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GENERALIZING THE BODE DIAGRAMS

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Abstract. **This paper presents the generalization of the Bode plot construction rules in the case where s ($s=j\omega$) does not lie on the imaginary axis but in general on a smooth curve γ of the complex s-plane. Some rules for generalized Bode plot construction are investigated. Some examples are also included.**

I. INTRODUCTION.

Bode plots construction rules have been developed for rational transfer function $H(s)$ having the form

$$H(s)=\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)} \tag{1}$$

where $z_i,\ p_i \in C$. Since $z_i,\ p_i \in C$, the numerator and denominator of $H(s)$ are factored into products of first-degree factors. $z_i,\ p_i$ are called zeros and poles respectively. In real systems, z_i, as well as p_i are appeared as complex conjugate pairs if $z_i,\ p_i \in C-R$. It is well known that the amplitude (in dB) and the phase (in degrees) are defined as follows

$$|H(s)|_{db}=20\,log_{10}\frac{\prod_{i=1}^{m}|(s-z_i)|}{\prod_{i=1}^{n}|(s-p_i)|}=$$

$$\sum_{i=1}^{m}20\,log_{10}|s-z_i|-\sum_{i=1}^{n}20\,log_{10}|s-p_i| \tag{2}$$

or

$$arg(H(s))=\sum_{i=1}^{m}arg(s-z_i)-\sum_{i=1}^{n}arg(s-p_i) \tag{3}$$

The functions $|H(s)|_{db}$ and $\arg H(s)$ have the important property to convert a product form into a summation form. This property is used in the construction of a Bode diagram by adding Bode diagrams of the elementary terms $(s-z_i)$ and $(s-p_i)$.

Classical Bode plots refer in the case that $s=j\omega$ and they are the plots of $|H(s)|_{db}$ and $arg(H(s))$ versus $log_{10}\,\omega$. However, in practice, there exists some interesting cases where s lies on a smooth curve γ of the s-plane.

A first interesting case is: Suppose that a linear (time invariant) system is described by the transfer function $H(s)$ or equivalently by the impulse response $h(t)$. If we desire to modify some properties of the system, we define the system $H(s+a)$ or equivalently $e^{-at} \cdot h(t)$. In a sinus steady state $s = j\omega$ and therefore we obtain $H(a+j\omega)$. So starting from the original transfer function $H(s)$ a substitution $s \longrightarrow \sigma + j\omega$ seems to take place (where $\sigma = a$). If some relation exists between σ and ω, then, for the Bode plot, s lies on a curve of the s-plane

A second very interesting case arises from the discrete time linear systems analysis. In these systems, the transfer function $H(z)$ is used (z is the complex variable of the utilised z-transform). In a sinus steady state $z = e^{j\omega}$. Therefore z lies on the unit circle.

Other special cases can be obtained in multidimensional systems theory. See for example the excellent books [1÷3].

In a recent paper, [4], C.S.Lindguist examined the Bode plots in the case that $s = \sigma \in R$. This very interesting case is applied in several cases in networks, electronics, control, communications and distributed systems. In [4], some important remarks are made and several examples are given.

In this paper, the Bode plots are investigated in the case that s lies on a smooth curve γ of the complex plane.

$20\log_{10}|H(s)|$ and $arg(H(s))$ are plotted versus $\log_{10}\theta$ where θ is the parameter of the smooth curve γ on which s lies. These Bode plots are a generalization of the classical Bode plots as well as of the plots of [4].

We focus our analysis on the study of the factor $(s-p_i)$. It is not difficult to modify it for the factors $(s-z_i)$. In reality, the amplitude or the phase Bode diagram of a factor $(s-a)$ is the mirror diagram of the diagram of $1/(s-a)$.

Suppose that $p_i = a+jb$ and the complex variable s lies on the smooth curve γ having the parameter equations: $\sigma_1 = \sigma_1(\theta)$ and $\omega_1 = \omega_1(\theta)$ where $\sigma_1(\theta)$ and $\omega_1(\theta)$ are differentiable functions and

$$\left(\frac{d\sigma_1(\theta)}{d\theta}\right)^2 + \left(\frac{d\omega_1(\theta)}{d\theta}\right)^2 \neq 0 \qquad \forall\, \theta.$$

Since

$$(s-p_i)=$$

$$\sigma_1(\theta)+j\omega_1(\theta)-a-jb=\sigma(\theta)+j\omega(\theta)-k-jl$$

where

$$\sigma(\theta)=\sigma_1(\theta)-\sigma_1(0)\ , \quad \omega(\theta)=\omega_1(\theta)-\omega_1(0)$$

and

$$k+jl=a+jb-\sigma_1(0)-j\omega_1(0)$$

we can say that the curve starts from the origin of the axes, i.e.

$\sigma(0)=0$ and $\omega(0)=0$. The new pole, after this displacement in the complex plane, is $k+jl$. So, $s=\sigma(\theta)+j\omega(\theta)$ where $\theta \in R_+$ and $\sigma(0)=0,\ \omega(0)=0$.

In the next section, we refer to the behaviour of the Bode plot a) when $\theta << 1$, b) when $\theta \to \infty$. c-d) when $|s-p_i|$ is minimum or maximum.

2. RESULTS.

Suppose that $\theta = 10^u$. Therefore $u = \log_{10}\theta$.

It is repeated that the Bode plot of the factor $(s-p_i)$ is examined. It should be noted that our analysis has the advantage to be restricted only on first order factors.

a). Obviously, $\theta \to 0 \ \Leftrightarrow\ u \to -\infty$. However

$$\theta \to 0 \ \Rightarrow\ \sigma(\theta) \to 0 \ \text{and}\ \omega(\theta) \to 0$$

For this reason one can write

$$-20\log_{10}|s-p_i| \cong -20\log_{10}|-p_i| \cong -20\log_{10}|p_i|$$

as well as

$$-arg(s-p_i) \cong -arg(-p_i) = 180^0 - arg(p_i)$$

The conclusion is that the amplitude plot for $\theta \to 0$ is approximated by the asymptote (straight line) $\left|H(s)\right|_{db} = -20\log_{10}\left|p_j\right|$ having *0 db/dec* slope and the phase plot for $\theta \to 0$ is approximated by the asymptote (straight line)

$$arg(H(s)) = -arg(-p_i) = 180^0 - arg(p_i)$$

having *0 degrees/dec* slope.

b). If $\theta \to \infty \Leftrightarrow u \to \infty$. We distinguish two cases:

b1). Our curve on which s lies is a closed curve. In this case

$$\theta \to \infty \Rightarrow \sigma(\theta) \xrightarrow{not} \infty \text{ and } \omega(\theta) \xrightarrow{not} \infty$$

If $\sigma(\theta)$ and $\omega(\theta)$ are considered periodical functions, then the amplitude and the phase plot appeared to be periodical plots too. If $\sigma(\theta)$ and $\omega(\theta)$ are considered bounded functions, then the amplitude and the phase plot appeared to be bounded plots too. The proof is trivial.

b2). Our curve on which s lies is such that $\sigma(\theta) \xrightarrow{\theta\to\infty} \infty$ or $\omega(\theta) \xrightarrow{\theta\to\infty} \infty$. In this case, one can write $-20\log_{10}\left|s - p_i\right| \cong -20\log_{10}\left|s\right|$ and $-arg(s-p_i) \cong -arg(s)$. Various cases can be obtained if $\left|s\right| = \sqrt{\sigma^2(\theta)+\omega^2(\theta)}$ is approximated, for example, by a polynomial of θ or by an exponential function of θ or by a logarithmic function of θ in big θ. So if $\sigma(\theta)$ behaves as a d_1-th degree polynomial, and $\omega(\theta)$ behaves as a d_2-th degree polynomial, then $\left|s\right| = \sqrt{\sigma^2(\theta)+\omega^2(\theta)}$ behaves as a *d-th* degree polynomial where $d = max(d_1, d_2)$. Therefore the asymptote is the straight line $y(\theta) =$

$$20\log_{10}\frac{1}{c\theta^d} = -20\cdot d\cdot\log_{10}\theta - 20\log_{10}c$$

where c is a positive constant. More analytically

$$c = \begin{cases} \left|c_1\right| & \text{if } d_1 > d_2 \\ \left|c_2\right| & \text{if } d_1 < d_2 \\ \sqrt{c_1^2 + c_2^2} & \text{if } d_1 = d_2 \text{ and } \left|d_1\right| + \left|d_2\right| \neq 0 \end{cases}$$

where c_1 is the coefficient of the maximum power of the polynomial that approximates $\sigma(\theta)$ in big θ and c_2 is the coefficient of the maximum power of the polynomial that approximates $\omega(\theta)$ in big θ too. So an asymptote with a slope *-20d* db/decade is obtained. The phase plot asymptote is

$$y(\theta) = -arg(s)$$

$$= \begin{cases} -arg(c_1) & \text{if } d_1 > d_2 \\ -90^0 - arg(c_2) & \text{if } d_1 < d_2 \\ -arg(c_1 + jc_2) & \text{if } d_1 = d_2 \text{ and } \left|d_1\right| + \left|d_2\right| \neq 0 \end{cases}$$

It is reminded that *arg(c)* =0 if *c*>0 and *arg(c)=180* if *c*<0.

Remark: If

$\sigma(\theta)$ is bounded (for $\theta \to \infty$)

<u>or</u> $\omega(\theta)$ is bounded (for $\theta \to \infty$) (<u>or</u> is *exclusive* or), we define

$d_1 = 0$ when $\sigma(\theta)$ is bounded (for $\theta \to \infty$)

as well as we define

$d_2 = 0$ when $\omega(\theta)$ is bounded (for $\theta \to \infty$)

If a more sensible approximation in big θ is

$$\sigma(\theta) \approx c_1 \cdot Exp(\varepsilon_1 \cdot \theta) \text{ or } \omega(\theta) \approx c_2 \cdot Exp(\varepsilon_2 \cdot \theta)$$

where ε_1 and ε_2 are positive constants, then we have the asymptote (no-straight line)

$$y(\theta) = 20\log_{10}\frac{1}{ce^{\varepsilon\theta}}$$

$$= -20\cdot\varepsilon\cdot\theta\cdot\log_{10}e - 20\log_{10}c$$

$$= -20\cdot\varepsilon\cdot\log_{10}e\cdot 10^{\log_{10}\theta} - 20\log_{10}c$$

where

$$c=\left\{\begin{array}{lr} |c_1| & \text{if } \varepsilon_1>\varepsilon_2 \\ |c_2| & \text{if } \varepsilon_1<\varepsilon_2 \\ \sqrt{c_1^{\ 2}+c_2^{\ 2}} & \text{if } \varepsilon_1=\varepsilon_2 \ and \ |\varepsilon_1|+|\varepsilon_2|\neq 0 \end{array}\right\}$$

and $\varepsilon = max(\varepsilon_1, \varepsilon_2)$. Therefore the amplitude plot asymptote is an exponential curve (no-straight line). Furthermore, the phase plot asymptote is

$$y(\theta)=-arg(s)=\left\{\begin{array}{lr} -arg(c_1) & \text{if } \varepsilon_1>\varepsilon_2 \\ -90^{0}-arg(c_2) & \text{if } \varepsilon_1<\varepsilon_2 \\ -arg(c_1+jc_2) & \text{if } \varepsilon_1=\varepsilon_2 \ and \ |\varepsilon_1|+|\varepsilon_2|\neq 0 \end{array}\right\}$$

Therefore, the phase plot asymptote is a straight line.

Remark: If

$\sigma(\theta)$ is bounded (for $\theta \to \infty$)

<u>or</u> $\omega(\theta)$ is bounded (for $\theta \to \infty$) (<u>or</u> is *exclusive* or), we define

$\varepsilon_1 = 0$ when $\sigma(\theta)$ is bounded (for $\theta \to \infty$)

as well as we define

$\varepsilon_2 = 0$ when $\omega(\theta)$ is bounded (for $\theta \to \infty$)

If the approximation in big θ, is

$\sigma(\theta) \approx c_1 \cdot log_{10}(\lambda_1 \cdot \theta)$

and $\omega(\theta) \approx c_2 \cdot log_{10}(\lambda_2 \cdot \theta)$ where λ_1 and λ_2 are positive constants, then we have the asymptote

$y(\theta) =$

$$= 20log_{10}\frac{1}{c\,log_{10}(\lambda\theta)} = -20log_{10}\left[c(log_{10}\theta) + c\,log_{10}\lambda\right]$$

where

$$c=\left\{\begin{array}{lr} |c_1| & \text{if} \lambda_1>\lambda_2 \\ |c_2| & \text{if} \lambda_1<\lambda_2 \\ \sqrt{c_1^{\ 2}+c_2^{\ 2}} & \text{if } \lambda_1=\lambda_2 \ and \ |\lambda_1|+|\lambda_2|\neq 0 \end{array}\right\}$$

and $\lambda = max(\lambda_1, \lambda_2)$. Thus the asymptote is also a "logarithmic" curve. Furthermore the phase plot asymptote is

$$y(\theta)=-arg(s)=\left\{\begin{array}{lr} -arg(c_1) & \text{if } \lambda_1>\lambda_2 \\ -90^{0}-arg(c_2) & \text{if } \lambda_1<\lambda_2 \\ -arg(c_1+jc_2) & \text{if } \lambda_1=\lambda_2 \ and \ |\lambda_1|+|\lambda_2|\neq 0 \end{array}\right\}$$

thus this asymptote is a straight line.

Remark:
If

$\sigma(\theta)$ is bounded (for $\theta \to \infty$)

<u>or</u> $\omega(\theta)$ is bounded (for $\theta \to \infty$) (<u>or</u> is *exclusive* or), we define

$\lambda_1 = 0$ when $\sigma(\theta)$ is bounded (for $\theta \to \infty$)

as well as we define

$\lambda_2 = 0$ when $\omega(\theta)$ is bounded (for $\theta \to \infty$)

c). Suppose that a local minimum of the distance $|\sigma(\theta) + j\omega(\theta) - k - jl|$ is obtained at the point θ^*. This can be obtained simply by a geometric inspection on our work chart. In order to avoid problems with respect to differentiability of the quantity $|\sigma(\theta) + j\omega(\theta) - k - jl|$, the square of this quantity, i.e. $|\sigma(\theta) + j\omega(\theta) - k - jl|^2$, is considered . Henceforth, we denote $f(\theta) = |\sigma(\theta) + j\omega(\theta) - k - jl|^2$. So θ^* is a (local) minimum of $f(\theta)$. A Taylor expa-nsion of $f(\theta)$ yields

$$f(\theta) \cong f(\theta^*) + f''(\theta^*)(\theta - \theta^*)^2 / 2 \tag{4}$$

since $f'(\theta^*) = 0$. It can been shown that after simple algebraic manipulation

$$f(\theta) = a\left[1 - 2\zeta\left(\frac{\theta}{\theta^*}\right) + \zeta\left(\frac{\theta}{\theta^*}\right)^2\right]$$

where

$$a = \frac{f''(\theta^*)\theta^{*2}}{2} + f(\theta^*)$$

and

$$\zeta = \frac{f''(\theta^*)\theta^{*2}/2}{f''(\theta^*)\theta^{*2}/2 + f(\theta^*)}$$

Obviously

$$20\log_{10}|\sigma(\theta) + j\omega(\theta) - k - jl| = 10\log_{10} f(\theta)$$

Thus

$$10\log_{10} f(\theta) = 10\log_{10} a + 10\log_{10}\left[1 - 2\zeta\left(\frac{\theta}{\theta^*}\right) + \zeta\left(\frac{\theta}{\theta^*}\right)^2\right]$$

therefore a standardisation can be made, since the usual second-order Bode plot factor is locally obtained. See Example 3. The same is also true for the Bode phase plot.

d). In the case which a local maximum of the distance $|\sigma(\theta) + j\omega(\theta) - k - jl|$ is obtained at the θ^* point, a similar analysis can be followed . This maximum is also obtained simply by a geometric inspection on our work chart. The square of this distance is also considered:

$f(\theta) = |\sigma(\theta) + j\omega(\theta) - k - jl|^2$. The same Taylor expansion of $f(\theta)$ is also true, since $f'(\theta^*) = 0$ too. The analysis follows the same steps as in the case of the local minimum. As a conclusion one can also say that in both (amplitude and phase) plots, a standardisation can be made based on ζ and θ.

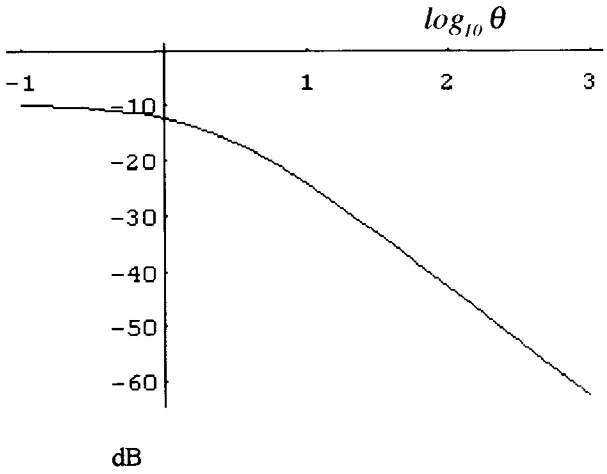
3. EXAMPLES

Some numerical examples are now presented. Here, the exact plots were derived by using the *Mathematica* 2.2 software package in a Windows 3.0 environment. However one can see that an important part of the information of these plots can also be extracted by using the rules discussed above.

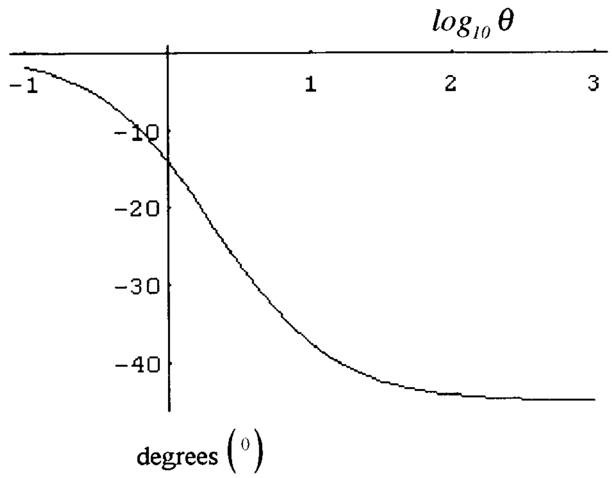
Example 1. Plot the amplitude and phase Bode plot for the transfer function

$H(s) = \frac{1}{s+3}$ when $s = \theta + j\theta$.

Amplitude Bode Plot (dB versus $\log_{10}\theta$):



Phase Bode Plot (degrees versus $\log_{10}\theta$):

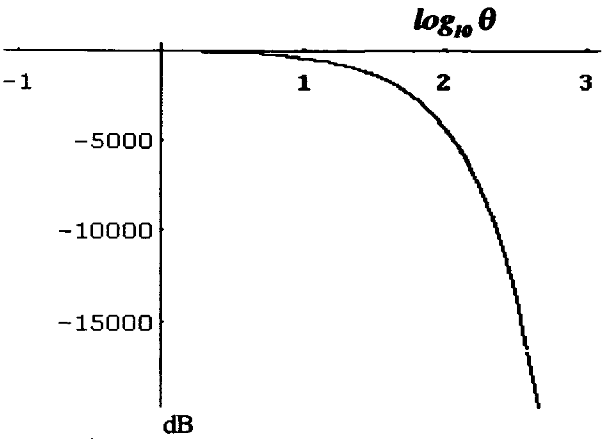


Example 2. Plot the amplitude and phase Bode plot for the transfer function

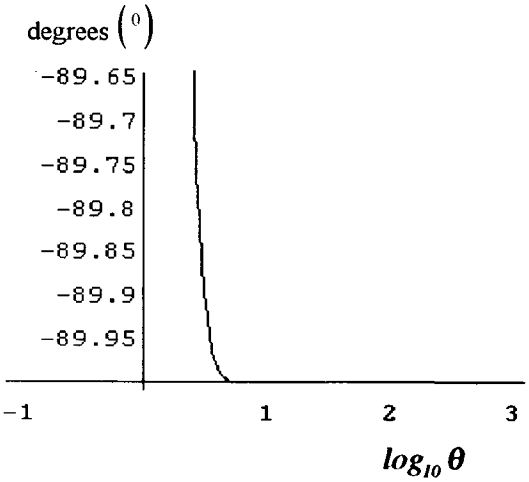
$$H(s) = \frac{1}{s+3} \text{ when }$$

$$s = Exp(3\theta) + jExp(5\theta).$$

Amplitude Bode Plot (dB versus $log_{10}\,\theta$):



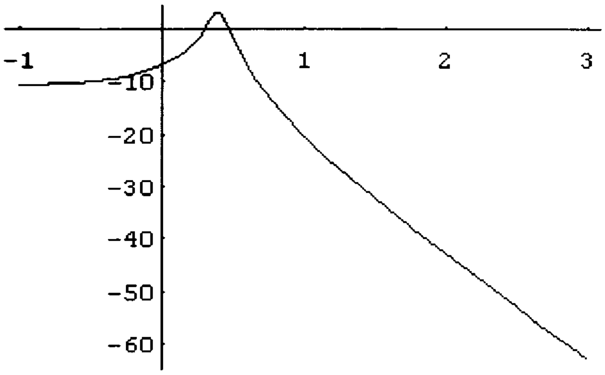
Phase Bode Plot (degrees versus $log_{10}\,\theta$):



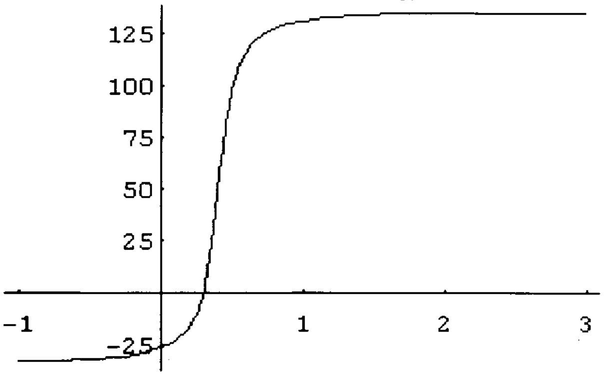
Example 3. Plot the amplitude and phase Bode plot for the transfer function

$$H(s) = \frac{1}{s-3-2j} \text{ when } s = \theta + j\theta$$

Amplitude Bode Plot (dB versus $log_{10}\,\theta$):



Phase Bode Plot (degrees versus $log_{10}\,\theta$):



4. CONCLUSION:

In this paper, an attempt is made to develop Bode plot construction rules in cases in which s lies on a smooth curve of the s-plane. The necessity for this generalization is presented. Some examples are also included.

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[4] C.S.Lindquist, "Bode Plot Constructions for Real-valued Rational Functions", Journal of the Franklin Institute, Vol. 330, No.2, pp.315-331, 1993.