On implicit linear n-D systems

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The I/O relation of a linear n-D discrete system can be given as follows

$$\sum_{\beta\in\mathcal{B}} a_\beta(\alpha)y(\alpha+\beta)=x(\alpha), \quad \alpha\in\mathcal{D}\subset\mathbf{Z}^n, \quad \mathrm{a}_\beta\neq\mathbf{0}, \tag{1}$$

$$y(\alpha)=c(\alpha),\ \alpha\in G\subset\mathcal{D}+\mathcal{B}.$$

Here, $1<|\mathcal{B}|<+\infty$, the sequence $x:\mathcal{D}\to\mathbf{C}$ is supposed to be given; any sequence $y:\mathcal{D}+\mathcal{B}\to\mathbf{C}$ satisfying the equation (1) is called its solution. The set G, called the initial set, is supposed to guarantee the existence and uniqueness of the solution of equation (1), with a given mapping $c:G\to\mathbf{C}$ which defines the initial values.

It has been shown earlier [1], that to any finite set $\mathcal{B}$ there always exists an initial set G. These conclusions hold true also in the more general case when the coefficients a_β in equation (1) are square matrices and the input x and output y are vectors of corresponding dimensions. The proof of the corresponding statement constructs the solution recursively from a 'leading' term, provided its coefficient is invertible, i.e. either nonzero in the case of a single equation, or regular in the case of a matrix equation. It has to be noted that this 'leading' term may, but need not be fixed for all values $\alpha\in\mathcal{D}$ depending on the mask $\mathcal{B}$.

The single equation or the system of equations (1) is commonly called singular, if the conditions of existence and/or uniqueness of the solution are not satisfied. Since in this setting for $n>1$ the common way of establishing EU conditions is the recursive construction of the solution, although in special cases indirect methods are also possible, we are concerned with this leading coefficient in equation (1), i.e. with the coefficient of the term recursively computed from the values of the solution which are already known. A feasible method to obtain EU conditions for singular equations is to find an equivalent recursively computable equation.

A single equation (1) would be called singular if its leading coefficient, say $a_{\beta*}(\alpha_0)=0$ for some $\alpha_0\in\mathcal{D}$. Since recursion implies a certain ordering $\prec$ of the set $\mathcal{D}+\mathcal{B}$, the solution is unique for all $\alpha\prec\alpha_0$. As for the remaining values of α a 'new' equation is created and its EU conditions must be separately investigated.

For systems of equations a similar situation arises if $\det a_{\beta*}(\alpha)=0$ for some or all $\alpha\in\mathcal{D}$, or, in a simpler case, if equation (1) has constant coefficients and $\det \alpha_{\beta*}=0$.

The latter case seems to be important. Equation (1) is, in fact, an equation containing partial backward or forward differences of arbitrary finite order.

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Equations containing first order differences only, can be considered canonical and, similarly as in the 1-D case, an algorithm can be given such that the equation (1) with $|\mathcal{B}| > 3$ can be reduced to such first order partial difference equation by introducing additional unknown sequences. The corresponding algorithm is the n-D generalization of the well-known Horner's scheme. To describe this procedure shortly, let $n = 2$ be considered. In the rigth-hand side if the equation (1) introduce shift operators p, q denoting forward shifts in the directions of both axes. Then (1) can be rewritten as a two-variable polynomial P. Now, in

$$P(p,q) = p\,P_1(p,q) + qP_2(p,q) + a_{00}$$

we may continue this process with P_1, P_2 obtaining P_{11}, P_{12}, P_{21}, P_{22} until all the polynomials so obtained will be constants. Put now $y = z_0$ and replace py by z_1 and qy by z_m. Similarly, $p\,z_i = z_{i+1}, q\,z_k = z_{k-1}$. If m is the least number of variables then the obtained system will be of the form

$$A\,p\,Z + B\,q\,Z + C = X,$$

where A, B, C will be matrices of order $m+1$, Z will be a column vector of order m+1, $Z = (z_0, z_1, z_2, \ldots, z_m)^T$ and the vector X has only one nonzero element. Moreover, it can easily be seen that, except for trivial cases, both matrices A, B have at least one row consisting of zeros only, hence, they are singular.

It can be shown, that the number m of unknown sequences must satisfy the inequality $3 + 2(m-1) \geq |\mathcal{B}|$ for $n = 2$, where m is closely connected to the minimal number of multiplications needed to evaluate a two-variable polynomial. The correspondence between initial conditions for a single equation and those for the first order system becomes nontrivial. Since details must be omitted here, a simple example could show the way of reasoning.

Let the mask $\mathcal{B}$ consist of 8 points as follows:

$$\mathcal{B} = (0,0),(0,1),(1,0),(1,1),(0,2),(1,2),(2,0),(2,1)$$

with constant coeficients a_{ik}, and $\mathcal{D} = \{(i,k) : i \geq 0, k \geq 0\}$. Introducing new unknown sequences

$Z = [z_{00}, z_{01}, z_{10}, z_{11}]^T$ by $z_{00} = y$ and

$$z_{10}(i,k) = z_{00}(i+1,k),\ z_{01}(i,k) = z_{00}(i,k+1),\ z_{11}(i,k) = z_{10}(i,k+1)$$

we obtain for the homogeneous case the following matrix equation

$$A\,Z(i+1,k) + B\,Z(i,k+1) + C\,Z(i,k) = 0, \tag{2}$$

where the square matrices A, B, C of order 4 are successively,

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_{20} & 0 & a_{21} \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & a_{02} & a_{12} \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_{00} & a_{10} & a_{01} & a_{11} \end{bmatrix}.$$

The single equation has a unique solution if the values $y(0,k)$, $y(1,k)$, $y(i,0)$ are given. For the system of equations, which has to be equivalent to the single equation, the 'leading' matrix A is singular. Therefore the 'selfevident' recursion with the initial values $Z(0,k)$ given cannot be applied.

Existence and uniqueness conditions for some of these and similar systems have recently been published [2]. According to these results EU conditions depend, in case of equation (2), on the behaviour of the matrix pair (A, B).

Taking two matrix pairs equivalent, $(G, H) \sim (K, L)$ iff there exists two regular matrices P, Q such that $K = PGQ$, $L = PHQ$, the following Theorem is known:

Theorem 1. *Any ordered pair (G, H) of nonzero $n \times n$ matrices uniquely determines three nonnegative integers r, p, q; $r + p + q = n$ such that there exists a pair $(G_1, H_1) \sim (G, H)$ of the following form*

$$(G_1 H_1) = \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & N_p & 0 \\ 0 & G_{32} & 0 \end{bmatrix}, \begin{bmatrix} H_{11} & 0 & H_{13} \\ 0 & I_p & 0 \\ 0 & 0 & O_q \end{bmatrix} \right). \tag{3}$$

Here, if any of the integers r, p, $q = 0$, the corresponding blocks are considered to be empty.

In this theorem we denoted by I_k, N_k, O_k successively the unit, nilpotent and zero matrix of order k and therefore the dimensions of the blocks are defined. The symbol 0 stands for zero blocks. We shall also use the notation $ch(G, H) = (r, p, q)$.

Our first result considers the case $\mathcal{D} = \{(i,k) : i \geq 0, k \geq 0\}$. For equation (2) the following Theorem holds [2]:

Theorem 2. *Suppose that in the equation (2) A, B are square matrices of order m such that $ch(A, B) = (m - p, p, 0)$. Then there exist two linear subspaces $L_p, L_{m-p} \subset L_m$ of dimension p and $m - p$, respectively, such that for any vectors $v, w : Z_+ \to R^m$ with $v(0) = w(0)$ such that $v \in K_{m-p}$, $w \in K_p$ there exists a unique solution $y : \{(i,k) : i \geq 0, k \geq 0\} \to R^2$ of the equation (2) satisfying the initial conditions $y(i,0) = v(i)$, $y(0,k) = w(k)$. Here K_s denotes a coset of the linear subspace L_s.*

Consider again the equation (2), but now with $\mathcal{D} = \{(i,k) : i \geq 0, k \in \mathbf{Z}\}$. The previous result cannot be directly applied, but its method remains effective. Assuming that the two pairs if matrices (A, B) and (A, C) have canonical forms of equal type then the following result can be proved.

Theorem 3. *Suppose that in the equation (2) A, B, C are square matrices of order m such that $ch(A, B) = ch(A, C) = (m - p, p, 0)$ Then there exist two linear subspaces $L_p, L_{m-p} \subset L_m$ of dimension p and $m-p$, respectively, such that for any vectors $v : Z \to R^m$, $w : Z_+ \to R^m$ with $v(0) = w(0)$ such that $v \in K_{m-p}$, $w \in K_p$ there exists a unique solution $y : \{(i,k) : i \geq 0, k \in Z\} \to R^2$ of the equation (2) satisfying for an arbitrarily fixed integer k_0 the initial conditions*

$y(i, k_0) = w(i)$, $y(0, k) = v(k)$. *Here K_s denotes a coset of the linear subspace L_s.*

These results illustrate how the canonical forms of matrix pairs can be used to establish recursive forms of singular 2-D systems. It has been shown [2], that in this way also some stability results can be obtained. On the other hand, their use heavily depends on an algorithm of finding the transforming matrices denoted P, Q above. A closely related algorithm has been published [3], where its computational complexity is reduced to $O(m^3)$. Here also extensive references to this problem are given.

To make the above results more transparent the part of their proof could be given here, in which the introduction of linear subspaces and their cosets becomes necessary. This should explain, why and how the 'usual' initial conditions must be restricted so as a unique solution exists.

The transformation of the matrix pair (A, B) by matrices P, Q in Theorem 1. means a multiplication of the equation by the matrix P and an introduction of a new sequence, say $x = Q\,y$. It turns out that $y(0, k)$, $y(i, 0)$ cannot be arbitrarily given. The conditions, which they have to satisfy, can best be formulated in terms of the matrix Q^{-1}. Let this matrix be partitioned in blocs $R_{ik}, i, k = 1, 2$ so that the diagonal blocs are of dimension $m - p$ and p, respectively. Then it must be

$$x_1(0, k) = [R_{11}|R_{12}]\,y(0, k)$$

$$x_2(i, 0) = [R_{21}|R_{22}]\,y(i, 0),$$

where x_1, x_2 denotes the $m - p$, p components of the vector sequence x. The first of these equations imposes $m - p$ conditions to be satisfied by the vector $y(0, k)$ and similarly the second one. These conditions are expressed in Theorems 1. and 2. using the concept of cosets, i.e. by the use of sets of 'shifted' elements of a linear subspace.

To use the results as they are formulated here some additional calculations would be necessary, but an effective method of solution of singular systems becomes possible.

References

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