Stabilization of the rigid body about the middle axis

R. Outbib J.C. Vivalda

INRIA Lorraine -Projet CONGE
and URA CNRS 399 - MMAS Université de METZ
Technopôle METZ 2000 - CESCOM
Metz - FRANCE
e-mail : {outbib,vivalda}@ilm.loria.fr
phone: 33 87 20 35 12
fax: 33 87 76 39 77

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Statement of the problem

There has been much interest in the problem of stabilizing the Euler angular velocity of the rigid body with $n \leq 2$ torques. Bloch and Marsden consider in [3] the problem of the stabilization of a rigid body around the middle axis with one control torque applied about its minor axis. This problem has also been considered differently by Aeyels in [1]. Notice that this problem is different from the one which is concerned with the stabilizability of the *origin* of the Euler equations and which has been studied in [6, 5, 1, 4]. In this communication, we analyze more closely this question and prove results about the stability. To be more precise, the rigid body equations with a single torque about the minor axis are:

$$(1) \quad \left\{ \begin{array}{rcl} \dot{\omega}_1 & = & a_1\omega_2\omega_3 \\ \dot{\omega}_2 & = & a_2\omega_3\omega_1 \\ \dot{\omega}_3 & = & a_3\omega_1\omega_2 + u \end{array} \right.$$

whrere $a_1 = \dfrac{I_2 - I_3}{I_1}$, $a_2 = \dfrac{I_3 - I_1}{I_2}$ and $a_3 = \dfrac{I_1 - I_2}{I_3}$, I_1, I_2, and I_3 being the principal moments of inertia (we assume $I_1 > I_2 > I_3$). We study the stability The free rigid body equation (system (1) with $u = 0$) has relative equilibria when

$$\begin{array}{rcl} (\omega_1, \omega_2, \omega_3) & = & (\Omega, 0, 0), \\ (\omega_1, \omega_2, \omega_3) & = & (0, 0, \Omega), \end{array}$$

and when

$$(\omega_1, \omega_2, \omega_3) = (0, \Omega, 0).$$

The first two cases, which correspond to rotation about the major or minor axis, are well known to be stable, while the last case, rotation about the intermediate axis, is unstable.

We study the stability when restricted to the invariant manifolds: more precisely, it is clear that the trajectories of system (1) evolve on cylinders which are the level surfaces of the function $\omega \mapsto -a_2\omega_1^2 + a_1\omega_2^2$; these trajectories satisfy on the cylinder the following equations:

$$(2) \quad \left\{ \begin{array}{rcl} \dot{r} & = & 0 \\ \dot{\theta} & = & -\sqrt{-a_1 a_2}\,\omega_3 \\ \dot{\omega}_3 & = & \dfrac{a_3}{\sqrt{-a_1 a_2}} r^2 \cos\theta \sin\theta + u \end{array} \right.$$

The intersection set $\mathcal{S}$ between the middle axis and the cylinder is constituted by two points A and B, the question is: can we find a feedback law in such a manner that point A is an asymptotically stable equilibrium point for closed-loop system (1) when restricted to the cylinder.

Main result

We are looking for feedbacks for which the singular points of the closed-loop system (2) are in $\mathcal{S}$ and this for each cylinder $-a_2\omega_1^2 + a_1\omega_2^2 = c$. Since we consider only analytic feedbacks, the set of singular points is exactly equal to $\mathcal{S}$.

We give a family of analytic feedbacks such that the closed-loop system (2) verifies the following properties on the cylinder:

- system (2) has two singular points A and B which belongs to the plane $(O\omega_1\omega_2)$,
- one point, say A, is locally asymptotically stable and its basin of attraction is the cylinder minus two trajectories,
- these two trajectories have their ω-limit sets constituted by point B,

From these properties, it follows obviously that the closed-loop system (1) is stable in $\mathbb{R}^3$ about an equilibrium point of the form $(0, \omega_2^0, 0)$ and even locally asymptotically stable. The stabilizing feedbacks are given explicitely: they are polynomials and we can stabilize about any axis in the plane $(O\omega_1\omega_2)$. Obviously, if the control torque is applied applied about the major axis, the same results are true.

Moreover, we prove that, according to the stabilization about the middle-axis, our feedbacks are the best possible in the following sense. We prove that if the closed-loop system (2) has two singular points, it is impossible that the ω-limit sets of all the trajectories (excepted the point B itself) are equal to point A while the α-limit sets (excepted point A itself) are equal to B.

We prove also that system (1) cannot be stabilized by means of a linear feedback.

In [2], Aeyels and Szafranski consider an analogous problem, i.e. the control torque is applied about the middle axis. They have proven that, when restricted to the invariant manifolds which are hyperbolic cylinders, the closed-loop system is asymptotically stable about the intersection between the major (or the minor) axis and a connected component of this invariant manifold.

There is a major topological difference between our problem and the one considered by Aeyels and Szafranski: in their case, the invariant manifolds are composed with two connected components diffeomorphic to $\mathbb{R}^2$. In our case, the invariant manifolds are cylinders which are connected and, as explained above, we caanot avoid to have two singular points on the cylinder.

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