Robust Steady-State Tracking in the Presence of Time-Varying Uncertainty*

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Abstract

This paper addresses the robust performance problem when the performance measure is the "steady-state" value of an error signal. Necessary and sufficient conditions are derived for robust steady-state tracking of fixed (possibly unbounded) inputs in the presence of *structured* time-varying uncertainty are derived. These conditions are easily computable and fit well with existing conditions on stability robustness and performance robustness when the performance measure is the level of disturbance rejection. Using these conditions, it is shown that time-varying perturbations of a nominal LTI plant can result in large steady-state tracking errors to fixed inputs even if the nominal plant and controller give zero steady-state tracking errors. The derived expressions for the worst-case steady-state tracking error give insight into how the time-variation in the plant affect tracking errors and suggest that certain transfer function norms should be minimized to reduce the effect of these perturbations on the steady-state value of error signals.

Keywords: Steady-state Tracking, Robust Assymptotic Tracking, Stability Robustness, Finite Memory Perturbations, Fading Memory Perturbations, Structured Uncertainty.

1 Introduction

The robust control literature contains various results on the robust stability and performance of systems under differing assumptions on the uncertainty. Different signal norms are also considered by various authors such as the L_2 signal norm which gives rise to $H_\infty/\mu/k_m$ theory, or the L_∞ signal norm which gives rise to L_1 control methodology and the associated robustness tools. In both of these design methodologies, when robust performance is considered, the performance measure is almost always taken to be some induced system transfer function norm. This corresponds directly to the problem of disturbance attenuation when a *class* of input signals, or disturbances, are assumed to enter the system. Although this is a useful performance measure, it is by no means the only one of interest. Other widely used performance measures whose robustness properties are of great importance from a practical viewpoint include such time-domain measures as steady-state tracking errors of fixed inputs, rise-time, overshoot, undershoot, etc. Unfortunately, the robustness of these "classical" performance measures to plant uncertainty is not widely studied, and more work remains to be done.

This paper addresses the problem of steady-state tracking of fixed inputs in the presence of linear norm-bounded structured perturbations. Steady-state tracking, or asymptotic tracking and regulation have been addressed in the literature by various authors since the 1970's. See [1, 2, 4, 5, 6, 11, 12] and the references therein. Given a linear time-invariant plant, the Internal Model Principle provides (when possible) a way for obtaining controllers which achieve zero steady-state tracking errors. However, in the presence of time variation in the plant, it can be shown that the steady-state error may no longer be zero, and can in fact be quite large. The objective of this work is to provide tools for the analysis of steady-state errors of linear time-invariant systems in the presence of time-varying perturbations. First, we given nonconservative conditions for stability robustness of systems in the presence of structured norm-bounded finite/fading memory perturbations. Next, computable necessary

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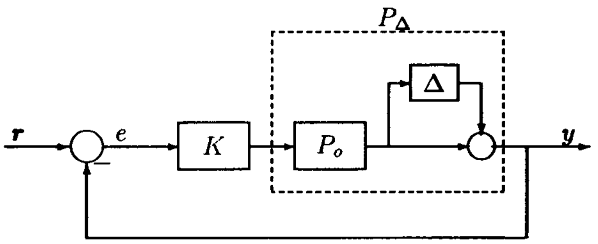

Figure 1: Single-loop Tracking Problem

and sufficient conditions for robust tracking are given. These conditions require the computation of a certain hybrid matrix containing both the ℓ_1 norm of certain subsystems and the steady-state semi-norm of certain signals in the system.

2 Notation

We use Z^+ to denote the nonnegative integers. ℓ_∞ is the space of sequences $x = \{x(k)\}_{k=0}^\infty$ such that $\|x\|_\infty := \sup_k |x(k)| < \infty$. c_0 is the subspace of ℓ_∞ of sequences x satisfying $x(k) \to 0$ as $k \to \infty$. We define the truncation operator P_k as follows:

$$P_k y = x, \qquad x(t) = \begin{cases} y(t) & t \le k \\ 0 & \text{otherwise} \end{cases}$$

We define the "tail" operator L_k as follows:

$$L_k y = x, \qquad x(t) = \begin{cases} y(t) & t > k \\ 0 & \text{otherwise} \end{cases}$$

LTI operators on ℓ_∞ will be viewed as elements of ℓ_1, the space of sequences $\{M(k)\}_{k=0}^\infty$ such that $\sum_{k=0}^\infty |M(k)| < \infty$. Every element M in ℓ_1 defines an LTI operator on ℓ_∞ which acts by convolution, i.e. $y = M * x$ where $x,\ y \in \ell_\infty$. For the remainder of the paper, we shall drop the $*$ and simply write $y = Mx$. In this case, the induced operator norm will be equal to $\|M\|_1$.

3 Motivation

In this section, we provide the motivation for investigating the robust steady-state tracking problem. Consider the system in Figure 1. P_o is a linear time-invariant plant, while K is a linear time-invariant stabilizing controller. Δ is a causal norm-bounded uncertainty which belongs to a certain class of admissible perturbations, say $\underline{\Delta}$. The exact characterization of the class $\underline{\Delta}$ will be discussed later. r is a fixed reference input to be tracked by the output y, and e is the tracking error. The objective is to make the error e as small as possible in the "steady-state". One possible way one might pose the robust tracking question is as follows: What is the worst case "steady-state value of the error" as Δ is varied over the class of admissible uncertainty $\underline{\Delta}$? The most common measure of the steady-state value of e is given by $\lim_{k\to\infty} |e(k)|$. This is both reasonable and practical, but the limit may not always exist. In fact, this is the case in many important cases (e.g. $e(k) = \sin(\theta k)$). We will generalize this definition to include the cases when the limit does not exist by defining *the steady-state semi-norm of* e as follows:

$$\|e\|_{SS} := \lim_{k\to\infty} \sup |e(k)|.$$

Note that when $\lim_{k\to\infty} |e(k)|$ exists, it will be equal to $\|e\|_{SS}$. It should be emphasized that $\|.\|_{SS}$ is a semi-norm on the space ℓ_∞. On the other hand, $\|.\|_{SS}$ *does* define a norm on the space ℓ_∞ / c_0 (ℓ_∞ modulo c_0). The definition of $\|.\|_{SS}$ can be extended to the space ℓ_∞^p as follows: $\|e\|_{SS} = \max_i \|e_i\|_{SS}$. With this definition, $\|e\|_{SS}$ is a measure of the *maximum persistent peak* of the signal e. One consequence of this definition is that $\|e\|_{SS} \le \|e\|_\infty$ for all $e \in \ell_\infty$.

Using the $\|.\|_{SS}$ definition, our tracking problem becomes that of determining the quantity $\sup_{\Delta\in\underline{\Delta}} \|e\|_{SS}$ for a given robustly stabilizing controller.

The next question to be addressed relates to the class of perturbations to be considered. The class of perturbations commonly used is the class of norm-bounded uncertainty, i.e.

$$\underline{\Delta} := \{\Delta : \ell_\infty \to \ell_\infty :\ \Delta \text{ is linear, causal and } \|\Delta\| \le 1\}$$

where $\|\Delta\|$ is the induced operator norm. For this class of perturbations the stability and performance robustness problems have been completely resolved [3, 8, 9], and computable necessary and sufficient conditions for robustness exist. However, for our purposes this class of perturbations is "too large" in some sense. More precisely, a perturbation $\Delta \in \underline{\Delta}$ can have *infinite* memory. For example suppose Δ has the following matrix representation:

$$\Delta = \begin{pmatrix} 1 & & & \\ 1 & 0 & & 0 \\ 1 & 0 & 0 & \\ \vdots & & & \ddots \end{pmatrix}.$$

It is easy to see that Δ is admissible. But Δ maps $\{1,0,0,\ldots\}$ to the the sequence $\{1,1,1,\ldots\}$, and the effect of the input at time $k = 0$ affects the output for all time. Clearly, this is not a very realistic model for uncertainty. A much more reasonable model for uncertainty is the class of perturbations which have fading or finite memory. See [10] and the references therein. We will call a bounded linear operator Δ a *fading-memory* operator if Δ maps c_0 into c_0. Thus inputs which go to zero are mapped to outputs which also go to zero. Similarly we will call a bounded linear operator $\Delta : \ell_\infty \to \ell_\infty$ a finite memory operator if Δ maps finite sequences into finite sequences. Let $\underline{\Delta}_{FD}$ denote the subset of $\underline{\Delta}$ of fading memory operators, and let $\underline{\Delta}_F$ denote the set of finite memory operators. Clearly $\underline{\Delta}_F \subset \underline{\Delta}_{FD} \subset \underline{\Delta}$. For the rest of the paper we shall take $\underline{\Delta}_F$ to be the class of uncertainty. All the results obtained will apply equally well to the class $\underline{\Delta}_{FD}$.

4 Robustness Against Finite Memory Perturbations

Before the question of robust steady-state tracking in the presence of finite memory perturbations can be addressed, the stability robustness against this class of uncertainty must be addressed first. As mentioned earlier, stability and performance robustness conditions when the class of perturbations is $\underline{\Delta}$ are known (see [3, 8, 9]). Since $\underline{\Delta}_F \subset \underline{\Delta}$, these conditions remain sufficient when the perturbations are restricted to the class $\underline{\Delta}_F$. It is an interesting fact that these same conditions remain necessary when the class of perturbations is $\underline{\Delta}_F$. This is the first main result of the paper, and will be addressed next.

The standard robust stability problem in the presence of structured uncertainty can be stated with the aid of Figure 2. In the figure, $\mathbf{M}$ is a linear time-invariant stable system with n inputs and n outputs. It represents the nominal part of the system and is typically composed of the LTI plant(s) and stabilizing LTI controller(s). Δ represents the system uncertainty and will be assumed to belong to the class:

$$\mathcal{D}_F(n) := \{diag(\Delta_1, \ldots, \Delta_n) : \Delta_i \in \underline{\Delta}_F.\}$$

So $\mathcal{D}_F(n)$ is the class of *structured* norm-bounded finite memory perturbations.

Definition 1 (Robust Stability) *The linear time invariant system* $\mathbf{M}$ *in figure 2 is is said to be robustly stable (against finite-memory perturbations) if* $\forall \Delta \in \mathcal{D}_F(n)$, *the map* $(r_1, r_2) \mapsto (e_1, e_2, y_1, y_2)$ *takes bounded sequences to bounded sequences and has a bounded induced norm.*

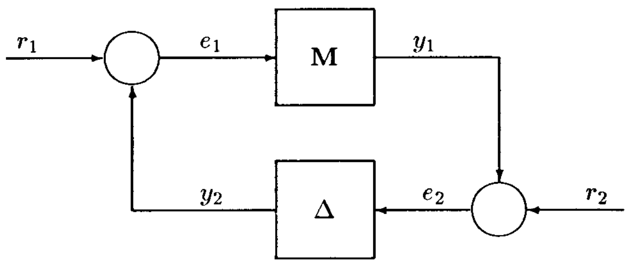

Figure 2: Robust Stability Problem

Equivalently, $\mathbf{M}$ is robustly stable if $(I - \mathbf{M}\Delta)^{-1}$ is ℓ_∞-stable for all $\Delta \in \mathcal{D}_F(n)$.

Define $\widehat{\mathbf{M}}$ as the following $n \times n$ nonnegative matrix:

$$\widehat{\mathbf{M}} := \begin{pmatrix} \|M_{11}\|_1 & \ldots & \|M_{1n}\|_1 \\ \vdots & & \vdots \\ \|M_{n1}\|_1 & \ldots & \|M_{nn}\|_1 \end{pmatrix}.$$

We now give the following simple condition for robust stability:

Theorem 1 (Robust Stability)
$\mathbf{M}$ *is robustly stable (against finite-memory perturbations) if and only if* $\rho(\widehat{\mathbf{M}}) < 1$, *where* $\rho(.)$ *is the spectral radius.*

Proof. The proof will be omitted.

Remark: The above theorem should be viewed in connection with a related theorem given in [9] where it was shown that $\rho(\widehat{\mathbf{M}}) < 1$ is necessary and sufficient for robust stability when the perturbation class is

$$\mathcal{D}(n) := \{diag(\Delta_1, \ldots, \Delta_n) : \Delta_i \in \underline{\Delta}\}.$$

It can be shown that when performance is measured by the level of ℓ_∞ disturbance attenuation, robust performance against finite memory perturbations can be reduced to robust stability, and thus robust performance conditions can be computable. Again this "equivalence" among the two notions of robustness has been shown to hold [8] when the perturbation class is $\mathcal{D}(n)$.

5 Steady-State Tracking

The tracking problem motivated in section 1 is a special case of a more general class of tracking problems

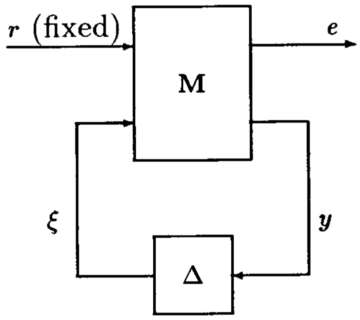

Figure 3: Robust Steady-State Tracking Problem

which we consider next. Consider the system in Figure 3 which represents the general steady-state tracking problem.

In the figure, $\mathbf{M}$ is a linear time-invariant stable system representing the nominal part of the system. Δ represents the system uncertainty and will be assumed to belong to the class:

$$\mathcal{D}_F(n) := \{diag(\Delta_1, \ldots, \Delta_n) : \Delta_i \in \underline{\Delta}_F.\}$$

r is a fixed reference input, while e is the output whose steady-state value is of interest (typically the tracking error). $\mathbf{M}$ is $(n+1) \times (n+1)$, and can be partitioned accordingly

$$\mathbf{M} = \begin{pmatrix} M_{11} & \ldots & M_{1,n+1} \\ \vdots & \ddots & \vdots \\ M_{n+1,1} & \ldots & M_{n+1,n+1} \end{pmatrix}$$

It is also convenient to partition $\mathbf{M}$ as follows:

$$\begin{pmatrix} e \\ y \end{pmatrix} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix} \begin{pmatrix} r \\ \xi \end{pmatrix}.$$

In this general setting, we give the definition for robust steady-state tracking:

Definition 2 (Robust Steady-State Tracking) *The linear time-invariant system* $\mathbf{M}$ *in Figure 3 is said to achieve robust steady-state tracking if*

1. *The interconnection of* $\mathbf{M}$ *and* Δ *is* ℓ_∞*-stable for all* $\Delta \in \mathcal{D}_F(n)$ *(Robust Stability).*

2. $\sup_{\Delta \in \mathcal{D}_F(n)} \|e\|_{SS} < 1.$

Remark: Note that robust steady state tracking does not necessarily imply that the tracking error will always go to zero. Rather it implies that the steady-state semi-norm of the error will be less than a certain prescribed value (here normalized to one) for all admissible perturbations.

In what follows, we will *not* restrict r to be bounded. This allows us to consider such fixed inputs as ramps, parabolic functions, or any signal which is polynomial in the time-variable. For the remainder of this section, however, we will impose the requirement that

$$\|M_{i,1}r\|_{SS} < \infty \qquad i = 1, \ldots, n+1.$$

The case when some of these quantities are infinite is addressed in the next section.

The results that follow will be based on a fundamental $(n+1) \times (n+1)$ nonnegative matrix derived from the problem data. We will refer to this matrix as the *Steady-State Norm Matrix*. It is defined as follows:

$$M_{SS} :=$$
$$\begin{pmatrix} \|M_{11}r\|_{SS} & \|M_{12}\|_1 & \ldots & \|M_{1,n+1}\|_1 \\ \|M_{21}r\|_{SS} & \|M_{22}\|_1 & \ldots & \|M_{2,n+1}\|_1 \\ \vdots & \vdots & \vdots & \vdots \\ \|M_{n+1,1}r\|_{SS} & \|M_{n+1,2}\|_1 & \ldots & \|M_{n+1,n+1}\|_1 \end{pmatrix}.$$

In the sequel we shall provide necessary and sufficient conditions for robust steady-state tracking in the terms of the steady-state norm matrix defined above. In proving these results, we will refer to the following two lemmas:

Lemma 1 *Let* $H : \ell_\infty \rightarrow \ell_\infty$ *be any bounded linear fading memory operator. Let* $x \in \ell_\infty$*. Then* $\|Hx\|_{SS} \leq \|H\| \, \|x\|_{SS}$*, where* $\|H\|$ *is the induced* ℓ_∞ *operator norm.*

Proof. For any integer n, $x = L_n x + P_n x$. Now for any m

$$\begin{aligned} \|L_m H x\|_\infty &= \|L_m H L_n x + L_m H P_n x\|_\infty \\ &\leq \|L_m H L_n x\|_\infty + \|L_m H P_n x\|_\infty \\ &\leq \|H\| \, \|L_n x\|_\infty + \|L_m H P_n x\|_\infty \end{aligned}$$

Taking the limit of both sides, first as m goes to infinity and then as n goes to infinity we have $\|Hx\|_{SS} \leq \|H\| \, \|x\|_{SS}$. ∎

The next lemma is related to the properties of square nonnegative matrices. Although its proof is provided elsewhere (e.g. [9]), we will provide a short proof here for convenience.

Lemma 2 *Let* A *be a square nonnegative matrix (i.e.* $a_{ij} \geq 0$*). Then* $\rho(A) < 1$ *if and only if* $x \geq 0$ *and* $x \leq Ax$ *imply* $x = 0$*, where the inequalities* $\leq$ *and* $\geq$ *are taken componentwise.*

Proof. To prove sufficiency, suppose $\rho(A) \geq 1$. By Perron-Frobenius theory for nonnegative mateices (see [7]) $\rho(A)$ is an eigenvalue of A and has a corresponding nonnegative eigenvector, say $\bar{x}$. Thus $A\bar{x} = \rho(A)\bar{x} \geq \bar{x}$. This proves sufficiency.
To show necessity, suppose $\rho(A) < 1$. This implies that $(I - A)^{-1} = I + A + A^2 + \ldots$ exists and is nonnegative. Now suppose $x \geq 0$ and $x \leq Ax$. This implies $(I - A)x \leq 0$. Multiplying this last inequality by $(I - A)^{-1}$ we get that $x \leq 0$. Thus $x = 0$ and the necessity is proved. ∎

We are now prepared to provide sufficient conditions for robust tracking.

Theorem 2 *If $\rho(M_{SS}) < 1$, then $\mathbf{M}$ is robustly stable and $\|e\|_{SS} < 1$ for all $\Delta \in \mathcal{D}_F(n)$.*

Proof. Applying Lemma 2 above, we can see that $\rho(M_{SS}) < 1$ implies $\rho(\widehat{\mathbf{M}}_{22}) < 1$ which is necessary and sufficient for robust stability. Now suppose $\|e\|_{SS} \geq 1$ for some $\Delta \in \mathcal{D}_F(n)$. For such a Δ, define

$$\xi := \Delta(I - \mathbf{M}_{22}\Delta)^{-1}\mathbf{M}_{21}r.$$

Now e is given by

$$e = \mathbf{M}_{11}r + \mathbf{M}_{12}\xi$$

Since $\|\cdot\|_{SS}$ satisfies the triangle inequality, we have

$$\begin{aligned} 1 &\leq \|e\|_{SS} \\ &\leq \|M_{11}r\|_{SS} + \|M_{12}\|_1 \|\xi_1\|_{SS} \\ &\quad + \ldots + \|M_{1,n+1}\|_1 \|\xi_n\|_{SS}. \end{aligned} \tag{1}$$

Now, define

$$y := (I - \mathbf{M}_{22}\Delta)^{-1}\mathbf{M}_{21}r.$$

Clearly

$$\begin{aligned} y &= \mathbf{M}_{22}\Delta y + \mathbf{M}_{21}r \\ &= \mathbf{M}_{22}\xi + \mathbf{M}_{21}r. \end{aligned}$$

Using the fact that $\|\Delta\| \leq 1$, and applying the triangle inequality we have

$$\begin{aligned} \|\xi_i\|_{SS} &\leq \|y_i\|_{SS} \\ &\leq \|M_{i+1,1}r\|_{SS} + \|M_{i+1,2}\|_1 \|\xi_1\|_{SS} \\ &\quad + \ldots + \|M_{i+1,n+1}\|_1 \|\xi_n\|_{SS}. \end{aligned} \tag{2}$$

Equations (1) and (2) together imply that $(1, \|\xi_i\|_{SS}, \ldots, \|\xi_n\|_{SS})$ is a solution to the system of inequalities $x \leq M_{SS}x$. By Lemma 2, this is equivalent to $\rho(M_{SS}) \geq 1$. The proof is thus complete. ∎

The next lemma shows the effect of adding an input in c_0 on the value of the steady-state tracking seminorm (see Figure 4).

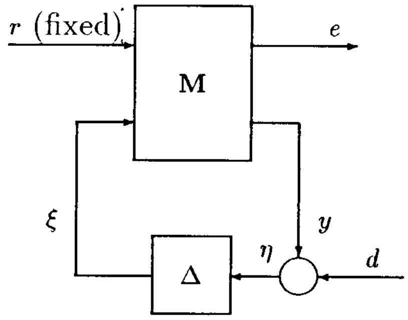

Figure 4: Auxiliary System used in Lemma 3.

Lemma 3 *Suppose the interconnection in Figure 4 is stable for all $\Delta \in \mathcal{D}_F(n)$. Then for any $\Delta \in \mathcal{D}_F(n)$, $\|e\|_{SS}$ remains unchanged for any $d \in c_0^n$.*

Proof.
We only need to show $\left\|\Delta(I - \mathbf{M}_{22}\Delta)^{-1}d\right\|_{SS} = 0$ for $d \in c_0^n$. Because Δ is finite-memory, this will be the case if $\left\|(I - \mathbf{M}_{22}\Delta)^{-1}d\right\|_{SS} = 0$.

Let $\tilde{\eta} := (I - \mathbf{M}_{22}\Delta)^{-1}d$. Then $\tilde{\eta} = \mathbf{M}_{22}\Delta\tilde{\eta} + d$.
Now

$$\begin{aligned} \|\tilde{\eta}_i\|_{SS} &= \|(\mathbf{M}_{22})_i\Delta\tilde{\eta} + d_i\|_{SS} \\ &= \|(\mathbf{M}_{22})_i\Delta\tilde{\eta}\|_{SS} \\ &= \|(\mathbf{M}_{22})_{i1}\Delta_1\tilde{\eta}_1 + \ldots + (\mathbf{M}_{22})_{in}\Delta_n\tilde{\eta}_n\|_{SS} \\ &\leq \|(\mathbf{M}_{22})_{i1}\|_1 \|\tilde{\eta}_1\|_{SS} \\ &\quad + \ldots + \|(\mathbf{M}_{22})_{in}\|_1 \|\tilde{\eta}_n\|_{SS}. \end{aligned}$$

Therefore, $x := (\|\tilde{\eta}_1\|_{SS}, \ldots, \|\tilde{\eta}_n\|_{SS})$ solves the system of inequalities $x \leq \widehat{\mathbf{M}}_{22}x$. But by robust stability, $\rho(\widehat{\mathbf{M}}_{22}) < 1$. This, together with $x \leq \widehat{\mathbf{M}}_{22}x$ imply (by Lemma 2) that $x = 0$. This completes the proof. ∎

Lemma 4 *Given any two sequences of real numbers η and ξ. There exists $\Delta \in \underline{\Delta}_F$ satisfying $\Delta\eta = \xi$ if and only if*

1. $\|P_k\xi\|_\infty \leq \|P_k\eta\|_\infty \quad \forall k.$
2. *For any $m \in Z^+$, there exists $\tilde{m} \in Z^+$ such that*

$$\|P_k L_{\tilde{m}}\xi\|_\infty \leq \|P_k L_m \eta\|_\infty \qquad \forall k.$$

Proof. The proof of this lemma will be omitted here.

Theorem 3 *Suppose $\mathbf{M}$ is robustly stable and that $\|e\|_{SS} < 1$, for all $\Delta \in \mathcal{D}_F(n)$. Then, $\rho(M_{SS}) < 1$.*

Proof. Suppose $\rho(M_{SS}) \geq 1$. It follows by Lemma 2 that the system of inequalities:

$$x \leq M_{SS}x \tag{3}$$

has a nonzero solution $x \geq 0$. If $x_1 = 0$ then the system of inequalities $y \leq \widehat{\mathbf{M}}_{22}y$ has a nonzero solution $y \geq 0$, which is equivalent to $\rho(\widehat{\mathbf{M}}_{22}) \geq 1$. Hence the system will not be robustly stable. If on the other hand $x_1 \neq 0$, we will show that for some admissible perturbation, Δ, $\|e\|_{SS} \geq 1$, and the proof would be complete. Since the solutions of the system of inequalities $x \leq M_{SS}x$ form a cone, we may without loss of generality take $x_1 = 1$. Given $\Delta \in \mathcal{D}_F(n)$, e is defined by

$$\begin{aligned} \begin{pmatrix} e \\ y \end{pmatrix} &= \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix} \begin{pmatrix} r \\ \xi \end{pmatrix} & (4) \\ \xi &= \Delta y & (5) \end{aligned}$$

These equations have unique solutions y and ξ. While replacing equation 5 by

$$\xi = \Delta(y + d) \tag{6}$$

for a given $d \in c_o^n$ will in general change the resulting e, according to lemma 3 the steady-state value of e will be unchanged. With this in mind, we proceed by constructing ξ, $\Delta \in \mathcal{D}_F(n)$, and $d \in c_o^n$ which satisfy equations (4) and (6), and which result in e satisfying $\|e\|_{SS} \geq 1$. This construction is described next.

The idea in the construction is to utilize the solution of the system of inequalities: $x \leq M_{SS}x$ in order to choose ξ. The choice of ξ will be such that

1. When $\mathbf{M}_{12}\xi$ is added to $\mathbf{M}_{11}r$ the resulting signal, e, has the largest possible steady-state value, which we show to be larger than 1.

2. When the signal $\mathbf{M}_{22}\xi$ is added to $\mathbf{M}_{12}r$, the resulting signal, y, will have the largest possible steady-state value, and will be such that it can be (modulo a signal $d \in c_o^n$) mapped back to ξ by an addmissible Δ.

Here are the details. Given a sequence of positive real numbers $E = \{\epsilon_1, \epsilon_2, \ldots\} \in c_o$, a vector signal ξ can be chosen such that:

$$|\xi_i(k)| = x_{i+1}, \qquad \forall k \tag{7}$$

and for some positive integers $N_0 < N_1 < N_2 < \ldots$ (which depend on E)

$$\begin{aligned} |e(N_0)| &\geq x_1 - \epsilon_1, \quad |y_1(N_1)| \geq x_2 - \epsilon_1 \\ & \qquad \ldots |y_n(N_n)| \geq x_{n+1} - \epsilon_1 & (8) \\ |e(N_{n+1})| &\geq x_1 - \epsilon_2, \quad |y_1(N_{n+2})| \geq x_2 - \epsilon_2 \\ & \qquad \ldots |y_n(N_{2n+1})| \geq x_{n+1} - \epsilon_2 & (9) \\ & \quad \vdots \end{aligned}$$

More specifically, one can choose a suffiently large integer $N_0 > 0$ and then specify $\xi(k)$ for $0 \leq k \leq N_0$ in such a way that $|\xi_i(k)| = x_{i+1}$ *and*

$$\begin{aligned} |e(N_0)| &= |(M_{11}r + M_{12}\xi_1 + \ldots + M_{1,n+1}\xi_n)(N_0)| \\ &\geq \|M_{11}r\|_{SS} + \|M_{12}\|_1 x_2 \\ &\quad + \ldots + \|M_{1,n+1}\|_1 x_{n+1} - \epsilon_1 \end{aligned}$$

If follows that $|e(N_0)| \geq 1 - \epsilon_1$, since by the first inequality in (3)

$$\begin{aligned} x_1 &= 1 \\ &\leq \|M_{11}r\|_{SS} + \|M_{12}\|_1 x_2 + \ldots + \|M_{1,n+1}\|_1 x_{n+1}. \end{aligned}$$

Next, one can choose $N_1 > N_0$, and $\xi(k)$, for $N_1 + 1 \leq k \leq N_2$ such that $|\xi_i(k)| = x_{i+1}$ and

$$\begin{aligned} |y_1(N_1)| &= |(M_{21}r + M_{22}\xi_1 + \ldots + M_{2,n+1}\xi_n)(N_1)| \\ &\quad -\epsilon_1 \\ &\geq \|M_{21}r\|_{SS} + \|M_{22}\|_1 x_2 \\ &\quad + \ldots + \|M_{2,n+1}\|_1 x_{n+1} - \epsilon_1 \end{aligned}$$

Invoking the second inequality in (3), we establish that $|y_1(N_1)| \geq x_2 - \epsilon_1$, and so on.

Proceeding in this manner, ξ can be constucted so that all the constraints in (8) are satisfied. In the same way, ξ can be further extended so that the constraints in (9) are satisfied, and so on.

Now we can construct $d \in c_0^n$ by specifying its ith component:

$$d_i(k) := \begin{cases} \|\xi_i\|_\infty sgn(y_i(0)) & k = 0 \\ \epsilon_1 \; sgn(y_i(k)) & 1 \leq k \leq N_n \\ \epsilon_2 \; sgn(y_i(k)) & N_n + 1 \leq k \leq N_{2n+1} \\ \vdots & \vdots \end{cases}$$

This d was constructed so that

$$\begin{gathered} \|P_k\xi_i\|_\infty \leq \|P_k(y + d)\|_\infty \qquad \forall k \text{ and} \\ \forall m \in Z^+, \exists \tilde{m} \in Z^+ \text{ such that} \\ \|P_k L_{\tilde{m}}\xi_i\|_\infty \leq \|P_k L_m(y_i + d_i)\|_\infty \qquad \forall k \end{gathered}$$

By Lemma 4, there exists $\Delta \in \mathcal{D}_F(n)$ such that $\xi = \Delta(y+d)$. This completes the proof. ■

Remark: Note that if any of the entries $\|M_{i1}r\|_{SS}$ in M_{SS} is replaced by a smaller number to form the matrix M'_{SS}, then from the proof it is seen that $\rho(M'_{SS}) \geq 1$ implies the existence of a destabilizing perturbation. This fact will be used later in treating the case when some of the entries of M_{SS} are infinite.

Combining the last two theorems we have the main result of the section:

Corollary 1 *$\rho(M_{SS}) < 1$ is both necessary and sufficient for robust steady-state tracking.*

6 Robust Tracking Conditions in the Case of Unbounded Signals

We have previously made the assumption that the entries of M_{SS} are finite. It is possible, however, that this condition is not met. This would be the case when one or more of the terms $\|M_{i1}r\|_{SS}$ are infinite due to the application of an unbounded reference input r. In this case, will provide a condition for robust tracking which using concepts from graph theory. See [7].

We begin by defining the following objects.

Definition 3 *Let A be a $n \times n$ nonnegative matrix (with ∞ an allowable entry). The directed graph of A, denoted by $\Gamma(A)$, is the directed graph on n nodes $P_1, P_2, \ldots, P_n$ such that there is a directed arc in $\Gamma(A)$ from P_i to P_j if and only if $A_{ij} \neq 0$.*

Definition 4 *A directed path in a graph Γ is a sequence of arcs $P_{i_1} \rightarrow P_{i_2}$, $P_{i_2} \rightarrow P_{i_3}$, $\ldots P_{i_{m-1}} \rightarrow P_{i_m}$. The number of successive arcs in the directed path is the length of the directed path.*

Example 1 *The graph of the matrix A defined by*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \\ \infty & 0.2 & 0.5 & 1 \\ 0 & 0 & 0 & 0.7 \end{pmatrix}$$

is shown in Fig 5.

Definition 5 *Let $\Gamma(A)$ be the directed graph of A. The weight of a directed path $P_{i_1} \rightarrow P_{i_2}$, $P_{i_2} \rightarrow P_{i_3}$, $\ldots P_{i_{m-1}} \rightarrow P_{i_m}$ in Γ is the product $A_{i_1 i_2} \cdot A_{i_2 i_3} \cdots A_{i_{m-1} i_m}$.*

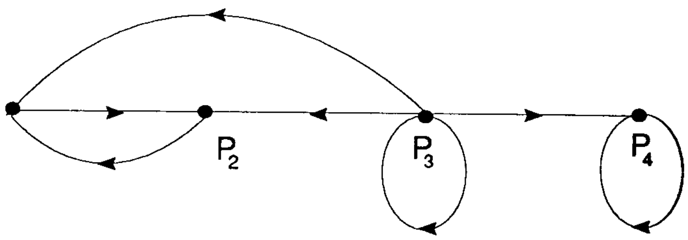

Figure 5: Example Graph

The following lemma relates the entries of A^m to directed paths of length m in the directed graph of A. It is a small modification of a theorem which appears in [7] (see pg. 358).

Lemma 5 *Given an $n \times n$ nonnegative matrix A. Let $\Gamma(A)$ be the directed graph of A with associated nodes $P_1, \ldots, P_n$. There exists a directed path of length m from node i to node j if and only if $(A^m)_{ij} \neq 0$, in which case $(A^m)_{ij}$ is the sum of the weights of all the directed paths of length m from node i to node j.*

Proof. The proof is by induction. For $m = 1$, the lemma is immediate. Suppose, the assertion holds for $m = q$. Now

$$(A^{q+1})_{ij} = \sum_{k=1}^{n} (A^q)_{ik} A_{kj} \neq 0$$

if and only if $(A^q)_{ik}$ and A_{kj} are both nonzero for some P_k. This means that there exists a directed path from node P_i to node P_k of length q and a path from node P_k to node P_j of length 1. This is equivalent to having a directed path from node P_i to node P_j of length $q+1$. Now, since $(A^q)_{ik}$ is the sum of the weights of all the directed paths of length q from node P_i to node P_k, then $(A^q)_{ik}A_{kj}$ is the sum of all the directed paths of length $q+1$ from node P_i to node P_j which pass through node k immediately before terminating in node j. Summing over all k we get the sum of the weights of all the directed paths of length $q+1$ from node P_i to node P_j. ■

We now return to the tracking problem. The following theorem characterizes the worst-case tracking error when some the steady-state value of some of the signals in the loop are infinite.

Theorem 4 *Consider the steady-state tracking problem for the system in Fig. 3. Let $\mathbf{M}$ be robustly stable. Suppose $\|M_{i_1 1}r\|_{SS}, \ldots, \|M_{i_p 1}r\|_{SS}$ all equal ∞. Let $\Gamma(M_{SS})$ be the directed graph associated with M_{SS}. Then*

1. *If there is a directed path from node P_1 to any of the nodes $P_{i_1}, \ldots, P_{i_p}$ then $\sup_\Delta \|e\|_{SS} = \infty$.*

2. *If, on the other hand, no directed path exists from node P_1 to any of the nodes $P_{i_1}, \ldots, P_{i_p}$, then $\sup_\Delta \|e\|_{SS} < 1$ if and only if $\rho(M'_{SS}) < 1$, where M'_{SS} is obtained from M_{SS} by replacing the ∞ entries with 0s.*

Proof. We start by proving the first assertion. Suppose that in $\Gamma(M_{SS})$, there is a directed path from node P_1 to P_q where $P_q \in \{P_{i_1}, \ldots, P_{i_p}\}$. Let $A(K)$ be the matrix obtained from M_{SS} by replacing $\|M_{i_1 1} r\|_{SS}, \ldots, \|M_{i_p 1} r\|_{SS}$ by $K \in \mathbb{R}$. We will show that $\lim_{K\to\infty} \rho(A(K)) = \infty$. This will imply that $\sup_\Delta \|e\|_{SS} = \infty$, since the remarks following the proof of Theorem 3 indicate that $\sup_\Delta \|e\|_{SS} \geq 1$ whenever $\rho(A(K)) \geq 1$.

For $K > 0$, there is a directed path from node P_1 to node P_q in the graph $\Gamma(A(K))$. It is easy to see that in this case, one can find a directed graph having length $m \leq n$ between these two nodes. Let $\alpha > 0$ be the weight of such a path. Since $A_{q1}(K) = K > 0$, it follows by definition that there is a directed path of length 1 from node P_q to node P_1, and hence there is a directed path from node P_1 to itself of length $m + 1$. The weight of such a path is equal to $K\alpha$. Applying lemma 5, we have $(A^{m+1})_{11} \geq K\alpha$. Now since the diagonal entries of a nonnegative matrix bound from below its spectral radius we obtain $\rho^{m+1}(A(K)) = \rho(A^{m+1}(K)) \geq K\alpha$. Taking the limit as $K \to \infty$, we have $\lim_{K\to\infty} A(K) = \infty$.

We now prove the second assertion. Suppose there is no directed path between P_1 and any of the nodes $P_{i_1}, \ldots, P_{i_p}$ in $\Gamma(M_{SS})$. We will show that the worst-case error will not change if we replace the system $\mathbf{M}$ by another system $\mathbf{M}'$ which is identical to $\mathbf{M}$ with the exception of certain entries which are specified as follows: $M'_{i_1,1} = \cdots = M'_{i_p,1} = 0$. Since M'_{SS} is finite, $\rho(M'_{SS})$ will determine the robust steady-state tracking properties of the system.

The relation between r and e is described by

$$e = M_{11} r + \mathbf{M}_{12}\Delta(I - \mathbf{M}_{22}\Delta)^{-1}(w_2 + \cdots + w_{n+1})$$

where $w_2 := \begin{pmatrix} M_{21} r \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, w_{n+1} := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ M_{n+1,1} r \end{pmatrix}$.

Let $m \in \{i_1, \ldots, i_p\}$. Clearly, $m \neq 1$: otherwise there would be a path between node P_1 and itself. The component of e which is due only to $M_{m,1} r$ is given by

$$e_{w_m} := \mathbf{M}_{12}\Delta(I - \mathbf{M}_{22}\Delta)^{-1} w_m$$

The proof will be complete by showing that $e_{w_m} = 0$, since this implies that $M_{m,1} r$ does not affect e at all.

Since $\|\Delta\| \leq 1$, it follows that $\forall k$:

$$\|P_k e_{w_m}\|_\infty \leq \begin{pmatrix} \|M_{12}\|_1 & \cdots & \|M_{1,n+1}\|_1 \end{pmatrix} \begin{pmatrix} \|P_k f_1\|_\infty \\ \vdots \\ \|P_k f_n\|_\infty \end{pmatrix}$$

where $f := (I - \mathbf{M}_{22}\Delta)^{-1} w_m$. If we define $S := (I - \widehat{\mathbf{M}}_{22})^{-1}$ (the inverse exists because $\rho(\widehat{\mathbf{M}}_{22}) < 1$ by robust stability), it is not difficult to show that

$$\|P_k f_i\|_\infty \leq S_{i,m-1} \|P_k M_{m1} r\|_\infty,$$

where we have used the fact that all but the $m-1$ entry of w_m are zero. So we now have

$$\|P_k e_{w_m}\|_\infty \leq \|P_k M_{m1} r\|_\infty \sum_{k=1}^{n} \|M_{1,i+1}\|_1 \, S_{i,m-1}.$$

Since

$$S = (I - \widehat{\mathbf{M}}_{22})^{-1} = I + \widehat{\mathbf{M}}_{22} + \widehat{\mathbf{M}}_{22}^2 + \cdots. \tag{10}$$

Note that $\widehat{\mathbf{M}}_{22}$ completely determines those paths in the graph $\Gamma(M_{SS})$ which do not begin or end in node P_1. Hence the (i,j) entry of $\widehat{\mathbf{M}}_{22}$ is nonzero if and only if there is a directed path from node P_{i+1} to node P_{j+1} which does not pass through the node P_1 in the graph $\Gamma(M_{SS})$. With this in mind, it follows from equation (10) above and from Lemma 5 that $S_{k,m-1} \neq 0$ if and only if either $i = m + 1$ or in the graph $\Gamma(M_{SS})$ there is a directed path (of any lenght) from node P_{k+1} to node P_m which does not pass through node P_1. Consequently, $\|M_{1,i+1}\|_1 S_{i,m-1} \neq 0$ only if there is a directed path from node P_1 to node P_m. Since such a path cannot exist by assumption, it follows that $\|M_{1,i+1}\|_1 S_{i,m-1} = 0$, for all i. Therefore $e_{w_m} = 0$ and the result follows. ■

Example 2 *Suppose for a given system $\mathbf{M}$, M_{ss} is given by*

$$M_{ss} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \\ \infty & 0.2 & 0.5 & 1 \\ 0 & 0 & 0 & 0.7 \end{pmatrix}.$$

The graph of this matrix appears in Fig. 5. Clearly, robust stability is achieved since $\rho(\widehat{\mathbf{M}}_{22}) < 1$. As there is no directed path from node P_1 to P_3, robust tracking is determined by

$$M'_{ss} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.2 & 0.5 & 1 \\ 0 & 0 & 0 & 0.7 \end{pmatrix}.$$

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