LOW *A-PRIORI* INFORMATION SIGNAL MODELLING FOR OPTIMAL SMOOTHING AND DIFFERENTIATION

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Abstract

Every appropriate filtering procedure of a noisy signal should reduce the noise present in the original recording while affecting the actual shape of the signal as little as possible. The main restrictions of classical methods (e.g. frequential filters), is the impossibility of reconciling the foregoing filtering requirements that are contradictory owing to noise superimposed on the signal band. Therefore a criterion for optimal filtering is needed.

Methods based on Kalman filtering theory have been widely investigated in the literature (see e.g. the special issue [1]) because of their appealing characteristics like the possibility of recursive implementation.

The main drawback related to Kalman filtering methods is that they require a lot of *a-priori* information. The signal model must be expressed in a suitable state-space form where all parameters as well as the statistics of both state and measurement noises have to be known. In many situations of practical interest it is not realistic to assume that such *a-priori* information is available or that it can be reliably obtained from noisy data.

For this reason we propose a signal model based on a very low *a-priori* statistical information about the signal generating process. The basic hypotheses are:

i) **Smoothness assumption**: *the signal and its first derivatives up to a suitably chosen order m are nontinuously differentiable;*

ii) **Stochastic assumption:** *the derivative of order m+1 is modelled by means of a zero-mean white gaussian process.*

Under the above hypotheses we are able to obtain a suitable state-space form of the signal and to define an adaptive procedure for a reliable identification of state and measurement noise covariances. This can be done as follows:

By the smoothness assumption it is possible to define a state vector X(t) composed of the signal x(t) and its first derivatives $x^{(i)}(t)$ up to m-th order:

$$X(t):=[d^i x(t)/dt^i,\ i=0,...,m]^T$$

By the stochastic assumption, the derivative $x^{(m+1)}(t)$ can be modelled by a white gaussian noise $\sim N(0, \sigma^2_w)$ and according to [2], the following state-space form of the sampled signal is obtained:

$$X(t_{i+1}), = A\, X(t_i) + \sigma_w w(t_{i+1}) \qquad (1)$$

$$y(t_i) = C\, X(t_i) + v(t_i) \qquad i = 1,2,...,n \qquad (2)$$

where:

- $A = \exp(G\Delta)$, $G = \begin{pmatrix} 0_{m-1} & I_{m-1} \\ a_0 \cdots & a_{m-1} \end{pmatrix}$, $\Delta = t_{i+1} - t_i$, $C = [1, 0^T_{m-1}]$,

being 0_{m-1} and I_{m-1} the m-1 column vector of zeros and the (m-1)x(m-1) identity matrix respectively

- $w(t_i)$ is an m-vector, white noise sequence $\sim \mathcal{N}(0,Q)$, with

$$Q = \int_0^{\Delta} \underline{a}_m(s)\underline{a}_m(s)^T ds,$$

being $\underline{a}_m(s)$ the m-th column of exp(Gs);

- $v(t_i)$ is a scalar, white noise sequence $\sim \mathcal{N}(0, \sigma^2_v)$.

State-space model (1), (2) is amenable to Kalman smoothing or filtering implementation once estimates of σ^2_w and σ^2_v have been obtained.

The proposed optimal estimation method of these variances is based on the matching between the theoretical and the observed autocorrelation function of Kalman filter innovation process.

The algorithm can be briefly summarized by the three following points:

a) It is shown that, for system (1),(2), the optimal Kalman filter gain K_i (i=1,2,...), is a function of the ratio σ_v/σ_w, and not of σ_w and σ_v separately.This allows us to save much of the computation time involved in the numerical minimization of the functional giving the $\mathcal{L}^2$ norm distance between the theoretical and observed autocorrelation function of the innovation process;

b) the steady-state filter gain K_∞ is computed for different values of the parameter σ_v/σ_w belonging to a predetermined range. In the light of point **a)**, the value of the filter gain is obtained by implementing the Kalman filter equations for a system *equivalent* to the original system (1),(2). Such *equivalent* system is obtained dividing equations (1) and (2) by σ_w ;

c) the value of K_∞ is used to compute the innovation process. Then, the pair producing the minimum $\mathcal{L}^2$ norm distance between the theoretical and the experimentally observed autocorrelation function of the innovation process is chosen as optimal estimate of σ_w and σ_v.

Remark: The necessity of using K_∞ (and not K_i) is justified by the fact that the *equivalent* system produces a dynamical sequence of filter gains K'_i (i=1,2,...) exactly equal to the sequence K_i produced by the actual system (1),(2), only if such *equivalent* system is properly initialized. On the other hand, proper initialization requires the knowledge of σ_w (and σ_v). Thus the steady state value K_∞ is used because it is independent of initial conditions.

The proposed signal model has been applied in a simulation context to the optimal smoothing and differentiation of the following band-limited test function:

$$g(t) = \sum_{i=1}^{5} A_i \sin(i\omega_0 t + \phi_i)$$

where $\omega_0=2\pi/T$, T is the signal period, and A_i and ϕ_i are reported in [2]. A sequence of 91 data points was generated with a sampling frequency $f_c = 91$ Hz, and a signal period T = 1s was assumed. Two noisy versions of the signal were obtained superimposing white gaussian noise sequences with standard deviation $\sigma_v=0.5$ and 1.0 respectively.

The following performance index was defined

$$\sigma(k) = [(1/n) \sum_{i=1}^{n} r^2_{ik}]^{1/2} \quad k=0,1,2$$

where n is the number of samples, and where r_{ik} are the residuals computed as the difference between the true signal and the filtered one. Index k refers to the derivative order, i.e. k=0 relates to the original data, k=1 to its first derivative, and k=2 to its second derivative.

The present approach has been compared to results abtained with cubic spline smoothing [3] using the generalized cross validation method [4]. A fixed lag Kalman smoother [5] was adopted chatacterized by a value of the signal model order m=3 and of the lag equal to 15.

The following table shows the comparison of the results

	KALMAN	SMOOTHER	CUBIC	SPLINES
	(σ_v=0.5)	(σ_v=1.0)	(σ_v=0.5)	(σ_v=1.0)
$\sigma_{(0)}$	0.27	0.51	0.28	0.54
$\sigma_{(1)}$	9.34	14.99	9.62	15.46
$\sigma_{(2)}$	491.3	643.2	502.4	620.1

As a final comment we remark that the proposed signal model cannot be implemented on-line because it requires off-line identification of noise covariances. Nevertheless, in the light of point a) the method results to be very fast and the signal can be processed shortly after it is acquired.

References

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