Stability of Infinite-Dimensional Sampled-Data Systems

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Abstract

This paper considers the stability of the interconnection of a possibly infinite dimensional continuous time plant with a sampled-data controller. First, the well known Kalman–Ho–Narendra (KHN) conditions, which guarantee that a plant which can be stabilized by a continuous time controller can also be stabilized by a sample-data controller, are reviewed from an input-output perspective. This perspective allows a straightforward extension from the case of idealized sampling and zero-order hold usually consider in the literature, to the case of non-standard sampling and hold.

Then, the class of plants which can be stabilized by a sampled-data controller is characterized. It is shown that, from a control viewpoint, the fact that the controller is sampled-data imposes essentially no *a priori* constraint on the plant. The characterization also allow the extension of KHN conditions to the infinite dimensional case.

Finally, stability of the sampled-data configuration is addressed. It is shown that stability of the sampled-data interconnection is equivalent to the stability of the discrete-time model obtained by sampling the output and holding the input of the continuous time plant. This coincides with the well-known result for finite dimensional plants, but requires a completely different treatment.

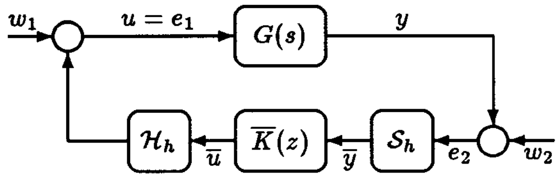

Figure 1: Sampled-data feedback configuration

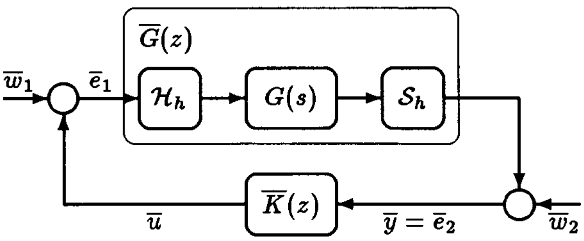

Figure 2: Discrete-time equivalent

1. Introduction

Consider the sampled-data feedback configuration illustrated in Fig. 1. Here a continuous time plant $\hat{G}(s)$ is controlled by a discrete-time controller through a sample and a hold device. The sampler $\mathcal{S}_h$ takes the continuous time output of the plant $y(t)$ and produces an output $\overline{y}_k$ every h units of time, where h denotes the sampling period which is assumed to be fixed. The hold device $\mathcal{H}_h$ takes the sequence of values $\overline{u}_l$, $l \leq k$ and generates the control signal $u(t)$, $kh < t \leq (k+1)h$. For example, if $\mathcal{S}_h$ is the idealized *impulse-train modulation model* (hereof refer to as the "idealized sampler"), then $\overline{y}_k = y(kh)$, and if $\mathcal{H}_h$ is the zero-order hold then $u(t) = \overline{u}_k$ on the given interval, but we do not want to exclude the possibility of having generalized sampling and hold devices. In sample-data control, the loop is closed by a *discrete-time* controller which generates the discrete time control signal based on the discrete time measurement as rendered by the sampling device. The resulting interconnection of a sampler, a controller and a hold is called a sampled-data controller. The interconnection of the hold, the continuous time plant and the sampler is refer to as the discrete-time model for the plant. Note that while the latter has a discrete-time transfer matrix, the former is a time-varying operator [2] and hence does not have a transfer matrix.

When using a sampled-data configuration, it is of interest to know whether this a priori assumption in the structure of the controller imposes some limitations on the set of plants that can be stabilized in closed-loop, i.e., whether a continuous time plant which is input-output stabilizable by a continuous time controller, can also be stabilized by a sampled-data one. The expression "input-output stabilizable" is used here to stress that this is not stabilizability in a state-space sense, but rather refers to the existence of a controller which achieves internal stability of the closed-loop. If the plant under consideration is finite dimensional, then the answer is generically positive: a plant that can be stabilized by a continuous-time controller can also be stabilized by a sampled-data one, except for a finite set of sampling frequencies. This was established in [9] for $\mathcal{L}^\infty$ stability and subsequently extended to $\mathcal{L}^p$ stability in [5] under the condition that the plant be strictly proper or else the signals are pre-filtered before being sampled, and to state-space models with output delay in [6]. However, the general case of an infinite dimensional plant with a (possibly infinite dimensional) discrete-time controller was left unanswered, and it is not apparent how the techniques employed in [5, 6, 9] can be modified to cope with this case, unless some restrictive assumptions are imposed in the model.

In order to address the problem of stability of a sampled-data interconnection, one begins by redrawing the feedback interconnection as in Fig. 2, and studies the input-output stabilizability of the resulting discrete time interconnection. This question is usually formulated in a state-space setting, where the plant $\hat{G}(s)$ is modeled by the state space equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t); \end{aligned} \tag{1}$$

stabilizability in continuous time is then equivalent to having the triple (A, B, C) stabilizable and detectable. It is then possible (see, e.g., [16]) to derive a *discrete-time* state-space model for $\overline{G}(z)$, and hence the question is whether the resulting discrete time system is stabilizable and detectable. This problem was consider in [15] where, under the assumption of idealized sampling and zero-order hold, sufficient conditions where given in terms of the eigenvalues of A and the sampling period h. The original result was subsequently extended to more general state-space models in [10] (there a necessary conditions was also given) and independently re-discovered in [9]; a detailed discussion is given in [16]. The conditions in [15], called the "KHN" conditions in the sequel, are as follows.

Lemma 1 (KHN Conditions) *Consider a system with state-space model as in 1 and assume that:*

(a) None of the points $j\frac{2\pi}{h}k$, $k \neq 0$ is an eigenvalue of A.

(b) If a_1 and a_2 denote two unstable eigenvalues of A, then $a_1 - a_2 \neq j\frac{2\pi}{h}k$, $k \neq 0$

Then if the continuous time system is stabilizable and detectable so is the zero-order hold discrete equivalent.

The KHN conditions imply that for almost all sampling rates, the discrete-time equivalent of a continuous-time system with a zero-order hold will be input-output stabilizable whenever the continuous time system is so. The rates at which the KHN conditions do not hold are referred to as pathological samplings.

Note that in spite of the fact that the two KHN conditions reduce to a single one whenever the system under consideration is real, they are usually stated separately, probably because they refer to two different although related consequences of sampling. While the first one appears because of the zeros introduced by the zero-order hold (recall that the transfer function of the hold in this case is equal to $H_{zoh} = \frac{1-e^{-sh}}{s}$), the second is related to an "aliasing" effect, by which the high-frequency poles are folded back into lower frequencies. Although the KHN conditions provide a satisfactory answer to the stabilizability question, the approach in [15, 10, 16, 9] falls short of providing insight into the mechanism that produces the loss of stabilizability at the critical frequencies. The conditions are then hard to extend to other setups since they have to be essentially re-derived, and no extension to infinite dimensional systems is apparent.

In the first part of this paper we provide a detailed input-output interpretation of the KHN conditions. Such an approach was suggested earlier in terms of hidden oscillations in [13] (i.e., before the original publication of the KHN conditions) and mentioned explicitly in [2] but to the best of our knowledge, a detailed analysis is lacking in the literature. For simplicity, we will only consider the case of poles with order one, although the *degree* may be larger than one (see [14, p. 447]). Extension to the case of poles with order larger than one is not straightforward using the techniques to be discussed below, although one can argue that an arbitrary small perturbation will reduce any plant to the case considered.

When properly formulated, the input-output approach provides a natural explanation for the pathological sampling rates in terms of pole-zero cancellations. It also appears to be very rewarding, since it allows a straightforward generalization of the KHN conditions to the case of generalized sampler and hold devices. Perhaps more interesting, the approach also provides a way of extending the conditions to the case when the plant under study is infinite dimensional. This leads us to investigate which set of infinite dimensional plants, when discretized, can be stabilized by a discrete time controller. As it turns out, plants on this set can only have a finite number of unstable poles (i.e., they are meromorphic in a half plane which contains the closed right-half plane), which provides some interesting connections with the notions of ill-posed distributed parameter systems raised in [11]. This is one of the main results of the paper.

Although stabilizability of the discrete-time model is clearly required, the question one is interested in when considering the interconnection of a continuous time-plant with a discrete-time controller is the stability of the hybrid configuration. Stability analysis is complicated by the fact that, because of the sampling device, the system is periodically *time-variant* and hence has no transfer function. In the finite-dimensional state-space setting, this analysis was carried out in [9] and subsequently extended in [5]; although input-output stability was discussed, it was based on the fact that an appropriately defined state-space model of the hybrid system was exponentially stable. This fact, which is very natural for finite state-space models, has no apparent translation to the general class of models contained in the transfer function theory of infinite dimensional systems developed in [3, 7, 4]. Stability results established in these and related works (see, for instance, [17]) cannot be applied to the present case because of its time-varying nature. However, we will show that the stability analysis can be nevertheless carried out in an input-output framework although not with purely algebraic tools. Together with the characterization of the set of plants stabilizable by a sampled-data controller, this stability result provides a satisfactory answer to the fundamental question posed in our opening remarks.

The paper is organized as follows. In the next two sections we will investigate when a plant that can be stabilized by a continuous-time controller results in a discrete-time model which can be stabilized by a discrete-time controller. Section 2 is devoted to finite dimensional plants, but as opposed to most of the available literature, we will consider an input output setting. This allows to consider generalized sampling and hold devices with little extra work. Section 3 contains the generalization to infinite dimensional systems and describes the set of "good" infinite dimensional plants from a sampled-data point of view. The stability analysis for the hybrid interconnection is addressed in Section 4. Section 5 contains the conclusions.

Preliminaries and Notation

The following notation and definitions will be used when discussing the infinite dimensional case. Readers are referred to [3, 7] for a detailed treatment.

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a function. Then $f \in \mathcal{A}_-(\sigma_0)$ for some σ_0 if and only if $f(t) = f_a(t) + \sum_0^\infty f_i\delta(t-t_i)$ such that the function $e^{-\sigma t}f_a(t) \in L_1$; $t_0 = 0$, $t_i > 0$ for $i > 0$ and $f_i e^{-\sigma t_i} \in l_1$ for some $\sigma < \sigma_0$. $\hat{f}$ denotes the Laplace transform of f and the corresponding set is $\hat{\mathcal{A}}_-(\sigma_0)$. Here L_1 (l_1) denote the standard Lebesgue space space of absolutely integrable (summable) functions. It is well known [3] that $\mathcal{A}_-(\sigma_0)$ ($\hat{\mathcal{A}}_-(\sigma_0)$) is a commutative convolution Banach algebra (with pointwise product). If $f_i = 0$ for each i, then the function belongs to $L_-(\sigma_0)$. To prevent the proliferation of notation, the names refer both to functions and to matrices of functions (i.e., transfer functions and transfer matrices).

The convolution algebra $\mathcal{A}_-(\infty) \doteq \cup_\sigma \mathcal{A}_-(\sigma)$ is the standard choice in the literature for modeling transfer function of possibly infinite dimensional system; see [11] for some examples. It is standard to define a system as stable whenever it belongs to the sub-set $\mathcal{A}_-(0)$. This is a very general notion of stability which encompasses BIBO and exponential stability. It is worth stressing that a system may belong to $\mathcal{A}_-(0)$ and be stable in the above sense but still not be exponentially stable in a strong sense; see [4] for a counterexample. Of special interest will be the set $\mathcal{B}(0)$ which denotes the set of functions that can be written as $f = f_s + f_u$ where $f_s \in \mathcal{A}_-(0)$ and f_u is a rational function with at most a finite number of poles in the closed right half plane.

The definitions above have natural counterpart in discrete time. Given a sequence $\overline{f} = \{f_k\}$, then $\overline{f} \in l_{1,-}(\rho_0)$ if for some $0 < \rho < \rho_0$ the sequence $\{f_k\rho^{-k}\} \in l_1$. A system is stable if it belongs to $l_{1,-}(1)$. If $f_0 = 0$ then $\overline{f} \in l^0_{1,-}(\rho_0)$. Similarly one can define $l_{1,-}(\infty)$ and $b(0)$. The reader is refer to [7] for details. Note that $l_{1,-}(\rho_0)$ is by itself endowed with an exponential decay rate.

Finally, linear, possibly *time-varying* operators in the time domain will be denoted by *script* letters. The domain and range of the operators will be apparent from the context.

2. Finite dimensional plants

The purpose of this section is to re-derive the KHN conditions for linear time-invariant, finite dimensional plants. This is the same setup as consider by the references cited above, and allows us to introduce the main idea in a simple and intuitive manner, without the mathematical technicalities required to address a more complicated problem.

For an interpretation of the first KHN condition, let $G(t)$ denote the impulse response of $\hat{G}(s)$. Then a straightforward derivation shows that $\overline{G}_k = \int_0^h \hat{G}(kh-\tau)d\tau$ denotes the impulse response of the discrete time system $\overline{G}(z)$ whenever $\mathcal{H}_h$ is the zero-order hold device. Since:

$$G(kh-\tau) = \frac{1}{2\pi j}\oint_C \hat{G}(s)e^{s(kh-\tau)}ds$$

where C contains all the poles in $\hat{G}(s)$, it follows that

$$\begin{aligned}\overline{G}_k &= \frac{1}{2\pi j}\int_0^h \oint_C \hat{G}(s)e^{s(kh-\tau)}dsd\tau \\ &= \frac{1}{2\pi j}\oint_C \hat{G}(s)\int_0^h e^{s(kh-\tau)}d\tau ds \\ &= \frac{1}{2\pi j}\oint_C \hat{G}(s)e^{skh}\left[\frac{1-e^{-sh}}{s}\right]ds,\end{aligned}$$

where the assumptions on $\hat{G}(s)$ allow to interchange the order of integration. If $\hat{G}(s)$ has any pole at $s = j\frac{2\pi}{h}k$ for some $k \neq 0$, then it will be cancelled by a zeros of $\frac{1-e^{-sh}}{s}$ and hence will not affect the impulse response $\overline{G}_k$. More specifically, if D_C denotes a disk containing all points of the form e^{sh}, $s \in C$, then for each z outside D_C, $\overline{G}(z)$ can be computed as

$$\begin{aligned}\overline{G}(z) &= \sum_{k=1}^{\infty}\overline{G}_k z^{-k} \\ &= \frac{1}{2\pi j}\oint_C \hat{G}(s)\left[\frac{1-e^{-sh}}{s}\right]\frac{e^{sh}}{z-e^{sh}}ds.\end{aligned}$$

Assume that $\hat{G}(s) = \frac{G_0}{s-a}$ and let C denote a circle centered at the origin with radius larger that $|a|$; then if $a \neq j\frac{2\pi}{h}k$, $k \neq 0$, then

$$\overline{G}(z) = G_0\frac{e^{ah}-1}{a}\frac{1}{z-e^{ah}}$$

(where the second term should be interpreted appropriately if $a = 0$) but $\overline{G}(z) = 0$ whenever the condition does not hold. This clearly shows that the first KHN condition rules out an illegal pole-zero cancellation between the poles of the plant and the zeros of the hold device.

To interpret the second condition, assume that the plant satisfies condition a) and can be factorize as

$$\hat{G}(s) = \sum_{i=1}^{N}\frac{G_i}{s-a_i}$$

where by simplicity we have assumed that $\hat{G}(s)$ is strictly proper. Then from the previous derivation,

$$\overline{G}(z) = \sum_{i=1}^{N} G_i\frac{e^{a_ih}-1}{a_i}\frac{1}{z-e^{a_ih}}.$$

If $a_m - a_n \neq j\frac{2\pi}{h}k$, $k \neq 0$ for each m and n, then $\overline{G}(z)$ will have N poles located at e^{a_ih}, and hence the KHN conditions imply that if a continuous time system is input-output stabilizable, then the resulting discrete-time model assuming a zero-order hold also is.

If for some indices $n \neq m$ and $k \neq 0$, $a_n - a_m = j\frac{2\pi}{h}k$, then $e^{a_mh} = e^{a_nh}$ and hence the n-th and m-th terms add to $(G_n/a_n + G_m/a_m)\frac{e^{a_nh}-1}{z-e^{a_nh}}$. If the system is single-input, single-output, the degree of $\overline{G}(z)$ appears to be lower than that of $\hat{G}(s)$ since $\overline{G}(z)$ has a *hidden mode*. If $\Re(a_n) \geq 0$[1] this implies that $\overline{G}(z)$ has an unstable hidden mode, which precludes its input-output stabilizability. It follows that the KHN conditions are also necessary in the scalar case (see [16] for a state-space proof). In the multivariable case, with G_n and G_m matrices, the order may or may not drop, depending on these residues. Taking this observation into account, it is possible to formulate the following necessary and sufficient condition for stabilizability.

Lemma 2 *Let $\hat{G}(s)$ be a continuous time system with poles at a_i and corresponding residues G_i, $i = 1,\cdots,N$, $\overline{G}(z)$ the discrete time model resulting from a zero-order hold. Then the continuous-time and discrete-time stabilizability are equivalent for a given h if and only if*

(a) None of the poles a_i is of the form $j\frac{2\pi}{h}k$, $k \neq 0$.

(b) For any two poles a_n, a_m with non negative real part, $a_n - a_m \neq j\frac{2\pi}{h}k$, $k \neq 0$ or *$a_n - a_m = j\frac{2\pi}{h}k$, $k \neq 0$ and $\mathrm{rank}\,(G_n + G_m) = \mathrm{rank}(G_n) + \mathrm{rank}(G_m)$.*

2.1. Generalized Hold and Sampler

Although the ideal sampler and first-order hold are the most extensively studied and used in practice, nonconventional samplers and holds have received attention recently because they arise naturally in many situations (e.g., in multirate systems) and may offer some improvements over standard devices. For a tutorial on this subject and a pointer to the literature the reader is referred to [1]; for possible limitations and a word of warning see [8]. The purpose of this section is to extend the KHN conditions to the the non-conventional setting. Suppose that the device $\mathcal{H}_h$ is a generalized hold, such that

$$(\mathcal{H}_h\overline{u})(kh+\tau) \doteq f_H(\tau)\overline{u}_k \quad \forall k \in \mathbb{Z}^+, \quad \tau \in [0,\ h),$$

and the device $\mathcal{S}_h$ is a generalized sampler, such that

$$(\mathcal{S}_hy)_{k+1} \doteq \int_0^h f_S(h-t)y(t+kh)dt \quad \forall k \in \mathbb{Z}^+.$$

[1] $\Re(x)$ denotes the real part of x

Then, repeating the calculations above we can get

$$\overline{G}(z) = \overline{G}_0 z^{-1} + \frac{1}{2\pi j}\oint_C \hat{F}_S(s)\hat{G}(s)\hat{F}_H(s)\frac{1}{z - e^{sh}},$$

where

$$\hat{F}_S(s) \doteq \int_0^h f_S(h-\tau)e^{s\tau}d\tau,$$

$$\hat{F}_H(s) \doteq \int_0^h e^{-s\tau} f_H(\tau)d\tau,$$

and $\overline{G}_0$ is some matrix. This suggests the following extension of the KHN conditions on generalized hold and sampler case.

Lemma 3 (KHN Conditions – GH&S) *Let $\hat{G}(s)$ be a continuous time system with poles at a_i and corresponding residues G_i, $i = 1, \cdots, N$. Let $\mathcal{H}_h$, $\mathcal{S}_h$ be a generalized hold with hold function f_H and a generalized sampler with sampling function f_S such that the sets $\mathcal{Z}_{\mathcal{H}} = \{z : \Re(z) \geq 0,\ \hat{F}_H(z) \text{ has reduced row rank}\}$ and $\mathcal{Z}_{\mathcal{S}} = \{z : \Re(z) \geq 0,\ \hat{F}_S(z) \text{ has reduced col. rank}\}$ have no finite accumulation point. Assume that:*

(a) For any pole a_i, $a_i \notin \mathcal{Z}_{\mathcal{H}} \cup \mathcal{Z}_{\mathcal{S}}$.

(b) For any two poles a_n, a_m with nonnegative real part, $a_n - a_m \neq j\frac{2\pi}{h}k$, $k \neq 0$.

Then if the continuous-time system is input-output stabilizable, so is the discrete-time system.

Remark: The conditions in Lemma 3 are also necessary in the SISO case. A necessary condition for the MIMO case may be given as in Lemma 2:

$$rank\left(\sum_{j=1}^{n_i} \hat{F}_S(a_j)G_j\hat{F}_H(a_j)\right) = \sum_{j=1}^{n_i} rank\,(G_j)$$

for all i and all j such that $e^{a_j h} = e^{a_i h}$.

3. Infinite Dimensional Plants

In this section we characterize the class of infinite dimensional systems which can be stabilized using a sampled-data controller. $\mathcal{A}_-(\infty)$, $\mathcal{A}_-(0)$ and $\mathcal{B}(0)$ denote the set of all models under consideration, all stable models and all models with at most a finite number of poles in the closed right half plane respectively. Corresponding sets for discrete-time systems are $l_{1,-}(\infty)$, $l_{1,-}(\rho_0)$ and $b(0)$. The reader is referred to the section following the introduction for precise definitions.

We will first argue that not every infinite dimensional system can be stabilized by means of a sample-data controller and describe the set in which stability can be achieved. We will relate this to the well-posedness notion of Helmicki et al. [11] to show that the class considered is large. Finally we will address the problem of extending the KHN conditions to the infinite dimensional case, assuming for simplicity an ideal sampler and a zero-order hold. Since the class of discrete time systems resulting from sampling the class of infinite dimensional systems enjoys nice properties [12], the extension is straightforward.

3.1. Sampling Infinite-Dimensional Systems

We now proceed to characterize the class of plants that can be stabilized by a sampled-data controller. We begin by considering the following Lemma, essentially borrowed from [11].

Lemma 4 *Let the plant $\hat{G}(s)$ and controller $\hat{K}(s)$ both belong to $\mathcal{A}_-(\infty)$ and assume that the closed-loop system is internally stable (in the sense that any closed-loop transfer matrix belongs to $\mathcal{A}_-(0)$). Then $\hat{G}(s)$, $\hat{K}(s)$ are meromorphic in a region containing the closed right half plane.*

Remark: Lemma 4 says that if a plant is to be internally stabilized by linear time-invariant feedback, then it can have no essential singularity in the right half plane.

By changing the domains of definition, one can write the discrete-time version of this lemma.

Lemma 5 *Let the plant $\overline{G}(z)$ and controller $\overline{K}(z)$ both belong to $l_{1,-}(\rho_0)$ and assume that the closed-loop system is internally stable (in the sense that any closed-loop transfer matrix belongs to $l_{1,-}(1)$). Then $\overline{G}(z)$, $\overline{K}(z)$ are meromorphic in the exterior of a disk contained in the open unit disk.*

Now suppose that a system with transfer function $\hat{G}(s) \in \mathcal{A}_-(\infty)$ is to be controlled by means of a sample-data controller. The discrete-time equivalent when using idealized sample and a first order hold is $\overline{G}(z) \in l_{1,-}(\rho_0)$, where $\rho_0 = e^{\sigma_0 h}$ [12]. Suppose that this system is internally stabilized by some controller $\overline{K} \in l_{1,-}(\rho_1)$, $\rho_1 > \rho_0$. Then, from Lemma 5 $\overline{G}$ is meromorphic outside a disk contained in the open unit disk. Since the function is analytic outside a disk of radius ρ_0 it follows that it has a finite number of poles outside the unit disk, i.e., a finite number of unstable poles. Under the assumption of internal stability there cannot be unstable hidden modes in $\overline{G}$, and hence $\hat{G}(s)$ can only have a finite number of unstable poles. This is summarized in the following theorem.

Theorem 1 *Let $\hat{G}(s) \in \mathcal{A}_-(\sigma_0)$ and assume that its discrete time model computed an idealized sampler and zero-order hold $\overline{G}(z)$ can be internally stabilized by a controller $\overline{K}(z) \in l_{1,-}(\rho_1)$. Then $\hat{G}(s) \in \mathcal{B}(0)$, i.e., the continuous time plant has only a finite number of unstable poles.*

Since the set $\mathcal{B}(0)$ is a proper subset of $\mathcal{A}_-(\infty)$, one is tempted to conclude that there are some linear time-invariant systems which may be stabilized by means of a linear controller but cannot be stabilized by using a combination of sampling, hold and discrete-time controller. Recent work by Helmicki et al. [11] suggests that this is not the case, since it is shown there that any plant which can be stabilized by a strictly proper controller or otherwise gives rise to a strictly proper closed-loop, belongs to the set $\mathcal{B}(0)$. It follows that, from a control viewpoint, the assumption required by the fact that the controller is discrete-time does not introduce an *a priori* restriction on the set of models that can be considered.

3.2. The KHN Conditions

We are now ready to extend the KHN conditions to the set $\mathcal{B}(0)$ of possibly infinite dimensional systems. Let $\hat{G}(s) \in \mathcal{B}(0)$ and factorize

$$\hat{G}(s) = \hat{G}_s(s) + \hat{G}_u(s), \tag{2}$$

where $\hat{G}_s(s) \in \mathcal{A}_-(0)$ and $\hat{G}_u(s)$ is a rational anti-stable transfer functions with poles at a_i, $i = 1, \cdots, n$. Suppose now that a discrete time model is computed, assuming an ideal sampler and a zero order hold. Then, by linearity (see also [12]), it is possible to compute this model by considering the discrete time ones for $\hat{G}_s$ and $\hat{G}_u$, say $\overline{G}_s(z)$ and $\overline{G}_u(z)$ respectively. Clearly $\overline{G}_u(z)$ is a rational function with all its poles outside the unit disk, and from [12, Theorem 1], $\overline{G}_s(z) \in l_{1,-}(1)$.

Lemma 6 (KHN Conditions - IDS) *Let $\hat{G}(s)$ be a continuous time system, $\hat{G}(s) \in \mathcal{B}(0)$, such that its necessarily finite dimensional unstable part $\hat{G}_u(s)$ has poles at a_i. Assume that:*

(a) None of the points $a_i = j\frac{2\pi}{h}k,\ k \neq 0$.

(b) If a_{i_1} and a_{i_2} denote two unstable poles of $\hat{G}_u(s)$, then $a_{i_1} - a_{i_2} \neq j\frac{2\pi}{h}k,\ k \neq 0$

Then if the continuous time system is input-output stabilizable so is the zero-order hold discrete equivalent.

Remark: Stability in the Lemma is in the $\mathcal{A}_-(0)$ and $l_{1,-}(1)$ sense for the continuous and discrete time respectively.

4. Stability of Feedback Configuration

We will now show that if a controller $\overline{K}(z)$ stabilizes the discrete time model, then the hybrid configuration will also be stable, in the sense that bounded signals w_1, w_2 will map into bounded signals e_1, e_2 (see Fig. 1). For finite dimensional plants, this was established in [9] for $\mathcal{L}^\infty$ stability and in [5] for $\mathcal{L}^p$ stability, by considering a state-space realization for the plant and the discrete-time controller and defining a state for the hybrid system. Stability of the discrete-time configuration is then equivalent to exponential decay of the sampled state, and the values at the intersampling are generated by the open loop plant; since the sampling period is finite, boundedness follows. $\mathcal{L}^p$ stability requires the inclusion of a strictly proper anti-aliasing filter. The approach has no apparent extension to the infinite dimensional case, specially if we do not want to assume that a state space model is available. A proof in the input-output setting is made harder by the fact that the hybrid configuration is time-varying, which violates the starting assumption of most algebraic approaches to closed-loop stability (e.g., [17] and the references therein). To circumvent this difficulty, we will need at some point to manipulate state-space representations; this will not conflict with our general framework since existence of this realizations is guaranteed by Theorem 1.

Let $\hat{G} \in \mathcal{B}(0)$ be the plant (by Theorem 1 this is without loss of generality because otherwise the plant cannot be stabilized) and write $\hat{G}(s) = \hat{G}_s(s) + \hat{G}_u(s)$, where $\hat{G}_s(s) \in \mathcal{A}_-(0)$ and $\hat{G}_u(s)$ is an n-dimensional strictly proper transfer function, which has all its poles in the right half plane. Consider a minimal state space representations $(A, B, C, 0)$ for $\hat{G}_u(s)$. Using standard state-space tools [17] it is possible to write a *left coprime factorization* (LCF) of $\hat{G}_u(s) = \hat{D}_c(s)^{-1}\hat{N}_c(s)$, where $\hat{D}_c(s)$, $\hat{N}_c(s)$ have realizations $(A + LC, L, C, I)$, $(A + LC, B, C, 0)$ for some L such that $A + LC$ is Hurwitz. By construction, $\hat{G}(s) = \hat{D}_c(s)^{-1}\left(\hat{D}_c(s)\hat{G}_s(s) + \hat{N}_c(s)\right)$ is a LCF of $\hat{G}(s)$. Recall that $\hat{N}_c(s)$, $\hat{D}_c(s) \in \mathcal{A}_-(0)$ are left coprime in $\mathcal{A}_-(0)$ if there exist $\hat{U}(s)$, $\hat{V}(s) \in \mathcal{A}_-(0)$ such that $\hat{N}_c(s)\hat{U}(s) + \hat{D}_c(s)\hat{V}(s)$ is unimodular in $\mathcal{A}_-(0)$, i.e., its inverse is also in $\mathcal{A}_-(0)$.

Consider now the discrete time model $\mathcal{S}_h\mathcal{G}\mathcal{H}_h$ with transfer function $\overline{G}(z) = \overline{G}_s(z) + \overline{G}_u(z)$. We will assume for simplicity that $\mathcal{S}_h$ is the idealized sampler and $\mathcal{H}_h$ is a first order hold. The discrete-time transfer matrix $\overline{G}_u(z)$ has a state space realization $\overline{G}_u(z) = \left(e^{Ah}, \int_0^h e^{A\tau}d\tau B, C, 0\right)$ with a LCF

$\overline{G}_u(z) = \overline{D}_d(z)^{-1}\overline{N}_d(z)$ where

$$\begin{aligned} \overline{N}_d(z) &= \left(e^{Ah} + L_d C, \int_0^h e^{A\tau} d\tau B, C, 0\right) \\ \overline{D}_d(z) &= \left(e^{Ah} + L_d C, L, C, I\right). \end{aligned}$$

Here L_d is such that $e^{Ah} + L_d C$ has all its eigenvalues inside the open unit disk. Under the KHN conditions, the realization for $D_d(z)$ is minimal and $\left(\overline{D}_d(z)\overline{G}_s(z) + \overline{N}_d(z), \overline{D}_d(z)\right)$ is left coprime.

Discrete-time stability

Let $\overline{K}(z)$ denote a controller achieving internal stability for the *discrete-time* configuration illustrated in Fig. 2. Here stability is to be understood in the sense that the transfer matrix between $\overline{w}_1$, $\overline{w}_2$ and $\overline{e}_1$, $\overline{e}_2$, i.e.,

$$\begin{bmatrix} \overline{e}_1 \\ \overline{e}_2 \end{bmatrix} = \begin{bmatrix} I + \overline{KQG} & \overline{KQ} \\ \overline{QG} & \overline{Q} \end{bmatrix} \begin{bmatrix} \overline{w}_1 \\ \overline{w}_2 \end{bmatrix} \qquad (3)$$

belongs to $l_{1,-}(1)$, which implies l^∞ stability and hence l^p stability since the system is linear time invariant. Here

$$\overline{Q}(z) \doteq (I - \overline{G}(z)\overline{K}(z))^{-1}.$$

For later use, note that

$$\overline{QG} = \overline{QD}_d^{-1}\left(\overline{D}_d\overline{G}_s + \overline{N}_d\right),$$

which implies, by the left coprimeness of the corresponding transfer matrices, that the transfer matrix $\overline{Q}(z)\overline{D}_d^{-1}(z)$ is also stable.

Continuous to discrete stability

Consider now the time-varying operator from w_1, w_2 to $\overline{u}_k$, $\overline{y}_k$ (note that the operator notation is enforced):

$$\begin{aligned} \begin{bmatrix} \overline{u}_k \\ \overline{y}_k \end{bmatrix} &= \mathcal{T}^{dc} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ &= \begin{bmatrix} \overline{KQ}\mathcal{S}_h\mathcal{G} & \overline{KQ}\mathcal{S}_h \\ \overline{Q}\mathcal{S}_h\mathcal{G} & \overline{Q}\mathcal{S}_h \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \end{aligned} \qquad (4)$$

Since $\mathcal{S}_h : \mathcal{L}^\infty \to l^\infty$, it follows that $\mathcal{T}_{12}^{dc}$ and $\mathcal{T}_{22}^{dc}$ are bounded. To prove that $\mathcal{T}_{21}^{dc}$ is bounded, write $\mathcal{T}_{21}^{dc} = \overline{Q\mathcal{D}}_d^{-1}\overline{\mathcal{D}}_d\mathcal{S}_h\mathcal{D}_c^{-1}(\mathcal{D}_c\mathcal{G}_s + \mathcal{N}_c)$; by a previous remark it suffices to show that $\overline{\mathcal{D}}_d\mathcal{S}_h\mathcal{D}_c^{-1}$ is a bounded operator. To show this, call $\overline{v} = \mathcal{S}_h\mathcal{D}_c^{-1}w_1$, $\hat{v} = \overline{\mathcal{D}}_d\overline{v}$ and bring the following state space representation:

$$\begin{aligned} \overline{x}_{k+1} &= e^{Ah}\overline{x}_k + \int_0^h e^{A\tau} L w_1(kh+\tau) d\tau \\ \overline{v}_k &= C\overline{x}_k + w_1(kh) \\ \hat{x}_{k+1} &= \left(e^{Ah} + L_d C\right)\hat{x}_k + L_d C\overline{x}_k + L w_1(kh) \\ \hat{v}_k &= C\hat{x}_k + C\overline{x}_k + w_1(kh). \end{aligned}$$

Introducing a new state $\acute{x}_k \doteq \overline{x}_k + \hat{x}_k$:

$$\begin{aligned} \acute{x}_{k+1} &= \left(e^{Ah} + L_d C\right)\acute{x}_k + \textstyle\int_0^h e^{A\tau} L w_1(kh+\tau) d\tau \\ \hat{v}_k &= C\acute{x}_k + w_1(kh), \end{aligned}$$

from which the boundedness from $\mathcal{L}^\infty \to l^\infty$ follows. A similar reasoning also shows that $\mathcal{T}_{22}^{dc}$ is bounded.

Continuous-time stability

From Fig. 1, the closed-loop operator between w_1, w_2 and e_1, e_2 is:

$$\begin{aligned} \mathcal{T} &= \begin{bmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{bmatrix} \\ &= \begin{bmatrix} I + \mathcal{H}_h\overline{KQ}\mathcal{S}_h\mathcal{G} & \mathcal{H}_h\overline{KQ}\mathcal{S}_h \\ \mathcal{G} + \mathcal{G}\mathcal{H}_h\overline{KQ}\mathcal{S}_h\mathcal{G} & I + \mathcal{G}\mathcal{H}_h\overline{KQ}\mathcal{S}_h \end{bmatrix}. \end{aligned} \qquad (5)$$

The objective is now to show that discrete-time stability implies boundedness of $\mathcal{T}$ as an operator from $\mathcal{L}^\infty$ to $\mathcal{L}^\infty$. Since the operators from w_1, w_2 to $\overline{e}_1$ are bounded and $e_1 = \mathcal{H}_h\overline{e}_1 + w_1$, one only needs to care for the mapping to e_2. Consider, for instance, $\mathcal{T}_{21}$. By the stability from continuous to discrete-time, the operator $\mathcal{S}_h\mathcal{T}_{21}$ is bounded. Write

$$\begin{aligned} \mathcal{T}_{21} &= \mathcal{T}_{21s} + \mathcal{T}_{21u} \\ &= (\mathcal{G}_s + \mathcal{G}_u)\left(I + \mathcal{H}_h\overline{KQ}\mathcal{S}_h\mathcal{G}\right); \end{aligned}$$

Stability will then follows from the stability of $\mathcal{T}_{21u}$. Consider the signal

$$w^{eq} = \left(I + \mathcal{H}_h\overline{KQ}\mathcal{S}_h\mathcal{G}\right) w_1$$

which, by the previous observation, belongs to $\mathcal{L}^\infty$. It then suffices to show that if $\mathcal{S}_h\mathcal{G}_u w^{eq} \in l^\infty$ then $\mathcal{G}_u w^{eq} \in \mathcal{L}^\infty$.

Lemma 7 *Let P be a strictly proper finite dimensional continuous time system such that for some signal $w \in \mathcal{L}^p$, $\mathcal{S}_h\mathcal{P}w \in l^p$, where the sampling rate is such that the KHN conditions hold. Then $\mathcal{P}w \in \mathcal{L}^p$.*

So we are now in a position to set the main result of this section:

Theorem 2 *Consider the setup in Fig. 1 and assume that the KHN conditions given in Lemma 6 hold. Then the discrete-time feedback interconnection in Fig. 2 is stable (in the $l_{1,-}(1)$ sense) if and only if the interconnection in Fig. 1 is stable, in the sense that the operator from w_1, w_2 to e_1, e_2 is bounded as a mapping from $\mathcal{L}^\infty$ to $\mathcal{L}^\infty$.*

Remark 1: If the output signal $y(t)$ is filtered by a stable, strictly proper filter before being sampled, then $\mathcal{L}^p$ stability can be deduced following exactly the same reasoning.

Remark 2: The assumption that the sample and the hold devices are an ideal sampler and a first-order hold can be easily lifted, subject to the constraint that $\mathcal{S}_h : \mathcal{L}^\infty \to l^\infty$ and $\mathcal{H}_h : l^\infty \to \mathcal{L}^\infty$. The proof detailed above remains the same[2] except for slight modification of the continuous to discrete part and Lemma 7.

5. Conclusions and Further Work

In this paper we have investigated the control of a possibly infinite dimensional continuous-time system by a sampled-data controller. We have shown that an infinite dimensional plant can be stabilized by a sampled-data controller only if it has at most a finite number of unstable poles and that the KHN conditions also hold in the infinite dimensional case. We have argued that the set of stabilizable plants is "large", since plants which fail to satisfy this conditions cannot be stabilized by a strictly proper controller or cannot result in strictly proper closed loop transfer matrix. Finally we have shown that under the KHN conditions, stability of the sampled-data configuration is equivalent to stability of the corresponding discrete-time model in discrete-time. Our arguments are carried over on an input-output framework and they apply with little extra work to non-conventional sampling and hold devices.

Our original interest in the stability problem of sampled-data system was in the context of time-varying plants described by their convolution kernel, which arise in some adaptive control problems. It was soon clear that finding a solution was made harder by the fact that most stability anaylsis of hybrid configuration relied heavily on underlying state-space representations. We believe that the content of this paper bridges this gap. Moreover, as preliminary calculations show, our results pave the way for solving more general and potentially interesting problems. This is a topic of current research.

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