Robust Stability
Via Polytopic Set Covering

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Summary

The recent advances in the robust stability analysis of time-invariant uncertain system evidenced that a dramatic reduction of computational complexity is achievable when dealing with family of polynomials having the structure of a polytope [1].

In real world situations the family of polynomials which represent an uncertain system has seldom a polytopic structure because of the complex relationship between physical parameters and coefficient of the caracteristic polynomial.

It is possible, however, generate polytopic approximations to this family, paying the price of an increased conservatism of the analysis (see for istance [3], [6], [7], [4], [5]).

In this paper we present an algorithm which enables the covering of the image of a given function by a polytope of know vertices. This algorithm works under quite general assumptions on the nature of the function. Successively a "domain splitting" algorithm is discussed in order to refine the immersion. Finally applications of this procedure to robust stability via Kharitonov's Theorem and "zero-exclusion" are discussed.

Decsription of the Algorithm

We consider a function $a : \mathcal{P} \to \mathbf{R}^q$, where $\mathcal{P} \subset \mathbf{R}^{n_p}$ is a bounded polytope. As mentioned before, the problem we deal with is to find a polytope $\mathcal{A}$ including the set $a(\mathcal{P})$ or, equivalently, to "immerse" the set $a(\mathcal{P})$ into a polytope $\mathcal{A}$.

Assumption. There exist known *affine* functions $\underline{a}, \overline{a}$ s.t., for all $p \in \mathcal{P}$,

$$\underline{a}(p) \leq a(p) \leq \overline{a}(p). \tag{1}$$

The following algorithm constructs $2^q\mu$ points in $\mathbf{R}^q$. Theorem 1 states that the convex hull of these points includes $a(\mathcal{P})$.

Algorithm. The algorithm is composed of three steps.

Step 1 Define the hyperrectangle $\mathcal{D} \triangleq \{\delta \in \mathbf{R}^q \mid \delta_i \in [0,1],\ i = 1, \ldots, q\}$, and the polytope $\Omega \triangleq \mathcal{P} \times \mathcal{D}$; compute the vertices $\omega_{(i)}$, $i = 1, 2, \ldots, 2^q\mu$, of Ω;

Step 2 Construct the function

$$a_m(p, \delta) \triangleq \left(I_q - \mathrm{diag}(\delta)\right)\underline{a}(p) + \mathrm{diag}(\delta)\overline{a}(p); \tag{2}$$

Step 3 Determine the points $a_{m(i)} \triangleq a_m(\omega_{(i)})$, $i = 1, 2, \ldots, 2^q\mu$.

Theorem 1 $a(\mathcal{P}) \subseteq \mathrm{Conv}\,\{a_{m(i)},\ i = 1, 2, \ldots, 2^q\mu\}$.

Proof. A proof can be found in (Garofalo *et al.*).

If the function a is continuous, then the affine functions $\underline{a}$ and $\overline{a}$ can be chosen to be constant, e.g. $\underline{a}_i(p) = \min_{p \in \mathcal{P}} a_i(p)$, $\overline{a}_i(p) = \max_{p \in \mathcal{P}} a_i(p)$, $i = 1, 2, \ldots, q$. On the other hand it should be clear that the better the functions $\underline{a}, \overline{a}$ fit a, the less conservative the immersion will be.

Generally speaking, the determination of "good" functions $\underline{a}$, $\overline{a}$ is not straightforward and could require an optimization algorithm by itself. It can be greatly eased if *the mapping a is convex and differentiable on $\mathcal{P}$* (for a discussion on this point [5]).

Remark. In many situations the function a can be written as sum of functions, i.e.,

$$a(p) = \sum_{i=1}^{r} \alpha_i(p),$$

where the immersion of each $\alpha_i(\mathcal{P})$ in a polytope $\mathcal{A}_i$ is relatively simple. Then one defines a function

$$s(v) = \sum_{i=1}^{r} v_i, \quad \text{with } v_i \in \mathcal{A}_i, \text{ for } i = 1, \ldots, r.$$

It is readily seen that $a(\mathcal{P})$ is enclosed into the polytope $s(\mathcal{A}_1 \times \cdots \times \mathcal{A}_r)$.

As said, the goodness of the fitting of the set $a(\mathcal{P})$ by the constructed polytope largely depends on the choice of the functions $\underline{a}$, $\overline{a}$. However, this fitting can be improved at will by immersing the given set in the union of a number of polytopes.

Let $\mathcal{T}$ be a covering of the polytope $\mathcal{P}$ into $k(\mathcal{T})$ polytopes, i.e. $\mathcal{T} = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k(\mathcal{T})}\}$, $\cup_{r=1}^{k(\mathcal{T})} \mathcal{P}_r = \mathcal{P}$. For the sake of brevity we will call $\mathcal{T}$ a *polytopic* covering, and define $\mathbf{T}$ as the set of all polytopic coverings of $\mathcal{P}$. For *each* polytope $\mathcal{P}_r \in \mathcal{T}$, one can apply the algorithm, i.e. first determine affine functions $\underline{a}^{(r)}(\cdot), \overline{a}^{(r)}(\cdot)$ s.t.

$$\underline{a}^{(r)}(p) \leq a(p) \leq \overline{a}^{(r)}(p), \qquad \forall p \in \mathcal{P}_r;$$

then construct the multilinear functions $a_m^{(r)}(\cdot, \cdot)$ according to (2). In view of Theorem 1, one has

$$\begin{aligned} a(\mathcal{P}) = a(\cup_r \mathcal{P}_r) \;&\subseteq\; \bigcup_{r=1}^{k(\mathcal{T})} a_m^{(r)}(\mathcal{P}_r \times \mathcal{D}) \\ &\subseteq\; \bigcup_{r=1}^{k(\mathcal{T})} \operatorname{Conv} a_m^{(r)}(\mathcal{P}_r \times \mathcal{D}), \end{aligned} \tag{3}$$

where the terms in the last union are polytopes computable as illustrated in the Algorithm.

It is possible to prove that the RHS of (3) approaches its LHS as the covering gets finer and finer, provided the functions $\underline{a}^{(r)}, \overline{a}^{(r)}$ are suitably chosen.

Summary of The Applications to the Robust Stability Problem

The proposed algorithm can be used to test the Hurwitz stability of a family of polynomials

$$\pi(s,a) = s^n + a_1 s^{n-1} + \cdots + a_n, \quad s \in \mathbf{C}, \tag{4}$$

where $a = (a_1, \ldots, a_n)^T : \mathcal{P} \to \mathbf{R}^n$ is a *continuous* function of parameter p.

Suppose that $a(\mathcal{P})$ is a polytope. Correspondingly, the family of polynomials $\mathcal{F}_\pi \stackrel{\triangle}{=} \pi(\,\cdot\,, a(\mathcal{P}))$ will be a polytope as well in the space of n-th degree polynomials. We recall here that, under the above hypothesis, the so-called "edge theorem" assures that a necessary and sufficient condition for the Hurwitz stability of the family (4) is the stability of all the polynomials on the edges of $\mathcal{F}_\pi$.

In general, $a(\mathcal{P})$ is not a polytope (unless mapping a is affine) nor is $\mathcal{F}_\pi$. Application of the Algorithm yields a polytope including $\mathcal{F}_\pi$, to which one can apply the edge theorem.

Another application of the preeceding results can be obtained recalling the "zero exclusion" principle.

Let us consider again the polynomial family described by (4). Let $\hat{\pi}(\omega, a) \stackrel{\triangle}{=} \pi(j\omega, a)$. The following result holds ([2]).

Fact 1 *The polynomial family $\mathcal{F}_\pi$ is Hurwitz* iff *(*a*) for some $p^* \in \mathcal{P}$ the polynomial $\pi(\,\cdot\,, a(p^*))$ is Hurwitz; and (*b*) $0 \notin \hat{\pi}(\mathbf{R} \times \mathcal{P})$).*

Application of Fact 1 can be eased by a suitable use of our Algorithm. Indeed, good algorithms exist to establish whether the origin belongs to a given polygon in the complex plane ([9]). Thus, even if $a(\mathcal{P})$ is not a polytope, using the Algorithm it is possible to determine a polytope $\mathcal{A} \supseteq a(\mathcal{P})$. Since $\hat{\pi}(\omega, \mathcal{A})$ is a polygon, after a suitable gridding of the imaginary axis, one can apply the above-mentioned algorithm to check the condition stated in Fact 1.

This procedure can be improved using a result in (Sideris, 1989). We first assume to have constructed, if necessary, a polytope $\mathcal{A} \supseteq a(\mathcal{P})$, whose μ vertices are denoted by $a_{(i)}$. Thus, we can consider the vertex polynomials (restricted on the imaginary axis) $\hat{\pi}_i(\omega) \stackrel{\triangle}{=} \hat{\pi}(\omega, a_{(i)})$, $i = 1, \ldots, \mu$. In turn, for each couple of vertices $a_{(i)}, a_{(j)}$ we can consider the edge polynomials (again restricted on the imaginary axis)

$$\hat{\pi}_{ij}(\omega, \lambda) \stackrel{\triangle}{=} (1-\lambda)\hat{\pi}_i(\omega) + \lambda\hat{\pi}_j(\omega) \qquad \text{for} \lambda \in [0,1].$$

Let Ω_e be the (finite) set of finite positive real solutions of the $\mu(\mu-1)/2$ edge equations:

$$\Re\hat{\pi}_i(\omega)\Im\hat{\pi}_j(\omega) - \Re\hat{\pi}_j(\omega)\Im\hat{\pi}_i(\omega) = 0, \quad \text{for } i, j = 1, \ldots, \mu. \tag{5}$$

where we have indicated with $\Re(\sigma)$ and $\Im(\sigma)$ are the real and imaginary part of the complex number σ, respectively.

The following holds, [8]

Fact 2 *The family $\mathcal{F}_\pi = \pi(\,\cdot\,, \mathcal{A})$ is Hurwitz* iff *(*a*) for some $a^* \in \mathcal{A}$, the polynomial $\pi(\,\cdot\,, a^*)$ is Hurwitz; and (*b*) $0 \notin \hat{\pi}(\Omega_e \times \mathcal{A})$.*

We will use the following corollary.

Corollary. The family $\mathcal{F}_\pi = \pi(\,\cdot\,, \mathcal{A})$ is Hurwitz *iff* (a) for some $a^* \in \mathcal{A}$, the polynomial $\pi(\,\cdot\,, a^*)$ is Hurwitz; and (b) $0 \notin \hat{\pi}(\Omega \times \mathcal{A})$, where Ω is any interval containing Ω_e.

By estimating the maximum modulus of the roots of each polynomial equation in (5), e.g. by Lehmer method, and taking the maximum, say $\tilde{\omega}_{\max}$, of all these estimates, we obtain a suitable set $\Omega = [0, \tilde{\omega}_{\max}]$. Our Algorithm can then be used to immerse the set $\hat{\pi}(\Omega \times \mathcal{A})$ into a polytope (actually a polygon in the complex plane) and verify the satisfaction of condition (b) in the Corollary.

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