PERIODIC SOLUTIONS TO 2-D CONSTANT LINEAR SYSTEMS

Tadeusz Kaczorek
Warsaw University of Technology
Faculty of Electrical Engineering
Institute of Control and Industrial Electronics
00-662 Warszawa, Koszykowa 75, Poland

The 2-D linear systems described by the equation

$$x_{i+1,j+1} = A_o x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + Bu_{ij} \qquad i,j \in N \tag{1}$$

will be considered, where $x_{ij} \in R^n$ is the local state vector at the point $(i,j) \in N^2$, N is the set of natural numbers including zero, $u_{ij} \in R^m$ is the input vector and A_o, A_1, A_2, B are real matrices of appropriate dimensions.
Boundary conditions for (1) are given by

$$x_{io} \text{ for } i \in N \quad \text{and} \quad x_{oj} \text{ for } j \in N \tag{2}$$

A solution x_{ij} of (1) is called periodic if

$$x_{i+kp_1, j+lp_2} = x_{ij} \quad \text{for all} \quad i,j,k,l \in N \tag{3}$$

A smallest pair of nonnegative integers (p_1, p_2) satisfying (3) such that $p_1 + p_2$ is minimal will be called the period of x_{ij}.

It is assuming that:
1) the input u_{ij} is periodic with the period (p_1, p_2), i.e.

$$u_{i+kp_1, j+lp_2} = u_{ij} \text{ for } i,j,k,l \in N \tag{4}$$

2) the boundary conditions (2) are periodic with the period (p_1, p_2), i.e.

$$x_{i+kp_1, o} = x_{io} \text{ and } x_{o, j+lp_2} = x_{oj} \quad \text{for all} \quad i,j,k,l \in N \tag{5}$$

Necessary and sufficient conditions under which there exist a periodic solution x_{ij} to (1) for periodic input u_{ij} and periodic boundary conditions (2) are given by the following.

Theorem

The equation (1) witch periodic input (4) and periodic boundary conditions (5) has a periodic solution x_{ij} with the same period (p_1, p_2) if and only if

$$\operatorname{rank} M < (p_1+p_2-1)n$$

where

$$M:=\begin{bmatrix}
M^0_{p_1 0}+M^1_{p_1 0}-I, & M^1_{10} & M^1_{20} & \dots M^1_{p_1-1,0} & 0 & 0 & \dots & 0 \\
M^0_{p_1 1}+M^1_{p_1 1}, & M^1_{11} & M^1_{21} & \dots M^1_{p_1-1,1} & M^2_{p_1 1}-I & 0 & \dots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
M^0_{p_1,p_2-1}+M^1_{p_1,p_2}, & M^1_{1,p_2-1} & & \dots M^1_{p_1-1,p_2-1} & M^2_{p_1 1} & M^2_{p_1 2} & \dots & M^2_{p_1,p_2-1}-I \\
M^0_{0,p_2}+M^2_{0 p_2}-I, & 0, & 0 & \dots 0 & M^2_{01} & M^2_{02} & \dots & M^2_{0,p_2-1} \\
M^0_{1p_2}+M^2_{1p_2}, & M^1_{1p_2}-I, & 0 & \dots 0 & M^2_{11} & M^2_{12} & \dots & M^2_{1,p_2-1} \\
M^0_{p_1-1,p_2}+M^2_{p_1-1,p_2}, & M^1_{1p_2} & M^1_{2p_2} & \dots M^1_{p_1-1,p_2}-I, & M^2_{p_1-1,1} & & \dots & M^2_{p_1-1,p_2-1}
\end{bmatrix}$$

$$M^1_{kj}:=T_{i-k-1,j-1}A_0+T_{i-k,j-1}A_1$$

$$M^2_{il}:=T_{i-1,j-l-1}A_0+T_{i-1,j-1}A_2$$

$$M^0_{ij}:=T_{i-1,j-1}A_0$$

and

$$T_{00} = I \text{ (the indentity matrix)}$$

$$T_{ij} = A_0 T_{i-1,j-1}+A_1 T_{i,j-1}+A_2 T_{i-1,j} \qquad \text{for } i,j \in N \tag{7}$$

$$T_{ij} = 0 \quad \text{(the zero matrix)} \quad \text{for} \quad i<0 \quad \text{or/and} \quad j<0$$