ON THE SOLVABILITY OF MORGAN'S PROBLEM:

A NECESSARY AND SUFFICIENT CONDITIONS FOR DECOUPLING

BY STATE FEEDBACK AND A CONSTANT SINGULAR INPUT TRANSFORMATION

by

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ABSTRACT

An algebraic approach based on the system matrix description of the linear systems is developed and is applied to establish a necessary and sufficient condition for the solvability of Morgan's problem that was stated as follows.

"Given the linear, time-invariant and controllable system S that is described by the state equations

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{1a}$$

$$\underline{y} = C\underline{x} \tag{1b}$$

where $\underline{x}\in R^n$, $\underline{u}\in R^l$, $\underline{y}\in R^m$ and $l\geq m$, find the control law

$$\underline{u} = F\underline{x} + K\underline{w} \tag{2}$$

such that the compensated system S_c has transfer function matrix

$$H(s) = C[sI-A-BF]^{-1}BK = \mathrm{Diag}[n_i(s)/d_i(s)] \tag{3}$$

and

$$\det H(s) \neq 0 \text{ ".} \tag{4}$$

The existence of the polynomials $n_i(s)$, $d_i(s)$, $i=1,2,\ldots,m$

introduces an unnecessary complication. It is proved that the existence of a solution to the Morgan's problem is ensured if and only if H(s) can be brought to the form

$$\hat{H}(s) = C[sI-A-B\hat{F}]^{-1}B\hat{K} = \mathrm{Diag}[s^{-k_i}] \tag{5}$$

where

$$k_i = \deg[d_i(s)] - \deg[n_i(s)]. \tag{6}$$

The existence of the nonsquare matrix K in the control law of equation (2) introduces a great difficulty to the solvability of the problem. Apart from the controllability property of the system that can be destroyed, its structure is changing and especially its controllability indices λ_i. This difficulty is overcomed by reformulating the problem in an appropriate manner at the sequel.

Suppose that the initial system S is described by the system matrix P(s) as follows

$$P(s) = \begin{bmatrix} T(s) & I_l \\ -V(s) & 0 \end{bmatrix}, \tag{7}$$

where T(s) is column proper and the degree of its jth column is equal to the jth controllability index λ_j of the system. Then if a state feedback F in combination with a constant nonsingular input transformation $\tilde{K}$ is applied to the system S, the system matrix of the compensated system will be

$$P_c(s) = \begin{bmatrix} \tilde{K}^{-1}[T(s)+V'(s)] & I_l \\ -V(s) & 0 \end{bmatrix}, \tag{8}$$

where the column degrees of the matrix $V'(s)$ are strictly less than the column degrees of the matrix T(s). If $\hat{F}$, $\hat{K}$ are as in equation (5), because of relation (4), $\hat{K}$ has full column rank rank and therefore $l-m$ vectors $\underline{k}_i$ can be chosen such that the matrix

$$\tilde{K} = [\ \hat{K} \mid \underline{k}_1, \underline{k}_2, \ldots, \underline{k}_{l-m}\ ] \tag{9}$$

is nonsingular. If the control law

$$\underline{u} = \hat{F}\underline{x} + \tilde{K}\underline{w} \tag{10}$$

is applied to S, the resulting transfer function matrix will be

$$\tilde{H}(s) = [\mathrm{Diag}(s^{-k_i}) \mid Q(s)], \tag{11}$$

where $Q(s)$ is an $m\times(l-m)$ matrix with rational entries. Then the Morgan's problem can be formulated as follows.

"Given the system S, find the state feedback $\hat{F}$ and the nonsingular input transformation $\tilde{K}$ (or equivalently the matrices $V'(s)$, $\tilde{K}$ in equation (8) so that the transfer function matrix of the compensated system is as in equation (11)".

In the main theorem it is proved that if λ represents the maximum controllability index of the system S and $V(s)$ be the submatrix of $P(s)$, as in equation (7), the decoupling problem will have a solution if and only if there exist integers k_i such that the relation

$$\mathrm{Diag}[s^{k_i}]V(s)\mathrm{Diag}[s^{\lambda-\lambda_j}] = [I_m \quad B(s)]M(s)\mathrm{Diag}[s^{\lambda-\lambda_j}] \tag{12}$$

is satisfied with $M(s)$, $B(s)$ polynomial matrices and $M(s)\mathrm{Diag}[s^{\lambda-\lambda_j}]$ row proper having row degrees equal to λ. Furthermore, a method is proposed for determining the control law of equation (2) when relation (12) is satisfied.

Relation (12) establishes a necessary and sufficient condition for the solvability of the Morgan's problem and is deduced that the solvability of the decoupling problem depends upon the possibility of factorizing the matrix $\mathrm{Diag}[s^{k_i}]V(s)\mathrm{Diag}[s^{\lambda-\lambda_i}]$ as in equation (12). From a geometric point of view relation (12) indicates that for every $i=1,2,\ldots,m$, the ith row vector of $V(s)$ over the field of the rational functions must belong to the vector space spanned by the ith row vector of and the last $\ell-m$ row vectors of the matrix $M(s)$.

The main difficulty in applying the above theorem is the lack of knowledge of the integers k_i. These are related to a set of indices of the initial system, the essential orders, that can be defined as follows. From the matrix

$$U(s) = V(s)\mathrm{Diag}[s^{\lambda-\lambda_j}] \tag{13}$$

that can be written

$$U(s) = U_1 s^{\lambda-1} + U_2 s^{\lambda-2} + \dots + U_\lambda . \tag{14}$$

since its column degrees are strictly less than λ, the matrix Δ_μ is formed as follows

$$\Delta_\mu = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_\mu \end{bmatrix} = \begin{bmatrix} U_1 & 0 & . & . & . & 0 \\ U_2 & U_1 & . & . & . & 0 \\ \vdots & \vdots & & & & \vdots \\ U_\mu & U_{\mu-1} & . & . & . & U_1 \end{bmatrix} \tag{15}$$

There exists an integer n_1 such that

$$\mathrm{rank}\Delta_{n_1+k+1} = m + \mathrm{rank}\Delta_{n_1+k} \qquad k\geq 0 \tag{16}$$

and if $\delta_{i\mu}$ represents the ith row of the matrix δ_μ and n_{ie} is equal to the infimum μ such that $\delta_{i\mu}$ is linearly independent of the other rows of the matrix Δ_{n_1}, the integers n_{ie} are called essential orders of the system.

The above defined indices are equal to the essential orders of the system defined earlier by Commault & Others from the transfer function matrix $G(s)$. The essential orders are invariant under the control law of equation (2) when K is nonsingular. Furthermore, it is obvious from the definition that the essential orders of the matrix $\mathrm{Diag}[s^{k-k_i}]U(s)$ are

$$n'_{ie} = n_{ie} + k_i . \tag{17}$$

It is proved that if k_i, $i=1,2,\ldots,m$, are as in equation (5),

$$k_i \geq n_{ie} \,. \tag{18}$$

Then the last theorem imposes bounds for the zeros structure at infinity of the decoupled system, since this is expressed by k_i.

The decoupled system is not necessarily controllable and observable and $\sum_{i=1}^{m} k_i$ represents the least order for a minimal realization of H(s). Since $\sum_{i=1}^{m} k_i \geq \sum_{i=1}^{m} n_{ie}$, $\sum_{i=1}^{m} n_{ie}$ determines the minimum possible among all the least orders that realize the transfer function matrices of the decoupled compensated systems. A system is called minimal delay decoupled if $k_i = n_{ie}$. It is to be noted that using the control law of equation (2) the possibility of decoupling a system does not imply the possibility for minimal delay decoupling while the converse is always true. This renders difficult the solvability of the decoupling problem since the degrees k_i are not known and have to be determined by trial.

Finally, an example is provided and the proposed method is applied for the determination of the decoupling control law for a system described by the system matrix

$$P(s) = \left[\begin{array}{ccc|ccc} s^4 & 0 & 0 & 1 & 0 & 0 \\ 0 & s^4 & 0 & 0 & 1 & 0 \\ 0 & 0 & s^3 & 0 & 0 & 1 \\ \hline -s^3 & 0 & -s & 0 & 0 & 0 \\ -s^3 & 1 & s & 0 & 0 & 0 \end{array}\right] \tag{19}$$

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