Robustness with respect to delays for stabilization of diffusion equations *

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When a system is stabilized by a feedback, it is of interest whether the stability is destroyed when small delays are introduced into the feedback loop. There are many examples in the partial differential equation literature of systems which are stabilized by feedback but destabilized by arbitrarily small delays - see Datko [3] for several such examples. There are also examples of systems with delays in the feedback loop for which there are no unstable modes as long as the delay is small enough (see Datko/Lagnese/Polis [2]). A mode is a solution of the form $e^{s_0 t}\phi$, where ϕ is a function of the space variable. The mode is called unstable if Re $s_0 \geq 0$, and stable otherwise. We say that a system is *modally stable* if it has no unstable modes, and we say that the modal stability of a feedback system is robust with respect to delays if the system is modally stable for all small enough delays in the feedback loop. In this paper we show that for a class of diffusion equations in R^n with boundary control, if a feedback modally stabilizes the system, then the modal stability is robust with respect to delays.

There are two types of results in the literature on robustness with respect to delays for distributed parameter systems. There are results about robustness of modal stability for specific partial differential equations with specific feedbacks - see [2], [3] for some of the earliest results of this type. More general systems can be studied with a frequency domain approach, where robustness of input-output stability is studied - see Barman/Callier/Desoer [1], Logemann/Rebarber/Weiss [8], Logemann/Rebarber [9], and Georgiou/Smith [4] for such results. The results in these papers are external in the sense that they are about boundedness (or analyticity) of transfer functions in the right half plane. In [9] these external results are used to obtain results about robustness of modal stability for a class of partial differential equations.

Most of the examples in both types of literature are for systems with one space variable in the partial differential equation. While systems-theoretic results for distributed parameter systems are rarely explicitly restricted to one space variable, most of the systems which have been shown to fit into an appropriate systems-theoretic framework are for one space variable. One of the purposes of this paper is to use systems-theoretic methods to obtain results about systems with space variable in R^n.

Let Ω be a bounded open domain in R^n with boundary Γ, assumed to be C^∞ (or a parellelpiped). Let $m, p \in Z^+$, $\langle \cdot, \cdot \rangle$ denote the real inner product in $L^2(\Omega)$, $\dot{w}$ denote differentiation with respect to time $t \geq 0$, and ∂ denote differentiation with respect to the spatial variable $x \in R^n$. Unless otherwise stated, assume that x denotes a variable in Ω, ζ denotes a variable in Γ, j ranges over the integer set $1, \ldots p$,

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and k ranges over the integer set $1, \ldots m$. Let

$$A(x,\partial) = \sum_{|\alpha| \leq 2p} a_\alpha(x)\partial^\alpha,$$

with $a_\alpha \in C^\infty(\bar{\Omega})$, and assume that $A(x,\partial)$ is strongly elliptic. Let $p_j < 2p$ be positive integers, and define the boundary operators

$$B_j(\zeta,\partial) = \sum_{|\eta| \leq p_j} b^j_\eta(\zeta)\partial^\eta,$$

with $b^j_\eta \in C^\infty(\Gamma)$. We will assume that the elliptic system

$$A(x,\partial)w(x,t) = 0,$$

$$B_j w(\zeta,t) = 0$$

is of *Agmon type.* The technical conditions for a system to be of Agmon type are given in Goldstein [5], page 137. We note here that if A is the Laplacian and the boundary operators define Dirichlet, Neumann, or mixed Dirichlet-Neumann conditions, then the system is of Agmon type. We now state the properties of this system which are important in this paper.

Define the operator $A : \mathcal{D}(A) \to L^2(\Omega)$ by $Aw(\cdot) = A(\cdot,\partial)w(\cdot)$, where $\mathcal{D}(A)$ is the $H^{2p}(\Omega)$ completion of $\{u \in C^{2p}(\Omega) \mid B_j u(\zeta) = 0 \text{ for } \zeta \in \Gamma,\ j = 1,\ldots p\}$. Then $-A$ generates an analytic semigroup on $L^2(\Omega)$ (see [5], p. 138). In particular, there exists $\theta \in (0,\pi/2]$, $M \geq 1$ and $a \in R$ such that

$$\|(sI + A)^{-1}\| \leq \frac{M}{|s-a|}, \tag{1}$$

$$|\arg\,(s-a)| < \theta + \pi/2, \quad s \neq 0.$$

If f is defined on Γ, define $\mathcal{B}_l f = R$ if R is the (distributional) solution to

$$-A(x,\partial)R(x) = 0,$$

$$B_j(\zeta,\partial)R(\zeta) = 0, \quad j \neq l,$$

$$B_l(\zeta,\partial)R(\zeta) = f,$$

where the boundary values are to be understood in the sense of trace. Then (Lion/Magenes [7], p. 188-189)

$$\mathcal{B}_j \in \mathcal{L}(H^{s+p_j-(1/2)}(\Gamma), H^s(\Omega)), \quad s \in R. \tag{2}$$

Consider now the following class of open loop systems:

$$\dot{w}(x,t) = -A(x,\partial)w(x,t), \tag{3}$$

$$B_j(\zeta,\partial)w(\zeta,t) = \sum_{i=1}^m g^j_i(\zeta)u^j_i(t), \tag{4}$$

$$y^j_k(t) = \langle \eta^j_k(\cdot), w(\cdot,t)\rangle. \tag{5}$$

We first wish to identify conditions on g^j_k and η^j_k which guarantee that the transfer function for (3) - (5) is *regular*, as defined in Weiss [11]. A transfer function $\mathbf{H}(s)$ is regular if $\mathbf{H}(s)$ is bounded in $\{\text{Re } s > \alpha\}$ for some $\alpha \in R$, and has a strong limit D (called the *feedthrough*) as $s \to \infty$ along the real axis.

Theorem 1 *If*

$$g^j_k \in H^{p_j-(1/2)}(\Gamma), \quad \eta^j_k \in L^2(\Omega), \tag{6}$$

then the transfer function $\mathbf{H}(s)$ *for the system (3) - (5) is regular with feedthrough* $D = 0$. *Furthermore,*

$$\limsup_{|s|\to\infty,\, s \in C_0 \setminus P_{\mathbf{H}}} \|\mathbf{H}(s)\| = 0, \tag{7}$$

where $P_{\mathbf{H}}$ *is the pole set of* $\mathbf{H}$.

Proof: Assume that there exists $\rho \in R$ such that

$$e^{-\rho\cdot}u^j_i \in L^2[0,\infty). \tag{8}$$

Upon formal Laplace transformation with respect to t (which will be justified once we show that $\hat{w}(x,s)$ is inverse transformable for s in some right half plane),

$$s\hat{w}(x,s) = -A(x,\partial)\hat{w}(x,s), \tag{9}$$

$$B_j(\zeta,\partial)\hat{w}(\zeta,s) = \sum_{i=1}^m g^j_i(\zeta)\hat{u}^j_i(s), \tag{10}$$

$$\hat{y}^j_k(s) = \langle \eta^j_k(\cdot), \hat{w}(\cdot,s)\rangle. \tag{11}$$

Let

$$v(x,s) = \hat{w}(x,s) - \sum_{j=1}^p \mathcal{B}_j\Big(\sum_{i=1}^m g^j_i(\zeta)\hat{u}^j_i(s)\Big).$$

Then, using the definition of $\mathcal{B}_j$ we see that $\hat{w}(x,s)$ satisfies (9) - (10) if and only if v satisfies

$$(s + A(x,\partial))v(x,s) = - s\sum_{j=1}^{p} \mathcal{B}_j(\sum_{i=1}^{m} g_i^j(\zeta)\hat{u}_i^j(s)). \quad (12)$$

$$B_j v(\zeta, s) = 0. \quad (13)$$

In particular, equation (13) implies that $v(s) := v(\cdot, s) \in \mathcal{D}(-A)$, so for $s \in \rho(-A)$

$$v(s) = -s(sI + A)^{-1}\sum_{j=1}^{p} \mathcal{B}_j(\sum_{i=1}^{m} g_i^j\hat{u}_i^j(s)).$$

Therefore $\hat{w}(s) := \hat{w}(\cdot, s)$ satisfies

$$\hat{w}(s) = (I - s(s + A)^{-1})\sum_{j=1}^{p} \mathcal{B}_j(\sum_{i=1}^{m} g_i^j\hat{u}_i^j(s))$$

$$= A(s + A)^{-1}\sum_{j=1}^{p} \mathcal{B}_j(\sum_{i=1}^{m} g_i^j\hat{u}_i^j(s)).$$

We see from (2, (6), (8) that $\hat{w}(x,s) \in L^2(\Omega)$ is inverse transformable for $s \in \{\text{Re } s > \rho\}$. Hence for $l = 1, \dots p$,

$$\hat{y}_k^l(s) = \langle \eta_k^l, \hat{w}(s)\rangle = \sum_{j=1}^{p}\sum_{i=1}^{m}\langle \eta_k^l, A(s + A)^{-1}\mathcal{B}_j g_i^j\rangle \hat{u}_i^j. \quad (14)$$

Denote $\langle \eta, w\rangle$ by $\eta^* w$ and define

$$\mathcal{B}^i g = [\mathcal{B}_i g_1^i, \mathcal{B}_i g_2^i, \dots, \mathcal{B}_i g_m^i],$$

$$\mathcal{B}g = [\mathcal{B}^1 g, \mathcal{B}^2 g, \dots \mathcal{B}^p g],$$

$$u^i = [u_1^i, u_2^i, \dots, u_m^i],$$

$$u = [u^1, u^2, \dots u^p]^T,$$

$$\eta^{i*} = [\eta_1^{i*}, \eta_2^{i*}, \dots, \eta_m^{i*}],$$

$$\eta^* = [\eta^{1*}\eta^{2*}, \dots, \eta^{p*}]^T,$$

$$y^i = [y_1^i, y_2^i, \dots, y_m^i],$$

$$y = [y^1, y^2, \dots y^p]^T.$$

Then (14) becomes

$$y = \eta^* A(sI + A)^{-1}\mathcal{B}g\ u.$$

Therefore, the transfer function for (3) - (5) is

$$\mathbf{H}(s) = \eta^* A(sI + A)^{-1}\mathcal{B}g. \quad (15)$$

To analyze $\mathbf{H}(s)$, note that it is a matrix with entries of the form

$$a(s) = \langle \eta_k^l, A(sI + A)^{-1}\mathcal{B}_j g_i^j\rangle$$

Since $g_i^j \in H^{p_j-(1/2)}(\Gamma)$, by (2) we see that $\mathcal{B}_j g_i^j \in L^2(\Omega)$, so $a(s)$ is defined for any $s \in \rho(-A)$. Furthermore, using (1), for $|\text{arg }(s - a)| < \theta + \pi/2$, $s \neq 0$,

$$\|(I - s(sI + A)^{-1})\mathcal{B}_j g_i^j\| \leq$$

$$\|\mathcal{B}_j g_i^j\|_{L^2(\Omega)}(1 + |s/(s - a)|M), \ .$$

Hence $a(s)$ is bounded in $\{|\text{arg }(s - b)| < \theta + \pi/2\}$, for every $b > a$. Since

$$\lim_{s\to\infty, s\in R^+}(I - s(sI + A)^{-1}) = 0$$

as a uniform limit (see Pazy [10], p. 9), we see that $a(s)$ is regular with feedthrough 0.

Let

$$\mathcal{W}(\psi) = \{re^{i\delta} \mid r \in (0,\infty), \delta \in (-\psi, \psi)\}.$$

The following property of regular transfer functions, which follows from [11], section 5, will be useful here: If $\mathbf{H}(s)$ is regular with feedthrough D, then for any $\psi \in (0, \pi/2)$,

$$\lim_{|s|\to\infty, s\in\mathcal{W}(\psi)} \mathbf{H}(s) = D. \quad (16)$$

Let $\phi \in [0, \theta)$, $b > a$ and $\mathbf{H}_\phi(s) = \mathbf{H}(e^{i\phi}(s-b))$. Then $\mathbf{H}_\phi(s)$ is holomorphic and bounded in $\{\text{Re }(s-b) > 0\}$, with feedthrough 0. Applying (16) to $\mathbf{H}_\phi(s)$ we can see that

$$\lim_{s\to\infty, s\in R^+} \mathbf{H}(is) = 0.$$

Similarly, applying (16) to $\mathbf{H}_{-\phi}(s) = \mathbf{H}(e^{-i\phi}(s - b))$ we get

$$\lim_{s\to\infty, s\in R^+} \mathbf{H}(-is) = 0.$$

This verifies (7), finishing the proof of Theorem 1. □

Let $\varepsilon_i^j \in [0, \infty)$,

$$\varepsilon^j = [\varepsilon_1^j, \varepsilon_2^j, \ldots, \varepsilon_m^j],$$

$$\vec{\varepsilon} = [\varepsilon^1, \varepsilon^2, \ldots \varepsilon^p],$$

and

$$\mathbf{E}_{\vec{\varepsilon}}(s) := \text{diag } \vec{\varepsilon}.$$

Let

$$\mathbf{G}_{\vec{\varepsilon}} = \mathbf{H}(s)(I + \mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s))^{-1}, \qquad (17)$$

the closed loop transfer function for (3) - (5) under the feedback $u_i^j(t) = -y_i^j(t - \varepsilon_i)$.

The delayed closed-loop system without control or observation is given by (3) and

$$\mathcal{B}_j(\zeta, \partial)w(\zeta, t) =$$

$$-\sum_{i=1}^{m} g_i^j(\zeta)\langle \eta_i^j(\cdot), w(\cdot, t - \varepsilon_i^j)\rangle = 0. \qquad (18)$$

Proposition 2 *If $s \in \rho(-A)$, then $e^{st}\phi$ is a mode of (3), (18) if and only if*

$$\det[I + \mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s)] = 0. \qquad (19)$$

Proof: Suppose $s \in \rho(-A)$. Let

$$v(s) = \phi(s) + \sum_{j=1}^{p}\sum_{i=1}^{m} \mathcal{B}_j g_i^j \langle \eta_i^j, \phi\rangle e^{-\varepsilon_i s}. \qquad (20)$$

Then $e^{st}\phi$ is a solution of (3), (18) if and only if

$$(s + A(\cdot, \partial))v(s) =$$

$$+ s\sum_{j=1}^{p}\sum_{i=1}^{m} \mathcal{B}_j g_i^j \langle \eta_i^j, \phi\rangle e^{-\varepsilon_i s}, \qquad (21)$$

$$\mathcal{B}_j v(\zeta) = 0. \qquad (22)$$

Note that $v \in \mathcal{D}(-A)$. Therefore

$$v = s(sI + A)^{-1}\sum_{j=1}^{p}\sum_{i=1}^{m} \mathcal{B}_j g_i^j \langle \eta_i^j, \phi\rangle e^{-\varepsilon_i s}$$

and

$$\phi = -A(sI + A)^{-1}\sum_{j=1}^{p}\sum_{i=1}^{m} \mathcal{B}_j g_i^j \langle \eta_i^j, \phi\rangle e^{-\varepsilon_i s}$$

$$= -A(sI + A)^{-1}\mathcal{B}g\mathbf{E}_{\vec{\varepsilon}}(s)\eta^*\phi.$$

Hence, we see that $e^{st}\phi$ is a mode of (3), (18) if and only if

$$[I + A(s + A)^{-1}\mathcal{B}g\mathbf{E}_{\vec{\varepsilon}}(s)\eta^*]\phi = 0. \qquad (23)$$

Suppose now that (19) holds. Then there exists $v \in R^{m+p}$ such that

$$[I + \mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s)]v = 0. \qquad (24)$$

Let

$$\phi = A(s + A)^{-1}\mathcal{B}gv.$$

Then

$$[I + A(s + A)^{-1}\mathcal{B}g\mathbf{E}_{\vec{\varepsilon}}(s)\eta^*]\phi =$$

$$A(sI + A)^{-1}\mathcal{B}g[[I + \mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s)]v = 0,$$

so (23) holds and $e^{st}\phi$ is a mode of (3), (18).

Conversely, assume $s \in \rho(-A)$ and $e^{st}\phi$ is a mode of (3), (18). Then (23) holds, so (taking η^* of both sides),

$$\eta^*\phi + \eta^* A(s + A)^{-1}\mathcal{B}g\mathbf{E}_{\vec{\varepsilon}}\eta^*\phi = 0,$$

which implies that

$$[I + \mathbf{H}(s)\mathbf{E}_{\vec{\varepsilon}}(s)]\eta^*\phi =$$

$$[I + \mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s)]\eta^*\phi = 0. \qquad (25)$$

If $\eta^*\phi = 0$, then $e^{st}\phi$ is a mode of (3) and (4) with $u = 0$. This would imply that $s \in \sigma(-A)$, which is a contradiction. Hence (19) holds, finishing the proof. □

Proposition 3 *Fix $\vec{\varepsilon}$. Suppose* Re $s \geq 0$, *$e^{st}\phi$ is a mode of (3), (18), and there are no unstable modes of (3), (18) with $\vec{\varepsilon} = 0$. Then there exists $v \neq 0$ such that (24) holds.*

Proof: If $s \in \rho(-A)$, then this follows from Proposition (2). Therefore assume that $s \in \sigma(-A)$, which consists only of eigenvalues, since A has compact resolvent. If $s = 0$, then it is easy to see that if $e^{st}\phi$ is a solution of (3), (18) for some choice of $\vec{\varepsilon}$, then it is a solution for every choice of $\vec{\varepsilon}$. In particular, this would be true for $\vec{\varepsilon} = 0$, contradicting the hypotheses. Therefore assume that $s \neq 0$. Let v be as in (20), so $e^{st}\phi$ is a solution of (3), (18) if and only if (21), (22) holds. Let Φ be the eigenspace of

$-A$ associated with s. Then this implies that $\alpha := \sum_{j=1}^{p}\sum_{i=1}^{m} \mathcal{B}_j g_i^j \langle \eta_i^j, \phi\rangle e^{-\varepsilon_i s}$ is in $\Phi^\perp$, the orthogonal complement of Φ in $L^2(\Omega)$. Hence we can define a partial inverse of $(sI + A)$ on α, which we denote by $(sI + A)^{-1}\alpha$. Furthermore, since we can define $(sI + A)^{-1}\alpha$, we can use an analogous argument to define $\mathbf{H}(s)$ by (15). By the same argument as in Proposition 2 we see that (25) holds. If $\eta^*\phi = 0$, then $e^{st}\phi$ is a solution of (3), (18) for every choice of $\vec{\varepsilon}$, in particular for $\vec{\varepsilon} = 0$. This contradicts the hypothesis that (3), (18) has no solutions of the form $e^{st}\phi$ with Re $s \geq 0$ when $\vec{\varepsilon} = 0$. Hence (24) holds with $v = \eta^*\phi \neq 0$. □

We now assume that the g_k^j and η_k^j are such that the closed loop, undelayed system (3), (18) with $\vec{\varepsilon} = 0$, is modally stable. For instance, in Lasiecka/Triggiani [6], theorem 1.2, the case where A is the the Laplacian and the boundary conditions are Dirichlet is considered, and conditions are given on these functions which guarantee that the feedback $u_i(t) = -y_i(t)$ exponentially stabilizes (hence modally stabilizes) the system. The following theorem shows that the modal stability of (3), (18) is robust with respect to delays for the class of parabolic equations considered in this paper.

Theorem 4 *Suppose (3), (18) is modally stable with* $\varepsilon_i = 0$, $i = 1, \ldots, m$. *Then there exists* ε^* *such that (3), (18) is modally stable when* $\varepsilon_i^j \in [0, \varepsilon^*]$.

Proof: In theorem 3.11 in [9] it is shown that when the open loop transfer function $\mathbf{H}$ is regular, condition (7) is sufficient to guarantee that there exists $\varepsilon^* > 0$ such that if $\mathbf{G}_0$ is holomorphic in $\{\text{Re } s \geq 0\}$, then $\mathbf{G}_{\vec{\varepsilon}}$ is holomorphic in $\{\text{Re } s \geq 0\}$ if $0 \leq \varepsilon_i \leq \varepsilon^*$. This is called robustness of *spectral stability*. However, the lack of right half plane poles of $\mathbf{G}_{\vec{\varepsilon}}$ does not immediately imply the lack of unstable modes of (3), (18), since there may be some pole zero cancellation.

Suppose that $0 \leq \varepsilon_i \leq \varepsilon^*$, $e^{st}\phi$ is a solution of (3), (18) with this choice of (ε_i^j), and Re $s \geq 0$. By Proposition 3, there exists $v \neq 0$ such that (24) holds. Then

$$\mathbf{H}(s)v \neq 0. \qquad (26)$$

By (17), for every $z \in \rho(-A)$

$$\mathbf{H}(z)v = \mathbf{G}_{\vec{\varepsilon}}(z)(I + \mathbf{H}(z)\mathbf{E}_{\vec{\varepsilon}}(z))v.$$

Since $\mathbf{G}_{\vec{\varepsilon}}$ is holomorphic at s,

$$\mathbf{H}(s)v = \mathbf{G}_{\vec{\varepsilon}}(s)(I + \mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s))v,$$

contradicting (24) and (26) and finishing the proof of Theorem 4. □

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