ROBUST H_∞-ESTIMATION OF NONLINEAR DISCRETE-TIME PROCESSES

M. Shergei and U. Shaked

Tel-Aviv University, Faculty of Engineering
Tel-Aviv 69978, ISRAEL

Keywords: Nonlinear H_∞–Estimation, Robust Estimation, Discrete-Time Nonlinear Control.

ABSTRACT

This paper investigates the problem of H_∞estimation for uncertain, discrete-time, nonlinear systems with nonlinear time-varying parameter uncertainty. A discrete-time nonlinear estimator is introduced so that an H_∞-norm-like measure of the estimation error is guaranteed to be bounded by a prescribed level, for all admissible uncertainties. An auxiliary estimation problem is considered which results from the original estimation problem by converting the family of norm-bounded parameter uncertainties into a set of exogenous energy bounded signals. Sufficient conditions for the existence of an H_∞ nonlinear estimator for the auxiliary problem are obtained, and it is shown that the resulting estimator, if exists, guarantees the required performance, when applied to the original problem, for all the admissible parameters.

1. INTRODUCTION

One of the main reasons for the considerable effort that has been devoted in the past decade to the development of H_∞ control and estimation is the good robustness properties of the resulting designs [1]. Not less important, however, is the fact that H_∞ theory can be readily extended to deal with systems with norm-bounded structural parameter uncertainties [2], [4].

One of the methods to cope with parameter uncertainty is to exploit the fact that the H_∞ design does not require the exact knowledge of the properties of the exogenous inputs to the system. The bounded uncertainties can thus be translated into fictitious disturbances in an auxiliary system [2]. H_∞-control or estimation that is designed for the auxiliary system is guaranteed to achieve an H_∞-type performance value when applied on the original uncertain system.

While H_∞-design techniques have been widely applied lately on nonlinear systems [5],[6],[7], only little effort has been made to extend the robust H_∞-design, that had been used so successfully on linear uncertain systems, also in the nonlinear regim. In [8] the method of [2] has been applied to obtain an H_∞-estimator for uncertain continuous-time nonlinear systems with nonlinear time-varting parameter uncertainties. A robust estimation technique is introduced there that is based on the results of [6], for the corresponding nonlinear problem with no uncertainty. The resulting estimator secures an upper-bound to the ratio between the energy of the estimation error and the energy of the noise inputs, for all the admissible parameters.

In the present paper, we develop a similar robust estimation method for nonlinear discrete–time systems. In the continuous-time case of [8], the theory for systems without uncertainty of [6] was readily applicable to the control of the auxiliary system. In the discrete-time case the nonlinear control problem has been solved in a general setting by [7]. In order to solve our estimation problem, and to be consistent with our notations and way of presentation, we bring in Section 3, after formulating the problem in Section 2, a short derivation of discrete-time nonlinear H_∞-control that we need for the solution of the robust estimation problem in Section 4. The obtained results may not always

This work was supported by the C. & M. Maus Chair at the Tel-Aviv University

lead to a feasible solution. This is why we consider in Section 5 the Extended Robust H_∞ Filter which is obtained by linearizing the equations around the zero error vector. A demonstrative example is given in Section 6.

2. PROBLEM FORMULATION

We consider the following uncertain discrete-time nonlinear system

$$\begin{aligned} x_{i+1} &= f_i(x_i) + H_{1i}(x_i)F_i(x_i)E_i(x_i) + g_{1i}(x_i)w_i, \\ y_i &= h_{2i}(x_i) + H_{2i}(x_i)F_i(x_i)E_i(x_i) + k_{21i}(x_i)w_i, \end{aligned} \quad (1)$$

where $x_i \in \mathcal{R}^n$ is the system state, x_0 is an unknown initial condition, $y_i \in \mathcal{R}^p$ is the measurement, and $w_i \in \mathcal{R}^r$ describes an unknown disturbance which is assumed to be in $l_2[0, N-1]$. The functions f_i, g_{1i}, h_{2i}, k_{21i}, E_i, H_{1i} and H_{2i} are known matrix functions that are assumed to be smooth in their arguments. The set $\{F_i\}$, $F_i \in \mathcal{R}^{i\times j}$ is a family of unknown matrix functions satisfying:

$$F_i^T(x_i)F_i(x_i) \le I, \quad \forall\, i \in [0, N] \quad \forall \text{ possible } x_i. \quad (2)$$

The mappings f_i, g_{1i}, h_{2i} and k_{21i} describe the nominal system of (1).

Our aim is to derive a filter for a priori estimating of the state x_i :

$$\hat{x}_i = \mathcal{F}(\mathcal{Y}_i), \quad (3)$$

where $\mathcal{Y}_i \triangleq \{y_k : k \le i-1\}$, and $\mathcal{F} : l_2[0, N, \mathcal{R}^p] \rightarrow l_2[0, N, \mathcal{R}^n]$, so that the estimation error satisfies a certain H_∞-type requirement for the whole admissible set of uncertainties that is expressed by F_i of (2).

The H_∞ requirement is defined by determining first an objective vector, $z_i \in \mathcal{R}^s$, which may be interpreted as an estimation error, namely,

$$z_i = h_{1i}(x_i) - h_{1i}(\hat{x}_i) \quad (4)$$

where h_{1i} is a known smooth matrix function. We also select some positive functional $N(x_0)$ on $\mathcal{R}^n$, and require that for a given $\gamma > 0$

$$\sum_{i=0}^{N-1} \|z_i\|^2 \le \gamma^2[N(x_0) + \sum_{i=0}^{N-1} \|w_i\|^2], \quad (5)$$

$\forall w_i \in l_2[0, N-1]$ and $\forall x_0 \in \mathcal{R}^n$.

We consider the following robust H_∞ nonlinear estimation problem for (1) – (4):

Problem 1. *For a given scalar $\gamma > 0$, find an estimator (3) that satisfies (5) for all the admissible uncertainties that satisfy (2).*

We make the following assumption:

A1: $k_{21i}(x)k_{21i}(x)^T > 0, \quad \forall x \in \mathcal{R}^n$.

The analysis of the above problem consists of two parts. In the next section we consider the first part which is a problem of discrete-time H_∞ nonlinear control via measurement feedback. This problem has been solved in [7] in a very general framework. In the present work we suggest a simple method to generate the solution of the H_∞ problem for some common practical cases. Our results will be used in Section 4 to solve Problem 1.

3. DISCRETE-TIME H_∞ NONLINEAR CONTROL

Consider a system that is described by

$$\begin{aligned} x_{i+1} &= f_i(x_i) + g_{2i}(x_i)u_i + g_{1i}(x_i)w_i \\ y_i &= h_{2i}(x_i) + k_{21i}(x_i)w_i \\ z_i &= h_{1i}(x_i) + k_{12i}(x_i)u_i \end{aligned} \quad (6)$$

where $x_i \in \mathcal{R}^n$, $y_i \in \mathcal{R}^p$, and $z_i \in \mathcal{R}^s$; $u_i \in \mathcal{R}^m$ is the control input, $w_i \in \mathcal{R}^r$ is an unknown disturbance which is assumed to be a member of $l_2[0, N-1]$. The mappings f_i, g_{1i}, g_{2i}, h_{2i}, k_{21i}, h_{1i} and k_{12i} are known matrix functions that are smooth in their arguments. The objective is to design a control law $\{u_i\}$ that achieves the H_∞-type requirement in the form of (5).

There are two major approaches to solve this control problem: one that is based on the theory of dissipative systems (see e.g. [10]) and the other is a dynamic game approach [11].

¿From the game theory point of view we define the pay-off function

$$J(w_i, u_i, x_0) \triangleq \sum_{i=0}^{N-1} \|z_i\|^2 - \gamma^2[N(x_0) + \sum_{i=0}^{N-1} \|w_i\|^2] \quad (7)$$

Satisfying (5) is equivalent to J being nonpositive. Following the approach of [6] and [12] this may be viewed as a zero-sum dynamic game played by two adversaries in which one minimizes J with respect to u_i, and the other maximizes it with respect to w_i and x_0. The control problem is to find a minimizing saddle-point strategy $\{u_i^*\}$ for $J(w_i, u_i, x_0)$.

On the other hand, we can utilize the notion of dissipative systems for discrete-time systems and define a supply rate $S_i = \gamma^2\|w_i\|^2 - \|z_i\|^2$. The system (6) with the control law $\{u_i^*\}$ is said to be dissipative with respect to $\{S_i\}$ if there exists a nonnegative family $\{V_i(x)\}$, $V_i(x) : \mathcal{R}^n \to \mathcal{R}$ such that $\forall x_0 \in \mathcal{R}^n$

$$V_{i+1}(x_{i+1}) - V_i(x_i) \leq S_i \quad \forall w_i \in l_2[0, N-1] \qquad (8)$$

We observe that (8) implies (5), provided that $V_0 \leq \gamma^2 N(x_0)$. Thus, the sufficient condition for $\{u_i\}$ to solve the control problem is that $\{u_i\}$ makes (6) dissipative with respect to some nonnegative $\{V_i\}$.

Similar to the continuous-time case (see e.g [6]) we define

$$H_i(u_i, w_i, x_i) \triangleq V_{i+1}(x_{i+1}) - \gamma^2\|w_i\|^2 + \|z_i\|^2 \qquad (9)$$

and $(u_i^*, w_i^*, x_0^*) \triangleq arg\{\min_{u_i}\max_{w_i}\max_{x_0} H_i(u_i, w_i, x_i)\}$. Our aim is to obtain conditions for the existence of such a set of nonnegative storage functions V_i that satisfy

$$-V_i(x_i) + H_i(u_i^*, w_i^*, x_i) \leq 0, \quad \forall x_i. \qquad (10)$$

If the set $\{u_i^*, w_i^*, x_0^*\}$ constitutes a saddle-point, we say that the sequence $\{u_i^*\}$ solves the control problem.

The control problem is first solved in the state-feedback case, where all the components of x_i are accessible. We consider

$$V_i(x_i) = x_i^T Q_i x_i \qquad (11)$$

where $\{Q_i\}$ is a family of positive semi-definite matrices. We argue that this requirement is valid in many cases, at least locally. Lemma 1 below provides a sufficient condition for the set $\{Q_i\}$. Before we state the lemma we introduce some notations and make additional assumptions.

We define

$$\bar{Q}_{i+1}(x) \triangleq I - \gamma^{-2} g_{1i}^T(x) Q_{i+1} g_{1i}(x),$$

$$\tilde{Q}_{i+1}(x) \triangleq [I - \gamma^{-2} Q_{i+1} g_{1i}(x) g_{1i}^T(x)]^{-1} Q_{i+1},$$

$$R_{1i}(x) \triangleq k_{12i}^T(x) k_{12i}(x).$$

$$M_{i+1}(x) \triangleq R_{1i}(x) + g_{2i}^T(x)\tilde{Q}_{i+1}(x) g_{2i}(x).$$

Assumptions: For all $x_i \in \mathcal{R}^n$

A2: $R_{1i} > 0 \quad \forall i \in [0, N]$

A3: $\bar{Q}_i > 0 \quad \forall i \in [0, N-1]$.

Lemma 1. *Consider the case where in (5) $N(x_0) = x_0^T P_0 x_0$, $P_0 \geq 0$, consider the family of positive semi-definite matrices $\{Q_i\}_{i=0}^N$ that satisfy* **A3** *and consider the following inequality for all $x \in \mathcal{R}^n$:*

$$-x^T Q_i x + h_{1i}^T(x) h_{1i}(x) + f_i^T(x)\tilde{Q}_{i+1}(x) f_i(x)$$
$$-[h_{1i}^T(x) k_{12i}(x) + f_i^T(x)\tilde{Q}_{i+1}(x) g_{2i}(x)] M_{i+1}^{-1}(x)$$
$$[k_{12i}^T(x) h_{1i}(x) + g_{2i}^T(x)\tilde{Q}_{i+1}(x) f_i(x)] \leq 0 \qquad (12)$$

$\forall i \in [0, N-1]$, $Q_N \equiv 0$ and $Q_0 \leq \gamma^2 P_0$. Then the pair (u_i^*, w_i^*) with

$$u_i^* = -M_{i+1}^{-1}(x_i)[k_{12i}^T(x_i) h_{1i}(x_i) + g_{2i}^T(x_i)\tilde{Q}_{i+1}(x_i) f_i(x_i)], \qquad (13)$$

$$w_i^* = \gamma^{-2} g_{1i}^T(x_i)\tilde{Q}_{i+1}(x_i)\,[f_i(x_i) + g_{2i}(x_i) u_i(x_i)] \qquad (14)$$

constitutes, together with $x_0^* = 0$, a saddle-point for the game with the objective function of (7).

Proof: Substituting (11) in (9) and replacing x_i by x we arrive at the following representation for H_i (all the arguments have been omitted below for convenience).

$$H_i = (f_i^T + u_i^T g_{2i}^T + w_i^T g_{1i}^T)\, Q_{i+1}(f_i + g_{2i} u_i + g_{1i} w_i)$$
$$+ h_{1i}^T h_{1i} + 2 h_{1i}^T k_{12i} u_i + u_i^T R_{1i} u_i - \gamma^2 w_i^T w_i$$
$$= f_i^T \tilde{Q}_{i+1} f_i + h_{1i}^T h_{1i}$$
$$-\gamma^2 \{w_i^T - \gamma^{-2}(f_i^T + u_i^T g_{1i}^T) Q_{i+1} g_{1i} \bar{Q}_{i+1}^{-1}\}\, \bar{Q}_{i+1}$$
$$\{w_i^T - \gamma^{-2}(f_i^T + u_i^T g_{1i}^T) Q_{i+1} g_{1i} \bar{Q}_{i+1}^{-1}\}^T$$
$$+2[h_{1i}^T k_{12i} + f_i^T \tilde{Q}_{i+1} g_{2i}] u_i + u_i^T (R_{1i} + g_{2i}^T \tilde{Q}_{i+1} g_{2i}) u_i.$$

A completion to the squares then leads to

$$H_i = -\gamma^2 \{w_i^T - \gamma^{-2}(f_i^T + u_i^T g_{1i}^T) Q_{i+1} g_{1i} \bar{Q}_{i+1}^{-1}\}$$
$$\bar{Q}_{i+1}\{w_i - \gamma^{-2}\bar{Q}_{i+1}^{-1} g_{1i}^T Q_{i+1}(f_i + g_{1i} u_i)\}$$
$$+[u_i + M_{i+1}^{-1}(k_{12i}^T h_{1i} + g_{2i}^T \tilde{Q}_{i+1} f_i]^T M_{i+1}$$
$$[u_i + M_{i+1}^{-1}(k_{12i}^T h_{1i} + g_{2i}^T \tilde{Q}_{i+1} f_i)]$$
$$-[h_{1i}^T k_{12i} + f_i^T \tilde{Q}_{i+1} g_{2i}] M_{i+1}^{-1} [k_{12i}^T h_{1i} +$$

$$g_{2i}^T \tilde{Q}_{i+1}(x) f_i] + f_i^T \tilde{Q}_{i+1} f_i + h_{1i}^T h_{1i}. \quad (15)$$

It follows from the last result that u_i of (13) and w_i of (14) minimizes and maximizes H_i, respectively. It also follows from the structure of H_i and assumptions **A2,A3** that H_i is a concave function in w_i for all u_i and a convex function in u_i for arbitrary w_i. It is easily verified then that $\max_{w_i} H_i(u_i^*, w_i) = H_i(u_i^*, w_i^*)$ and $\min_{u_i} H_i(u_i, w_i^*) = H_i(u_i^*, w_i^*)$, which leads to

$$H_i(u_i^*, w_i) \leq H_i(u_i^*, w_i^*) \leq H_i(u_i, w_i^*).$$

The result for x_0 follows from the fact that the pay-off function of (7) can be written, using (9), as

$$J = \sum_{i=0}^{N-1} H_i - \sum_{i=0}^{N} V_i + V_0 - \gamma^2 N(x_0) = \sum_{i=0}^{N-1} H_i - \sum_{i=0}^{N} V_i$$

$$-x_0^T(\gamma^2 P_0 - Q_0)x_0$$

and the maximizing initial condition is thus zero. $\nabla\nabla\nabla$

Remark 1 The inequality of (12) reduces to the discrete-time Riccati inequality when the linear case is considered (see e.g. [3]).

In the output-feedback case we want to examine, similarly to [6], whether x_i in the feedback control law strategy of (13) may be replaced by some estimate $\hat{x}_i$ and still keep the dissipativity of the system. We look for an estimator of the form

$$\hat{x}_{i+1} = f_i(\hat{x}_i) + g_{1i}(\hat{x}_i) w_i^*(\hat{x}_i) + g_{2i}(\hat{x}_i) u_i^*(\hat{x}_i)$$

$$+ G_i(\hat{x}_i) \left[y_i - h_{2i}(\hat{x}_i) - k_{21i}(\hat{x}_i) w_i^*(\hat{x}_i) \right], \hat{x}_0 = 0. \quad (16)$$

where $\{G_i\}$ is a family of matrix functions to be determined.

In order to simplify the notation we write

$$\alpha_{1i}(x_i) \triangleq w_i^*(x_i), \quad \alpha_{2i}(x_i) \triangleq u_i^*(x_i),$$

$$\bar{h}_{2i}(x_i) \triangleq h_{2i}(x_i) + k_{21i}(x_i)\alpha_{1i}(x_i),$$

$$\bar{f}_i(x_i) \triangleq f_i(x_i) + g_{1i}(x_i)\alpha_{1i}(x_i),$$

$$\bar{w}_i \triangleq \bar{Q}_{i+1}^{\frac{1}{2}} \left[w_i - \alpha_{1i}(x_i) \right].$$

We obtain the following description for the of the closed-loop system (6)-(16):

$$\begin{bmatrix} x \\ \hat{x} \end{bmatrix}_{i+1} = \begin{bmatrix} \bar{f}_i(x_i) + g_{2i}(x_i)\alpha_{2i}(\hat{x}_i) \\ \bar{f}_i(\hat{x}_i) + g_{2i}(\hat{x}_i)\alpha_{2i}(\hat{x}_i) + G_i(\hat{x}_i) K_i \end{bmatrix}$$

$$+ \begin{bmatrix} g_{1i}(x_i) \\ G_i(\hat{x}_i) k_{21i}(x_i) \end{bmatrix} \bar{Q}_{i+1}^{-\frac{1}{2}} \bar{w}_i \quad (17)$$

where $K_i(x_i, \hat{x}_i) \triangleq \bar{h}_{2i}(x_i) - \bar{h}_{2i}(\hat{x}_i)$.

Denoting $\bar{v}_i \triangleq M_{i+1}^{\frac{1}{2}} \left[\alpha_{2i}(x_i) - \alpha_{2i}(\hat{x}_i) \right]$ we obtain from (15) that for $u_i = \alpha_{2i}(\hat{x}_i)$ we require that

$$\sum_{i=0}^{N-1} \|\bar{v}_i\|^2 \leq \gamma^2 \sum_{i=0}^{N-1} \|\bar{w}_i\|^2 + x_0^T \left(\gamma^2 P_0 - Q_0 \right) x_0. \quad (18)$$

This is again an H_∞-type requirement of the form of (5). Also here the controller looks for a minimizing $\{\bar{v}_i\}$ while his adversary can deviate from his strategy of Lemma 1 by choosing nonzero $\bar{w}_i$ and x_0. To satisfy (18) we are looking for storage functions of the form $\{U_i\}: \quad U_i = (x_i - \hat{x}_i)^T W_i (x_i - \hat{x}_i)$.

We define

$$F_i(x_i, \hat{x}_i) \triangleq \bar{f}_i(x_i) - \bar{f}_i(\hat{x}_i) + [g_{2i}(x_i) - g_{2i}(\hat{x}_i)]\alpha_{2i}(\hat{x}_i),$$

$$R_{2i}(x_i) \triangleq k_{21i}(x_i) \bar{Q}_{i+1}^{-1}(x_i) k_{21i}^T(x_i),$$

$$\bar{W}_{i+1}(x_i, \hat{x}_i) \triangleq I - \gamma^{-2} \bar{Q}_{i+1}^{-\frac{1}{2}}(x_i)[g_{1i}(x_i) - G_i(\hat{x}_i)$$

$$k_{21i}(x_i)]^T W_{i+1} \left[g_{1i}(x_i) - G_i(\hat{x}_i) k_{21i}(x_i) \right] \bar{Q}_{i+1}^{-\frac{1}{2}}(x_i),$$

$$\tilde{W}_{i+1}(x_i) \triangleq [I + \gamma^{-2} W_{i+1} g_{1i}(x_i) \bar{Q}_{i+1}^{-1}(x_i)(k_{21i}^T(x_i)$$

$$R_{2i}^{-1} k_{21i}(x_i) \bar{Q}_{i+1}^{-1}(x_i) - I) g_{1i}^T(x_i)]^{-1} W_{i+1},$$

and assume that for all $x \in \mathcal{R}^n$ and $\forall i \in [0, N-1]$

A4: $\tilde{W}_{i+1}$ exists and $I - \gamma^{-2} \tilde{W}_{i+1}(G_i - g_{1i}\bar{Q}_{i+1}^{-1} k_{21i}^T R_{2i}^{-1}) R_{2i}(G_i - g_{1i}\bar{Q}_{i+1}^{-1} k_{21i}^T R_{2i}^{-1})^T > 0$.

A5: $R_{2i} > 0 \quad \forall i \in [0, N]$.

We then obtain the following result:

Lemma 2: *Consider the system of (6). Suppose that:*

1: Assumptions **A2-A5** *are satisfied.*

2: Inequality (12) is satisfied for some set of positive semi-definite matrices $\{Q_i\}_{i=0}^N$ for all $i \in [0, N-1]$ and all $x \in \mathcal{R}^n$.

3: There exists a family of positive semi-definite matrices $\{W_i\}_{i=0}^N$ that satisfies the following inequality for all $x \in \mathcal{R}^n$, $\forall i \in [0, N-1]$:

$$-(x - \hat{x})^T W_i (x - \hat{x}) + \bar{v}_i(x, \hat{x})^T \bar{v}_i(x, \hat{x}) + \{F_i(x, \hat{x})$$

$$- g_{1i}(x) \bar{Q}_{i+1}^{-1}(x) k_{21i}^T(x) R_{2i}^{-1}(x) K_i(x, \hat{x})\}^T \tilde{W}_{i+1}$$

$$\{F_i(x, \hat{x}) - g_{1i}(x) \bar{Q}_{i+1}^{-1}(x) k_{21i}^T(x) R_{2i}^{-1}(x) K_i(x, \hat{x})\}$$

$$-\gamma^2 K_i^T(x, \hat{x}) R_{2i}^{-1}(x, \hat{x}) K_i(x, \hat{x}) \leq 0, \quad (19a)$$

$$W_0 = \gamma^2 P_0 - Q_0. \tag{19b}$$

4: The following holds for some set $\{G_i^*\}_{i=0}^N$ *and for* $i \in [0, N-1]$

$$\left(G_i^*(\hat{x}) - g_{1i}(x)\bar{Q}_{i+1}^{-1}(x)k_{21i}^T(x)R_{2i}^{-1}(x)\right)^T \tilde{W}_{i+1}$$

$$\{F_i(x,\hat{x}) - g_{1i}(x)\bar{Q}_{i+1}^{-1}(x)k_{21i}^T(x)R_{2i}^{-1}(x)K_i(x,\hat{x})\}$$

$$= \gamma^2 R_{2i}^{-1}(x)K_i(x,\hat{x}). \tag{20}$$

Then, the control problem is solved by the output feedback $u_i = \alpha_{2i}(\hat{x}_i)$ *for*

$$\hat{x}_{i+1} = \hat{f}_i(\hat{x}_i) + g_{2i}(\hat{x}_i)\alpha_{2i}(\hat{x}_i) + G_i^*(\hat{x}_i)[y_i - \hat{h}_{2i}(\hat{x}_i)]. \tag{21}$$

Proof: Similarly to the proof of the Lemma 1 we define

$$\mathcal{H}_i(w_i, G_i(\hat{x}_i), x_i, \hat{x}_i) \triangleq U_{i+1}(x_{i+1}) + \|\bar{v}_i\|^2 - \gamma^2\|\bar{w}_i\|^2.$$

We omit all arguments for clarity and rewrite the last equation in the form

$$\mathcal{H}_i = [F_i^T - K_i^T G_i^T + \bar{w}_i^T \bar{Q}_{i+1}^{-\frac{1}{2}}(g_{1i} - G_i k_{21i})^T]W_{i+1}$$
$$[F_i - G_iK_i + (g_{1i} - G_i k_{21i})\bar{Q}_{i+1}^{-\frac{1}{2}}\bar{w}_i] + \bar{v}_i^T\bar{v}_i - \gamma^2\bar{w}_i^T\bar{w}_i$$
$$= -\gamma^2\{\bar{w}_i - \gamma^{-2}\bar{Q}_{i+1}^{-\frac{1}{2}}(g_{1i} - G_i k_{21i})^T[I - \gamma^{-2}W_{i+1}$$
$$(g_{1i} - G_i k_{21i})\bar{Q}_{i+1}^{-1}(g_{1i} - G_i k_{21i})^T]^{-1}W_{i+1}$$
$$(F_i - G_iK_i)\}^T W_{i+1}\{\bar{w}_i - \gamma^{-2}\bar{Q}_{i+1}^{-\frac{1}{2}}(g_{1i} - G_i k_{21i})^T$$
$$[I - \gamma^{-2}W_{i+1}(g_{1i} - G_i k_{21i})\bar{Q}_{i+1}^{-1}(g_{1i} - G_i k_{21i})^T]^{-1}$$
$$W_{i+1}(F_i - G_iK_i)\} + (F_i^T - K_i^T G_i^T)[I - \gamma^{-2}W_{i+1}(g_{1i}$$
$$-G_i k_{21i})\bar{Q}_{i+1}^{-1}(g_{1i} - G_i k_{21i})^T]^{-1}W_{i+1}(F_i - G_iK_i) + \bar{v}_i^T\bar{v}_i.$$

Note that

$$[I - \gamma^{-2}W_{i+1}(g_{1i} - G_i k_{21i})\bar{Q}_{i+1}^{-1}(g_{1i} - G_i k_{21i})^T]^{-1}W_{i+1}$$
$$= [I - \gamma^{-2}\tilde{W}_{i+1}\bar{G}_i R_{2i}\bar{G}_i^T]^{-1}\tilde{W}_{i+1},$$

where $\bar{G}_i = G_i - g_{1i}\bar{Q}_{i+1}^{-1}k_{21i}^T R_{2i}^{-1}$ and where by **A3** and **A4** the above inverses are all well defined.
Defining

$$\bar{F}_i \triangleq F_i - g_{1i}\bar{Q}_{i+1}^{-1}k_{21i}^T R_{2i}^{-1}K_i,$$

$$\Gamma_i \triangleq \left[R_{2i}^{-1} - \gamma^{-2}\bar{G}_i^T\tilde{W}_{i+1}\bar{G}_i\right]^{-1}$$

we obtain

$$\mathcal{H}_i = \bar{v}_i^T\bar{v}_i - \gamma^2\{\bar{w}_i - \gamma^{-2}\bar{Q}_{i+1}^{-\frac{1}{2}}(g_{1i} - G_i k_{21i})^T$$
$$[I - \gamma^{-2}\tilde{W}_{i+1}\bar{G}_i R_{2i}\bar{G}_i^T]^{-1}\tilde{W}_{i+1}(\bar{F}_i - \bar{G}_iK_i)\}^T W_{i+1}$$
$$\{\bar{w}_i - \gamma^{-2}\bar{Q}_{i+1}^{-\frac{1}{2}}(g_{1i} - G_i k_{21i})^T[I - \gamma^{-2}\tilde{W}_{i+1}\bar{G}_i R_{2i}\bar{G}_i^T]^{-1}$$
$$\tilde{W}_{i+1}(\bar{F}_i - \bar{G}_iK_i)\} + \bar{F}_i^T\tilde{W}_{i+1}\bar{F}_i - \gamma^2 K_i^T R_{2i}^{-1}K_i + \gamma^2[$$
$$\gamma^{-2}\bar{G}_i^T\tilde{W}_{i+1}\bar{F}_i - R_{2i}^{-1}K_i]^T\Gamma_i[\gamma^{-2}\bar{G}_i^T\tilde{W}_{i+1}\bar{F}_i - R_{2i}^{-1}K_i].$$

Similar to the proof of Lemma 1 it follows that

$$\bar{w}_i^* = \gamma^{-2}\bar{Q}_{i+1}^{-\frac{1}{2}}(g_{1i} - G_i k_{21i})^T[I - \gamma^{-2}\tilde{W}_{i+1}\bar{G}_i R_{2i}^{-1}\bar{G}_i^T]^{-1}$$
$$\tilde{W}_{i+1}(F_i - G_iK_i),$$

together with $x_0^* = 0$ and G_i^* of (19), constitute a saddle-point for the game with the objective function of (18). The strategy of $u_i^* = \alpha_{2i}(\hat{x}_i)$, with $\hat{x}_i$ of (21), then solves the control problem. ▽▽▽

4. DISCRETE-TIME H_∞-ESTIMATION OF UNCERTAIN NONLINEAR PROCESSES

We solve the robust estimation problem by applying the approach that has been proposed in [2], for the linear case, and has been used in [8] for the corresponding nonlinear estimation problem in the continuous-time case. According to this approach we convert the set of parameter uncertainties into an exogenous, bounded energy signal in an auxiliary system that does not encounter any parameter uncertainty. We shall show that the solution to the H_∞-estimation problem that is associated with the latter system guarantees a solution to the original problem.

The performance index of (7) that is associated with Problem 1 is a function of operator $\mathcal{F}$ of (3), the uncertain part F_i of (2), and x_0, namely:

$$J(w_i, \mathcal{F}, x_0, F_i) \triangleq \sum_{i=0}^{N-1}\|z_i\|^2 - \gamma^2[N(x_0) + \sum_{i=0}^{N-1}\|w_i\|^2] \tag{22}$$

The robust estimation problem is to find the saddle-point operator $\mathcal{F}^*$ that minimizes $\sup_{w_i, F_i} J(w_i, \mathcal{F}, x_0, F_i)$, where $w_i \in l_2[0, N-1, \mathcal{R}^r]$ and $x_0 \in \mathcal{R}^n$.

The above problem has been solved in [9] for the corresponding case without parameter uncertainty. Utilizing its approach we introduce the following auxiliary system:

$$\begin{aligned} x_{a i+1} &= f_i(x_{ai}) + \tfrac{\gamma}{\epsilon_i}H_{1i}(x_{ai})w_{ai} + g_{1i}(x_{ai})w_i, \\ y_{ai} &= h_{2i}(x_{ai}) + \tfrac{\gamma}{\epsilon_i}H_{2i}(x_{ai})w_{ai} + k_{21i}(x_{ai})w_i, \end{aligned} \tag{23}$$

where $x_{ai} \in \mathcal{R}^n$ is the state, $w_{ai} \in \mathcal{R}^i$ is the additional disturbance which belongs to $l_2[0, N-1, \mathcal{R}^i]$, $y_{ai} \in \mathcal{R}^p$ is the measurement, the mappings $f_i, g_{1i}, h_{2i}, k_{21i}, E_i$, H_{1i} and H_{2i} are as in (1) and (4), and $\{\varepsilon_i\}$ is a sequence of nonzero scalars to be chosen later.

Associated with the system of (23) we introduce an estimate $\hat{x}_{a_i} = \mathcal{F}_a(\mathcal{Y}_i^a)$, where $\mathcal{Y}_i^a \triangleq \{y_{a_s} : s \leq i-1\}$, together with a penalty vector $z_{ai} \in \mathcal{R}^{s+j}$ given by

$$z_{ai} = \begin{bmatrix} h_{1i}(x_{ai}) - h_{1i}(\hat{x}_{a_i}) \\ \varepsilon_i E_i(x_{ai}) \end{bmatrix}, \tag{24}$$

where the mappings h_{1i} and E_i are the same as in (1).

We define the following performance index for the estimation problem for the auxiliary system (23)-(24):

$$J_a(w_i, w_{ai}, \mathcal{F}_{ai}, x_{a_0}, \varepsilon_i) \triangleq \sum_{i=0}^{N-1} \|z_{ai}\|^2 - \gamma^2[N(x_{a_0}) + \sum_{i=0}^{N-1} \|w_i\|^2 + \sum_{i=0}^{N-1} \|w_{ai}\|^2], \tag{25}$$

where $N(\cdot)$ is the same as in (22). We look for an estimator $\hat{x}_{a_i} = \mathcal{F}_a(\mathcal{Y}_i^a)$ with $\hat{x}_{a_0} = 0$ that satisfies $J_a \leq 0$ for all $w_i \in l_2[0, N-1, \mathcal{R}^r]$, $w_{ai} \in l_2[0, N-1, \mathcal{R}^i]$ and $x_{a0} \in \mathcal{R}^n$.

Applying the same arguments used in [8] we readily arrive at the following result:

Lemma 3. *Consider the systems of (1) and (23)-(24) together with the performance indices (22) and (25), respectively. Let $\mathcal{F}_a(\cdot)$ be a given operator such that $\sup_{w_{ai}, w_i} J_a(w_i, w_{ai}, \mathcal{F}_a, x_{a_0}, \varepsilon_i)$ is bounded $\forall x_{a_0} \in \mathcal{R}^n$. Then, it follows that for any admissible set $\{\varepsilon_i\}$*

$$\sup_{w_i, F_i} J(w_i, \mathcal{F}_a, x_0, F_i) \leq \sup_{w_{ai}, w_i} J_a(w_{ai}, w_i, \mathcal{F}_a, x_0, \varepsilon_i)$$

for all $x_0 \in \mathcal{R}^n$.

It follows from Lemma 3 that a solution of the estimation problem (23)-(25) provides a solution to the estimation Problem 1. In view of this, we will solve the problem of (23)-(25) instead of solving the problem for (1) with the performance index (22). This leads to the following estimation problem:

Problem 2. *For a given scalar $\gamma > 0$, find a sequence $\{\varepsilon_i\}, \varepsilon_i \neq 0$ and $\mathcal{F}_a(\cdot)$ so that (25) with $\hat{x}_{a_0} = 0$ remains nonpositive for all $w_{ai} \in l_2[0, N-1, \mathcal{R}^i]$, $w_i \in l_2[0, N-1, \mathcal{R}^r]$, and $x_{a_0} \in \mathcal{R}^n$.*

Remark 2 The operator $\mathcal{F}_a(\cdot)$ which solves the above auxiliary estimation problem provides a solution to Problem 1, namely, the estimator $h_{1i}(\hat{x}_i)$ with $\hat{x}_i = \mathcal{F}_a(\mathcal{Y}_i)$ ensures the performance of (5) for all admissible $F_i(x_i)$ satisfying (2).

The solution to Problem 2 can be obtained by solving an H_∞ nonlinear control problem for a related system. Defining

$$g_{a_i}(x_i) \triangleq \left[\frac{\gamma}{\varepsilon_i} H_{1i}(x_i) \quad g_{1i}(x_i) \right],$$

$$k_{a_i}(x_i) \triangleq \left[\frac{\gamma}{\varepsilon_i} H_{2i}(x_i) \quad k_{21i}(x_i) \right],$$

it is easy to see that Problem 2 is equivalent to the following :

Problem 3. *Given the system*

$$\begin{aligned} x_{a_i} &= f_i(x_{a_i}) + g_{a_i}(x_{a_i}) w_{c_i}, \\ y_{c_i} &= h_{2i}(x_{a_i}) + k_{a_i}(x_{a_i}) w_{c_i} \\ z_{c_i} &= \begin{bmatrix} h_{1i}(x_{a_i}) \\ \varepsilon_i E_i(x_{a_i}) \end{bmatrix} + \begin{bmatrix} -I \\ 0 \end{bmatrix} u_{c_i} \end{aligned} \tag{26}$$

where $x_{a_i} \in \mathcal{R}^n$ is the state, $u_{c_i} \in \mathcal{R}^s$ is a control input, w_{c_i} is the disturbance input, $y_{c_i} \in \mathcal{R}^p$ is the measured output, $z_{c_i} \in \mathcal{R}^{s+j}$ is the controlled output, and $\{\varepsilon_i\}$ is a set of nonzero scaling parameter to be chosen. Find a control law $\{u_{c_i}\}, u_{c_i} = h_{1i}(\hat{x}_{a_i})$, with $\hat{x}_{a_i} = \mathcal{F}_a(\mathcal{Y}_i^c)$, such that

$$\sum_{i=0}^{N-1} \|z_{c_i}\|^2 \leq \gamma^2[N(x_0) + \sum_{i=0}^{N-1} \|w_{c_i}\|^2] \tag{27}$$

for all $w_{c_i} \in l_2[0, N-1, \mathcal{R}^{i+r}]$ and $x_{a_0} \in \mathcal{R}^n$, where $\mathcal{Y}_i^c \triangleq \{y_{c_k} : k \leq i-1\}$.

In light of Remark 2, it follows from the above that if the control law operator $\mathcal{F}_a(\cdot)$ solves Problem 3 the estimator $h_{1i}(\hat{x}_i)$ with $\hat{x}_i = \mathcal{F}_a(\mathcal{Y}_i)$ solves Problem 1.

Problem 3 can be solved using the results of Section 3. Thus, defining

$$\bar{Q}_{ai+1}(x_i) \triangleq I - \gamma^{-2} g_{a_i}^T(x_i) Q_{i+1} g_{a_i}(x_i), \tag{28a}$$

$$R_{2a_i}(x_i) \triangleq k_{a_i}(x_i) \bar{Q}_{ai+1}^{-1}(x_i) k_{a_i}^T(x_i), \tag{28b}$$

$$\tilde{Q}_{ai+1}(x_i) \triangleq [I - \gamma^{-2} Q_{i+1} g_{a_i}(x_i) g_{a_i}^T(x_i)]^{-1} Q_{i+1}, \tag{28c}$$

$$\tilde{W}_{ai+1}(x_i) \triangleq \{I - \gamma^{-2} W_{i+1} g_{a_i}(x_i) \bar{Q}_{ai+1}^{-1}(x_i)[I-$$

$$k_{a_i}^T(x_i)R_{2a_i}^{-1}(x_i)k_{a_i}(x_i)\bar{Q}_{a_{i+1}}^{-1}(x_i)]g_{a_i}^T(x_i)\}^{-1}W_{i+1}, \tag{28d}$$

we obtain the following:

Theorem 1. *Consider the system of (1) with the estimation error of (4). Suppose that*
1: There exists a family of positive semi-definite matrices $\{Q_i\}_{i=0}^N$ *that satisfies, for a nonzero* ε_i, *the following inequality for all* $x \in \mathcal{R}^n$ *and* $\forall i \in [0, N-1]$:

$$-x^T Q_i x + \varepsilon_i^2 E_i^T(x)E_i(x) + f_i^T(x)\tilde{Q}_{a_{i+1}}(x)f_i(x) \le 0,$$

$$Q_N = 0, \quad Q_0 \le \gamma^2 P_0. \tag{29a - c}$$

2: There exists a family of positive semi-definite matrices $\{W_i\}_{i=0}^N$ *that satisfies for all* $x \in \mathcal{R}^n$ *and* $\forall i \in [0, N-1]$ *the following:*

$$-(x-\hat{x})^T W_i(x-\hat{x}) + (h_{1_i}(x) - h_{1_i}(\hat{x}))^T$$

$$(h_{1_i}(x) - h_{1_i}(\hat{x})) + \bar{F}_{c_i}^T(x,\hat{x})\tilde{W}_{a_{i+1}}(x)\bar{F}_{c_i}(x,\hat{x})$$

$$-\gamma^2 K_{c_i}^T(x,\hat{x})R_{2a_i}^{-1}(x)K_{c_i}(x,\hat{x}) \le 0 \tag{30a}$$

$$W_0 = \gamma^2 P_0 - Q_0 \tag{30b}$$

where

$$\bar{F}_{c_i}(x,\hat{x}) \triangleq F_{c_i}(x,\hat{x})$$

$$-g_{a_i}(x)\bar{Q}_{a_{i+1}}^{-1}(x)k_{a_i}^T(x)R_{2a_i}^{-1}(x)K_{c_i}(x,\hat{x}),$$

$$F_{c_i}(x,\hat{x}) \triangleq [I - \gamma^{-2}g_{a_i}(x)g_{a_i}^T(x)Q_{i+1}]^{-1}f_i(x)$$

$$-[I - \gamma^{-2}g_{a_i}(\hat{x})g_{a_i}^T(\hat{x})Q_{i+1}]^{-1}f_i(\hat{x}),$$

$$K_{c_i}(x,\hat{x}) \triangleq h_{2i}(x) - h_{2i}(\hat{x}) + \gamma^{-2}k_{a_i}(x)g_{a_i}^T(x)$$

$$\tilde{Q}_{a_{i+1}}(x)f_i(x) - \gamma^{-2}k_{a_i}(\hat{x})g_{a_i}^T(\hat{x})\tilde{Q}_{a_{i+1}}(\hat{x})f_i(\hat{x}).$$

3: **A1**, **A2** *are satisfied,* $R_{2a_i} > 0$, $\bar{Q}_{a_i} > 0$, $\tilde{W}_{a_{i+1}}$ *exists and* $\forall i \in [0, N-1]$ *and* $x \in \mathcal{R}^n$.

$$I - \gamma^{-2}\tilde{W}_{a_{i+1}}(G_i - g_{a_i}\bar{Q}_{a_{i+1}}^{-1}k_{a_i}^T R_{2a_i}^{-1})R_{2a_i}(G_i$$

$$- g_{a_i}\bar{Q}_{a_{i+1}}^{-1}k_{a_i}^T R_{2a_i}^{-1})^T > 0 \tag{31}$$

4: The following holds:

$$\left(G_i^*(\hat{x}) - g_{a_i}(x)\bar{Q}_{a_{i+1}}^{-1}(x)k_{a_i}^T(x)R_{2a_i}^{-1}(x)\right)^T \tilde{W}_{a_{i+1}}(x)$$

$$\bar{F}_{c_i}(x,\hat{x}) = \gamma^2 R_{2a_i}^{-1}(x)K_{c_i}(x,\hat{x}). \tag{32}$$

Then the estimation $u_{c_i} = h_{1i}(\hat{x}_{i+1})$ *with*

$$\hat{x}_{i+1} = [I - \gamma^{-2}g_{a_i}(\hat{x}_i)g_{a_i}^T(x_i)Q_{i+1}]^{-1}f_i(\hat{x}_i) + G_i^*(\hat{x}_i)$$

$$[y_i - h_{2i}(\hat{x}_i) - \gamma^{-2}k_{a_i}(\hat{x}_i)g_{a_i}^T(\hat{x}_i)\tilde{Q}_{i+1}(\hat{x}_i)f_i(\hat{x}_i)] \tag{33}$$

solves Problem 1.

Remark 3 Note that inequality (29a) is a nonlinear analogous of the Discrete Bounded Real Lemma and that it is equivalent to the dissipativity property between the signals w_i and $\varepsilon_i E_i(x_i)$.

5. THE ROBUST EXTENDED H_∞ FILTER

Similar to the derivation of the well known extended Kalman filter [13] the condition of (30a) can be linearized in the neighborhood of the estimated trajectory $\{\hat{x}_i\}$, or equivalently, about $\{e_i\} = 0$.
We first introduce the following notations which we shall use in the sequel:

$$A_i(\hat{x}_i) \triangleq \left[\frac{\partial\{[I - \gamma^{-2}g_{a_i}(x)g_{a_i}^T(x)Q_{i+1}]^{-1}f_i(x)\}}{\partial x}\right]_{x=\hat{x}_i}$$

$$- g_{a_i}(\hat{x}_i)\bar{Q}_{a_{i+1}}^{-1}(\hat{x}_i)k_{a_i}^T(\hat{x}_i)R_{2a_i}^{-1}(\hat{x}_i)C_{2_i}(\hat{x}_i), \tag{34a}$$

$$C_{1_i} \triangleq \left[\frac{\partial h_{1_i}(x)}{\partial x}\right]_{x=\hat{x}_i}, \quad C_{2_i}(\hat{x}_i) \triangleq \left[\frac{\partial h_{2_i}(x)}{\partial x}\right]_{x=\hat{x}_i}$$

$$+ \left[\frac{\partial\gamma^{-2}k_{a_i}(x)g_{a_i}^T(x)\tilde{Q}_{a_{i+1}}(x)f_i(x)\}}{\partial x}\right]_{x=\hat{x}_i} \tag{34b, c}$$

We define

$$J_w^i \triangleq -W_i + C_{1_i}^T(\hat{x}_i)C_{1_i}(\hat{x}_i) + A_i^T(\hat{x}_i)\tilde{W}_{a_{i+1}}(\hat{x}_i)A_i(\hat{x}_i)$$

$$-\gamma^2 C_{2_i}^T(\hat{x}_i)R_{2a_i}^{-1}(\hat{x}_i)C_{2_i}(\hat{x}_i),$$

where $\tilde{W}_{a_{i+1}}$ is defined in (28d) and assume that there existx $G_i^*(\hat{x}_i)$ that satisfies

$$A_i^T(\hat{x}_i)\tilde{W}_{a_{i+1}}^T(\hat{x}_i)G_i^*(\hat{x}_i) = \gamma^2 C_{2_i}^T(\hat{x}_i)R_{2a_i}^{-1}(\hat{x}_i)$$

$$+ A_i^T(\hat{x}_i)\tilde{W}_{a_{i+1}}^T(\hat{x}_i)g_{a_i}(\hat{x}_i)\bar{Q}_{a_{i+1}}^{-1}(\hat{x}_i)k_{a_i}^T(\hat{x}_i)R_{2a_i}^{-1}(\hat{x}_i) \tag{35}$$

where $\{\hat{x}_i\}$ is given by (33). We obtain, in the next lemma, the conditions that guarantee the dissipativity of the system of Problem 3, with respect to the index of (27), at the neighborhood of $\{e_i\} = 0$. These conditions are therefore sufficient for the solution of Problem 1.

Lemma 4. *Assume that* **A1**, **A2** *hold and that (35) has a solution, and suppose there exists a family of positive semi-definite matrices* $\{Q_i\}_{i=0}^N$ *and a nonzero sequence* $\{\varepsilon_i\}$ *that satisfy (29a-c)* $\forall i \in [0, N-1]$ *and*

$\forall x \in \mathcal{R}^n$. *In addition, suppose that the family of positive semi-definite matrices* $\{W_i\}_{i=0}^N$ *that satisfies condition 3 in Theorem 1 satisfies also* $\forall i \in [0, N-1]$ *the following:*

$$J_w^i + \delta I \leq 0, \quad W_0 = \gamma^2 P_0 - Q_0, \tag{36}$$

for some $\delta > 0$, *where* $\hat{x}_i$ *is defined in (33) and* $G_i^*(\hat{x}_i)$ *is given by*

$$G_i^*(\hat{x}_i) = \gamma^2 \tilde{W}_{a_{i+1}}^{-T}(\hat{x}_i)(A_i^+)^T(\hat{x}_i)C_{2_i}^T(\hat{x}_i)R_{2a_i}^{-1}(\hat{x}_i)$$
$$+ g_{a_i}(\hat{x}_i)\bar{Q}_{a_{i+1}}^{-1}(\hat{x}_i)k_{a_i}^T(\hat{x}_i)R_{2a_i}^{-1}(\hat{x}_i), \tag{37}$$

and where D^+ *is a left pseudo inverse of* D. *Then the estimate* $u_{c_i} = h_{1_i}(\hat{x}_{i+1})$ *solves Problem 1, at the neighborhood of* $\{e_i\} = 0$.

Proof: By expanding (30a) about $e = 0$ we readily obtain the following requirement for $\{W_i\}$

$$e^T\{-W_i + C_{1_i}^T(\hat{x}_i)C_{1_i}(\hat{x}_i) + A_i^T(\hat{x}_i)\tilde{W}_{a_{i+1}}(\hat{x}_i)A_i(\hat{x}_i)$$
$$-\gamma^2 C_{2_i}^T(\hat{x}_i)R_{2a_i}^{-1}(\hat{x}_i)C_{2_i}(\hat{x}_i)\}e + d(||e||^3, \hat{x}_i),$$

where $d(||e||^3, \hat{x}_i)$ is of the order of $||e||^3$ for a given $\hat{x}_i$, that is, $d(||e||^3, \hat{x}_i)/||e||^2 \to 0$ as $e \to 0$. Since our horizon is finite, it follows that there is some neighborhood of the origin in $\mathcal{R}^n$, Ω, such that $d(||e||^3, \hat{x}_i) < \delta||e||^2$ for all $e \in \Omega$ and for all $\hat{x}_i, i \in [0, N-1]$. Using now (29a-c) and (36) we find that as long as $e_i \in \Omega$ for all $i \in [0, N]$ we can apply Theorem 1 to conclude the proof. $\nabla\nabla\nabla$

6. EXAMPLE

In order to demonstrate the use of the above theory we consider the Robust Extended H_∞ Filter for a simple second order problem.

Consider the time-invariant system of [9]

$$x_{i+1} = Ax_i + Bw_i, \tag{38a}$$

where

$$A = \begin{bmatrix} 0.569 & 0.763 \\ -0.763 & 0.416 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1.267 \\ -1.625 \end{bmatrix}.$$

We assume here that the measurement is described by

$$y_i = Cx_i + H_2 F(x_i)E(x_i) + \sin(2x_{1_i}) + 0.5n_i, \tag{38b}$$

where x_{1_i} is the first component of the two dimentional state-vector x_i, $C = [3 \quad 0]$, $H_2 = 2$, $E(x_i) = [1 \quad 0]x_i$, $|F(x_i)| \leq 1 \quad \forall i \in [0, N]$ and $\forall x_i \in \mathcal{R}^2$. We consider the time-interval $[0, N]$, where $N = 500$, and we are looking for an estimate of $C_1 x_i$, where $C_1 = [0 \quad 1]$.

We have chosen $\gamma = 10.9$, $\varepsilon = 0.1$ and have introduced

$$k_a = \begin{bmatrix} \frac{\gamma}{\varepsilon}H_2 & 0 & 0.5 \end{bmatrix} \quad \text{and} \quad g_a = [0 \quad B \quad 0].$$

Using the theory of Section 5 we have to solve (29) and (36). Inequality (29) does not depend on the current estimate $\hat{x}_i$ and we can thus find a solution $\{Q_i\}$ off-line. After 30 steps we obtained that the solution of the equation, that results from (29) by taking the equality sign, converges to the constant matrix

$$Q = \begin{bmatrix} 0.0311 & 0.0041 \\ 0.0041 & 0.0250 \end{bmatrix}.$$

The matrix Q thus solves (29) for all $i \leq 470$.

In (36), the only term that depends on $\{\hat{x}_i\}$ is

$$C_{2_i}(\hat{x}_i) = [3 + 2\cos(2\hat{x}_{1_i}) \quad 0].$$

Using monotonicity arguments that are similar to those of [3], we can guarantee that $\{W_i\}$ solves (36) if it satisfies (36) for $C_2 = [1 \quad 0]$. Since the latter matrix does not depend on $\{\hat{x}_i\}$ we can solve (36) off-line. The solution of the equation, that results from (36) by taking the equality sign and $\delta = 0$, slowly converges to

$$W = \begin{bmatrix} 5.94 & 0.63 \\ 0.63 & 5.90 \end{bmatrix}.$$

Thus, with the assumption that $W_0 = W$, the matrix W solves (36) for $\delta = 0$, $\forall i \in [0, N]$.

We compare the result of our robust estimator with the one obtained by the H_∞ nonlinear estimation method of [9], for the near-minimum value of $\gamma = 2.7$, and with the result that is obtained by the EKF procedure, where we designed the two filters for the nominal system of (38), namely for $F \equiv 0$.

We describe below the simulation results for $\{w_i\}$ and $\{n_i\}$ that are uncorrelated standart Gaussian white noise processes. We simulate the above three estimators for the worst values of the uncertainty F, namely, for each estimator we describe the estimation error in the worst case, where we take $F = -1$ for the Robust extended H_∞ estimator and $F = 1$ for the estimator of [9] and the EKF.

The advantage the Robust extended H_∞ filter is far more significant in the case where, say, $w_i = \sqrt{2}\sin(0.1i)$ and $\{n_i\}$ is still standart white noise. In Fig. 1 and Fig. 2 we compare the worst case errors, that are obtained in the case of $F = 1$, for the three estimators. The resulting l_2-norms of the estimation errors are displayed in Table 1.

Estimator	l_2-norm of the estimation error	
	$\{w_i\}$ is white	$w_i = \sqrt{2}\sin(0.1i)$
Robust	77.17	45.46
Filter of [9]	102.84	62.80
EKF	140.49	330.81

Table 1. Comparison between the l_2 error norms of the three estimator designs.

7. CONCLUSIONS

In the present paper we have introduced a robust H_∞ estimation method for nonlinear time-varying processes with norm-bounded time-varying uncertainties. We have formulated an auxiliary nonlinear control problem and showed that the H_∞ control solution for this problem ensures the solution to the original problem. A sufficient condition has been derived for the solvability of the auxiliary system. This condition leads to an estimation procedure that extends the recent results of [9] to systems with parameter uncertainties.

In the present work we have chosen a specific quadratic structure for the various storage functions. This, of course, may not be the optimal choice, and it may thus lead to an overdesign. Moreover, the minimizing G is not always feasible, in the sense that it is not a function of $\hat{x}_i$ only. This is why a linearization around the zero error vector is suggested which leads to the Extended Robust H_∞ Filter.

The approach we have adopted is one of a priori estimation type, where the current measurement is not available for estimation. Similar results can be derived also for the a posteriori case.

8. REFERENCES

[1] G. Zames, "Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative Seminorms and Approximate Inverses". *IEEE Trans. on Automat. Contr.* **AC-26**, pp. 301-320, 1981.

[2] L. Xie, C. E. de Souza and M. Fu, "H_∞ Estimation for Discrete-Time Linear Uncertain Systems". *Int. J. of Robust and Nonlinear Control*, Vol. 1, pp. 111-123, 1991 .

[3] M. Green and D. J. N. Limebeer, *Linear Robust Control*, Prentice Hall, New Jersy 1995.

[4] I. R. Petersen and C. V. Hollot, "A Riccati Eq. Approach to the Stabilization of Uncertain Linear Systems". *Automatica*, Vol. 22, pp. 397-411, 1986.

[5] A. J. Van der Schaft, "L_2 -Gain Analysis of Nonlinear Systems and Nonlinear State Feedback H_∞ Control". *IEEE Trans. on Automat. Contr.*, **AC-37**, pp. 770-784, 1992.

[6] A. Isidori and A. Astolfi, "Disturbance Attenuation and H_∞ Control Via Measurement Feedback in Nonlinear Systems". *IEEE Trans. Automat. Control*, **AC-37**, pp. 1283-1293, 1992.

[7] J. A. Ball and J. W. Helton "Nonlinear H_∞ Control Theory for Stable Plants" . *Math. of Control Signals and Systems*, Vol. 5, pp. 233-261, 1992.

[8] M. Shergei, U. Shaked and C. E. de Souza, "Robust H_∞ Nonlinear Estimation" . *Proc. of the 4th Workshop on Adaptive Control: Applications to Nonlinear Systems and Robotics*, Cancun, Mexico, December 1994.

[9] U. Shaked and N. Berman, "H_∞ Nonlinear Filtering of Discrete-Time Processes". *IEEE Trans. on Signal Processing*, 1994 in press.

[10] J. C. Willems, "Dissipative Dynamical Systems, Part I. General Theory". *Arch. Rational Mechanics and Analysis* , Vol. 45, pp. 321-351, 1972.

[11] A. Isidori, "H_∞-Control Via Measurement Feedback for Affine Nonlinear Systems", Preprint, November, 1992.

[12] N. Berman and U. Shaked, "H_∞ Nonlinear Filtering".*Int. J. of Robust and Nonlinear Control*, in press.

[13] F. L. Lewis, *Optimal Estimation* , John Wiley & Sons, 1986.

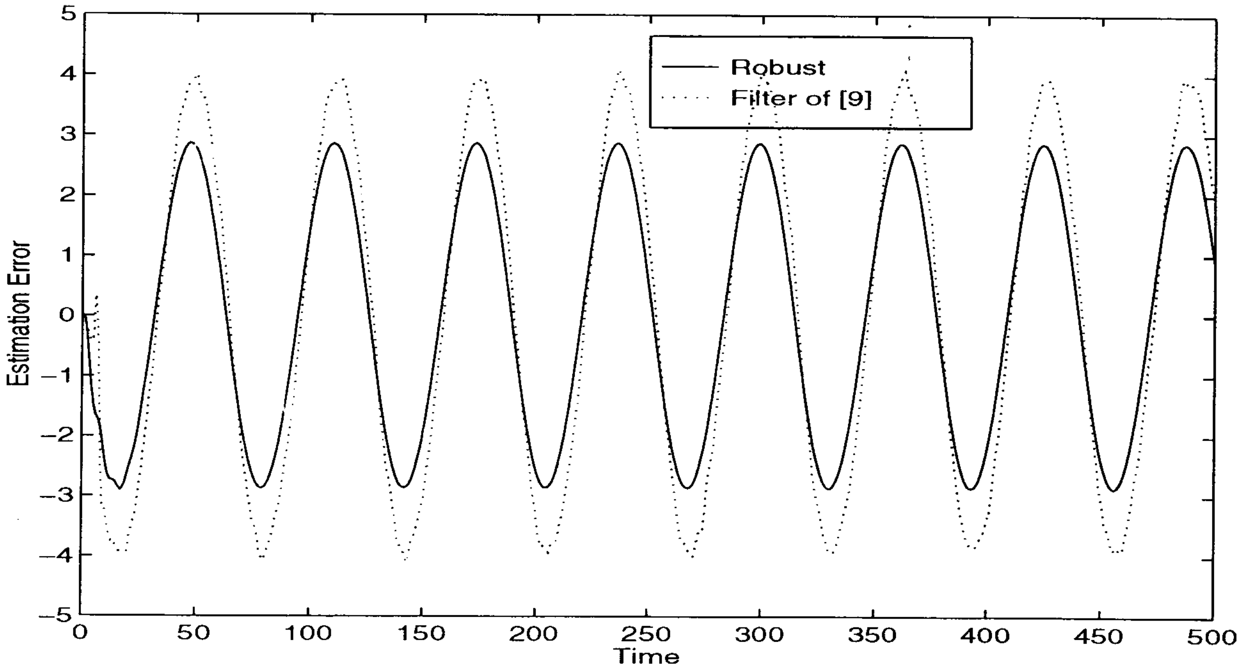

Figure 1: Comparison between the worst case results of the Robust Extended H_∞ Filter and the filter of [9] in the case where the driving disturbance is $\sqrt{2}\sin(0.1i)$.

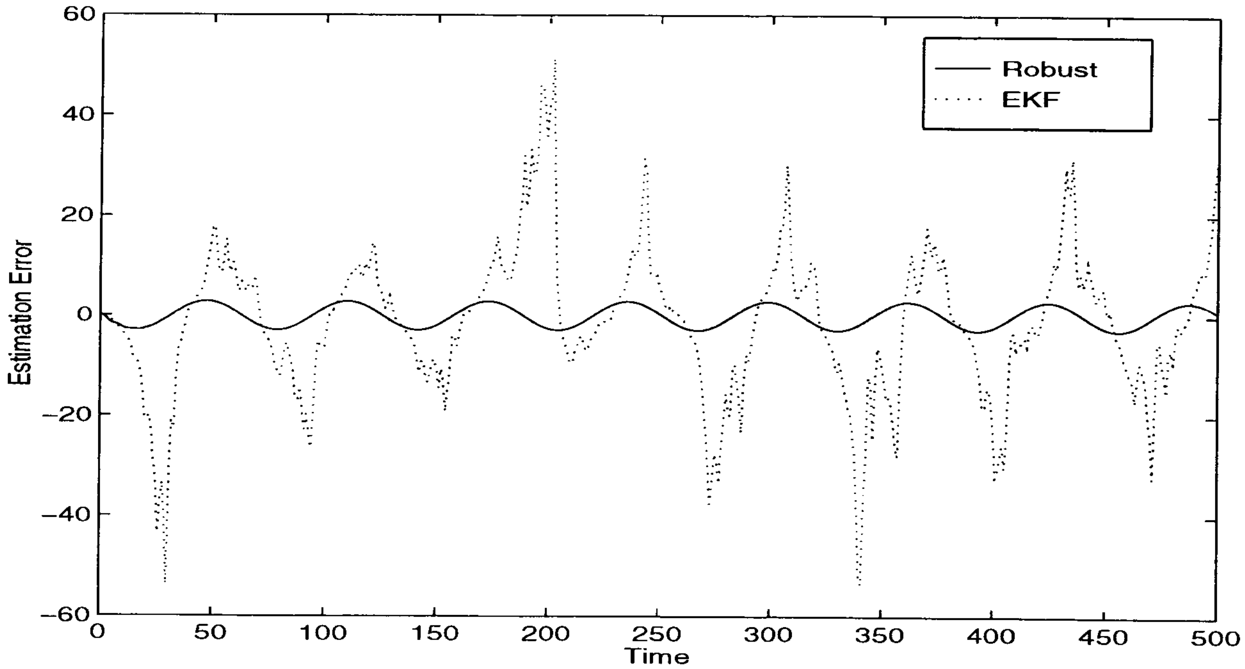

Figure 2: Comparison between the results of the Robust Extended H_∞ Filter and the EKF in the case where the driving disturbance is $\sqrt{2}\sin(0.1i)$.