Combined Solutions of Optimal Control and System Parameter Identification Problems for Parallel Machines

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Extended Abstract

Frick and Stech [1, 2] recently employed the Epsilon-Ritz method for solving optimal control problems on parallel computers. The approach provided an open-loop solution since there was no dynamic feedback control law involved. In this paper the Epsilon-Ritz method is extended to calculate the feedback law and also to perform the system parameter identification. The solutions of the dynamic feedback and system parameter identification problems are then combined to provide an adaptive optimal control computational method on parallel computers.

Central to the calculations is the use of the Epsilon Technique, originally proposed by Balakrishnan [3]. This method is discussed in more detail below. Its main feature is that it transforms the optimal control problem into an optimization problem in L_2 and thus avoid having to solve any differential equations (either forward or backward) in time. By approximating the state and control functions by a finite series of orthogonal polynomials (Legendre or Chebyshev) or Piecewise constant functions (Walsh or Block-Pulse) as the basis the resulting Ritz formulation of the problem now transforms the optimal control problem to a nonlinear programming problem. By adopting a *VEC* notation, first introduced by Brewer [4], the parameterized problem is transformed into a fairly simple vector-matrix form that can be solved on parallel machines using standard matrix methods for parallel computations.

For readers unfamiliar with the Epsilon method, the process is illustrated below for optimal control problems of the form where

$$V(\underline{x}, \underline{u}) = \int_0^T G(\underline{x}, \underline{u}, t)dt$$

represents the cost functional that is to be minimized with respect to the choice of the control function $\underline{u}$, for the dynamic system

$$\dot{\underline{x}}(t) = f(\underline{x}, \underline{u}, t); \qquad \underline{x}(0) = \underline{x}_s.$$

The Integral Epsilon Problem [5] is obtained by defining the composite cost functional

$$J(\varepsilon, \underline{x}, \underline{u}) = \int_0^T \left\{ \frac{1}{2\varepsilon} \|\underline{e}(t)\|^2 + G(\underline{x}, \underline{u}, t) \right\} dt,$$

where the scalar $\varepsilon > 0$ and the error function is defined as

$$\underline{e}(t; \varepsilon) = \underline{x}(t) - \underline{x}_s - \int_0^t f(\underline{x}(\tau), \underline{u}(\tau), \tau)d\tau.$$

A solution to the optimal control problem is obtained by simultaneously minimizing the composite cost functional J with respect to both $\underline{x}$ and $\underline{u}$ for each member of a monotonically decreasing sequence $\{\varepsilon_j\}$ of scalars.

Convergence, subject to the usual boundedness, continuity and convexity assumptions, is assured by the following result [5].

Convergence Result. Consider the sequence of scalars $\{\varepsilon_j\} \downarrow 0$ monotonically decreasing. For the corresponding sequence of minimizing solutions to the integral ε problem [that is $J(\varepsilon; \underline{x}, \underline{u})$ above] denoted by $\{\underline{x}_o(\varepsilon_j), \underline{u}_o(\varepsilon_j)\}$ and the associated $\{\underline{\varepsilon}_o(\varepsilon_j)\}$, we have

$$\begin{aligned}
\underline{u}_o(\varepsilon_j) &\rightarrow \underline{u}^*, \\
\underline{x}_o(\varepsilon_j) &\rightarrow \underline{x}^*, \\
\frac{\underline{\varepsilon}_o(\varepsilon_j)}{\varepsilon_j} &\rightarrow \underline{\lambda}^*, \\
\underline{\varepsilon}_o(\varepsilon_j) &\downarrow 0, \\
J(\varepsilon_j, \underline{x}_o(\varepsilon_j), \underline{u}_o(\varepsilon_j)) &\uparrow V(\underline{x}^*, \underline{u}^*).
\end{aligned}$$

as $\varepsilon \downarrow 0$. Here $\underline{x}^*, \underline{u}^*$ and $V(\underline{x}^*, \underline{u}^*)$ are the optimal control, state and cost for the optimal control problem.

Dynamic Feedback Control Law

The problem of linear systems with quadratic optimal criteria is chosen in this study, since it is well known to have both analytical and numerical solutions via methods for solving the Riccati matrix equation [7]. The linear system is represented by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{1}$$

where $x(t) \in R^n$ and $u(t) \in R^m$ and the initial condition $x(0) = x_s$.
The quadratic performance index is

$$J = \int_0^{t_f} \{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\}dt \tag{2}$$

where $Q(t)$ is semi-positive definite and $R(t)$ is positive definite.
It is well known that the solution of the linear quadratic regulator problem is the control $u(t)$ having the form of a linear combination of the state $x(t)$ i.e.

$$u^*(t) = L(t)x(t) \tag{3}$$

with $L(t)$ is a time varying linear feedback, obtained from the solution the solution $K(t)$ of the Riccati equation.

The matrix Riccati equation is time varying and nonlinear

$$\dot{K}(t) = K(t)A(t) + A^T(t)K(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t) + Q(t), \tag{4}$$

its solution $K(t)$ is often found by numerical techniques. Having $K(t)$ then the optimal control $u(t)$ is of the form

$$u^*(t) = -R^{-1}B^T(t)K(t)x(t). \tag{5}$$

The product $K(t)x(t)$ is the costate of the system and is designated here as $y(t)$. Hence, if the costate can be found by another method beside solving for the solution of the Riccati equation, then the feedback and the control law are subsequently determined.

In this new approach to calculate the feedback law, the costate of the system is approximated by integrating the result of $\underline{\varepsilon}_o(\varepsilon_j)/\varepsilon_j \rightarrow \underline{\lambda}^*$ backward in time. This provides a constructive way to formulate an expression to calculate the feedback law which depends only on the system matrices ($A(t)$ and $B(t)$) and the performance weighted matrices ($Q(t)$ and $R(t)$). This method is implemented using the base proposed by Stech and Frick [1], that is, a parametric approximation to the costate in terms of the Walsh parameters of the state variables. Here, the costate is denoted by the vector quantity $\underline{y}(t) \in R^n$, and like the states are approximated by

$$\underline{y}(t) = Y\Psi(t) and \quad \underline{x}(t) = X\Psi(t)$$

where $\Phi_m(t) = [\phi_0(t), \phi_1(t), \cdots, \phi_{m-1}]^T$ is an array of Walsh functions, say. Then

$$Vec(Y) = \left[I_{nm} - (S^T \otimes A^T)\right]^{-1} (S^T \otimes Q) Vec(X) \tag{6}$$

where m is the number of Walsh functions used in the approximation, n is the order system and S is the Walsh operational matrix for backward integration [6].

This in turn provides a simple mechanism to devise an expression for the optimal feedback controller. Since the algebraic form of the optimal controller is well known as

$$\underline{u}(t) = -R^{-1}B^T\underline{y}(t),$$

the corresponding parametric form is given by

$$Vec(U) = (I_n \otimes R^{-1}B^T)Vec(Y).$$

Replacing Vec(Y) in the above by the expression in equation 6 yields

$$Vec(U) = (I_n \otimes R^{-1}B^T)\left[I_{nm} - (S^T \otimes A^T)\right]^{-1} (S^T \otimes Q) Vec(X), \tag{7}$$

providing a very simple algebraic (matrix-vector) expression for the optimal feedback control law for the system.

System Parameter Identification

New results are presented illustrating how the parameterized problem formulation can also be used to perform **system parameter identification.** Here, the performance index is the square of the system's dynamic error signal which is then minimized with respect to the parameters contained in the system matrices in an analogous way to the optimal control calculation. The calculations is summarized below.

Consider the performance index

$$J(\varepsilon, A, B) = \int_0^{t_f} \frac{1}{2\varepsilon}\{e(t,\varepsilon)\}^2 \tag{8}$$

where $e(t)$ is the error function, the difference between the true state $x(t)$ and its estimate. In Walsh domain, the above equation is approximate by

$$J = \int_0^1 \{\Phi_m^T(t)E^T E\Phi_m(t)\}dt. \tag{9}$$

Now, using the VEC notation, Frick and Stech [1] found that J can be rewritten as

$$J = \frac{1}{2\varepsilon}Vec(E)^T Vec(E). \tag{10}$$

The error function in Vec notation $Vec(E)$ is a function of the system matrix parameters of $A(t)$ and $B(t)$. Subsequently, J of equation (10) is also a function of the system matrix parameters. Hence, the gradients of J with respect to $Vec(A)$ and $Vec(B)$ are calculated and set equal to zero. This process resulted to a simple algebraic matrix equation

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{pmatrix} VecA \\ VecB \end{pmatrix} = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} \tag{11}$$

where

$$k_{11} = \frac{1}{\varepsilon}[(XP) \otimes I_n][(XP)^T \otimes I_n],$$

$$k_{12} = \frac{1}{\varepsilon}[(XP) \otimes I_n][(UP)^T \otimes I_n],$$

$$k_{21} = \frac{1}{\varepsilon}[(UP) \otimes I_n][(XP)^T \otimes I_n],$$

$$k_{22} = \frac{1}{\varepsilon}[(UP) \otimes I_n][(UP)^T \otimes I_n],$$

$$D_1 = \frac{1}{\varepsilon}[(XP) \otimes I_n][Vec X - VEC X_0],$$

$$D_2 = \frac{1}{\varepsilon}[(UP) \otimes I_n].$$

with X and U are the Walsh appoximations of the state $x(t)$ and the input $u(t)$, I_n is the identity matrix of the dimension n (the system order), X_0 is the Walsh approximation of the initial condition x_0 and P is the Walsh integration matrix.

Adaptive Optimal Control Solution

The optimal feedback and the system matrix identification solutions are then combined to produce a solution to the adaptive optimal control problem. The closed-loop feedback control calculation is periodically updated based on the most current estimate of the system parameters. A parallel machine is employed to not only perform the calculations for the parameter estimation and the updated optimal control law in parallel, but to conduct the two processes simultaneously. Examples for the linear quadratic case, with Walsh function as basis in the Ritz method is presented. Other examples which are extended to time varying and the linearization of nonlinear systems are also discussed.

References

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