Sampled-Data H^2-Optimal Control with Mixed Discrete/Continuous Specifications

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Abstract

This paper introduces a new approach to the H^2 design for sampled-data systems, that is systems consisting of a continuous-time plant and a discrete-time controller connected via sampling and hold devices. The main feature of the proposed approach is the presense of not only continuous-time, but also of discrete-time performance specifications, which can be asynchronized with the control loop, hence mixed specifications. It is shown that this problem can be reduced to a finite-dimensional pure discrete-time constrained H^2 problem. In the paper the full state-space solution of the latter problem is derived. The benefits of the proposed approach are demonstrated by a numerical example.

Notations

$\xi(t)$, $\bar{\xi}[k]$:	continuous-time and discrete-time signals, respectively;
$\mathcal{G}$, $G(s)$:	linear operator in continuous time and its transfer matrix (if the latter exists);
$\overline{\mathcal{G}}$, $\bar{G}(z)$:	linear operator in discrete time and its transfer matrix;
$\mathcal{S}_h^\tau$, $\mathcal{S}_h$:	sampling operators with period h $((\mathcal{S}_h^\tau \xi)[k] = \xi(kh+\tau),\ \mathcal{S}_h = \mathcal{S}_h^0)$;
$\mathcal{H}_h$:	(zero order) hold operator with period h $((\mathcal{H}_h \bar{\xi})(t) = \bar{\xi}[k],\ \forall t \in [kh, (k+1)h))$;
A' :	transpose of a matrix A;
$\mathcal{F}(\mathcal{P}, \mathcal{C})$:	closed-loop mapping between disturbances and controlled signals when $\mathcal{P}$ is in feedback interconnection with $\mathcal{C}$;
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$:	transfer function in terms of its state-space realization;

1. Introduction and motivations

Consider the sampled-data (SD) control system setup in Fig. 1. $\mathcal{P}$ is a continuous-time linear time-invariant (LTI) generalized plant; $\overline{\mathcal{K}}$ is a discrete-time linear controller; the measured plant output y is sampled by the sampling device $\mathcal{S}_h$ with a period h; the control input u is generated by the hold device $\mathcal{H}_h$; w_c

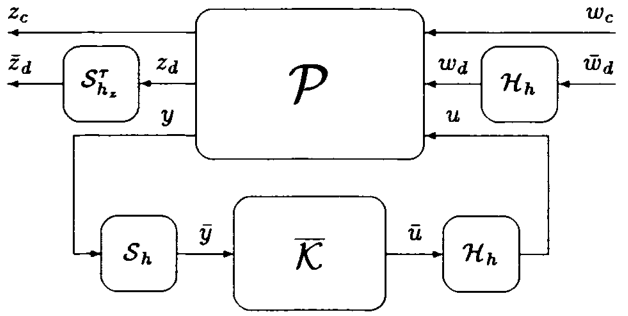

Figure 1: Sampled-data control system setup.

is a continuous-time disturbance; and $\bar{w}_d$ is a discrete-time disturbance (measurement noise, for example). The main feature of this setup is the presence of both continuous-time (z_c) and discrete-time ($\bar{z}_d$) regulated outputs[1], that implies *stressing* discrete-time performance of some signals. In other words, we are concerned with system performance at the intersampling time *as well as* at the sampling instances kh_z. This is in contrast with previous approaches to the H^2 problem and is motivated as follows.

First, consideration of only a discrete-time objective (by discretizing the plant) may be undesirable, since H^2-optimal controllers designed by this criterion may produce bad continuous-time behavior. This is so because of the complicated zero structure of discretized models and the appearance of lightly damped zeros after discretization [2]. It is in principle possible to prevent the cancellation of discrete-time lightly damped zeros by, for example, adding an extra penalty on the control variable [2] or minimizing the L^2-norm of the signals after pre-filtering [12]. However, such approaches are *ad hoc* and hence add extra "fuzzyness" into the design. Moreover, by using generalized hold functions [8] or multirate digital controllers [11] it is possible to achieve almost arbitrary good performance in the sampled time instances at the expense of dete-

[1] Note, that h_z, the sampling period for $\bar{z}_d$, needs not coincide with h, the control loop sampling period.

riorating the intersampling performance. This observation prompted researchers to treat the continuous-time performance for SD systems *directly*; for SD H^2-optimization see [6, 9, 3].

However pure continuous-time specifications also are not always what is needed. Necessity in considering (or emphasising) discrete-time performance, even for the case when plant dynamic is continuous-time, arises naturally in a good deal of practical problems, such as fire control and so on. But if we will be concerned only with continuous-time performance, we can obtain unnecessary poor discrete behavior. Discrete-time performance achievable with a controller, which is designed on the basis of a pure continuous-time criterion, can be essentially worse than is possible to get using other approaches. Losely speaking, instantaneous performance "dissolves" in a continuous-time one and it might be undesirable.

Also, mathematical formulation of any practical control problem always involves some auxiliary variables, which are needed to provide desirable behavior of signals of interest. Frequently such an auxiliary signals can be naturally introduced in discrete time[2].

So one ought to realize that SD design should include, beside others, some tradeoff between sampled and intersampled performance. Starting from two extremes — a pure continuous-time and a pure discrete-time treatments — it is reasonable to look for a solution, which takes into account both of these sides. The setup in Fig. 1 reflects this desirement.

The problem to be solved in this paper is the problem of minimizing the H^2 norm of the closed-loop mapping between the external signals w_c and $\bar{w}_d$ and the regulated outputs z_c and $\bar{z}_d$ in the setup in Fig. 1. It will be shown that this problem is equivalent to a pure discrete H^2 problem with a periodic controller. The latter problem, in turn, can be reduced to a time invariant H^2 problem with *constrained* controller feedthrough matrix [10]. Our treatment of this constraint is different from those that have been proposed in the literature. Rather than handle it via some intermediate steps, we will do it *directly* in terms of controller parameters. To this end the ideas of Trentelman and Stoorvogel [13] are extended to the constrained case. The resulting formulae are very simple, both analytically and computationally.

The paper is organized as follows. In Section 2 we formulate the problem to be considered and discuss its main features. Section 3 contains some preliminary material: in Subsection 3.1 we review the notion of the H^2-norm for periodic systems, while Subsections 3.2 and 3.3 are devoted to the separate treatment of the slightly simplified issues of continuous-time and discrete-time performance respectively. The mixed problem is considered in Section 4, where a state-space solution is given. In Section 5 an illustrative example demonstrating the potential benefits of the proposed approach is considered. Finally, in the last section concluding remarks are given.

[2] The illustrative example bellow demonstrates this point.

2. Problem formulation

In this section we will describe the plant under consideration, state some assumptions used in the sequel and pose the problem to be considered.

We start from the state-space realization for the generalized plant in Fig. 1:

$$P(s) = \left[\begin{array}{c|ccc} A & B_{1c} & B_{1d} & B_2 \\ \hline C_{1c} & 0 & D_{11cd} & D_{12c} \\ C_{1d} & 0 & D_{11dd} & D_{12d} \\ C_2 & 0 & D_{21d} & 0 \end{array}\right], \tag{1}$$

where the partitioning is compatible with Fig. 1. The assumptions that $D_{11dc} = 0$ and $D_{21c} = 0$ are made to guarantee L^2-stability of the sampling operations, $D_{11cc} = 0$ to provide boundness of the H^2 norm of the closed-loop system. Similarly, $D_{22} = 0$, to make closed-loop operator well defined. In addition we make the following assumptions:

(A1): The triple (C_2, A, B_2) is stabilizable and detectable;

(A2): The control loop sampling period h is non-pathological with respect to A (see [6]).

The above assumptions guarantee the existence of discrete-time stabilizing controllers. Moreover, for the sake of simplicity we will assume that

(A3): $h_z = Nh$ for some $N \in \mathbb{Z}_+$ and $\tau \in [0, h)$.

Now let us define our requirements for the system in Fig. 1. To this end we write the closed-loop operator $\mathcal{F}(\mathcal{P}, \mathcal{H}_h \overline{\mathcal{K}} \mathcal{S}_h)$ from w_c and w_d to z_c and z_d as:

$$\begin{bmatrix} z_c \\ z_d \end{bmatrix} = \begin{bmatrix} \mathcal{T}_{z_c w_c} & \mathcal{T}_{z_c w_d} \\ \mathcal{T}_{z_d w_c} & \mathcal{T}_{z_d w_d} \end{bmatrix} \begin{bmatrix} w_c \\ w_d \end{bmatrix}.$$

Since the disturbance signal w_d is assumed to be of the form $w_d = \mathcal{H}_h \bar{w}_d$ and the controlled output is $\bar{z}_d = \mathcal{S}^{\tau}_{h_z} z_d$ rather than z_d, we are interested in the following operator

$$\mathcal{T}_{zw} \doteq \begin{bmatrix} \mathcal{T}_{z_c w_c} & \mathcal{T}_{z_c w_d} \mathcal{H}_h \\ \mathcal{S}^{\tau}_{h_z} \mathcal{T}_{z_d w_c} & \mathcal{S}^{\tau}_{h_z} \mathcal{T}_{z_d w_d} \mathcal{H}_h \end{bmatrix}. \tag{2}$$

The operator $\mathcal{T}_{zw}$ is quite complex since its domain and image spaces contain both continuous-time and discrete-time signals. But if the discrete-time controller $\overline{\mathcal{K}}$ is N-periodic we can consider the $\mathcal{T}_{zw}$ as Nh-periodic. To see this, let us define the continuous-time σ-delay operator $\mathcal{D}_\sigma$ ($\zeta = \mathcal{D}_\sigma \xi \;\Leftrightarrow\; \zeta(t) = \xi(t-\sigma)$) and the discrete-time backward shift operator $\overline{\mathcal{U}}$ ($\bar{\zeta} = \overline{\mathcal{U}} \bar{\xi} \;\Leftrightarrow\; \bar{\zeta}[k] = \bar{\xi}[k-1]$). Then it is not difficult to verify (since $\mathcal{S}^{\tau}_{h_z} = \mathcal{S}_{Nh} \mathcal{D}_{-\tau}$) that

$$\begin{bmatrix} \mathcal{D}_{Nh} & 0 \\ 0 & \overline{\mathcal{U}}^N \end{bmatrix} \mathcal{T}_{zw} = \mathcal{T}_{zw} \begin{bmatrix} \mathcal{D}_{Nh} & 0 \\ 0 & \overline{\mathcal{U}}^N \end{bmatrix},$$

which justifies considering the operator $\mathcal{T}_{zw}$ as Nh-periodic operator. But in this case it is natural to seek discrete controllers among the class of N-periodic operators and, since for periodic systems the notion of H^2-norm is well defined (see [9, 3] for detailed discussion), we can pose the following optimal control problem:

OP: Find a finite-dimensional discrete-time N-periodic controller $\overline{\mathcal{K}}$, which internally stabilizes the plant $\mathcal{P}$ and minimizes the performance index

$$J \doteq \|\mathcal{T}_{zw}\|_{H^2}^2, \tag{3}$$

which will be referred to throughout the paper as the sampled-data H^2 problem with *mixed discrete/continuous* specifications.

The problem OP is more general than those that have been considered in the literature. First, we treat both continuous-time and discrete-time performance issues simultaneously. Next, the shift included in the operator $\mathcal{S}_{h_z}^{\tau}$ allows one to "asynchronize" control loop operations and signals whose performance we are concerned with, which may be useful in many applications. Finally, we allow the external disturbances to be both continuous-time and discrete-time[3]. The problem considered in [3] did not explicitly contain discrete-time disturbances and therefore the approach in [3] allows to consider only a restricted class of prefiltered (via *strictly proper* filters) discrete measurement noise (since the sampling operator is unbounded at L^2). This disadvantage was partly overcome in [9] (see also [4]), where discrete sensor noise was incorporated via a nonzero matrix D_{21d}. However since the matrix B_{1d} was still zero, the treatment in [9] did not allow sensor noise to be correlated with plant disturbances. Moreover, there are other sources of discrete-time disturbance in sampled-data systems, digital actuator disturbances for instance, that might be more naturally modeled in discrete time. Hence, the introduction of the B_{1d} is justified.

Before considering the OP some preliminaries are required. These are introduced in the next section.

3. Preliminaries

3.1. H^2-norm of sampled-data systems

In this subsection we briefly recall the ideas of [9] and [3] concerning the H^2-norm for periodic systems. Given an Nh-periodic stable system $\mathcal{G}$, denote its response to the Dirac δ-function $\delta(t-\theta)$, $0 \le \theta < Nh$, by $g_\theta(t)$. Then the H^2-norm of $\mathcal{G}$ is defined as

$$\|\mathcal{G}\|_{H^2}^2 \doteq \frac{1}{Nh}\int_0^{Nh} \|g_\theta\|_2^2 d\theta.$$

[3]Note that the presence of the $\bar{w}_d$ may also be treated as the combination of the approaches of Chen and Francis [5] and Khargonekar and Sivashankar, and Bamieh and Pearson [9, 3].

Since we are interested in the sampled-data system (2), assume that $\mathcal{G}$ results from the feedback interconnection of a time-invariant continuous-time plant, a sampling and a hold devices with a sampling period h, and a discrete-time N-periodic controller. Then it is natural to expand

$$\|\mathcal{G}\|_{H^2}^2 = \frac{1}{N}\left(\sum_{i=0}^{N-1} \frac{1}{h}\int_0^h \|g_{ih+\theta}\|_2^2 d\theta\right). \tag{4}$$

In the case of shift-invariant discrete controller $\overline{\mathcal{K}}$, (4) can be reduced [9, 3] to

$$\|\mathcal{G}\|_{H^2}^2 = \|\mathcal{F}(\overline{\mathcal{G}},\overline{\mathcal{K}})\|_{H^2}^2, \tag{5}$$

where $\overline{\mathcal{G}}$ is some *finite-dimensional discrete shift-invariant* plant, which does not depend on $\overline{\mathcal{K}}$. As is shown in subsections 3.2 and 3.3, the same result is valid for each term in (4), namely for

$$\|\mathcal{G}\|_{H^2,i}^2 \doteq \frac{1}{h}\int_0^h \|g_{ih+\theta}\|_2^2 d\theta, \quad (i = 1,\ldots,N). \tag{6}$$

Hence, the norm in (4) can be treated just as H^2-norm of an N-periodic discrete-time system. Treating H^2-norm of discrete-time periodic systems is simpler than its sampled-data counterparts. This suggests a separate consideration of $\|\mathcal{G}\|_{H^2,i}^2$.

Using the definition of H^2-norm for periodic systems it can be easily shown that the criterion J in (3) is the weighted sum[4] of a continuous-time (intersample)

$$J_c \doteq \left\| \begin{bmatrix} \mathcal{T}_{z_c w_c} & \mathcal{T}_{z_c w_d}\mathcal{H}_h \end{bmatrix} \right\|_{H^2}^2 = \|\mathcal{T}_{z_c w}\|_{H^2}^2 \tag{7a}$$

and a discrete-time (sample)

$$J_d \doteq \left\| \mathcal{S}_{h_z}^{\tau} \begin{bmatrix} \mathcal{T}_{z_d w_c} & \mathcal{T}_{z_d w_d}\mathcal{H}_h \end{bmatrix} \right\|_{H^2}^2 \tag{7b}$$

performance indices. For the sake of clarity we will consider in this section the issues of continuous and discrete performances separately. Moreover, for the discrete-time case we will treat the following simplified index

$$J_d^s \doteq \left\| \mathcal{S}_h^{\tau} \begin{bmatrix} \mathcal{T}_{z_d w_c} & \mathcal{T}_{z_d w_d}\mathcal{H}_h \end{bmatrix} \right\|_{H^2}^2 = \|\mathcal{T}_{z_d w}^s\|_{H^2}^2 \tag{7c}$$

(that is $h_z = h$), rather than (7b). In the next section (7b) will be considered.

3.2. Continuous-time performance

In this subsection we will be concerned with performance index J_c (7a). Hence we can extract the "intersample" part $\mathcal{P}_c$ from $\mathcal{P}$:

$$P_c(s) = \left[\begin{array}{c|ccc} A & B_{1c} & B_{1d} & B_2 \\ \hline C_{1c} & 0 & D_{11cd} & D_{12c} \\ C_2 & 0 & D_{21d} & 0 \end{array}\right]. \tag{8}$$

[4]All weights are assumed to be absorbed in the generalized plant $\mathcal{P}$.

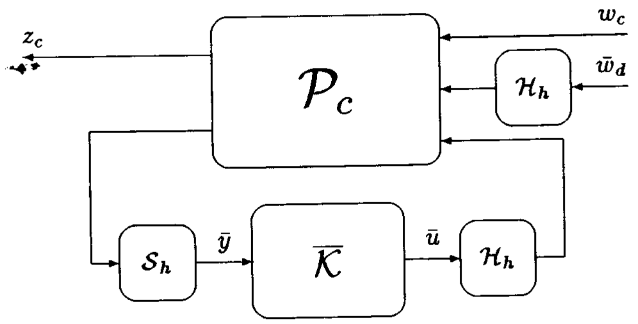

Figure 2: Continuous-time performance setup.

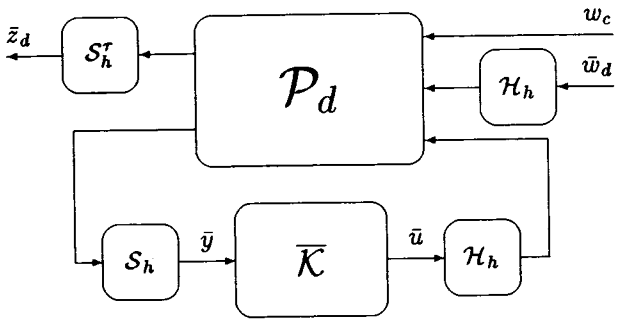

Figure 3: Discrete-time performance setup.

$\mathcal{P}_c$ is depicted in Fig. 2. For this plant we consider minimization of J_c by discrete-time N-periodic controller $\overline{\mathcal{K}}$. Such a problem is very close to the ones considered in [9, 4] except that here the controller is periodic and $B_{1d} \neq 0$. Thus we give here only the final result:

Lemma 1 *Given the continuous-time plant $\mathcal{P}_c$, (8), form the discrete-time generalized plant*

$$\bar{P}_c(z) = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_{1c} & \bar{D}_{11c} & \bar{D}_{12c} \\ \bar{C}_2 & \bar{D}_{21} & 0 \end{array}\right], \tag{9}$$

where, denoting $\Psi(\sigma) \doteq \int_0^\sigma e^{At} dt$,

$$\bar{A} = e^{Ah}, \quad \bar{B}_2 = \Psi(h)B_2, \quad \bar{C}_2 = C_2, \tag{10a}$$

$$\bar{B}_1 = \left[\, \bar{B}_{1c} \quad \Psi(h)B_{1d} \,\right], \quad \bar{D}_{21} = \left[\, 0 \quad D_{21d} \,\right], \tag{10b}$$

$$\bar{D}_{11c} = \left[\, 0 \quad \bar{D}_{11cd} \,\right], \tag{10c}$$

$\bar{B}_{1c}$ is any matrix such that

$$\bar{B}_{1c}\bar{B}_{1c}' = \int_0^h e^{At} B_{1c} B_{1c}' e^{A't}\, dt \tag{10d}$$

and $\bar{C}_{1c}$, $\bar{D}_{11cd}$ and $\bar{D}_{12c}$ are any matrices such that

$$\begin{bmatrix} \bar{C}_{1c}' \\ \bar{D}_{11cd}' \\ \bar{D}_{12c}' \end{bmatrix} \left[\, \bar{C}_{1c} \; \bar{D}_{11cd} \; \bar{D}_{12c} \,\right] = \frac{1}{h} \int_0^h \Gamma_c'(t)\Gamma_c(t) dt, \tag{10e}$$

where

$$\Gamma_c(t) \doteq \left[\, C_{1c} \; D_{11cd} \; D_{12c} \,\right] \exp\left\{ \begin{bmatrix} A & B_{1d} & B_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} t \right\}.$$

Then a discrete-time N-periodic controller $\overline{\mathcal{K}}$ stabilizes $\mathcal{P}_c$ iff it stabilizes $\overline{\mathcal{P}}_c$, and for any $\overline{\mathcal{K}}$ and $i = 1, \ldots, N$

$$\|\mathcal{T}_{z_c w}\|_{H^2,i}^2 = \|\mathcal{F}(\overline{\mathcal{P}}_c, \overline{\mathcal{K}})\|_{H^2,i}^2 + \frac{1}{h} \operatorname{tr}\{M_c\},$$

where

$$M_c \doteq C_{1c} \int_0^h \int_0^{h-t} e^{As} B_{1c} B_{1c}' e^{A's} ds\, dt\, C_{1c}'.$$

It is seen that the minimization of $\|\mathcal{T}_{z_c w}\|_{H^2,i}^2$ is equivalent to the minimization of H^2-norm of a pure discrete-time system. Also, Lemma 1 indicates that the discrete-time generalized plant $\overline{\mathcal{P}}_c$ *does not* depend on i, which is of great importance and allows us to establish the following

Corollary 1 *Given the continuous-time generalized plant $\mathcal{P}_c$, (8), and the discrete-time generalized plant $\overline{\mathcal{P}}_c$, (9), a discrete-time N-periodic controller $\overline{\mathcal{K}}$ stabilizes $\mathcal{P}_c$ iff it stabilizes $\overline{\mathcal{P}}_c$, and for any $\overline{\mathcal{K}}$*

$$\|\mathcal{T}_{z_c w}\|_{H^2}^2 = \|\mathcal{F}(\overline{\mathcal{P}}_c, \overline{\mathcal{K}})\|_{H^2}^2 + \frac{1}{h} \operatorname{tr}\{M_c\}.$$

3.3. Discrete-time performance

Here we consider the discrete-time performance index J_d^s (7c) for the plant $\mathcal{P}$. Hence we can extract the "sampled" part $\mathcal{P}_d$ from $\mathcal{P}$:

$$P_d(s) = \left[\begin{array}{c|ccc} A & B_{1c} & B_{1d} & B_2 \\ \hline C_{1d} & 0 & D_{11dd} & D_{12d} \\ C_2 & 0 & D_{21d} & 0 \end{array}\right], \tag{11}$$

which is shown in Fig. 3. Although such a problem, to the best of the authors' knowledge, has not been considered yet, the ideas of [9, 3] can be easily extended to this case.

Lemma 2 *Given the continuous-time plant $\mathcal{P}_d$, (11), form the discrete-time generalized plant*

$$\bar{P}_d(z) = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_{1d} & \bar{D}_{11d} & \bar{D}_{12d} \\ \bar{C}_2 & \bar{D}_{21} & 0 \end{array}\right], \tag{12}$$

where the matrices $\bar{A}$, $\bar{B}_1$, $\bar{B}_2$, $\bar{C}_2$ and $\bar{D}_{21}$ are defined in (10a) and (10b), and

$$\bar{C}_{1d} = C_{1d} e^{A\tau}, \tag{13a}$$

$$\bar{D}_{12d} = D_{12d} + C_{1d} \int_0^\tau e^{At} dt\, B_2, \tag{13b}$$

$$\bar{D}_{11d} = \left[\, 0 \quad D_{11dd} + C_{1d} \int_0^\tau e^{At} dt\, B_{1d} \,\right]. \tag{13c}$$

Then a discrete-time N-periodic controller $\overline{\mathcal{K}}$ stabilizes $\mathcal{P}_d$ iff it stabilizes $\overline{\mathcal{P}}_d$, and for any $\overline{\mathcal{K}}$ and $i = 1, \ldots, N$

$$\|\mathcal{T}^s_{z_d w}\|^2_{H^2,i} = \|\mathcal{F}(\overline{\mathcal{P}}_d, \overline{\mathcal{K}})\|^2_{H^2,i} + \operatorname{tr}\{M_d\},$$

where

$$M_d \doteq C_{1d} \int_0^T e^{At} B_{1c} B'_{1c} e^{A't} \mathrm{d}t\, C'_{1d}.$$

It is not surprising that this case is also reducible to a finite-dimensional discrete-time H^2 problem and the following corollary can be established:

Corollary 2 *Given the continuous-time generalized plant $\mathcal{P}_d$, (11), and the discrete-time generalized plant $\overline{\mathcal{P}}_d$, (12), then a discrete-time N-periodic controller $\overline{\mathcal{K}}$ stabilizes $\mathcal{P}_d$ iff it stabilizes $\overline{\mathcal{P}}_d$, and for any $\overline{\mathcal{K}}$*

$$\|\mathcal{T}^s_{z_d w}\|^2_{H^2} = \|\mathcal{F}(\overline{\mathcal{P}}_d, \overline{\mathcal{K}})\|^2_{H^2} + \operatorname{tr}\{M_d\}.$$

Now we are in a position to consider the mixed problem posed in Section 2.

4. Solution of the mixed problem

To combine the results of Corollaries 1 and 2 let us define a discrete-time generalized plant $\overline{\mathcal{P}}$ having the following realization:

$$\bar{P}(z) = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_{1c} & \bar{D}_{11c} & \bar{D}_{12c} \\ \bar{C}_{1d} & \bar{D}_{11d} & \bar{D}_{12d} \\ \bar{C}_2 & \bar{D}_{21} & 0 \end{array}\right], \tag{14}$$

where the parameters are defined in (10) and (13). Then, partitioning the feedback interconnection of $\overline{\mathcal{P}}$ and $\overline{\mathcal{K}}$ as:

$$\mathcal{F}(\overline{\mathcal{P}}, \overline{\mathcal{K}}) = \begin{bmatrix} \overline{\mathcal{T}}_c \\ \overline{\mathcal{T}}_d \end{bmatrix},$$

it is not difficult to see (using the results of the preceding section) that

$$\|\mathcal{T}_{zw}\|^2_{H^2} = \left\| \begin{bmatrix} \overline{\mathcal{T}}_c \\ \overline{\mathcal{V}}_N \overline{\mathcal{T}}_d \end{bmatrix} \right\|^2_{H^2} + \operatorname{tr}\{\tfrac{1}{h} M_c + M_d\}, \tag{15}$$

where the expansion operator $\overline{\mathcal{V}}_N$ is defined as

$$\overline{\mathcal{V}}_N : \quad \bar{\zeta} = \overline{\mathcal{V}}_N \bar{\xi} \;\Leftrightarrow\; \bar{\zeta}[k] = \begin{cases} \bar{\xi}[k] & \text{if } k \bmod N = 0 \\ 0 & \text{elsewere.} \end{cases}$$

Thus, we can see that our mixed discrete/continuous sampled-data H^2 problem is reduced to a discrete-time periodic H^2 problem. To handle the latter we will use the lifting technique of [7], which allows to convert a periodic problem to an equivalent time-invariant one. To this end let us define the discrete-time sampler $\overline{\mathcal{S}}_N$ and the discrete-time lifting operator $\overline{\mathcal{W}}_N$ (see [7] for detailed discussion of their properties):

$$\overline{\mathcal{S}}_N : \quad \bar{\zeta} = \overline{\mathcal{S}}_N \bar{\xi} \;\Leftrightarrow\; \bar{\zeta}[k] = \bar{\xi}[Nk];$$

$$\overline{\mathcal{W}}_N : \quad \zeta = \overline{\mathcal{W}}_N \xi \;\Leftrightarrow\; \zeta[k] = \begin{bmatrix} \xi[Nk] \\ \xi[Nk+1] \\ \vdots \\ \xi[Nk+N-1] \end{bmatrix}.$$

Since the operator $\overline{\mathcal{W}}_N$ is an isometric isomorphism in ℓ^2 and $\overline{\mathcal{W}}_N \overline{\mathcal{V}}_N = \overline{\mathcal{S}}_N$ [7] we have:

$$\left\| \begin{bmatrix} \overline{\mathcal{T}}_c \\ \overline{\mathcal{V}}_N \overline{\mathcal{T}}_d \end{bmatrix} \right\|^2_{H^2} = \left\| \begin{bmatrix} \overline{\mathcal{W}}_N \overline{\mathcal{T}}_c \\ \overline{\mathcal{S}}_N \overline{\mathcal{T}}_d \end{bmatrix} \overline{\mathcal{W}}^*_N \right\|^2_{H^2},$$

and the operators $\overline{\mathcal{W}}_N \overline{\mathcal{T}}_c \overline{\mathcal{W}}^*_N$ and $\overline{\mathcal{S}}_N \overline{\mathcal{T}}_d \overline{\mathcal{W}}^*_N$ are *shift-invariant* [7]. However the controller $\overline{\mathcal{K}}$ is still periodic, that is, our problem is not a standard H^2-optimization problem yet. To transform it to a time-invariant one let us define the following operators

$$\breve{\mathcal{P}} \doteq \begin{bmatrix} \overline{\mathcal{W}}_N & 0 & 0 \\ 0 & \overline{\mathcal{S}}_N & 0 \\ 0 & 0 & \overline{\mathcal{W}}_N \end{bmatrix} \overline{\mathcal{P}} \begin{bmatrix} \overline{\mathcal{W}}^*_N & 0 \\ 0 & \overline{\mathcal{W}}^*_N \end{bmatrix}$$

(the partitioning is compatible with that for $\overline{\mathcal{P}}$) and

$$\breve{\mathcal{K}} \doteq \overline{\mathcal{W}}_N \overline{\mathcal{K}} \overline{\mathcal{W}}^*_N.$$

Since the operator $\overline{\mathcal{P}}$ is shift-invariant, so is the operator $\breve{\mathcal{P}}$, and its realization is

$$\breve{P}(z) = \begin{bmatrix} \breve{P}_{11} & \breve{P}_{12} \\ \breve{P}_{21} & \breve{P}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \bar{A}_P & \acute{B}_1 & \acute{B}_2 \\ \hline \grave{C}_1 & \breve{D}_{11} & \breve{D}_{12} \\ \grave{C}_2 & \breve{D}_{21} & \breve{D}_{22} \end{array}\right], \tag{16}$$

where

$$\bar{A}_P = \bar{A}^N, \tag{17a}$$

$$\acute{B}_i = \begin{bmatrix} \bar{A}^{N-1}\bar{B}_i & \bar{A}^{N-2}\bar{B}_i & \ldots & \bar{B}_i \end{bmatrix}, \tag{17b}$$

$$\grave{C}_1 = \begin{bmatrix} \bar{C}_{1c} \\ \bar{C}_{1c}\bar{A} \\ \vdots \\ \bar{C}_{1c}\bar{A}^{N-1} \\ \bar{C}_{1d} \end{bmatrix}, \quad \grave{C}_2 = \begin{bmatrix} \bar{C}_2 \\ \bar{C}_2\bar{A} \\ \vdots \\ \bar{C}_2\bar{A}^{N-1} \end{bmatrix}, \tag{17c}$$

$$\breve{D}_{1i} = \begin{bmatrix} \bar{D}_{1ic} & 0 & \ldots & 0 \\ \bar{C}_{1c}\bar{B}_i & \bar{D}_{1ic} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{C}_{1c}\bar{A}^{N-2}\bar{B}_i & \bar{C}_{1c}\bar{A}^{N-3}\bar{B}_i & \ldots & \bar{D}_{1ic} \\ \bar{D}_{1id} & 0 & \ldots & 0 \end{bmatrix}, \tag{17d}$$

$$\breve{D}_{2i} = \begin{bmatrix} \bar{D}_{2i} & 0 & \ldots & 0 \\ \bar{C}_2\bar{B}_i & \bar{D}_{2i} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{C}_2\bar{A}^{N-2}\bar{B}_i & \bar{C}_2\bar{A}^{N-3}\bar{B}_i & \ldots & \bar{D}_{2i} \end{bmatrix}, \tag{17e}$$

with $i = 1, 2$ and $\bar{D}_{22} = 0$, of course.

The lifted controller $\breve{\mathcal{K}}$ is also shift-invariant and hence it can be represented by a state-space realization, e. g.

$$\breve{K}(z) = \left[\begin{array}{c|c} \bar{A}_K & \bar{B}_K \\ \hline \bar{C}_K & \bar{D}_K \end{array}\right]. \tag{18}$$

However, due to causality requirements [10], the matrix $\bar{D}_K$ in (18) can no longer be considered as arbitrary, but rather it should be constrained as

$$\bar{D}_K \in \mathbb{T}^N, \tag{19}$$

where the set $\mathbb{T}^N$ consists on $N \times N$ block matrices of the following form:

$$\mathbb{T}^N \doteq \left\{ M : M = \begin{bmatrix} M_{11} & 0 & \cdots & 0 \\ M_{21} & M_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ M_{N1} & M_{N2} & \cdots & M_{NN} \end{bmatrix} \right\}.$$

Now, simple manipulations with the lifting operator $\overline{\mathcal{W}}_N$ yield [7]:

$$\begin{bmatrix} \overline{\mathcal{W}}_N & 0 \\ 0 & \overline{\mathcal{S}}_N \end{bmatrix} \mathcal{F}(\overline{\mathcal{P}}, \overline{\mathcal{K}}) \overline{\mathcal{W}}_N^* = \mathcal{F}(\breve{\mathcal{P}}, \breve{\mathcal{K}}).$$

and thus we can reformulate our sampled-data H^2 problem as a discrete-time constrained H^2 problem:

OP_e: Given the plant $\breve{\mathcal{P}}$ with realization (16), find an LTI controller $\breve{\mathcal{K}}$, which internally stabilizes $\breve{\mathcal{P}}$ and minimizes the performance index

$$\breve{J} \doteq \|\mathcal{F}(\breve{\mathcal{P}}, \breve{\mathcal{K}})\|_{H^2}^2 \tag{20}$$

subject to (19).

If this problem did not contain the constraints, it would be a standard discrete-time H^2 problem, the solution of which could be found using well established machinery (see the books [1, 2], for instance, and [13] for a comprehensive treatment of various singular cases). Fortunately, since the only controller parameter to be constrained is the feedthrough "D" matrix, the approach in [13] can be extended quite directly to our case. The solution of the OP_e is outlined bellow.

First, let us make the following standard assumption, which guarantees the uniqueness of the solution of OP_e:

(A4): The transfer matrices $\breve{P}_{12}(z)$ and $\breve{P}_{21}'(z)$ are left invertible on the unit circle.

Second, note that our case differs from that considered in [13] not only by the presence of the constraint (19), but also by the presence of nonzero matrices $\breve{D}_{11}$ and $\breve{D}_{22}$. To get rid of the latter matrix we will use the following well known "trick": the action of a controller $\breve{\mathcal{K}}$ on the plant $\breve{\mathcal{P}}$ is equivalent to the action of the controller $(I - \breve{\mathcal{K}}\breve{D}_{22})^{-1}\breve{\mathcal{K}}$ on a plant, which is $\breve{\mathcal{P}}$ with $\breve{D}_{22} = 0$. It is therefore clear that in solving the OP_e we can take $\breve{D}_{22} = \emptyset$. Having solved for $\breve{\mathcal{K}}$ we then implement $\breve{\mathcal{K}}_{impl}$[5]:

$$\breve{\mathcal{K}}_{impl} \doteq \breve{\mathcal{K}}(I + \breve{D}_{22}\breve{\mathcal{K}})^{-1}.$$

[5]The inversion is always well defined, since the feedthrough part $I + \breve{D}_{22}\bar{D}_K$ of the operator under inversion is block lower triangular with identity diagonal blocks.

Finally, let us consider $\breve{D}_{11}$. As is shown in [13] for the case $\breve{D}_{11} = 0$, the choice of the parameter $\bar{D}_K$ in the H^2 optimization problem is independent of the other controller parameters and is based on the following parameteric optimization problem:

$$\min_{\bar{D}_K} \operatorname{tr}\left\{(\bar{D}_X \bar{D}_K \bar{D}_Y)'(\bar{D}_X \bar{D}_K \bar{D}_Y) + 2 M_1 \bar{D}_K\right\}, \tag{21}$$

where $\bar{D}_X$ and $\bar{D}_Y$ are any square matrices such that[6]

$$\bar{D}_X' \bar{D}_X = \breve{D}_{12}' \breve{D}_{12} + \acute{B}_2' X \acute{B}_2, \tag{22a}$$

$$\bar{D}_Y \bar{D}_Y' = \breve{D}_{21} \breve{D}_{21}' + \grave{C}_2 Y \grave{C}_2', \tag{22b}$$

where X and Y are the stabilizing solutions of the following algebraic Riccati equations:

$$X = \bar{A}_P' X \bar{A}_P + \grave{C}_1' \grave{C}_1 - (\grave{C}_1' \breve{D}_{12} + \bar{A}_P' X \acute{B}_2) (\breve{D}_{12}' \breve{D}_{12} + \acute{B}_2' X \acute{B}_2)^{-1} (\breve{D}_{12}' \grave{C}_1 + \acute{B}_2' X \bar{A}_P), \tag{23a}$$

$$Y = \bar{A}_P Y \bar{A}_P' + \acute{B}_1 \acute{B}_1' - (\bar{A}_P Y \grave{C}_2' + \acute{B}_1 \breve{D}_{21}') (\breve{D}_{21} \breve{D}_{21}' + \grave{C}_2 Y \grave{C}_2')^{-1} (\breve{D}_{21} \acute{B}_1' + \grave{C}_2 Y \bar{A}_P'), \tag{23b}$$

and

$$M_1 = \bar{D}_{21} \bar{B}_1' X \bar{B}_2 + \bar{C}_2 Y (\bar{C}_1' \bar{D}_{12} + \bar{A}' X \bar{B}_2).$$

Recall that for any stable discrete-time system

$$\left\| \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \right\|_{H^2}^2 = \operatorname{tr}\{D'D\} + \left\| \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right] \right\|_{H^2}^2,$$

which implies (since when $\breve{D}_{22} = 0$ the feedthrough term of the operator $\mathcal{F}(\breve{\mathcal{P}}, \breve{\mathcal{K}})$ is $\breve{D}_{11} + \breve{D}_{12}\bar{D}_K\breve{D}_{21}$) that in our case the presence of $\breve{D}_{11}$ changes the H^2 cost only by the quantity:

$$\operatorname{tr}\left\{\breve{D}_{11}' \breve{D}_{11}\right\} + 2\operatorname{tr}\left\{\breve{D}_{11}' \breve{D}_{12} \bar{D}_K \breve{D}_{21}\right\}.$$

Thus we have that for nonzero matrix $\breve{D}_{11}$ the controller feedthrough matrix $\bar{D}_K$ should be found not from the problem (21), but rather from the problem

$$\min_{\bar{D}_K} \Phi(\bar{D}_K), \tag{24}$$

where

$$\Phi(\bar{D}_K) \doteq \operatorname{tr}\left\{(\bar{D}_X \bar{D}_K \bar{D}_Y)'(\bar{D}_X \bar{D}_K \bar{D}_Y)\right\} + 2\operatorname{tr}\left\{(M_1 + \breve{D}_{21} \breve{D}_{11}' \breve{D}_{12}) \bar{D}_K\right\}. \tag{25}$$

The foregoing discussion delt with the unconstrained problem. Now we are in the position to take into account the constraits (19). Because the matrix $\bar{D}_K$ is sought as the solution of the problem (24) independently of the other controller parameters, we should just modify (24) as follows:

$$\min_{\bar{D}_K \in \mathbb{T}^N} \Phi(\bar{D}_K). \tag{26}$$

[6]According to the assumption (A4) the matrices $\bar{D}_X$ and $\bar{D}_Y$ are nonsingular [13].

The standard completing to square arguments give

$$\begin{aligned}\Phi(\bar{D}_K) &= \mathrm{tr}\Big\{\big(\bar{D}_X\bar{D}_K\bar{D}_Y+M_a\big)'\big(\bar{D}_X\bar{D}_K\bar{D}_Y+M_a\big)\Big\} \\ &\quad -\mathrm{tr}\{M_a'M_a\},\end{aligned}$$

where

$$M_a = \left(\bar{D}_X'\right)^{-1}\left(M_1 + \check{D}_{21}\check{D}_{11}'\check{D}_{12}\right)'\left(\bar{D}_Y'\right)^{-1}.$$

It is clear that we can always choose the matrices $\bar{D}_X$ and $\bar{D}_Y$ from (22) such that $\bar{D}_X \in \mathbb{T}^N$ and $\bar{D}_Y \in \mathbb{T}^N$. Then $\bar{D}_K \in \mathbb{T}^N \Leftrightarrow \bar{D}_X\bar{D}_K\bar{D}_Y \in \mathbb{T}^N$, and since the matrix M_a does not depend on $\bar{D}_K$, the problem (26) has the following solution:

$$\bar{D}_K = -\bar{D}_X^{-1}\left(\Pi_{\mathbf{T}}M_a\right)\bar{D}_Y^{-1}, \qquad (27)$$

where $\Pi_{\mathbf{T}}$ denotes the orthogonal projection on the space $\mathbb{T}^N$.

Having solved for an optimal $\bar{D}_K$, the other controller parameters can be calculated as in [13]:

$$\bar{A}_K = \bar{A}_P + \acute{B}_2F_P + H_P\grave{C}_2 - \acute{B}_2\bar{D}_K\grave{C}_2, \qquad (28a)$$

$$\bar{B}_K = \acute{B}_2\bar{D}_K - H_P, \qquad (28b)$$

$$\bar{C}_K = F_P - \bar{D}_K\grave{C}_2, \qquad (28c)$$

where the state feedback matrix F_P and the output injection matrix H_P are given by

$$F_P = -(\bar{D}_X'\bar{D}_X)^{-1}(\check{D}_{12}'\grave{C}_1 + \acute{B}_2'X\bar{A}_P), \qquad (28d)$$

$$H_P = -(\bar{A}_PY\grave{C}_2' + \acute{B}_1\check{D}_{21}')(\bar{D}_Y\bar{D}_Y')^{-1}. \qquad (28e)$$

As can be seen, the only difference between the constrained and unconstrained [13] cases is the orthogonal projection operator $\Pi_{\mathbf{T}}$ in (27) that does not complicate the computations.

5. Example

To illustrate the possibilities of the approach proposed in this paper let us consider an example. Let the continuous-time plant $y(t) = \mathcal{G}u(t)$ have the following transfer function

$$G(s) = \frac{1}{s(s+0.1)} = \left[\begin{array}{c|c} \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 0 \end{bmatrix} & 0 \end{array}\right].$$

A sampled-data controller with a sampling period $h = 3/4$ is to be designed. The control goal is to provide good transient against arbitrary initial conditions $y(0)$ in the plant output. It is not difficult to verify that arbitrary $y(0)$ can be modeled as an discrete-time impulse at the input, connected with the state vector through the vector[7]

$$B_{1d} = \left(\int_0^h e^{-At}dt\right)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 0 \end{bmatrix}.$$

[7] Since in this case $B_{1c} = 0$, we will suppose in the sequel the presence of only the discrete-time external input $\bar{w}_d$.

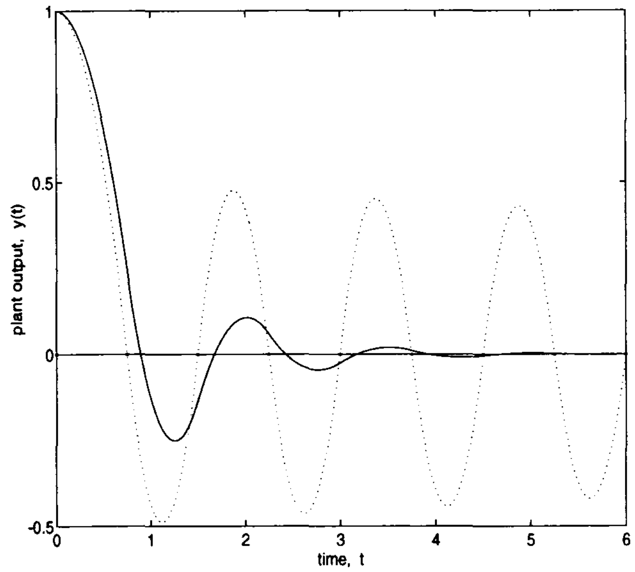

Figure 4: Plant output under a discrete (dotted line) and a sampled-data (solid line) designs.

First, we minimize the ℓ^2-norm of the discretized plant output $\bar{y}[k]$, which corresponds to the generalized plant

$$P(s) = \left[\begin{array}{c|cc} \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} & \begin{bmatrix} 4/3 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \end{array}\right] \qquad (29)$$

and a pure discrete design. The result is shown in Fig. 4 by the dotted line. As it is seen, though at the sampling points the plant output behaves in an excellent manner, significant intersample ripples take place. Next we minimize the L^2-norm of $y(t)$. This situation corresponds to the generalized plant (29) and a sampled-data design. The result is shown in Fig. 4 by the solid line and looks significantly better than the previous one. However the behavior is still slightly oscillatory, which is an inherent problem of the H^2 approach. The natural idea for preventing these oscillations is to add some penalty of the signal $\dot{y}(t)$ into the criterion. However, intuitively it is clear that in order to prevent the intersample oscillations of an output signal for second order systems, it is justified to penalize the signal $\dot{y}(t)$ *only* at the points kh. To check this point let us consider two approaches: the sampled-data design and the mixed design proposed in this paper. For the former we consider the generalized plant

$$P_\lambda^c(s) = \left[\begin{array}{c|cc} \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} & \begin{bmatrix} 4/3 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \hline \begin{bmatrix} 1-\lambda & 0 \\ 0 & \lambda \end{bmatrix} & 0 & 0 \\ \begin{bmatrix} 1 & 0 \end{bmatrix} & 0 & 0 \end{array}\right]$$

and minimize the L^2-norm of the continuous-time 2-dimensional controlled output, while for the latter we

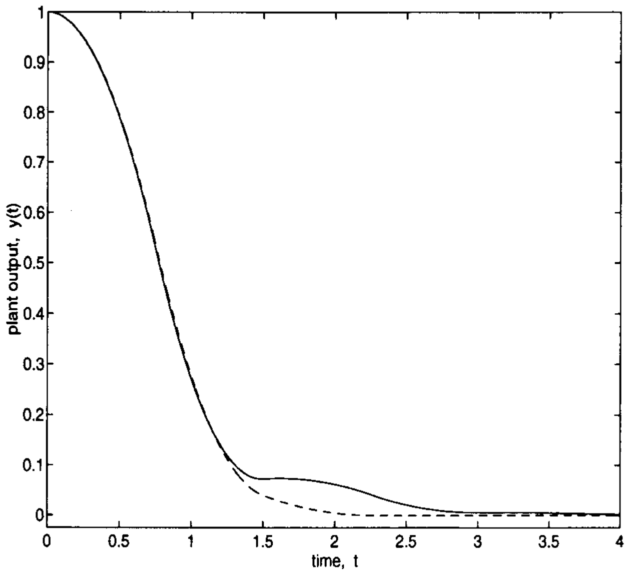

Figure 5: Plant output under a sampled-data (solid line) and a mixed (dashed line) designs.

consider the generalized plant

$$P_\lambda^m(s) = \left[\begin{array}{c|cc} \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} & \begin{bmatrix} 4/3 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \hline \begin{bmatrix} 1-\lambda & 0 \\ 0 & h\lambda \\ 1 & 0 \end{bmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{array}\right]$$

and make the mixed design. The parameter $\lambda \in [0,1]$ can be adjusted to provide desirable tradeoff between penalties of y and $\dot{y}$. For a fair comparison of these two approaches we compare results for values of λ, which provide equal values of the quantity $\int_0^\infty y(t)^2 dt$ for both approaches. Then in all of the cases the mixed approach led to better responses of $y(t)$. One typical case is shown in Fig. 5. Both cases in Fig. 5 gave $\int_0^\infty y(t)^2 dt = 0.6$, were $\lambda = 0.368$ and $\lambda = 0.153$ were used for the sampled-data and the mixed designs respectively.

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6. Concluding remarks

In this paper a new approach to the H^2 design for sampled-data systems has been introduced. The main feature of this approach is the presense of *both* continuous-time *and* discrete-time (hence *mixed*) performance specifications. It has been shown that this problem can be reduced to an equivalent discrete H^2 problem with constraints on controller structure. A state-space solution for the latter problem has been provided. This solution makes it possible to handle controller constraints *directly* in terms of controller parameters. Finally, a numerical example has demonstrated the potential benefits of the approach.

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