Stability for time-variant differential equations

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For autonomous differential equations there exists a well developed Liapunov function approach for the examination of stability problems. In particular Lasalle's principle has proved to be an important tool in cases where the derivative of the Liapunov function along solutions is negative **semi** definite. Unfortunately, a complete extension of this principle to nonautonomous differential equations seems unlikely te become available. Basically this comes about by the particular properties exhibited by the limit sets of autonomous differential equations as opposed to nonautonomous equations. In fact, a crucial step in the proof of Lasalle's principle (or closely related formulations like Barbashin's theorem) is based on the fact that limit sets are invariant under the flow. For periodic differential equations this property holds true (with an appropriate definition of the notion of invariance) and so does Lasalle's principle. Extensions to asymptotically constant or asymptotically periodic systems, as well as to almost periodic systems have been developed in the literature, leading to weaker statements in general. For differential equations with more general types of time-variance no definite statements can be made; however, there are theorems capturing **some** features of the invariance principle.

Time-varying equations arise quite naturally in different applications. We will concentrate on the following equation

$$\dot{x}(t)=-m(t)\,m(t)^{T}x(t)$$

with $x(t)$ and $m(t)$ n-dimensional.

This equation or related versions arise in adaptive identification and control problems and have been studied extensively (Kreisselmeier, Anderson, Narendra,...).

The equation arises also in other contexts such as pattern recognition, associative memory, and in many questions of numerical methematics where e.g. algorithms are to be constructed converging to solutions of linear algebraic equations, or in computing pseudo inverses.

The differential equation above has also been studied in the context of the so-called novelty detector, introduced by Kohonen, where $x(t)$ represents the "weights" or the "memory". The change of weights is then brought about by the product of the output $m^T x$ with the so-called input $m(t)$, (this is a particular case of the adaptive laws encountered in (linear) neural networks). Notice that in this framework one is not in general interested in $x(t)$ tending to zero for large t, since $x(\infty)$ is the "novelty" of the to be recognized x^* with respect to its initial value $x(o)$. On the other hand, in adaptive identification $x(t)$ should tend to zero for large t. Indeed $x(t)$ then represents the parameter error that is driven to zero ,based on an observation of the error $m^T(t)x(t)$.

It is perhaps worthwhile to notice that the differential equation is linear and has (n-1) eigenvalues equal to zero, with the last eigenvalue $-m^T(t)\ m(t)$.
The stability study when $m(t)$ is constant, is based on the Liapunov fuction $V(x) = x^T x$ and is quite trivial but not interesting from the point of view of applications. With $m(t)$ periodic, (asymptotic) stability can be investigated quite directly with the help of the Liapunov function $V(x) = x^T x$ and Lasalle's invariance principle. These results are well known. One should also add that for **any** $m(t)$ a Liapunov study quickly leads to **stability** of the origin.

A set of new results is related to the notion of persistency of excitation of m(t). Let m(t) be a regulated function. It is called persistently exciting if there exists $T > o$ such that for all s

$$\beta I \geq \int_{s}^{s+T} m(t)\, m^{T}(t)\; dt \geq \alpha\, I$$

with $\alpha \geq o$ and $\beta \geq o$.

It has been shown (Kreisselmeier, Anderson,...) that this is a necessary and sufficient condition for exponential stability, and not faster than exponential stability. The importance of this result is that it accomodates for a wide class of signals, beyond the (almost) periodicity constraint. Several remarks are in order. One notices that α and β are taken independent of s. With Anderson one notices that if the lower bound fails, there may or may not be convergence, and if there is convergence, it will not be exponential. If the upper bound fails, one should expect convergence at least as fast as exponential. Of course V(x(t)) would then be unbounded. Also notice that T is independent of s. Can this not be relaxed while still guaranteeing (a weaker form of) asymptotic stability ?

When examining the proofs of exponential stability as they appear in the literature, it is not entirely clear how they could be altered so as to accomodate for the remarks raised above. We are currently investigating these questions, trying to come up with weaker conditions that would still ensure asymptotic stability.