Weighting Selection and Robustness of H_∞ Designs

by

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Abstract

The choice of dynamic cost-function weightings in H_∞ design is considered. Simple design examples are presented which illustrate pitfalls which can arise in the selection of weightings. The lessons which can be learned are discussed and design rules are formulated. The design of industrial control systems using H_∞ methods depends upon the selection of the weightings to satisfy the given specifications. The aforementioned design rules provide a first step to the development of a formalized design procedures for industrial applications such as gas turbine control.

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1. Introduction

A number of problems are considered in the choice of H_∞ cost functions weightings. The results of simple examples are presented and discussed highlighting difficulties and resulting design guidelines. The selection of cost function weights involves a range of requirements and design guidelines and only a few particular properties will be considered.

It is easy to select examples which seem to demonstrate poor robustness properties for H_∞ designs. However, in most cases when the related physical problem description is considered it is found to be impractical and unrealistic. There are some very obvious rules which should be followed and simple methods of detecting when unrealistic problem descriptions are defined. The examples presented seem at first sight to be based on reasonable problem descriptions but they lead to unrealistic results. On closer inspection the problems are found to be physically unrealistic and the method of detecting this situation is described.

There are a number of misconceptions in the literature. In two recent published papers some of the above features are included. Woodgate (1991 [1]) has noted that the generalised H_∞ problem first proposed by Grimble (1987 [2]) has robustness problems. GH_∞ problem is in fact related to the one block designs of Glover and MacFarlane in that it provides a one shot simple H_∞ calculating procedure. Although robustness properties are not so well defined as in the usual mixed-sensitivity H_∞ problem by following simple guidelines robustness problems of the type discussed can normally easily be avoided. In fact the examples reveal that the same robustness problems apply for the example given to the mixed sensitivity problem which is that most often considered in the literature. A very similar property also applies in LQG design. It is therefore important to emphasise that robustness is not automatically provided by any of the H_∞ solution methods and some effort of the choice of cost weights and system models is necessary to obtain a good result.

It has also been noted in the literature that the mixed-sensitivity problem solution (Foley and Harris 1992 [4]) also has very particular robustness difficulties. In this particular case the authors use constant cost function weightings. This appears to be the worst possible choice of weighting which is a second issue discussed.

It is not the contention in the following H_∞ deign automatically provides good robust solutions. The main point is that some of the criticisms are ill-founded since they are based on using very poor choices of cost-functions and system models. The design guidelines which are proposed here will avoid most of the very obvious robustness problems but there will of course be a need to build up weighting selection rules to address many other design requirements.

Structure of the Paper

The usual cost function minimisation problem is first considered in Section 2. An example is presented illustrating that for a simple system description and an apparently reasonable choice of weightings very poor robustness properties are obtained. The reasons for the poor results are considered and explained. In section 3 the GH_∞ control problem is discussed and a similar example presented with the same poor robustness results. Again the lessons to be learned and the obvious errors in system definition are explained. In Section 4 the use of constant cost weightings in H_∞ design is briefly discussed and shown to be almost the worst possible weighting choice. In Section 5 some simple design rules which would have avoided the difficulties are recorded. Finally Conclusions are drawn in Section 6.

2. The H_∞ Control Problem

There are basically two types of cost-function commonly employed in H_∞ design. Examples might be written as:

Mixed Sensitivity Cost : $\qquad J_\infty = \|(S^*QS + M^*RM)\|_\infty$

GH_∞ Cost : $\qquad J_\infty = \|((P_cS + F_cM)^*(P_cS + F_cM))\|_\infty$

where $P_c(z^{-1})$ and $F_c(z^{-1})$, and $Q(z^{-1})$ and $R(z^{-1})$ are dynamic cost-function weightings.

The cost function minimised in the usual H_∞ control problem (2 or 4 block) leads to relatively complicated algorithms, whether using state-space or polynomial system models. This is particularly important in multivariable design problems where calculations are always more complex. Computational complexity is also important when an adaptive controller is to be developed. A so called *Generalized H_∞* (denoted GH_∞) controller was therefore developed (Grimble, 1987 [2]) with a special cost function which leads to much simpler controllers.

The GH_∞ cost function leads to a problem solution involving a straightforward linear eigenproblem. The usual mixed sensitivity cost function results in a nonlinear eigenproblem and this is the main reason that H_∞ control calculations are often complicated. The generalized problem is a *one block* problem, however, it has properties similar to the *two block* or *mixed sensitivity* problems usually considered.

2.1 Examples of Poor Robustness due to Unrealistic Weightings

There follows an example which demonstrates that poor robustness can result for the usual mixed sensitivity criterion and the simplest systems unless care is taken with the cost weighting definition.

Example 1 : ***Mixed Sensitivity Problem***

Consider the continuous-time single DOF system shown in Figs 1 and 2, with the following models:

$$W = A^{-1}B = 1 \Rightarrow A = B = 1$$

$$W_d = A_d^{-1}C_d = 1 \Rightarrow C_d = 1,\ W_r = 0$$

Let the cost-weightings $Q_c = P_c^*P_c$ and $R_c = F_c^*F_c$ be defined so that $P_c = 1/s$ has a high gain at low frequency and $F_c = \theta^2/(1+\varepsilon^2 s)$ has a gain which is higher at high frequencies

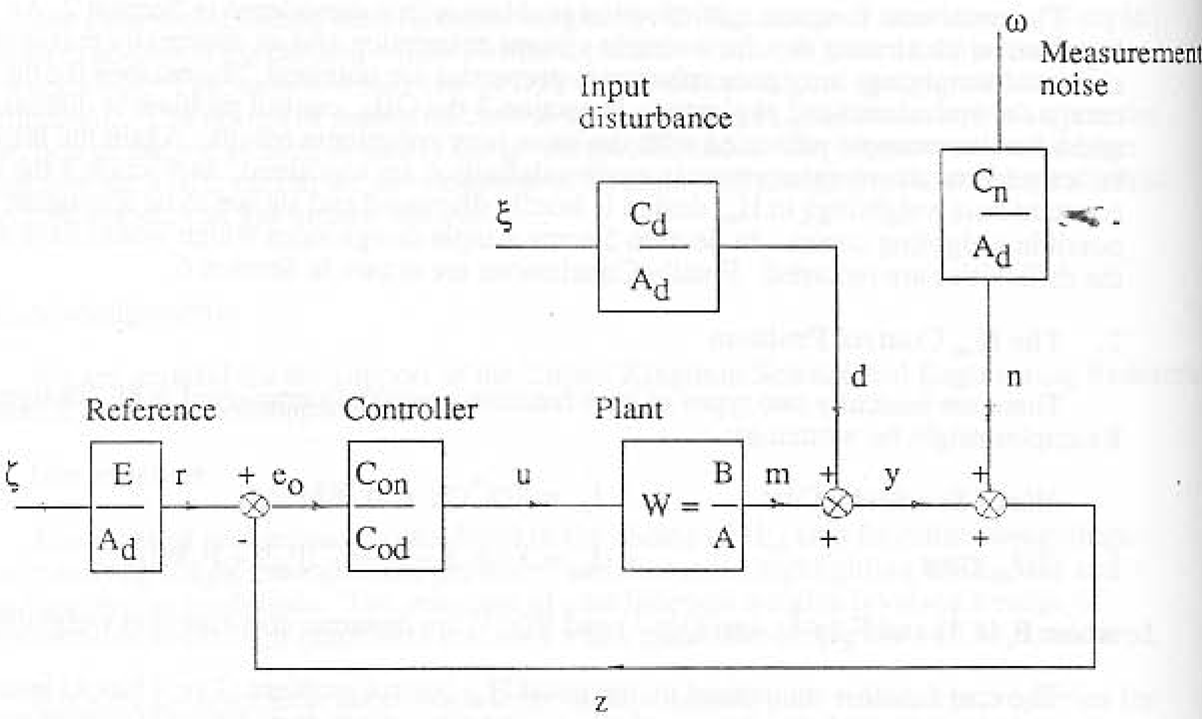

Fig. 1 : Canonical Feedback System With Disturbances and Reference Model

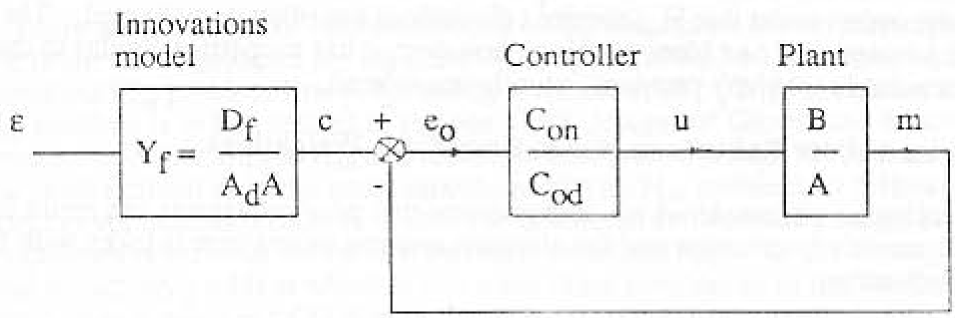

Fig. 2 : Innovations Form of Feedback System

($|\theta/\varepsilon| > 1$). Thence, let the weighting terms:

$$A_w \triangleq s(1+\varepsilon^2 s), \quad Q_n \triangleq (1-\varepsilon^4 s^2) \text{ and } R_n \triangleq -s^2\theta^4.$$

The H_∞ controller may now be computed from the results of Grimble (1986 [7]).

$$D_c^* D_c = Q_n + R_n = 1 - (\varepsilon^4+\theta^4)s^2 = 1 - m^2 s^2$$

where $m \triangleq (\varepsilon^4 + \theta^4)^{1/2}$, so that $D_c = (1 + ms)$. Also,

$$A_\sigma^* A_\sigma = D_c^* D_c A_w^* A_w \lambda^2 - R_n Q_n$$
$$= \{(1-m^2s^2)\lambda^2 - \theta^4\} A_w^* A_w = A_{\sigma_1}^* A_{\sigma_1} A_w^* A_w$$

If $\deg(D_c) = 1$ and $\deg F = 0$ and $F = F_s = 1$ then the equations to be solved, noting $A_\sigma = A_{\sigma_1} A_w$

$$(1-m^2s^2)G + A_{\sigma_1} s(1+\varepsilon^2 s) = (1-\varepsilon^2 s)(1+\varepsilon^2 s)$$

$$(1-m^2s^2)H - A_{\sigma_1} s(1+\varepsilon^2 s) = -s^2\theta^4$$

Thence G and H can be written as:

$$G = (1 + \varepsilon^2 s) \quad \text{and} \quad H = sH_1$$

and the above equation for A_σ then gives:

$$A_{\sigma_1} = m^2 s - \varepsilon^2 \quad \text{and} \quad A_{\sigma_1}^* A_{\sigma_1} = (-m^4 s^2 + \varepsilon^4)$$

hence $\lambda^2 = \theta^4 + \varepsilon^4 = m^2$, and $H_1 = -\varepsilon^2$.

The controller follows as $C_o = H^{-1}G = -(1+\varepsilon^2 s)/(s\varepsilon^2)$. Thence, the sensitivity and control sensitivity functions become:

$$S = -\varepsilon^2 s \quad \text{and} \quad M = (1+\varepsilon^2 s)$$

and

$$P_c S = -\varepsilon^2 \quad \text{and} \quad F_c M = \theta^2$$

The minimum cost-function spectrum is equalizing:

$$S^* Q_c S + M^* R_c M = \varepsilon^4 + \theta^4 = \lambda^2$$

The return-difference $1 + WC_o = -1/(\varepsilon^2 s)$ and the only zero is for $s \to \infty$. The controller $C_o(s) = -(1+1/(\varepsilon^2 s))$ is therefore of PI form and stabilizes the closed-loop system.

The controller obtained is not physically realistic since a pure PI controller has too high a gain at high frequencies. The problem arises because the slope of both weightings is the same at high frequencies. The control weighting term should really have more lead terms to ensure the controller rolls off at high frequencies. However, PI controllers are found in real applications and hence the solution is acceptable, if not desirable.

The main problem arises due to robustness. Let $W = k$ where $k > 0$ is a scalar gain then the return-difference becomes:

$$1 + WC_0 = (\varepsilon^2 s - k(1+\varepsilon^2 s))/(\varepsilon^2 s) = -k(1+(1-1/k)\varepsilon^2 s)/(\varepsilon^2 s)$$

Clearly the system remains stable for all $k > 1$ but is unstable for an arbitrarily small fall in the gain below the nominal plant gain of unity.

Thus, the mixed sensitivity H_∞ problem can provide non-robust solutions when care is not taken over the weighting choice. Note that the system model is also important. Even though this problem appeared realistic the disturbance model W_d was assumed to be a unity gain. In stochastic terms this corresponds to designing a regulator for a system with a white noise disturbance input. Such disturbances cannot be countered effectively and the problem is unrealistic and hence it is not surprising that the design is impractical.

3. Cost Weighting Selection for Generalized Cost Functions

The GH_∞ cost-function is very attractive computationally. The criterion has some special performance and robustness properties which require particular care to be taken when choosing cost-function weightings P_c and F_c.

Example 2 : ***GH_∞ Control Problem***

Consider the continuous-time single DOF system, similar to that above, with the constant plant and disturbance models:

$$W = A^{-1}B = 1 \Rightarrow A = B = 1$$

$$W_d = A^{-1}C_d = 1 \Rightarrow C_d = 1$$

Let the GH_∞ cost-function weightings:

$$P_c(s) = \rho(s+1)/(s+\beta^2)$$

$$F_c(s) = \theta^2(s+1)/(\varepsilon^2 s+1)$$

where $0 < \varepsilon^2 \leq 1$. Identify $P_n = \rho(s+1)(\varepsilon^2 s+1)$, $F_n = \theta^2(s+1)(s+\beta^2)$ and $P_d = (s+\beta^2)(\varepsilon^2 s+1)$.

The equations to be solved which determine the GH_∞ controller, from Grimble (1987 [2]) become:

$$L = P_n B - F_n A = (s+1)(s(\rho\varepsilon^2 - \theta^2) + (\rho - \theta^2\beta^2))$$

which is minimum-phase if $(\theta^2 - \rho\varepsilon^2)/(\theta^2\beta^2 - \rho) > 0$ and hence the solution becomes $\lambda = 0$, $F_2 = 0$, $G_2 = P_n$ and $H_2 = -F_n$. Thence, the controller is obtained as:

$$C_0(s) = -P_n(s)/F_n(s) = \frac{-\rho(\varepsilon^2 s+1)}{\theta^2(s+\beta^2)}$$

Now let the plant gain $W(s) = k$ then the characteristic polynomial:

$$\rho_c(s) = \theta^2(s+\beta^2) - \rho k(\varepsilon^2 s+1) = s(\theta^2 - \rho k\varepsilon^2) + (\theta^2\beta^2 - \rho k)$$

Clearly if the plant has the nominal gain $k = 1$ and the above assumption that L is minimum-phase is satisfied the system is stable. Now consider the case where ε^2 is very small and $(\theta^2\beta^2-\rho) = \delta^2$ which is arbitrarily small. An arbitrarily small change in k will then result in the constraint:

$$(\theta^2-\rho k\varepsilon^2)/(\theta^2\beta^2-\rho k) > 0$$

being violated.

Woodgate (1992 [1]) noted that the GH_∞ controller is not robust for this simple form of plant uncertainty, namely a variation in the loop gain, and that there was nothing particularly special about the above problem. Closer inspection reveals this is not the case.

The example reveals how an unrealistic system and weighting definition leads to unrealistic results. The following points should be noted:

(i) If $\theta^2\beta^2 - \rho = \delta^2$ then $\theta^2\beta^2 > \rho$ and the DC gain of the controller $C_0(0) = \dfrac{-\rho}{\theta^2\beta^2}$. Thus, in this example of poor robustness the loop gain has a nominal value of less then unity. Such a system is clearly impractical resulting in steady-state step-response errors of greater than 50%.

(ii) The sign of the weighting elements in the GH_∞ criterion is crucial. The signal $\varphi = P_c e + F_c u = -P_c y + F_c u$ and if for example P_c and F_c are constants it is clear that both H_2 and H_∞ problems are ill-posed in this case. Clearly the signals y and u are not limited when either the variance of φ, or the H_∞ norm of $\Phi_{\varphi\varphi}$, is small. When P_c and F_c are dynamical and have frequency responses as previously recommended ($|P_c|$ dominates at low frequency and $|F_c|$ dominates at high frequency) this situation does not arise. However, if at low frequency φ is to be small and the feedback is to be negative then P_c and F_c should have opposite signs.

(iii) For the above example, if the gain of $P_c(s) : \rho = -\rho_1 < 0$ then the DC gain of the controller : $C_0(0) > 0$ and the characteristic polynomial $\rho_c(s) = s(\theta^2+\rho_1 k\varepsilon^2) + (\theta^2\beta^2+\rho_1 k)$. The system is clearly stable for all positive variations of the plant gain k. Moreover, this is true for the more realistic situation where $\rho_1/(\theta^2\beta^2) >> 1$ and hence the controller and nominal loop gains are much greater than unity.

(iv) When the sign of the weighting elements is chosen correctly the controller provides a negative feedback solution and good robustness which was not the case in the impractical design where $\rho > 0$.

(v) The GLQG controller for this problem which minimizes the variance of the signal φ, is the same as given above and suffers the same robustness problem if weights are not chosen reasonably.

(vi) The disturbance model in stochastic terms represents white noise and it is by definition impossible to reject such a disturbance. To make the above example a physically realistic problem $W_d(s)$ should be a filter specifying the frequency response characteristics of the disturbance.

GH_∞ Design

The last example suggested several problems with GH_∞ design but gave a rather misleading impression since:

(a) The example presented was for an incorrect choice of weighting sign. Robustness was totally recovered when the sign was corrected and negative feedback at low frequencies was obtained.

(b) The other optimal cost-functions have similar poor robustness properties in very similar situations when weights are unrealistic.

(c) There are in fact robustness results for the GH_∞ solution considered in Chapter 3.

(d) The example was for a very special case. The disturbance model was a constant, the weights led to positive feedback at low frequency and did not ensure the controller rolled off at high frequencies.

There is certainly a need to be cautious about the robustness properties of the GH_∞ design approach. However, experience has revealed that for many real design studies the results for GH_∞ design are almost identical to those obtained from the mixed sensitivity problem. Since the GH_∞ algorithm is much simpler to implement than the mixed sensitivity algorithm, it remains a strong candidate for use in, for example, adaptive control systems.

Robustness is not a right!

An early paper by Doyle (1978 [5]) demonstrated that robustness is not guaranteed with LQG design and the Examples have shown the same result, that robustness cannot be taken for granted. If weights and system models are unrealistic it is not unreasonable that controllers should also be undesirable. However, it was very obvious that the designs obtained were unrealistic and the remedy was straightforward. The above situation therefore falls under the old maxim that *ask a silly question* and you *get a silly answer*. It also emphasises that simple analysis methods can detect obvious flaws in a design which can often easily be corrected by reappraising the choice of disturbance models and cost weightings.

4. Use of Constant Cost Weighting in H_∞ Design

Although constant weightings are often used in LQ and LQG controller designs very successfully they are almost the worst possible choice for H_∞ design. It is crucial that frequency shaped weightings are used in H_∞ design and the basic form needed is suggested by the problems which arise in the constant weighting case. These may now be considered. Recall that in the scalar case the mixed-sensitivity equalizing solution has the form:

$$J_\infty = Y_f^*(S^*Q_cS + M^*R_cM)Y_f = Y_f^*S^*(Q_c + C_0^*R_cC_0)SY_f = \lambda^2$$

If the disturbance weighting Y_f is low pass which is normally the case, then the controller frequency response (C_0) must increase with frequency. The result is a very wide bandwidth and large measurement noise amplification (the remedy is to ensure R_c increases with frequency at the appropriate rate).

At low frequencies the controller gain will become constant and depend upon the ratio of Q_c/R_c. Hence the controller will not include integral action which is normally desirable for systems containing low-frequency disturbances. If of course Q_c/R_c is very large the low frequency gain may be reasonable but the problem is then almost the limiting case of only sensitivity minimization. The controller will therefore normally attempt to provide high gain at high frequencies again resulting in an unacceptable performance.

Asymptotic properties:

The limiting cases of $Q_c \to 0$ and $R_c \to 0$ graphically illustrate the problems which are bound to arise when constant weightings are employed.

$Q_c = 0$:

In this case $|C_0 S| = |\lambda / R_c^{1/2}| / |Y_f|$ or $|M| = 1/|Y_f|$. Thus, the control sensitivity function $M = C_0 S$ has a magnitude which is inversely proportional to the disturbance spectral factor Y_f. If the plant W is open-loop stable the trivial solution is obtained that the optimal control is zero and $\lambda = 0$. If W is unstable such a design would not be acceptable since it would imply huge measurement noise amplification at high frequencies.

$R_c = 0$:

In this case $|S| = |\lambda / Q_c^{1/2}| / |Y_f|$ or $|S| = 1/|Y_f|$. Thus, the sensitivity-function has a magnitude spectrum which is increasing with frequency. If, for example, the disturbance spectral-factor denotes an integrator the sensitivity function will have a frequency-response which increases uniformly with frequency. If the plant W is minimum-phase the optimal control will ensure λ is zero in the continuous-time case (small in discrete-time), by introducing infinite loop-gain. If W is non-minimum phase the sensitivity-function will have increasing gain with frequency. The designs are clearly impractical in either case.

Foley and Harris (1992 [4]) recently showed that H_∞ design gives an inferior performance to LQ solutions in some problems. However, they employed constant cost weightings for the comparison and the poor performance of the H_∞ solution was therefore predictable.

5. Cost Function Weighting Selection for Scalar Systems

The selection of cost weighting functions does not involve precise rules but general guidelines. It is difficult to give rules which ensure a given behaviour is obtained, since in most cases a number of criteria must be satisfied at the same time and trade-offs must be made. The following guidelines will however provide a basis for selecting and changing cost function weightings for both LQG and H_∞ problems.

1. *Integral error weighting*
   An integrator on the error weighting function will often result in integral action in the controller. There are a few cases where integral action is not introduced automatically when integral error weightings are used. For example, when two degrees of freedom designs are considered, inferential control is used or when noise models cause a change in the controller response so that pure integral action is not included. The general affect of introducing integral error weighting is, however, to introduce high gain into the controller at low frequencies.

2. *Integral sensitivity weighting*
   When integral weighting is used on the sensitivity function, this has a similar effect to Case 1. However, sensitivity costing normally arises in mixed sensitivity problems where measurement noise is not present in the system description and hence integral action in the controller normally occurs (again not necessarily for 2 DOF or inferential control problems).

3. *Lead terms on the control weighting*
By introducing a high gain at high frequencies on the control weighting term, the controller is normally made to roll off in the frequency range where the gain is high (relative to error weighting terms). The use of a weighting function with high gain at high frequency is more important in H_∞ design than in H_2 minimization problems. This weighting provides one mechanism of ensuring the controller will roll off at high frequencies. It ensures the usual wide bandwidth property of H_∞ designs does not lead to unacceptable measurement noise amplification problems. Controller roll-off at high frequencies occurs naturally in LQG or H_2 designs due to the use of a measurement noise model. If a measurement noise model is not included, LQG designs can give too high a gain at high frequencies.

4. *Lead terms on the control sensitivity costing*
The control sensitivity function plays a similar role to the control weighting term referred to in Case 3. In mixed sensitivity problems where a *control sensitivity* term is present, high weighting gain at high frequency is normally advisable for H_∞ designs.

5. *Complementary sensitivity costing*
In H_2 or LQG problems complementary sensitivity terms are not normally present. In early H_∞ designs these terms were introduced commonly, but the disadvantages have recently been recognised Complementary sensitivity weighting has an identical effect to combining a weig[illegible] function together with the plant transfer-function acting on a *control sensitivity* term. Multiplying the control sensitivity function by the plant transfer function does of course give th[illegible] complementary sensitivity function. Since there are generally disadvantages in using a complementary sensitivity weighting, this term is normally neglected.

6. *Effects of the weighting functions on the cross-over frequency*
When large or small error weightings are discussed, this is of course relative to the size of the control weighting terms. In this context *large* is only in the relationship to the other weighting functions. Although the weighting functions do have an affect which depends upon the scaling of the system model, it is also true that the point at which the frequency response plots of the error weighting (sensitivity weighting) and the control weighting (control sensitivity weighting) cross often determines the bandwidth point for the system. Indeed a starting point in H_∞ design, for choosing the relative gain sizes, is to choose the cross-over point to coincide with the desired bandwidth. In LQG design the crossover frequency between the plots of $W^* Q_c W$ and R_c may give a better indication of the bandwidth which will be achieved.

7. *Angle between the weightings*
In general the angle between the frequency responses of the weighting should be limited at the crossover point. Recall that this point is often close to the unity-gain crossover frequency for the system, and the weightings should not therefore introduce rapid unnecessary phase changes unless this is important for stability.

8. *Lead terms on the error weighting*
A lead term can be introduced on the error weighting function or sensitivity weighting functio[illegible] an attempt to improve transient responses. If integral action is used on the error term and a lead term is used on the control weighting, the crossover of the magnitude diagrams will involve a difference in slope of 40 dB per decade. This can result in the system being particularly sensitivity in the mid frequency range. By adding a lead term on the error weighting function, the change in slope can be made 20 dB's per decade and the resulting more gradual phase shifts often lead to a design with better step response characteristics. Similar remarks apply to sensitivity weighting functions where a lead term on the cost weighting may be necessary to reduce the rate of change of gain and phase in the mid frequency region.

9. *Robustness weighting function*
Instead of penalising each of the cost terms independently, it is sometimes more beneficial to multiply each term by the same weighting function W_{σ}. This is particularly true when trying to reduce the peak level on sensitivity functions which occur in the mid-frequency range. At low frequencies a high penalty on the sensitivity function will cause a high controller gain which results in a small sensitivity function magnitude. In the frequency range where the loop gain has a magnitude of approximately unity, this rule (that heavy penalities will force down the sensitivity function magnitude) no longer holds. A more effective way of reducing the peak on the sensitivity function, in this case, is to reduce the loop gain so that a frequency response peak of greater than unity does not occur.

6. Conclusions

The importance of selecting the dynamic cost-function weighting terms in H_∞ design has been illustrated through examples. The use of constant cost weightings is the worst possible design strategy. However, it is also important that the frequency responses of the weighting are appropriate. Some design guidelines have been presented which provide a starting point for a more formalized design procedure.

7. References

1. Woodgate, K., 1992, *On the generalized H_∞ control problem*, American Control Conference, Chicago, Illinois, pp. 2261-2265.
2. Grimble, M.J., 1987, *H_∞ robust controller for self-tuning applications*, Int. J. Control, Part I, Vol. 46, No. 3, pp. 1429-1444 ; Part II, Vol. 46, No. 5, pp. 1819-1840.
3. Glover, K., and McFarlane, D., 1989, *Robust stabilization of normalized coprime factor plant descriptions with H_∞ bounded uncertainty*, IEEE Trans. on Autom. Control, 34, 8, pp. 821-830.
4. Foley, M.W. and Harris, T.J., 1992, *Structure and performance of discrete H_∞ optimal control*, Optimal Control Application and Methods, Vol. 13, 1-27.
5. Doyle, J.C., 1978, *Guaranteed margins for LQG regulators*, IEEE Trans. on Auto. Control, AC-23, 4, pp. 756-757.
6. Grimble, M.J. and Fragopolous, D., 1991, *Relationship between H_∞ and H_2 optimal control robust design advantages and limitations*, Proc. of Symposium "Robust Control System Design Using H_∞ and Related Methods" Ed. P.H. Hammond, Published Inst. M. and Control, pp. 105-37.
7. Grimble, M.J., 1986, *Optimal H_∞ robustness and the relationship to LQG design problems*, IJC, Vol. 43, No. 2, pp. 351-372.