Noninvertibility and its Role in the Dynamics of Adaptively-Controlled Systems

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Abstract

When we try to adaptively control a system, an interesting interplay develops between the dynamics of the system itself, and the dynamics "imposed" by the adaptation process. The overall process is inherently nonlinear, and, for the case of discrete-time control, we find that it is also *noninvertible*. This property, which arises naturally in adaptive systems, makes them very different from "typical" systems (e.g., those described solely by ODEs in time): given the present state, the state of the system at the previous time step may not be uniquely determined.

The nonlinear nature of the process gives rise to phenomena like multistability: the locally-stable set point can coexist with other, undesirable attractors. Quantifying the stability of the set point therefore involves constructing its basin of attraction. Because basin boundaries often consist of stable manifolds of saddle-type solutions (computed by following states backward in time), noninvertibility can have a profound effect on the shape and interactions of basins of attraction. They may now consist of disconnected pieces (possibly infinitely many), distorted by being repeatedly stretched out to infinity and back, etc. In this paper we concentrate on the complicated dynamics exhibited by noninvertible systems, using various adaptive control configurations as illustrative examples.

The analysis of global stability characteristics begins with local stability and bifurcation analysis of the system attractors, starting with the set point and its instabilities. The bifurcation parameters represent the mismatch between the system and its model (following Golden and Ydstie, 1988), as well as design parameters of the controller. Basin boundaries are computed using local approximations of stable manifolds of saddle points and their continuation in phase space (iteration backward in time). The *preimages* of a current state are the (possibly several or none) initial conditions this state can be coming from. Since the preimage behavior will vary depending on where we are located in phase space, our main tool is understanding the inverse (backward in time) map, and its "bifurcations", delineating regions of phase space with qualitatively different preimage behavior. As saddle invariant manifolds cross the boundaries between these regions, new geometric features emerge on the manifolds (Gumowski and Mira, 1980).

The other model reduction approach we follow here is the method of empirical eigenfunctions (Karhunen-Loeve expansion or proper orthogonal decomposition). Consider the instantaneous spatial solution profiles of a PDE ("snapshots" as they are referred to by Sirovich). The basic premise of the POD (the orthogonal decomposition of the spatial covariance of these snapshots) is part of classical pattern recognition methodologies. It consists of finding a linear, low-dimensional subspace of the phase space in which the data approximately lie, and a set of coordinates for this susbpace (a hierarchy of modes, resulting from the diagonalization of the two-point correlation matrix of the ensemble of snapshots). A combination of this hierarchy of structures with a Galerkin weighted residual discretization of the fundamental model equations (e.g. the Navier- Stokes) will then provide a spatially and temporally accurate model of the PDE dynamics, provided that a sufficient number of modes has been retained.

Almost by definition of the empirical eigenfunctions, it is indeed difficult to rigorously associate the behavior of the reduced models with that of the original PDEs. There is increasing evidence, however, through the study of several model problems, that this methodology could be a crucial engineering algorithmic tool for the reduction, analysis, design and control of distributed systems, beyond the context of fluid flow.

The POD itself is an *a posteriori* technique for extracting important structures from a known (computational or experimental) data set. The theory of approximate inertial manifolds, on the other hand, gives an *a priori* estimate of where the dynamics are occurring. This estimate is generally not sharp. However, POD analysis of the trajectories of points on this manifold can be used to sharpen the estimate of the important structures for the dynamics. This procedure leads to a reduced model for all of the dynamics of a given system, not just for the dynamics from the evolution of one initial condition, as described above.

We discuss the application of these techniques to a number of examples, including:

a. The Kuramoto-Sivashinsky equation, a model for interfacial instabilities of thin films and of flame front propagation;

b. Experimental data showing patterns formed during the oxidation of carbon monoxide on a single crystal of platinum (these data come from a collaboration with the group of Prof. G. Ertl in Berlin).

c. A reaction-diffusion model of a chemical system containing an autocatalytic species and an inhibitor of the autocatalytic reaction.

Stable manifolds may break off to several segments, giving rise to "polka-dots" in the basin of attraction. This results from a stable manifold crossing a "turning-point bifurcation" of the inverse map. They also can stretch off to infinity and back again; this results from a stable manifold crossing a "bifurcation from infinity" of the inverse map. Unstable manifolds may start crossing *themselves*, or also break off to several disconnected or multiply connected pieces; this again involves their interaction with a "turning-point bifurcation" of the inverse map. A wealth of complicated scenaria can be built from the interaction of a few such elementary transitions, some of which can lead to chaotic dynamics.

The dynamics of noninvertible systems (except for the 1-dimensional maps like the logistic map) have not been extensively studied, and their generic behavior is therefore largely unknown. The picture we describe is incomplete, and mainly guided by the dynamics of the particular examples we study. However, even the simplest adaptive control schemes (low-order, or even scalar linear plant) will exhibit these phenomena, and we have obtained partial experimental verification of theoretical predictions in a stirred mixing tank experiment.

The phenomenon of noninvertibility also appears as an artifact in discrete-time approximations of continuous-time systems. Typical examples include explicit numerical integrators as well as time-T maps identified from experimental time series using artificial neural netwoks. We discuss the effects of noninvertibility on the validity of the attractors (long-term dynamics) predicted by such discrete-time models.

References:

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