Stabilization of a class of discrete-time bilinear systems

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1 Introduction

This paper is a contribution to the stabilizability problem of discrete-time bilinear systems of the form

$$x_{k+1} = Ax_k + u_k^1 B_1 x_k + u_k^2 B_2 x_k \tag{1}$$

where $x_k \in \mathbb{R}^n$, $u_k = (u_k^1, u_k^2)^{\mathrm{T}} \in \mathbb{R}^2$ and A, B_1 and B_2 are constant $n \times n$ matrices.

The most general result in the case of continuous-time systems follows from the Jurdjevic-Quinn theorem (see [3]) where sufficient conditions are given for bilinear systems with dissipative drift. The analogous situation for discrete-time systems has been adressed by Tarbouriech and Burgat [2] in the case where the matrix A is critically stable in the sense of LaSalle (i.e. the open-loop system is stable but with some eigenvalues $\lambda_i(A)$ such that $|\lambda_i(A)| = 1$).

More recently homogeneous feedback of degree zero have been introduced by Chabour and Vivalda [1] to stabilize continuous-time bilinear systems which are not stabilizable by any continuous feedback at the origin. In our point of view, the most important properties of these feedback are that they are bounded and constant on the directions. The aim of this paper is to use these properties to stabilize discrete-time bilinear systems. In a first part we give the stabilizing feedback law for any initial condition which lies in an open dense subset $E \subset \mathbb{R}^n$, then we have just to remark that a transversality condition on the triple (A, B_1, B_2) with respect to $\mathbb{R}^n - E$ (which is an algebraic surface) allows to stabilize the system (1) on $\mathbb{R}^n$.

2 Main theorem

For any constant $2 \times n$ matrix C such that $\mathrm{rank}(C) = 2$, one can define the characteristic numbers ρ_1, ρ_2 of the system (1) by

$$\rho_i = \inf \{ m \in \mathbb{N} / \exists 1 \leq j \leq 2,\ C_i A^m B_j \neq 0 \}$$

where C_i is the i-th row of C, $i \in \{1, 2\}$. If $\rho_i < +\infty$ for $1 \leq i \leq 2$, denote by $\Omega(x)$ the 2×2 matrix defined for all $x \in \mathbb{R}^n$ by

$$\Omega(x) = \begin{pmatrix} C_1 A^{\rho_1} B_1 x & C_1 A^{\rho_1} B_2 x \\ C_2 A^{\rho_2} B_1 x & C_2 A^{\rho_2} B_2 x \end{pmatrix}.$$

Assume that there exists a full rank constant $2 \times n$ matrix C such that

(H1) $\rho_1 + \rho_2 = n - 2$.

(H2) The $n \times n$ matrix $T = \begin{pmatrix} C_1 \\ \vdots \\ C_1 A^{\rho_1} \\ C_2 \\ \vdots \\ C_2 A^{\rho_2} \end{pmatrix}$ is regular.

(H3) The homogeneous quadratic form $q(x) = \det \Omega(x)$ is non identically equal to zero on $\mathbb{R}^n$.

Given $\lambda \in \mathbb{R}^*$, set for all $x \in \mathbb{R}^n$ such that $q(x) \neq 0$

$$u(x) = \begin{pmatrix} u^1(x) \\ u^2(x) \end{pmatrix} = \Omega^{-1}(x) \begin{pmatrix} C_1 A^{\rho_1} \\ C_2 A^{\rho_2} \end{pmatrix} (\lambda \mathbf{I}_n - A)\, x$$

and

$$F(x) = Ax + u^1(x) B_1 x + u^2(x) B_2 x$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix, and let E be the open dense subset of $\mathbb{R}^n$ defined by

$$E = \{x \in \mathbb{R}^n \,/\, q(F^m(x)) \neq 0,\ 0 \leq m \leq \alpha\}$$

with $\alpha = \max\{\rho_1, \rho_2\} + 1$, $F^0 = Id$ and for all $m \geq 1$, $F^m = F \circ F^{m-1}$.

Proposition *Under the assumptions* (H1), (H2) *and* (H3), *the open subset E is invariant by F, i.e. $F(E) \subset E$.*

Proof : Using the linear change of coordinates on $\mathbb{R}^n$ $\xi = Tx$ one has

$$TF(T^{-1}\xi) = \tilde{A}\xi, \text{ with } \tilde{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where A_i is the $(\rho_i + 1) \times (\rho_i + 1)$ matrix given by

$$A_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}.$$

Set $\tilde{q}(\xi) = q(T^{-1}\xi)$. Hence, it suffices to prove that $\tilde{A}\tilde{E} \subset \tilde{E}$ where

$$\tilde{E} = TE = \left\{\xi \in \mathbb{R}^n \,/\, \tilde{q}(\tilde{A}^m \xi) \neq 0,\ 0 \leq m \leq \alpha\right\}.$$

A simple computation shows that $\tilde{A}^{\alpha+1} = \lambda \tilde{A}^{\alpha}$ which implies that for any homogeneous quadratic form on $\mathbb{R}^n$ $p(z) = z^{\mathrm{T}} S z$, $S = S^{\mathrm{T}}$,

$$p(\tilde{A}^{\alpha+1} z) = \lambda^2 p(\tilde{A}^{\alpha} z),\ \forall z \in \mathbb{R}^n, \tag{2}$$

and one can deduce that for all $\xi \in \tilde{E}$, $\tilde{q}(\tilde{A}^{\alpha+1}\xi) = \lambda^2 \tilde{q}(\tilde{A}^{\alpha}\xi) \neq 0$, and so $\tilde{A}\xi \in \tilde{E}$. ∎

The above proposition allows to state the following stabilization result for the system (1).

Theorem *For any $\lambda \in \mathbb{R}^*$ such that $|\lambda| < 1$, the feedback law defined on E by*

$$u_k = u(x_k) = \Omega^{-1}(x_k) \begin{pmatrix} C_1 A^{\rho_1} \\ C_2 A^{\rho_2} \end{pmatrix} (\lambda I_n - A) x_k, \quad k \in \mathbb{N}$$

is a bounded feedback which stabilizes the system (1) on E.

Proof : Notice that $u^1(x)$ and $u^2(x)$ are homogeneous functions of degree zero of the form

$$u^i(x) = \frac{p_i(x)}{q(x)}, \ \forall x \in \mathbb{R}^n \text{ such that } q(x) \neq 0 \ (i = 1, 2),$$

where p_1 and p_2 are homogeneous quadratic forms on $\mathbb{R}^n$. Setting $\tilde{p}_i(\xi) = p_i(T^{-1}\xi)$, it follows from (2), by using the linear change of coordinates $\xi = Tx$, that for all $x_0 = T^{-1}\xi_0 \in E$

$$u^i_{\alpha+1} = \frac{\tilde{p}_i(\tilde{A}^{\alpha+1}\xi_0)}{\tilde{q}(\tilde{A}^{\alpha+1}\xi_0)} = \frac{\lambda^2 \tilde{p}_i(\tilde{A}^{\alpha}\xi_0)}{\lambda^2 \tilde{q}(\tilde{A}^{\alpha}\xi_0)} = u^i_\alpha, \ i = 1, 2$$

and recursively $u_k = u_\alpha$, $\forall k \geq \alpha$ which allows to state the boundedness of u_k on E.

Besides, since $|\lambda| < 1$, one can deduce from the above proposition the asymptotic stability of the closed-loop system on E. ∎

Remark : If in addition of (H1), (H2) and (H3) one assume the following transversality condition

$$\forall x \in \mathbb{R}^n - E, \ \exists (a(x), b(x)) \in \mathbb{R}^2 \text{ such that } \Big(A + a(x)B_1 + b(x)B_2\Big)x \in E$$

then the system (1) is globally asymptotically stabilizable by means of the feedback law

$$\begin{cases} u_0 = (a(x_0), b(x_0))^T \text{ and } u_k = u(x_k) \text{ for } k \geq 1, \text{ if } x_0 \in \mathbb{R}^n - E \\ u_k = u(x_k) \text{ for } k \geq 0, \text{ if } x_0 \in E. \end{cases}$$

References

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