mathbf{P}_{\alpha}\mathbf{W}_{\alpha} + \mathbf{G}_{\alpha}^{-1}]^{-1}\mathbf{P}_{\alpha}. \tag{16}$$

Successful application of the proposed design algorithm depends on two points: (i) Does there exist a controller $\mathbf{G}$ that stabilizes the uncertain plant $\mathcal{P}\mathbf{W}$ and satisfies the *Bounds* as calculated in each design step? A solution for this problem for a large class of plants has been given by Yaniv (1991) and by Perez *et al.* (1993); and (ii) Is $\sigma_k(\omega)$ a *Legitimate Bound* (see definition 3.1) for all k and ω?

Definition 3.1 *We shall say that $\sigma_k(\omega)$ is a Legitimate Bound if there exists a positive scalar $c_k(\omega)$ such that $g_k \in \sigma_k(\omega)$ if $|g_k(\omega)| > c_k(\omega)$.*

Design to QFT specs must have *Legitimate Bounds* in low frequencies because the low-frequency specs force large control efforts. This is true also for the specs 2, where the lower permitted deviation of the plant input from the optimal input forces larger control efforts. To see this, substitute equation 16 into equation 11 to get

$$\begin{aligned}\lim_{g_i \to \infty, i=1,..,n} \tilde{\mathbf{u}}_k - \mathbf{u}_k &= [\mathbf{I} - \mathbf{W}[\mathbf{P}\mathbf{W}]^{-1}\mathbf{P}]\mathbf{u}_k \\ &= \mathbf{u}_k - \mathbf{U}[\mathbf{P}\mathbf{U}]^{-1}\mathbf{P}\mathbf{u}_k = 0 \end{aligned} \tag{17}$$

The next Lemma proves that *Bounds* calculated by the proposed algorithm to satisfy the specs 2 are *Legitimate Bounds*.

Lemma 3.3 *The Bounds $\sigma_k(\omega)$ as calculated from inequalities 15 are Legitimate Bounds.*

Proof: By induction. For $k = 1$, $g_2, ..., g_n$ are assumed to be ∞ in the definition of $\mathbf{C}_{1i}$, and so, from equation 17, $\sigma_1(\omega)$ is a *Legitimate Bound*. If $\sigma_k(\omega)$ is a *Legitimate Bound* then inequalities 15 are valid if in the definition of $\mathbf{C}_{ki}$, $g_m = \infty$ for $m = k+1,..,n$. Hence there exists $c_{k+1}(\omega)$ such that if $|g_{k+1}(\omega)| > c_{k+1}(\omega)$, inequalities 15 are valid.

4 Example

In Fig. 1 the uncertain plant family is

$$\mathbf{P} = \frac{1}{s(s+2)} \begin{bmatrix} k_{11} & k_{12} & 1.5(s+a) \\ k_{21} & k_{22} & 2.5(s+a) \end{bmatrix}, \tag{18}$$

where $k_{11} \in [1.5, 2.5]$, $k_{12} \in [3.5, 4.5]$, $k_{21} \in [4.5, 5.5]$, $k_{22} \in [2.0, 3.0]$ and $a \in [0.5, 1.5]$. Let

$$\mathbf{P}_1 = \frac{1}{s(s+2)} \begin{bmatrix} 2.0 & 4.0 & 1.5(s+1) \\ 5.0 & 2.5 & 2.5(s+1) \end{bmatrix}, \tag{19}$$

for which the two optimal inputs that cancel the disturbances

$$\begin{aligned} \mathbf{d}_1 &= \frac{1}{s(s+2)(s+3)} \begin{bmatrix} -1.5 \\ -1.5 \end{bmatrix} \quad and \\ \mathbf{d}_2 &= \frac{1}{s(s+2)} \begin{bmatrix} -0.9 \\ -1.5 \end{bmatrix}, \end{aligned} \tag{20}$$

are

$$\mathbf{u}_1 = \frac{1}{s+3} \begin{bmatrix} 0.15 \\ 0.3 \\ 0 \end{bmatrix} \quad and \quad \mathbf{u}_2 = \frac{1}{s+1} \begin{bmatrix} 0.5 \\ 0.2 \\ 0.6 \end{bmatrix}. \tag{21}$$

The closed-loop specs 2 are the same for all channels, that is, the plant input vector $\tilde{\mathbf{u}}_1$ for disturbance $\mathbf{d}_1$ and plant input vector $\tilde{\mathbf{u}}_2$ for disturbance $\mathbf{d}_2$ should satisfy

$$|\tilde{\mathbf{u}}_i(i\omega) - \mathbf{u}_i(i\omega)| \leq \mathbf{a}_i, \tag{22}$$

where the vector $\mathbf{a}_i$ has the same entry in each of its element for a given ω. These are 0.01, 0.02, 0.05, 0.1, 0.2, 0.4 and 0.5 for the frequencies 1, 2, 5, 10, 15, 20 and 30 respectively. From equation 6, using $\mathbf{H} = \mathbf{I}$,

$$\mathbf{W} = \frac{1}{s+3} \begin{bmatrix} 0.15 & 0.5 \\ 0.3 & 0.2 \\ 0 & 0.6 \end{bmatrix}. \tag{23}$$

4.1 Design

Fig. 3 shows the *Bounds* which solve inequality 15 for $k = 1$, assuming $g_2 = \infty$, and which intersect with the usual QFT *Bounds* that guarantee stability for all plant uncertainty with margin condition (Yaniv 1992):

$$|1 + \mathbf{L}_i| > -4db; \tag{24}$$

Fig. 3 also presents the open loop for the nominal plant derived from the chosen nominal $\mathbf{P}_1\mathbf{W}$. The controller g_1 is

$$g_1 = \frac{34(1+s/0.4)(1+s/3.2)(1+s/10.5)}{(1+s/2.5)(1+s/70)(1+s/70+s^2/70^2)}. \tag{25}$$

Fig. 4 shows the *Bounds* which solve inequalities 15 for $k = 2$ for g_2 (now g_1 is known, as given in equation 25) and which intersect with the usual QFT *Bounds* that guarantee stability for all plant uncertainty with the condition 24. Fig. 4 also presents the open loop for the nominal plant derived from the chosen nominal $\mathbf{P}_1\mathbf{W}$. The controller g_2 is

$$g_2 = \frac{113(1+s/2.2)(1+s/20)(1+s/16.5)}{(1+s/5)(1+s/10)(1+s/83+s^2/83^2)}. \tag{26}$$

Fig. 5 presents the Bode plot of 5 elements of the nondiagonal controller $\mathbf{WG}$.
(the element 31 is 0).

4.2 Simulations

Simulations that validate Performance 22 are presented in Fig. 6. The * mark is the maximum allowed value for all 6 plant elements. Clearly these are all below the * sign in each frequency. In Fig. 4 the open loop touches the *Bounds* at $\omega = 5, 10$, so that no overdesign should be expected. This is shown in the simulation in Fig. 6, where the * sign touches the highest plant input. This fact guides the designer in eliminating overdesign, i.e., by shaping the controller to be as close as possible to the *Bounds*. Simulations that validate Performance 24 are presented in Fig. 7.

Time domain simulations that represent the output and the control efforts of the plant $\mathbf{P}_1$ for plant output disturbances $\mathbf{d}_1$ and $\mathbf{d}_2$ are given in Fig. 8 and Fig. 9. The optimal control efforts shown by the dot lines. Fig. 10 and Fig. 11 show the output and the control efforts of the plant $\mathbf{P}_1$ for plant output disturbances $\mathbf{d}_1$ and $\mathbf{d}_2$ for 33 cases of the plant uncertainty, dot lines coresponded to nominal plant $\mathbf{P}_1$.

5 Conclusions

This paper extends the QFT design technique for MIMO nxn systems to nxm systems ($n < m$) with output feedback, to have either (i) control efforts for n pairs of plants and disturbance commands to lie in a preferred set, or (ii) control efforts for one plant and n disturbance commands to lie in a preferred set. It was shown that the controller must have a special structure of the form $\mathbf{WG}$, where $\mathbf{W}$ is a function of the optimal control efforts and $\mathbf{G}$ is a square nxn controller. The algorithm presented here, to calculate the QFT *Bounds* to satisfy the control effort tolerances in each step of the sequential design process, must succeed in the sense that it result in *Legitimate Bounds*. An example of a highly uncertain $2x3$ plant which emphasized the efficiency of the technique in leading the designer to solutions with almost no overdesign was presented.

Appendix

The proof of equation (13) is a trivial consequence of Kailath (1980, p. 655), applied on the right side of

$$\begin{aligned}[\mathbf{I}+\mathbf{WGP}]^{-1} &= [\mathbf{I}+\mathbf{W}(\mathbf{G}-\mathbf{G}_k)\mathbf{P}+\mathbf{WG}_k\mathbf{P}]^{-1} \\ &= [\mathbf{C}+g_k\mathbf{u}_k\mathbf{v}]^{-1} \end{aligned} \tag{27}$$

where $\mathbf{u} = \mathbf{w}_k$ and $\mathbf{v}'$ is the kth column of $\mathbf{P}$.
Proof of equation (16): Since rows and columns α of $\mathbf{G}^\alpha$ are zero and $\mathbf{G}^\alpha$ diagonal

$$\mathbf{I}+\mathbf{WG}^\alpha\mathbf{P} = \mathbf{I}+\mathbf{W}_\alpha\mathbf{G}^\alpha\mathbf{P}_\alpha = \mathbf{I}+\mathbf{W}_\alpha\mathbf{G}_1\mathbf{P}_\alpha, \tag{28}$$

where $\mathbf{G}_1$ is the same as $\mathbf{G}_\alpha$, except for the diagonal zero elements, which are replaced by 1. Now use Kailath (1980, p. 656) and substitute $\mathbf{G}_1^{-1} = \mathbf{G}_\alpha^{-1}$.

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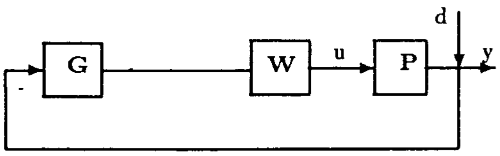

Figure 1: A MIMO one DOF feedback structure

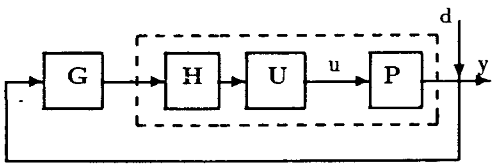

Figure 2: The modified feedback structure

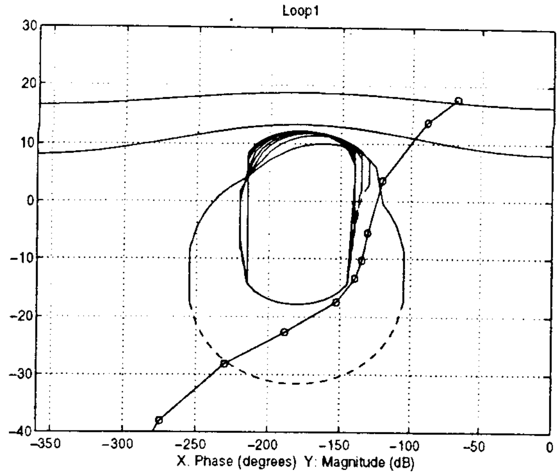

Figure 3:*Bounds* and open loop for L_1 (g_1)

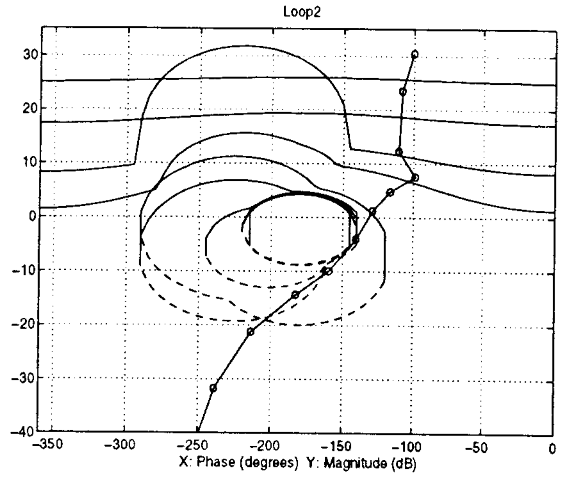

Figure 4:*Bounds* and open loop for L_2 (g_2)

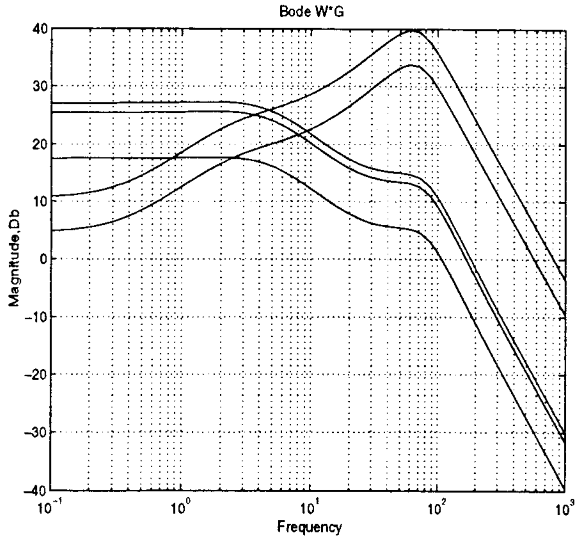

Figure 5:Bode functions for 5 elements of nondiagonal controller **WG**

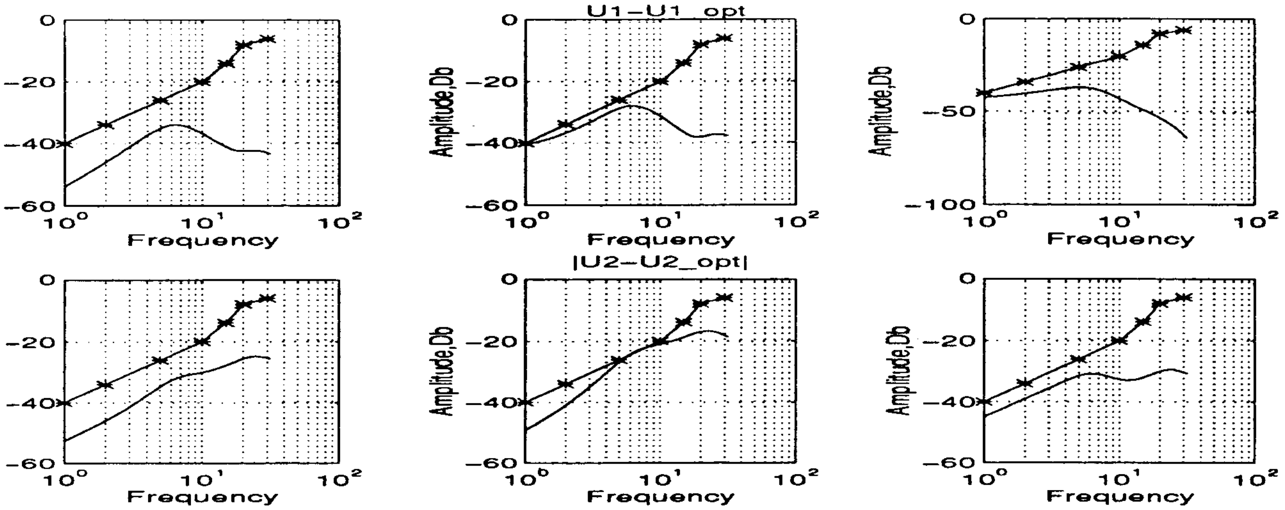

Figure 6: Simulation to validate closed-loop specs for control efforts

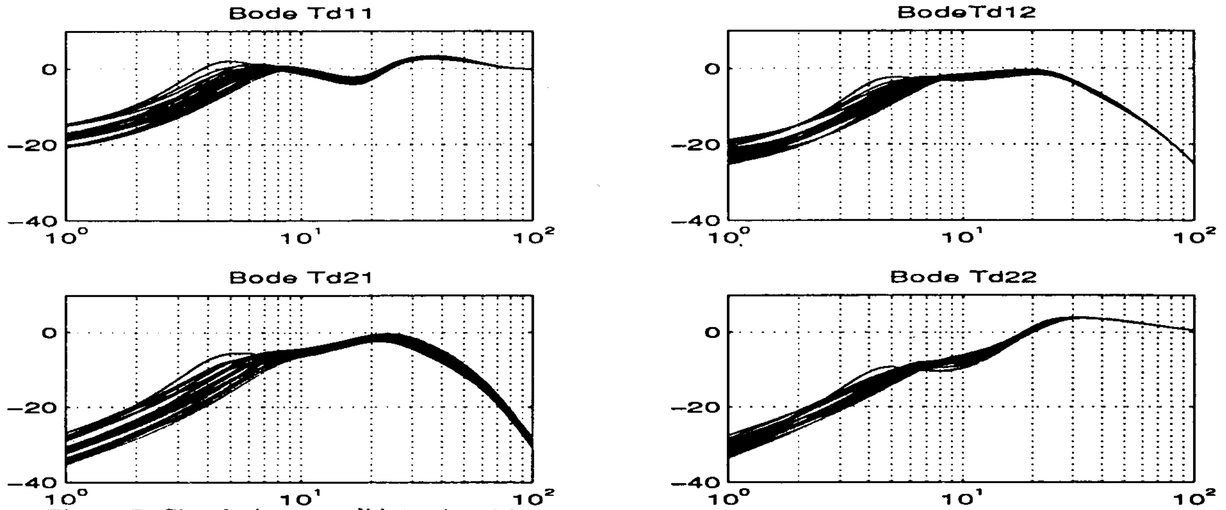

Figure 7: Simulation to validate closed-loop margin specs

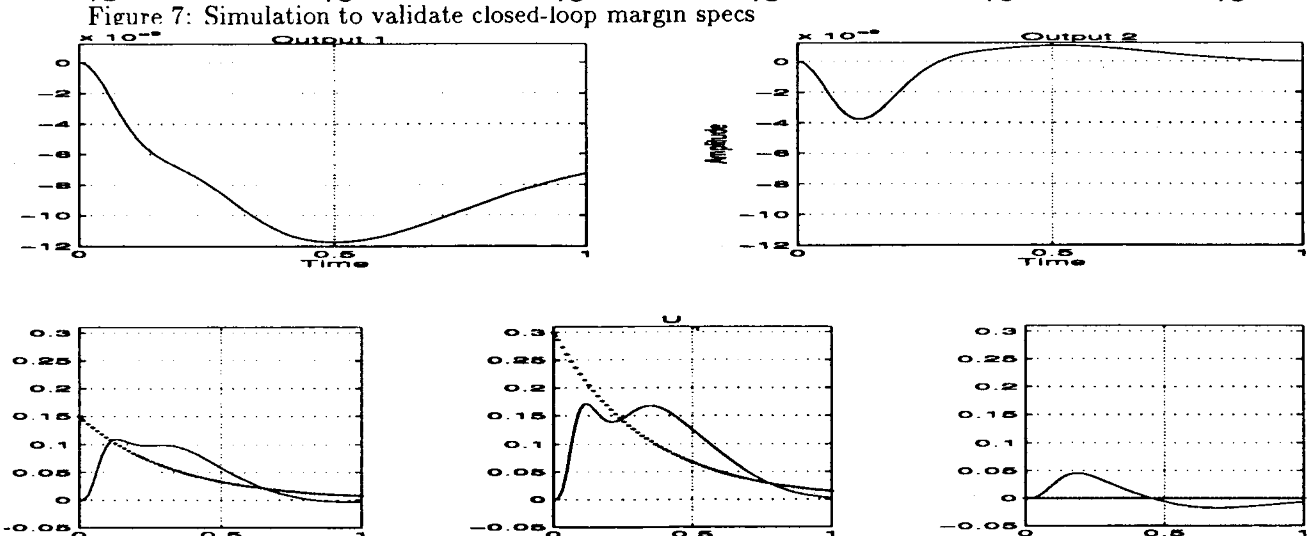

Figure 8: Time domain simulation for d_1(dot lines for optimal control efforts U_1)

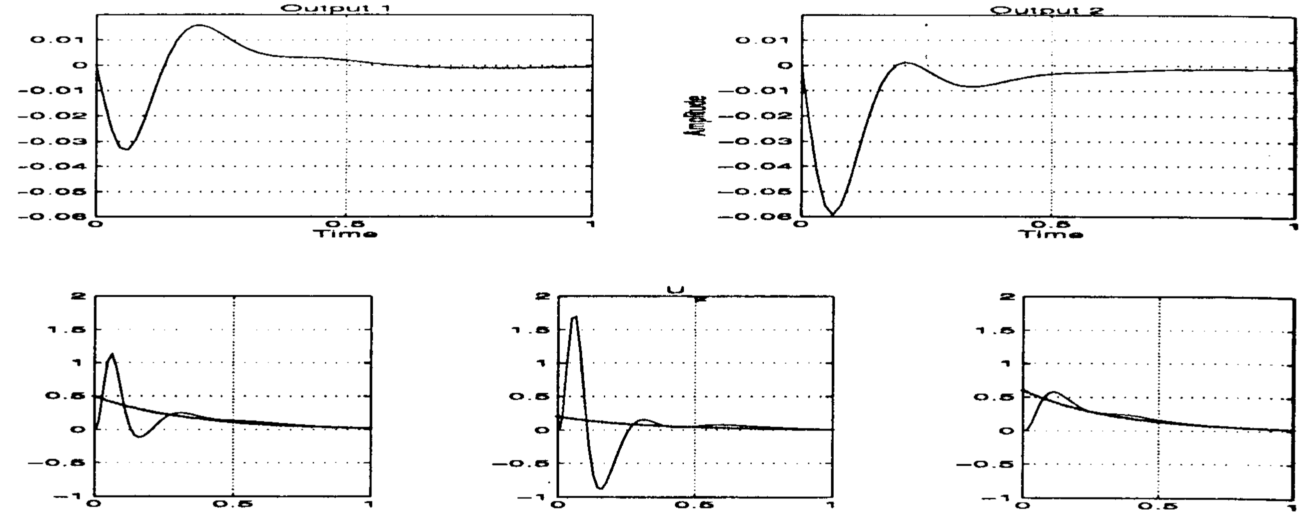

Figure 9: Time domain simulation for d_2(dot lines for optimal control efforts U_2)

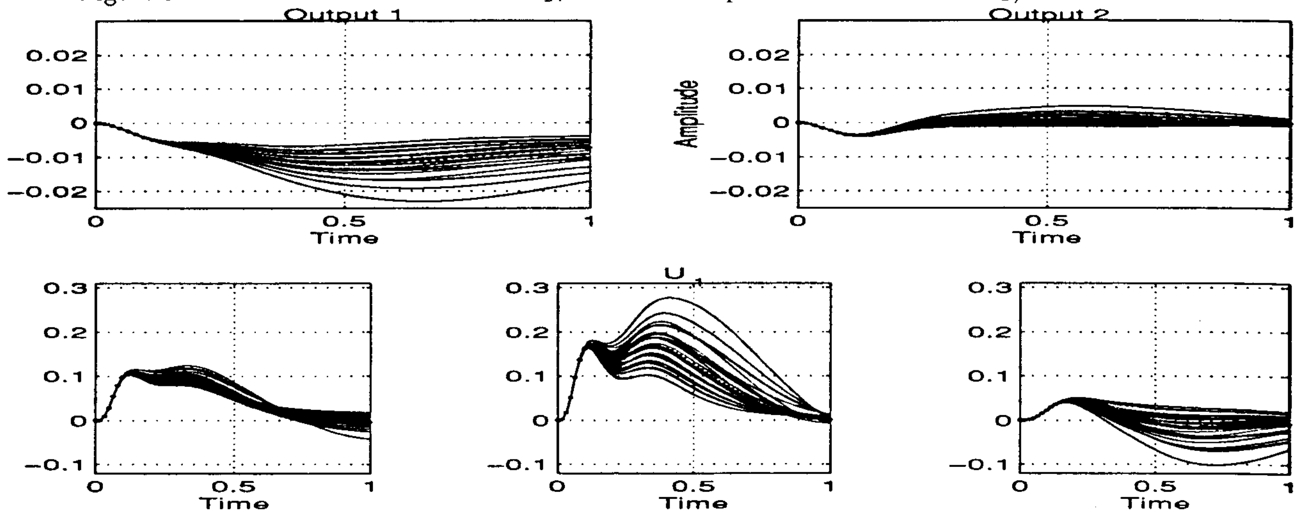

Figure 10: Time domain simulation for d_1 for 33 cases of the plant(dot lines for nominal plant) .

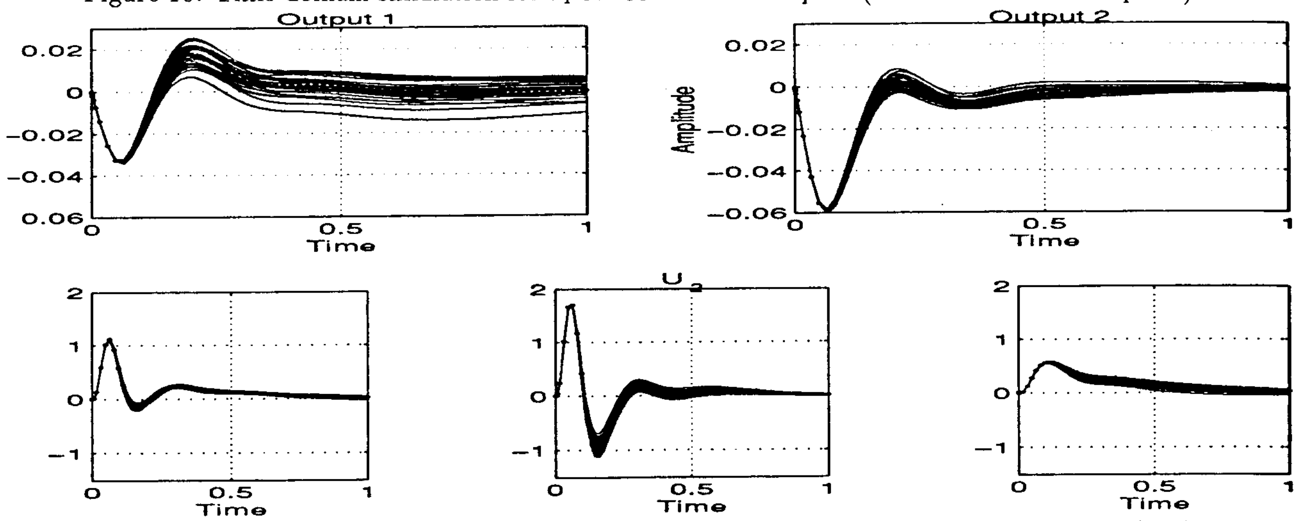

Figure 11: Time domain simulation for d_2 for 33 cases of the plant(dot lines for nominal plant) .