Results on the Adaptive Estimation of Stiffness in Nonlinear Beam Models *

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1. Introduction

Current results on the adaptive (on-line) parameter identification of nonlinear beam models are presented along with summarized theoretical results concerning the state and parameter estimators of such nonlinear systems. The nonlinearity involved is a piecewise linear stiffness function meant to provide a simple model of damage. The estimation scheme identifies the time at which the system becomes nonlinear and it provides an estimate of the stiffness function.

This problem involves approximation at several levels. The plant is modeled by a partial differential equation, which must be solved numerically. The identification algorithm considered here requires "full state feedback" which means we must estimate the infinite dimensional state from finite dimensional observations. We give a complete description of our algorithm, along with a numerical example which illustrates its utility.

The paper is organized as follows. In Section 2 we give the basic problem statement. In Section 3, we outline the adaptive algorithm and state appproximation. Estimates of the full state from partial observation is the topic of discussion of Section 4, and in Section 5, we give our numerical results. Finally, some conclusions and plans for future study are given in Section 6.

2. Problem Statement

We consider the Euler-Bernoulli beam with Kelvin-Voigt viscoelastic damping

$$w_{tt}(t,x)+[EI(t,w_{xx}(t,x))+c_D I w_{txx}(t,x)]_{xx} = f(t,x)$$

with the boundary and initial conditions given by

$$\left\{ \begin{array}{l} w(0,t) = w_x(0,t) = 0 \\ w(l,t) = w_x(l,t) = 0 \end{array} \right. , \quad 0 \le x \le l,$$

$$\left\{ \begin{array}{l} w(\cdot,0) = w_0 \in H_0^2(0,l), \\ w_t(\cdot,0) = w_1 \in L_2(0,l). \end{array} \right.$$

The nonlinear term $EI(t, w_{xx}(t,x))$ is given by

$$EI(t,w_{xx}(t,x)) = \left\{ \begin{array}{ll} EI_1 w_{xx}(t,x) & \text{if } w_{xx}(t,x) > 0 \\ EI_2 w_{xx}(t,x) & \text{otherwise.} \end{array} \right.$$

The goal here is to identify the beam parameters adaptively. It is assumed that the beam displacement and velocity are available for measurement at each time t. In the next section, we give our algorithm for estimating the beam parameters from state observations.

3. Adaptive Estimator and Convergence

In this section we will assume that the stiffness function is the sum of a linear term and a nonlinear one. Additionally, we will assume that the damping coefficient could also be considered unknown. This is done in order to have a more general parameter estimator for the plant. For our numerical studies in §5, we use this general framework for the identification of the nonlinear stiffness function by treating the damping coefficient as known.

Before we proceed with the abstract formulation of the beam model, we need to provide some notation for the abstract spaces involved. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $|\cdot|$, and let V be a reflexive Banach space with norm denoted by $\|\cdot\|$. We assume that V is embedded densely and continuously in H. It follows that (see, for example [14, 15])

$$V \hookrightarrow H = H^* \hookrightarrow V^*, \tag{3.1}$$

where H^* and V^* denote the continuous duals of H, and V, respectively. All of the embeddings in (3.1) are dense and continuous. In particular, we assume that there exists a positive constant K_V such that $|\varphi| \le K_V \|\varphi\|$, $\varphi \in V$. We identify the space $L_2(0,l)$ (see, [1]) with the Hilbert space H, and the Sobolev space $H_0^2(0,l) = \{\varphi \in H^2(0,l) : \varphi(x) = \varphi_x(x) = 0,\ x = 0, l\}$ with V; then (3.1) is valid, see [1]. The dual of the Sobolev space V is $V^* = H^{-2}(0,l)$, see [7, 15]. Here we use $\langle \cdot, \cdot \rangle$ to denote the usual duality product obtained as the extension by continuity of the H- inner product from $H \times V$ to $V^* \times V$, see [2, 15].

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We consider the nonlinear beam equation given in variational form

$$\langle w_{tt}, \varphi\rangle + \langle q_1 w_{xx} + g(q_2, w_{xx}), \varphi_{xx}\rangle + \langle q_3 w_{txx}, \varphi_{xx}\rangle = \langle Bu(t), \varphi\rangle, \quad (3.2)$$

with the boundary and initial conditions given by

$$\begin{cases} w(0,t) = w_x(0,t) = 0 \\ w(l,t) = w_x(l,t) = 0 \end{cases}, \quad 0 \le x \le l,$$
$$\begin{cases} w(\cdot,0) = w_0 \in H_0^2(0,l), \\ w_t(\cdot,0) = w_1 \in L_2(0,l). \end{cases} \quad (3.3)$$

The input operator $B \in (U, V^*)$ is assumed to be known, and the nonlinear function $g(q_2, w_{xx})$ is given by

$$g(q_2, w_{xx}(t,x)) = \begin{cases} q_{2p} w_{xx}(t,x) & \text{if } w_{xx}(t,x) > 0 \\ q_{2n} w_{xx}(t,x) & \text{otherwise.} \end{cases} \quad (3.4)$$

The unknown parameters are $q = \{q_1, q_{2p}, q_{2n}, q_3\} \in Q$, where $\{Q, \langle\cdot,\cdot\rangle_Q, |\cdot|_Q\}$ is the parameter space. For this specific problem, the parameter space is identified with the Euclidean space $\mathbf{R}^4$, i.e. $Q \equiv \mathbf{R}^4$. In order to simplify the above beam equation and include all the parameters in the equation, we define the indicator function $\alpha(t,x)$ to be

$$\alpha(t,x) = \begin{cases} 1 & \text{if } w_{xx}(t,x) > 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.5)$$

and rewrite the above beam equation in weak form

$$\langle w_{tt}, \varphi\rangle + \langle q_1 w_{xx} + q_{2p}\alpha w_{xx} + q_{2n}(1-\alpha) w_{xx}, \varphi_{xx}\rangle + \langle q_3 w_{txx}, \varphi_{xx}\rangle = \langle Bu(t), \varphi\rangle, \quad (3.6)$$

It is desired to identify the parameter q on line (adaptively), given that measurements of the state $(w(t), w_t(t))$ of the plant (beam) are available.

We define, in a similar fashion to the linear case presented in [9], the (state and parameter) estimator in the form of an initial-value problem

$$\begin{aligned} &\langle v_{tt}, \varphi\rangle + \langle q_1^* v_{xx}, \varphi_{xx}\rangle + \langle q_3^* v_{txx}, \varphi_{xx}\rangle \\ &+ \langle \hat{q}_1(t) w_{xx} + \hat{q}_{2p}(t)\alpha w_{xx} + \hat{q}_{2n}(t)(1-\alpha) w_{xx}, \varphi_{xx}\rangle \\ &+ \langle \hat{q}_3(t) w_{txx}, \varphi_{xx}\rangle = \langle Bu(t), \varphi\rangle + \langle q_1^* w_{xx}, \varphi_{xx}\rangle \\ &+ \langle q_3^* w_{txx}, \varphi_{xx}\rangle, \quad \varphi \in V, \end{aligned} \quad (3.7)$$

where $v(t)$ is the state estimate of $w(t)$,

$$\langle D_t \widehat{q}(t), p\rangle_Q = \left\langle p \cdot \vec{W}, (v_{xx} - w_{xx}) + \gamma(v_{txx} - w_{txx}) \right\rangle, \quad (3.8)$$

for $p \in Q$, $\gamma > 0$,

$$v(0) = v_0 \in V, \;\; v_t(0) = v_1 \in H, \;\; \widehat{q}(0) = \widehat{q}_0 \in Q, \quad (3.9)$$

where the *regressor* vector $\vec{W}$ is given by

$$\vec{W} = [w_{xx} \;\; \alpha w_{xx} \;\; (1-\alpha) w_{xx} \;\; w_{txx}]^T,$$

$\widehat{q}(t) = [\hat{q}_1(t), \hat{q}_{2p}(t), \hat{q}_{2n}(t), \hat{q}_3(t)]$ is the vector of the (adaptively) estimated parameters and $p = [p_1, p_{2p}, p_{2n}, p_3] \in Q$. The above estimator has the *series-parallel* configuration (see [11]), where filtered values of the plant state and estimates of the parameters are used in the state estimator equation.

Remark 3.1 *The well posedness of the plant equation has been treated in the paper by Banks, Gilliam and Shubov in [4].*

Remark 3.2 *It should be noted that $q^* = (q_1^*, q_{2p}^*, q_{2n}^*, q_3^*) \in Q$ is a vector of design parameters chosen to satisfy certain conditions as in* Lemma 3.4 *below. In this case q_{2p}^*, q_{2n}^* can be chosen as $q_{2p}^* = q_{2n}^* = q_2^*$ and we have in this case a linear estimator. A much simpler case would be to choose $q_{2p}^* = q_{2n}^* = 0$, i.e. $q^* = (q_1^*, 0, 0, q_3^*)$. Either choice would give a linear equation for the state estimator. If, on the other hand, the design parameters (q_{2p}^*, q_{2n}^*) are not set equal, then the state estimator becomes nonlinear and the analysis more involved. For simplicity, we take throughout this paper $q_{2p}^* = q_{2p}^* = 0$.*

Using the more compact notation for the *regressor vector* and the comments of Remark 3.2, we now (re)write the plant, and state and parameter estimator equations in variational form

Plant:

$$\langle w_{tt}, \varphi\rangle + \langle q \cdot \vec{W}, \varphi_{xx}\rangle = \langle Bu(t), \varphi\rangle \quad (3.10)$$

State Estimator:

$$\langle v_{tt}, \varphi\rangle + \langle q^* \cdot \vec{U}, \varphi_{xx}\rangle = \langle Bu(t), \varphi\rangle + \langle [q^* - \widehat{q}(t)] \cdot \vec{W}, \varphi_{xx}\rangle \quad (3.11)$$

Parameter Estimator:

$$\langle D_t \widehat{q}(t), p\rangle_Q = \langle p \cdot \vec{W}, (v_{xx} - w_{xx}) + \gamma(v_{txx} - w_{txx})\rangle, \quad (3.12)$$

where the vector $\vec{U}$ is given by

$$\vec{U} = [v_{xx} \;\; \alpha v_{xx} \;\; (1-\alpha) v_{xx} \;\; v_{txx}]^T.$$

As was observed above, despite the fact that the plant is nonlinear, the state and parameter estimators are linear. This would then simplify the finite dimensional approximation theory necessary for the implementation of the above proposed estimators.

In order to guarantee convergence of the adaptive estimator, we must impose a form of a boundedness condition on the beam state, namely the *admissible plant.*

Assumption 3.3 (Boundedness of plant) *A plant is a pair (q, w) with $q \in Q$ and w a solution to the initial-value problem* (3.2), (3.3) *with $w, w_t \in V$, a.e. $t > 0$, for which there exists a constant $\mu > 0$ such that*

$$\left|\langle p \cdot \vec{W}, \phi_{xx}\rangle\right| \leq \mu \, |p|_Q \, |\phi|_V$$

for almost every $t > 0$, $p \in Q$ and all $\phi \in V$.

We establish the convergence of the state estimator and, with the additional assumption of *persistence of excitation*, parameter convergence. We assume throughout this section that the boundedness assumption of the plant, Assumption 3.3, is satisfied. Using equations (3.10) and (3.11) for the plant and estimator states, respectively, and denoting by $r(t) = \widehat{q}(t) - q$ the parameter error, we have that e, where $e = v - w$, and r satisfy the initial value problem

$$\langle e_{tt}, \varphi\rangle + \langle q_3^* e_{txx}, \varphi_{xx}\rangle + \langle q_1^* e_{xx}, \varphi_{xx}\rangle$$
$$+\langle r \cdot \vec{W}(t), \varphi_{xx}\rangle = 0, \quad \varphi \in V \;\; t > 0, \quad (3.13)$$

$$\langle D_t \widehat{q}(t), p\rangle_Q = \langle p \cdot \vec{W}, e_{xx} + \gamma e_{txx}\rangle, \quad p \in Q \;\; t > 0, \quad (3.14)$$

$$e(0) \in V, \;\; e_t(0) \in H, \;\; r(0) \in Q. \quad (3.15)$$

It should be noted that the state error initial conditions given by $e(0) = v(0) - w(0)$ and $e_t(0) = v_t(0) - w_t(0)$ are not necessarily known, since the beam initial displacement and velocity are not assumed to be known. We now establish a Lyapunov-like estimate for the system (3.13) - (3.15).

Lemma 3.4 *If γ is chosen to satisfy*

$$\gamma > \max\left\{K_V, \frac{K_V}{q_1^*}, \frac{K_V^2}{q_3^*}\right\}, \quad (3.16)$$

then there exist constants $\rho, \sigma > 0$ such that for all $t > 0$

$$\|e(t)\|^2 + |e_t(t)|^2 + |r(t)|_Q^2$$
$$+\rho \int_0^t \left\{\|e(\tau)\|^2 + \|e_\tau(\tau)\|^2\right\} d\tau \leq \xi$$

where

$$\xi = \sigma\left\{\|e(0)\|^2 + |e_t(0)|^2 + |r(0)|_Q^2\right\}.$$

Remark 3.5 *In the case that the plant initial conditions, $(w(0), w_t(0))$, are known, then the state estimator's initial conditions could be taken identical to the plant initial conditions, i.e. $e(0) = 0$, $e_t(0) = 0$, thus making the bound ξ of the Lyapunov estimate of* Lemma 3.4, *only a function of* $r(0)$.

The convergence of the state estimate is easily established. It is essentially an infinite dimensional extension of Barbălat's lemma (see [12]), and was used for the adaptive estimation of time invariant parameters of second order distributed parameter systems in [9].

Theorem 3.6 *Assume that the plant satisfies the boundedness condition. If γ satisfies the bound* (3.16), *then the energy functional given by*

$$E(t) = \gamma\left\{\langle q_1^* e_{xx}(t), e_{xx}(t)\rangle + |e_t(t)|^2\right\}$$
$$+2\langle e(t), e_t(t)\rangle + \langle q_3^* e_{xx}(t), e_{xx}(t)\rangle + |r(t)|_Q^2 \quad (3.17)$$

is nonincreasing and

$$\lim_{t\to\infty} \|e(t)\|^2 = 0 \quad \textit{and} \quad \lim_{t\to\infty} |e_t|^2 = 0.$$

Parameter convergence is established via the additional assumption of *persistence of excitation*, a richness condition that is imposed on the plant.

Definition 3.7 ([9, 10]) The plant is said to be *persistently excited* if there exists $T_0, \delta_0, \epsilon_0 > 0$, and a sequence of positive real numbers $\{t_k\}_{k=1}^{\infty}$ with $\lim_{t\to\infty} t_k = \infty$, such that for each $\mathbf{p} = (p_1, p_{2p}, p_{2n}, p_3) \in Q$ with $|\mathbf{p}|_Q = 1$ and each positive integer k, there exists a $t_k^* \in [t_k, t_{k+T_0}]$ such that

$$\int_{t_k^*}^{t_k^*+\delta_0} \mathbf{p} \cdot \vec{W}(\tau)\, d\tau \in V^*$$

and

$$\left\|\int_{t_k^*}^{t_k^*+\delta_0} \mathbf{p} \cdot \vec{W}(\tau)\, d\tau\right\|_{V^*} \geq \epsilon_0.$$

We can now prove parameter convergence by imposing the persistence of excitation condition onto the plant information.

Theorem 3.8 *If the plant is persistently excited then*

$$\lim_{t\to\infty} |r(t)|_Q = 0$$

Proof. The proof of this theorem is similar to the one given for the linear case in [9]. □

4. Finite Dimensional Approximation

The estimator given by (3.7) - (3.9) is infinite dimensional and consequently its implementation requires some finite dimensional approximation. We briefly outline a Galerkin approach and present some convergence results.

For each $n = 1, 2, \ldots$, let H^n be a finite-dimensional subspace of H with $H^n \subset V$, $n =$

$1, 2, \ldots$, and let Q^n be a finite dimensional subspace of Q. Consider the Galerkin equations for v^n and r^n in H^n and Q^n that correspond to (3.7) - (3.9)

$$\begin{aligned}
&\langle v_{tt}^n, \varphi\rangle + \langle q_1^* v_{xx}^n, \varphi_{xx}^n\rangle + \langle q_3^* v_{txx}^n, \varphi_{xx}^n\rangle \\
&\quad + \langle \hat{q}_1^n(t) w_{xx} + \hat{q}_{2p}^n(t)\alpha w_{xx} + \hat{q}_{2n}^n(t)(1-\alpha) w_{xx}, \varphi_{xx}^n\rangle \\
&\quad + \langle \hat{q}_3^n(t) w_{txx}, \varphi_{xx}^n\rangle = \langle B^n u(t), \varphi^n\rangle + \langle q_1^* w_{xx}, \varphi_{xx}^n\rangle \\
&\quad + \langle q_3^* w_{txx}, \varphi_{xx}^n\rangle, \quad \varphi^n \in H^n, \qquad (4.1)
\end{aligned}$$

$$\begin{aligned}
&\langle D_t \hat{q}^n(t), p^n\rangle_Q = \\
&\left\langle p^n \cdot \vec{W}, (v_{xx}^n - w_{xx}) + \gamma(v_{txx}^n - w_{txx})\right\rangle, \qquad (4.2)
\end{aligned}$$

for $p^n \in Q^n$,

$$v^n(0),\ v_t^n(0) \in H^n, \quad \hat{q}^n(0) \in Q^n. \qquad (4.3)$$

In order to present the convergence results we make the following standard Galerkin assumption made similarly for the linear case in [9].

Assumption 4.1 *The solution to the initial value problem* (3.7)-(3.9) *is such that* $v_t(\cdot) \in L_2(0,T;V)$ *and* $v_{tt}(\cdot) \in L_2(0,T;H)$ *and the subspaces* H^n *and* Q^n *are such that there exist functions* $v_n(\cdot) \in L_2(0,T;H^n)$ *and* $\hat{q}_n(\cdot) \in L_2(0,T;Q^n)$ *such that* $v_n \to v$ *in* $C(0,T;V)$, $\dot{v}_n \to \dot{v}$ *in* $C(0,T;H)$ *and* $L_2(0,T;H)$, $\ddot{v}_n \to \ddot{v}$ *in* $L_2(0,T;H)$, $\hat{q}_n \to \hat{q}$ *in* $C(0,T;Q)$, *and* $\dot{\hat{q}}_n \to \dot{\hat{q}}$ *in* $L_2(0,T;Q)$.

Theorem 4.2 *We assume that* Assumption 4.1 *is satisfied and that the plant satisfies the boundedness condition,* Assumption 3.3. *Let the pair* $(\hat{q}, v)$ *be the solution to the initial value problem* (3.7) - (3.9), *and for each* $n = 1, 2, \ldots$, *let* $(\hat{q}^n, v^n)$ *be the solution to the initial value problem* (4.1) - (4.3) *with*

$$v^n(0) = v_n(0),\ \dot{v}^n(0) = \dot{v}_n(0),\ \hat{q}^n(0) = \hat{q}_n(0).$$

Then

$$v^n \to v \ \text{ in } C(0,T;V),$$

$$\dot{v}^n \to \dot{v} \ \text{ in } C(0,T;H) \text{ and } L_2(0,T;V),$$

and

$$\hat{q}^n \to \hat{q}, \ \text{ in } C(0,T;Q).$$

Proof. The proof of this theorem is rather standard for the online parameter estimation of infinite dimensional systems and, due to the linearity of the parameters for this problems, most of the arguments leading to its proof are identical to the ones used in the linear case, [9].

The above theorem uses a finite dimensional approximation of the state and parameter estimator and the full infinite dimensional state of the plant w. From an implementational point of view, it is more convenient to replace w in the approximating estimator (4.1) - (4.3) by a finite dimensional approximation w_n. We require the following additional assumption (as in [9]).

Assumption 4.3 *For the plant given by* (3.2) *(or* (3.10)*), there exists* $w_n(\cdot) \in C^1(0,T;H^n)$ *such that* $w_n \to w$ *and* $\dot{w}_n \to \dot{w}$ *in* $C(0,T;V)$.

Theorem 4.4 *Assume that* (q, w) *satisfies the boundedness condition and that* Assumption 4.1 *and* Assumption 4.3 *hold. Let* $(\hat{q}, v)$ *be the solution to the initial value problem* (3.7) - (3.9) *and for each* $n = 1, 2, \ldots$, *let* $(\hat{q}^n, v^n)$ *be the solution to the initial value problem* (4.1) - (4.3) *with* w *replaced by* w_n. *Then*

$$v^n \to v \ \text{ in } C(0,T;V),$$

$$\dot{v}^n \to \dot{v} \ \text{ in } C(0,T;H) \text{ and } L_2(0,T;V),$$

and

$$\hat{q}^n \to \hat{q}, \ \text{ in } C(0,T;Q).$$

Proof. Once again, the proof of this theorem follows from the linear case presented in [9].

5. Example and Numerical Results

In this section we present some of our numerical findings. We consider the Euler-Bernoulli beam with piezoceramic actuator, [5, 6, 7], given by

$$\int_0^l \{\rho_b w_{tt}\phi + (EI\,(w_{xx}) + c_D I w_{txx})\,\phi_{xx}\}\,dx$$
$$= \left(\int_0^l \mathcal{K}_B \chi_{[\alpha_1,\alpha_2]}(x)\phi_{xx}\,dx\right) u_{patch}(t), \quad (5.1)$$

where ρ_b is the beam linear mass density, $u_{patch}(t)$ is the voltage applied to the patch, $\mathcal{K}_B$ is a parameter which depends on the geometry and piezoceramic material properties (e.g., see, [3]) and $\chi_{[x_1,x_2]}(x)$ is the characteristic function over the interval $[x_1, x_2]$. Using the Galerkin scheme outlined in §4, we discretize the beam in terms of spline expansions (see [13]). Modified cubic splines on the interval $(0, l)$ with respect to the uniform mesh $\{0, \frac{1l}{n}, \frac{2l}{n}, \ldots, l\}$ were used to approximate (3.7) - (3.9). We denote the 1-D cubic splines by $\{B_i^n\}_{i=1}^{n-1}$ and the approximating subspace $H^n = \text{span}\{B_i^n\}_{i=1}^{n-1}$. For each $n = 1, 2, \ldots$, let P^n denote the orthogonal projection of $L_2(0,l)$ onto H^n and set $v_n = P^n v$. We also let P_n be the orthogonal projection of $H_0^2(0,l)$ onto H^n with respect to the $H_0^2(0,l)$ inner product, and set $w_n = P_n w$. As was noted in [9] we have

$$\langle P_n \phi_{xx}, \psi_{xx}^n\rangle_{L_2} = \langle \phi_{xx}, \psi_{xx}^n\rangle_{L_2}, \quad \psi^n \in H^n$$

and by letting

$$w_n(t) = P_n w(t) = \sum_{j=1}^{n-1} W_j^n(t) B_j^n(x),$$

where $W^n(t) \in \mathbb{R}^{n-1}$ is the coordinate vector for $w_n(t)$ with respect to the spline basis $\{B_j^n\}_{j=1}^{n-1}$, we have that

$$W^n(t) = (K^n)^{-1} \int_0^l w_{xx}(t,x) D_x^2 B_j^n(x) dx,$$

$j = 1, 2, \ldots, n-1$ where K^n is the $(n-1) \times (n-1)$ *stiffness* matrix and is given by

$$K^n = [K^n]_{ij} = \int_0^l D_x^2 B_i^n(x) D_x^2 B_j^n(x) dx.$$

Now we let $V^n(t) \in \mathbb{R}^{n-1}$ be the vector representation of the state estimator $v^n(t)$,

$$v^n(t) = \sum_{j=1}^{n-1} V_j^n(t) B_j^n(x).$$

Then the finite dimensional (approximated) state estimator equation corresponding to (4.1) is given by

$$\begin{aligned} &M^n D_t^2 V^n(t) + q_3^* K^n D_t(V^n(t) - W^n(t)) \\ &\quad + q_1^* K^n (V^n(t) - W^n(t)) \\ &\quad + (\hat{q}_1^n(t) + \hat{q}_{2p}^n(t)\alpha^n + \hat{q}_{2n}^n(t)(1-\alpha^n)) K^n W^n(t) \\ &\quad + \hat{q}_3^n(t) K^n \dot{W}^n(t) = \mathcal{K}^B F^n(t) \qquad (5.2) \end{aligned}$$

where the $(n-1) \times (n-1)$ *mass* matrix M^n is given by

$$M^n = [M^n]_{ij} = \int_0^l B_i^n(x) B_j^n(x) dx,$$

and $F^n(t)$ is given by

$$\begin{aligned} F^n(t) &= [F^n]_i \\ &= \left[\int_0^l \chi_{[\alpha_1,\alpha_2]}(x) D_x^2 B_i^n(x)\, dx \right] u_{patch}(t) \\ &= \left[\int_{\alpha_1}^{\alpha_2} D_x^2 B_i^n(x)\, dx \right] u_{patch}(t). \end{aligned}$$

The parameter estimator equation corresponding to (4.2) is given by

$$\dot{\hat{q}}_1^n(t) = \lambda_1 [W^n(t)]^T K^n G_n(t), \qquad (5.3a)$$

$$\dot{\hat{q}}_{2p}^n(t) = \lambda_{2p} [W^n(t)]^T K^n \alpha^n(t) G_n(t), \qquad (5.3b)$$

$$\dot{\hat{q}}_{2n}^n(t) = \lambda_{2n} [W^n(t)]^T K^n (1-\alpha^n(t)) G_n(t), \qquad (5.3c)$$

$$\dot{\hat{q}}_3^n(t) = \lambda_3 [\dot{W}^n(t)]^T K^n G_n(t), \qquad (5.3d)$$

where $G_n(t)$ is given by $G_n(t) = \left[E_n(t) + \gamma \dot{E}_n(t)\right]$ with $E_n(t) = W^n(t) - V^n(t)$ and $\lambda_1, \lambda_{2p}, \lambda_{2n}, \lambda_3$ are positive constants acting as *adaptive gains* (see [9]).

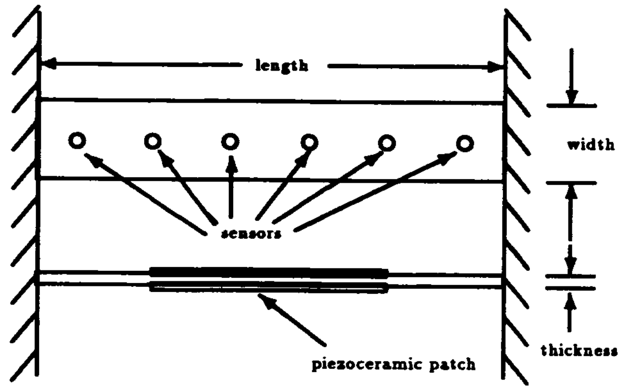

Figure 5.1: Beam with piezoceramic actuator

Remark 5.1 *The plant displacement and velocity that are used in the above state and parameter estimator are obtained via an interpolation of the discrete (spatially) measurements provided by a finite number, $(m-1)$, of sensors assumed to be mounted at equal intervals along the length of the beam as seen in Figure 5.1 below.*

If for $t > 0$, we let $w_m(t,\cdot)$ denote the interpolate to the $(m-1)$ displacement measurements, $\{w(t, \frac{j}{m}l)\}_{j=1}^{m-1}$, then we have

$$w_m(t) = w_m(t,\cdot) = \sum_{j=1}^{m-1} \zeta_j^m(t) B_j^m(x), \quad t \geq 0, \quad (5.4)$$

where $\zeta_j^m(t) = (L^m)^{-1} W_m(t)$ with the $(m-1) \times (m-1)$ matrix L^m given by

$$L^m = [L^m]_{ij} = B_j^m \left(\frac{il}{m} \right), \quad i, j = 1, \ldots, m-1, \quad (5.5)$$

and the $(m-1)$-vector $W_m(t)$ is given by $[W_m(t)]_i = w(t, \frac{il}{m})$, $i = 1, \ldots, m-1$. It then follows that

$$W^n(t) = (K^n)^{-1} K^{n,m} (L^m)^{-1} W_m(t) \qquad (5.6)$$

$$\dot{W}^n(t) = (M^n)^{-1} M^{n,m} (L^m)^{-1} \dot{W}_m(t) \qquad (5.7)$$

where the $(n-1) \times (m-1)$ matrices $K^{n,m}$ and $M^{n,m}$ are given by

$$[K^{n,m}]_{ij} = \langle D^2 B_i^n, D^2 B_j^m \rangle, \qquad (5.8)$$

$$[M^{n,m}]_{ij} = \langle B_i^n, B_j^m \rangle, \qquad (5.9)$$

for $i = 1, 2, \ldots n-1$, $j = 1, \ldots, m-1$. Here, the matrices $K^{n,m}$ and $M^{n,m}$ denote the $V-$ and $H-$ inner products of the estimator and interpolationg splines, respectively. The vectors $W_m(t) = [w(t, \frac{1}{m}l), \ldots, w(t, \frac{m-1}{m}l)]$ and $\dot{W}_m(t) = [w_t(t, \frac{1}{m}l), \ldots, w_t(t, \frac{m-1}{m}l)]$ used above are the vectors of beam displacements and velocities at the points $(\frac{1}{m}l, \ldots, \frac{m-1}{m}l)$. These beam displacements and velocities are the observations taken from experimental data via the output of proximity sensors placed along the beam, as shown in Figure 5.1. For simulation purposes we implemented these displacements and velocities by simulating numerically the nonlinear beam plant, given by equation (5.1), via a finite dimensional approximation scheme similar to the one used for the state estimator equation (5.2). Thus, we have

$$w(t,x) \approx w^N(t,x) = \sum_{i=1}^{N-1} W_i^N(t) B_i^N(x), \qquad (5.10)$$

with $N >> n$. Therefore the vectors $W_m(t)$ and $\dot{W}_m(t)$ are given by

$$\begin{aligned} W_m(t) &= \left[w\left(t, \frac{1}{m} l\right), \dots, w\left(t, \frac{m-1}{m} l\right) \right] \\ &\approx \left[B_j^N \left(\frac{i}{m} l \right) \right]_{ij} W^N(t), \end{aligned} \qquad (5.11)$$

and

$$\dot{W}_m(t) \approx \left[B_j^N \left(\frac{i}{m} l \right) \right]_{ij} \dot{W}^N(t), \qquad (5.12)$$

for $i = 1, \dots, m-1$, $j = 1, \dots, N-1$, where $W^N(t)$, $\dot{W}^N(t)$ are the infinite (approximated) dimensional vectors of generalized coordinates of the beam displacement and velocity and are given by

$$W^N(t) = \left[W^N(t)\right]_i = \left[W_1^N(t), \dots, W_{N-1}^N(t)\right].$$

$$\dot{W}^N(t) = \left[\dot{W}^N(t)\right]_i = \left[\dot{W}_1^N(t), \dots, \dot{W}_{N-1}^N(t)\right].$$

Therefore, the beam displacement and velocity generalized coefficient vectors $W^n(t)$ and $\dot{W}^n(t)$ are then approximated by

$$W^n(t) \approx (K^n)^{-1} K^{n,m} (L^m)^{-1} \left[B_j^N \left(\frac{i}{m} l \right) \right]_{ij} W^N(t).$$

$$\dot{W}^n(t) \approx (M^n)^{-1} M^{n,m} (L^m)^{-1} \left[B_j^N \left(\frac{i}{m} l \right) \right]_{ij} \dot{W}^N(t).$$

Remark 5.2 *Because the plant information is required to be infinite dimensional, in this case we take the index N much larger than the parameter estimator's discretization index, n. Due to implementation restrictions and computational efficiency, we have the number of observations to be less than the estimator's index n. Thus, we have*

$$N \gg n \geq m.$$

We summarize the above numerical implementation below:

Step 1. Generate the $(N-1)$-dimensional ("approximated infinite dimensional") plant displacement and velocity vectors $W^N(t)$ and $\dot{W}^N(t)$ using

$$\begin{aligned} & M^N D_t^2 W^N(t) + q_3 K^N D_t W^N(t) \\ & +(q_1 + q_{2p}\alpha^N + q_{2n}(1-\alpha^N)) K^N W^N(t) \\ & = \mathcal{K}^B F^N(t), \end{aligned} \qquad (5.13)$$

$$\begin{aligned} W_0^N &= (K^N)^{-1} \int_0^1 D_x^2 w_0(x) D_x^2 B_i^N(x)\, dx \\ W_1^N &= (M^N)^{-1} \int_0^1 w_1(x) B_i^N(x)\, dx \end{aligned} \qquad (5.14)$$

where K^N and M^N are the $(N-1) \times (N-1)$ *stiffness* and *mass* matrices, respectively. The forcing (input) term is similarly given by

$$F^N(t) = \left[\int_{\alpha_1}^{\alpha_2} D_x^2 B_i^N(x)\, dx \right] u_{patch}(t).$$

Step 2. Use this "approximated" infinite dimensional information at the $(m-1)$ sensors to project them onto the estimator $(n-1)$ splines via

$$(K^n)^{-1} K^{n,m} (L^m)^{-1} \left[B_j^N \left(\frac{i}{m} l \right) \right]_{ij} W^N(t).$$

and

$$(M^n)^{-1} M^{n,m} (L^m)^{-1} \left[B_j^N \left(\frac{i}{m} l \right) \right]_{ij} \dot{W}^N(t).$$

Step 3. Implement the state estimator (5.2) and the parameter estimator (5.3) using the approximated measured beam displacement and velocity, $W^n(t)$ and $\dot{W}^n(t)$.

Remark 5.3 *Because of the terms $\alpha^N(t)$ and $(1-\alpha^N(t))$, equation (5.13) was actually generated by*

$$\begin{aligned} & M^N D_t^2 W^N(t) + q_3 K^N D_t W^N(t) + q_1 K^N W^N(t) \\ & \quad + \sum_{\xi=0}^{\Xi} \left(q_{2p} K_{\xi,+}^{N,\Xi}(t) + q_{2n} K_{\xi,-}^{N,\Xi}(t) \right) W^N(t) \\ & = \mathcal{K}^B F^N(t), \end{aligned} \qquad (5.13)'$$

where the family of $(N-1) \times (N-1)$ matrices $\left\{K_{\xi,+}^{N,\Xi}(t)\right\}_{\xi=0}^{\Xi}$ and $\left\{K_{\xi,-}^{N,\Xi}(t)\right\}_{\xi=0}^{\Xi}$ are given by

$$\left[K_{\xi,+}^{N,\Xi}(t)\right]_{ij} = \int_0^l \alpha^N(t, x_\xi) D_x^2 B_i^N(x) D_x^2 B_j^N(x)\, dx$$

$$\left[K_{\xi,-}^{N,\Xi}(t)\right]_{ij} = \int_0^l [1 - \alpha^N(t, x_\xi)] D_x^2 B_i^N(x) D_x^2 B_j^N(x)\, dx$$

for $i, j = 1, \dots N-1$, $\xi = 0, 1, \dots, \Xi$ and the points x_ξ are given by

$$x_\xi = \frac{\xi}{\Xi} l, \quad \xi = 0, 1, \dots, \Xi.$$

The indicator function $\alpha^N(t, x_\xi)$ is given by

$$\alpha^N(t, x_\xi) = \begin{cases} 1 & \textit{if } w_{xx}(t, x_\xi) > 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Note that

$$K^N = \sum_{\xi=0}^{\Xi} \left(K_{\xi,+}^{N,\Xi}(t) + K_{\xi,-}^{N,\Xi}(t) \right).$$

Similarly, equations (5.2) *and* (5.3*b*), (5.3*c*) *are given by*

$$
\begin{aligned}
&M^n D_t^2 V^n(t) - q_3^* K^n D_t E_n(t) \\
&\quad -q_1^* K^n E_n(t) + \hat{q}_1^n(t) K^n W^n(t) \\
&\quad + \sum_{\xi=0}^{\Xi} \left(\hat{q}_{2p}^n(t) K_{\xi,+}^{n,\Xi}(t) + \hat{q}_{2n}^n(t) K_{\xi,-}^{n,\Xi}(t) \right) W^n(t) \\
&\quad + \hat{q}_3^n(t) K^n \dot{W}^n(t) = \mathcal{K}^B F^n(t) \qquad (5.2)'
\end{aligned}
$$

$$\dot{\hat{q}}_{2p}^n(t) = \lambda_{2p} (W^n)^T(t) \left(\textstyle\sum_{\xi=0}^{\Xi} K_{\xi,+}^{n,\Xi}(t) \right) G_n(t), \quad (5.3b)'$$

$$\dot{\hat{q}}_{2n}^n(t) = \lambda_{2n} (W^n)^T(t) \left(\textstyle\sum_{\xi=0}^{\Xi} K_{\xi,-}^{n,\Xi}(t) \right) G_n(t), \quad (5.3c)'$$

where $E_n(t) = W^m(t) - V^n(t)$ *and the family of* $(n-1)\times(n-1)$ *matrices* $\left\{K_{\xi,+}^{n,\Xi}(t)\right\}_{\xi=0}^{\Xi}$ *and* $\left\{K_{\xi,-}^{n,\Xi}(t)\right\}_{\xi=0}^{\Xi}$ *are given in an analogous manner by*

$$\left[K_{\xi,+}^{n,\Xi}(t)\right]_{ij} = \int_0^l \alpha^n(t, x_\xi) D_x^2 B_i^n(x) D_x^2 B_j^n(x)\, dx$$

$$\left[K_{\xi,-}^{n,\Xi}(t)\right]_{ij} = \int_0^l [1 - \alpha^n(t, x_\xi)] D_x^2 B_i^n(x) D_x^2 B_j^n(x)\, dx$$

for $i, j = 1, \ldots n-1$, $\xi = 0, 1, \ldots, \Xi$ *with the indicator function* $\alpha^n(t, x_\xi)$ *given in an analogous manner.*

For our numerical simulations we assumed that $q_1 \equiv 0$ and that the damping coefficient, q_3 is known. The nonlinear stiffness ($EI(w_{xx}) = g(w_{xx})$) is given by

$$g(w_{xx}(t,x)) = \begin{cases} 75 w_{xx}(t,x) & \text{if } w_{xx}(t,x) > 0 \\ 70 w_{xx}(t,x) & \text{otherwise} \end{cases},$$

for $0 \le x \le l$, $t > 0$, the damping parameter is $c_D I(x) = q_3(x) = 0.001$, $0 \le x \le l$ and the linear mass density is $\rho_b = 1.35$. The tuning parameters (see [10]) q_1^* and q_3^* are chosen to be

$$q_1^*(x) = 80, \;\; q_3^*(x) = 0.02, \quad 0 \le x \le l.$$

The adaptive gains $\lambda_{2p}, \lambda_{2n}$ in (5.3b)', (5.3c)' are

$$\lambda_{2p} = \lambda_{2n} = 1 \times 10^5,$$

the parameter γ in (5.2)' is $\gamma = 1 \times 10^3$, the initial guesses for the parameter estimates are

$$\hat{q}_{2p}(0) = \hat{q}_{2n}(0) = 65$$

and the plant and estimator states are

$$v(0,x) = v_t(0,x) = 0,$$

$$w(0,x) = 2 \times 10^{-3} x^2 (x-l)^2,$$

$$w_t(0,x) = 1 \times 10^{-2} \sin^2(2\pi x/l) \cos(2\pi x/l),$$

for $0 \le x \le l$. The beam length is $l = 0.60$ and the (centered) patch covers a half of the beam length, i.e.

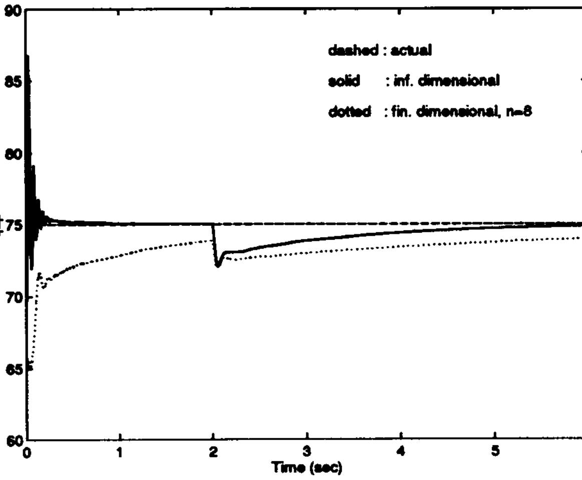

Figure 5.2: Parameter $\hat{q}_{2p}(t)$

$x_1 = 0.15$ and $x_2 = 0.45$. The piezoceramic constant is $\mathcal{K}_B = 0.002331655$ and the voltage is

$$
\begin{aligned}
u_{patch}(t) &= \sin(150\pi t) + \sin(650\pi t) \\
&\quad + \sin(400\pi t) + \sin(800\pi t).
\end{aligned}
$$

We now describe the results of our numerical simulations. We simulated the plant (3.7) with

$$
\begin{aligned}
g(w_{xx}) &= 75 w_{xx} & 0 \le t \le 2, \\
g(w_{xx}) &= \begin{cases} 75 w_{xx} & \text{if } w_{xx} > 0 \\ 70 w_{xx} & \text{otherwise} \end{cases} & 2 < t \le 6.
\end{aligned}
$$

This stiffness simulates a plant that initially ($0 \le t \le 2$) has a linear stiffness parameter that becomes nonlinear for $2 < t \le 6$ and assumes different values depending on the sign of the curvature ($\alpha(t) = 1$ if $u_{xx} > 0$). In Figures 5.2, 5.3 we plot the actual values of the parameters ($q_{2p} = 75$, and $q_{2n} = 75$ for $t \le 2$, $q_{2n} = 70$ for $t > 2$), their estimates based on the *ideal* estimator (infinite dimensional estimator using infinite dimensional plant information) and the parameter estimates based on the *approximated* estimator (the finite dimensional parameter estimator using finite dimensional interpolated plant information). We observe that both parameters $\hat{q}_{2p}$ and $\hat{q}_{2n}$ are identified and that the time ($t = 2$) that the nonlinearity occurs is *sensed* by the estimator. Specifically, we observe that for the *ideal* case , we have better convergence than the *approximated* case (that is, $n = 8$ dimensional estimator with $m = 8$ interpolation data and an $N = 16$ "infinite" dimensional plant) as expected.

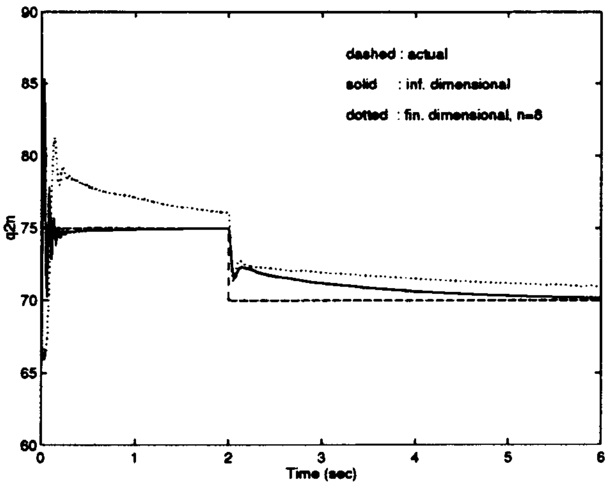

Figure 5.3: Parameter $\hat{q}_{2n}(t)$

The finite dimensional estimator does provide an acceptable estimate of the parameters given the fact that only a small number of displacement and velocity information is available. This is important from the implementation point of view, as it is seldom the case that full (infinite dimensional) plant velocity and displacement is available.

6. Conclusions and Future Research

An extensive study is undertaken to test this estimator numerically and specifically the relaxation of the requirement of the full knowledge of the plant state ($w(t,x)$ and $w_t(t,x)$) in the state and parameter estimator, (3.11), (3.12), via finite dimensional approximation. Using results from our previous work, [8], we use the finite dimensional approximation framework developed there and present some of our more recent numerical findings which use finite dimensional approximation of the infinite dimensional plant state ($w(x,t)$ and $w_t(x,t)$). Thus, the finite dimensional approximation of the state and parameter estimator, an easier system to implement numerically, uses a finite dimensional approximation of the plant state. The results are comparable to the *ideal case* thus suggesting a succesfull implementation of this finite dimensional parameter estimator for the infinite dimensional plant.

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