Modelling and control of a class of systems with hysteresis.
Application to friction compensation.

P.-A. BLIMAN M. SORINE*

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## 1. Motivations

Some high precision controllers for mechanical systems require dry friction model-based compensators in order to achieve precise positionning and suppression of the limit cycles; see [1] and the bibliography therein. The search for accurate models of these nonlinear effects is at the origin of an important experimental dry friction modelling activity. In particular, the models must agree with the observed dependence of friction on the relative speed of the two parts in contact and on the relative displacement when the relative speed is crossing zero [1, 2]. Independently, modelling dry friction or more general hysteresis effects occurring in mechanics, magnetism, chemistry, biology ..., has recently received a great deal of attention in Nonlinear Analysis [3, 4, 5].

We propose a class of hysteresis models, we believe to be useful for control purposes. These models present the following advantages when applied to friction modelling:

- They are in good accordance with recent experiments on dry friction at low speed. For example, Coulomb friction during switching exhibits an elastic behavior (this is the so called Dahl effect) while the usual Yosida regularization for sign of speed has a viscous behavior. Elastic regularizations of the multivalued sign operator are available in the proposed class of models.
- They lead to mathematicaly well posed equations of motion; this property, useful to derive sound numerical schemes, seems to lack to some previously proposed models based on variations of the multivalued sign function, classical model of Coulomb friction.
- They become *linear* differential equations after a change of time variable, the new time being the total distance covered by the relative motion, or, more generally, by the total variation of the input. This makes easier model shaping (in particular ensuring passivity), fast numerical solving (a useful property for real time computation), and the design of compensators (including parameters identification and algorithm discretization).

## 2. A class of systems with hysteresis

We consider a single-degree-of-freedom system, associated to the real variable $u$, and involving hysteresis effects, through the variable $y$. We shall say that the relation $H : u \to y$ is a *hysteresis operator*.

For example when modelling friction, $u$ and $y$ denote respectively the relative position of two pieces in contact and the resulting friction force: the hysteresis operator expresses a behavior law of the contact.

Following [4], hysteresis operators $H$ are defined as *causal rate-independent* operators:

$$\begin{cases} \bullet\ y(t) \text{ is a function of } u(\tau),\ \tau \le t \\ \bullet\ H(u \circ \varphi) = H(u) \circ \varphi \text{ for any non-decreasing diffeomorphism } \varphi \text{ on } [0, T] \end{cases}$$

We shall also use (causal) *geometric operators* $G$, that we define as:

$$\bullet\ G(u \circ \varphi) = G(u) \text{ for any non-decreasing diffeomorphism } \varphi \text{ on } [0, T]$$

*Both authors are with Institut National de Recherche en Informatique et Automatique (I.N.R.I.A.), Domaine de Voluceau, Rocquencourt B.P. 105, 78153 Le Chesnay Cedex, France. phone: 33/1/39 63 55 68, fax: 33/1/39 63 53 30, e-mail: pierre-alexandre.bliman@inria.fr, michel.sorine@inria.fr.

The following two properties are straightforward: let $H_1, H_2$ be hysteresis operators, $G$ a geometric operator; then $u \mapsto G(u) \circ H(u)$ is a hysteresis operator and $u \mapsto H_1(u) \circ H_2^{-1}(u)$ a geometric operator (the inverse and the composition are in the sense of graphs).

For absolutely continuous $u$, two basic examples of hysteresis operators are:

- $S$, where $S(u)(t) \triangleq \int_0^t |\dot{u}(t)|.dt$ is the total variation of $u$ over $[0, t]$.
- $\mathrm{sgn}\frac{d}{dt}$, where sgn is the usual multivalued operator $\mathbb{R} \to \mathbb{R}$, $z \mapsto -1$ if $z < 0$, $+1$ if $z > 0$, $[-1, +1]$ if $z = 0$.

Now, define the following geometric operator:

$$\sigma(u) = \mathrm{sgn}\dot{u} \circ S^{-1}(u)$$

$\sigma(u)$ is defined by its graph, but is in fact singlevalued for a.e. $s$; it can be considered as an element of $L^\infty(0, S(u)(T))$, and fulfills the following properties:

$$|\sigma(u)| = 1 \; s - \text{a.e. and } \dot{u} = (\sigma(u) \circ S(u)).\dot{S}(u) \; t - \text{a.e.}$$

$\sigma(u)$ interprets as the tangent unit vector to the trajectory of $u$, directed towards the increasing $s$'s.

From $\sigma$ can be derived the geometric operator $\Sigma$, given by $\Sigma(u)(s) \triangleq \int_0^s \sigma(u)(s).ds$ (the position on the curve $u$ expressed wrt the curvilinear abscissa). It is such that $\Sigma(u) \circ S(u) = u$.

Very general rate-independent causal operators can then be generated as:

$$H(u) = \tilde{G}(\sigma(u)) \circ S(u) + D\mathrm{sgn}\dot{u} \tag{1}$$

where $D \in \mathbb{R}^+$ and $\tilde{G}$ is an operator such that $\tilde{G}(\sigma(u))$ is continuous.

Finally, the $H$'s we propose are those obtained choosing for $\tilde{G}$ a linear filter: let $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times 1}$, $C \in \mathbb{R}^{1\times p}$ and define:

$$\begin{cases} \dfrac{dx_S}{ds} = Ax_S + B\sigma(u) \;\; s - \text{a.e.}, \; x_S(0) = x_0 \in \mathbb{R}^p \\ H(u) = Cx_S \circ S(u) + D\mathrm{sgn}\dot{u} \end{cases} \tag{2}$$

It can be proved (*linear representation* result) that

$$H(u)(t) = (Cx_S + D\sigma(u)) \circ S(u) \tag{3}$$

$s$-a.e. on any finite time-interval on which $S(u)$ is non constant, and (energy representation) that

$$\int_0^T y(t)\dot{u}(t).dt = \int_0^{S(u)(T)} (Cx_S + D\sigma(u))(s).\sigma(u)(s).ds \tag{4}$$

This structure is central, as it permits the use of linear-systems theory techniques. Due to (3), the models (2) are called *Linear Space Invariant (LSI)*.

When modelling dry friction, we have the following interpretation: $t \mapsto s = S(u)(t)$ is the curvilinear abscissa associated to the trajectory of the relative motion and $\sigma(u)$ is the tangent unit vector, direction of Coulomb's friction forces. $H(u)$ hence appears as the result of a *linear filtering of the Coulomb's friction* $\mathrm{sgn}\dot{u}$ by finite-dimensional LSI filter.

## 3. Contents

In the applications we have in mind, the LSI model $u \mapsto H(u)$ is coupled to a *Linear Time Invariant (LTI)* system. For example when modelling friction, the LTI system is the set of Euler equations.

We give an appropriate functional framework to study the coupled LSI/LTI problem. Some qualitative properties are given (including a simple *dissipativeness* [6] criterion deduced from (4) and Kalman-Yakubovich-Popov lemma). With the help of asymptotic and singular perturbation results, Coulomb model for example is limit case of fast first-order LSI models.

The LSI models can be shaped easily, by use of the well-known linear control techniques of *transfer function shaping*. We show e.g. how to reproduce various observed behaviors in friction modelling, namely those described by Coulomb model, Dahl model (showing spring-like behavior around positions where the speed crosses zero), the sticktion effect (leading to stick-slip), Stribeck effect (characterized by decrease of the friction with respect to velocity at low velocity), the appearance of limit cycles in speed-control loop, the existence of a lag between the beginning of the sliding and the response of the friction. Numerical simulations are provided, trying to reproduce those phenomena.

Finally, we propose a simple friction compensation algorithm using the special structure of LSI model.

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