THE LYAPUNOV EQUATION FOR UNBOUNDED CONTROL OPERATORS

George Weiss
Dept. of Electrical Engineering
Ben-Gurion University
Beer Sheva 84105, Israel
weiss@bguvm.bgu.ac.il

1. Some background

The aim of this paper is to clarify the relationship between the admissibility of unbounded control operators for strongly continuous operator semigroups and the solvability of certain Lyapunov equations. Moreover, we derive certain stability results for such semigroups, in terms of the associated Gramian. In this section we give some general facts about admissible and infinite-time admissible control operators, following Hansen and Weiss [5], Ho and Russell [6], Salamon [9], [10] and Weiss [11], [12] (our notation follows [5] and [11]). The material on the connection with Lyapunov equations and stability is in Section 2.

We need a notation for some spaces which will be used. Suppose A is the generator of a strongly continuous semigroup $\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$ on the Hilbert space X. For any $n \in \mathbb{N}$ define the space X_{-n} as the completion of X with respect to the norm

$$\|x\|_{-n} = \|(\beta I - A)^{-n}x\|,$$

where $\beta \in \rho(A)$ (X_{-n} does not depend upon β). We put $X_0 = X$. Then $(\beta I - A)^{-1}$ extends to an isomorphism from X_{-n} to X_{-n+1}. $\mathbb{T}$ extends to a strongly continuous semigroup on X_{-n}, whose generator is an extension of A, with domain X_{-n+1}. The extended semigroup is isomorphic to the initial one. We denote the extensions of $\mathbb{T}$ and A by the same symbols. Let Z_n be the Hilbert space obtained by endowing $D((A^*)^n)$ with the norm

$$\|x\|_n = \|(\beta I - A^*)^n x\|.$$

We identify X with X^*. It follows that $Z_n^* = X_{-n}$ for any $n \in \mathbb{N}$. For $z \in Z_n$ and $x \in X_{-n}$, we denote by $\langle z, x\rangle$ the duality pairing which reduces to the usual scalar product on X if $x \in X$.

Definition 1.1. *With the above notation, let U be a Hilbert space and $B \in \mathcal{L}(U, X_{-1})$. Then B is said to be an admissible control operator for $\mathbb{T}$, if for some $\tau > 0$ and any $u \in L^2([0,\infty), U)$ we have $\tilde{\Phi}_\tau u \in X$, where $\tilde{\Phi}_\tau u$ is defined by*

$$\tilde{\Phi}_\tau u = \int_0^\tau \mathbb{T}_\sigma B u(\sigma)\, d\sigma .$$

If B is admissible then for any $\tau \geq 0$, $\tilde{\Phi}_\tau$ defined above is a bounded linear operator from $L^2([0,\infty), U)$ to X (this follows from the closed graph theorem). In other words, for each $\tau \geq 0$ there is a $k_\tau \geq 0$ such that

$$\|\tilde{\Phi}_\tau u\|_X \leq k_\tau \|u\|_{L^2} \qquad \forall u \in L^2([0,\infty), U). \tag{1.1}$$

The concept of admissibility is important because it is equivalent to the solvability, in a reasonable sense, of the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{1.2}$$

More precisely, if B is admissible, then for any $x_0 \in X$ and any $u \in L^2_{loc}([0,\infty), U)$, the X-valued function x defined on $[0,\infty)$ by

$$x(t) = \mathbb{T}_t x_0 + \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma$$

is continuous (in X), and it is a strong solution of (1.2) (in X_{-1}). Any *abstract linear control system* may be represented in the form (1.2), with admissible $B \in \mathcal{L}(U, X_{-1})$, see [10], [11] for the definition and for details.

The space $\mathcal{B}(U, X, \mathbb{T})$ of all admissible control operators for $\mathbb{T}$ with domain U is a subspace of $\mathcal{L}(U, X_{-1})$. This space becomes a Banach space with the norm

$$|||B|||_\tau = \sup_{\|u\|_{L^2}\leq 1} \|\tilde{\Phi}_\tau u\|_X ,$$

where the choice of $\tau > 0$ is unimportant for the topology of $\mathcal{B}(U, X, \mathbb{T})$.

Definition 1.2. *With the above notation, an operator $B \in \mathcal{L}(U, X_{-1})$ is infinite-time admissible for $\mathbb{T}$ if $B \in \mathcal{B}(U, X, \mathbb{T})$ and for any $u \in L^2([0,\infty), U)$, the function $\tau \to \tilde{\Phi}_\tau u$ (from $[0,\infty)$ to X) is bounded.*

If B is infinite-time admissible then the constant k_τ appearing in (1.1) can be chosen to be independent of τ (this follows from the uniform boundedness theorem):

$$\|\tilde{\Phi}_\tau u\|_X \leq k\|u\|_{L^2} \qquad \forall\, u \in L^2([0,\infty), U). \quad (1.3)$$

It is easy to see that an admissible B is infinite-time admissible if and only if, for any $u \in L^2([0,\infty), U)$, the strong solution x of (1.2) with $x(0) = 0$ is bounded (in the space X).

We denote by $\tilde{\mathcal{B}}(U, X, \mathbb{T})$ the space of all infinite-time admissible control operators for $\mathbb{T}$ with domain U. $\tilde{\mathcal{B}}(U, X, \mathbb{T})$ becomes a Banach space with the norm

$$|||B|||_\infty = \lim_{\tau\to\infty} |||B|||_\tau . \qquad (1.4)$$

(The completeness of this space follows from the completeness of $\mathcal{B}(U, X, \mathbb{T})$.) If the semigroup $\mathbb{T}$ is exponentially stable, then $\tilde{\mathcal{B}}(U, X, \mathbb{T}) = \mathcal{B}(U, X, \mathbb{T})$. Otherwise (even if $\mathbb{T}$ is strongly stable) the two notions of admissibility are not equivalent.

A simple but important fact is that $B \in \tilde{\mathcal{B}}(U, X, \mathbb{T})$ if and only if there exists $\tilde{\Phi} \in \mathcal{L}(L^2([0,\infty), U), X)$ such that for any $u \in L^2([0,\infty), U)$

$$\tilde{\Phi} u = \lim_{\tau\to\infty} \tilde{\Phi}_\tau u \qquad (\text{in } X). \qquad (1.5)$$

In this formula, by writing "in X" we mean that the limit converges in X. In order to prove (1.5), we use (1.3) to show that for $0 \leq \tau \leq t$

$$\|\tilde{\Phi}_t u - \tilde{\Phi}_\tau u\|_X \leq k\|u\|_{L^2([\tau,t],U)} .$$

Clearly $\|\tilde{\Phi}\| \leq k$, k being the constant appearing in (1.3).

Let $\alpha \geq 0$ be such that for any $\beta > \alpha$, $e^{-\beta t}\|\mathbb{T}_t\| \to 0$ as $t \to \infty$. Let $\mathbb{C}_\alpha$ denote the set of complex numbers s with Re $s > \alpha$. By taking in (1.5) $u(t) = \mathrm{v}e^{-st}$, where $\mathrm{v} \in U$ and $s \in \mathbb{C}_\alpha$, and estimating $\|\tilde{\Phi} u\|$, we obtain that for $K = k^2/2$

$$\|(sI - A)^{-1}B\|^2_{\mathcal{L}(U,X)} \leq \frac{K}{\text{Re s}}, \qquad \forall s \in \mathbb{C}_\alpha .$$

Thus, the above estimate is a necessary condition for the infinite-time admissibility of B. It has been conjectured in Weiss [12] that it is sufficient as well, and various partial results in this direction have been obtained in [5] and [12] (these papers assume that the semigroup is exponentially stable, but this is not a significant restriction). Related material and extensions are contained in the recent papers of Grabowski [3] and Grabowski and Callier [4]

We give now the dual formulation of the concepts introduced above. For any $B \in \mathcal{L}(U, X_{-1})$ and any $\tau > 0$, the dual of $\tilde{\Phi}_\tau$ is an operator in $\mathcal{L}(Z_1, L^2([0,\infty), U))$ (we make the identification $U = U^*$) which is given by

$$(\tilde{\Phi}^*_\tau x)(t) = \begin{cases} B^*\mathbb{T}^*_t x & t \leq \tau, \\ 0 & t > \tau, \end{cases} \qquad \forall x \in Z_1 .$$

It follows that $B \in \mathcal{B}(U, X, \mathbb{T})$ if and only if for some (hence for any) $\tau > 0$, $\tilde{\Phi}^*_\tau$ extends continuously to X. In other words, there is a $k_\tau \geq 0$ such that

$$\int_0^\infty \|(\tilde{\Phi}^*_\tau x)(t)\|^2_U \, dt \leq k^2_\tau \|x\|^2_X , \qquad \forall x \in Z_1 .$$

We have $B \in \tilde{\mathcal{B}}(U, X, \mathbb{T})$ if and only if B is admissible and the constant k_τ appearing above can be chosen to be independent of τ. Assume $B \in \mathcal{L}(U, X_{-1})$ and define for every $x \in Z_1$ the function Ψx on $[0,\infty)$ by

$$(\Psi x)(t) = B^*\mathbb{T}^*_t x , \qquad \forall x \in Z_1 .$$

It is now clear that $B \in \tilde{\mathcal{B}}(U, X, \mathbb{T})$ if and only if there is a $k \geq 0$ (in fact the same as in (1.3)) such that

$$\int_0^\infty \|(\Psi x)(t)\|^2_U \, dt \leq k^2 \|x\|^2_X , \qquad \forall x \in Z_1. \quad (1.6)$$

Equivalently, $B \in \tilde{\mathcal{B}}(U, X, \mathbb{T})$ if and only if Ψ has an extension to X which is bounded as an operator from X to $L^2([0,\infty), U)$. This extension, still denoted Ψ, is the adjoint of the operator $\tilde{\Phi}$ defined in (1.5):

$$\Psi = \tilde{\Phi}^* . \qquad (1.7)$$

A formula for Ψ which is valid on X will be given in Remark 2.6.

2. The Lyapunov equation and the controllability Gramian

In this section we describe the relationship between infinite-time admissibility and the Lyapunov equation. This connection has been investigated by Levan [7] (who assumed that $B \in \mathcal{L}(U, X)$) and by Grabowski [2]. Related results have appeared in Russell and Weiss [8]. The main result of this section is the following theorem, parts of which are contained in [2]. We use the notation of the last section.

Theorem 2.1. *Let $\mathbb{T}$ be a strongly continuous semigroup on the Hilbert space X, with generator A. Let U be a Hilbert space and assume $B \in \mathcal{L}(U, X_{-1})$. Then the following three statements are equivalent:*

(i) *B is an infinite-time admissible control operator for $\mathbb{T}$.*

(ii) *There exists an operator $P \in \mathcal{L}(X)$ such that for any $x \in Z_1$,*

$$Px = \lim_{\tau\to\infty} \int_0^\tau \mathbb{T}_t BB^* \mathbb{T}_t^* x \, dt \qquad (\text{in } X). \qquad (2.1)$$

(iii) *There exist operators $\Pi \in \mathcal{L}(X)$, $\Pi \geq 0$, which satisfy the following equation with terms in $\mathcal{L}(Z_1, X_{-1})$:*

$$A\,\Pi + \Pi\, A^* = -BB^*. \qquad (2.2)$$

Moreover, if B is infinite-time admissible, then the following two statements are true:

(I) *P defined in (2.1) is the smallest positive solution of (2.2). In other words, $P \geq 0$, P satisfies (2.2) and, if $\Pi \in \mathcal{L}(X)$, $\Pi \geq 0$ and (2.2) holds, then $P \leq \Pi$.*

(II) *For any $x \in X$,*

$$\lim_{t\to\infty} P^{\frac{1}{2}} \mathbb{T}_t^* x = 0 \qquad (\text{in } X).$$

Proof. First we shall prove that (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). As in Section 1, we denote $(\Psi x)(t) = B^* \mathbb{T}_t^* x$, for any $x \in Z_1$ and any $t \geq 0$.

(i) $\Longrightarrow$ (ii): Assume (i) holds. Then $\tilde{\Phi}$ defined in (1.5) is a bounded operator from $L^2([0,\infty), U)$ to X. We define $P = \tilde{\Phi}\tilde{\Phi}^*$, so that $P \in \mathcal{L}(X)$. Then (1.5) and (1.7) show that for any $x \in Z_1$, Px is given by (2.1), so that (ii) holds.

(ii) $\Longrightarrow$ (i): Let $P \in \mathcal{L}(X)$ be defined by (2.1) (this formula defines P since Z_1 is dense in X). Since for any $x \in Z_1$, $\|\Psi x\|^2 = \lim_{\tau\to\infty} \|\Psi x\|^2_{L^2([0,\tau],U)}$, we get

$$\|\Psi x\|^2 = \langle Px, x\rangle, \qquad \forall x \in Z_1. \qquad (2.3)$$

This shows that (1.6) holds (with $k^2 = \|P\|$), so B is infinite-time admissible.

(ii) $\Longrightarrow$ (iii): Let $P \in \mathcal{L}(X)$ be defined by (2.1). We show that (2.2) is satisfied for $\Pi = P$. Let $x, y \in Z_2$ and for $t \geq 0$ define $f(t) = \langle B^*\mathbb{T}_t^* x, B^*\mathbb{T}_t^* y\rangle$. Then f is continuously differentiable and

$$\frac{d}{dt} f(t) = \left\langle B^*\mathbb{T}_t^* A^* x, B^*\mathbb{T}_t^* y\right\rangle + \left\langle B^*\mathbb{T}_t^* x, B^*\mathbb{T}_t^* A^* y\right\rangle.$$

Integrating both sides on $[0,\tau]$ gives

$$f(\tau) - f(0) = \left\langle \int_0^\tau \mathbb{T}_t BB^*\mathbb{T}_t^* A^* x\, dt, y\right\rangle + \left\langle \int_0^\tau \mathbb{T}_t BB^*\mathbb{T}_t^* x\, dt, A^* y\right\rangle. \qquad (2.4)$$

Since $A^*x \in Z_1$, by (ii) each of the above integrals converges (in X) as $\tau \to \infty$. Hence $\lim_{\tau\to\infty} f(\tau)$ also exists. Since by (ii) the integral $\int_0^\tau f(t)dt$ has a finite limit as $\tau \to \infty$, we must have $f(\tau) \to 0$ as $\tau \to \infty$. We then let $\tau \to \infty$ in (2.4) to find that

$$\langle PA^*x, y\rangle + \langle APx, y\rangle = -\langle BB^*x, y\rangle.$$

Since Z_2 is dense in Z_1, the above equality holds for any $y \in Z_1$. This implies $(AP + PA^*)x = -BB^*x$, for any $x \in Z_2$. Since both $AP + PA^*$ and BB^* are in $\mathcal{L}(Z_1, X_{-1})$, again by a density argument P satisfies (2.2).

(iii) $\Longrightarrow$ (i): Assume $\Pi \in \mathcal{L}(()X)$, $\Pi \geq 0$ and Π satisfies (2.2). For any $x \in X$ and any $t \in [0,\infty)$, we define $E_t(x)$ by $E_t(x) = \langle \Pi\mathbb{T}_t^* x, \mathbb{T}_t^* x\rangle$. Then $E_t(x) \geq 0$ and for any $x \in Z_1$, $E_t(x)$ is a continuously differentiable function of t. By (2.2) we have that for any $x \in Z_1$,

$$\frac{d}{dt} E_t(x) = -\langle BB^*\mathbb{T}_t^* x, \mathbb{T}_t^* x\rangle = -\|B^*\mathbb{T}_t^* x\|^2 \leq 0, \qquad (2.5)$$

so that $E_t(x)$ is nonincreasing. Since $E_t(x)$ is a continuous function of x, from the density of Z_1 in X we conclude that for any $x \in X$, $E_t(x)$ is nonincreasing. This can be written in the following form: for $0 \leq \tau \leq t$,

$$\mathbb{T}_t \,\Pi\, \mathbb{T}_t^* \leq \mathbb{T}_\tau \,\Pi\, \mathbb{T}_\tau^*.$$

It is a well known fact that any decreasing positive operator-valued function has a strong limit. Thus, there exists $\Pi_\infty \in \mathcal{L}(X)$, $\Pi_\infty \geq 0$, such that for all $x \in X$

$$\lim_{t\to\infty} \mathbb{T}_t \,\Pi\, \mathbb{T}_t^* x = \Pi_\infty x \qquad (\text{in } X). \qquad (2.6)$$

It is clear that $0 \leq \Pi_\infty \leq \Pi$. Integrating (2.5) on $[0,\infty)$ we get that for $x \in Z_1$

$$\langle \Pi x, x\rangle - \langle \Pi_\infty x, x\rangle = \int_0^\infty \|B^*\mathbb{T}_t^* x\|^2 dt = \|\Psi x\|^2. \qquad (2.7)$$

¿From here we see that (1.6) is satisfied, so that (i) holds.

Now assume that B is infinite-time admissible and let us prove statement (I). We see from (2.3) that $P \geq 0$. We have seen earlier that P satisfies (2.2). If $\Pi \in \mathcal{L}(X)$, $\Pi \geq 0$ and (2.2) holds, then by (2.3) and (2.7) we have that for all $x \in Z_1$

$$\langle Px, x\rangle = \langle \Pi x, x\rangle - \langle \Pi_\infty x, x\rangle, \qquad (2.8)$$

so that $P \leq \Pi$, as claimed in (I). Finally, to prove (II) we take $\Pi = P$ in (2.6) and (2.8) and obtain $\Pi_\infty = 0$.

By (2.6) this implies $\lim_{t\to\infty}\langle \mathbb{T}_t P \mathbb{T}_t^* x, x\rangle = 0$ for any $x \in X$, which is precisely (II). $\square$

If B is infinite-time admissible for $\mathbb{T}$, then P defined in (2.1) is called the *controllability Gramian* of $\mathbb{T}$ and B. Equation (2.2) is called a *Lyapunov equation* (this name is also used for slightly different equations). In several papers, the connection between the solvability of a Lyapunov equation and the stability of $\mathbb{T}$ was investigated; see, e.g., Levan [7] and the references therein. We shall briefly discuss this connection.

Suppose that B is infinite-time admissible for $\mathbb{T}$, so that the Gramian P defined in (2.1) exists. It is clear from (II) of Theorem 2.1 that if P is invertible (i.e., $P \geq \varepsilon I > 0$) then $\mathbb{T}^*$ is *strongly stable*, i.e., $\mathbb{T}_t^* x \to 0$ as $t \to \infty$, for any $x \in X$. This is the best possible result under the given assumptions, see Example 2.4 below.

With $\mathbb{T}$, B and P as above, it is clear from (2.6) and (2.8) that if $\mathbb{T}^*$ is strongly stable, then P is the unique positive solution of (2.2). (In [7] it is claimed that $\mathbb{T}$ uniformly bounded implies the uniqueness of P, which is wrong even if X is one dimensional.) If $\mathbb{T}$ is strongly stable but $\mathbb{T}^*$ is not, then (2.2) may have many positive solutions. For example, if $\mathbb{T}$ is the left shift semigroup on $L^2[0,\infty)$ and $B = 0$, then any multiple of the identity I satisfies (2.2).

¿From the preceding two paragraphs it follows that if B is infinite-time admissible and P is invertible, then P is the unique positive solution of (2.2).

The following proposition is a slight generalization of a result in [7] (where only the case $B \in \mathcal{L}(U, X)$ is considered).

Proposition 2.2. *With the notation of Theorem* 2.1, *assume that B is an infinite-time admissible control operator for $\mathbb{T}$. If $P > 0$ and $\mathbb{T}$ is uniformly bounded, then $\mathbb{T}$ is weakly stable, i.e., $\langle \mathbb{T}_t x, y\rangle \to 0$ as $t \to \infty$, for any $x, y \in X$.*

Weak stability is the strongest possible conclusion under the assumptions of the proposition, as Example 2.5 shows.

Proof. Denote $V = \operatorname{Ran} P^{\frac{1}{2}}$, then V is dense in X (because $\overline{V}$ is the orthogonal complement of $\operatorname{Ker} P^{\frac{1}{2}} = \operatorname{Ker} P = \{0\}$). It follows from (II) of Theorem 2.1 that for any $x \in X$ and any $v \in V$, $\lim_{t\to\infty}\langle \mathbb{T}_t x, v\rangle = 0$. Let $x, y \in X$ be fixed. We claim that for any $\varepsilon > 0$ we can find $T \geq 0$ such that $\langle \mathbb{T}_t x, y\rangle \leq \varepsilon$ for each $t \geq T$. Indeed, let $v \in V$ be such that $\langle \mathbb{T}_t x, y - v\rangle \leq \frac{\varepsilon}{2}$ for all $t \geq 0$ (this is possible by the uniform boundedness of $\mathbb{T}$). Now if T is such that $\langle \mathbb{T}_t x, v\rangle \leq \frac{\varepsilon}{2}$ for all $t \geq T$, then T is the desired number. The existence of such a T for any $\varepsilon > 0$ means that $\langle \mathbb{T}_t x, y\rangle \to 0$. $\square$

Remark 2.3. After Proposition 2.2 it is worth mentioning the following facts: Suppose that $\mathbb{T}$ is a weakly stable (hence uniformly bounded) semigroup on the Hilbert space X, and let A denote its generator.

(a) If for some (hence for any) $s \in \rho(A)$, $(sI - A)^{-1}$ is compact, then $\mathbb{T}$ and $\mathbb{T}^*$ are strongly stable. This is well known and easy to prove.

(b) If $\sigma(A) \cap i\mathbb{R}$ is at most countable, then $\mathbb{T}$ and $\mathbb{T}^*$ are strongly stable. This follows from the stability theorem of Arendt and Batty [1].

Example 2.4. Let $\mathbb{T}$ be the right shift semigroup on $X = L^2[0,\infty)$. We take $U = \mathbb{C}$ and $B = \delta_0$, i.e., $B^* x = x(0)$ for each $x \in Z_1 = H^1[0,\infty)$. Then it is not difficult to see that B is infinite-time admissible and $P = I$. Since P is invertible, we have that $\mathbb{T}^*$ is strongly stable, but $\mathbb{T}$ is not strongly stable.

Example 2.5. This is a refinement of the preceding example. Let $\mathbb{T}$ be the right shift semigroup on $X = L^2(\mathbb{R})$. We take $U = l^2$ and decompose $Bv = b_1 v_1 + b_2 v_2 + b_3 v_3 \ldots$, for any $v = (v_1, v_2, v_3 \ldots) \in l^2$. We define the components of B by $b_k = 2^{-k/2}\delta_{-k}$, i.e., $b_k^* x = 2^{-k/2} x(-k)$ for each $x \in Z_1 = H^1(\mathbb{R})$. Then it is not difficult to see that B is infinite-time admissible and for any $x \in X$, $(Px)(\xi) = \varphi(\xi)x(\xi)$, where $\varphi(\xi) = \sum_{k \geq -\xi} 2^{-k}$. In particular, $\varphi(\xi) = 1$ for $\xi \in [-1,\infty)$ and $\varphi(\xi)$ decreases rapidly as $\xi \to -\infty$. Since $\varphi(\xi) > 0$ everywhere, we have $P > 0$. By Proposition 2.2 $\mathbb{T}$ is weakly stable, but no stronger stability concepts are true for $\mathbb{T}$, since it is unitary.

Remark 2.6. Assume B is an admissible control operator for $\mathbb{T}$, and let Ψ be as in (1.7). We have seen in Section 1 that for any $x \in Z_1$, $(\Psi x)(t) = B^* \mathbb{T}_t^* x$. In order to obtain a formula valid for any $x \in X$, we may replace B^* by its *Λ-extension* B_Λ^*. This operator is defined as follows:

$$B_\Lambda^* x_o = \lim_{\lambda\to+\infty} B^* \lambda(\lambda I - A^*)^{-1} x_o \qquad (2.9)$$

(λ is real), for all x_o in the domain

$$D(B_\Lambda^*) = \{\, x_o \in X \mid \text{the limit in (2.9) exists} \,\}.$$

For more details about this operator we refer to the paper [13]. We have that for any $x_o \in X$ and almost every $t \geq 0$,

$$(\Psi x_o)(t) = B_\Lambda^* \mathbb{T}_t^* x_o .$$

Together with $P = \tilde{\Phi}\Psi$ this leads to the following expression for the controllability Gramian P (valid for any $x_o \in X$):

$$P x_o = \lim_{\tau\to\infty} \int_0^\tau \mathbb{T}_t B B_\Lambda^* \mathbb{T}_t^* x_o \, dt \qquad (\text{in } X).$$

References

[1] W. Arendt and C.J.K. Batty: Tauberian theorems and stability of one-parameter semigroups, *Trans. Amer. Math. Soc.* **306,** pp. 837–852, 1988.

[2] P. Grabowski: On the spectral Lyapunov approach to parametric optimization of distributed-parameter systems, *IMA J. of Math. Control & Information* **7,** pp. 317–338, 1991.

[3] P. Grabowski: Admissibility of observation functionals, *Intern. Jour. of Control,* to appear in 1995.

[4] P. Grabowski and F. Callier: Admissible observation operators. Duality of observation and control, preprint, Facultés Universitaires de Namur, 1994.

[5] S. Hansen and G. Weiss: The operator Carleson measure criterion for admissibility of control operators for diagonal semigroups on l^2, *Systems and Control Letters* **16,** pp. 219–227, 1991.

[6] L.F. Ho and D.L. Russell: Admissible input elements for systems in Hilbert space and a Carleson measure criterion, *SIAM J. Control & Optim.* **21,** pp. 614–640, 1983.

[7] N. Levan: Stability, stabilizability and the equation $[PAx, x] + [x, PAx] = -\|B^*x\|^2$, *Recent Advances in Communication and Control Theory,* editors: R.E. Kalman, G.I. Marchuk, A.E. Ruberti, A.J. Viterbi, Optimization Software, Inc., New York, pp. 214–226, 1987.

[8] D.L. Russell and G. Weiss: A general necessary condition for exact observability, *SIAM J. Control & Optim.* **32,** 1994, pp. 1–23, 1994.

[9] D. Salamon: Infinite dimensional systems with unbounded control and observation: A functional analytic approach, *Trans. Amer. Math. Soc.* **300,** pp. 383–431, 1987.

[10] D. Salamon: Realization theory in Hilbert space, *Math. Systems Theory* **21,** pp. 147–164, 1989.

[11] G. Weiss: Admissibility of unbounded control operators, *SIAM J. Control & Optim.* **27,** pp. 527–545, 1989.

[12] G. Weiss: Two conjectures on the admissibility of control operators, *Estimation and Control of Distributed Parameter Systems,* editors: W. Desch, F. Kappel, K. Kunisch, Birkhäuser-Verlag, Basel, pp. 367–378, 1991.

[13] G. Weiss: Regular linear systems with feedback, *Mathematics of Control, Signals and Systems* **7,** pp. 23–57, 1994.