

The Pole Placement Equation - A Survey

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We consider the linear equation

$$AX + BY = C \quad (1)$$

where A, B and C are given polynomials from $K[s]$, the ring of polynomials in the indeterminate s over a field K , and X and Y are unknown polynomials in $K[s]$.

MOTIVATION

Equation (1) has found application in several design problems for linear control systems, including the pole placement design. This problem consists in the following: given a plant with real-rational proper transfer function

$$P(s) = \frac{B(s)}{A(s)}$$

where A and B are coprime polynomials, one seeks a dynamic output feedback controller with a real-rational proper transfer function, say

$$Q(s) = -\frac{Y(s)}{X(s)}$$

such that the closed-loop system has prespecified poles.

Provided A is the characteristic polynomial of the plant and X is that of the controller, then the characteristic polynomial of the closed-loop system, say $C(s)$, which specifies the poles desired, is given by $C = AX + BY$.

Thus the pole placement design is based on equation (1). However not all solution pairs X, Y are of interest: one must take the one in which Y has least degree. This leads to a proper controller

whenever one exists.

REVIEW OF THEORY

It is well known [1] that $K[s]$ is a principal ideal domain. Thus (1) is solvable if and only if any greatest common divisor of A and B divides C . Writing D for a greatest common divisor of A and B and denoting

$$\bar{A} = \frac{A}{D}, \quad \bar{B} = \frac{B}{D}, \quad \bar{C} = \frac{C}{D}$$

one concludes that (1) has a solution if and only if \bar{C} is a polynomial. Therefore if A and B are coprime then (1) is solvable for any C .

Suppose that \bar{X}, \bar{Y} is a particular solution pair of (1). Since the equation is linear, any and all solution pairs of (1) are given by

$$X = \bar{X} - \bar{B}T, \quad Y = \bar{Y} + \bar{A}T$$

where T varies over $K[s]$. Thus the solution class of (1) is parametrized through T in a simple manner.

It is well known [1] that $K[s]$ is a euclidean domain. Therefore if (1) is solvable and $B \neq 0$ there is a unique solution pair $X_{1\min}, Y_1$ of (1) such that either $X_{1\min} = 0$ or $\deg X_{1\min} < \deg \bar{B}$. Further if (1) is solvable and $A \neq 0$ then there is a unique solution pair $X_2, Y_{2\min}$ of (1) such that either $Y_{2\min} = 0$ or $\deg Y_{2\min} < \deg \bar{A}$. These two *least-degree solution* pairs coincide [4] whenever $\deg \bar{A} + \deg \bar{B} > \deg \bar{C}$.

As a result, equation (1) with $A \neq 0$ and $B \neq 0$ can possess solution pairs X, Y of arbitrarily high degree, limited only from below by $\deg X_{1\min}$ and

$\deg Y_{2\min}$.

FIXED DEGREE SOLUTIONS

We shall study the class of solutions whose degrees are limited from above. We suppose that A, B and C in (1) are non-zero polynomials from $K[s]$ with A and B coprime. Hence (1) is solvable. Let

$$p = \deg A, \quad q = \deg B, \quad r = \deg C.$$

If

$$A = a_0 + a_1 s + \dots + a_p s^p$$

then, for any integer $k \geq p$, we denote

$$\text{vec}_k A = [a_0 \ a_1 \dots a_p \ \underbrace{0 \dots 0}_{k-p}].$$

Here is the existence result [5]. Let m, n be non-negative integers and $d = \max(m + p, n + q, r)$. Then a solution pair X, Y of (1) exists such that

$$X = 0 \text{ or } \deg X \leq m, \quad Y = 0, \text{ or } \deg Y \leq n \quad (2)$$

if and only if $\text{vec}_d C$ is a K -linear combination of $\text{vec}_d A, \text{vec}_d sA, \dots, \text{vec}_d s^m A, \text{vec}_d B, \dots, \text{vec}_d s^n B$.

A special case of particular interest concerns the constant solutions of (1). Putting $m = n = 0$ we deduce [6] that a solution pair X, Y of (1) exists in K if and only if $\text{vec}_d C$ is a K -linear combination of $\text{vec}_d A$ and $\text{vec}_d B$.

The set of solutions whose degrees are limited from above can be parametrized as follows [5]. Let $m \geq q$ and $n \geq p$. If $n \geq r - q$ then the set of solutions X, Y of (1) that satisfy (2) is given as

$$X = X_{1\min} - BT_1, \quad Y = Y_1 + AT_1 \quad (3)$$

where T_1 varies over $K[s]$ and

$$\deg T_1 \leq \min(m - q, n - p);$$

if $m \geq r - p$ then the set of solutions X, Y of (1) that satisfy (2) is given as

$$X = X_2 - BT_2, \quad Y = Y_{2\min} + AT_2 \quad (4)$$

where T_2 varies over $K[s]$ and

$$\deg T_2 \leq \min(m - q, n - p).$$

We note that at least one of the two conditions, $m \geq r - p$ and $n \geq r - q$, is always satisfied. Of course (3) can be used to parametrize the solution set (2) even if $n < r - q$. Then, however, T_1 has a higher degree than shown and is not completely free in $K[s]$. An analogous statement is true for (4) when $m < r - p$. To illustrate, we parametrize the solution class of

$$X + sY = s^2$$

such that $\deg X \leq 1$ and $\deg Y \leq 1$. Using (3),

$$X = -sT_1, \quad Y = s + T_1, \quad T_1 \text{ constant}$$

while using (4),

$$X = s^2 - T_2, \quad Y = T_2, \quad T_2 = s + \tau, \quad \tau \text{ constant.}$$

EXAMPLES

Can the double integrator

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad y = x_1$$

be converted into an harmonic oscillator using a proportional output feedback?

The double integrator gives rise to the transfer function

$$P(s) = \frac{1}{s^2}$$

and any harmonic oscillator has the characteristic polynomial

$$C(s) = s^2 + \omega^2$$

for some real constant $\omega > 0$. Thus the answer depends on the polynomial equation

$$s^2 X + Y = s^2 + \omega^2$$

having a constant solution pair X, Y .

Since

$$\text{vec}_2 A = [0 \ 0 \ 1]$$

$$\text{vec}_2 B = [1 \ 0 \ 0]$$

$$\text{vec}_2 C = [\omega^2 \ 0 \ 1]$$

the answer is an affirmative: the output feedback $u = -\omega^2 y$ will do the job. The resulting system equations read

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u - \omega^2 x_1, \quad y = x_1.$$

On the other hand, the double integrator cannot be stabilized via proportional output feedback: the polynomial $s^2 X + Y$ is not Hurwitz for any real numbers X and Y .

As the second example, we consider the plant

$$\dot{x}_1 = u - x, \quad y = x$$

and find all output feedback controllers that will alter its characteristic polynomial $s + 1$ to $s^2 + 3s + 2$.

These controllers possess the transfer functions

$$Q(s) = -\frac{Y(s)}{X(s)}$$

where X, Y is the solution set of the equation

$$(s + 1)X + Y = s^2 + 3s + 2$$

such that $\deg X = 1$ and $\deg Y \leq 1$.

The condition $m \geq r - p = 1$ is verified. Therefore the solution set is given by

$$X = s + 2 - T_2, \quad Y = (s + 1)T_2$$

where T_2 is any real polynomial of degree at most $\min(m - q, n - p) = 0$, hence any real constant.

A realization of the parametrized controller set is

$$\dot{w} = (T_2 - 2)w + (T_2 - 1)y$$

$$-u = T_2 w + T_2 y.$$

The case $T_2 = 0$ leads to an unobservable realization while $T_2 = 1$ leads to an uncontrollable realization. A PI controller is obtained when $T_2 = 2$.

If desired, the parameter T_2 can be chosen so that a specific goal is achieved. For example, if the H_∞ -norm of the sensitivity function

$$S(s) = \frac{s + 2}{s + 2 - T_2}$$

is not to exceed 1, we should avoid the values $0 < T_2 < 4$.

METHODS OF SOLUTION

Equation (1) can be solved in several ways [4]. One can distinguish *parametric* methods (where the polynomials are represented by their coefficients) and *non-parametric* ones (where the polynomials are represented by their functional values.) We shall describe three major parametric methods.

We suppose that A, B and C in (1) are non-zero real polynomials with A and B coprime. Hence (1) is solvable. For the sake of simplicity let

$$\deg A = \deg B = N, \quad \deg C = 2N - 1.$$

The *Method of Indeterminate Coefficients* [4] converts equation (1) into a system of $2N$ linear equations over the field of real numbers. Suppose we seek the least-degree solution pair X, Y :

$$\deg X \leq N - 1, \quad \deg Y \leq N - 1$$

The $2N$ coefficients of X, Y satisfy the system of equations

$$[\text{vec}_{N-1} X \quad \text{vec}_{N-1} Y] \begin{bmatrix} \text{vec}_{2N-1} A \\ \dots \\ \text{vec}_{2N-1} s^{N-1} A \\ \text{vec}_{2N-1} B \\ \dots \\ \text{vec}_{2N-1} s^{N-1} B \end{bmatrix} = \text{vec}_{2N-1} C.$$

The system matrix is a Sylvester matrix and it has full rank since A and B are coprime.

The *Method of Polynomial Reductions* [3] reduces equation (1) to a polynomial equation that is much easier to solve. It consists of the substitutions

$$C' = C - A \frac{c_{\deg C}}{a_{\deg A}} s^{\deg C - \deg A}$$

$$C' = C - B \frac{c_{\deg C}}{b_{\deg B}} s^{\deg C - \deg B}$$

$$B' = B - A \frac{b_{\deg B}}{a_{\deg A}} s^{\deg B - \deg A}$$

$$A' = A - B \frac{a_{\deg A}}{b_{\deg B}} s^{\deg A - \deg B}$$

each reducing the degree of one of the polynomials A, B, C . The substitutions are repeated for the new polynomials A', B', C' and will ultimately reduce all A, B, C but one to zero. The resulting equation has a solution $X' = 0, Y' = 0$ and the solution pair X, Y of (1) is obtained through the backward substitutions

$$X = X' + \frac{c_{\deg C}}{a_{\deg A}} s^{\deg C - \deg A}$$

$$Y = Y' + \frac{c_{\deg C}}{b_{\deg B}} s^{\deg C - \deg B}$$

$$X = X' - Y' \frac{b_{\deg B}}{a_{\deg A}} s^{\deg B - \deg A}$$

$$Y = Y' - X' \frac{a_{\deg A}}{b_{\deg B}} s^{\deg A - \deg B}.$$

The process involves the euclidean algorithm for A, B and leads to the least-degree solution pair X, Y .

The *Method of State-space Realization* [2] combines matrix and polynomial operations. We write (1) as

$$X + \frac{B}{A} Y = \frac{C}{A}$$

and determine a reachable state-space realization (F, G, H, J) of the rational function B/A . The N coefficients of Y satisfy the system of equations

$$\text{vec}_{N-1} Y \begin{bmatrix} H \\ HF \\ \dots \\ HF^{N-1} \end{bmatrix} = \text{vec}_{N-1} (C \bmod A)$$

and the corresponding X is recovered from (1); it is the least-degree solution pair. The system matrix is an observability matrix and it has full rank since A and B are coprime.

NUMERICAL EXPERIENCE

The method of indeterminate coefficients is straightforward and leads directly to a system of linear equations for the coefficients of the unknown polynomials. The method of polynomial reductions solves the polynomial equation by polynomial means and is not suitable for pencil-and-paper calculations, for it requires a large number of logical operations. The method of state-space realization combines the two above: one unknown polynomial is obtained by solving a system of linear equations while the other results from polynomial manipulations.

The comparison of the methods with respect to the arithmetic complexity is quite clear. The fastest is the method of polynomial reductions, where the operations count is proportional to N^2 . For the other two methods the arithmetic complexity is proportional to N^3 . The slowest method, however, is that of indeterminate coefficients because it leads to a larger system of linear equations than the method of state-space realization.

The comparison of the methods from the precision point of view is not that simple, however. Provided the polynomials A and B have no (especially multiple) roots close to each other, the precision of all three methods is alike. The ill-conditioned data, however, make the method of polynomial reductions fail more often than that of indeterminate coefficients. The method of state-space realization shows no clear-cut tendency, it stays between the two preceding methods.

To conclude, polynomial reductions are fast but sensitive to data, indeterminate coefficients are robust but slow, and the method of state-space realization is universal but second best in each single aspect.

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