ENTROPIC FUNCTIONALS AND APPLICATIONS IN IMAGE RESTORATION

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1. INTRODUCTION

Some of the most prominent linear restoration techniques are based on the concept of regularization of ill-posed problems. Typical regularized approaches, such as the constrained least-squares (CLS) and the Tikhonov-Miller formulations, attempt to compensate for the ill-posedness of the pseudo-inverse solution by utilizing *smoothness* information in the restoration process [1-3]. These approaches employ quadratic objective functions that reflect the Gaussian assumption for both the prior and the posterior signal statistics. Quadratic functions are attractive, because they enable the analytic derivation of the corresponding estimators and provide cost-efficient implementation. The Gaussian model, however, does not cover most realistic noise sources, which are characterized by Poison, Laplacian, or impulsive noise distributions [4]. Moreover, the Gaussian distribution cannot characterize the vast majority of images. Recent efforts in image restoration acknowledge the presence of different types of noise [4] and address the need for accurate models characterizing the prior signal statistics [5].

In this paper, we address the aspects of robust estimation in regularized image restoration, with the utilization of non-quadratic objective functions. The approaches introduced aim to achieve accurate representation of both the signal and the noise distributions through the concept of robust M-estimation [4]. We exploit the structural flexibility of generalized maximum likelihood functions in order to provide accurate representation of a wide class of posterior (noise) distribution functions. Moreover, we present a different view to the stochastic representation of edges and we address the utilization of non-quadratic smoothing functionals in the regularization process. In this consideration, the existence of sharp edges manifests some uncertainty regarding the assumed (Gaussian) distribution of the signal. Expressing our intention to tolerate such uncertainty in the signal distribution, we modify the stabilizing functional in the regularized approach as to reflect the structure of M-estimation schemes. Thus, an influence function is utilized to restrain the contribution of large signal deviations in the stabilizing term.

The overall robust approach introduced in this paper can be interpreted as a generalized MAP formulation of image restoration. In the context of robust estimation, we introduce novel entropic functionals which operate on a high-pass version of the original image and can accurately characterize a wide ensemble of images. The entropy functionals proposed describe the distribution of only the high-frequency content of the original signal. The corresponding prior distributions permit large signal deviations and enable the reconstruction of sharp edges.

2. ROBUST OBJECTIVE FUNCTIONALS

Consider the linear space-invariant image formation model with additive noise, which is expressed as:

$$g = H\ f + n\ , \tag{1}$$

where f, g, and n denote the original image, the degraded data, and the noise process, respectively, whereas the block-Toeplitz matrix H represents the point-spread function of the system.

The so called *robust constrained least-squares* (RCLS) approach requires the satisfaction of the following constraints:

$$R_n(g-Hf) \triangleq \sum_{k=1}^{N} r_n(\, g_k - \sum_{j=1}^{N} H_{kj} f_j \,) \ \le\ \varepsilon\ , \tag{2a}$$

and

$$R_f(Cf) \triangleq \sum_{k=1}^{N} r_f(\, \sum_{j=1}^{N} C_{kj} f_j \,) \ \le\ E\ , \tag{2b}$$

where ε denotes a measure of the noise statistics and E is a prescribed constant determining the smoothing influence of the robust regularizing functional on the estimate. Moreover C is a linear highpass operator employed in the stabilizing functional. The signal kernel function $r_f(x)$ and the noise kernel function $r_n(x)$ are defined in terms of their derivatives, which in a robust estimation environment are referred to as the *influence functions*.

Through the robust functional $R_n(g-Hf)$, the posterior distribution can represent the statistics of a wide variety of medium and long-tailed noise processes. With an absolute-value metric this functional represents the Laplacian distribution, while it can still reflect the Gaussian distribution with a quadratic metric. Alternatively, the robust metric $R_f(Cf)$ on the signal space can be selected as to reflect long tails in the signal distribution and allow the accurate representation of the detailed structure.

According to the Tikhonov-Miller formulation [2], the constraints in (2) can be combined into a single criterion that incorporates the regularization parameter α. The minimization of this criterion can be performed through an iterative gradient technique. The overall implementation will be presented in the main body of this paper. It is proved that the convergence of the robust algorithm is guaranteed, provided that the influence functions employed are non-decreasing.

2.1 Robust Absolute-Entropy Measure

Entropic functionals have been successfully utilized in the overall representation of smooth images with sharp impulsive detail [6]. In order to expand this structural feature to a wide ensemble of images, we define an entropic measure on the detailed structure of the image rather than on the image itself. Thus, the new entropic functional is applied on a high-pass version of the original image. The domain of its kernel function must cover the entire real space, so that negative values resulting from the high-pass operator are appropriately treated. In order to satisfy this requirement while preserving the structure of the conventional entropy measure, we introduce a completely new entropy function. For a variable x, the so-called *absolute entropy* function is expressed as:

$$r_e(x) = (\, |x| + e^{-1} \,) \ln(\, |x| + e^{-1} \,)\ . \tag{3}$$

The design of the absolute entropy function involves a displacement of the negated entropy function $\{\, x \ln(x) \,\}$ shifting the minimum to the origin, and a subsequent mirror expansion of the section

{ $x>0$ } over the origin. By taking the left and right limits at the origin, it is verified that this function is twice continuously differentiable. Due to its continuity, its robust characteristics, and its relationship to entropy issues, the absolute entropy function forms a good kernel for the definition of the stabilizing functional $R_f(Cf)$.

2.2 Robust Absolute-Information Measure

The performance of regularized algorithms can be improved through the use of *a priori* information regarding the original image [6, 7]. In order to exploit prior information in robust regularized formulations along with the absolute entropy criterion, we introduce the robust *absolute-information* measure, which is motivated by the minimum information principle. Within an information theoretic framework, this principle minimizes the distance between the prior distribution state and the posterior distribution assignment within the space consistent with the problem's constraints, or the space that embodies no more than the available information. The absolute information measure preserves the advantages of the robust formulation and incorporates prior structural information in the restoration algorithm.

2.3 Examples

The properties of robust algorithm are demonstrated through restoration examples in different noise environments. In conclusion, the robust approach yields impressive improvement over the quadratic regularized scheme in the case of noise processes mixed with outliers. Considering single noise processes, either Gaussian or Laplacian, the robust approach improves significantly on the resolution of the estimate in moderate to low signal-to-noise ratios.

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