A Dynamic Game Approach to Discrete-Time Mixed H_∞ / H_2 Filtering

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Abstract

Estimators that achieve bounded H_∞-error norm are useful in cases where no statistical knowledge of the disturbances is available. An H_∞-optimal estimator guarantees that $\|T_{rd}\|_\infty$, the induced ℓ_2 norm of the operator T_{rd} which relates the disturbances to the estimation errors, will assume its lowest possible value γ_0. The requirement of $\|T_{rd}\|_\infty = \gamma_0$ leads, however, to filters with unacceptable H_2 performance [1]. This is the reason why so much work has been devoted lately to the design of mixed H_∞ / H_2 estimators, which achieve acceptable performance both in the H_∞ and the H_2 senses [2][3][4]. The basic idea behind mixed H_∞ / H_2 filtering is that for any $\gamma > \gamma_0$, there exist many filters that achieve the performance level of γ, namely $\|T_{rd}\|_\infty < \gamma$. The set of all filters that achieve a given H_∞ performance level was parameterized in the continuous-time case by a single bounded causal operator $\mathcal{U}$, $\|\mathcal{U}\|_\infty \le \gamma$, in [5]. A similar parameterization is given in the present paper for the discrete-time case. One may select $\mathcal{U}$ to optimize an H_2 performance measure. The mixed H_∞ / H_2 filtering problem is trivially solved by the H_2-optimal (Kalman) estimator for $\gamma > \gamma_2$, where γ_2 denotes the H_∞ performance level that is achieved by the Kalman filter.

No exact solution for the mixed H_∞ / H_2 filtering problem has yet been found. Most of the existing works on the topic are devoted to the optimization of the Berenstein and Haddad bound on the H_2 performance of the filter [2],[3], subject to $\|T_{rd}\|_\infty < \gamma$. This optimization of full order filters leads to the so called 'central filter'. This filter is obtained by selecting the parameter $\mathcal{U}$ to be identically zero. The central filter is also obtained as a result of minimizing the expected value of the exponent of the power seminorm of the estimation error, when the disturbances are standard white [4]. The central filter is, however, not H_∞ / H_2 optimal. This is easily seen by noting that it differs from the Kalman filter for $\gamma > \gamma_2$. In this paper we propose, for the first time, a method for selecting a non-zero $\mathcal{U}$ that brings the estimate as close as possible to the H_2 optimal estimate while complying with the required H_∞ performance of $\|T_{rd}\|_\infty < \gamma$.

A different meaning has been assigned to the mixed H_∞ / H_2 filtering problem in [6], where two different sets of disturbances have been considered. The first set has to be rejected optimally in the H_2 sense, while the rejection of the second set must achieve a prescribed H_∞ performance level. The approach there is different from the one we use here by the fact that in [6] the response of the estimation filter is optimized separately (in different senses) for each of the system disturbance inputs. It cannot, therefore, guarantee a minimum performance level for the combined effect of the two types of disturbances.

In the present paper we consider the following H_∞ aposteriori discrete-time filtering problem: Given the linear, discrete time-varying system

$$x_{k+1}=Ax_k+Bw_k \quad , \; y_k=C_2x_k+v_k \tag{1}$$

where we omit the explicit time dependence of the system matrices for the sake of simplicity. We are looking for the estimator $\hat{z}_k$ that will use the measurements $\{y_i \, , \, i \le k\}$, to guarantee the performance level of $\gamma > \gamma_0$, with a given initial conditions weighting matrix $R_0 > 0$, in the sense that

$$\|C_1x_{k+1}-\hat{z}_{k+1}\|_2^2 \le \gamma^2(x_0^t R_0^{-1}x_0+\|w_k\|_2^2+\|v_k\|_2^2) \quad \forall \quad \{w_k\},\{v_k\} \in \ell_2[0,N] \tag{2}$$

where $\|d_k\|_2^2=\Sigma_0^{N-1}d_k^t d_k$.

The requirement of (2) can be put in a dynamic game context where the estimator $\{\hat{z}_k\}$ can be considered playing against nature that can choose any x_0 , $\{w_k\}$ and $\{v_k\}$ in $\ell_2[0,N]$ so as to maximize

$$J_\infty=-\gamma^2\|x_0\|_{R_0^{-1}}^2+\|C_1x_{k+1}-\hat{z}_{k+1}\|_2^2-\gamma^2(\|w_k\|_2^2-\|y_{k+1}-C_2x_{k+1}\|_2^2) \tag{3}$$

The estimator wishes to minimize J_∞ for the worst possible selection of nature. It is well known [7] that there exists a saddle-point to the above game iff there exists a solution $\Sigma_k > 0$ for the following recursion on $[0, N]$.

$$M_{k+1}=A\,\Sigma_k A^t+BB^t \quad , \; \Sigma_{k+1}^{-1}+\gamma^{-2}C_1^tC_1=M_{k+1}^{-1}+C_2^tC_2 \; , \; \Sigma_0=R_0^{-1}$$

In the latter case, one filter that achieves (2) is the following central filter:

$$\hat{z}_k^{(\infty)}=C_1x_k^{(\infty)} \quad , \; x_{k+1}^{(\infty)}=Ax_k^{(\infty)}+K_{1,k+1}(y_{k+1}-C_2Ax_k^{(\infty)}) \; , \; K_{1,k+1}=M_{k+1}C_2^t(I+C_2M_{k+1}C_2^t)^{-1} \tag{4}$$

The central filter is not the only one that satisfies (2). Based on the treatment of the general 4-block problem in [8], we obtain that all the aposteriori filters that satisfy (2) have the following structure

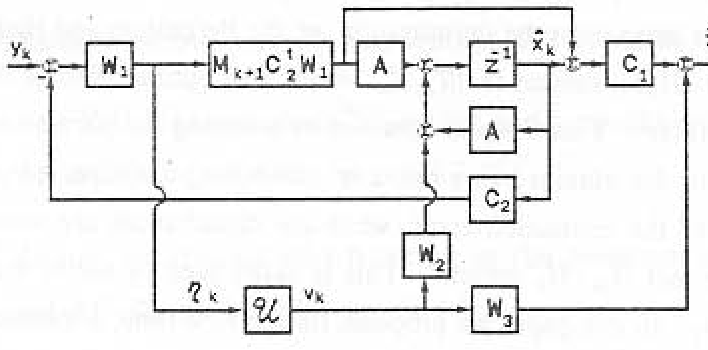

Fig. 1: Parameterization of all the aposteriori filters that satisfy (2)

where

$$W_1=(I+C_2M_{k+1}C_2^t)^{-1/2} \quad , \; W_2=-\gamma^2AM_{k+1}(I+C_2^tC_2M_{k+1})^{-1}C_1^tW_3^{-1},$$
$$W_3=[I-\gamma^{-2}C_1M_{k+1}(I+C_2^tC_2M_{k+1})^{-1}C_1^t]^{1/2},$$

and where $\mathcal{U}$ is an arbitrary linear causal operator that satisfies

$$\|\mathcal{U}\|_\infty \le \gamma. \tag{5}$$

The objective of our paper is to find the operator $\mathcal{U}$ of (5) that minimizes $\|\hat{z}_{k+1}-C_1x_{k+1}\|_2^2$. It is easily verified that there exists no linear causal $\mathcal{U}$ which minimizes the above. This is the reason why we consider the following alternative objective

$$J_a = -\delta^2\|z_{k+1}^{(\infty)}-z_{k+1}^{(2)}\|_2^2 + \|\hat{z}_{k+1}-z_{k+1}^{(2)}\|_2^2 \tag{6}$$

where $z_k^{(\infty)}$ is the above central aposteriori H_∞ estimate, $z_k^{(2)}$ is the corresponding aposteriori Kalman estimate, and $\hat{z}_k$ is the estimate obtained for $\mathscr{U} \neq 0$ in Fig. 1. Note that in the case where $\{w_k\}$ is a standard white sequence, the second term in the right side of (6) denotes the increase in the estimation error variance of our filter in comparison with the one obtained by the corresponding Kalman filter. We want to find $\mathscr{U}$ that, for the smallest possible δ, will assure $J_a \leq 0$ for all y_k. If such a $\mathscr{U}$ is found for say δ_0, it implies that the ℓ_2-norm of the difference between the H_∞ and the H_2 estimates is minimized and that the ratio between the error variance increase of the estimator and the error variance increase of the H_∞ central filter is bounded by δ_0. It can be shown that for $\gamma_0 < \gamma < \gamma_2$, the optimal $\mathscr{U}$ satisfies (5) with equality. We can thus incorporate the norm restriction (5) on $\mathscr{U}$ into J_a by augmenting the objective function as follows:

$$J = J_a + \kappa[\|v_k\|_2^2 - \|\eta_k\|_2^2] \qquad (7)$$

where η_k is the signal at the input to $\mathscr{U}$, and v_k is its output, as in Fig. 1. The minimizing strategy for v_k is obtained by writing the state space equations for the system whose state vector is $\bar{x}_k^{\,t} = [\hat{x}_k^{\,t} \;\; x_k^{(\infty)t} \;\; x_k^{(2)t}]$. We write these equations so that η_k plays the role of the external disturbance, and v_k stands for the control input. The optimization of (7), subject to the latter state equations, is a standard conflicting objectives H_∞ state-feedback problem which can be solved by modifying the techniques of [8].

We start the search for δ_0 with $\delta=1$, since the value of $\delta=1$ can be trivially achieved by $\mathscr{U}=0$. We then search for a value of κ for which $\|v_k\|_2^2 = \|\eta_k\|_2^2$. We repeat this search of κ for with monotonously decreasing values of δ, and stop at the lowest value of δ for which there still exists such κ. The obtained optimal strategy for v_k is then substituted in the configuration of Fig. 1 to obtain $\hat{z}_k$. Note that if $\gamma > \gamma_2$, the optimization problem of (6) can be solved for $\delta=0$. In this case no κ will achieve $\|v_k\|_2^2 = \|\eta_k\|_2^2$ and the resulting estimator will be identical to the Kalman filter.

Numerical example : Consider the following steady state mixed H_∞ / H_2 filtering problem. Given the system

$$x_{k+1} = \begin{bmatrix} 1 & 0.01 \\ -0.01 & 0.996 \end{bmatrix} x_k + \begin{bmatrix} 0.0092 \\ 0.0184 \end{bmatrix} w_k \;, \quad y_k = [0 \;\; 54.3] x_k + v_k$$

It is required to estimate $[0.3 \;\; -0.3] x_k$ with a unity H_∞ performance level, and with as good as possible H_2 performance. We obtained a mixed-norm filter according to the above theory with $\delta_0 = 0.4372$ and $\kappa = 0.3405$. A comparison of the singular value Bode plots of T_{rd} for the proposed filter and for the corresponding H_2-optimal and H_∞-central filters is depicted in Fig. 2.

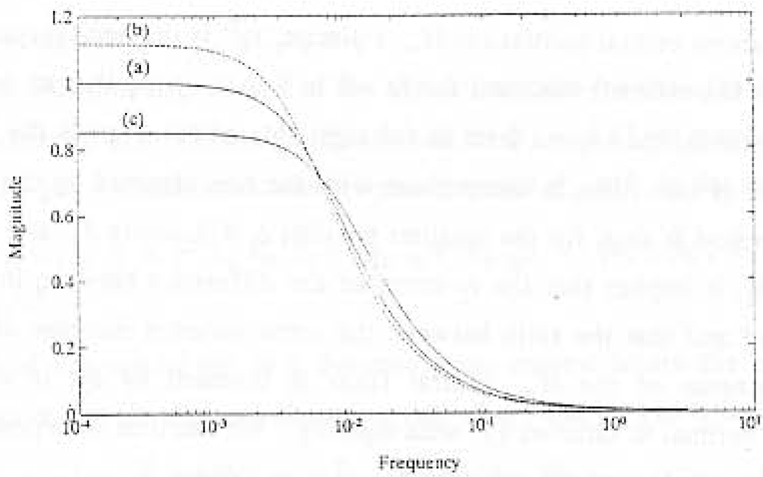

Fig. 2: Maximal singular value of T_{rd} for: (a)proposed, $(b)H_2$-optimal, $(c)H_\infty$-central, $\gamma=1$, filters.

Both the proposed and the H_∞-central filters achieve $\|T_{rd}\|_\infty \leq 1$, but in the case where $\{w_k\}$ is a standard, unit intensity, white noise sequence, the power-seminorm of the estimation error of the proposed filter is 5.59×10^{-2}, only slightly more then the value of 5.55×10^{-2} that is achieved by the Kalman filter. This result is significantly lower than the value of 5.86×10^{-2} that is achieved by the H_∞ central filter for $\gamma=1$.

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