An Algebra for Modelling Manufacturing Processes

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1. INTRODUCTION

1.1. The modelling problem of discrete production processes

The formal description of the real world by means of a mathematical language, what is usually meant as construction of a mathematical model, is the indispensable premise to the design of any management and control system of the real world itself.

At this date, a formal language having been developed *ad hoc* or, in any case, looking fully appropriate to describe discrete manufacturing precesses, does not exist. Two approaches are usually adopted: *Queueing Theory* and *Petri Nets*. Besides them, other approaches exist, employing typical models of continuous dynamical systems, like differential equations. None of such methodologies, which have been developed for other fields of science, is suitable for providing an accurate description, as far as needed, of a discrete production process. They impose instead on process description approximations, which are neither desired by the modelling engineer nor yield always accepatable results, but are inherent to the specific mathematical formalism, owing to its lack of descriptive capabilities, adequate to the kind of physical phenomena under description.

In summary, by using such tools, investigators are prevented from establishing the right degree of approximation they retain necessary in the process modelling; the tool instead imposes on its own some simplifying assumptions independently of the experimenter desires. To overcome such limits of the available methodologies, investigators are brought to select the most appropriate approach, according to the specific case and problem to be solved. Very often they are forced to introduce heuristic variants as a result of the effort for overcoming the inherent limitations of the adopted methodology. The natural consequence is a great variety of methods, as a boundless bibliography evidences, but highly customized, and hence difficult to be transferred from case to case.

For the same reasons, a success can hardly be claimed by all the actions that have been undertaken, during the last ten years, by suppliers, universities and customers, either autonomously or promoted by the European Community, and mainly directed to develop open hardware and software architectures in the field of CIM. The reference architecture CIMOSA is one of the best known examples: notwithstanding the validity of the concepts and the strong effort made, its potentialities are very far from being recognized and used.

Several people think that the aforementioned problems could remarkably benefit by the availability of a descriptive methodology for discrete production processes which is at the same time mathematically rigorous and capable of treating the whole set of possible cases, either working at very detailed levels or with very sinthetic aggregations. The authors are working since a few years to provide a mathematical methodology, having the precise objective of becoming a suitable tool for solving the management problems of discrete production processes, from production planning to real-time production control [1,2]. This work is under development within the project HIMAC, a Basic Research Project, part of the ESPRIT project of the European Community. The project is at the end of its first year and the first results, presented in this paper, look quite promising.

The presentation is subdivided into two papers: the first one is concerned with the Manufacturing Algebra, the second one [3] with the Factory Dynamics. The papers are strictly interconnected, since Dynamics is founded on Algebra.

1.2. Aim of the Algebra

The Algebra aims at describing the production process as an ordered sequence of manufacturing operations, which starting from raw materials, through the production of semifinished objects, provides at the end few finished products.

The Algebra approach is axiom-based; i.e. although one is keeping an eye on the manufacturing process technology, the authors mean to formulate a self-consistent mathematics, were all the results can be derived from few axioms through a logical sequence of theorems and corollaries. The eye which is kept open on the physical reality should guarantee that the developing Algebra be a sound instrument for *modelling* production processes in a form whhich is particularly suitable for solving management, planning and control problems. An important remark, however, is that the Algebra is not meant to be a discipline of Physics, devoted to studying and analysing the reality of the production processes, but instead, as the name itself suggests, a mathematical discipline. Specifically it is meant to be a discipline in the hands of investigators for *modelling*, in other words for describing production processes at any detail level as imposed by the study goals.

Algebra will use as far as possible the same terms currently used for describing manufacturing processes (f.i. raw materials, reusable objects, semifinished and finished products, manufacturing operations, operation time, production cycles,...) to denote those mathematical elements which have been conceived to describe physical entities of the same name. Let us however point out that it is not allowed to confuse a mathematical element, rigorously defined on an axiomatic base, with the physical reality, which behind the same name hides something complex and changing from case to case. It shall be the investigator's responsibility to ascertain whether a specific algebraic element is suitable or not for describing reality according to his study objectives.

In this note having only the scope of an introductory presentation, all demonstrations have been omitted and the mathematical formalism has been reduced as well to not burden the reading of people willing to get only a preliminary idea of the work under development.

2. THE ELEMENTS OF THE ALGEBRA

2.1. The set of the objects

A manufacturing process has to do with material parts of different kinds, like raw materials, tools, fixtures, components, semifinished and finished products, etc. A common property is that all of them, independently of their use, can be moved all around the factory. They will be denoted generically as *manufacturing objects* or simply *objects.* A basic assumption is that they form a countable and finite set $\boldsymbol{O}$, whose cardinality is n_k. Thus the elements of $\boldsymbol{O}$ can be ordered and related one-to-one with the subset $Z\{1,n_k\}$ of the integer numbers; i.e. any object can be identified by its index k.

2.2. The space of the object quantities

A generic quantity of objects belonging to $\boldsymbol{O}$, is described by an n_k-dimensional vector $\boldsymbol{q}$, whose component q_k gives the quantity of the object k. Only integer quantities are considered, then q_k belongs to the set of integer numbers **Z**. The quantity q_k can be either positive or negative: a positive component of the object k will mean *availability* of that object. Instead, a negative quantity will mean *need* of that object.

The set $\boldsymbol{Q}$ of the quantity vectors $\boldsymbol{q}$, equipped with the following elements and operations, will be *the space of the object quantities* or simply *the quantity space.*

Null vector $\boldsymbol{o}$. It is defined as $o_k = 0$ for $k \in \mathbf{Z}\{1, n_k\}$.

Vector addition:

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z} \tag{1}$$

The vector addition is defined by $x_k = y_k + z_k$ for $k \in \boldsymbol{Z}\{1, n_k\}$.

Negative quantity vector. $\boldsymbol{z}$ is the negative quantity of $\boldsymbol{x}$ iff $\boldsymbol{z} + \boldsymbol{x} = \boldsymbol{o}$; since $z_k = - x_k$ holds, the following notation will be used:

$$\boldsymbol{z} = - \boldsymbol{x} \tag{2}$$

Integer scalar multiplication. A vector $\boldsymbol{z}$ is the scalar multiplication of the integer α times the vector $\boldsymbol{x}$, iff $z_k = \alpha x_k$ for $k \in \mathbf{Z}\{1, n_k\}$; it will be denoted by

$$\boldsymbol{z} = \alpha \boldsymbol{x} \tag{3}$$

Logic union. A vector $\boldsymbol{z}$ is the union of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ iff $z_k = max\ (x_k, y_k)$ for $k \in \boldsymbol{Z}\{1, n_k\}$; it will be denoted by

$$\boldsymbol{z} = \boldsymbol{x} \cup \boldsymbol{y} \tag{4}$$

Logic intersection. A vector $\boldsymbol{z}$ is the intersection of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ iff $z_k = min\ (x_k, y_k)$ for $k \in \boldsymbol{Z}\{1, n_k\}$; it will be denoted by

$$\boldsymbol{z} = \boldsymbol{x} \cap \boldsymbol{y} \tag{5}$$

2.3. The space of the manufacturing operations

Definition

Using the quantity space $\boldsymbol{Q}$, a very generic definition of manufacturing operation is now possible; the definition allows to encompass all the operations being stage of any discrete manufacturing process, like assembling, disassembling, part machining and transformation, setting-up, fixturing, transporting, quality control, repairing, ...

A *manufacturing operation A* is defined as an element of the Cartesian product $A = Q \times Q$; i.e.

$$A = (u, y),\ u \in Q,\ y \in Q \tag{6}$$

where u and y denote respectively the *input quantity vector* and the *output quantity vector* of the *manufacturing operation A*.

Graphical symbol of an operation

In Figure 1 an operation A is represented by a box and by arrows connecting it to circles, representing input and output objects. Arrow direction goes from input circles to operation and from operation to output circles; the object quantity is written on the side of the corresponding arrow.

$$u = \left|\begin{matrix} u_1 \\ u_2 \\ u_3 \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right| \quad y = \left|\begin{matrix} 0 \\ 0 \\ y_3 \\ y_4 \\ 0 \\ \vdots \\ 0 \end{matrix}\right|$$

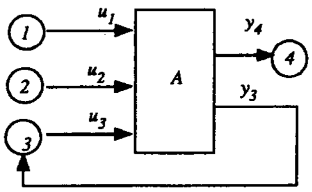


Figure 1. Symbol of an operation.

As an example consider the operation A in Figure 1. It has three kinds of input objects; two of them, the objects *1* and *2*, are assembled into the product *4*; the third one, the object *3*, is used to perform the operation, but a quantity y_3 appears among the output objects. Several kinds of manufacturing objects behave like the object *3*; think for example to fixtures, tools,... Note that depending on the quantity ratio u_3/y_3 different uses of the fixture can be described:

- if $u_3/y_3=1$, all the fixtures used by the operation A are reusable at its end;
- if $u_3/y_3>1$, part of the fixtures, u_3-y_3 are assembled in the product *4*;
- if $u_3/y_3<1$, one or both the input objects *1* and *2* are assemblies including y_3-u_3 fixtures.

Hence the operation A describes an assembling operation which includes the disassembling of a fixture-like object.

Graphical interaction of operations

Manufacturing operations interact each other as far as they operate on the same objects. Using the above symbols, interactions can be put in evidence graphically. An important type of interaction, shown in Figure 2, happens when the same objects that are requested as input by an operation are provided as output by another operation. As an example, a pair of interacting operations, describing a two-stage assembling, is illustrated in Figure 2: the operation $B=(p,q)$ has an input object, the object *4*, which is also the output object of the operation $A=(u,y)$; it means that the quantity vectors y and p have the same component *4* which is different from zero; this fact can be generically formulated using the logic intersection, i.e. $y \cap p \neq o$.

$$u = \left|\begin{matrix} u_1 \\ u_2 \\ u_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right| = \left|\begin{matrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right| \quad y = \left|\begin{matrix} 0 \\ 0 \\ 0 \\ y_4 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right| = \left|\begin{matrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right| \quad p = \left|\begin{matrix} 0 \\ 0 \\ 0 \\ p_4 \\ p_5 \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right| = \left|\begin{matrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right| \quad q = \left|\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ q_6 \\ 0 \\ \vdots \\ 0 \end{matrix}\right| = \left|\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{matrix}\right|$$

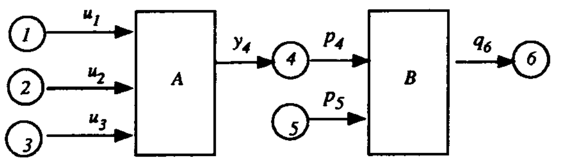


Figure 2. Graphical interconnection of two assembling operations.

Addition and multiplication of operations

The set A of the manufacturing operations is equipped with the following elements and operations.

The *Null operation O* is defined as $O = (o, o)$; i.e., input and output quantity vectors are zero.

The *Identity operation I* is defined as $I = (q, q)$ for any $q \in Q$; i.e., the identity does not alter the input quantities. The identity operation is not unique, but there exist as many identities as the number of quantity vectors in the space Q. Such an operation will be mainly used to describe *transport operations*.

Addition (or *parallel composition)*. Given a pair of manufacturing operations $B = (q, p)$ and $C = (r, s)$, their sum A is defined as the quantity pair $A = (u, y)$, where $u = q + r$ and $y = p + s$. The sum is indicated as

$$A = B + C \tag{7}$$

The addition models the aggregation of two or more manufacturing operations when the output objects of the component operations are not used as input to either operation. Then the addition is suitable to model manufacturing operations which are performed simultaneously or in parallel.

Negative operation. Given an operation B, its negative A is the operation such that their parallel composition equals the null operation O; i.e. $A + B = O$. We shall use the formalism

$$A = -B \tag{8}$$

Integer scalar multiplication (or *parallel repetition*). Given any operation $A = (\boldsymbol{u}, \boldsymbol{y})$, the product of A times the scalar integer α is the α-fold parallel composition of A; i.e.

$$A = \alpha B = B + B + \ldots + B \;\; (\alpha \text{ times}) \tag{9}$$

where $A = (\alpha \boldsymbol{u}, \alpha \boldsymbol{y})$.

Multiplication (or *series composition*). Given a pair of manufacturing operations $B = (\boldsymbol{p}, \boldsymbol{q})$, and $C = (\boldsymbol{r}, \boldsymbol{s})$, the *left multiplication* of B times C, denoted either with CB or $C \cdot B$, is defined by

$$A=(\boldsymbol{u},\boldsymbol{y})=CB, \;\; \boldsymbol{u}=\boldsymbol{p}+\boldsymbol{r}-\boldsymbol{q}\cap\boldsymbol{r}, \;\; \boldsymbol{y}=\boldsymbol{q}+\boldsymbol{s}-\boldsymbol{q}\cap\boldsymbol{r} \tag{10}$$

which means that the output of the operation B is made available as input to the operation C. The *right multiplication* of B times C, denoted either with BC or $B \cdot C$, is defined similarly by

$$D=(\boldsymbol{w},\boldsymbol{z})=BC, \;\; \boldsymbol{w}=\boldsymbol{p}+\boldsymbol{r}-\boldsymbol{s}\cap\boldsymbol{p}, \;\; \boldsymbol{z}=\boldsymbol{q}+\boldsymbol{s}-\boldsymbol{s}\cap\boldsymbol{p} \tag{11}$$

The multiplication models the aggregation of manufacturing operations when the output objects of the first operation are used as input to the second operation. Then the multiplication is suitable to model manufacturing operations which are performed sequentially so that the input objects of the second operation can include output objects of the first one.

The inverse with respect to multiplication. Given any operation $A=(\boldsymbol{u},\boldsymbol{y})$, there exists a unique inverse operation denoted by A^{-1} and such that

$$A^{-1} = (\boldsymbol{y},\boldsymbol{u}) \tag{12}$$

Power (or *series repetition*) The β-th power $A = (\boldsymbol{u}, \boldsymbol{y})$ of an operation $B = (\boldsymbol{p}, \boldsymbol{q})$ is defined as the α-fold series composition of B, i.e.,

$$A = B^{\beta} = B \cdot B \cdot \;\ldots\; \cdot B \;\; (\beta \text{ times}) \tag{13}$$

where it is easy to show that

$$\boldsymbol{u} = \beta \boldsymbol{p} - (\beta-1)(\boldsymbol{p} \cap \boldsymbol{q}), \;\; \boldsymbol{y} = \beta \boldsymbol{q} - (\beta-1)(\boldsymbol{p} \cap \boldsymbol{q}) \tag{14}$$

<u>Example</u>

As an example of the different ways addition and multiplication work, consider the pair of the assembling operations A and B illustrated in Figure 2. The operation $D=(\boldsymbol{r},\boldsymbol{s})$ will be the composition result, which is illustrated in Figure 3. One of the objectives driving the algebraic composition is to balance the quantity of objects produced by some operations and used by other operations. Objects to be balanced are usually called semifinished objects; here there is only one semifinished object, the object *4*, whose balance requires the operation A to be repeated twice. The same operation can be repeated either in series or parallel; here since the input and output quantity vectors of A do not share components, i.e. $\boldsymbol{u}\cap\boldsymbol{y}=\boldsymbol{o}$, the parallel and series repetition provide the same result, which means $C=2A=A^2$. This will be not the case if C and B are composed: by applying the multiplication $BA^2=B(2A)$, the semifinished object *4* disappears from the input and output quantities of the resulting operation D.

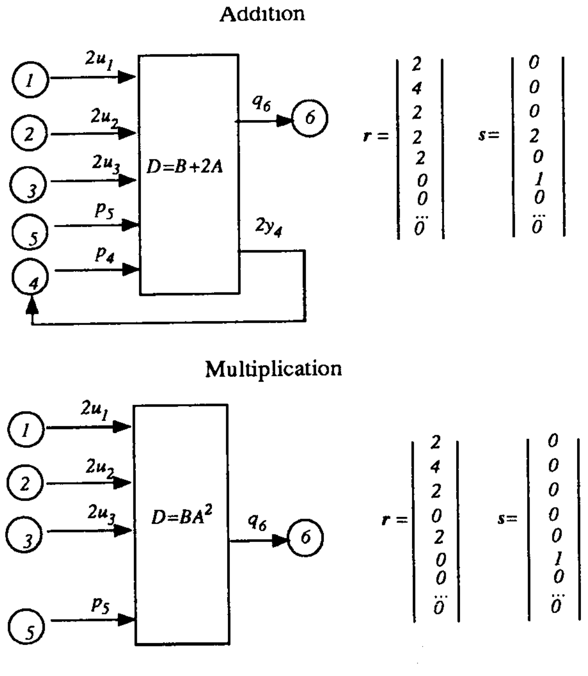


Figure 3. Addition and multiplication results.

2.4. *Properties of the manufacturing operation Algebra*

In this section the properties of addition and multiplication are shortly introduced.

The addition is commutative and associative. Let $A_1=(\boldsymbol{u}_1,\boldsymbol{y}_1)$, $A_2=(\boldsymbol{u}_2,\boldsymbol{y}_2)$ and $A_3=(\boldsymbol{u}_3,\boldsymbol{y}_3)$ be manufacturing operations.

- The *commutative property*, i.e. $A_1 + A_2 = A_2 + A_1$, follows immediately from the commutative property of the vector addition in the quantity space $\boldsymbol{Q}$.
- The *associative property*, i.e. $(A_1 + A_2) + A_3 = A_1 + (A_2 + A_3)$ follows immediately from the associative property of the vector addition in the quantity space $\boldsymbol{Q}$.

The multiplication is associative, but not commutative. Let $A_1=(\boldsymbol{u}_1,\boldsymbol{y}_1)$, $A_2=(\boldsymbol{u}_2,\boldsymbol{y}_2)$ and $A_3=(\boldsymbol{u}_3,\boldsymbol{y}_3)$ be manufacturing operations.

- The *associative property*, i.e. $(A_1 \; A_2)\, A_3 = A_1 \; (A_2$

A_3), is proven by verifying the equality between the common quantities that are subtracted from the input and output vectors.

- The *commutative property* does not hold in general, i.e. $A_1 \cdot A_2 \neq A_2 \cdot A_1$; hence it is of interest to find out the conditions for the commutative property holds.

Lemma. Given a pair of operations $A_1=(\boldsymbol{u}_1,\boldsymbol{y}_1)$ and $A_2=(\boldsymbol{u}_2,\boldsymbol{y}_2)$, a sufficient condition for the commutative property holds is that:

$$\boldsymbol{u}_1 \cap \boldsymbol{y}_2 = \boldsymbol{u}_2 \cap \boldsymbol{y}_1 = \boldsymbol{o} \tag{15}$$

i.e. if no output object of one operation can be used as input to the other one.

Definition. A pair of operations $A_1=(\boldsymbol{u}_1,\boldsymbol{y}_1)$ and $A_2=(\boldsymbol{u}_2,\boldsymbol{y}_2)$ satisfying equation (14) will be called *mutually not interacting*. Mutually not interacting operations can be composed both by multiplication and by addition, since it holds

$$B \cdot C = C \cdot B = C + B \tag{16}$$

Example. The operation A in Figure 2 is not interacting with itself as already pointed out; instead, the operations A and B in the same Figure are mutually interacting and their addition and multiplication are different.

For what concerns distributive properties the following results can be easily proved:

- The *distributive property* of the addition, with respect to the multiplication, does not hold in general, i.e. $A_1 + (A_2A_3) \neq (A_1+A_2)\,(A_1+A_3)$.
- The *distributive property* of the multiplication, with respect to addition, does not hold in general, i.e. $A_1(A_2+A_3) \neq (A_1A_2)+(A_1A_3)$.

2.5. Stochastic manufacturing operations

A manufacturing operation is said to be *stochastic* when some or all of its variables are stochastic. In particular the following variables can be stochastic:

- The quantities of objects required as *inputs* of manufacturing operations. An example is given by the *repair* operations. Whenever the kind of failure to be repaired is not identified *a priori*, it is impossible to ascertain which parts will have to be replaced, therefore the input of the operation turns out to be stochastic.
- The quantities of objects produced as outputs of manufacturing operations. An example is given by the quality control operations: the input, consisting of an object *still to be qualified*, is deterministic, but the output can be - with probability p - an object *qualified as good* or - with probability $(1 - p)$ - an object *qualified as damaged*. Note that, from the viewpoint of the algebra, the fact that an object has not yet been qualified, or that it has been qualified as good or as damaged, will define its belonging to one out of three different types of objects.
- A functional, associated with the manufacturing operation. An example is the manufacturing time, which could be defined as a stochastic variable, whose mean value and variance are given.

The composition of stochastic manufacturing operations is a stochastic operation. Usually either the hypothesis is made that all operations be deterministic, or the whole process is treated as a stochastic one. It is a choice of modelling, to be made in connection with the objects of the study.

3. INDEPENDENT OPERATIONS AND PRODUCTION CYCLES.

3.1. The balance vector of a manufacturing operation

Given any set $S=\{A_1,\ldots,A_h,\ldots,A_n\}$ of operations, new operations can be obtained by applying addition and multiplication. Denote a generic algebraic composition with the following equation:

$$A=f(A_1,\ldots,A_h,\ldots,A_n;\alpha_1,\ldots,\alpha_h,\ldots,\alpha_n) \tag{17}$$

where each operation A_h is repeated a total of $\alpha_h \geq 0$ times, either in series or in parallel. If the operations are mutually interacting, the input and output vectors of A will be different and depending on the kind of the compositions employed and on their order. There exists however a quantity vector which is independent of the kind and order of the compositions and only dependent on the set $\boldsymbol{S}$ and the repetition factors α_h.

Definition. The *balance* of a manufacturing operation is defined as the difference between the vector of the object quantities produced (output) and the vector of the object quantities required (input). Considering the operation $A = (\boldsymbol{u},\boldsymbol{y})$, the balance vector is defined as

$$\boldsymbol{b} = \boldsymbol{y} - \boldsymbol{u} \tag{18}$$

In the balance vector, the quantity of objects required by the operation appears with the negative sign, whereas the quantity of objects generated by the operation appears with the positive sign.

The following results can be easily proved.

Theorem. Given a set $\boldsymbol{A}=\{A_1,\ldots,A_h,\ldots,A_n\}$ of operations, the balance vector $\boldsymbol{b}$ of any algebraic composition $A=f(A_1,\ldots,A_h,\ldots,A_n;\alpha_1,\ldots,\alpha_h,\ldots,\alpha_n)$ equals the linear combination of the balance vectors $\boldsymbol{b}_h$ of the component operations

$$\boldsymbol{b}=\Sigma_{h=1,\ldots,n}\boldsymbol{b}_h\alpha_h \tag{19}$$

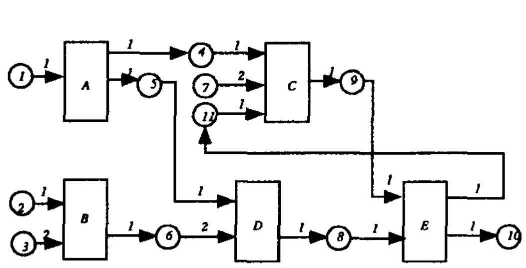
Corollary. Given a set of manufacturing operations, independently of their composition in parallel or in series and of the order of the series operations, the resulting equivalent operation has always the same balance vector.

3.2. Set of independent operations.

In this section the problem of defining a set of independent manufacturing operations is afforded. The goal is to separate between operations which are not technologically replaceable even if arbitrarily composed using addition and multiplication. A very generic definition is possible using algebraic compositions; here we shall limit to define independency on the base of the balance vector.

Definition. A finite set of n operations $S=\{A_1,\dots,A_h,\dots,A_n\}$ defined on an object set $\boldsymbol{O}$ is said to be *independent,* whenever none of their balance vectors $\boldsymbol{b}_h$ can be expressed as a linear combination of the other ones. Hence a sufficient condition for independency is that the balance vectors form a vector basis or the balance matrix $\boldsymbol{B}=[\boldsymbol{b}_1\dots\boldsymbol{b}_h\dots\boldsymbol{b}_n]$ be of full rank.

Example. The set of five operations A,B,C,D,E graphically illustrated in Figure 4 are independent, as it can be easily verified on the balance matrix $\boldsymbol{B}$. The operations are shown interconnected through the common objects.



$$A = \left|\begin{matrix}1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{matrix}\right|\left|\begin{matrix}0\\0\\0\\1\\1\\0\\0\\0\\0\\0\\0\end{matrix}\right| \quad B = \left|\begin{matrix}0\\1\\2\\0\\0\\0\\0\\0\\0\\0\\0\end{matrix}\right|\left|\begin{matrix}0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\end{matrix}\right| \quad C = \left|\begin{matrix}0\\0\\0\\1\\0\\0\\2\\0\\0\\0\\1\end{matrix}\right|\left|\begin{matrix}0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\end{matrix}\right| \quad D = \left|\begin{matrix}0\\0\\0\\0\\1\\2\\0\\0\\0\\0\\1\end{matrix}\right|\left|\begin{matrix}0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\end{matrix}\right| \quad E = \left|\begin{matrix}0\\0\\0\\0\\0\\0\\0\\1\\1\\0\\0\end{matrix}\right|\left|\begin{matrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\1\end{matrix}\right|$$

$$\boldsymbol{B}=\left|\begin{matrix}-1 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & -2 & 0 & 0 & 0\\ 1 & 0 & -1 & 0 & 0\\ 1 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & -2 & 0\\ 0 & 0 & -2 & 0 & 0\\ 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 & 1\end{matrix}\right|$$

Figure 4. A set of independent operations.

The algebraic compositions of the manufacturing operations belonging to an independent set $\boldsymbol{S}$ define a numerable and infinite set $\boldsymbol{A(S)}$ of manufacturing operations, which is called the *set spanned by the operations of the independent set* $\boldsymbol{S}$.

3.3. The manufacturing operation time

Let us consider the set $\boldsymbol{A(S)}$ spanned by a set $\boldsymbol{S}$ of n independent operations A_h, $h=1,\dots,m$. A positive real number $\tau(A_h)$, called *manufacturing operation time,* and representing the time required to execute the operation, is associated to each operation $A_h \in \boldsymbol{S}$.

By applying the following rules, the manufacturing operation time is defined for any manufacturing operation belonging to the set $\boldsymbol{A(S)}$.

Let us consider the addition $A = B+C$: the time $\tau(A)$ is defined by

$$\tau(A) = max\,\{\tau(B), \tau(C)\} \tag{20}$$

Let us consider the multiplication $A= BC$: the time $\tau(A)$ is defined by

$$\tau(A) = \tau(B) + \tau(C) \tag{21}$$

3.4. The manufacturing operation cost

Let us consider the set $\boldsymbol{A(S)}$ spanned by a set $\boldsymbol{S}$ of n independent operations A_h, $h=1,\dots,m$. A positive real number $\rho(A_h)$, called *manufacturing operation cost*, and representing the cost required to carry out the operation, is associated with each operation belonging to the independent set $\boldsymbol{S}$.

By applying the following rules, the manufacturing operation cost is defined for any manufacturing operation belonging to the set $\boldsymbol{A(S)}$.

Let us consider the addition $A = B+C$: the cost $\rho(A)$ is defined by

$$\rho(A) = \rho(B) + \rho(C) \tag{22}$$

Let us consider the multiplication $A = BC$: the cost $\rho(A)$ is defined by

$$\rho(A) = \rho(B) + \rho(C)$$

3.5. Object classification

The objects concerned in a production process are subdivided, with respect to their technological characteristics, into *raw materials, semifinished products, finished products, reusable tools.* A similar subdivision of the mathematical objects making up the set $\boldsymbol{O}$ can be obtained, with respect to their algebraic properties, as it follows.

Definition. The following subsets of objects are defined.

- *Non-used objects* are the objects whose quantity is always zero both in the input and in the output quantity vectors of all the manufacturing operations belonging to $A(S)$.
- As *raw materials* we define those objects - different from the non-used objects - whose quantity is always zero in the output quantity vectors of all the manufacturing operations belonging to $A(S)$.
- As *finished products* we define those objects - different from the non-used objects - whose quantity is always zero in the input quantity vectors of all manufacturing operations belonging to $A(S)$.
- All the other objects having the property to be at the same time input and output objects of operations in $A(S)$, are either *semifinished* or *reusable.*

Although on the technological level the distinction between semifinished and reusable be clear and never ambiguous, on the algebraic level it is not immediate. In the following, it will be apparent that it is easy to find out whether or not a production cycle (whose precise definition will soon be given) requires to employ a reusable object, whereas instead it is difficult to find out which objects have to be considered reusable and which have to be considered semifinished. In fact, it can occur that a *semifinished* (in the technological significance of the word) be assembled together with a *reusable tool* (again in the technological significance of the word). In such a situation, while the two objects are distinct on the technological level, on the algebraic level, being assembled together, they make up one object. The separation of the two, if required, can take place through an operation devoted to this purpose, which has therefore the *hybrid* object as input and, as output, the *semifinished* (or *finished product)* and the *reusable tool.*

Consequently one forgets about giving a precise distinction between semifinished and reusable objects, since it was established the possible existence of objects which include semifinished products and reusable tools. For convenience, all objects which are present without distinction both in the input and in the output vectors of the operations will be called *semifinished.* Later on the condition will be given, stating that reusable tools exist among the semifinished products included in a production cycle.

The following lemma which can be immediately verified, gives necessary and sufficient conditions to assign any object $k \in \boldsymbol{O}$ to the four subsets defined by an independent operation set S.

Lemma. The following conditions hold:

- Any k object is a *non used object* iff its quantity is always zero in the input and output quantity vectors of all the manufacturing operations belonging to $\boldsymbol{S}$.
- Any k object is a raw material iff its quantity is always zero in the output quantity vectors of all the manufacturing operations belonging to $\boldsymbol{S}$.
- Any k object is a finished product iff its quantity is always zero in the input quantity vectors of all manufacturing operations belonging to S.

Example. Consider the set $\boldsymbol{S}$ illustrated in Figure 4 and built over an object set $\boldsymbol{O}$ including $n_k=11$ elements denoted by the indices $k=1,...,11$. No *non-used* object exists. The objects $\{1,2,3,7\}$ are *raw materials.* Object 10 is the only *finished product.* All the other objects are *semifinished objects.*

3.6. Minimal manufacturing operations

Definition. A manufacturing operation $A=(\boldsymbol{u},\boldsymbol{y})$ belonging to the set $A(S)$, spanned by the set of independent manufacturing operations $\boldsymbol{S}$, is said to be *minimal,* when no manufacturing operation $B=(\boldsymbol{r},\boldsymbol{s})$ exists in the set $A(S)$, whose balance vector $\boldsymbol{q}=\boldsymbol{s}-\boldsymbol{r}$ is an integer submultiple of the balance vector of A; i.e. no operation $B \in A(S)$ and no integer $\alpha>1$ exist such that $\boldsymbol{b}=\alpha\boldsymbol{q}$.

The following theorem can be easily proved.

Theorem. If $A=(\boldsymbol{u},\boldsymbol{y})$ is a minimal operation in $A(S)$, the equation

$$A = \alpha_1 B + \alpha_2 B^2 + \,..+\alpha_h B^h+.... + \alpha_m B^m \qquad (23)$$

does not admit solution for any positive integers α_h, $h=1,...,m$, any positive integer m and any $B=(\boldsymbol{r},\boldsymbol{s}) \in A(S)$, with the constraint $\alpha_1>1$.

3.7. Production cycles

Definitions

Definition. A *production cycle* is any operation $A=(\boldsymbol{u},\boldsymbol{y})$ belonging to the set $A(S)$, whose balance vector $\boldsymbol{b}=\boldsymbol{y}-\boldsymbol{u}$ contains only raw materials (in negative quantities) and finished products (in positive quantities).

The production cycle of an object k is a production cycle whose balance vector does not contain other finished products different from k.

A *mix* of objects is a nonnegative quantity vector $\boldsymbol{m}$ such that no other nonnegative quantity vector $\boldsymbol{n}$ and positive integer $\alpha>1$ exist satisfying the equality $\boldsymbol{m}=\alpha\boldsymbol{n}$; i.e. the unique gretest common divisor of the vector $\boldsymbol{m}$ is the unity. The production cycle of a given mix $\boldsymbol{m}$ of objects is the production cycle which has, in its balance vector, the finished products of the mix in the same quantity ratio defined by the mix.

Definition. The *Bill-of-Materials* of a production cycle is defined as its balance vector $\boldsymbol{b}$. It lists with the negative sign the quantities of raw materials required to produce the quantities of finished products, listed with the positive sign.

Definition. A *minimal production cycle* is a production cycle which is a *minimal* manufacturing operation in the set $A(S)$.

Minimal production cycles have the following properties:

- For a given object or mix $\boldsymbol{m}$ of objects, more then one minimal production cycles exist, all characterized by the same balance vector, but with input and output vectors including nonzero quantities of semifinished products.
- Each non-minimal production cycle has a balance vector which is an integer multiple of the balance vector of the corresponding minimal production cycle.

Definition. A production cycle will be said to not require reusable objects, whenever at least one out of the different implementing solutions (all with the same balance vector) has input and output quantity vectors which do not include any object other than raw materials and finished products.

A production cycle is said to require reusable objects, when they can not be eliminated by any reordering of the manufacturing operations from the input and output quantity vectors of the production cycle, even if they can turn out to be included in a semifinished product, hence forming a single object.

Example

Let us consider again the set S illustrated in Figure 4. Several minimal production cycles can be implemented, for instance:

$$W_1 = (A+2B+C+D+E) = \begin{vmatrix} 1 \\ 2 \\ 4 \\ 1 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \\ 0 \\ 1 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 2 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} \quad W_2 = (E\,CDAB^2) = \begin{vmatrix} 1 \\ 2 \\ 4 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{vmatrix} \qquad (24)$$

The cycle described by the operation W_1 assumes the performance in parallel of all the operations required to produce object *10*, starting from the raw materials $\{1,2,3,7\}$. In order to perform the operations in parallel, it is necessary that a given quantity of semifinished products be available, whereas an equal quantity will be produced.

The cycle described by operation W_2 assumes the performance of the operations necessary to produce object 10 in an ordered sequence, such that no initial availability of semifinished products will have to be required.

Note that the object *11* is a reusable object. The balance vector turns out to be the same for all minimal cycles:

$$\boldsymbol{b} = | -1, -2, -4, 0, 0, -2, 0, 0, 1, 0 |^T \qquad (25)$$

The time required by the two operations is different. It results:

$$\begin{aligned} \tau(W_1) &= max\ \{\tau(A), \tau(B), \tau(C), \tau(D), \tau(E)\} \\ \tau(W_2) &= \tau(A) + 2\tau(B) + \tau(C) + \tau(D) + \tau(E) \end{aligned} \qquad (26)$$

It is possible to see immediately advantages and disadvantages of both production cycles, which represent two extreme cases. By composing in parallel all the manufacturing operations, the production cycle time is minimized, but some semifinished products must be stored and, of course, all the machines needed for carrying out at the same time all manufacturing operations have to be made available. By composing in an ordered series the operations, the maximum economy is obtained (when all the operations can be performed by a single machine) to the detriment of the cycle time which will become greater.

It is a task of the production control to set up the optimal production cycle, with reference to specific optimization criteria and with regard to specific constraints imposed by the production plant. A simple example of optimization is given by another production cycle having the same input and output vectors as the operation W_2, but with a smaller manufacturing time:

$$W_2 = E\,(C + D)\,(A + B^2) \qquad (27)$$

It results:

$$\tau(W) = \tau(E) + max\{\tau(C), \tau(D)\} + max\{\tau(A) + 2\tau(B)\} \qquad (28)$$

4. REFERENCES

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