Identification of Parameters in Hereditary Systems: A Numerical Study

Ferenc Hartung†, Terry L. Herdman‡ and Janos Turi†

† Programs in Mathematical Sciences, University of Texas at Dallas, Richardson, TX 75083
‡ Interdisciplinary Center for Applied Mathematics, VPI & SU, Blacksburg, VA 24061

Abstract

In this paper we study the numerical performance of parameter identification techniques, based on Euler-type approximation schemes, for non-singular and singular neutral delay differential equations.

1. Introduction

In a recent paper [6] we presented case studies for a parameter identification method, based on approximation by equations with piecewise constant arguments, on various classes of hereditary systems, including state-dependent delay systems and non-singular neutral equations. In this paper we continue experimenting with this method for more general non-singular neutral equations (NFDEs), and also for singular neutral equations (SNFDEs).

Consider e.g., the initial value problem (IVP) corresponding to the NFDE

$$\frac{d}{dt}\Big(x(t)+\sum_{i=1}^{m} q_i(t)x(t-\tau_i(t))\Big) = f\Big(t, x(t), x(t-\sigma(t))\Big) \tag{1.1}$$

for $t \in [0, T]$, with initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{1.2}$$

We assume that certain parameters, γ, in IVP (1.1)-(1.2) are not known explicitly, but some information is available on their values via measurements $(X_0, X_1, \ldots, X_l)$ of the solution, $x(t)$, at discrete time values $(t_0, t_1, \ldots, t_l)$. The goal is to find the parameter value, which minimizes the least squares fit-to-data criterion

$$J(\gamma) = \sum_{i=0}^{l} |x(t_i;\gamma) - X_i|^2, \qquad \gamma \in \Gamma,$$

i.e., which is the best-fit parameter for the measurements. (Denote this problem by $\mathcal{P}$). Problem $\mathcal{P}$ has been studied by many authors, for different classes of differential equations (see e.g. [1] and the references therein), including delay equations (see e.g. [2] and [8]).

All the above cited papers use the same idea to find the solution of the optimization problem $\mathcal{P}$:

Step 1) First take finite dimensional approximations of the parameters, γ^N, (i.e., $\gamma^N \in \Gamma^N \subset \Gamma$, $\dim \Gamma^N < \infty$, $\gamma^N \to \gamma$ as $N \to \infty$).

Step 2) Consider a sequence of approximate initial value problems ($\mathrm{IVP}_{M,N}$) corresponding to a discretization of IVP (1.1)-(1.2) for some fixed parameter $\gamma^N \in \Gamma^N$ with solutions $y^M(\cdot;\gamma^N)$ satisfying that $y^M(t,\gamma^N) \to x(t,\gamma)$ as $N, M \to \infty$, uniformly on compact time intervals.

Step 3) Define the least square minimization problems ($\mathcal{P}^{N,M}$) for each $N, M = 1, 2, \ldots$, i.e., find $\gamma^{N,M} \in \Gamma^N$, which minimizes the least squares fit-to-data criterion

$$J^{N,M}(\gamma^N) = \sum_{i=0}^{l} |y^M(t_i;\gamma^N) - X_i|^2, \qquad \gamma^N \in \Gamma^N.$$

Step 4) Assuming that the actual parameters belong to a compact subset of Γ, argue, that the sequence of solutions, $\gamma^{N,M}$ $(N, M = 1, 2, \ldots)$, of the finite dimensional minimization problems $\mathcal{P}^{N,M}$, has a convergent subsequence with limit $\bar\gamma \in \Gamma$.

Step 5) Show that $\bar\gamma$ is the solution of the minimization problem $\mathcal{P}$.

Note, that step 4) and 5) can be argued without using the particular approximation method of the initial value problem, using only compactness arguments and step 2) above (see e.g. in [8]).

In Section 2 we define an Euler-type approximation scheme for a class of non-singular neutral equations. (This scheme is a natural generalization of that in [4] for linear neutral equations with constant delays.) In

This research was supported in part by the Air Force Office of Scientific Research under Grant AFOSR F49620-93-1-0280

Section 2 we present numerical examples for identification of coefficient $q_i(t)$, delays τ_i and the initial function $\varphi(t)$. In Section 5 we apply an Euler-type approximation technique for identification of parameters in some singular neutral equations.

2. An Approximation Framework

Consider the vector NFDE

$$\frac{d}{dt}\Big(x(t)+\sum_{i=1}^{m} q_i(t)x(t-\tau_i(t))\Big) = f\Big(t, x(t), x(t-\sigma(t))\Big) \tag{2.1}$$

for $t \in [0,T]$, with initial condition

$$x(t)=\varphi(t), \qquad t\in[-r,0]. \tag{2.2}$$

Here $q_i : [0,T] \to \mathbf{R}$, $\tau_i : [0,T] \to [\varepsilon, \infty)$ $(i=1,\ldots,m)$ (for some $\varepsilon > 0$), $f : [0,T]\times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, $\sigma : [0,T] \to [0,\infty)$, and $\varphi : [-r,0] \to \mathbf{R}^n$ are continuous functions, where $r \geq \tau_1(t),\ldots,\tau_m(t),\sigma(t)$ for $t\in[0,T]$.

In this paper we concentrate on identifying parameters in the left hand side of the equations, such as coefficients q_i, delays τ_i, and the initial function φ. Parameters in the right hand side of the equation can be treated similarly, see e.g. [6]. We define $\gamma \equiv (q_1,\ldots,q_m,\tau_1,\ldots,\tau_m,\varphi)$ for our parameter vector, and $\Gamma \equiv C^m([0,T];\mathbf{R}^n)\times C^m([0,T];\mathbf{R})\times C([-r,0];\mathbf{R}^n)$ for our parameter space.

Following the general method described in the Introduction, first we consider finite dimensional approximations $\gamma^N = (q_1^N,\ldots,\tau_1^N,\ldots,\varphi^N) \in \Gamma^N$ of parameter $\gamma\in\Gamma$. In the numerical examples we shall use linear spline approximations with equidistant mesh points. It is known (see e.g. [9]) that linear splines can be used to approximate piecewise smooth functions uniformly on compact time intervals.

The second step is to define discretizations of IVP (2.1)-(2.2) with parameter γ^N. We use the natural generalization of the numerical scheme introduced in [4]:

Let h be a positive number. Throughout this paper we shall use the notation $[t]_h \equiv [t/h]h$, where $[\cdot]$ is the greatest integer function. Elementary estimates give that $t-h < [t]_h \leq t$ and therefore $[t]_h \to t$ as $h\to 0$.

We associate the following NFDE with piecewise constant right-hand side to (2.1):

$$\begin{aligned}\frac{d}{dt}\Big(y_h(t)+\sum_{i=1}^{m} q_i^N([t]_h)y_h(t-[\tau_i^N([t]_h)]_h)\Big) \qquad (2.3)\\ = f\left([t]_h, y_h([t]), y_h([t]_h - [\sigma([t]_h)]_h\right).\end{aligned}$$

The subscript h of $y_h(t)$ emphasizes that $y_h(t)$ is the solution of (2.3) corresponding to the discretization parameter h. The associated initial condition to (2.3) is

$$y_h(t) = \varphi^N(t), \qquad t\in[-r,0]. \tag{2.4}$$

By a solution of the initial value problem (2.3)-(2.4) we mean a function $y_h : [-r,T] \to \mathbf{R}^n$, which is defined on $[-r,0]$ by (2.4) and satisfies the following properties on $[0,T]$:

(i) it is continuous on $[0,T]$,

(ii) the function $y_h(t)+\sum_{i=1}^m q_i^N([t]_h)y_h(t-[\tau_i^N([t]_h)]_h)$ is differentiable at each point $t\in(0,T)$ with the possible exception of the points kh $(k=0,1,2,\ldots)$ where finite one-sided derivatives exist,

(iii) y_h satisfies (2.3) on each interval $[kh,(k+1)h)\cap[0,T]$ for $k=0,1,2,\ldots$.

Using the method of steps it can be verified that IVP (2.3)-(2.4) has a unique solution on $[0,\infty)$. We introduce the notation $a(k)\equiv y_h(kh)$. It is easy to see, using that the right hand side of (2.3) is constant on the intervals $[kh,(k+1)h)$, that the sequence $a(k)$ satisfies

$$\begin{aligned} a(k+1) &= a(k)+\sum_{i=1}^{m}\left(q_i^N(kh)a\left(k-\left[\frac{\tau_i^N(kh)}{h}\right]\right)\right.\\ &\quad \left. - q_i^N((k+1)h)a\left(k+1-\left[\frac{\tau_i^N((k+1)h)}{h}\right]\right)\right)\\ &\quad + hf\left(kh, a(k), a\left(k-\left[\frac{\sigma(kh)}{h}\right]\right)\right)\\ &\quad \text{for } k=0,1,\ldots, \qquad (2.5)\\ a(k) &= \varphi^N(kh), \quad \text{for } -r\leq kh\leq 0. \qquad (2.6)\end{aligned}$$

Therefore computing $a(k)$ is a simple numerical task. Note, that this scheme uses approximate solution values only at mesh points.

We conjecture the following:

Theorem 2.1 *Assume that the function f is locally Lipschitz-continuous in its second and third arguments. If $\gamma^N \to \gamma$ (in a product norm), then $y_h(t;\gamma^N)\to x(t;\gamma)$ uniformly on compact time intervals, as $h\to 0^+$, $N\to\infty$, where $x(t;\gamma)$ and $y_h(t;\gamma^N)$ are the solutions of IVP (2.1)-(2.2) and IVP (2.3)-(2.4) corresponding to parameter γ and γ^N, respectively.*

The proof of this theorem for state-dependent retarded delay equations and for a very similar approximating scheme can be found in [5].

In practice we proceed as follows: We select "small enough" $h>0$ and "large enough" N, and consider the least square criterion

$$J_h^N(\gamma^N) = \sum_{i=0}^{l} |y_h(t_i;\gamma^N)-X_i|^2, \qquad \gamma^N\in\Gamma^N,$$

then solve the (finite dimensional) minimization problem numerically, and use the solution of it as an approximation of the solution of the original identification problem.

3. Case Studies

In this section we present some numerical examples to illustrate the identification method described in Sections 1 and 2. We note, that in Examples 3.5, 5.1 and 5.2 we used a simple golden section search method for solving the one dimensional minimization problems. In the other examples, where we had high dimensional optimization problems, we used a nonlinear least square minimization code, based on a secant method with Dennis-Gay-Welsch update, combined with a trust region technique. See Section 10.3 in [3] for detailed description of this method.

Example 3.1 Consider the vector NFDE

$$\frac{d}{dt}\left(\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}+q\begin{pmatrix}x_1(t-1)\\x_2(t-1)\end{pmatrix}\right)=2\pi\begin{pmatrix}-x_2(t)\\x_1(t-2)\end{pmatrix},\quad t\in[0,3], \tag{3.1}$$

$$\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}=\begin{pmatrix}\cos\pi t\\ \sin\pi t\end{pmatrix},\qquad t\in[-2,0]. \tag{3.2}$$

It is easy to check that the solution of IVP (3.1)-(3.2) for $q=-1$ is $x_1(t)=\cos\pi t$ and $x_2(t)=\sin\pi t$. We generated the measurements $X_i=(X_{i,1},X_{i,2})^T$ at $t_i=0.25i$ $(i=0,1,\ldots,12)$ using the true solution. The corresponding approximate IVP is

$$\frac{d}{dt}\left(\begin{pmatrix}y_{h,1}(t)\\y_{h,2}(t)\end{pmatrix}+q\begin{pmatrix}y_{h,1}(t-[1]_h)\\y_{h,2}(t-[1]_h)\end{pmatrix}\right) \tag{3.3}$$

$$= 2\pi\begin{pmatrix}-y_{h,2}([t]_h)\\y_{h,1}([t]_h-[2]_h)\end{pmatrix},\quad t\in[0,3]$$

$$\begin{pmatrix}y_{h,1}(t)\\y_{h,2}(t)\end{pmatrix}=\begin{pmatrix}\cos\pi t\\ \sin\pi t\end{pmatrix},\qquad t\in[-2,0]. \tag{3.4}$$

We minimize

$$J_h(q)=\sum_{i=0}^{12}\Big((y_{h,1}(t_i;q)-X_{i,1})^2+(y_{h,2}(t_i;q)-X_{i,2})^2\Big)$$

for $q\in[-3,3]$. The numerical results of the minimization problem, corresponding to initial guess $q=0$, are shown in Table 1.

Example 3.2 Consider a scalar NFDE with two delays

$$\begin{aligned}\frac{d}{dt}\Big(x(t)+q_1x(t-1)+q_2x(t-2)\Big)&= -t,\qquad t\ge 0,\\ x(t)&= t,\qquad t\in[-2,0].\end{aligned}$$

Table 1

h	$\bar q$	$J_h(\bar q)$
0.1000	-1.023867	9.5308382
0.0100	-1.002190	0.0948585
0.0010	-1.000218	0.0009480
0.0001	-0.999988	0.0000007

The solution of this IVP corresponding to $q_1=-1$ and $q_2=2$ is

$$x(t)=\begin{cases}-t-t^2/2, & t\in[0,1],\\ 3/2-2t-t^2, & t\in[1,2],\\ (-1-4t-t^2)/2, & t\in[2,3].\end{cases} \tag{3.5}$$

The corresponding approximate IVP is

$$\begin{aligned}\frac{d}{dt}\Big(y_h(t)+q_1y_h(t-[1]_h)+q_2y_h(t-[2]_h)\Big)&\\ = -[t]_h,&\qquad t\ge 0,\\ y_h(t) = -t,&\qquad t\in[-2,0].\end{aligned}$$

Using function (3.5) we generated measurements X_i at $t_i=0.1i$ $(i=0,\ldots,30)$. Consider the minimization problem

$$\min J_h(q_1,q_2)\equiv\sum_{i=0}^{30}(y_h(t_i;q_1,q_2)-X_i)^2,$$

for $(q_1,q_2)\in[-3,3]\times[-3,3]$. The numerical solutions of this problem for different h values are listed in Table 2.

Table 2

h	$\bar q_1$	$\bar q_2$	$J_h(\bar q_1,\bar q_2)$
0.1000	-1.015760	2.055376	9.32088e-04
0.0100	-1.001582	2.005536	9.46526e-06
0.0010	-1.000158	2.000554	9.47992e-08
0.0001	-1.000016	2.000055	9.48138e-10

Example 3.3 Our next example is the scalar NFDE

$$\begin{aligned}\frac{d}{dt}\Big(x(t)-2x(t-2)\Big)&= -x(t-1),\quad t\ge 0,\\ x(t)&=\varphi(t),\qquad t\in[-2,0].\end{aligned}$$

The solution of this IVP corresponding to $\varphi(t)=(t+1)^2$ is

$$x(t)=\begin{cases}(1-4t+2t^2-t^3)/3 & t\in[0,1],\\ (53-136t+78t^2-12t^3+t^4)/12 & t\in[1,2],\\ (3097-3320t+1310t^2-240t^3 & \\ \quad +20t^4-t^5)/60 & t\in[2,3].\end{cases}$$

With this function we generated measurements X_i at $t_i=0.1i$ $(i=0,\ldots,30)$. Since the initial function is infinite dimensional, first we approximate it using linear spline functions with equidistant mesh points. In the first case consider a 3 dimensional approximation, i.e., a linear spline with 3 mesh points at -2, -1 and at 0 with corresponding function values a_1, a_2 and a_3. We assume that $\gamma \equiv (a_1, a_2, a_3) \in \Gamma \equiv [-4,4]^3$, and we minimize the cost function

$$J_h(\gamma) = \sum_{i=0}^{30} (y_h(t_i;\gamma) - X_i)^2, \qquad \gamma \in \Gamma.$$

Table 3 presents our numerical findings, using initial guesses $a_i = 0$ $(i = 1,2,3)$. Table 4 and Table 5 contain the value of the cost function and the maximal error, (i.e., $\max_{i=0,\ldots,30} |y_h(t_i;\gamma) - X_i|$), respectively, for 3, 7, 11 and 15 dimensional spline approximations and several h values, using constant zero function as the initial guess for φ. We show the true initial function (solid line) and the identified initial functions (dashed lines) using 3, 5 and 7 dimensional spline approximations and discretization parameter $h = 0.0001$ in Figure 1.

Table 3

h	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$J_h(\bar{\gamma})$
0.1000	0.883875	-0.172934	0.750900	1.7943582
0.0100	0.876778	-0.136629	0.727087	1.7510779
0.0010	0.876124	-0.133062	0.724640	1.7473206
0.0001	0.876059	-0.132706	0.724395	1.7469508

Table 4 : $J_h(\bar{\gamma})$

h	3	7	11	15
0.1000	1.794358	0.022129	0.009508	0.003518
0.0100	1.751077	0.018978	0.006181	0.000076
0.0010	1.747320	0.019012	0.006189	0.000037
0.0001	1.746950	0.019019	0.006193	0.000036

Table 5 : Maximal error

h	3	7	11	15
0.1000	0.249100	0.068077	0.061564	0.058939
0.0100	0.272913	0.025331	0.011904	0.010888
0.0010	0.275360	0.023690	0.008381	0.006894
0.0001	0.275605	0.023846	0.008483	0.006646

Example 3.4 In our next example we consider

$$\frac{d}{dt}\Big(x(t) + q(t)x(t-1)\Big) = -t, \qquad t \geq 0,$$
$$x(t) = 1, \qquad t \in [-2,0].$$

The solution of this IVP with $q(t) = 1 - (t-1)^2/4$ is

$$x(t) = \begin{cases} 1 - t/2 - t^2/4, & t \in [0,1], \\ (13 - 10t + 2t^3 - t^4)/16, & t \in [1,2], \\ (52 - 40t + 24t^2 + 6t^3 - 21t^4 \\ \quad + 8t^5 - t^6)/64, & t \in [2,3]. \end{cases}$$

We used this function to generate the measurements at $t_i = 0.1i$ $(i = 0,1,\ldots,30)$. First we approximated $q(t)$ on $[0,3]$ by 3 dimensional linear splines. The corresponding numerical results are in Table 6. In Tables 7 and 8 we show the cost function and the maximal error for 3, 11, 19 and 27 dimensional cases. In Figure 2 we plotted out the true $q(t)$ (solid line) and the identified $q(t)$ (dashed lines) for 3, 5 and 7 dimensional cases, corresponding to $h = 0.001$ and constant zero initial guess.

Table 6

h	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$J_h(\bar{\gamma})$
0.1000	0.648225	1.000492	-0.190801	0.1584303
0.0100	0.751983	1.041864	-0.113512	0.1514255
0.0010	0.762359	1.046010	-0.105767	0.1507152
0.0001	0.763397	1.046424	-0.104992	0.1506441

Table 7 : $J_h(\bar{\gamma})$

h	3	11	19	27
0.1000	1.584e-1	2.200e-4	2.697e-06	5.884e-07
0.0100	1.514e-1	1.980e-4	2.817e-06	5.532e-09
0.0010	1.507e-1	1.966e-4	2.861e-06	5.226e-11
0.0001	1.506e-1	1.964e-4	2.866e-06	3.762e-11

Table 8 : Maximal error

h	3	11	19	27
0.1000	0.190801	0.123327	0.123439	0.123373
0.0100	0.113512	0.014406	0.012419	0.012337
0.0010	0.108510	0.007543	0.003063	0.001233
0.0001	0.108924	0.008260	0.003803	0.000782

Example 3.5 Consider the scalar NFDE

$$\frac{d}{dt}\Big(x(t) - 0.5x(t-\tau)\Big) = x(t-2), \quad t \geq 0, \tag{3.6}$$

$$x(t) = \begin{cases} t+1, & t \in [-2,-1], \\ t^2, & t \in [-1,0]. \end{cases} \tag{3.7}$$

We assume that $\tau \in [0.1, 2]$. The solution corresponding to $\tau = 1$ is

$$x(t) = \begin{cases} t^2 - t & t \in [0,1], \\ \frac{1}{3}t^3 - \frac{3}{2}t^2 + \frac{5}{2}t - \frac{4}{3} & t \in [1,2], \\ \frac{1}{2}t^3 - \frac{15}{4}t^2 + \frac{37}{4}t - \frac{43}{6} & t \in [2,3]. \end{cases}$$

We generate X_i at $t_i = 0.25i$ $(i = 0,\ldots,12)$ using this function. Consider the approximate IVP

$$\frac{d}{dt}\Big(y_h(t) - 0.5y_h(t - [\tau]_h)\Big) = y_h([t]_h - [2]_h), \tag{3.8}$$

$$y_h(t) = \begin{cases} t+1, & t \in [-2,-1], \\ t^2, & t \in [-1,0]. \end{cases} \tag{3.9}$$

and the minimization problem

$$\min_{\tau\in[0.1,2]} J_h(\tau) \equiv \sum_{i=0}^{12}(y_h(t_i,\tau)-X_i)^2. \qquad (3.10)$$

Unfortunately, a secant-type numerical minimization routine fails for (3.10), since $J_h(\tau)$ is piecewise constant, due to the discretization, $[\tau]_h$, of τ in (3.8). But (3.10) does have a solution, which we found by a golden section search method. Table 9 contains the results.

Table 9

h	$\bar\tau$	$J_h(\bar\tau)$
0.1000	1.000000	4.1461e-03
0.0100	1.000000	4.6388e-05
0.0010	1.000000	4.6825e-07
0.0001	1.000000	4.6870e-09

4. Modified Approximation Scheme

In Example 3.5 we have observed that the approximate scheme (2.5)-(2.6) is not appropriate for delay identification, since it discretizes the delay, and therefore makes the objective function to be piecewise constant. We modify (2.5)-(2.6) as follows: Given $h>0$, let $z_h(t)$ be a piecewise linear function on the intervals $[kh,(k+1)h]$ ($k=0,1,\ldots$), defined by

$$\begin{aligned} z_h((k+1)h) &= z_h(kh)+\sum_{i=1}^{m}\Big(q_i^N(kh)z_h(kh-\tau_i^N(kh)) \\ &\quad - q_i^N((k+1)h)z_h((k+1)h-\tau_i^N((k+1)h))\Big) \\ &\quad + hf\Big(kh, z_h(kh), z_h(kh-[\sigma(kh)]_h)\Big) \\ &\quad \text{for } k=0,1,\ldots, \qquad (4.1)\\ z_h(t) &= \varphi^N(t), \quad \text{for } -r\le t\le 0. \qquad (4.2)\end{aligned}$$

Note, that the difference between (2.5)-(2.6) and (4.1)-(4.2) is that in the latter scheme we use an interpolate value for the solution at $kh-\tau_i(kh)$, instead of using the value at the corresponding mesh point, $kh-[\tau_i(kh)]_h$. Also note, that on the initial interval, $[-r,0]$, we do not interpolate between mesh points. This interpolation slightly slows down the method, but the advantage is that the solution depends continuously on the delays, τ_i, and hopefully it is differentiable with respect to delays, so we can use the secant method for the numerical optimization.

Example 4.1 Here we redo Example 3.5 using the modified approximation scheme, (4.1)-(4.2), and our secant optimization method. The numerical results, using initial guess $\tau=0.5$, are presented in Table 10.

Table 10

h	$\bar\tau$	$J_h(\bar\tau)$
0.1000	1.017260	3.8472e-03
0.0100	1.001874	4.0117e-05
0.0010	1.000189	4.0263e-07
0.0001	1.000019	4.0278e-09

Example 4.2 Consider the scalar NFDE

$$\frac{d}{dt}\Big(x(t)+x(t-\tau(t))\Big)=x(t-1), \qquad t\ge 0,$$
$$x(t)=t, \qquad t\le 0.$$

The solution of this IVP corresponding to the delay function

$$\tau(t)=\begin{cases} -t^2+2t+2, & t\in[0,2],\\ -0.5t+3, & t\in[2,3]\end{cases}$$

is

$$x(t)=\begin{cases} -\frac12 t^2, & t\in[0,1],\\ -\frac16 t^3-\frac12 t^2+\frac12 t-\frac13, & t\in[1,2],\\ -\frac{1}{24}t^4+\frac52 t^2-\frac{55}{6}t+\frac{19}{3}, & t\in[2,2.5],\\ -\frac{1}{24}t^4+\frac43 t^3-\frac{11}{2}t^2+\frac{35}{6}t-2, & t\in[2.5,3].\end{cases}$$

We used this function to generate the measurements at $t_i=0.025i$ ($i=0,1,\ldots,120$). The numerical solutions of the corresponding minimization problems, using 3, 5 and 7 dimensional spline functions, discretization parameter $h=0.001$, and initial guess $\tau(t)=1.5$, are printed out in Figure 3. Tables 11 and 12 contain the cost function and the maximal error of the numerical solution, respectively, for dimensions 3, 5, 7 and 9.

Table 11 : $J_h(\bar\gamma)$

h	3	5	7	9
0.1000	5.627e+00	2.174e-01	5.052e-02	1.933e-02
0.0100	5.452e+00	2.501e-01	5.977e-02	2.646e-02
0.0010	5.405e+00	2.494e-01	6.001e-02	2.656e-02
0.0001	5.400e+00	2.493e-01	6.003e-02	2.713e-02

Table 12 : Maximal error

h	3	5	7	9
0.1000	0.311296	0.177627	0.084575	0.107679
0.0100	0.246514	0.141536	0.056191	0.083910
0.0010	0.244553	0.138008	0.053530	0.086728
0.0001	0.244352	0.137672	0.053275	0.066188

5. Singular Neutral Equations

In this section we experiment with identification methods for singular neutral equations using an Euler-type approximation scheme and the general identification method of Section 1. We illustrate our approximation method on two examples (see also [7]).

Example 5.1 Consider the scalar SNFDE

$$\int_{-1}^{0}(-s)^{\alpha}x(t+s)\,ds \;=\; 1, \qquad t>0, \tag{5.1}$$

$$x(t) \;=\; 0, \qquad t\in[-1,0], \tag{5.2}$$

where α is the parameter to be identified. The exact solution of IVP (5.1)-(5.2) corresponding to $\alpha=-1/2$ is

$$x(t)=\begin{cases} \frac{1}{\pi}t^{-1/2}, & 0<t\le 1,\\ \frac{2}{\pi}t^{-1/2}, & 1<t\le 2,\\ \frac{2}{\pi}t^{-1/2}\left(1+\frac{2}{\pi}\arctan\left(\frac{t-2}{t}\right)^{1/2}\right), & 2<t\le 3. \end{cases} \tag{5.3}$$

We hope to approximate the solution of IVP (5.1)-(5.2) by functions $y_N(t)$ as $N\to\infty$, which are linear on each interval $[k/N,(k+1)/N]$, $(k=-N,-N+1,\ldots)$, and satisfy

$$\int_{-1}^{0}(-s)^{\alpha}y_N(t+s)\,ds \;=\; 1, \qquad t>0, \tag{5.4}$$

$$y_N(k/N) \;=\; 0,\; k=-N,\ldots,0. \tag{5.5}$$

Introduce the notations $a(k)\equiv y_N(k/N)$ and

$$I(\alpha,k)\equiv\int_{k/N}^{(k+1)/N}s^{\alpha}\,ds. \tag{5.6}$$

Linearity of $y_N(t)$ yields for $t\in[k/N,(k+1)/N]$ that

$$y_N(t)=a(k)+N(a(k+1)-a(k))(s-k/N). \tag{5.7}$$

Elementary manipulations and (5.6) and (5.7) imply

$$\begin{aligned}
&\int_{-1}^{0}(-s)^{\alpha}y_N(n/N+s)\,ds\\
&= \sum_{k=0}^{N-1}\int_{k/N}^{(k+1)/N}s^{\alpha}y_N(n/N-s)\,ds\\
&= \sum_{k=0}^{N-1}\int_{k/N}^{(k+1)/N}s^{\alpha}\Big(a(n-k)\\
&\quad + N(a(n-k)-a(n-k-1))(s-k/N)\Big)\,ds\\
&= \sum_{k=0}^{N-1}\Big(a(n-k)+k(a(n-k)\\
&\qquad - a(n-k-1))\Big)I(\alpha,k)\\
&\quad + \sum_{k=0}^{N-1}N(a(n-k)-a(n-k-1))I(\alpha+1,k)\\
&= a(n-N)\Big(NI(\alpha+1,N-1)\\
&\qquad -(N-1)I(\alpha,N-1)\Big)\\
&\quad + \sum_{k=n-N+1}^{N-1}a(k)\Big((n-k+1)I(\alpha,n-k)\\
&\qquad - NI(\alpha+1,n-k)\\
&\qquad + NI(\alpha+1,n-k-1)\\
&\qquad -(n-k-1)I(\alpha,n-k-1)\Big)\\
&\quad + a(n)\Big(I(\alpha,0)-NI(\alpha+1,0)\Big).
\end{aligned} \tag{5.8}$$

Using relation (5.8) and equation (5.4), we can obtain a simple difference equation for $a(n)$, and conclude that IVP (5.4)-(5.5) has unique solution, which is numerically easy to obtain. We used this approximation scheme and the general identification method described in Section 1 to identify α in IVP (5.1)-(5.2). We generated the measurements X_i by the function (5.3) at $t_i=0.5i$ $(i=0,1,\ldots,6)$. We found the minimum of

$$J_N(\alpha)\equiv\sum_{i=0}^{6}(y_N(t_i;\alpha)-X_i)^2, \qquad \alpha\in[-0.9,3.0]$$

by the golden search method. Table 13 contains our numerical findings, which show good convergence to the true parameter, $\alpha=-1/2$.

Table 13

N	$\bar{\alpha}$	$J_N(\bar{\alpha})$
5	-0.594451	0.102035
10	-0.583831	0.053790
20	-0.559629	0.025588
50	-0.524100	0.003814
100	-0.500188	0.000051

Example 5.2 Our next example is the vector SNFDE

$$\frac{d}{dt}\left(x_1(t)+\int_{-1}^{0}x_2(t+s)\,ds\right)=x_2(t),\; t\ge 0, \tag{5.9}$$

$$\frac{d}{dt}\left(\int_{-1}^{0}(-s)^{\alpha}x_2(t+s)\,ds\right)=x_1(t),\; t\ge 0, \tag{5.10}$$

$$x_1(t)=1, \qquad t\in[-1,0], \tag{5.11}$$

$$x_2(t)=0, \qquad t\in[-1,0]. \tag{5.12}$$

The objective in this example is the identification of α. The solution of IVP (5.9)-(5.12) corresponding to $\alpha=-1/2$ is

$$x_1(t)=\begin{cases} 1, & 0<t\le 1,\\ 1+\frac{4}{3\pi}(t-1)^{3/2}, & 1<t\le 2,\end{cases}$$

and

$$x_2(t)=\begin{cases} \frac{2}{\pi}t^{1/2}, & 0<t\le 1,\\ \frac{2}{\pi}+\frac{4}{\pi}(t^{1/2}-1)+\frac{1}{2\pi}(t-1)^2, & 1<t\le 2.\end{cases}$$

The corresponding approximate IVP is

$$\frac{d}{dt}\left(y_{N,1}(t) + \int_{-1}^{0} y_{N,2}(t+s)\,ds\right) = y_{N,2}(t), \quad (5.13)$$

$$\frac{d}{dt}\left(\int_{-1}^{0} (-s)^{\alpha} y_{N,2}(t+s)\,ds\right) = y_{N,1}(t), \quad (5.14)$$

$$y_{N,1}(k/N) = 1, \qquad k = -N, \ldots, 0, \quad (5.15)$$

$$y_{N,2}(k/N) = 0, \qquad k = -N, \ldots, 0, \quad (5.16)$$

where we assume that $y_{N,1}(t)$ and $y_{N,2}(t)$ are piecewise linear functions on the intervals $[k/N, (k+1)/N]$ ($k = -N, -N+1, \ldots$). Integrating (5.13) from n/N to $(n+1)/N$ we get

$$\begin{aligned} y_{N,1}((n+1)/N) &+ \int_{-1}^{0} y_{N,2}((n+1)/N+s)\,ds \\ &= y_{N,1}(n/N) + \int_{-1}^{0} y_{N,2}(n/N+s)\,ds \\ &\quad + \int_{n/N}^{(n+1)/N} y_{N,2}(s)\,ds. \end{aligned}$$

Changing variables in these integrals yields

$$\begin{aligned} y_{N,1}((n+1)/N) &+ \int_{(n-N+1)/N}^{(n+1)/N} y_{N,2}(s)\,ds \\ &= y_{N,1}(n/N) + \int_{(n-N)/N}^{(n+1)/N} y_{N,2}(s)\,ds. \end{aligned}$$

Therefore, using the notations $a_1(k) \equiv y_{N,1}(k/N)$ and $a_2(k) \equiv y_{N,2}(k/N)$, we have the simple recursive formula for $a_1(n+1)$:

$$a_1(n+1) = a_1(n) + \Big(a_2(n-N+1) + a_2(n-N)\Big)/(2N).$$

Integrating (5.14) from n/N to $(n+1)/N$ gives

$$\begin{aligned} &\int_{-1}^{0} (-s)^{\alpha} y_{N,2}((n+1)/N+s)\,ds \\ &= \int_{-1}^{0} (-s)^{\alpha} y_{N,2}(n/N+s)\,ds + \int_{n/N}^{(n+1)/N} y_{N,1}(s)\,ds. \end{aligned}$$

From this equation, applying relation (5.8) and the fact that we have already obtained $a_1(n+1)$, we can find a recursive formula for $a_2(n+1)$, which we omit here. This shows that IVP (5.13)-(5.16) has a unique solution, and numerical testing proves that it approximates the solution of IVP (5.9)-(5.13).

We generated measurements $X_i = (X_{i,1}, X_{i,2})^T$ at $t_i = 0.2i$ ($i = 0, \ldots, 10$), using the solution of IVP (5.9)-(5.13) corresponding $\alpha = -1/2$. The approximate minimum of

$$J_N(\alpha) \equiv \sum_{i=0}^{10}\Big((y_{N,1}(t_i;\alpha) - X_{i,1})^2 + (y_{N,2}(t_i;\alpha) - X_{i,2})^2\Big)$$

for $\alpha \in [-0.9, 3]$ is presented in Table 14. The numerical result is surprisingly good, even for very small N values.

Table 14

N	$\bar{\alpha}$	$J_N(\bar{\alpha})$
5	-0.501357	1.3592e-03
10	-0.499989	2.6756e-05
20	-0.500019	1.0103e-07
50	-0.500004	8.5005e-09
100	-0.500001	5.3938e-10

6. References

[1] H. T. Banks and K. Kunisch, "Estimation Techniques for Distributed Parameter Systems", Birkhäuser, 1989.

[2] H. T. Banks and P. K. Daniel Lamm , *Estimation of delays and other parameters in nonlinear functional differential equations*, SIAM J. Control and Optimization, 21, No. 6, pp. 895–915, 1983.

[3] J. E. Dennis and R. B. Schnabel, "Numerical Methods for Unconstrained Optimization and Nonlinear Equations", Prentice-Hall, 1983.

[4] I. Győri, *On approximation of the solutions of delay differential equations by using piecewise constant arguments*, Internat. J. of Math. & Math. Sci., 14, No. 1, pp. 111-126, 1991.

[5] F. Hartung, "On classes of functional differential equations with state-dependent delays", Ph.D. Dissertation, University of Texas at Dallas, 1995.

[6] F. Hartung, T. L. Herdman and J. Turi, *Identifications of parameters in hereditary systems*, to appear in the Proceedings of ASME Fifteenth Biennial Conference on Mechanical Vibration and Noise, Boston, Massachusetts, September 1995.

[7] T. L. Herdman and J. Turi, *Singular neutral equations*, in Distributed Parameter Control Systems, G. Chen, E. B. Lee, W. Littman and L. Markus (ed.), Marcel Dekker, pp. 501–511, 1991.

[8] K. A. Murphy, *Estimation of time- and state-dependent delays and other parameters in functional differential equations*, SIAM J. Appl. Math., 50, No. 4, pp. 972–1000, 1990.

[9] M. H. Schultz, "Spline Analysis", Prentice-Hall, 1973.

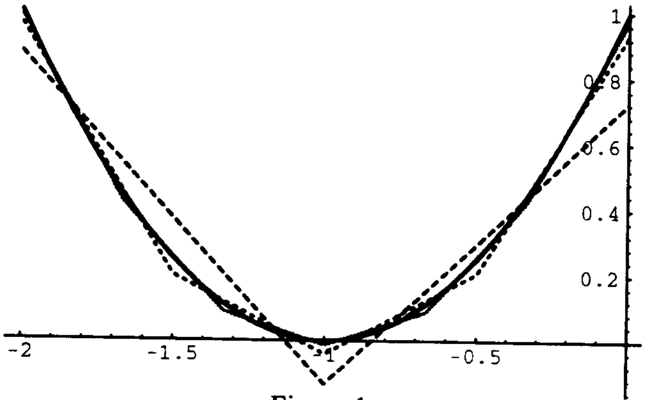

Figure 1

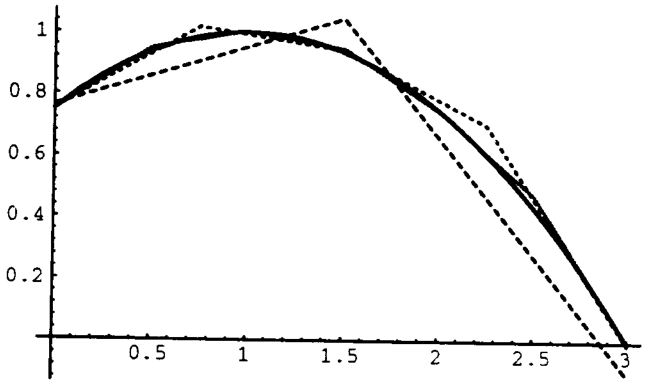

Figure 2

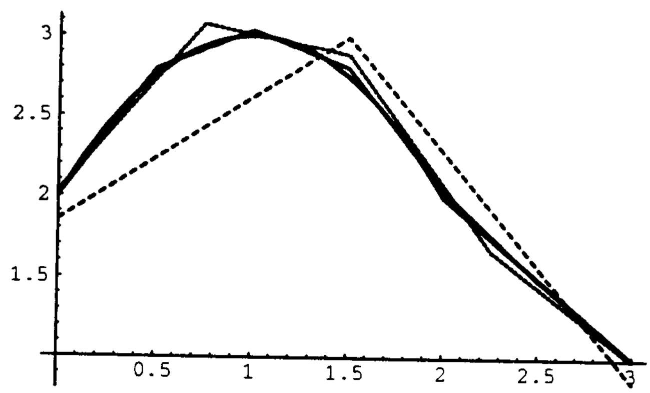

Figure 3