Numerical Behaviour of Linear Polynomial Operations or What Do Experiments Reveal?

(Extended Abstract)

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1 Introduction

The game of polynomial control design is played with linear polynomial equations[1]

$$ax + by = c, \tag{1}$$

with a, b, c given and x, y unknown polynomials in one indeterminate. When facing multi-input multi-output systems, one has to put to use a matrix version of (1) such as

$$AX + BY = C, \tag{2}$$

which is the subject of our paper. Here A, B, C are given while X, Y are unknown polynomial matrices at compatible sizes.

During every design procedure for multivariable systems, a number of equations (2) and alike is to be usually solved. Hence, to have an effective tool to crack (2) is a must.

Various numerical algorithms for (2) have appeared in last two decades. Due to the lack of numerical mathematics for polynomial equations, unfortunately, we are almost unable to compare them theoretically. Therefore, we have started the project of *an experimental* comparison which consists in computing a crowd of numerical examples by means of our software package [6] which is made up of MATLAB and C programs for basic polynomial operations.

[1] For its similarity to the equation over integers it is often called the Diophantine.

In our experimental work we have exercised EOM, the method using *elementary operations* [5], PIM, *the polynomial interpolations* method [1], SSM, the method based on *the state-space realization* [2], and finally the *the indeterminate coefficients* approach [6]. The first three procedures will be briefly described and illustrated on simple examples.

Some preliminary conclusions drawn from the first group of experiments will be described in this extended abstract. However, more details and, perhaps, the final judgment is to be presented at the conference.

Before focusing on the algorithms, let us recall a little of theory. Equation (1) has a solution if and only if the greatest common left divisor G_1 of A, B is a left divisor of C. Further, when given a particular solution X_0, Y_0, the general solution of (2) is

$$\begin{aligned} X &= X_0 - B_1 T \\ Y &= Y_0 + A_1 T, \end{aligned} \tag{3}$$

where A_1, B_1 are right coprime matrices of compatible dimensions satisfying $AB_1 = BA_1$ and T is an arbitrary matrix of an appropriate dimension. For more details, see [5].

2 The Elementary Operations Method

The first approach chosen to solve (2) is taken from [5]. The solution is calculated via elementary (unimodular) operations in the following steps:

1. By elementary column operations (unimodular matrix U), bring the composite matrix of A, B into the quasi-triangular form; i.e.,

   $$\begin{bmatrix} A & B \end{bmatrix} U = \begin{bmatrix} G_1 & 0 \end{bmatrix},$$

   with G_1 a triangular matrix. U may be partitioned into the form of

   $$U = \begin{bmatrix} P_1 & R_1 \\ V_1 & S_1 \end{bmatrix},$$

   such that

   $$\begin{aligned} AP_1 + BV_1 &= G_1 \\ AR_1 + BS_1 &= 0. \end{aligned}$$

   P_1, V_1 and R_1, S_1 are couples of right coprime matrices.

2. Extract G_1, the left divisor of C so that $C = G_1 C_1$

3. Assemble the general solution according to (2), namely

   $$\begin{aligned} X &= P_1 C_1 + R_1 T \\ Y &= V_1 C_1 + S_1 T. \end{aligned}$$

3 The Polynomial Interpolation Method

For full detail, the reader is referred to the authors-of-the-method paper presented within the same section and to [1].

Here, A is considered to be a square row reduced $(l \times l)$ matrix.

The solution degree r depends on the reachability index ν of the system represented by $A^{-1}B$, where A, B are left coprime matrices with A row reduced. Solution of r-th degree exists if r satisfies $\deg_{ri}[C] \leq d_i + r$ and $r \geq \nu - 1$ for $i = 1, 2, \dots$ Hence, for a sufficiently high r, the equation is solved as follows:

1. Compute the number of interpolation couples (z_j, α_j), where z_j are distinct scalars and α_j are nonzero constant l-vectors.

2. Choose interpolation couples. To ensure X is of the r-th degree, assume X_r, the matrix coefficient at z^r of X, to be identity.

3. For all interpolation couples, consider (1) in z_j-s and multiply it from the left by $\alpha(j)$. Solve linear equation in constant matrices.

4. Recover X, Y from the partitioned matrix $[X_0'Y_0'X_1'Y_1' \dots X_r'Y_r']'$, where X_i, Y_i are matrix coefficients at z^i of matrix polynomials X, Y, respectively.

4 The State-space Realization Method

The basic idea of this approach was given in [2]. The algorithm was worked out by [3].

Here, A is considered square nonsingular.

1. Split the matrix fractions $A^{-1}B$, $A^{-1}C$ into the polynomial and strictly proper parts.

2. Find state-space realisations of the above strictly proper matrix fractions. Construct $\mathcal{C}$, the reachability matrix of the realisation originating from $A^{-1}B$.

3. Solve $\mathcal{C}\hat{Y} = G_C$, the constant matrices equation[2], where G_C stands for the input matrix of the realisation originating from $A^{-1}C$. The i-th entry of the column vector $\hat{Y}$ is the matrix coefficient at z^i of Y. Hence, polynomial matrix Y is recovered through its matrix polynomial.

4. Multiply the strictly proper part of $A^{-1}B$ by Y and split the product into polynomial and strictly proper parts.

5. Multiply the polynomial part of $A^{-1}B$ by Y and add the product to the polynomial obtained in 4. To get X, subtract this sum from the polynomial part of $A^{-1}C$.

[2] It's solvable whenever (1) is solvable.

5 Numerical examples

It is perhaps of interest at this point to demonstrate the application results of the above solution methods. We consider

$$A = \begin{bmatrix} 1+z+z^2 & 1-z \\ z+z^2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2+2z & 2+4z \\ 1+2z & 2+4z \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4}$$

In one run, unlike EOM, PIM and SSM yield only particular solution. Of course, general solution may be recovered via (2) after solving the homogeneous equation. Solution to (3), as obtained by particular methods, follow:

EOM

$$X = \begin{bmatrix} 1+2z & -4 \\ -z & 2 \end{bmatrix} + \begin{bmatrix} -1-z+2z^2 & 2+4z \\ 1+z-z^2 & -2z \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

$$Y = \begin{bmatrix} -z-z^2 & 1+2z \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} z-z^3 & -2-2z-2z^2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

PIM

$$X = \begin{bmatrix} 0 & -4 \\ 1 & 3 \end{bmatrix} \qquad Y = \begin{bmatrix} z & 3z \\ -0.5z & 0.5-0.5z \end{bmatrix}$$

SSM

$$X = \begin{bmatrix} 0 & -4 \\ 0 & 2 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & 1+2z \\ -0.5 & 0 \end{bmatrix}$$

On grounds of (2) with B_1 and A_1 computed by EOM, the particular solutions, as obtained by the first two methods, may be converted into the one with the least row degrees of Y, as achieved by SSM, using

$$T_{EOM} = \begin{bmatrix} 0 & 0 \\ -0.5 & 0 \end{bmatrix} \qquad T_{PIM} = \begin{bmatrix} -1 & -1 \\ -0.5+0.5z & -0.5+0.5z \end{bmatrix}.$$

In one run, the last two methods can compute particular solutions that meet different constraints. This is possible under certain circumstances, namely when enough constant coefficients equations are picked up to accommodate additional degrees of freedom spent on the constraints.

6 Preliminary conclusions

Unfortunately, both by the method nature, each method results in a different particular solution. Namely, the degrees of EOM result are a priori unknown, while PIM produces a solution up to the chosen degree. As implemented, SSM produces a solution with minimum row degrees in Y. In fact, any particular solution may be converted into other one by (3). These additional operations are