Stochastic Adaptive Boundary Control of Some Linear Distributed Parameter Systems*

T.E. Duncan† and B. Pasik-Duncan†

Abstract

Some unknown linear stochastic distributed parameter systems are considered that can be described by analytic semigroups. A stochastic adaptive control problem is formulated and solved for these systems where the performance measure is an ergodic quadratic cost functional and the control occurs on the boundary. The "highest order" operator is assumed to be known but the "lower order" operators contain unknown parameters as well as the linear operators of the state and the control on the boundary. The noise in the system is a standard cylindrical white Gaussian noise. A diminishing excitation is used for the identification of the unknown parameters to ensure sufficient excitation but it has no effect on the ergodic cost. A family of least squares estimates is shown to be strongly consistent. The adaptive control using the certainty equivalence control with random switchings to the zero control is shown to be self-optimizing.

1 Introduction

An important class of models for linear distributed parameter systems is the family that is described by analytic semigroups. To model some perturbations or inaccuracies in these models, it is often reasonable to consider stochastic, linear distributed systems. In many applications of controlled linear distributed parameter systems it is natural to consider that the control occurs on the boundary or at discrete points because it is often unreasonable to expect that the control can be applied throughout the domain.

If there is an ergodic, quadratic cost functional, then under suitable assumptions the optimal control can be obtained from the solution of an algebraic or stationary Riccati equation. Typically the stochastic differential equation model for the stochastic, linear distributed parameter system contains some unknown parameters so there is the problem of stochastic adaptive control. It is assumed that the "highest order" operator is known but that "lower order" operators contain unknown parameters. The unknown operators include linear operators on the state on the boundary and on the control on the boundary or at discrete points in the domain.

For the identification of the unknown parameters that occur in the linear operator acting on the control, it is necessary to ensure that there is sufficient excitation. This is accomplished by a diminishing excitation for strong consistency. This strong consistency is obtained for a family of least squares estimates. It is assumed that the analytic semigroup is stable. The control at time t is required to be measurable with respect to the past (of the state process) until time $t - \Delta$ where $\Delta > 0$ is arbitrary but fixed. This assumption accounts for some natural delay in processing the information for the construction of the control. No boundedness assumptions are made on the range of the unknown parameters.

The adaptive control is the certainty equivalence control for the ergodic, quadratic cost functional with switchings to the zero control. These random switchings are determined to ensure stability of the estimated infinitesimal generator and to satisfy a suitable boundedness for the control. This adaptive control is shown to be self-optimizing.

The proofs of the results given here are contained in [6]. Some other results for adaptive control of stochastic linear distributed parameter systems are given in [4] and [5].

*Research partially supported by NSF Grant DMS-9305936.

†Mathematics Department, University of Kansas, Lawrence, KS 66045.

2 Preliminaries and Main Results

The stochastic system is described by the stochastic evolution equation

$$\begin{aligned} dX(t,\alpha) &= [A_0 + A_1(\alpha) + A_0BC(\alpha)]X(t,\alpha)dt \\ &\quad + A_0BD(\alpha)U(t)dt + GdW(t) \quad (2.1) \\ X(0,\alpha) &= x \end{aligned}$$

in a separable Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ where A_0 is the infinitesimal generator of an exponentially stable analytic semigroup $(S_0(t), t \geq 0)$ on H, $A_0 = A_0^*$, $\alpha \in \mathcal{K} \subset R^p$. Let D_A^γ for $\gamma \in R$ be the domain of the fractional power $(-A_0)^\gamma$ with the $(-A_0)^\gamma$ graph norm. Let $B \in \mathcal{L}(H_1, D_A^\epsilon)$ for some $\epsilon \in (0,1)$, $A_1^*(\alpha) \in \mathcal{L}(D_A^\eta, H)$ for some $\eta \in [0,1)$, $C(\alpha) \in \mathcal{L}(H, H_1)$ and $D(\alpha) \in \mathcal{L}(H_2, H_1)$ for each value of $\alpha \in \mathcal{K}$ where H_1 and H_2 are separable Hilbert spaces. General information about analytic semigroups can be found in [12]. The formal process $(W(t), t \geq 0)$ is a cylindrical Wiener process with the incremental covariance the identity, $I \in \mathcal{L}(H)$, that is defined on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t, t \geq 0)$. For $p \geq 2$ let $M_W^p(H_2) = \bigcap_{T>0} M_W^p(0.T, H_2)$ where

$$\begin{aligned} M_W^p(0,T,H_2) &= \{U | U : [0,T] \times \Omega \to H_2, \\ &\quad (U(t), t \geq 0) \quad (2.2) \\ &\quad \text{is } (\mathcal{F}_t) \text{ adapted and} \\ &\quad E\int_0^T |U(t)|^p \, dt < \infty\} \end{aligned}$$

The control process $(U(t), t \geq 0)$ in (2.1) is assumed to belong to the space $M_W^p(H_2)$ for some fixed $p > \max(\frac{1}{\epsilon}, \frac{1}{1-\eta})$ and $p \geq 2$.

For the control problem, the following ergodic, quadratic cost functional is used

$$J(\alpha, U) = \limsup_{t\to\infty} \frac{1}{t} J(t, x, \alpha, U) \quad (2.3)$$

where

$$J(t,x,\alpha,U) = \int_0^t [\langle Q_1 X(s,\alpha), X(s,\alpha)\rangle + \langle Q_2 U(s), U(s)\rangle] ds$$

and $Q_1 = Q_1^* \in \mathcal{L}(H)$, $Q_1 \geq 0$, $Q_2 = Q_2^* \in \mathcal{L}(H_2)$ and $Q_2 \geq cI$, $c > 0$.

A solution of (2.1) is understood as a mild solution, that is, an H-valued process $(X(t), t \geq 0)$ that satisfies (almost surely)

$$\begin{aligned} X(t) &= S_0(t)x + \int_0^t S_0(t-r)A_1X(r)dr \quad (2.4) \\ &\quad + \int_0^t A_0S_0(t-r)BDU(r)dr + \\ &\quad \int_0^t A_0S_0(t-r)BCX(r)dr + Z(t) \quad (2.5) \end{aligned}$$

where

$$Z(t) = \int_0^t S_0(t-r)GdW(t). \quad (2.6)$$

The operator $S_0(t-r)A_1$ is identified with its (unique) extension as an element of $\mathcal{L}(H)$ which exists because

$$|S_0(t-r)A_1x| \leq \frac{c}{(t-r)^\eta}|x|$$

for $x \in \mathcal{D}(A_0)$, $0 \leq r < t \leq T$ and some $c > 0$ where $A_1^*(\alpha) \in \mathcal{L}(D_A^\eta, H)$ and the analyticity of $S_0(\cdot)$ are used. To ensure that the stochastic integral (2.6) is a "nice" process it is assumed that the following condition is satisfied

(C1) $(-A_0)^{-\delta}G$ is Hilbert-Schmidt for some $0 \leq \delta < \frac{1}{2}$.

The following result verifies that there is a unique mild solution of (2.1) with continuous sample paths.

Proposition 2.1 *If (C1) is satisfied and $U \in M_W^p(H_2)$ then the equation (2.1) has a unique mild solution with H-valued continuous sample paths.*

In a similar way we can obtain the existence and the uniqueness for the solution of (2.1) with a feedback control $U(t) = K(t)X(t)$ where $K(t) = \tilde{K}(t, X(u), u \leq t - \Delta)$, $\Delta > 0$ is fixed, $K(\cdot)$ is deterministic on $[0, \Delta]$ and $K(\cdot) : R_+ \times \Omega \to \mathcal{L}(H, H_2)$ is uniformly bounded, measurable and adapted to $(\mathcal{F}_{t-\Delta}, t \in R_+)$. The equation

$$X(t) = S_0(t)x + \int_0^t S_0(t-r)A_1X(r)dr \quad (2.7)$$

$$+\int_0^t A_0S_0(t-r)BCX(r)dr+$$
$$\int_0^t A_0S_0(t-r)BDK(r)X(r)dr+Z(t)$$

can be treated similarly to (2.4). The feedback control is an element of $M_W^p(H_2)$ because from (2.7) we have that

$$E|X(t)|^p \leq c_1[1+E[\int_0^t (t-r)^{-\gamma}|X(r)|dr]^p+E[|Z(t)|^p] \tag{2.8}$$

for some $c_1 > 0$ and $\gamma = \max(\eta, 1-\epsilon)$. Applying the Hölder inequality to the integral on the right hand side of (2.8) and then the Gronwall inequality, it follows that $X(\cdot) \in M_W^p(H)$. If $U(t) = K(t)X(t-\Delta)$ then the mild solution of (2.1) is well defined and $X(\cdot) \in M_W^p(H)$.

The Riccati equation that is used to solve the ergodic, quadratic control problem for the system (2.1) is the one associated with the infinite time horizon deterministic control problem

$$\begin{aligned} \frac{dy}{dt}(t,\alpha) &= (A_0 + A_1(\alpha) + A_0BC(\alpha))y(t,\alpha) \\ &\quad +A_0BD(\alpha)u(t) \qquad\qquad (2.9) \\ y(0;a) &= x \end{aligned}$$

with the quadratic cost functional

$$\tilde{J}(x,\alpha,u) = \int_0^\infty \langle Q_1y(t,\alpha), y(t,\alpha)\rangle + \langle Q_2u(t), u(t)\rangle dt \tag{2.10}$$

where $Q_1 = Q_1^* \in \mathcal{L}(H), Q_1 \geq 0$ and $Q_2 = Q_2^* \in \mathcal{L}(H_2)$, $Q_2 \geq cI$, $c > 0$. This deterministic control problem has been investigated in [3], [7], [8], [9], [10]. The following conditions are used:

(C2) (Compactness of the resolvent) A_O^{-1} is compact.

(C3) (Continuous dependence on parameters) $A_1^*(\cdot), C(\cdot), D(\cdot)$ are continuous functions from the parameter set $\mathcal{K} \subset R^q$ into $\mathcal{L}(D_A^\eta, H), \mathcal{L}(H, H_1)$ and $\mathcal{L}(H_2, H_1)$ respectively.

(C4) (Uniform detectability and stabilizability) There are linear operators $F \in \mathcal{L}(H, H_2)$ and $K \in \mathcal{L}(H)$ and constants $c > 0$ and $r > 0$ such that

i) $|exp[t(A_0 + A_1^*(\alpha) + C^*(\alpha)\Psi + Q_1K)]|_{\mathcal{L}(H)} \leq ce^{-\rho t}$

ii) $|exp[t(A_0+A_1^*(\alpha)+C^*(\alpha)\Psi+F^*D^*(\alpha)\Psi]|_{\mathcal{L}(H)} \leq ce^{-\rho t}$

for all $t \geq 0$ and $\alpha \in \mathcal{K}$ where $\Psi \in \mathcal{L}(D_A^{1-\epsilon}, H_1)$ is the extension of B^*A_0.

For each $\alpha \in \mathcal{K}$ define $\mathcal{C}(\alpha) \in \mathcal{L}(H, D_A^{-\gamma})$ as $\mathcal{C}(\alpha) = A_1(\alpha) + [C^*(\alpha)\Psi]^*$ where $\gamma = \max(1-\epsilon, \eta)$ and $\mathcal{B}(\alpha) \in \mathcal{L}(H_2, D_A^{\epsilon-1})$ as $\mathcal{B}(\alpha) = [D^*(\alpha)\Psi]^*$. The solution of the equation (2.9) is defined as the mild solution

$$z(t,\alpha) = S(t,\alpha)x + \int_0^t S(t-s,\alpha)\mathcal{B}(\alpha)u(s)ds$$

for $t \geq 0$ where $(S(t,\alpha), t \geq 0)$ is the analytic semigroup generated by $A(\alpha) = A_0 + \mathcal{C}(\alpha)$. This semigroup is analytic because $C^*(\alpha)$ is $(-A_0)^\gamma$ bounded.

The following result [3], [7], [9] gives the solution of the deterministic control problem (2.9, 2.10).

Proposition 2.2 *If (C4) is satisfied then there is a unique, nonnegative, self-adjoint linear operator V on H such that $V \in \mathcal{L}(H, D_A^\gamma)$ for all $\gamma \in (0,1)$ and*

$$\begin{aligned} &\langle (A_0 + \mathcal{C}(\alpha))x, Vy\rangle + \langle (A_0 + \mathcal{C}(\alpha))y, Vx\rangle + \\ &\quad \langle Q_1x, y\rangle - \langle Q_2^{-1}\mathcal{B}^*(\alpha)Vx, \mathcal{B}^*(\alpha)Vy\rangle = 0 \end{aligned} \tag{2.11}$$

for all $x, y \in \mathcal{D}(A_0)$. The optimal control for the control problem (2.9, 2.10) has the feedback form $\hat{u}(t,x,\alpha) = -Q_2^{-1}\mathcal{B}^*(\alpha)V(\alpha)y(t,\alpha)$ and the optimal cost is

$$\tilde{J}(x,\alpha) = \min_u \tilde{J}(x,\alpha,u) = \langle V(\alpha)x, x\rangle. \tag{2.12}$$

The following result describes a continuity property of V.

Proposition 2.3 *If (C2 - C4) are satisfied then*

$$\lim_{\alpha\to\alpha_0} |V(\alpha) - V(\alpha_0)|_{\mathcal{L}(H, D_A^{1-\epsilon})} = 0 \tag{2.13}$$

where $V(\cdot)$ is the solution of (2.11).

To estimate the parameters of the unknown system (2.1) a family of least squares estimates is given that is shown to be strongly consistent. Some additional conditions are introduced.

(C5) The semigroup generated by $A_0+\mathcal{C}(\alpha)$ is stable for each α where $\mathcal{C}(\alpha) = A_1(\alpha) + [C^*(\alpha)\Psi]^*$ and $\Psi \in \mathcal{L}(D_A^{1-\epsilon}, H_1)$ is the extension of B^*A_0.

(C6) The linear operators $A_1(\alpha), C(\alpha)$ and $D(\alpha)$ have the following form:

$$A_1(\alpha) = A_{10} + \sum_{i=1}^{q_1} \alpha^i A_{1i}$$

$$C(\alpha) = C_0 + \sum_{i=1}^{q_1} \alpha^i C_i$$

$$D(\alpha) = D_0 + \sum_{i=q_1+1}^{q} \alpha^i D_i$$

where $A_{1i}^* \in \mathcal{L}(D_A^\eta, H), C_i \in \mathcal{L}(H, H_1)$ for $i = 0, ..., q_1$ and $D_i \in \mathcal{L}(H_2, H_1)$ for $i = 0, q_1+1, ..., q$. Define the linear operators $\mathcal{C}_i$ and $\mathcal{B}_i$ as follows: $\mathcal{C}_i = A_{1i} + [C_i^*\Psi]^*$ for $i = 0, ..., q_1$ and $\mathcal{B}_i = [D_i^*\Psi]^*$ for $i = 0, q_1 + 1, ..., q$. Clearly $\mathcal{C}_i \in \mathcal{L}(H, D_A^{-\gamma})$ for $i = 0, ..., q_1$, where $\gamma = \max(1-\epsilon, \eta)$ and $\mathcal{B}_i \in \mathcal{L}(H_2, D_A^{\epsilon-1})$.

(C7) There is a finite dimensional projection $\tilde{P} : D_A^{-1} \to \tilde{P}(D_A^{-1}) \subset H$ and $\tilde{P}\mathcal{B}_i, i = q_1 + 1, ..., q)$ are linearly independent and for each nonzero $\beta \in R^{q_1}$

$$tr \sum_{i=1}^{q_1} \beta_i(\tilde{P}(\mathcal{C}_i)) \int_0^\Delta S(r, \alpha_0)GG^*S^*(r, \alpha_0)dr \cdot$$

$$\cdot \sum_{i=1}^{q_1} \beta_i(\tilde{P}(\mathcal{C}_i))^* > 0$$

where $(S(t, \alpha_0), t \geq 0)$ is the C_0-semigroup with the infinitesimal generator $A_0 + \mathcal{C}(\alpha_0)$.

Let $(\Omega, \mathcal{F}, P)$ denote a probability space for (2.1) where P includes a measure induced from the cylindrical Wiener process and a family of independent random variables for a diminishingly excited control introduced subsequently. $\mathcal{F}$ is the P-completion of an appropriate σ-algebra on Ω and $(\mathcal{F}_t, t \geq 0)$ is a filtration so that the cylindrical Wiener process $(W(t), t \geq 0)$, the solution $(X(t), t \geq 0)$ of (2.1) and the diminishingly excited control are adapted to $(\mathcal{F}_t, t \geq 0)$.

For the adaptive control problem it is convenient to enlarge the class of controls to $\tilde{M}_W^p(H_2) = \bigcap_{T>0} \tilde{M}_W^p(0, T, H_2)$ where

$$\tilde{M}_W^p(0, T, H_2) = \{U | U : [0, T] \times \Omega \to H_2$$

$$(U(t), t \geq 0) \text{ is } (\mathcal{F}_t)$$

adapted and

$$\int_0^T |U(s)|^p ds < \infty \quad \text{a.s.}\}$$

It is elementary to verify that the regularity properties of the sample paths of the solution of (2.1) with $U \in M_W^p(H_2)$ carry over to $U \in \tilde{M}_W^p(H_2)$.

Define the $(\tilde{P}(D_A^{-1}))^q$-valued process $(\varphi(t), t \geq 0)$ by the equation

$$\begin{aligned}\varphi(t) = [&\tilde{P}(\mathcal{C}_1)X(t), ..., \tilde{P}(\mathcal{C}_{q_1})X(t), \\ &\tilde{P}\mathcal{B}_{q_1+1}U(t), ..., \tilde{P}\mathcal{B}_q U(t)]. \end{aligned} \tag{2.14}$$

for $\beta \in R^q$ and $\varphi(t) = [\varphi_1(t), ..., \varphi_q(t)]$ as above, define $\varphi \cdot \beta$ by the equation

$$\varphi(t) \cdot \beta = \sum_i \varphi_i(t)\beta_i.$$

If $a = (a_1, ..., a_\ell)$ is an ℓ-tuple of R^k vectors and $b = (b_1, ..., b_m)$ is an m-tuple of R^k vectors then define $a \times b \in \mathcal{L}(R^k, R^m)$ as

$$a \times b = (\langle a_i, b_j \rangle)$$

If $F \in \mathcal{L}(\tilde{P}D_A^{-1})$ then define $\tilde{F}\varphi$ by the equation

$$\tilde{F}\varphi = (F\varphi_i)$$

The stochastic differential equation for $(\tilde{P}X(t), t \geq 0)$ can be expressed as

$$\begin{aligned} d\tilde{P}X(t) = &[\tilde{P}(A_0 + \mathcal{C}_0)]X(t)dt + \\ &\tilde{P}\mathcal{B}_0U(t)dt + \varphi(t) \cdot \alpha dt + \tilde{P}GdW(t). \end{aligned} \tag{2.15}$$

Fix $a > 0$ and define the $\mathcal{L}(\tilde{P}D_A^{-1})$-valued process $(\Gamma(t), t \geq 0)$ as

$$\Gamma(t) = (\int_0^t \varphi(s) \times \varphi(s)ds + a^{-1}I)^{-1} \tag{2.16}$$

A family of least squares estimates $(\hat{\alpha}(t), t \geq 0)$ of the true parameter vector α_0 is defined as the solution of the following affine stochastic differential equation

$$\begin{aligned} d\hat{\alpha}(t) &= \Gamma(t)[\varphi(t) \times (d\tilde{P}(t)X(t) \\ &- \tilde{P}(A_0 + C_0)X(t)dt - \tilde{P}B_0U(t)dt \\ &- \varphi(t) \cdot \hat{\alpha}(t)dt] \qquad (2.17) \\ \hat{\alpha}(0) &= \alpha(0) \end{aligned}$$

where $U \in \tilde{M}^p_W(H_2)$.

Let $\tilde{\alpha}(t) = \alpha_0 - \hat{\alpha}(t)$ for $t \geq 0$. The process $(\hat{\alpha}(t), t \geq 0)$ satisfies the following stochastic differential equation

$$\begin{aligned} d\tilde{\alpha}(t) &= -\Gamma(t)[\varphi(t) \times \varphi(t) \cdot \tilde{\alpha}(t)dt \\ &+ \tilde{P}GdW(t)] \qquad (2.18) \\ \tilde{\alpha}(0) &= \alpha_0 - \hat{\alpha}(0). \end{aligned}$$

Since $\frac{d\Gamma}{dt} = -\Gamma(t)[\varphi(t) \times \varphi(t)]\Gamma(t), \Gamma(0) = aI$ we have that the solution of (2.18) is

$$\tilde{\alpha}(t) = -\Gamma(t)\Gamma^{-1}(0)\tilde{\alpha}(0) - \Gamma(t)\int_0^t \varphi(s) \times \tilde{P}GdW(s)). \qquad (2.19)$$

The control is a sum of a desired (adaptive) control and a diminishing excitation control. Let $(Z_n, n \in N)$ be a sequence of H_2-valued, independent, identically distributed, random variables that is independent of the cylindrical Wiener process $(W(t), t \geq 0)$. It is assumed that $EZ_n = 0$ and the covariance of Z_n is Λ for all n where Λ is positive and nuclear and there is a $\sigma > 0$ such that $|Z_n|^p \leq \sigma$ a.s. Choose $\tilde{\epsilon} \in (0, 1/2)$ and fix it. Define the H_2-valued process $(V(t), t \geq 0)$ as

$$V(t) = \sum_{n=0}^{[\frac{t}{\Delta}]} \frac{Z_n}{n^{\tilde{\epsilon}/2}} 1_{[n\Delta,(n+1)\Delta)}(t). \qquad (2.20)$$

Clearly we have that

$$\lim_{t\to\infty} |V(t)| = 0 \quad a.s. \qquad (2.21)$$

and for each $\ell_1, \ell_2 \in H_2^* = H_2$

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t^{1-\tilde{\epsilon}}} \int_0^t \langle \ell_1, V(s)\rangle\langle \ell_2, V(s)\rangle ds = \\ &\lim_{t\to\infty} \frac{1}{t^{1-\tilde{\epsilon}}} \sum_{i=1}^{[\frac{t}{\Delta}]} \frac{\langle \ell_1, Z_i\rangle\langle \ell_2, Z_i\rangle}{i^{\tilde{\epsilon}}}\Delta + o(1) \qquad (2.22) \\ &= \Delta^{\tilde{\epsilon}}(1-\tilde{\epsilon})^{-1}\langle \Lambda\ell_1, \ell_2\rangle \quad \text{a.s.} \end{aligned}$$

It is assumed that $Z_n \in \mathcal{F}_{n\Delta}$ and Z_n is independent of $\mathcal{F}_s$ for $s < n\Delta$ for all $n \in N$. A finite dimensional version of this diminishing excitation is given in [2].

The diminishingly excited control is

$$U(t) = U^d(t) + V(t) \qquad (2.23)$$

for all $t \geq 0$.

The following result verifies the strong consistency and provides a rate of convergence for the family of least squares estimates given by (2.17).

Theorem 2.4 *Let $\tilde{\epsilon} \in (0, 1/2)$ be determined from the definition of $(V(t), t \geq 0)$ in (2.20). If (C1-C7) are satisfied and the control process $(U(t), t \geq 0)$ for (2.1) is given by (2.23) where $U^d(t) \in \mathcal{F}((t-\Delta) \vee 0)$ for $t \geq 0, U^d \in \tilde{M}^p_W(H_2)$ and*

$$limsup_{t\to\infty} \frac{1}{t^{1+\delta}} \int_0^t |U^d(s)|^2 ds < \infty \quad a.s. \qquad (2.24)$$

for some $\delta \in [0, 1-2\tilde{\epsilon})$, then

$$|\alpha_0 - \hat{\alpha}(t)|^2 = O(\frac{\log t}{t^\beta}) \quad a.s. \qquad (2.25)$$

as $t \to \infty$ for each $\beta \in (\frac{1+\delta}{2}, 1-\tilde{\epsilon})$ and $(\hat{\alpha}(t), t \geq 0)$ satisfy (2.17).

A self-optimizing adaptive control is constructed for the unknown linear stochastic system (2.1) with the ergodic quadratic cost functional (2.3) using the family of least squares estimates $(\hat{\alpha}(t), t \geq 0)$ that satisfies (2.17).

The family of admissible controls $\mathcal{U}(\Delta)$ for the minimization of (2.3) is

$$\begin{aligned} &\mathcal{U}(\Delta) = \\ &\{U : U(t) = U^d(t) + U^1(t), U^d(t) \in \mathcal{F}((t-\Delta) \vee 0) \\ &\text{and } U^1(t) \in \sigma(V(s), (t-\Delta) \vee 0 \leq s \leq t) \qquad (2.26) \\ &\text{for all } t \geq 0, U \in \tilde{M}^p_W(H_2), \\ &\limsup_{t\to\infty} \frac{|X(t)|^2}{t} = 0 \quad \text{a.s., and} \\ &\limsup_{t\to\infty} \frac{1}{t}\int_0^t (|X(t)|^2 + |U(t)|^2) ds < \infty \quad a.s.\}. \end{aligned}$$

Since $A_0 + C(\alpha_0)$ is the infinitesimal generator of a stable analytic semigroup it is known that for the deterministic infinite time boundary control problem with $G = 0$ there is a solution P of the algebraic Riccati equation that is formally expressed as

$$A^*P + PA - P\tilde{B}Q_2^{-1}\tilde{B}^*P + Q_1 = 0 \qquad (2.27)$$

where $A = A_0 + \mathcal{C}(\alpha_0)$ and $\tilde{B} = \mathcal{B}(\alpha_0)$. This formal Riccati equation can be expressed as a precise inner product equation

$$\langle Ax, Py\rangle + \langle Px, Ay\rangle -$$
$$\langle Q_2^{-1}\tilde{B}^*Px, \tilde{B}^*Py\rangle + \langle Q_1 x, y\rangle = 0 \qquad (2.28)$$

where $x, y \in \mathcal{A}$. This solution P is the strong limit of the family of solutions of the differential Riccati equations as the final time tends to infinity. This solution is called the minimal solution of (2.27) or (2.28).

We can apply Itô formula to $(\langle PX(t), X(t)\rangle, t \geq 0)$ and use (2.28) to obtain

$$\langle PX(t), X(t)\rangle - \langle Px, x\rangle =$$
$$\int_0^t [2\langle U(s), \tilde{B}^*PX(t)\rangle + \langle Q_2^{-1}\tilde{B}^*PX(s), \tilde{B}^*PX(s)\rangle$$
$$- \langle Q_1X(s), X(s)\rangle]ds \qquad (2.29)$$
$$+tTr(-A_0)^{\delta}PGG^*(-A_0)^{-\delta} + 2\int_0^t \langle PX(s), GdW(s)\rangle.$$

Rewriting (2.29) we have

$$\langle PX(t), X(t)\rangle - \langle Px, x\rangle + \int_0^t \langle Q_1X(s), X(s)\rangle ds$$
$$= \int_0^t \langle U(s) + Q_2^{-1}\tilde{B}^*PX(s), Q_2(U(s)+$$
$$Q_2^{-1}\tilde{B}^*PX(s))\rangle ds \qquad (2.30)$$
$$+t\,Tr(-A_0)^{\delta}PGG^*(-A_0)^{-\delta}+$$
$$2\int_0^t \langle PX(s), GdW(s)\rangle.$$

Define the H-valued process $(\hat{X}(t), t \geq \Delta)$ by the equation

$$\hat{X}(t) = S(\Delta; \alpha_0)X(t-\Delta)+$$
$$\int_{t-\Delta}^t S(t-s; \alpha_0)\mathcal{B}(\alpha_0)V(s)ds. \qquad (2.31)$$

Clearly for $t \geq \Delta$

$$X(t) = \hat{X}(t) + \int_{t-\Delta}^t S(t-s; \alpha_0)\mathcal{B}(\alpha_0)U(s)ds+$$
$$\int_{t-\Delta}^t S(t-s; \alpha_0)\mathcal{B}(\alpha_0)GdW(s)$$

where $(X(t), t \geq 0)$ satisfies (2.1) and the input or control in (2.1) is a sum of V and $U \in \mathcal{U}(\Delta)$.

For any $U \in \mathcal{U}(\Delta)$ it follows that

$$\limsup_{t\to\infty} \frac{1}{t} J(x, U, \alpha_0, t) = Tr(-A_0)^{\delta}PGG^*(-A_0)^{-\delta}$$
$$+ \limsup_{t\to\infty} \frac{1}{t}\int_\Delta^t \langle U(s) + Q_2^{-1}\tilde{B}^*P\hat{X}(s)$$
$$+Q_2^{-1}\tilde{B}^*P(\int_{s-\Delta}^s S(s-r; \alpha_0)\tilde{B}U(r)dr+$$
$$\int_{s-\Delta}^s S(s-r; \alpha_0)GdW(r)),$$
$$Q_2(U(s) + Q_2^{-1}\tilde{B}^*P\hat{X}(s)+$$
$$Q_2^{-1}\tilde{B}^*P(\int_{s-\Delta}^s S(s-r; \alpha_0)\tilde{B}U(r)dr \qquad (2.32)$$
$$+ \int_{s-\Delta}^s S(s-r; \alpha_0)GdW(r))\rangle ds$$
$$\geq Tr(-A_0)^{\delta}PGG^*(-A_0)^{-\delta} + Tr\tilde{B}^*PR(\Delta)P\tilde{B}Q_2^{-1} \quad \text{a.s.}$$

where J is given by (2.3) and $R(\Delta)$ satisfies

$$R(\Delta) = \int_0^\Delta S(r; \alpha_0)GG^*S^*(r; \alpha_0)dr$$

It is clear that

$$U(s) = -Q_2^{-1}\tilde{B}^*P\hat{X}(s)-$$
$$\int_{s-\Delta}^s Q_2^{-1}\tilde{B}^*PS(s-r; \alpha_0)\tilde{B}U^d(r)dr + V(s) \in \mathcal{U}(\Delta)$$

and it minimizes the ergodic cost functional (2.3) for the family of controls $\mathcal{U}(\Delta)$.

Define the H_2-valued (control) process $(U^0(t), t \geq \Delta)$ by the equation

$$U^0(t) = -Q_2^{-1}\tilde{B}^*(t-\Delta)P(t-\Delta)(S(\Delta; t-\Delta)X(t-\Delta)$$
$$+ \int_{t-\Delta}^t S(t-s; t-\Delta)\tilde{B}(t-\Delta)U^d(s)ds) \qquad (2.33)$$

where $\tilde{B}^*(t) = (\mathcal{B}^*(\hat{\alpha}(t))^*, S(\tau;t) = e^{\tau A(t)}$ and $A(t)$ is defined as

$$A(t) = \begin{cases} A_0 + \mathcal{C}(\hat{\alpha}(t)) & \text{if } A_0 + \mathcal{C}(\hat{\alpha}(t)) \text{ is stable} \\ \tilde{A} & \text{otherwise} \end{cases} \tag{2.34}$$

and $\tilde{A}$ is a fixed stable infinitesimal generator (that is, the associated semigroup is stable) such that $\tilde{A} = A_0 + \mathcal{C}(\alpha_1)$ for some parameter vector α_1, $P(t)$ is the minimal solution of (2.28) using $A(t)$ and $\tilde{B}^*(t)$. It will be clear by the construction of U^d that $U^0 \in \mathcal{U}(\Delta)$.

Define two sequences of stopping times $(\sigma_n, n = 0, 1, ...)$ and $(\tau_n, n = 1, 2, ...)$ as follows:

$$\sigma_0 \equiv 0$$

$$\sigma_n =$$

$$\sup\{t \geq \tau_n; \int_0^s |U^0(r)|^p dr \leq \tau_n^\delta s \text{ for all } s \in [\tau_n, t)\} \tag{2.35}$$

$$\tau_n = \inf\{t > \sigma_{n-1} + 1; \int_0^t |U^0(r)|^p dr \leq t^{1+\delta/2} \text{ and}$$
$$|X(t-\Delta)|^2 \leq t^{1+\delta/2}\}.$$

where $\delta > 0$ is fixed and $\frac{1+\delta}{2} < 1 - \epsilon$ and U^0 is given by (2.33). It is clear that $(\tau_n - \sigma_{n-1}) \geq 1$ on $\{\sigma_{n-1} < \infty\}$ for all $n \geq 1$.

Define the adaptive control $(U^*(t), t \geq 0)$ by the equation

$$U^*(t) = U^d(t) + V(t) \tag{2.36}$$

for $t \geq 0$ where

$$U^d(t) = \begin{cases} 0 & \text{if } t \in [\sigma_n, \tau_{n+1}) \text{ for some } n \geq 0 \\ U^0(t) & \text{if } t \in [\tau_n, \sigma_n) \text{ for some } n \geq 1 \end{cases} \tag{2.37}$$

and $U^0(t), V(t)$ satisfy (2.33),(2.23) respectively. It is clear that $U^d \in \tilde{M}_W^p(H_2)$.

The adaptive control U^* that is the the sum of the certainty equivalence control with random switchings to the zero control and the diminishing excitation is self-optimizing.

Theorem 2.5 *If (C1-C7) are satisfied then the adaptive control $(U^*(t), t \geq 0)$ for (2.1) given by (2.36) is an element of $\mathcal{U}(\Delta)$ and is self-optimizing, that is,*

$$\inf_{U \in \mathcal{U}(\Delta)} \limsup_{t\to\infty} \frac{1}{t} J(x, U, \alpha_0, t) =$$

$$\limsup_{t\to\infty} \frac{1}{t} J(x, U^*, \alpha_0, t)$$

$$= Tr(-A_0)^\delta PGG^*(-A_0)^{-\delta} + Tr\tilde{B}^* PR(\Delta)P\tilde{B}Q_2^{-1} \quad a.s. \tag{2.38}$$

where J is given by (2.3).

An example of a differential operator that can be used in (2.1) is briefly described.

Let $A_\alpha(x, D)$ be a $2m$-order differential operator of the form

$$A_\alpha(x, D)y = \sum_{|p|,|q| \leq m} (-1)^{|p|} D^p(a_{pq}(\alpha, x) D^q y), x \in \mathcal{O} \tag{2.39}$$

where $\mathcal{O}$ a bounded domain in R^n whose boundary $\partial\mathcal{O}$ is infinitely smooth with $x \in \mathcal{O}$ locally on one side of the boundary. The coefficients $a_{pq}(\alpha, \cdot)$ are in $C^\infty(\bar{\mathcal{O}})$ for $\alpha \in \mathcal{K} \subset R^k$ and all values of the multi-indices p, q with the lengths $|p| \leq m, |q| \leq m$. Assume that $a_{pq}(\alpha, x) = a_{pq}(x)$ does not depend on α for $|p| = |q| = m$ and set

$$\bar{A}(x, D)y = \sum_{|p|=|q|=m} (-1)^m D^p(a_{pq}(x) D^q y), x \in \mathcal{O}.$$

Assume that $\bar{A}(x, D)$ is uniformly elliptic, i.e.,

$$|\sum_{|p|=|q|=m} (-1)^m a_{pq}(x) \lambda^{p+q}| \geq \hat{\mu}|\lambda|^{2m}, x \in \mathcal{O}, \lambda \in R^n.$$

for some $\hat{\mu} > 0$, where $\lambda^{p+q} = \lambda_1^{p_1+q_1} \lambda_2^{p_2+q_2} \cdots \lambda_n^{p_n+q_n}$. Furthermore, let $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}_0, \cdots, \tilde{\mathcal{B}}_{m-1})$ be a system of boundary operators

$$\tilde{\mathcal{B}}\varphi = \sum_{|h| \leq m_j} b_{jh}(x) D^h \varphi, x \in \partial\mathcal{O}$$

where $j = 0, 1, ..., m-1, 0 \leq m_0 < m_1 < ... < m_{m-1} \leq 2m-1, b_{jh}, \varphi \in C^\infty(\partial\mathcal{O})$. Assume that $a_{pq} = a_{qp}$ for $|p| = |q| = m$, the system $(\bar{A}(x, D), \tilde{\mathcal{B}}_j, j = 0, 1, \ldots, m-1)$ is formally self-adjoint and the system $\{\tilde{\mathcal{B}}_j\}$ is normal and covers $\bar{A}(x, D)$ and there exists a Green function for the problem $\dot{y} = \bar{A}(x, D)y, \tilde{\mathcal{B}}y = 0$ (cf. [1],[11]). For instance, we can consider the Dirichlet boundary problem in which case

$$\tilde{\mathcal{B}}_j = \frac{\partial^j}{\partial n^j}, j = 0, \cdots, m-1$$

is the j-th normal derivative, $n = (n^1, ..., n^n)$ is the outward normal.

References

[1] R. Arima, On general boundary value problem for parabolic equations, J. Math. Kyoto Univ. 4 (1964), 207-243.

[2] H. F. Chen, T. E. Duncan and B. Pasik-Duncan, Stochastic adaptive control for continuous time linear systems with quadratic cost, to appear in Appl. Math. Optim.

[3] G. DaPrato and A. Ichikawa, Riccati equations with unbounded coefficients, Ann. Math. Pura Appl. 140 (1985), 209-221.

[4] T. E. Duncan, B. Goldys and B. Pasik-Duncan, Adaptive control of linear stochastic evolution systems, Stochastics and Stochastic Reports 36 (1991), 71-90.

[5] T. E. Duncan, B. Maslowski and B. Pasik-Duncan, Adaptive boundary control of linear stochastic distributed parameter systems, SIAM J. Control Optim. 32 (1994), 648-672

[6] T. E. Duncan, B. Maslowski and B. Pasik-Duncan, Adaptive boundary control of linear distributed parameter systems described by analytic semigroups, to appear in J. Appl. Math. Optim.

[7] F. Flandoli, Direct solution of a Riccati equation arising in a stochastic control problem with control and observations on the boundary, Appl. Math. Optim. 14 (1986), 107-129.

[8] F. Flandoli, Algebraic Riccati equation arising in boundary control problems, SIAM J. Control Optim. 25 (1987), 612-636.

[9] I. Lasiecka and R. Triggiani, The regulator problem for parabolic equations with Dirichlet boundary control I, Appl. Math. Optim. 16 (1987), 147-168.

[10] I. Lasiecka and R. Triggiani, Numerical approximations of algebraic Riccati equations for abstract systems modelled by analytic semigroups and applications, Math. Comp. 57 (1991), 639-662.

[11] J. L. Lions and E. Magenes, Nonhomogeneous Boundary Value Problems and Applications, I, Springer-Verlag, Berlin, 1972.

[12] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.