REACHABILITY OF POLYNOMIAL MATRIX DESCRIPTIONS (PMD's)

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Abstract

We consider the concept of Reachability for Polynomial Matrix Descriptions (PMD's) i.e. systems of the form $\Sigma : A(\rho)\beta(t) = B(\rho)u(t)$, $y(t)=C(\rho)\beta(t)$ where $\rho := d/dt$ the differential operator , $A(\rho) = \sum_{i=0}^{q_1} A_i\rho^i \in \mathbb{R}^{r\times r}[\rho]$, $A_i \in \mathbb{R}^{r\times r}$, $i = 0,1,...q_1 \geq 1$ with $\mathrm{rank}_{\mathbb{R}} A_{q_1} \leq r$, $B(\rho) = \sum_{i=0}^{\sigma} B_i\rho^i \in \mathbb{R}^{r\times m}[\rho]$, $B_i \in \mathbb{R}^{r\times m}$, $i = 0,1,...\sigma \geq 0$,$C(\rho) = \sum_{i=0}^{\sigma_1} C_i\rho^i \in \mathbb{R}^{r\times m_1}[\rho]$, $C_i \in \mathbb{R}^{r\times m_1}$, $i = 0,1,...\sigma_1 \geq 0$, $\beta(t) : (0^-, \infty) \to \mathbb{R}^r$, is the pseudostate of (Σ) , $u(t) : [0 , \infty) \to \mathbb{R}^m$ the control input to (Σ) and $y(t)$ the output of the system (Σ) . Polynomial Matrix Descriptions are governed by singular differential equations which endow the systems with many special features that are not found in regular state space systems . Among these are impulse terms and input derivatives in the free and forced pseudo–state responce , nonproperness of the transfer function matrix, noncausality between input and pseudo–state (or input and output), inconsistent and admissible initial conditions and many others which make the study of PMD's more complicated than the study of the classical regular systems . Starting from the fact that generalized state space systems i.e. systems of the form $\Sigma_1 : E\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$, where $E \in \mathbb{R}^{r\times r}$, $\mathrm{rank}_{\mathbb{R}} E < r$, $A \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{r\times m}$,$C \in \mathbb{R}^{m_1\times r}$ represent a particular case of PMD's. , we generalize various known results regarding the smooth and impulsive solutions of the homogeneous and the non–homogeneous system (Σ_1) to the more general case of PMD's (Σ) .In recent papers (see [10],[9],[6]) various known results regarding the smooth and impulsive solutions of homogeneous generalized state space systems have been translated to the more general case of PMD's . Also relying heavily on the theory regarding the Smith–McMillan form of a rational matrix at infinity and applying it to the polynomial matrix $A(s) = L_-[A(\rho)]$ the theory of Weierstrass canonical form of a regular matrix pencil $Es-A$ under strict equivalence to the more general case of polynomial matrix $A(s)$ was generalized [9] .

Theorem 1 [9] Let $A(s) \in \mathbb{R}^{r\times r}[s]$ with Smith–McMillan form at $s = \infty$, given by $S^{\infty}_{A(s)}(s) = \mathrm{diag}[s^{q_1}, s^{q_2}, ..., s^{q_k}, \frac{1}{s^{\hat{q}_{k+1}}}, ..., \frac{1}{s^{\hat{q}_r}}]$. Write : $A(s)^{-1} = H_{pol}(s) + H_{spr}(s)$ where $H_{pol}(s)\ \mathbb{R}[s]^{r\times r}$ and $H_{spr}(s)\ \mathbb{R}^{r\times r}_{pr}(s)$ is strictly proper . Let $n := \deg|A(s)|$. Then $n = \delta_M(H_{spr}(s))$. Let $\mu = \sum_{i=k+1}^{r} (\hat{q}_i + 1)$; Then $\delta_M(H_{pol}(s)) = \mu$. Let $C \in \mathbb{R}^{r\times n}$, $J \in \mathbb{R}^{n\times n}$ $B \in \mathbb{R}^{n\times r}$ be a minimal realization of $C_\infty \in \mathbb{R}^{r\times\mu}$, $J_\infty \in \mathbb{R}^{\mu\times\mu}$, $B_\infty \in \mathbb{R}^{\mu\times r}$ be a minimal realization of $H_{pol}(s)$. Then C,J is <u>a finite Jordan pair of $A(s)$</u> and C_∞, J_∞ is <u>an infinite Jordan pair of $A(s)$</u> . Furthermore $A(s)^{-1}$ can be written : $A(s)^{-1} = C[sI_n - J]^{-1} + C_\infty[I_\mu - sJ_\infty]^{-1}$. The solution of the homogeneous matrix differential equation $A(\rho)\beta(t) = 0$ is :

$$\beta^h(t) = L_-^{-1}[\hat{\beta}(s)] = [C\ C_\infty]\begin{bmatrix} e^{Jt}x_s(0^-) \\ -\sum_{i=1}^{\hat{q}_r} \delta^{(i-1)} J_\infty^i x_f(0^-) \end{bmatrix} \quad (1)$$

where $x_s(0^-)$ is the "slow state at $t = 0^-$" and $x_f(0^-)$ is the "fast state at $t = 0^-$") . The solution of a non—homogeneous matrix differential equation $A(\rho)\beta(t) = B(\rho)u(t)$ is :

$$(2) \qquad \beta^n(t) = [C \ \ C_\infty] \begin{bmatrix} \int_0^t e^{Jt}\Omega\, u(\tau)d\tau \ + \ \sum_{i=0}^{\sigma-1} \Phi_{i+1}u^{(i)}(t) \\ \sum_{i=0}^{\hat{q}_r} J_\infty^i \bar{\Omega}\, u^{(\sigma+i)}(t) + \sum_{i=0}^{\sigma-1} Z_i u^{(i)}(t) \end{bmatrix}$$

where the superscript (i) means distributional derivative and Ω , $\bar{\Omega}$, Φ_i and Z_i are constant matrices with appropriate dimensions (see [7]) . Cobb in his research papers ([2]—[5]) using time—domain analysis considers the distributional solution of a singular system of the form (Σ) . We extend his theory and show how to cover the more general case evaluating the complete solution of a PMD using distributional derivatives .The complete solution of (Σ) is given by :

$$(3)\beta^c(t) = \beta^h(t) + \beta^n(t) = [C \ C_\infty] \begin{bmatrix} e^{Jt}x_s(0^-) \ + \int_0^t e^{J(t-\tau)}\Omega\, u(\tau)d\tau + \sum_{i=0}^{\sigma-1} \Phi_{i+1}u^{(i)}(t) \\ -\sum_{i=1}^{\hat{q}_r} \delta^{(i-1)}J_\infty^i x_f(0^-) + \sum_{i=0}^{\hat{q}_r} J_\infty^i \bar{\Omega}\, u^{(\sigma+i)}(t) \ + \sum_{i=0}^{\sigma-1} Z_i u^{(i)}(t) \end{bmatrix}$$

We extend the notions of Admissible Initial Conditions (A.I.C.) proposed by [11],[8] in a way to cover the more general case of PMD's .The set of A.I.C. is :

$$(4) \ H_{Iu} = \{\beta^c(0^-) = [C \ C_\infty] \begin{bmatrix} x_s^c(0^-) \\ x_f^c(0^-) \end{bmatrix} / \ x_s^c(0^-) \epsilon \mathbb{R}^n, \text{and } x_f^c(0^-) \epsilon \sum_{i=0}^{\hat{q}_r} J_\infty^i \operatorname{Im} \bar{\Omega} + \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i + \operatorname{Ker} J_\infty\}$$

We treat the differential equations which give rise to a PMD (Σ) using ordinary(regular) derivarives and we generalize the results of [1],[11] , evaluating the complete solutions of PMD's :

$$(5) \qquad \beta^c(t) = [C \ C_\infty] \begin{bmatrix} e^{Jt}x_s(0^-) \ + \int_0^t e^{J(t-\tau)}\Omega\, u(\tau)d\tau + \sum_{i=0}^{\sigma-1} \Phi_{i+1}u^{[i]}(t) \\ \sum_{i=0}^{\hat{q}_r} J_\infty^i \bar{\Omega}\, u^{[\sigma+i]}(t) + \sum_{i=1}^{\sigma-1} Z_i u^{[i]}(t) \end{bmatrix}$$

where [i] means ordinary(regular) derivative . We remark here that the complete solution (5) of a PMD lacks of impulsive components since we assume that the set of the initial conditions of our system belong to the set of Admissible Initial Conditions (4) . We generalize the notions of Reachability given in [11] , [8] in such a way to cover the case of PMDs .

<u>Definition 1</u> Given a point $\beta_0^c = \beta^c(0^-) \epsilon H_{Iu}$, we say that another point $\beta_T \epsilon \mathbb{R}^r$ is <u>Reachable</u> from β_0^c if there exists an input u(t) and T > 0 such that $\beta^c(t) = \beta^c(t\,;\,0^-,\,\beta_0^c\,,\,u(t))$ is impulse—free on [0,T] and holds : $\beta^c(T) = \beta_T$. □

We introduce also the notion of Reachable subspace for PMD's :

$$(6) \qquad R := [C \ C_\infty] \begin{bmatrix} \langle J \,/\, \operatorname{Im} \ \Omega \rangle \\ \langle J_\infty / \operatorname{Im} \bar{\Omega} \rangle \ + \ \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i \end{bmatrix}$$

and we provide a precise form for all the future(reachable) states of our system for $t \geq 0^-$. Furthermore we show how the future states of our system may be reached in any short period with a suitably chosen input u(t) . In [11] the author has provided an open loop control which eliminate the impulsive components of the solution of generalized state–space systems (Σ_1) . We extend this latter method in a way to obtain an open–loop control u(t) such that the complete solutions of a PMD have no impulsive terms without using linear feedback . We also give some useful Reachability tests for PMDs which are natural extensions of the corresponding tests for generalized state space systems .

Definition 2 The subspace : $R_s := \langle J/\mathrm{Im}\ \Omega\rangle \subset \mathbb{R}^n$ is called the slow–state reachable subspace of (Σ) . R_s is spanned by the linearly independent columns of the matrix : $Q_s = [\Omega, J\Omega, \dots, J^{n-1}\Omega]$ which is called slow–state reachability matrix . The subspace : $R_f := \langle J_\infty/\mathrm{Im}\ \bar{\Omega}\rangle + \sum_{i=0}^{\sigma-1} \mathrm{Im}\ Z_i \subset \mathbb{R}^{\mu}$ is called the fast–state reachable subspace of (Σ) . R_f is spanned by the linearly independent columns of the matrix : $Q_f = [\bar{\Omega}, J_\infty\bar{\Omega}, \dots, J_\infty^{\hat{q}} r\bar{\Omega}, Z_0, Z_1, \dots, Z_{\sigma-1}]$ which is called fast–state reachability matrix . From the form of R in (6) it follows that Reachable subspace R is spanned by the linearly independent columns of the matrix : $Q = [C\ C_\infty]\begin{bmatrix} Q_s & 0 \\ 0 & Q_f \end{bmatrix}$ which is called pseudo–state Reachability matrix of (Σ) . The system (Σ) is called : slow state reachable if $\mathrm{rank}[Q_s] = n$; fast state reachable if $\mathrm{rank}[Q_f] = \mu$.

Theorem 2 Every $\beta_T \in \mathbb{R}^r$ is Reachable iff : (i) $R \equiv \mathbb{R}^r$ or equivalently , (ii) $\mathrm{rank}[Q] = r$

Proposition 1 The following statements are equivalent : (i) The system (Σ) is slow state reachable ; (ii) $\mathrm{rank}[Q_s] = \mathrm{rank}[\Omega, J\Omega, \dots, J^{n-1}\Omega] = n$,(iii) A(s) and B(s) are coprime in $\mathbb{C}$ i.e. $\mathrm{rank}[A(s), B(s)] = r\ \forall\ s \in \mathbb{C}$

Proposition 2 The system (Σ) is fast state reachable if at least one of the following conditions hold : (i) $\mathrm{rank}[Q_{f1}] = [\bar{\Omega}, J_\infty\bar{\Omega}, \dots, J_\infty^{\hat{q}} r\bar{\Omega}] = \mu$; (ii) $\mathrm{rank}[Q_{f2}] = [Z_0, Z_1, \dots, Z_{\sigma-1}] = \mu$; $\mathrm{rank}[Q_{f1}] + \mathrm{rank}[Q_{f2}] = \mu_1 + \mu_2 \geq \mu$,where $\mu_1 := \mathrm{rank}[Q_{f1}]$ and $\mu_2 := \mathrm{rank}[Q_{f2}]$

Finally we have to point out that our definition of Reachability is equivalent and natural generalization of the notions of Controllability ([2]) , C–Controllability ([11]) and Reachability ([8]) .

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