# Approximate Optimal Control for Nonlinear Systems

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**Abstract** We consider the design of approximate optimal controllers for quadratic systems. We propose a nonlinear controller designed with a nonlinear transformation followed by a standard linear-quadratic controller. This is compared with the exact optimal controller in an example.

**Introduction** We consider nonlinear systems of the form $\dot{x} = Ax + xMx + Bu$ and the problem of finding an optimal control $u^*$ which drives the system from from an initial value to a desired equilibrium point (assumed 0), subject to a cost

$$J = \int_0^{t_f} (x^T Q x + u^T R u)\, dt.$$

If we were to use a linear model for the system, i.e. only $\dot{x} = Ax + Bu$, the optimal control would be calculated as $u^* = -R^{-1}B^T Kx$, with $K$ a solution of the appropriate Riccati equation. This could then be applied to the original system, but the resulting system may not be able to meet the design criteria. The presence of a quadratic term in the system leads us to try a second order expansion for the controller (i.e. a quadratic feedback law, as in [1],[3]), which improves the results, but is costly to calculate. We propose to apply a nonlinear transformation first and then design a controller that only requires the calculation of a standard linear-quadratic optimal control.

**The Controller** We use the transformation suggested by Krener in [2], namely

$$\begin{aligned} z &= x + x\Phi x \\ v &= v + x\alpha x + x\beta u \end{aligned}$$

giving a new approximate system, $\dot{x} = Fx + Bu + \phi_3$, where $\phi_3$ is a term of third order in $x$. The truncation of the new system to a first order is now an order of accuracy better than the earlier suggested truncation. The optimal controller based on this linear system, transforms readily back to the quadratic feedback law

$$u^* = -R^{-1}B^T K(x + x\Phi x) - x\alpha x + x\beta R^{-1} B^T Kx \tag{1}$$

This controller is both easy to implement and potentially more accurate than the one based on the linearization of the original system. We next apply this to an example.

**Example** We consider the following nonlinear model for an electrical motor

$$
\begin{aligned}
\dot{x}_1 &= -\frac{R_a}{L_a}x_1 - \frac{c_2}{L_a}x_2x_3 + \frac{1}{L_a}u_1 \\
\dot{x}_2 &= -\frac{D}{J}x_1 + \frac{c_2}{J}x_1x_3 \\
\dot{x}_3 &= -\frac{R_f}{L_f}x_3 + u_2
\end{aligned}
$$

where $x_1$ is the armature current, $x_2$ is the angular velocity, $x_3$ is the magnetic flux, and $u_1$ and $u_2$ are the armature and field voltages, respectively. In addition, we choose a criterion with R and Q diagonal, positive definite matrices.

For this system, we compute $\Phi$, $\alpha$, and $\beta$, as required for the transformation in the previous section. We then apply three different controllers: the controller based on direct linearization, the nonlinear controller (1), and a more accurate approximation of the exact optimal control (based on a full power series expansion, as in [1]). After extensive simulations, we observed that our nonlinear controller matches the "exact" one noticeably better than the one based on the linear model.

**Conclusions** We have presented a method for nonlinear controller design that results in a good approximation of the exact optimal controller for a class of nonlinear systems. It is superior to the approximation which uses the linearization in the original $x$ variables, without the computational costs involved in solving nonlinear optimal control problems. The approach can be easily generalized to systems including bilinear terms of the form $xNu$.

**References**

[1] Cebuhar, W.A. & Costanza, V., "Approximation Procedures for the Optimal Control of Bilinear and Nonlinear Systems," *Journal of Optimization Theory and Applications*, Vol. 43, 1984.

[2] Krener, A., "Approximate Normal Forms of Nonlinear Systems," *Proceedings of the 27th CDC Conference*, Austin, Texas, 1988.

[3] Tibken, B. and Hofer, E.P., "A novel computer approach to optimal feedback control of bilinear systems," *Proceedings of the IFAC Symposium on Nonlinear Control Systems Design*, Capri, Italy, 1989.