REDUCED ORDER OBSERVER WITH STEADY STATE GAIN PRESERVATION

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ABSTRACT

The problem of fixed-order observers for systems with deterministic , i.e. known, input is considered. The reduced order observer is required to preserve the steady-state gain matrix of the observed system and to minimize the transient estimation error. The solution consists of a modified Riccati equation and two modified Lyapunov equations coupled by three projection matrices. Explicit expressions for the optimal reduced order observer are given in terms of the solution of the modified Riccati equation and by the three projections.

1. INTRODUCTION

In this paper, we consider the problem of designing a reduced order observer for systems subjected to stochastic disturbances, (unknown), and deterministic input (known). It is well known, e. g. [8], that the estimation error is independent of the determinstic input if and only if an asymptotic observer is used. In the full order case, the optimal estimator, i. e. the Kalman filter is automatically an observer. However, the observer structure property is not preserved in the standard reduced order estimation. The reduced order estimators, [3, 5] can not be applied in presence of a deterministic input. In [4], the problem of reduced order optimal state estimation for systems subjected to colored noises and deterministic input was solved. The structure of an observer was imposed in addition to estimation optimization. It was assumed there that the input acts only on part of the system and the order of the estimator was pre-constrained to be at least equal to the order of the subsystem driven by the deterministic input. Under the same assumption, [2] have designed an optimal asymptotic observer for a pre-specified unstable subspace. These results are not applicable when the effect of the deterministic input is not restricted to a subsystem. Consider, for example, a system without noises, it is then clear that the known input acts on the entire system. Hence, special treatment is needed in presence of a deterministic input. A common assumption is to consider the deterministic input as generated by known dynamic stable system excited by white noises [1]. [10] gave a suboptimal solution to the reduced order estimation problem with known input.

In this paper, we approach the problem in a different way. We assume that the deterministic input consists of a series of step functions of different levels. We also assume that no error remains before the next change in the value of the step. It is therefore required that the steady-state estimation error is zero and the additional degrees of freedom will be used for the minimization of the transient estimation error. Hence the reduced order observer preserves the steady-state gain matrix of the observed system.This property is important for many physical applications such as navigation and guidance.

The problem of steady-state gain preservation in order reduction problems was first introduced by [11] for model order reduction. There are some similarities between the preliminary steps of the problem of [11] and that of the problem solved here. However, the solution procedures are different due to the essential difference that exists between model and observer order reduction problems.

This paper is organized as follows. Section 2 contains definitions and states the problem. In Section 3, we present the main results of the paper. The results of the paper are summarized in Section 4.

2. PRELIMINARIES AND PROBLEM FORMULATION

Consider the following linear continuous time-invariant n-th order system :

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.1}$$

$$y(t) = Cx(t), \tag{2.2}$$

where $x(t) \in \boldsymbol{R}^n$ is the state-space vector, $y(t) \in \boldsymbol{R}^r$ is the measured output and $u(t) \in \boldsymbol{R}^m$ is a deterministic input.

The objective is to design a $n_e \leq n$ order observer:

$$\dot{x}_e(t) = A_e x_e(t) + B_e u(t) + K_e y(t), \tag{2.3}$$

$$y_e(t) = C_e x_e(t) + D_{e1} u(t) + D_{e2} y(t), \tag{2.4}$$

We define the augmented state vector $\tilde{x}^T(t) \triangleq \left[\ x^T(t) \quad x_e^T(t)\ \right]^T$, $\tilde{x}(t) \in \boldsymbol{R}^{\tilde{n}}$, where $\tilde{n} = n + n_e$ and denote the observation error as

$$e(t) = Lx(t) - y_e(t),$$

then, the augmented state equation is given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_e(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ K_e C & A_e \end{bmatrix} \begin{bmatrix} x(t) \\ x_e(t) \end{bmatrix} + \begin{bmatrix} B \\ B_e \end{bmatrix} u(t)\,, \tag{2.5}$$

$$e(t) = \begin{bmatrix} L - D_{e2}C & -C_e \end{bmatrix} \begin{bmatrix} x(t) \\ x_e(t) \end{bmatrix} - D_{e1} u(t)\,. \tag{2.6}$$

It follows from (2.5)-(2.6) that the observation error is directly affected by the input, i.e.

$$e(s) = T_{eu}(s) u(s).$$

In the full order observer, [8],

$$A_e = A - KC,\ B_e = B,\ K_e = K,\ D_{e1,2} = 0$$

and this structure leads to $T_{eu}(s) \equiv 0$. The same holds for the Kalman filter since it has the structure of a full order observer. In the deterministic reduced order (Luenberger) observer of [9], $ne = n - r$, it is also obtained that $T_{eu}(s) \equiv 0$ by the choice

$$A_e = (C_2 - KC)AL_2,\ B_e = (C_2 - KC)B,$$

$$K_e = (C_2 - KC)A(L_1 + L2K),$$

$$C_e = L_2,\ D_{e1} = 0,\ D_{e2} = (L_1 + L2K),$$

where

$$\begin{bmatrix} C \\ C_2 \end{bmatrix}^{-1} = \begin{bmatrix} L_1 & L_2 \end{bmatrix} \tag{2.7}$$

and C_2 is an arbitrary matrix chosen such that the left side of (2.7) exists. In [2, 4, 6], $T_{eu}(s) = 0$ is required, but these works consider systems in which the deterministic input affects only part of the overall system. To satisfy this constraint, a full order observer is designed for the deterministic subsystem. In the other cases of reduced order observers, $T_{eu}(s) \neq 0$ and the problem is how to make $T_{eu}(s)$ 'small' in some sense. To deal with this problem, one needs to make some assumptions on $u(t)$. The reduced order estimation literature, e.g. [5], [3], generally disregards the influence of a deterministic input and deals only with the minimization of the estimation error due to stochastic disturbances. If $u(t)$ does exist, a common approach is to consider $u(t)$ as a stochastic signal with known spectrum. Technically, it can be considered as a measured output consisting of noise only ([1])

In this paper, we consider the problem in a different approach where we assume that the deterministic input $u(t)$ consists of a series of step functions where the value of the step may change only after the observer has reached steady-state. Based on this assumption, we therefore take $u(t)$ as a single step function and, in addition, in order to avoid a steady state error (before the next change in the value of the step), we require that $T_{eu}(0) = 0$. We can use the observer parameters remaining free for optimization to minimize the error during the transient period. The mathematical formulation of the problem is given as follows:

i. The steady-state observation error is set to zero.

ii. The following quadratic cost criterion is minimized:

$$J_u(A_e, B_e, K_e, C_e, D_{e1}, D_{e2}) =$$

$$\lim_{T\to\infty}\int_0^T \{Lx(t)-y_e(t)\}^T\{Lx(t)-y_e(t)\}\,dt, \tag{2.8}$$

We thus obtain that the problem is equivalent to the minimization of the step response from $u(t)$ to $e(t)$ in (2.5)-(2.5) under the constraint that the steady state error is set to zero.

In order to formulate this problem as an L_2 optimization problem, we introduce the following lemma:

Lemma 2.1 *Consider the asymptotically stable LTI system Σ_1 with state-space realization given by $\{A_1, B_1, C_1, D_1\}$. Let the state-space realization of the system Σ_2 given by $\{A_2, B_2, C_2, 0\}$ with*

$$A_2 = A_1\ ,\ B_2 = B_1\ ,\ C_2 = C_1A_1^{-1}\ ,$$

and define the constant matrix

$$D_2 = D_1 - C_1A_1^{-1}B_1.$$

Then,

$$s_{\Sigma_1}(t) = h_{\Sigma_2}(t) + D_2\ , \tag{2.9}$$

where $s_{\Sigma_1}(t)$ denotes the step response of Σ_1 and $h_{\Sigma_2}(t)$ denotes the impulse response of Σ_2.

(The step response matrix is defined similarly to the well known impulse response matrix definition.)

Proof: The proof readily follows from the definitions of step and impulse responses of Σ_1 and Σ_2.

We apply Lemma 2.1 to the system (2.1)-(2.2) and we get that the optimization problem we want to solve is equivalent to the minimization of the L_2-norm of the transfer function of the following auxiliary system:

$$\tilde{A} = \begin{bmatrix} A & 0 \\ K_eC & A_e \end{bmatrix},\ \tilde{B} = \begin{bmatrix} B \\ B_e \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} \hat{L} - \hat{D}_{e2}\hat{C} & -\hat{C}_e \end{bmatrix}, \tag{2.10a - c}$$

with the requirement that

$$\hat{D}_{e1} - \hat{D}_{e2}\hat{C}B - \hat{L}B = 0, \tag{2.11}$$

where

$$\hat{L} = LA^{-1}\ ,\ \hat{C} = CA^{-1}\ ,\ \hat{C}_e = C_eA_e^{-1},$$

$$\hat{D}_{e1} = D_{e1} - C_eA_e^{-1}B_e\ ,\ \hat{D}_{e2} = D_{e2} - C_eA_e^{-1}K_e\ .$$

The requirement (2.11) ensures the steady-state gain preservation. The optimization problem is meaningful only for stable minimal observers. We therefore assume that $(A_e,\ B_e,\ C_e)$ is minimal and A_e is stable. The latter implies the asymptotic stability of $\tilde{A}$. Thus the steady state covariance matrix

$$\tilde{Q} = lim_{t\to\infty}E\{\tilde{x}(t)\tilde{x}^T(t)\}$$

is the non-negative solution of the Lyapunov equation:

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{B}\tilde{B}^T = 0 \tag{2.12}$$

The L_2-norm of the auxiliary system (2.10a-c) may be written in terms of $\tilde{Q}$ as

$$J_a = trace\ \tilde{Q}\tilde{C}^T\tilde{C}\ . \tag{2.13}$$

Therefore, the optimization problem is the minimization of (2.13) under the constraint (2.12), where, given the relation (2.11), the parameters which remain free for the optimization are only $A_e, B_e, \hat{C}_e$ and $\hat{D}_{e2}$. The details of the optimization procedure are given in the proof of Theorem 3.1.

Before stating the solution of the optimization problem in the next section, the following lemma is needed:

Lemma 2.2 *Factorization lemma [7]:*

Let $\hat{Q},\ \hat{P}\ \in \mathbf{R}^{n\times n}$ be nonnegative definite matrices that satisfy

$$rank\ \hat{Q}\hat{P} = n_r, \tag{2.14}$$

there exist $G,\ \Gamma\ \in \mathbf{R}^{n\times n_r}$ and $M\ \in \mathbf{R}^{n_r\times n_r}$ nonsingular matrix, such that

$$\hat{Q}\hat{P} = G^TM\Gamma\ ,\ \Gamma G^T = I_{n_r}, \tag{2.15}$$

and $(G,\ M,\ \Gamma)$ is a factorization of $\hat{Q}\hat{P}$. The matrix $\tau \triangleq G^T\Gamma$ is an oblique projection matrix satisfying $\tau^2 = \tau$.

3. MAIN RESULT

The following Theorem states the necessary conditions for the reduced order observer that preserves the steady-state gain

Theorem 3.1 *Assume $(A_e, B_e, C_e, D_{e1}, D_{e2})$ solve the reduced order steady-state gain preserving observer problem. Then there exist $n \times n$ nonnegative definite matrices $Q, \hat{Q}, \hat{P}$ such that (A_e, B_e, K_e, C_e, D_{e1}, D_{e2}) are given by*

$$A_e = \Gamma A \nu_\perp G^T - (G\hat{P}G^T)^{-1}\hat{\nu}_\perp^T \hat{L}^T \hat{L}\hat{\nu}\nu G^T \ , \quad (3.1)$$

$$B_e = \Gamma B \ , \quad (3.2)$$

$$K_e = \left(\Gamma A + (G\hat{P}G^T)^{-1}\hat{\nu}_\perp^T \hat{L}^T \hat{L}\hat{\nu}_\perp\right) C^* \ , \quad (3.3)$$

$$C_e = \hat{L}\hat{\nu}_\perp G^T A_e \ , \quad (3.4)$$

$$D_{e1} = \hat{L}\hat{\nu}_\perp (B + G^T B_e) \ , \quad (3.5)$$

$$D_{e2} = \hat{L}(\hat{C}^* + \hat{\nu}_\perp G^T K_e) \ , \quad (3.6)$$

and Q, $\hat{Q}$, $\hat{P}$, satisfy the following equations:

$$\begin{aligned}&\left(A - \tau A\nu - (G\hat{P}G^T)^{-1}\hat{\nu}_\perp^T \hat{L}^T \hat{L}\hat{\nu}\nu\right) Q \\ &+ Q\left(A - \tau A\nu - (G\hat{P}G^T)^{-1}\hat{\nu}_\perp^T \hat{L}^T \hat{L}\hat{\nu}\nu\right)^T \\ &- \tau_\perp BB^T \tau_\perp^T = 0 \end{aligned} \quad (3.7)$$

$$\begin{aligned}&A\hat{Q} + \hat{Q}A^T + \left(\tau A\nu + (G\hat{P}G^T)^{-1}\hat{\nu}_\perp^T \hat{L}^T \hat{L}\hat{\nu}_\perp \nu\right) Q \\ &+ Q\left(\tau A\nu + (G\hat{P}G^T)^{-1}\hat{\nu}_\perp^T \hat{L}^T \hat{L}\hat{\nu}_\perp \nu\right)^T \\ &- BB^T + \tau_\perp BB^T \tau_\perp^T = 0 \end{aligned} \quad (3.8)$$

$$\begin{aligned}&\hat{P}A\nu_\perp + \nu_\perp^T A^T \hat{P} - \nu^T \hat{\nu}_\perp^T \hat{L}^T \hat{L}\hat{\nu}_\perp \tau - \tau^T \hat{\nu}_\perp^T \hat{L}^T \hat{L}\hat{\nu}_\perp \nu \\ &+ \hat{\nu}_\perp^T \hat{L}^T \hat{L}\hat{\nu}_\perp - \tau_\perp^T \hat{\nu}_\perp^T \hat{L}^T \hat{L}\hat{\nu}_\perp \tau_\perp = 0 \end{aligned} \quad (3.9)$$

$$rank\ \hat{Q}\hat{P} = rank\ \hat{Q} = rank\ \hat{P} = n_e \quad (3.10)$$

where

$$C^* = QC^T(CQC^T)^{-1} \ , \ \nu = C^*C \ , \ \nu_\perp = I_n - \nu, \quad (3.11)$$

$$\hat{C} = CA^{-1} \ , \ \hat{C}^* = Q\hat{C}^T(\hat{C}Q\hat{C}^T)^{-1} \ ,$$

$$\hat{\nu} = \hat{C}^*\hat{C} \ , \ \hat{\nu}_\perp = I_n - \hat{\nu}, \quad (3.12)$$

$$\tau = G^T\Gamma \ , \ \tau_\perp = I_n - \tau \quad (3.13)$$

and (G, M, Γ) is a factorization of $\hat{Q}\hat{P}$.

Proof: To minimize (2.13) under the constraint (2.12), we construct the Lagrangian:

$$\mathcal{L} = trace\ \left[\tilde{Q}\tilde{C}^T\tilde{C} + (\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{B}\tilde{B}^T)\tilde{P}\right] \quad (3.14)$$

where $\tilde{P} \ \in \boldsymbol{R}^{\tilde{n}\times\tilde{n}}$ is the Lagrange multiplier matrix. We partition $\tilde{Q}$ and $\tilde{P}$ into $n \times n$, $n \times n_e$ and $n_e \times n_e$ subblocks as

$$\tilde{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}$$

$$\tilde{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix} \quad (3.15)$$

$\frac{\partial \mathcal{L}}{\partial \tilde{P}} = 0$ yields (2.12) and

$$\frac{\partial \mathcal{L}}{\partial \tilde{Q}} = \tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} + \tilde{C}^T\tilde{C} = 0 \quad (3.16)$$

$$\frac{\partial \mathcal{L}}{\partial A_e} = P_{12}^T Q_{12} + P_2 Q_2 = 0 \quad (3.17)$$

$$\frac{\partial \mathcal{L}}{\partial B_e} = P_2 B_e + P_{12}^T B = 0 \quad (3.18)$$

$$\frac{\partial \mathcal{L}}{\partial K_e} = (P_2 Q_{12}^T + P_{12}^T Q_1)C^T = 0 \quad (3.19)$$

$$\frac{\partial \mathcal{L}}{\partial \hat{C}} = -(\hat{L} - \hat{D}_{e2}\hat{C})Q_{12} + \hat{C}_e Q_2 = 0 \quad (3.20)$$

$$\frac{\partial \mathcal{L}}{\partial \hat{D}_{e2}} = -(\hat{L} - \hat{D}_{e2}\hat{C})Q_1\hat{C}^T + \hat{C}_e Q_{12}^T\hat{C}^T = 0 \quad (3.21)$$

Expanding (2.12) and (3.16), we get

$$AQ_1 + Q_1A^T + BB^T = 0 \quad (3.22)$$

$$AQ_{12} + Q_{12}A_e^T + Q_1C^TK_e^T + BB_e^T = 0 \quad (3.23)$$

$$A_eQ_2 + K_eCQ_{12} + Q_2A_e^T + Q_{12}^TC^TK_e^T + B_eB_e^T = 0 \quad (3.24)$$

$$\begin{aligned}&P_1A + P_{12}K_eC + A^TP_1 + Q_2A_e^T + C^TK_e^TP_{12}^T \\ &+ (\hat{L} - \hat{D}_{e2}\hat{C})^T(\hat{L} - \hat{D}_{e2}\hat{C}) = 0 \end{aligned} \quad (3.25)$$

$$\begin{aligned}&P_{12}A_e + A^TP_{12} + Q_2A_e^T + C^TK_e^TP_2 \\ &+ (\hat{L} - \hat{D}_{e2}\hat{C})^T\hat{C}_e = 0 \end{aligned} \quad (3.26)$$

$$P_2A_e + A_e^TP_2 + \hat{C}_e^T\hat{C}_e = 0 \quad (3.27)$$

It follows from (3.27) and from the minimality of (2.3)-(2.4) that $P_2 > 0$ and we assume that $Q_2 > 0$. We thus define the following matrices:

$$\hat{Q} = Q_{12}Q_2^{-1}Q_{12}^T, \ Q = Q_1 - \hat{Q},$$

$$\hat{P} = P_{12}P_2^{-1}P_{12}^T,\ P = P_1 - \hat{P},$$

$$G^T = Q_{12}Q_2^{-1},\ \text{and}\ \Gamma = -P_2^{-1}P_{12}^T \qquad (3.28\text{a}-\text{f})$$

(3.2) derives from (3.18). We obtain from (3.17) that $\Gamma G^T = I_{n_e}$, thus (3.10) follows and the projection matrix τ is given by (3.13). (3.20) along with (3.28a-f) leads to an expression for $\hat{C}_e$:

$$\hat{C}_e = (\hat{L} - \hat{D}_{e2}\hat{C})G^T$$

(3.4),(3.6) and (3.12) are obtained by substituting that expression in (3.21) and assuming $\hat{C}Q\hat{C}^T > 0$. Substituting (3.6) in (2.11) yields (3.5). (3.19) leads to

$$CQ\Gamma^T = 0 \qquad (3.29)$$

Using (3.29), we compute $Q_2^{-1}\{(3.23)-(3.24)\}\Gamma^T$ and get

$$A_e = \Gamma AG^T - K_eCG^T \qquad (3.30)$$

Substitution of (3.4),(3.29) and (3.30) into $CQ(3.26)P_2^{-1}$, using the fact that $G^TP_2^{-1} = G^T(GP\hat{P}G^T)^{-1}$, yields K_e. The expression for A_e is obtained by substituting (3.3) into (3.30). (3.7) follows from
$G^T(3.24)G$+(3.22)-(3.23)G-$G^T(3.23)^T$.
(3.23)G+G^T(3.23)-τ(3.23)G yields (3.8). (3.9) is obtained from
$\Gamma^T(3.27)\Gamma+\Gamma^T(3.26)^T+(3.26)\Gamma$ using the identity:
$\hat{P}G^T(G\hat{P}G^T)^{-1}G = \tau$.

□

It should be noted that from the definition of ν, Q-equation (3.7) contains quadratic terms in Q and then (3.7) has the structure of a Riccati-like equation. Hence, the set of equations to be solved consists of a Riccati-like equation and two Lyapunov-like equations, all coupled. This is typical to L_2 order reduction problems [7, 3] that always results in a set of Riccati and Lyapunov equations.

From equations (3.11)÷(3.13) it is seen that τ, ν and $\hat{\nu}$ are idempotent matrices satisfying:

$$\tau^2 = \tau \ ,\ \nu^2 = \nu \ ,\ \hat{\nu}^2 = \hat{\nu} \qquad (3.31)$$

i.e. they are projection matrices.

τ is the optimal order reduction projection which appears throughout the optimal order reduction literature ,e.g. [7]. ν is the projection due to the singularity of the estimation problem [5]. $\hat{\nu}$ is a new projection arising from the definition of the auxiliary system. The following relation derived directly from the proof of the theorem,(3.29), show the disjointness of two of the projections:

$$\tau\nu = 0 \qquad (3.32)$$

It may be also noted that the reduced order observer obtained, (3.1)-(3.6), has the familiar structure of a 'model' part and a correcting term:

$$\dot{x}_e(t) = \Gamma AG^Tx_e(t) + \Gamma Bu(t) + K_e(y(t) - CG^Tx_e(t)), \qquad (3.33)$$

Here the 'model' part is replaced by its restriction to the range subspace of τ.

4. SUMMARY AND CONCLUSIONS

The problem of an optimal fixed order observer preserving the steady state gain of the system was considered and solved in this paper. The observer is given explicitly in terms of the nonnegative definite matrix Q and the three projections τ, ν and $\hat{\nu}$. Extensions of that problem, which are under work, are the design of a reduced order observer for systems with deterministic inputs which are not restricted to be step functions, and for systems with stochastic disturbances in addition to the step deterministic input.

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