Computational Techniques for Flow Control and Optimization

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1 Introduction

We are interested in problems wherein flows are controlled in order to achieve some desired objective. Our approach is unique in that it is a concerted, comprehensive, and innovative program of research into flow control problems. Our program is characterized by its broad coverage of a variety of objectives and control mechanisms, including the first rigorous analysis of boundary control problems. We are also engaged in a developmental process to design numerical algorithms for the computational approximation of flow control problems. In summary, we study these constrained optimal control problems through a systematic approach that contains the following components:

1. build mathematical models of the physical problems, invoking a minimum of assumptions about the physical phenomena;
2. rigorously analyze the mathematical models to answer questions on existence and regularity of solutions, to verify the existence of Lagrange multipliers to enforce constraints, and, most important, to derive necessary conditions that optimal controls must satisfy;
3. construct and analyze discretization methods for determining approximate solutions of the optimal control problems, including a rigorous derivation of error estimates; and

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4. develop computer codes implementing our discretization algorithms, first for the purpose of showing the efficacy of these methods, and ultimately, to solve problems of practical interest.

Our recent successes have demonstrated the validity of our strong belief that our approach, which combines mathematical rigor, algorithmic development, and computer implementation, is the best way to attack these difficult problems.

Here we briefly describe the various problems that we have formulated and which have been, or are about to be, subjected to our systematic attack. We first give some of the objectives that we are interesting in achieving through the control of flows; then we describe the constraints placed on the optimization problem. Subsequently, we describe methods of control.

2 Objectives

In this section we describe some of the objectives that we want to achieve by controlling flows. In each case, we will briefly discuss the physical situation, and then give a mathematical cost functional whose minimization will achieve the desired physical objective. For the sake of simplicity, we give, in each case, a stationary version of the problem; time dependent versions can also be easily defined. We also use the context of incompressible flows; similar objectives and mathematical realizations can be devised for compressible flows. Note that the objectives listed here are but a sample of those we have, or hope to, treat, and that, furthermore, the machinery that we are building will allow us to treat many other objectives as they are presented to us.

2.1 Flow tracking

Let $\mathbf{u}$ denote the velocity field. We want to control the flow so that the velocity field is close to a given flow field $\mathbf{u}_0$. In this case, it is natural to minimize the functional

$$\mathcal{J}_1(\mathbf{u}) = \|\mathbf{u} - \mathbf{u}_0\|^\alpha ,$$

where the choice of norm and exponent is governed by both mathematical and physical considerations. For example, one choice that we have considered is

$$\mathcal{J}_1(\mathbf{u}) = \frac{1}{4}\int_\Omega |\mathbf{u} - \mathbf{u}_0|^4 \, d\Omega$$

where Ω denotes the flow domain. Included in this class of problems (but with different types of functionals) are maneuvering problems wherein the objective is to steer a submerged body along a desired path.

2.2 Viscous drag minimization

An important objective in many applications is the minimization of drag. It is well known that the drag on a body can be computed from the integral of the dissipation function, *i.e.*,

$$\mathcal{J}_2(\mathbf{u}) = \frac{\mu}{2}\int_\Omega |(\mathrm{grad}\,\mathbf{u}) + (\mathrm{grad}\,\mathbf{u})^t|^2\, d\Omega\,,$$

where μ denotes the viscosity coefficient. Thus, if one wishes to minimize the drag on a submerged body, one merely minimizes the functional $\mathcal{J}_2(\mathbf{u})$.

2.3 Avoiding hot spots

In many applications it is desirable that temperatures and/or temperature gradients along flow boundaries, *e.g.*, structural components, not be allowed to exceed certain specified values. In particular, one would like to avoid "hot spots" along bounding surfaces, *i.e.*, places where temperature peaks occur, since often such phenomena lead to meltdown or to flexural failures. A candidate functional whose minimization would avoid such problems is

$$\mathcal{J}_3(T) = \int_{\Gamma_T} |\mathrm{grad}_s\, T|^2\, d\Gamma\,,$$

where T denotes the temperature, grad_s the surface gradient, and Γ_T the portion of the flow boundary along which one would like to avoid the above problems.

3 Constraints

The minimization of any of the functionals listed in §2 is subject to constraints, most notably, that the flow variables describe realizable flows, *i.e.*, that the flow variables satisfy the governing equations of fluid mechanics. We invoke no simplification of these equations, retaining the full nonlinearities. For example, for problems involving incompressible flows of a single fluid, with no temperature coupling, we require the velocity $\mathbf{u}$ and the pressure p to satisfy the continuity equation

$$\mathrm{div}\,\mathbf{u} = 0 \quad \text{in } \Omega$$

and the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\,\mathrm{div}\,\left((\mathrm{grad}\,\mathbf{u}) + (\mathrm{grad}\,\mathbf{u})^T\right) + \mathbf{u}\cdot\mathrm{grad}\,\mathbf{u} + \mathrm{grad}\,p = \mathbf{f} \quad \text{in } \Omega\,,$$

where ν denotes the kinematic viscosity, $\mathbf{f}$ a body force, and Ω the flow domain. In addition, initial conditions on the velocity must be specified, as well as some boundary conditions. For example, one could specify the velocity $\mathbf{u}$, or the stress vector $[-p\mathbf{n} + \nu(\operatorname{grad}\mathbf{u} + \operatorname{grad}\mathbf{u}^t) \cdot \mathbf{n}]$, or some components of each, along the boundary of the flow domain; indeed, in most applications different types of boundary conditions would be imposed along different portions of the boundary of Ω. (Here, $\mathbf{n}$ denotes the unit normal to the boundary of the flow domain.) For problems involving the temperature and/or fluid mixtures, additional constraints are provided by the energy equation and/or conservation of species equations, respectively, as well as some additional boundary and initial conditions.

4 Control mechanisms

The objectives listed in §2 are to be achieved by controlling the flow. Here, we discuss some of the control mechanisms that we use to achieve these goals. We note that if the size of the control (measured in an appropriate norm) is not *a priori* constrained to be bounded, then the functionals of §2 must be penalized with some norm of the control; otherwise, optimal controls would usually be unbounded, and therefore not physically realizable. We also note that many other control mechanisms are possible, and here we provide only a sampling of these possibilities.

4.1 Velocity along portions of the boundary

A very much used mechanism of control is to inject or suck fluid through orifices along bounding surfaces. (Such control mechanisms have long been used in experimental studies of boundary layer control and drag minimization.) Thus, if Γ_c denotes the portion of the boundary covered by the orifices, we would seek a control $\mathbf{g}$ such that one of the functionals of §2 is minimized, subject to the constraints of §3, and also

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_c .$$

4.2 Temperature and heating controls

Another common control mechanism is to adjust the temperature, or even more often, the heat flux, along portions of the boundary of the flow domain in order to achieve one of the desired objectives. Within this class of controls we find "heating" and "cooling" controls. For example, one could seek a control q such that one of the functionals of §2 is minimized, subject to the constraints of §3, and also

$$\frac{\partial T}{\partial n} = q \quad \text{on } \Gamma_T ,$$

where Γ_T denotes the portion of the boundary along which one allows the control to act and $\partial/\partial n$ denotes the normal derivative at the boundary.

4.3 Distributed controls

One could try to effect control through the body force in the Navier-Stokes equation of §3. Thus, one would seek a control $\mathbf{f}$, defined on the flow domain Ω or on a portion of Ω, such that one of the functionals of §2 is minimized and subject to the constraints of §3. Physically, one way to effect such control is through a magnetic field acting on an ionized fluid or on a liquid metal.

Another possible distributed control is a heat source in the energy equation. In this case, we would seek a control Q, defined on the flow domain Ω or on a portion of Ω, such that such that one of the functionals of §2 is minimized, subject to the constraints of §3, and also such that

$$\frac{\partial T}{\partial t} - k\Delta T + \mathbf{u} \cdot \operatorname{grad} T = Q \quad \text{in } \Omega .$$

Physically, one way to effect such a control is through radiation mechanisms, or through a targeted laser beam.

4.4 Shape control

The control mechanisms discussed so far are known as *value controls*; this refers to the fact that we try to effect control through the adjustment of the values of the data of the problem. Another class of controls are known collectively as *shape controls*; in this case control is effected by adjusting the shape of the flow domain. The shape of the flow domain may be changed in many ways. For example, one could use leading and/or trailing edge flaps, or movable walls, or rudders, or propeller pitch. A related problem is the *optimal design* problem. Here, we want to choose a flow domain, *e.g.*, the exterior of an airfoil, so that some objective is achieved. Of course, choosing the flow domain is tantamount to choosing its boundary, *i.e.*, in this case, the airfoil itself.

5 Bibliography

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