Explicit Solutions for Nonlinear Partially Observable Stochastic Control Problems

[†]Charalambos D. Charalambous and [‡]Robert J. Elliott

[†]Measurement and Control Research Center, College of Engineering
Idaho State University, Pocatello, ID 83209

[‡]Department of Mathematical Sciences
University of Alberta, Edmonton,
Alberta, Canada T6G 2G1

Abstract. Continuous time nonlinear partially observable stochastic control problems are solved by explicitly determining the optimal control laws, which are reminiscent of the linear-exponential-quadratic-Gaussian and the linear-quadratic-Gaussian tracking problems. Using these results we compute explicitly, the optimal control laws for parameter identification problems.

1. Introduction

Recent results (see [1, 2]) have revealed classes of nonlinear partially observable stochastic control problems having optimal control laws equivalent to that associated with either the linear-exponential-quadratic-Gaussian (LEQG) [3, 4, 5] or the linear-quadratic-Gaussian (LQG) [6, 4] tracking problems. Historically, related work can be found in [7, 8, 5, 9]. This program is executed in two steps. First, a sufficient statistic (information state) summarizing the information available to the controller is identified, and the partially observable control problem is converted to an equivalent completely observable control problem in infinite dimensions. The state of the completely observable control problem satisfies a stochastic partial differential equation SPDE, similar to that associated with the conditional density of the nonlinear filtering problem, known as the Zakai equation [4]. Second, this SPDE is solved explicitly for nonlinear systems in terms of only a finite dimensional number of ordinary differential equations, and the problem is then cast into one of complete observations in finite dimensions. The optimal feedback control laws are then derived, and are shown to be reminiscent of that associated with LEQG/LQG tracking problems. It is important to note that in general, nonlinear partially observable control problems are infinite dimensional.

In this paper, we extent the results on finite dimensional nonlinear control problems considered in [1, 2] (and explained above) to new classes. We then show how to solve certain parameter identification problems by deriving explicit expressions for implementing the optimal control law.

We shall henceforth refer to the information state associated with the usual integral cost function as the Zakai equation, and to the information state associated with the cost function featuring both the integral and the exponential of the integral (risk-sensitive) cost functions as the Feynman-Kac equation.

The classes of problems to be treated in this paper involve an $\Re^n$−valued unobservable process $\tilde{x}^u(\cdot)$ satisfying the stochastic differential equation

$$d\tilde{x}_t = f(t,\tilde{x}_t)dt + B_t u(t,y)dt + G_t dw_t, \ \ \tilde{x}(0). \qquad (1)$$

This is observed through an $\Re^d$−valued process $y(\cdot)$ which satisfies the stochastic differential equation

$$dy_t = h(t,\tilde{x}_t)dt + N_t^{\frac{1}{2}} db_t, \qquad y(0) = 0. \qquad (2)$$

$y(\cdot)$ is called the observation process, $w(\cdot), b(\cdot)$ are, respectively, $\Re^n$ and $\Re^d$−valued independent Wiener processes, independent of the random variable $\tilde{x}(0)$, $u(\cdot)$ is the control process, and $t \in [0,T]$, where $T \in \Re$ is fixed and finite. The cost function to be minimized over the controls u, which are non anticipating functionals of the observations y, is of the general form

$$\begin{aligned} J_G^\theta(u(\cdot)) &= E^u \left\{ \int_0^T \ell_2(t,\tilde{x}_t,u(t,y)) \right. \\ &\times \ \exp\theta \left(\int_0^t \ell_1(s,\tilde{x}_s,u(s,y))ds \right) dt \\ &+ \ \varphi_2(\tilde{x}_T)\exp\theta \left(\int_0^T \ell_1(t,\tilde{x}_t,u(t,y))dt \right. \\ &+ \ \left. \varphi_1(\tilde{x}_T)) \right\}, \qquad \theta > 0. \end{aligned} \qquad (3)$$

Here $\ell_i, \varphi_i, i = 1,2$ are real valued functions and E^u denotes expectation with respect to a certain probability measure P^u. Notice that the integral cost function is given by $J_I(u(\cdot)) \doteq \{J_G^\theta(u(\cdot));\ \theta = 0\}$, while the exponential of integral cost function is given by $J_{EI}^\theta(u(\cdot)) \doteq \{J_G^\theta(u(\cdot));\ \ell_2 = 0, \varphi_2 = 1\}$.

2. Problem Formulation

2.1 Dynamics

We start with a reference probability space $(\Omega, \mathcal{A}, P)$ with a complete filtration $\{\mathcal{F}_t; t \in [0,T]\}$, two adapted standard Wiener processes $\{w_t; t \in [0,T]\}$, $\{b_t; t \in [0,T]\}$, and an $\mathcal{F}_0$ measurable random variable $\tilde{x}(0)$ such that: $w : [0,T] \times \Omega \to \mathcal{R}^n$, $b : [0,T] \times \Omega \to \mathcal{R}^d$, $x(0) : \Omega \to \mathcal{R}^n$, are mutually independent, and an observation process $y(\cdot)$ given by

$$dy_t = N_t^{\frac{1}{2}} db_t, \qquad y(0) = 0. \tag{4}$$

Assumption 0.1 *1. $\mathcal{U}$ is a non-empty subset of $\mathcal{R}^m$; 2. $f : [0,T] \times \Re^n \to \Re^n$ is Borel measurable, $|f(t,\tilde{x}) - f(t,\tilde{z})| \leq k|x - z|^n$, where $n \equiv$integer; 3. $h : [0,T] \times \Re^n \to \Re^d$ is Borel measurable, $|h(t,\tilde{x})| \leq k(1 + |\tilde{x}|)$; 4. $N : [0,T] \to \mathcal{L}(\Re^d; \Re^d)$, $N = N^*$, $\exists\, \beta_1 > 0$ such that $N \geq \beta_1 I_d$; 5. $G : [0,T] \to \mathcal{L}(\Re^n; \Re^n)$, $\exists\, \beta_2 > 0$ such that $G \geq \beta_2 I_n$; 6. $\ell_i : [0,T] \times \Re^n \times \mathcal{U} \to \Re$, $\varphi_i : \Re^n \to \Re, i = 1,2$ are Borel measurable, $|\ell_i(t,\tilde{x},u)| \leq k(1 + |\tilde{x}|^q + |u|^q)$, $|\varphi_i(\tilde{x})| \leq k(1 + |\tilde{x}|^q)$, $q \geq 1, i = 1,2$; 7. the distribution of $\tilde{x}(0)$ is, $\Pi_0^\theta(d\tilde{x}) = q_0^\theta(\tilde{x})d\tilde{x}$, $\int |\tilde{x}|^2 d\Pi_0^\theta(\tilde{x}) < \infty$; 8. N, G, B, f, h are Borel measurable and bounded in t.*

Write $\{\mathcal{F}_t^y; t \in [0,T]\}$ for the complete filtration generated by $\{y_s; 0 \leq s \leq t \leq T\}$ and assume that $\{\mathcal{F}_t; t \in [0,T]\}$ is generated by $\{\tilde{x}_s, w_s, b_s; 0 \leq s \leq t \leq T\}$.

Definition 0.2 *The set of pre-admissible controls denoted by $\tilde{\mathcal{U}}$, consists of $\mathcal{F}_t^y$−predictable functions with values in $\mathcal{U}$. The set of admissible controls denoted by $\hat{\mathcal{U}}$ is defined by $\hat{\mathcal{U}} \doteq \{u \in \tilde{\mathcal{U}};\ E[\Lambda^u(T)] = 1\}$, where for $u \in \tilde{\mathcal{U}}$, $\Lambda^u(\cdot)$ is the $\{\mathcal{F}_t; t \in [0,T]\}$−adapted process*

$$\begin{aligned} \Lambda^u(t) &= \exp\left\{\int_0^t h(s,\tilde{x}_s).N_s^{-1} dy_s \right. \\ &- \left. \frac{1}{2}\int_0^t h(s,\tilde{x}_s).N_s^{-1} h(s,\tilde{x}_s) ds\right\}, \end{aligned}$$

such that for the system $(\Omega, \mathcal{A}, P; \mathcal{F}_t)$ and for each $u \in \hat{\mathcal{U}}$, when $w(\cdot)$ is replaced by $w^u(\cdot)$, there is a strong solution $\tilde{x}^u(\cdot)$ satisfying the Ito equation (2).

The definition of admissible controls implies that for $u \in \hat{\mathcal{U}}$ a new measure P^u can be defined through the Radon-Nikodym derivative $\frac{dP^u}{dP}|_{\mathcal{F}_T} \doteq \Lambda^u(T)$, (see [10]). Then, Girsanov's theorem states that P^u is a probability measure on $(\Omega, \mathcal{A}; \mathcal{F}_t)$ and that for the system $(\Omega, \mathcal{A}, P^u; \mathcal{F}_t)$ the stochastic processes $(\tilde{x}^u(\cdot), y(\cdot))$ are (weak) solutions of (1), (2) (see [10]).

2.2 Infinite Dimensional Control Problem

From [1] we have the following representation theorem.

Theorem 0.3 *Let $\phi : \Re^n \to \Re$ be a Borel measurable twice continuously differentiable function and assume that $q_t^\theta(\phi)$ has density function $q^\theta(\tilde{x},t) \equiv q^\theta(\tilde{x}, \{y(s); 0 \leq s \leq t\}, t)$ defined by $q_t^\theta(\phi) \doteq (\phi, q^\theta(t)) = \int_{\Re^n} \phi(\tilde{z}) q^\theta(\tilde{z},t) d\tilde{z}$. For each $u \in \mathcal{U}$ define*

$$\begin{aligned} A\phi &\doteq \frac{1}{2} Tr\left(GG^* \frac{\partial^2}{\partial \tilde{x}^2}\phi\right) + (f + Bu).\frac{\partial}{\partial \tilde{x}}\phi \\ &= \frac{1}{2} Tr\left(GG^* D_{\tilde{x}}^2 \phi\right) + (f + Bu).D_{\tilde{x}}\phi. \end{aligned}$$

Then $q_\cdot^\theta(\cdot)$ satisfies the SPDE

$$\begin{aligned} dq_t^\theta(\phi) &= q_t^\theta(A(t)\phi)dt + \theta q_t^\theta(\ell_1(t,x,u(t))\phi)dt \\ &+ q_t^\theta(h(t,x)\phi).N_s^{-1} dy_s,\ q_0^\theta(\tilde{x}) = q^\theta(\tilde{x},t), \end{aligned}$$

where $y(\cdot)$ is given by (4). Furthermore, for $u \in \hat{\mathcal{U}}$ the total cost function (3) is represented as

$$\begin{aligned} J_G^\theta(u(\cdot)) &= E\left\{\int_0^T (\ell_2(t,\cdot,u(t)), q^\theta(t))\, dt \right. \\ &+ \left. (\varphi_2 \exp\theta(\varphi_1), q^\theta(T))\right\}. \end{aligned} \tag{5}$$

¿From the representation of Theorem 0.3 we obtain an evolution equation for the Feynman-Kac information state given by

$$\begin{aligned} dq_t^\theta &= (A(t)^* + \theta\ell_1(t,\tilde{x},u(t)))\, q_t^\theta dt \\ &+ h(t,\tilde{x})q_t^\theta.N_t^{-1} dy_t,\ \ q_0^\theta(\tilde{x}), \end{aligned} \tag{6}$$

($A(t)^*$ denotes he formal adjoint of $A(t)$) and an evolution equation for the information state ($\theta = 0$) given by

$$dq_t = A(t)^* q_t dt + h(t,\tilde{x})q_t.N_t^{-1} dy_t,\ q_0(\tilde{x}). \tag{7}$$

3. Solutions of Feynman-Kac Equation

3.1 Control System $\mathcal{S}_G^1$

The objective of this section is to obtain explicit solutions for the infinite dimensional state process $q^\theta(\cdot)$ governed by the SPDE (6), in terms of a finite number of ODE's forming the finite dimensional sufficient statistics for the estimation problem. Thus, by carrying out the integration of inner product terms $(\cdot,\cdot)$ present in (5), (whenever possible), we recover a cost function which is of standard finite dimensional form, expressed in terms of the sufficient statistics. We shall use the notation $\alpha.\beta = \alpha^*\beta$.

Control System ($\mathcal{S}_G^1$): Suppose assumptions 0.1 hold, the dynamics, observations, cost function are

$$\begin{aligned} d\tilde{x}_t &= (F_t\tilde{x}_t + g_t(\tilde{x}) + f_t)\, dt + B_t u(t,y)dt + G_t dw_t, \\ dy_t &= (H_t\tilde{x} + h_t)\, dt + N_t^{\frac{1}{2}} db_t, \\ J_{\mathcal{S}_G^1} &\equiv J_G^\theta(u(\cdot)) = (3), \end{aligned}$$

respectively.

A1: $2\ell_1(t,x,u) = Q_t x.x + R_t u.u + 2m_t x + 2n_t u + \tilde{\ell}_1(t,\tilde{x},u)$; $Q_t \in \Re^{n\times n}$, $R_t \in \Re^{m\times m}$, $m_t \in (\Re^m)^*$; $Q = Q^* \geq 0$, $R = R^* > 0$.

If $u \in \hat{\mathcal{U}}$ and A1 hold the information state associated with control system $\mathcal{S}_G^1$ is given by

$$\begin{aligned} dq_t^\theta &= \frac{1}{2} Tr\left(G_t G_t^* D_{\tilde{x}^2} q_t^\theta\right) dt \\ &- \frac{\partial}{\partial \tilde{x}}\left(q_t^\theta\left(F_t\tilde{x} + g_t(\tilde{x}) + f_t + B_t u\right)\right) dt \\ &+ \frac{\theta}{2}\left(Q_t\tilde{x}.\tilde{x} + R_t u.u + 2m_t\tilde{x} + 2n_t u + \tilde{\ell}_1(t,\tilde{x},u)\right) \\ &+ q_t^\theta\left(H_t\tilde{x} + h_t\right).N_t^{-1} dy_t. \end{aligned} \tag{8}$$

Theorem 0.4 *Suppose assumptions 0.1 hold, $u \in \hat{\mathcal{U}}$, and there exists a function $\phi \in C_{x,t}^{2,1}(\Re^n \times [0,T])$ satisfying the partial differential equation*

$$\begin{aligned} \frac{\partial \phi_t}{\partial t} &+ \frac{1}{2} Tr\left(G_t G_t^* D_{\tilde{x}^2}\phi_t\right) + \frac{1}{2} D_{\tilde{x}}\phi_t..G_t G_t^* D_{\tilde{x}}\phi \\ &+ \left(F_t\tilde{x} + f_t\right).D_{\tilde{x}}\phi_t = \frac{1}{2}\tilde{x}.\tilde{\Lambda}_t\tilde{x} + \tilde{x}.\tilde{\sigma}_t + \delta_t \\ &+ \frac{\theta}{2}\tilde{\ell}_1(t,\tilde{x},u) - B_t u.D_{\tilde{x}}\phi_t, \end{aligned} \tag{9}$$

where the functions $\tilde{\ell}_1(\cdot), \tilde{\Lambda} : [0,T] \rightarrow \mathcal{L}(\Re^n;\Re^n)$, $\tilde{\Lambda} = \tilde{\Lambda}^*, \tilde{\sigma} : [0,T] \rightarrow \Re^n, \delta : [0,T] \rightarrow \Re$ are to be chosen so that (9) yields explicit solutions. Suppose A1 hold and there exists a $\theta \leq \theta^*$ such that

$$H_t^* N_t^{-1} H_t + \tilde{\Lambda}_t - \theta Q_t \geq 0, \qquad \forall t \in [0,T].$$

Then the control system $\mathcal{S}_G^1$ with nonlinear function

$$g_t(\tilde{x}) = G_t G_t^* D_{\tilde{x}}\phi(\tilde{x},t) \tag{10}$$

admits exact solutions for the Feynman-Kac equation given by

$$\begin{aligned} q^\theta(\tilde{x},t) &= \exp\left(\phi(\tilde{x},t) + c_t + \lambda_t\right) \\ &\times \frac{\exp\left(-\frac{1}{2}P_t^{-1}(\tilde{x}-r_t).(\tilde{x}-r_t)\right)}{(2\pi)^{\frac{n}{2}}|P_t|^{\frac{1}{2}}}, \end{aligned} \tag{11}$$

where $r(\cdot), P(\cdot), P = P^*, c(\cdot), \lambda(\cdot)$ are given by

$$\begin{aligned} dr_t &= \left\{F_t - P_t\left(-\theta Q_t + \tilde{\Lambda}_t\right)\right\} r_t dt \\ &+ \left(f_t - P_t\tilde{\sigma}_t + B_t u_t\right) dt + \theta P_t m_t^* dt \\ &+ P_t H_t^* N_t^{-1}\left(dy_t - H_t r_t dt - h_t dt\right), \ r(0) \end{aligned} \tag{12}$$

$$\begin{aligned} \dot{P}_t &= F_t P_t + P_t F_t^* - P_t\left(H_t^* N_t^{-1} H_t\right. \\ &+ \left.\tilde{\Lambda}_t - \theta Q_t\right) P_t + G_t G_t^*, \qquad P(0), \end{aligned} \tag{13}$$

$$\begin{aligned} \lambda_t &= \frac{\theta}{2}\int_0^t \left\{[Q_s - \frac{\tilde{\Lambda}_s}{\theta}]r_s.r_s + Tr\left(P_s[Q_s - \frac{\tilde{\Lambda}_s}{\theta}]\right)\right\} ds \\ &+ \frac{\theta}{2}\int_0^t \left(R_s u_s.u_s + 2r_s.[m_s^* - \frac{\tilde{\sigma}_s}{\theta}]\right. \\ &+ \left. 2[n_s u_s - \frac{\delta_s}{\theta}]\right) ds, \end{aligned} \tag{14}$$

$$\begin{aligned} c_t &= \int_0^t \left(H_s r_s + h_s\right).N_s^{-1} dy_s \\ &- \frac{1}{2}\int_0^t |N_s^{-\frac{1}{2}}\left(H_s r_s + h_s\right)|^2 ds. \end{aligned} \tag{15}$$

Proof. See [1]. □

Theorem 0.4 implies that whenever (9) admits explicit solutions the Feynman-Kac equation evolves on a finite dimensional manifold (i.e., the state space of $q^\theta(\cdot)$ is finite dimensional). If we set $\phi(\cdot) = 0$ (which implies that $\tilde{\Lambda}(\cdot) = 0, \tilde{\sigma}(\cdot) = 0, \delta(\cdot) = 0$), we recover the explicit solution of the Feynman-Kac equation corresponding to the LEQG regulator problem given in [3, 5], while by setting $\theta = 0$ we recover the conditional density of the LQG problem (unnormalized). The fundamental difficulty with obtaining explicit solutions of (9) is caused by the presence of the term $B_t u.D_{\tilde{x}}\phi_t$. One can eliminate this term by choosing $\tilde{\ell}_1(\cdot)$ appropriately. However, this approach will imply that the function $\ell_1(\cdot)$ should contain the term $\frac{2}{\theta}B_t u..D_{\tilde{x}}\phi_t$, and thus excludes the case when the function $\ell_1(\cdot)$ is quadratic in $\tilde{x}, u$. Towards eliminating this term we consider an alternative control system.

3.2 Control System $\mathcal{S}_G^2$

Control System ($\mathcal{S}_G^2$): Suppose assumptions 0.1 hold the dynamics, observations, and cost function are

$$\begin{aligned} \begin{bmatrix} dx_t \\ dz_t \end{bmatrix} &= \begin{bmatrix} F_{11}(t) & F_{12}(t) \\ F_{21}(t) & F_{22}(t) \end{bmatrix}\begin{bmatrix} x_t \\ z_t \end{bmatrix} dt + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} dt \\ &+ \begin{bmatrix} g_t(x_t) \\ 0 \end{bmatrix} dt + \begin{bmatrix} B_t^1 u(t,y) \\ B_t^2 u(t,y) \end{bmatrix} dt \\ &+ \begin{bmatrix} G_t^1 & 0 \\ 0 & G_t^2 \end{bmatrix}\begin{bmatrix} dw_t^1 \\ dw_t^2 \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ z(0) \end{bmatrix}, \\ &\equiv F_t\tilde{x}_t dt + g(t,\tilde{x}_t)dt + f_t dt \\ &+ B_t u(t,y)dt + G_t w_t, \end{aligned} \tag{16}$$

$$\begin{aligned} dy_t &= \left(H_1(t)x_t + H_2(t)z_t\right. \\ &+ \left. h_1(t) + h_2(t)\right) dt + N_t^{\frac{1}{2}} db_t, \ y(0) = 0, \\ &\equiv H_t\tilde{x}_t + h_t dt + N_t^{\frac{1}{2}} db_t. \end{aligned} \tag{17}$$

$$J_{\mathcal{S}_G^2}^\theta \doteq J_G^\theta(u(\cdot)) = (3),$$

respectively, where $\tilde{x}^* \equiv (x^*, z^*)$ and $x(\cdot), z(\cdot)$ are, respectively, $\Re^{n_1}, \Re^{n-n_1}$–valued processes.

A2: A1 hold, $\tilde{\ell}_1(t,\tilde{x},u) \doteq \tilde{\ell}_1(t,x,u), F_{12}.(GG)^{-1} = 0, B^1 = 0$.

Theorem 0.5 *Suppose assumptions 0.1 hold, $u \in \hat{\mathcal{U}}$, and there exists a function $\phi \in C^{2,1}_{x,t}(\Re^{n_1} \times [0,T])$ satisfying the partial differential equation*

$$\begin{aligned}
\frac{\partial \phi_t}{\partial t} &+ \frac{1}{2}Tr\left(G^1_t G^{1*}_t D^2_x \phi_t\right) + \frac{1}{2}D_x\phi_t..G^1_t G^{1*}_t D_x\phi \\
&+ \left(F_{11}(t)x + f_1(t)\right).D_x\phi_t \\
&= \frac{1}{2}x.\Lambda_t x + x.\sigma_t + \delta_t + \frac{\theta}{2}\tilde{\ell}_1(t,x,u), \qquad (18)
\end{aligned}$$

where the functions $\tilde{\ell}_1(\cdot), \Lambda(\cdot)$, $\Lambda = \Lambda^*$, $\sigma(\cdot)$, $\delta(\cdot)$ (defined similarly as in Theorem 0.4 are to be chosen so that (18) yield explicit solutions and

$$\tilde{\Lambda}_t \doteq \begin{bmatrix} \Lambda_t & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{\sigma}_t \doteq \begin{bmatrix} \sigma_t \\ 0 \end{bmatrix}.$$

Suppose A2 hold and there exists a $\theta \le \theta^*$ such that

$$H^*_t N^{-1}_t H_t + \tilde{\Lambda}_t - \theta Q_t \ge 0, \qquad \forall t \in [0,T].$$

Then the control system $\mathcal{S}^2_G$ with nonlinear function

$$g_t(x) = G^1_t G^{1,*}_t D_x\phi(x,t) \qquad (19)$$

admits exact solutions for the Feynman-Kac equation given by

$$\begin{aligned}
q^\theta(\tilde{x},t) &= \exp\left(\phi(x,t) + c_t + \lambda_t\right) \\
&\times \frac{\exp\left(-\frac{1}{2}P^{-1}_t(\tilde{x}-r_t).(\tilde{x}-r_t)\right)}{(2\pi)^{\frac{n}{2}}|P_t|^{\frac{1}{2}}}, \quad (20)
\end{aligned}$$

where the functions $r(\cdot), P(\cdot), \lambda(\cdot), c(\cdot)$ are given by (12)-(15), respectively.

Proof. See [2]. □

3.3 Examples of Finite Dimensional Systems

In this section we shall present certain classes of nonlinear functions $g(\cdot)$ that admit finite dimensional solutions for the Feynman-Kac equation.

Theorem 0.6 *Suppose $u \in \hat{\mathcal{U}}$ and there exists $\theta \le \theta^*$ such that*

$$H^*_t N^{-1}_t H_t + \tilde{\Lambda}_t - \theta Q_t \ge 0, \qquad \forall t \in [0,T].$$

Let $\Delta : [0,T] \to \mathcal{L}(\Re^n;\Re^n)$, $\zeta : [0,T] \to \Re^n$, $n : [0,T] \to \Re$, define $\Gamma_2(t,x) \doteq \frac{1}{2}\Delta_t x.x + x.\zeta_t + \eta_t$, $\tilde{\ell}_1(t,x,u) \doteq 0$, (i.e., $\ell_1(t,\tilde{x},u) \equiv$ quadratic in $\tilde{x},u$). The Feynman-Kac information state $q^\theta(\cdot)$ associated with system $\mathcal{S}^2_G$ given in Theorem 0.5 admits explicit representations, at least for the following three classes:

Class 1 (Rational Nonlinearities). *A solution of (18) is*

$$\phi_{R_2}(x,t) = \Gamma_2(x,t).$$

This implies that the nonlinear drift term $g(\cdot)$ is

$$g_t(x) = \frac{G^1_t G^{1*}_t}{\frac{1}{2}\Delta_t x.x + x.\zeta_t + \eta_t}\left(\Delta_t x + \zeta_t\right),$$

where

$$\begin{aligned}
\dot{\Delta}_t &+ F_{11}(t)^*\Delta_t + \Delta_t F_{11}(t) = \delta_t\Delta_t, \\
\dot{\zeta}_t &+ F_{11}(t)^*\zeta_t + \Delta_t f_1(t) = \delta_t\zeta_t, \\
\dot{\eta}_t &+ \frac{1}{2}Tr\left(G^1_t G^{1*}_t\Delta_t\right) + f_1(t).\zeta_t = \delta_t\eta_t, \\
\Lambda_t &= 0, \quad \sigma_t = 0, \quad \delta_t = \text{arbitrary}.
\end{aligned}$$

Class 2 (Exponential Nonlinearities). *Suppose $\gamma^1, \gamma^2 : [0,T] \to \Re$. A solution of (18) is*

$$\phi_{E_2}(x,t) = \log\left\{\gamma^1_t \exp\left(\Gamma_2(x,t)\right) + \gamma^2_t \exp\left(-\Gamma_2(x,t)\right)\right\}.$$

This implies that the nonlinear drift term $g(\cdot)$ is

$$g_t(x) = \frac{\gamma^1_t \exp\left(\Gamma_2(x,t)\right) - \gamma^2_t \exp\left(-\Gamma_2(x,t)\right)}{\gamma^1_t \exp\left(\Gamma_2(x,t)\right) + \gamma^2_t \exp\left(-\Gamma_2(x,t)\right)} G^1_t G^{1*}_t\left(\Delta_t x + \zeta_t\right),$$

where

$$\begin{aligned}
\dot{\Delta}_t &+ F_{11}(t)^*\Delta_t + \Delta_t F_{11}(t) = 0, \\
\dot{\zeta}_t &+ F_{11}(t)^*\zeta_t + \Delta_t f_1(t) = 0, \\
\dot{\eta}_t &+ \frac{1}{2}Tr\left(G^1_t G^{1*}_t\Delta_t\right) + f_1(t).\zeta_t = \frac{1}{2}\frac{d}{dt}\left(\log\frac{\gamma^1_t}{\gamma^2_t}\right), \\
\Lambda_t &= \Delta_t G^1_t G^{1*}_t \Delta_t, \quad \sigma_t = \Delta_t G^1_t G^{1*}_t \zeta_t, \\
\delta_t &= \frac{1}{2}\zeta^*_t G^1_t G^{1*}_t \zeta_t + \frac{1}{2}\frac{d}{dt}\left(\log\gamma^1_t\gamma^2_t\right).
\end{aligned}$$

Class 3 (Ratio of Sinusoidal Nonlinearities). *A solution of (18) is*

$$\phi_{S_2}(x,t) = \log\sin(\Gamma_2(x,t)).$$

This implies that the nonlinear drift term $g(\cdot)$ is

$$g_t(x) = G^1_1 G^{1*}_t\left(\Delta_t x + \zeta_t\right)\frac{\cos(\Gamma_2(x,t))}{\sin(\Gamma_2(x,t))},$$

where $\Delta(\cdot), \zeta(\cdot), \eta(\cdot), \delta(\cdot), \Lambda(\cdot), \sigma(\cdot)$ satisfy certain equations (similar to those of classes 1, 2).

Proof. Substitute the solutions into the evolution equation of $\phi(\cdot)$. □

Similar results hold if we consider the system $\mathcal{S}^1_G$ and we set $\tilde{\ell}_1(t,\tilde{x},u) = \frac{2}{\theta}B_t u_t..(G_t G_t)^{-1}g(t,\tilde{x})$. Additional classes of nonlinear systems with finite dimensional Feynman-Kac equation are given in [1, 2].

Example 0.7 *Suppose $x : [0,T]\times\Omega \to \Re$, $z : [0,T]\times\Omega \to \Re$, $y : [0,T]\times\Omega \to \Re$ and consider*

$$\begin{aligned}
dx_t &= -\Pi x^3_t dt + x_t dt + dw^1_t, \quad x(0), \Pi \ge 0, \\
dz_t &= x_t dt + z_t dt + u(t,y)dt + dw^2_t, \quad z(0), \\
dy_t &= x_t dt + z_t dt + db_t.
\end{aligned}$$

If we set $\tilde{\ell}_1(t,x,u) = \frac{1}{\theta}|x^3\Pi|^2$, the Feynman-Kac equation is finite dimensional and is obtained from Theorem 0.5 by setting $\phi(x,t) = -\frac{\Pi}{4}x^4$.

3.4 Control System $\mathcal{S}_G^3$

We have thus far presented classes of nonlinear systems that admit finite dimensional solutions for the Feynman-Kac equation. Whether or not the systems $\mathcal{S}_G^1, \mathcal{S}_G^2$ admit finite dimensional solutions for their associated Feynman-Kac equations depends on whether or not one can solve the partial differential equations (9), (18), respectively. Here, we shall show that by defining the function $\tilde{\ell}_1(\cdot)$ appropriately, if $g(\cdot)$ is the gradient of a potential function, that is, $g(t,\tilde{x}) \doteq G_t G_t^* D_{\tilde{x}}\phi_t$, then the Feynman-Kac equation corresponding to system $\mathcal{S}_G^1$ is finite dimensional (similar results hold for the system $\mathcal{S}_G^2$).

Control System ($\mathcal{S}_G^3$): Suppose assumptions 0.1 hold, consider the control system $\mathcal{S}_G^1$ with time invariant nonlinearities $g(\tilde{x})$, assume A1 hold, define

$$\begin{aligned}\tilde{\ell}_1(t,\tilde{x},u) &\doteq \frac{1}{\theta}|G_t^{-1}[F_t\tilde{x} + f_t + g(\tilde{x}) + B_t u(t,y)]|^2\\ &+ Tr\left(((G_tG_t^*)^{-1}D_{\tilde{x}}g(\tilde{x})\right),\end{aligned}$$

and consider the cost function $J_{\mathcal{S}_G^3} \doteq J_G^\theta(u(\cdot)) = (3)$.

Define

$$\begin{aligned}Q_t^\theta &\doteq Q_t + \frac{1}{\theta}F_t^*(G_tG_t^*)^{-1}F_t, \quad t \in [0,T),\\ R_t^\theta &\doteq R_t + \frac{1}{\theta}B_t^*(G_tG_t^*)^{-1}F_t, \quad t \in [0,T),\\ m_t^\theta &\doteq m_t + \frac{1}{\theta}f_t^*(G_tG_t^*)^{-1}F_t, \quad t \in [0,T),\\ n_t^\theta &\doteq n_t + \frac{1}{\theta}f_t^*(G_tG_t^*)^{-1}B_t, \quad t \in [0,T).\end{aligned}$$

Theorem 0.8 *Suppose assumptions 0.1 hold, $u \in \hat{\mathcal{U}}$, and there exists a $\theta \le \theta^*$ such that*

$$H_t^* N_t^{-1} H_t - \theta Q_t^\theta \ge 0, \qquad \forall t \in [0,T].$$

Assume (without loss of generality) $F_t.B_t = 0$.
Then the control system $\mathcal{S}_G^3$ with nonlinear function $g(\cdot)$ given by the gradient of some potential, that is,

$$g_t(\tilde{x}) = G_tG_t^* D_{\tilde{x}}\phi(\tilde{x},t) \tag{21}$$

admits exact solutions for the Feynman-Kac equation given by

$$\begin{aligned}q^\theta(\tilde{x},t) &= \exp\left(\phi(\tilde{x},t) + c_t + \lambda_t\right)\\ &\times \frac{\exp\left(-\frac{1}{2}P_t^{-1}(\tilde{x}-r_t).(\tilde{x}-r_t)\right)}{(2\pi)^{\frac{n}{2}}|P_t|^{\frac{1}{2}}},\end{aligned} \tag{22}$$

where $c(\cdot)$ is given by (15) and $r(\cdot), P(\cdot), P = P^*, \lambda(\cdot)$ are solutions of the following equations:

$$\begin{aligned}dr_t &= \left(F_t + \theta P_t Q_t^\theta\right) r_t dt + \theta P_t m_t^{\theta,*} dt\\ &+ \left(f_t dt + B_t u_t\right) dt + P_t H_t^* N_t^{-1}\left(dy_t\right.\\ &- \left.H_t r_t dt - h_t dt\right), \quad r(0),\end{aligned} \tag{23}$$

$$\begin{aligned}\dot{P}_t &= F_tP_t + P_tF_t^* - P_t\left(H_t^*N_t^{-1}H_t\right.\\ &+ \left.-\theta Q_t^\theta\right)P_t + G_tG_t^*, \qquad P(0),\end{aligned} \tag{24}$$

$$\begin{aligned}\lambda_t &= \frac{\theta}{2}\int_0^t \left\{Q_s^\theta r_s.r_s + Tr\left(P_sQ_s^\theta\right) + \frac{|G_t^{-\frac{1}{2}}f_t|^2}{\theta}\right\} ds\\ &+ \frac{\theta}{2}\int_0^t \left(R_s^\theta u_s.u_s + 2r_s.m_s^{\theta,*} + 2n_s^\theta u_s\right) ds.\end{aligned} \tag{25}$$

Proof: Write the Feynman-Kac equation and verify the results. □

4. Information State

4.1 Control System $\mathcal{S}_G^4$

In this section we shall show that the information state satisfying the Zakai equation admits finite dimensional solutions for certain nonlinear systems as well.

Control System ($\mathcal{S}_G^4$): Consider the control system $\mathcal{S}_G^2$ with cost function given by

$$J(u(\cdot)) = E^u\left\{\int_0^T \ell_2(t,\tilde{x}_t,u(t,y))dt + \varphi(\tilde{x}_T)\right\},$$

A3: A2 hold and $\tilde{\ell}_1 = 0$.

For the class of control systems $\mathcal{S}_G^4$ we know from Theorem 0.3 (see (5)) that for $u \in \hat{\mathcal{U}}$ the cost criterion has the equivalent representation $J(u(\cdot)) \doteq \{J_G^\theta(u(\cdot)) \equiv (7);\ \theta = 0\}$, where $q(\cdot)$ is governed by (5). Consequently, from Theorems 0.4, 0.5, 0.6, we deduce the next theorem.

Theorem 0.9 *Suppose assumptions 0.1 hold, $u \in \hat{\mathcal{U}}$, and there exists a function $\phi \in C_{x,t}^{2,1}(\Re^{n_1} \times [0,T])$ satisfying the partial differential equation*

$$\begin{aligned}\frac{\partial\phi_t}{\partial t} &+ \frac{1}{2}Tr\left(G_t^1G_t^{1*}D_x^2\phi_t\right) + \frac{1}{2}D_x\phi_t.G_t^1G_t^{1*}D_x\phi_t\\ &+ \left(F_{11}(t)x + f_1(t)\right).D_x\phi_t\\ &= \frac{1}{2}x\tilde{\Lambda}_t.x + x.\tilde{\sigma}_t + \delta_t,\end{aligned} \tag{26}$$

where $\tilde{\Lambda}(\cdot), \tilde{\sigma}(\cdot), \delta(\cdot)$ are defined as in Theorem 0.5 and A3 hold.
The Zakai equation associated with the control system $\mathcal{S}_G^4$ having nonlinear function $g(t,x) = G_t^1G_t^{1*}D_x\phi(x,t)$ admits exact solutions

$$\begin{aligned}q(\tilde{x},t) &= \exp\left(\phi(x,t)\right) \times \frac{\exp\left(-\frac{1}{2}P_t^{-1}(\tilde{x}-r_t).(\tilde{x}-r_t)\right)}{(2\pi)^{\frac{n}{2}}|P_t|^{\frac{1}{2}}}\\ &\times \exp\left(c_t + \lambda_t\right),\end{aligned}$$

where the $r(\cdot), P(\cdot), \lambda(\cdot), c(\cdot)$ correspond, respectively, to (23)-(25) with $\theta = 0$, and (15).
In addition, the information state $q(\cdot)$ admits explicit solutions for at least the classes of nonlinear functions $\phi(\cdot)$ considered in Theorem 0.6 corresponding to $\tilde{\ell}_1 = 0$.

Proof. Follows from the results of the previous sections by setting $\theta = 0$. □

5. Exact Optimal Control Laws

In Sections 3, 4 we have presented general theorems that render finite dimensional solutions of the Zakai and Feynman-Kac equations for general classes of nonlinear control systems. In this section we shall present sufficient conditions for identifying nonlinear control systems having optimal control laws reminiscent of LEQG and LQG tracking problems.

5.1 Equivalence of Nonlinear and LEQG

The developments of this section will be based on the results of Theorem 0.8 associated with system $\mathcal{S}_G^3$. Nevertheless, similar results will hold for the remaining systems $\mathcal{S}_G^1, \mathcal{S}_G^2$ as well.

Control System ($\mathcal{S}_{EI}^\theta$): Consider the control system $\mathcal{S}_G^3$, assume (without loss of generality) $F_t.B_t = 0$, and introduce the cost function $J_{EI}^\theta(u(\cdot)) \doteq \{J_{\mathcal{S}_G^3};\ \ell_2 = 0\}$.

Since the system $\mathcal{S}_{EI}^\theta$ is a special case of the control system $\mathcal{S}_G^3$, from Theorem 0.8, we know that the problem of minimizing $J_{EI}^\theta(u(\cdot))$ is equivalent to the following finite dimensional completely observable problem:
Minimize over $u \in \hat{\mathcal{U}}$ the cost function

$$\begin{aligned} J_{EI}^\theta(u(\cdot)) &= \tilde{\mathcal{I}}_{0,T} E\left\{\exp\frac{\theta}{2}\int_0^T (r_s Q_s^\theta . r_s\right.\\ &+ R_s^\theta u(s,y).u(s,y))\,ds\\ &\times \exp\frac{\theta}{2}\int_0^T (2r_s.m_s^{\theta,*} + 2n_s^\theta u(s,y))\,ds\\ &\times \left.\hat{\varphi}(r_T)\times\hat{\Lambda}_{0,T}\right\}, \end{aligned}$$

where $\hat{\Lambda}_{0,t} = \exp(c_{0,t}), c_{0,t} \equiv c_t$ (see (15)),

$$\tilde{\mathcal{I}}_{0,T} \doteq \exp\frac{\theta}{2}\int_0^T \left\{Tr\left(P_t Q_t^\theta\right) + \frac{f_t^* f_t}{\theta}\right\} dt,$$

$$\begin{aligned} \hat{\varphi}_2(r) &\doteq \int_{\Re^n} \varphi_2(\tilde{x})\exp\left(\frac{\theta}{2}[Q_T\tilde{x}.\tilde{x} + 2m_T\tilde{x}]\right)\\ &\times \frac{\exp\left(\phi(x,T) - \frac{1}{2}P_T^{-1}(\tilde{x}-r_T).(\tilde{x}-r_T)\right)}{(2\pi)^{\frac{n}{2}}|P_T|^{\frac{1}{2}}}d\tilde{x}, \end{aligned}$$

subject to observer dynamics given by (23), (24).
For $u \in \hat{\mathcal{U}}$ we now define a new measure P^u through the Radon-Nikodym derivative $\frac{dP^u}{dP}|_{\mathcal{F}_{0,T}^y} \doteq \hat{\Lambda}_{0,T}$. Then Girsanov's theorem states that P^u is a probability measure on $(\Omega, \mathcal{A}; \mathcal{F}_{0,t}^y)$ and that, if the stochastic process $\hat{b}^u(\cdot)$ is defined by

$$d\hat{b}_t^u \doteq dy_t - (H_t r_t + h_t)\,dt,$$

then $\hat{b}^u(\cdot)$ is a Wiener process with covariance $N(\cdot)$, when defined on the system $(\Omega, \mathcal{A}, P^u; \mathcal{F}_{0,t}^y)$. Therefore, we define for each $u \in \hat{\mathcal{U}}$ the cost-to-go function

$$\begin{aligned} S_{EI}(r,t) &= \frac{1}{\tilde{\mathcal{I}}_{t,T}}\inf_{u\in\hat{\mathcal{U}}} E^u\left\{\hat{\varphi}_2(r_T)\exp\frac{\theta}{2}\int_t^T (r_s Q_s^\theta . r_s\right.\\ &+ R_s^\theta u(s,y).u(s,y))\,ds\\ &\times \left.\exp\frac{\theta}{2}\int_t^T (2r_s.m_s^{\theta,*} + 2n_s^\theta u(s,y))\,ds|\mathcal{F}_{0,t}^y\right\}. \end{aligned}$$

and for $x \in \Re^n, p \in \Re^n, s \in \Re$ we introduce

$$\mathcal{H}_{EI}^\theta(x,p,s) = \inf_{u\in\hat{\mathcal{U}}}\left\{p.B_t u + \frac{\theta}{2}\left(R_t^\theta u.u + 2n_t^\theta u\right)s\right\}.$$

If we now define the second order operator

$$\begin{aligned} \tilde{A}_{EI} &\doteq \frac{1}{2}\sum_{i,j=1}^n \tilde{\alpha}_{ij}\frac{\partial^2}{\partial\tilde{x}_i\partial\tilde{x}_j} + \sum_{i=1}^n\left(\sum_{j=1}^n\left(\tilde{F}_{i,j}r_j\right.\right.\\ &+ \left.\left.\theta P_{i,j}m_j^{\theta,*}\right) + f_i\right)\frac{\partial}{\partial\tilde{x}_i} + \frac{\theta}{2}\left(rQ^\theta.r + 2r.m^{\theta,*}\right), \end{aligned}$$

where $\tilde{F}_t = F_t + \theta P_t Q_t^\theta$, $\tilde{\alpha}(t) = P_t H_t^* N_t^{-1} H_t P_t$, by using dynamic programming arguments (see [5, 1]) we deduce the 2^{nd} order Hamilton-Jacobi (HJ) equation

$$\begin{aligned} \frac{\partial}{\partial t}S_{EI}(r,t) &+ \mathcal{H}_{EI}^\theta(r, D_r S_{EI}(r,t), S_{EI}(r,t))\\ &+ \tilde{A}_{EI}(t)S_{EI}(r,t), &(27)\\ S_{EI}(r,T) &= \hat{\varphi}_2(r). &(28) \end{aligned}$$

Consequence we have the next verification theorem.

Theorem 0.10 *Consider the control system $\mathcal{S}_{EI}^\theta$ with admissible controls, which are of separated form $u(t) \equiv u(t,r)$. Suppose $S_{EI}(\cdot)$ denotes the solution of the HJ equation (27), (28) then*

$$S_{EQ}(r(0),0) \le \frac{J_{EQ}^\theta(u(\cdot))}{\tilde{\mathcal{I}}_{0,T}}, \quad \forall u \in \hat{\mathcal{U}}.$$

Further, letting $u^*(t) = u^*(t,r)$, where u^* is a Borel measurable function minimizing $\mathcal{H}_{EI}^\theta$ given by

$$u^*(t) = -\frac{R_t^{\theta,-1}B_t^*}{\theta}\frac{D_r S_{EI}(r,t)}{S_{EI}(r,t)} - \frac{R_t^{\theta,-1}n_t^{\theta,*}}{\theta}, \quad (29)$$

and $r(\cdot) \equiv r^{u^*}(\cdot))$ is the corresponding solution of

$$\begin{aligned} dr_t &= \left\{F_t + \theta P_t Q_t^\theta\right\} r_t dt + f_t dt\\ &+ B_t u^*(t)dt + \theta P_t m_t^{\theta,*}dt + P_t H_t^* N_t^{-1} d\hat{b}_t^u,\\ r(0), \hat{b}_t^u &\equiv \textit{Wiener process} \qquad (30) \end{aligned}$$

(*in the strong sense*) *we have*

$$S_{EI}(r(0),0)=\frac{J^{\theta}_{EI}(u^*(\cdot))}{\bar{\mathcal{I}}_{0,T}}=\inf_{u\in\hat{\mathcal{U}}}\frac{J^{\theta}_{EI}(u(\cdot))}{\bar{\mathcal{I}}_{0,T}}.$$

Proof. See [5]. □

It is important to note that the results of Theorem 0.10 imply that for any nonlinear function $\phi(\cdot)$ obtained from Theorem 0.8 (hence, $g_t(x)=G^1_tG^{1*}_tD_x\phi(x,t)$), the optimal observer dynamics are reminiscent of that associated with the LEQG tracking problem. If we set $\varphi_2=1$ we can not expect to be able to solve the HJ equation (27), (28) explicitly, and hence determine the optimal control law (see [1] for a detailed exposition). Thus, unless the terminal cost of the HJ equation is the exponential of quadratic form $S^{\theta}_{EQ}(r,T)=\hat{\varphi}_2(r)=\exp\frac{\theta}{2}[\hat{Q}r.r+2\hat{m}r]$, we might not be able to obtain explicitly the optimal control law. This observation gives rise to sufficient conditions for identifying nonlinear partially observable stochastic control problems, reminiscent of LEQG problems, stated in the next theorem.

Theorem 0.11 *Consider the control system $\mathcal{S}^{\theta}_{EI}$. The optimal control law $u^*\in\hat{\mathcal{U}}$ minimizing the total cost function $J^{\theta}_{EI}(u(\cdot))$ is linear feedback, reminiscent of the LEQG tracking problem, if the following hold:*

1. *The function $g(\cdot)$ is defined by*

$$g(t,\tilde{x})\doteq G_tG^*_tD_{\tilde{x}}\phi(\tilde{x},t),\qquad(31)$$

2. *The function $\varphi_2(\cdot)$ is defined by*

$$\varphi_2(\tilde{x})\doteq\exp(-\phi(\tilde{x},T)).\qquad(32)$$

Proof. If the conditions 1, 2 of the theorem are satisfied, we know that the terminal condition (28) of the HJ equation is an exponential of quadratic function. □

Next, we present the optimal control laws corresponding to the classes of nonlinear control systems defined by $\mathcal{S}^{\theta}_{EI}$, when the sufficient conditions of Theorem 0.11 are satisfied.

Theorem 0.12 (*Exact Optimal Control Laws*). *Consider the control system $\mathcal{S}^{\theta}_{EI}$, denote by $\bar{\rho}(AB)$ the spectral radius of AB, and define*

$$\theta^*\doteq\left\{\sup\theta;\quad P\geq0,\quad S\geq0,\quad\bar{\rho}(PS)<\frac{1}{\theta},\quad\forall t\in[0,T]\right\},$$

where $P(\cdot)$ is given in Theorem 0.8 and $S(\cdot)$ is the solution of the Riccati differential equation

$$\begin{aligned}\dot{S}_t&+&F^*_tS_t+S_tF_t-S_t\left(B_tR^{\theta,-1}_tB^*_t\right.\\&-&\left.\theta G_tG^*_t\right)S_t+Q^{\theta}_t,\ S_T=Q_T.\end{aligned}$$

If the conditions of Theorem 0.11 hold, then for $\theta\leq\theta^*$ the optimal control law corresponding to the class of control systems $\mathcal{S}^{\theta}_{EI}$ is given by

$$\begin{aligned}u^*(t)&=&-R^{\theta,-1}_tB^*_t(\Sigma_tr_t+k^*_t)-R^{\theta,-1}_tn^{\theta,*}_t=-R^{\theta,-1}_tn^{\theta,*}_t\\&-&R^{\theta,-1}_tB^*_t\left((I-\theta S_tP_t)^{-1}S_tr_t+k^*_t\right),\end{aligned}$$

where $r(\cdot)\equiv r^{u^*}(\cdot)$, $P(\cdot)$ satisfy (23), (24), respectively, while the control gains are

$$\begin{aligned}\dot{\Sigma}_t&+&\Sigma_t\left(F_t+\theta P_tQ^{\theta}_t\right)+\left(F^*_t+\theta Q^{\theta}_tP_t\right)\Sigma_t+Q^{\theta}_t\\&-&\Sigma_t\left\{B_tR^{\theta,-1}_tB^*_t-\theta P_tH^*_tN^{-1}_tH_tP_t\right\}\Sigma_t=0,\\\Sigma_T&=&\frac{1}{2}\left\{(I-\theta Q_TP_T)^{-1}Q_T+Q_T(I-\theta P_TQ_T)^{-1}\right\},\\\dot{k}_t&+&k_t\left(F_t+\theta P_tH^*_tN^{-1}_tH_tP_t\Sigma_t\right.\\&+&\left.\theta P_tQ^{\theta}_t-B_tR^{\theta,-1}_tB^*_t\Sigma_t\right)\\&+&m^{\theta}_t+\left(f^*_t+\theta m^{\theta}_tP_t-n^{\theta}_tR^{\theta,-1}_tB^*_t\right)\Sigma_t=0,\\k_T&=&m_T(I-\theta P_TQ_T)^{-1}.\end{aligned}$$

Furthermore, the optimal total cost associated with system $\mathcal{S}^{\theta}_{EI}$ is given by

$$\begin{aligned}J^{\theta}_{EI}(u^*(\cdot))&=&\mathcal{I}_{0,T}\times\exp\frac{\theta}{2}\left(\Sigma(0)r(0).r(0)\right.\\&+&\left.2k(0)r(0)+\rho(0)\right),\end{aligned}$$

where the deterministic functions $\mathcal{I}(\cdot),\rho(\cdot)$ are:

$$\begin{aligned}\dot{\rho}_t&+&Tr\left(P_tH^*_tN^{-1}_tH_tP_t\Sigma_t\right)+\theta k_tP_tH^*_tN^{-1}_tH_tP_tk^*_t\\&+&2k_t\left(f_t+\theta P_tm^{\theta,*}_t\right)\\&-&|R^{\theta,-\frac{1}{2}}_t\left(B^*_tk^*_t+n^{\theta,*}_t\right)|^2=0,\quad\rho_T=0,\\\mathcal{I}_{0,T}&=&\frac{1}{|I-\theta P_TQ_T|^{\frac{1}{2}}}\exp\left\{\frac{\theta^2}{2}n_T(I-\theta P_TQ_T)^{-1}P_Tn^*_T\right.\\&+&\left.\frac{\theta}{2}\int_0^T\left(Tr(P_tQ^{\theta}_t)+2\frac{f^*_tf_t}{\theta}\right)dt\right\}.\end{aligned}$$

Proof. Solve (27), (28) explicitly. □

5.2 Equivalence Between Nonlinear and LQG

Clearly, one can establish equivalence between nonlinear and LQG partially observable tracking problems, following the developments of Section 5.1 (see [2]).

6. Parameter Estimation and Control

In this section we suppose that systems $\mathcal{S}^i_G$, $i=1,2,3$ contain unknown constant parameters $\Theta_1,\Theta_2,\ldots,\Theta_{n-n_1}$ which we desire to estimate as well. Specifically, we have the situation when the $\Re^{n_1}$-valued unobservable process $x(\cdot)$ and observation process $y(\cdot)$ are described by

$$dx_t\;=\;(F_{11}(t)x_t+f_1(t))\,dt+g(x_t,\Theta)dt$$

$$
\begin{aligned}
&+ \quad B_t^1 u(t,y)dt + dw_t^1, \quad x(0), \qquad (33)\\
dy_t &= (H_1(t)x_t + H_2(t)\Theta + h_t)\,dt + N_t^{\frac{1}{2}} db_t,\\
&\equiv (H_t\tilde{x}_t dt + h_t)\,dt + N_t^{\frac{1}{2}} db_t, \quad y(0) = 0, \qquad (34)
\end{aligned}
$$

where $\tilde{x} = \begin{pmatrix} x \\ \Theta \end{pmatrix}$ and

$$
g(x,\Theta) = \sum_{i=1}^{n-n_1} \Theta_i A_i x, \quad \{A_i\}_{i=1}^{n-n_1}.
$$

Here, Θ is an $n - n_1$–dimensional vector (i.e., $\Theta \subset \Re^{n-n_1 \times n-n_1}$). Notice that since Θ is a constant vector, we can take this as part of the dynamics by introducing the equation

$$
d\Theta = 0, \qquad \Theta(0). \qquad (35)
$$

The parameter estimation and control problem of interest is defined as follows:

Parameter/Control System ($\mathcal{S}_\Theta$): Suppose assumption 0.1 hold (with $\tilde{x}^* = (x^*, \Theta^*)$), the dynamics are given by (33), (35), the observations are given by (34), and the cost function to be minimized over $u \in \hat{\mathcal{U}}$ is defined by

$$
\begin{aligned}
J_\Theta^\theta &\doteq \Big\{ J_G^\theta(u(\cdot)); \ \ \ell_2 = 0, \ \ \tilde{\ell}_1 = \frac{1}{\theta}|F_{11}(t)x + f_1(t)\\
&+ \quad g(x,\Theta) + B_t^1 u(t,y)|^2 + \frac{1}{\theta} Tr\,(D_x g(x,\Theta)) \Big\}.
\end{aligned}
$$

Although system $\mathcal{S}_\Theta$ is slightly different from systems $\mathcal{S}_G^i, 1 \leq i \leq 3$, the analysis of the previous sections goes through, by taking $g(x,\Theta) = D_x\phi(x,\Theta)$, where $\phi(x,\Theta) = \frac{1}{2}x.\sum_{i=1}^{n-n_1} \Theta_i A_i x$. The Feynman-Kac equation for the system $\mathcal{S}_\Theta$ is explicitly solvable and is obtained exactly as in Theorem 0.8. In addition, if we use Theorem 0.11, that is, set $\varphi_2(\tilde{x}) = \exp(-\phi(x,\Theta))$, the optimal control law is obtained exactly as in Theorem 0.12 and thus, it is linear feedback and equivalent to that associated with the LEQG tracking problem. It is important to note that one can introduce more general classes of parameter/control systems than $\mathcal{S}_\Theta$. These generalization follow from the results of the previous sections and should appear elsewhere.

7. Conclusion

In this paper we have considered stochastic control problems when the unobservable dynamics are nonlinear, the measurements are linear in the unobservable state, and the cost criterion features both the integral and the exponential of integral cost criteria. We have presented specific classes of unobservable nonlinearities leading to finite dimensional solutions for the Feynman-Kac equation and the Zakai equations, in terms of a finite number of ordinary differential equations forming the optimal observer dynamics. We have then derived sufficient conditions for constructing partially observable nonlinear stochastic optimal control problems which are equivalent to LEQG and LQG tracking problems. In addition, we have shown that a class of parameter identification problems leads to optimal control laws that can be implemented in real-time.

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