Asymptotic observers for continuous-time linear periodic systems[1]

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Abstract This paper considers the problem of the design of asymptotic observers for continuous-time linear periodic systems. First, under a simplifying assumption, necessary and sufficient conditions, involving the concept of right eigenvector, are derived for the observability of continuous-time linear periodic systems. Then, under the same simplifying assumption and under such necessary and sufficient conditions, a procedure is proposed for the observer design.

1. Introduction

The interest in considering periodic linear systems is motivated by the large variety of processes that can be modelled by (difference or differential) linear equations with periodic coefficients (see, e.g., [1]-[7] for the continuous-time ones and [8]-[18] for the discrete-time ones). A control theory is developing for periodic linear systems, and contributions on several control problems have been given, including state and output dead-beat control, disturbance localization, model matching, robust tracking and regulation, block decoupling, and adaptive control [19]-[31].

A nontrivial problem is the design of asymptotic observers for continuous-time linear periodic systems, and the aim of this paper is to give a partial answer to this problem.

The outline of the paper is as follows. Section 2 briefly recalls some preliminary results about the eigenvalues and the eigenvectors of continuous-time linear periodic systems, while Section 3 reports the necessary and sufficient conditions for the existence of an asymptotic observer for a class of continuous-time linear periodic systems, and a procedure is proposed for the observer design, which is effective for the same class of continuous-time linear periodic systems. Section 4 completes the paper with an illustrative example.

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2. Problem definition and preliminary results

The continuous-time linear periodic systems of period ω (briefly, ω-periodic) that are considered in this paper, are described by:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), && \text{(1a)}\\ y(t) &= C(t)x(t) + D(t)u(t), && \text{(1b)} \end{aligned}$$

where $\omega \in \mathbb{R}, \omega > 0$, $t \in \mathbb{R}, t \geq 0$, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the control input, $y(t) \in \mathbb{R}^q$ is the measured output, and $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, are real matrices that are ω-periodic continuous functions of $t \in \mathbb{R}$.

A dynamic system that is able to give an asymptotic estimate of $x(t)$, with an arbitrary fixed rate of convergence, on the basis of the available measurements of $y(t)$ and $u(t)$, is referred to as **asymptotic observer** for system (1). If any, an asymptotic observer for system (1) can be taken as described by the following equation:

$$\begin{aligned} \dot{\hat{x}}(t) &= A(t)\hat{x}(t) + H(t)\left(C(t)\hat{x}(t) - y(t)\right) \\ &+ \left(B(t) + H(t)D(t)\right)u(t), \quad \forall t \in \mathbb{R},\, t \geq 0, \quad (2) \end{aligned}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$, and $H(t)$ is a suitable ω-periodic real matrix; it is easy to see that the corresponding estimation error $\tilde{x}(t) := \hat{x}(t) - x(t)$ satisfies the following error dynamics:

$$\dot{\tilde{x}}(t) = \left(A(t) + H(t)C(t)\right)\tilde{x}(t), \quad \forall t \in \mathbb{R},\, t \geq 0. \quad (3)$$

The aim of this paper is to propose a simple algorithm for the choice of matrix $H(t)$, so that the error dynamics (3) are exponentially stable, with an arbitrary assigned rate of convergence of the free motions.

In order to formally state this problem, a basic definition and a preliminary result are given with reference to an ω-periodic homogeneous linear system, such as the one that is obtained from (1a) by setting $u(t) = 0, \forall t \in \mathbb{R}$, which is described by the following equation:

$$\dot{x}(t) = A(t)x(t), \quad \forall t \in \mathbb{R}, t \geq 0. \tag{4}$$

Definition 1 (See [32]). *A complex λ is an* **eigenvalue** *of the ω-periodic matrix $A(t)$ if and only if there exists an ω-periodic differentiable vector function $v(\cdot) \in \mathbb{C}^n$ of $t \in \mathbb{R}$, $v(t) \neq 0$ for all $t \in \mathbb{R}$, which is referred to as a* **right eigenvector** *of $A(t)$, such that the vector function $\xi(\cdot)$ of $t \in \mathbb{R}$, defined as follows*

$$\xi(t) := v(t)e^{\lambda t}, \quad \forall t \in \mathbb{R}, t \geq 0, \tag{5}$$

is solution of (4) from the initial time $t = 0$; such a vector function $\xi(\cdot)$ is called an **eigensolution** *of (4) with eigenvalue λ.*

The following lemma is reported from [32] without proof.

Lemma 1 *System (4) is exponentially stable if and only if all the eigenvalues of matrix $A(t)$ have negative real part.*

For each given $\rho \in \mathbb{R}, \rho < 0$, if system (4) is exponentially stable with all its eigenvalues having real part less then ρ, then the free motions of (4) go to zero faster than $e^{\rho t}$.

Now, the problem that will be studied in this paper, under the subsequent Assumption 1, is formally stated.

Problem 1 *Find, if any, an ω-periodic continuous matrix function $H(\cdot) \in \mathbb{R}^{n \times q}$, such that all the eigenvalues of the dynamic matrix $A(t) + H(t)C(t)$ of the error dynamics (3) are arbitrarily placed, with negative real part.*

The remainder of this section is devoted to give some preliminary results (which are reported without proof), stating properties of the eigenvalues and eigenvectors of an ω-periodic homogeneous system. These properties will be useful for giving a simple solution of Problem 1.

Lemma 2 (See [32]). *Let $v(\cdot) \in \mathbb{C}^n$ be an ω-periodic differentiable vector function of $t \in \mathbb{R}$, different from the zero vector for all $t \in \mathbb{R}$. Then, $v(t)$ is a right eigenvector of $A(t)$ with eigenvalue $\lambda \in \mathbb{C}$ if and only if the following relation holds:*

$$\dot{v}(t) = [A(t) - \lambda I]v(t), \quad \forall t \in \mathbb{R}, t \geq 0. \tag{6}$$

Remark 1 By Lemma 2, for some $\lambda \in \mathbb{C}$, a solution $v(t)$ of (6) is a right eigenvector of $A(t)$ with eigenvalue λ if and only if it is an ω-periodic function of $t \in \mathbb{R}$ different from the zero vector for all $t \in \mathbb{R}$. □

Let $\Phi(t, \tau)$, $t, \tau \in \mathbb{R}$, be the state transition matrix of (4). Then, the following lemma gives conditions, based on $\Phi(t, \tau)$, for a complex λ to be an eigenvalue of $A(t)$.

Lemma 3 *The complex λ is an eigenvalue of $A(t)$ if and only if the following relation holds:*

$$\det[\Phi(\omega, 0) - \eta I] = 0, \tag{7}$$

where $\eta := e^{\lambda\omega}$.

Definition 2 (See [32]). *The following polynomial*

$$p(\eta) := \det[\Phi(\omega, 0) - \eta I]$$

is referred to as the **characteristic polynomial** *of the ω-periodic matrix $A(t)$, and the n roots of $p(\eta) = 0$ are referred to as the* **characteristic multipliers** *of $A(t)$.*

If η is a characteristic multiplier of $A(t)$, then a number λ such that $\eta = e^{\lambda\omega}$ is referred to as eigenvalue of $A(t)$.

Let Λ be a matrix in Jordan form. Matrix Λ is referred to as the **Jordan form** of matrix $A(t)$ if and only if there exists a differentiable ω-periodic matrix function $V(\cdot) \in \mathbb{C}^{n \times n}$ of time $t \in \mathbb{R}$ that is nonsingular for all $t \in \mathbb{R}$, i.e.

$$V(\cdot) \in C^1(\mathbb{R}, \mathbb{C}^{n \times n}), \quad V(t + \omega) = V(t),$$
$$\det(V(t)) \neq 0, \quad \forall t \in \mathbb{R}, \tag{8}$$

such that

$$\dot{V}(t) = A(t)V(t) - V(t)\Lambda, \quad \forall t \in \mathbb{R}, t \geq 0. \tag{9}$$

If Λ is diagonal with diagonal entries λ_i, $i = 1, 2, \ldots, n$, equation (9) can be rewritten as follows:

$$\dot{v}_i(t) = (A(t) - \lambda_i I)\, v_i(t),$$
$$\forall t \in \mathbb{R},\ t \geq 0,\ i = 1, 2, \ldots, n, \tag{10}$$

where the i-th column $v_i(t)$ of $V(t)$ is just the right eigenvector of matrix $A(t)$ with eigenvalue λ_i.

Since $V(t)$ is nonsingular, ω-periodic, and differentiable for all $t \in \mathbb{R}$, then $x(t) = V(t)z(t)$ qualifies as a Floquet-Lyapunov ω-periodic state space transformation, and Λ qualifies as the dynamic matrix of the ω-periodic system (4) expressed in the z-coordinates:

$$\dot{z}(t) = \Lambda z(t), \quad t \in \mathbb{R}, t \geq 0. \tag{11}$$

The following lemma is given in order to introduce the concept of left eigenvector, with the aim of extending the similar concept that is well known in the case of time-invariant linear systems.

Lemma 4 *Let $V(t)$ be a differentiable nonsingular ω-periodic matrix satisfying* (9), *and let $W(t)$ be the inverse of $V(t)$. Then, $W(t)$ is time differentiable, nonsingular, ω-periodic and satisfies the following relation*

$$\dot{W}(t) = -W(t)A(t) + \Lambda W(t), \quad \forall t \in \mathbb{R}, t \geq 0. \quad (12)$$

If Λ is a diagonal matrix with diagonal entries λ_i, $i = 1, 2, \ldots, n$, equation (12) can be rewritten as follows:

$$\begin{aligned} \dot{w}_i(t) &= w_i(t)(\lambda_i I - A(t)), \\ &\forall t \in \mathbb{R}, t \geq 0, \ i = 1, 2, \ldots, n, \quad (13) \end{aligned}$$

where $w_i(t)$ is the i-th row of matrix $W(t)$ and is referred to as the **left eigenvector** of matrix $A(t)$ with eigenvalue λ_i.

Remark 2 By Lemma 4, for some $\lambda \in \mathbb{C}$, a solution $w_i(t)$ of (13) is a left eigenvector of $A(t)$ with eigenvalue λ_i if and only if it is an ω-periodic differentiable function of $t \in \mathbb{R}$ different from the zero vector for all $t \in \mathbb{R}$. □

3. The proposed procedure

The procedure proposed in this paper for the design of matrix $H(t)$ is based on the notion of right eigenvector $v(t)$ of matrix $A(t)$ with eigenvalue λ, as defined in (6), and on the following Lemma 5, which is implied by a similar result given in [33] with reference to time-varying systems. For the sake of simplicity, this lemma will be stated under the following simplifying assumption.

Assumption 1 *The Jordan form Λ of matrix $A(t)$ is diagonal, with its diagonal entries λ_i, $i = 1, 2, \ldots, n$, being real and distinct.*

It is stressed that, the subsequent results should be properly modified if Assumption 1 is removed, thus allowing matrix Λ to be block diagonal, with diagonal entries possibly complex and coincident.

Lemma 5 *Under Assumption 1, the ω-periodic system* (1) *is observable if and only if there is no eigenvalue λ_i of $A(t)$, $i \in \{1, 2, \ldots, n\}$, such that:*

$$C(t)v_i(t) = 0, \quad \forall t \in [0, \ \omega], \quad (14)$$

where $v_i(t)$ is the right eigenvector of $A(t)$ with eigenvalue λ_i, $i = 1, 2, \ldots, n$.

Proof. By virtue of Theorem 5-9 of [33], the ω-periodic system (1) is observable if and only if all the columns of matrix $M(t) = C(t)\Phi(t, 0)$ are linearly independent functions of t over the field of complex numbers, where $\Phi(\cdot, \cdot)$ is the state transition matrix of system (1). By virtue of Assumption 1, matrix $M(t)$ can be rewritten as $M(t) = C(t)V(t)\mathrm{e}^{\Lambda t}W(0)$, where Λ is the diagonal Jordan form of $A(t)$, the rows of $W(t)$ are the left eigenvectors of $A(t)$ and the columns of $V(t)$ are the right eigenvectors of $A(t)$, with eigenvalues being the real and distinct diagonal entries of Λ, respectively. Since $\det W(0) \neq 0$, and Λ is diagonal with distinct diagonal entries by Assumption 1, the columns of $M(t)$ are linearly independent over the field of complex numbers if and only if all the columns $\mathrm{e}^{\lambda_i t}C(t)v_i(t)$, $i \in \{1, 2, \ldots, n\}$, of $C(t)V(t)\mathrm{e}^{\Lambda t}$ are not identically zero in $[0, \ \omega]$; since $\mathrm{e}^{\lambda_i t}$, $i \in \{1, 2, \ldots, n\}$, is non zero for all $t \in \mathbb{R}$, this condition implies that (14) must not hold, as was to be proved. □

It is stressed that if there is no eigenvalue λ_i of $A(t)$, $i = 1, 2, \ldots, n$, such that (14) holds, then for each $i = 1, 2, \ldots, n$, there exists a time $t_i \in [0, \ \omega]$ such that

$$C(t_i)v_i(t_i) \neq 0; \quad (15)$$

then, the continuity of functions $v_i(\cdot)$, and $C(\cdot)$ implies that if system (1) is observable, then the following relation holds for each right eigenvector $v_i(t)$ of $A(t)$ with eigenvalue λ_i:

$$\int_0^{\omega} [C(\tau)v_i(\tau)]^T [C(\tau)v_i(\tau)] d\tau \neq 0, \ i = 1, 2, \ldots, n. \quad (16)$$

Under Assumption 1, consider the following procedure (which can be completed if system (1) satisfies condition (16) with $i = 1$), for the computation of an ω-periodic matrix $H_1(t)$, such that the eigenvalues of $A(t) + H_1(t)C(t)$ are exactly the ones of $A(t)$, except the real eigenvalue λ_1 that is shifted into the new real location γ_1, $\gamma_1 < 0$, as stated and proved in the subsequent Proposition 1.

Procedure 1 (Step 1). Compute a real eigenvalue λ_1 of $A(t)$ to be shifted, and compute the corresponding real right eigenvector $v_1(t)$ (it is noted that $v_1(t)$ can always be computed real, by the assumptions). Let γ_1, $\gamma_1 < 0$, be the real eigenvalue desired for $A(t) + H_1(t)C(t)$, instead of λ_1, $\gamma_1 \neq \lambda_i$, $i = 2, 3, \ldots, n$.

(Step 2). Taking into account that (16) holds with $i = 1$ (by the assumptions), compute the following real continuous ω-periodic function of t:

$$\begin{aligned} \alpha_1(t) &:= \frac{(\gamma_1 - \lambda_1)\omega}{\displaystyle\int_0^{\omega} [C(\tau)v_1(\tau)]^T [C(\tau)v_1(\tau)] \mathrm{d}\tau} [C(t)v_1(t)]^T, \\ &\forall t \in \mathbb{R}, t \geq 0. \quad (17) \end{aligned}$$

(Step 3). Define the following ω-periodic matrix:

$$H_1(t) := v_1(t)\alpha_1(t), \quad \forall t \in \mathbb{R}, t \geq 0. \tag{18}$$

Proposition 1 *Under Assumption 1, and under the assumption that condition* (16) *holds with* $i = 1$, *denote by* $\hat{A}(t)$ *the* ω*-periodic matrix* $A(t)+H_1(t)C(t)$, *where* $H_1(t)$ *is given in* (18). *Then,*

(i) the pair $(C(t), \hat{A}(t))$ *is observable if system* (1) *is observable,*

(ii) the characteristic polynomial $\hat{p}(\eta)$ *of matrix* $A(t) + H_1(t)C(t)$ *is obtained from the characteristic polynomial* $p(\eta) = \prod_{i=1}^{n}(\eta - e^{\lambda_i \omega})$ *of* $A(t)$ *by replacing the factor* $(\eta - e^{\lambda_1 \omega})$ *with the factor* $(\eta - e^{\gamma_1 \omega})$.

(iii) Assumption 1 holds with matrix $A(t)$ *replaced by* $A(t) + H_1(t)C(t)$.

Proof. The first part of the proposition can be trivially proved by taking into account that an ω-periodic output injection, such as the one represented by $H_1(t)$, does not alter the observability property of an ω-periodic system.

In order to show the second part of the proposition, first it will be shown that γ_1 is an eigenvalue of $A(t) + H_1(t)C(t)$ with right eigenvector

$$\hat{v}_1(t) := d(t)v_1(t), \quad \forall t \in \mathbb{R}, t \geq 0, \tag{19}$$

where the ω-periodic differentiable function $d(t), d(t) \neq 0$ for all $t \in \mathbb{R}$, is given by:

$$d(t) := \exp\left(\int_0^t \lambda_1 - \gamma_1 + \alpha_1(\tau)[C(\tau)v_1(\tau)]\, \mathrm{d}\tau\right).$$

This is easily proved by showing that the following relation (which is obtained from (10) by replacing $v_i(t)$, λ_i, and $A(t)$ by $\hat{v}_1(t)$, γ_1, and $A(t) + H_1(t)C(t)$, respectively), holds:

$$\dot{\hat{v}}_1(t) = (A(t) + H_1(t)C(t) - \gamma_1 I)\, \hat{v}_1(t), \quad \forall t \in \mathbb{R}, t \geq 0. \tag{20}$$

Since the following relations hold:

$$\begin{aligned} \dot{\hat{v}}_1(t) &= \dot{d}(t)v_1(t) + d(t)\dot{v}_1(t) \\ &= d(t)v_1(t)\left(\lambda_1 - \gamma_1 + \alpha_1(t)[C(t)v_1(t)]\right) \\ &+ d(t)A(t)v_1(t) - d(t)\lambda_1 v_1(t) \\ &= -d(t)\gamma_1 v_1(t) + d(t)v_1(t)\alpha_1(t)[C(t)v_1(t)] \\ &+ d(t)A(t)v_1(t), \end{aligned} \tag{21}$$

and

$$\begin{aligned} &(A(t) + H_1(t)C(t) - \gamma_1 I)\, \hat{v}_1(t) = d(t)A(t)v_1(t) \\ &+ d(t)v_1(t)\alpha_1(t)[C(t)v_1(t)] - d(t)\gamma_1 v_1(t), \end{aligned} \tag{22}$$

one can easily see, by comparing the right hand sides of (21) and (22), that (20) holds.

For the other eigenvalues, denote by $w_i(t)$ the left eigenvector of $A(t)$ with eigenvalue λ_i, $i = 2, 3, \ldots, n$; for each time $t \in \mathbb{R}, t \geq 0$, the vectors $w_i(t)$, $i = 2, 3, \ldots, n$, are linearly independent over the field of complex numbers and satisfy:

$$w_i(t)v_1(t) = 0, \quad \forall t \in \mathbb{R}, t \geq 0, \quad i = 2, 3, \ldots, n. \tag{23}$$

In order to show that the real numbers λ_i, $i = 2, 3, \ldots, n$, are still eigenvalues of matrix $A(t)+H_1(t)C(t)$, it is sufficient to show that the following relations (which are obtained from (13) by replacing $A(t)$ by $A(t) + H_1(t)C(t)$) hold:

$$\begin{aligned} \dot{w}_i(t) &= w_i(t)(\lambda_i I - A(t) - H_1(t)C(t)), \\ &\forall t \in \mathbb{R}, t \geq 0, \quad i = 2, 3, \ldots, n. \end{aligned} \tag{24}$$

Relations (24) are yielded by the following equalities ($i = 2, 3, \ldots, n$), which are obtained by taking (18) and (23) into account:

$$\dot{w}_i(t) = w_i(t)(\lambda_i I - A(t)), \tag{25a}$$

$$\begin{aligned} w_i(t)(\lambda_i I - A(t) - H_1(t)C(t)) &= \\ w_i(t)(\lambda_i I - A(t)) + w_i(t)v_1(t)\alpha_1(t)C(t) &= \\ w_i(t)(\lambda_i I - A(t)). \end{aligned} \tag{25b}$$

Therefore, relations (24) show that the real numbers λ_i, $i = 2, 3, \ldots, n$, are still eigenvalues of the matrix $A(t) + H_1(t)C(t)$, with left eigenvectors $w_i(t)$, $i = 2, 3, \ldots, n$. The proof is completed taking into account taht γ_1 has been chosen as $\gamma_1 \neq \lambda_i$, $i = 2, 3, \ldots, n$, and $\lambda_i \neq \lambda_j$ if $i \neq j$, by Assumption 1. □

Under the assumption that condition (16) holds with $i = 1$, and under Assumption 1, the algorithm given in Procedure 1 allows the real eigenvalue λ_1 to be shifted in another arbitrary location given by the real γ_1, $\gamma_1 \neq \lambda_i$, $i = 2, 3, \ldots, n$, preserving the positions of the other eigenvalues and the observability property of the closed-loop system, as well as Assumption 1 properly re-written. With an iterative application of a properly modified version of Procedure 1, it seems that all the eigenvalues λ_i, $i \in \{1, 2, \ldots, n\}$, of $A(t)$ could be shifted in arbitrary (real and distinct) new locations γ_i, $i \in \{1, 2, \ldots, n\}$, provided that system (1) is observable. Actually, under the assumption that system (1) is observable and under Assumption 1, the output injection, represented by $H(t)$, that shifts the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ in the new locations $\gamma_1, \gamma_2, \ldots, \gamma_n$, respectively, can be designed by an iterative computation of the following ω-periodic functions:

$$\alpha_i(t) \; := \; \frac{(\gamma_i - \lambda_i)\omega}{\displaystyle\int_0^{\omega} [C(\tau)\bar{v}_i(\tau)]^T [C(\tau)\bar{v}_i(\tau)] \mathrm{d}\tau} [C(\tau)\bar{v}_i(\tau)]^T, \quad \forall t \in \mathbb{R}, t \geq 0, \; i = 1, 2, \ldots, n, \tag{26}$$

and by defining

$$H_i(t) := \bar{v}_i(t)\alpha_i(t), \quad i = 1, 2, \ldots, n, \tag{27}$$

where $\bar{v}_i(t)$ denotes, with an abuse of notation, the right eigenvector of the matrix $A(t) + \sum_{j=1}^{i-1} H_j(t)C(t)$ with eigenvalue λ_i, $i = 1, 2, \ldots, n$; it is stressed that, matrix $A(t)+\sum_{j=1}^{i-1} H_j(t)C(t)$ is obtained from $A(t)$ after having shifted the eigenvalues λ_1, $\lambda_2, \ldots$, λ_{i-1} of $A(t)$ into the new locations γ_1, $\gamma_2, \ldots$, γ_{i-1} by means of $H_1(t)$, $H_2(t)$, $\ldots$, $H_{i-1}(t)$, respectively.

It is recalled that the overall iterative procedure represented by (26), (27) is based on the assumption that, at each step of the procedure involving the computation of (26), (27), both λ_i and γ_i are real and distinct from the other λ_j and γ_j, respectively, with $j \in \{1, 2, \ldots, n\}, i \neq j$. If such an assumption does not hold, then the procedure should be properly modified, with an increase of the complexity.

The main result of the paper is summarized by the following theorem, whose proof can be easily derived from the above discussion.

Theorem 1 *If Assumpiton 1 holds, then*

(a) Problem 1 can be solved if and only if there is no eigenvalue λ_i of $A(t)$, $i \in \{1, 2, \ldots, n\}$, such that (14) *holds;*

(b) if the necessary and sufficient condition of item (a) holds, then a solution of Problem 1 is

$$H(t) := \sum_{i=1}^{n} H_i(t), \quad \forall t \in \mathbb{R}, t \geq 0, \tag{28}$$

with the $H_i(t)$'s given by (26), (27).

4. Illustrative example

In order to illustrate the procedure proposed for the design of asymptotic observers for continuous-time linear ω-periodic systems, a simple example is illustrated.

Consider the periodic system (1), characterized by the following $A(\cdot)$ and $C(\cdot)$ matrices of period $\omega = 2\pi$:

$$A(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad C(t) = \begin{bmatrix} 1 & \cos(t) \end{bmatrix}, \tag{29}$$

whereas the expression of matrices $B(\cdot)$ and $D(\cdot)$ may be omitted, because they have not influence on the error dynamics (3).

It is easy to see that Assumption 1 holds, since matrix $A(t)$ is already in diagonal Jordan form, and its diagonal entries ($\lambda_1 = 1$, $\lambda_2 = 2$) are real and distinct; the right eigenvectors $v_1(t)$ and $v_2(t)$ of $A(t)$ with eigenvalues λ_1 and λ_2, respectively, are given by:

$$v_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{30}$$

The necessary and sufficient condition for the solvability of Problem 1, which is reported in item (a) of Theorem 1, holds because both of the following scalar functions are not identically zero in $[0, 2\pi]$:

$$C(t)v_1(t) = 1, \;\; C(t)v_2(t) = \cos(t), \;\; \forall t \in \mathbb{R}, \;\; t \geq 0. \tag{31}$$

Then, by item (b) of Theorem 1, a solution $H(t)$ of Problem 1 is given by (28). In order to compute such a solution $H(t)$, the eigenvalues to be obtained for the closed-loop system are chosen as $\gamma_1 = -1$, $\gamma_2 = -2$; they are real, negative, distinct, and distinct from the eigenvalues λ_1 and λ_2 of matrix $A(t)$.

The design procedure can be detailed as follows. The eigenvalue λ_1 is initially shifted in γ_1 and the eigenvalue λ_2 is left unchanged, with the 2π-periodic matrix $H_1(t)$ carried out through the following Steps 1.1-3.1.

(Step 1.1). Let $\lambda_1 = 1$, $\bar{v}_1(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, and $\gamma_1 = -1$.

(Step 2.1). On the basis of equation (26), since

$$C(t)\bar{v}_1(t) \;=\; 1, \tag{32a}$$

$$\int_0^{2\pi} [C(\tau)\bar{v}_1(\tau)]^T [C(\tau)\bar{v}_1(\tau)] \mathrm{d}\tau \;=\; 2\pi, \tag{32b}$$

it follows:

$$\alpha_1(t) = -2. \tag{33}$$

(Step 3.1). On the basis of (27) with $i = 1$, the 2π-periodic matrix $H_1(t)$ is given by:

$$H_1(t) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}. \tag{34}$$

After having shifted the eigenvalue λ_1 into γ_1, the matrix $A(t) + H_1(t)C(t)$ takes the form:

$$A(t) + H_1(t)C(t) = \begin{bmatrix} -1 & -2\cos(t) \\ 0 & 2 \end{bmatrix} \tag{35}$$

Now, the second eigenvalue λ_2 is shifted in γ_2 and the eigenvalue γ_1 is left unchanged, with the 2π-periodic matrix $H_2(t)$ carried out through the following Steps 1.2-3.2.

(Step 1.2). Let $\lambda_2 = 2$ and let $\gamma_2 = -2$. As for the right eigenvector $\bar{v}_2(t)$ of matrix $A(t) + H_1(t)C(t)$ with eigenvalue λ_2, a simple computation gives $\bar{v}_2(t) = \left[\begin{array}{cc} (\sin(t) - 3\cos(t)) & 5 \end{array}\right]^T$.

(Step 2.2). On the basis of equation (26), since

$$C(t)\bar{v}_2(t) = \sin(t) + 2\cos(t), \tag{36a}$$

$$\int_0^{2\pi} [C(\tau)\bar{v}_2(\tau)]^T [C(\tau)\bar{v}_2(\tau)] \mathrm{d}\tau = 5\pi, \tag{36b}$$

it follows:

$$\alpha_2(t) = -\frac{8}{5}\left(\sin(t) + 2\cos(t)\right). \tag{37}$$

(Step 3.2). On the basis of (27) with $i = 2$, the 2π-periodic matrix $H_2(t)$ is given by:

$$H_2(t) = \begin{bmatrix} -\frac{8}{5} + \frac{8}{5}\sin(t)\cos(t) + \frac{56}{5}\cos(t)^2 \\ -8\left(\sin(t) + 2\cos(t)\right) \end{bmatrix}. \tag{38}$$

Now, by equation (28) of item (b) of Theorem 1, the overall compensator $H(t)$ is given by:

$$H(t) = \begin{bmatrix} -\frac{18}{5} + \frac{8}{5}\sin(t)\cos(t) + \frac{56}{5}\cos(t)^2 \\ -8\left(\sin(t) + 2\cos(t)\right) \end{bmatrix} \tag{39}$$

and the matrix $A(t) + H(t)C(t)$ takes the form:

$$A(t) + H(t)C(t) = \begin{bmatrix} \frac{1}{5}\left(-13 + 8\sin(t)\cos(t) + 56\cos^2(t)\right) & -8\left(\sin(t) + 2\cos(t)\right) \\ \frac{1}{5}\left(-18\cos(t) + 8\sin(t)\cos^2(t) + 56\cos^3(t)\right) & 2\left(1 - 4\sin(t)\cos(t) - 8\cos^2(t)\right) \end{bmatrix} \tag{40}$$

the corresponding state transition matrix $\Phi(2\pi, 0)$, numerically computed over $[0, 2\pi]$, is given by:

$$\Phi(2\pi, 0) = 10^{-3} \cdot \begin{bmatrix} 5.4231 & 3.2517 \\ -5.9259 & -3.5520 \end{bmatrix}; \tag{41}$$

the characteristic polynomial of $A(t) + H(t)C(t)$ is given by $p(\eta) = \eta^2 - 1.8710 \times 10^{-3}\eta + 6.3978 \times 10^{-9}$ and the resulting characteristic multipliers are $\eta_1 = 1.8676 \times 10^{-3}$, $\eta_2 = 3.4839 \times 10^{-6}$; the corresponding real eigenvalues are $\frac{\log(1.8676 \times 10^{-3})}{2\pi} = -1$ and $\frac{\log(3.4839 \times 10^{-6})}{2\pi} = -2$, as desired.

5. Conclusions

This paper has considered the problem of the design of asymptotic observers for continuous-time linear periodic systems. Under the assumption that the Jordan form of the dynamic matrix of the ω-periodic system is diagonal, with real and distinct eigenvalues, necessary and sufficient conditions (involving the concept of right eigenvector) have been given for the observability of ω-periodic systems; in addition, under such an assumption, a procedure has been given for the design of asymptotic observers for continuous-time linear ω-periodic systems, provided that they are observable.

Future work will face the same problem, allowing the Jordan form of the dynamic matrix of the ω-periodic system to be block-diagonal, with diagonal entries possibly being complex and coincident.

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