# A GLOBAL STABILIZATION THEOREM FOR PLANAR NONLINEAR SYSTEMS

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**Abstract**
We provide sufficient conditions for global stabilization of planar nonlinear systems by means of feedback law which is continuous on the whole state space.

## 1. Introduction

We consider planar system of the form

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}f(x,y)\\ u\end{pmatrix}, \quad (x,y)\in\mathbb{R}^2,\ u\in\mathbb{R}, \tag{1.1}$$

where the map $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ is continuous ($C^0$) with $f(0,0)=0$ and further is **analytic** in a neighborhood of $0\in\mathbb{R}^2$.

We now give our main sufficient conditions for global stabilization of (1.1).

**Assumption 1.1.** Suppose that there exist a **connected** closed subset $M\subset\mathbb{R}^2$ which contains zero $0\in\mathbb{R}^2$ such that

(i) $xf(x,y)<0,\ \forall\ (x,y)\in M,\ x\neq 0;$ (1.2)

(ii) the projection I of M on the x-axis along the y-axis is the whole real line:

$$I = \mathbb{R}\ ;$$

(iii) for each compact set $Q\subset\mathbb{R}$ the set

$$M_Q=\{(x,y)\in Q\times\mathbb{R}\}\cap M \text{ is compact.}$$

(iv) there exist a pair of disjoint opens sets $U^+,U^-\subset\mathbb{R}^2$ such that

$$M\cup U^+\cup U^-=\mathbb{R}^2$$

and further if we denote by $Y^+$ and $Y^-$ the intersection of the y-axis with $\mathbb{R}^+\setminus\{0\}$ and $\mathbb{R}^-\setminus\{0\}$ respectively, then

$$Y^+\subset U^+;\ Y^-\subset U^-$$

## 2. Main result

Before state and prove our main theorem we need the following lemma which is a special case of Artstein's theorem [1].

**Lemma 2.1.** Consider the affine in the control system

$$\dot{x} = F(x)+uG(x),\ x\in\mathbb{R}^n,\ u\in\mathbb{R}, \tag{2.1}$$

where F and G are $C^0$ and $F(0)=0$. Suppose that there exist:

(i) a control Lyapunov function (clf) $\Phi$ of zero $0\in\mathbb{R}^n$, namely $\Phi$ is positive definite (i.e. $\Phi(0)=0$; $\Phi(x)>0$ otherwise), is uniformly unbounded on $\mathbb{R}^n$ (i.e. $\Phi(x)\to+\infty$ as $\|x\|\to+\infty$) is $C^1$ and satisfies

$$(D\Phi G)(x)=0,\ x\neq 0 \Rightarrow (D\Phi F)(x)<0, \tag{2.2}$$

(ii) a $C^0$ map $u_0:\mathbb{R}^n\to\mathbb{R}$ such that $u_0(0)=0$ and

$$(D\Phi F+u_0 D\Phi G)(x)<0 \tag{2.3}$$

for all $x\neq 0$ near zero.

Then there exists a feedback law $u=u(x)$ that is $C^0$ on the whole state space $\mathbb{R}^n$ with $u(x)=u_0(x)$ for x near zero and which globally asymptotically stabilizes (2.1) at $0\in\mathbb{R}^n$.

**Theorem 2.2.** If Assumption 1.1 is satisfied, then the planar system (1.1) is G.A.S. by means of a feedback law which is continuous on the whole state space $\mathbb{R}^2$ vanishing at zero $0\in\mathbb{R}^2$.

**Proof.** Since f is analytic near $0\in\mathbb{R}^2$ there exist integers $k_i$, $l_i$, $i=0,1,2,\dots$ and real constants $c_i$ such that

$$f(x,y)=\sum_{i=0}^{+\infty} c_i x^{l_i} y^{k_i},\ (x,y \text{ near zero}).$$

Let $l=\min\{l_j\ 0\leq j\leq+\infty\}$. Then f is written

$$\begin{aligned} &f(x,y)=x^l a(x,y);\\ &a(x,y)=\sum_{i=0}^{+\infty} c_i x^{l_i-l} y^{k_i} \end{aligned} \tag{2.4}$$

Then either $a(0,0)\neq 0$ or there exists an integer k such that

$$a(0,0)=\ldots=\frac{\partial^{k-1}a}{\partial^{l-1}y}(0,0)=0, \text{ whereas} \tag{2.5}$$

$$\frac{\partial^k a}{\partial y^k}(0,0)\neq 0. \tag{2.6}$$

Hence by (2.4),(2.7),(2.8) and the Weierstrass preparation theorem f is written

$$f(x,y)=x^l q(x,y)\prod_{i=1}^{k}(y-\varphi_i(x)), \tag{2.7}$$

for x,y near zero with $q(0,0)\neq 0$ and the factors $\varphi_i$ are in general complex valued $C^0$ functions having the form

$$\varphi_i(x)=b_i\, x^{\frac{s_i}{n_i}}(1+c_{i_1}x^{\frac{1}{n_i}}+c_{i_2}x^{\frac{2}{n_i}}+\ldots). \tag{2.8}$$

for suitable integers $n_i$ and $s_i$ with $\frac{s_i}{n_i}\geq\frac{1}{k}$ and constants $b_i$ and $c_{i_j}$. Since f is real the factors $\varphi_i$ should occur in complex conjurate pairs in the factorization (2.7). Without any loss of generality we may assume that $k\geq 1$ and q is positive definite near zero. From (1.2) and (2.7) we get

$$x^{l+i}\prod_{i=1}^{k}(y-\varphi_i(x))<0,\ (x,y)\in M,\ x\neq 0 \text{ near zero.} \tag{2.9}$$

By (2.9) and condition (iv) it follows that there exist a positive constant $\delta$ and a $C^0([-\delta,\delta])$ real function $\varphi$ with $\varphi(0)=0$ such that

$$\{(x,y)\in\mathbb{R}^2,\ x\in[-\delta,\delta],\ y=\varphi(x)\}\subset M$$

and therefore

$$x^{l-1}\prod_{i=1}^{k}(\varphi(x)-\varphi_i(x))<0,\ x\neq 0 \text{ near zero.}$$

Consequently there exist functions $\varphi_{\alpha_1},\varphi_{\beta_1},\varphi_{\alpha_2},\varphi_{\beta_2}$ of the form (2.8) taking real values such that

$$\varphi_{\alpha_1}(x)<\varphi(x)<\varphi_{\beta_1}(x),\ x>0;$$

$$\varphi_{\alpha_2}(x)<\varphi(x)<\varphi_{\beta_2}(x),\ x<0, \text{ near zero;}$$

$$x^{l+1}\prod_{i=1}^{k}(y-\varphi_i(x))<0,$$

$$\forall\ (x,y)\in\{(x,y)\in\mathbb{R}^2 : x>0,\ \varphi_{\alpha_1}(x)<y<\varphi_{\beta_1}(x)\}$$

$$\cup\{(x,y)\in\mathbb{R}^2 : x<0,\ \varphi_{\alpha_2}(x)<y<\varphi_{\beta_2}(x)\},\ ((x,y) \text{ near zero}). \tag{2.10}$$

Since the functions $\varphi_{\alpha_i},\varphi_{\beta_i}$, $i=1,2$ have the form (2.8) it follows that there exist real constants $\sigma_1$ and $\sigma_2$ and a pair of positive real numbers $\rho_1$ and $\rho_2$ with $\rho_1,\rho_2\geq\frac{1}{k}$ such that

$$\begin{aligned}&\varphi_{\alpha_1}(x)<\sigma_1 x^{\rho_1}<\varphi_{\beta_1}(x),\ x>0\\ &\varphi_{\alpha_2}(x)<\sigma_2 x^{\rho_2}<\varphi_{\beta_2}(x),\ x<0 \text{ near zero.}\end{aligned} \tag{2.11}$$

Then we can construct a $C^0$ function $\sigma:[-b,b]\to\mathbb{R}$; $\sigma(0)=0$, $b$ being a suitably small positive constant with $b\leq\delta$, and a $C^0$ function $\lambda:\mathbb{R}^+\to\mathbb{R}^+$ with $\lambda(s)=0$ for $s\in[-b/2,b/2]$, $\lambda(s)=1$ for $s\in[-b,-3b/4]\cup[3b/4,b]$ and $0\leq\lambda(s)\leq1$ otherwise, such that

$$\sigma(x)=\begin{cases}\lambda(x)\varphi(x)+(1-\lambda(x))\sigma_1 x^{\rho_1}, & b\geq x\geq 0\\ \\ \lambda(x)\varphi(x)+(1-\lambda(x))\sigma_2 x^{\rho_2}, & -b\leq x\leq 0;\end{cases} \tag{2.12}$$

and further conditions (i)-(iii) of Assumption 1.1 are satisfied with $\hat{M}=M\cup\{(x,y)\in\mathbb{R}^2 : x\in[-b,b],\ y=\sigma(x)\}$ instead of M. In particular from (2.10)-(2.12) we have

$$xf(x,\sigma(x))<0,\ \forall\ x\in[-b,b]\setminus\{0\}. \tag{2.13}$$

We consider now the mappings

$$W_1(x,y)=\begin{cases}\frac{y^{k+1}}{k+1}-\sigma^k(x)y+\frac{k}{k+1}\sigma^{k+1}(x); & x\in[-b,b],\ y\in\mathbb{R}\\ 0 & \text{, otherewise}\end{cases} \tag{2.14a}$$

$$\Phi_1(x,y)=\frac{1}{m+1}x^{m+1}+W_1(x,y)\ ;$$

$$
u_0(x,y)=\begin{cases} -y^k, & x=0 \\ \dfrac{f(x,y)D\sigma^k(x)-x^m(\partial f/\partial y)(x,y)}{k\sigma^{k-1}(x)}, & x\in[-b,]\setminus\{0\}, y=\sigma(x) \\ \dfrac{y-\sigma(x)}{y^k-\sigma^k(x)}fD\sigma^k-(y^k-\sigma^k)-x\dfrac{f(x,y)-f(x,\sigma(x))}{x^k-\sigma^k(x)}, & x\in[-b,b]\setminus\{0\}, \\ & y\neq\sigma(x) \end{cases} \tag{2.14b}
$$

m being any suitably large odd integer. Then similar to [2] it can be shown that the map $\Phi_1$ is $C^1$, and positive definite on the region $(-b,b)\times\mathbb{R}$. Furthermore by (2.12) it follows that $\sigma(x)=\sigma_1 x^{p_1}$ for $x>0$ and $\sigma(x)=\sigma_2 x^{p_2}$ for $x<0$ near zero. From Lemma 1 in [2] it follows that the function $u_0$ is continuous for $(x,y)\neq 0$, $x\in(-b,b)$ vanishing at zero and also locally asymptotically stabilizes (1.1) at $0\in\mathbb{R}^2$. In particular

$$
\left(f\frac{\partial\Phi_1}{\partial x}+u_0\frac{\partial\Phi_1}{\partial y}\right)(x,y)<0,\ (x,y)\neq 0,\ x\in(-b,b) \tag{2.15}
$$

and the following condition holds:

$$
\frac{\partial\Phi_1}{\partial y}(x,y)=\frac{\partial W_1}{\partial y}(x,y)=0,\ (x,y)\neq 0,\ x\in(-b,b) \Rightarrow \left(f\frac{\partial\Phi_1}{\partial x}\right)(x,y)<0 \tag{2.16}
$$

Notice also that

$$
\frac{\partial W_1}{\partial y}(x,y)=0,\ x\in(-b,b) \Leftrightarrow y=\sigma(x); \tag{2.17}
$$

$$
\frac{\partial W_1}{\partial y}(x,y)=0,\ x\in(-b,b) \Rightarrow W_1(x,y)=\frac{\partial W_1}{\partial x}(x,y)=0. \tag{2.18}
$$

Next we proceed to the construction of a global clf guaranteeing global stabilization. Invoking Assumption 1.1 and following exactly the same procedure with that of [3, Theorem 1] we can establish the existence of a nonnegative $C^1$ mapping $W_2:\mathbb{R}^2\to\mathbb{R}$ such that $W_2(x,y)=0$ for all $(x,y)$ belonging to the region $R\doteq\{(x,y)\in\mathbb{R}^2: x\in(-b/2,+b/2)\}$, whereas

$$
\Phi_2(x,y)\doteq\frac{1}{m+1}x^{m+1}+W_2(x,y)>0,\ \forall\ (x,y)\in\mathbb{R}^2\setminus R.
$$

Furthermore the following properties are satisfied:

$$
\frac{\partial\Phi_2}{\partial y}(x,y)=\frac{\partial W_2}{\partial y}(x,y)=0,\ (x,y)\in\mathbb{R}^2\setminus R,\ |x|>b/2
$$

$$\Rightarrow (x,y)\in\hat{M} \; ; \; \left(\frac{\partial\Phi_2}{\partial x}f\right)(x,y)<0; \tag{2.19}$$

in particular

$$\frac{\partial W_2}{\partial y}(x,y)=0, \; (x,y)\in\mathbb{R}^2\setminus R, \; x\in I\div[-b,-b/2)\cup(b/2,b]$$

$$\Leftrightarrow y=\sigma(x), \; x\in I; \tag{2.20}$$

$$\frac{\partial W_2}{\partial y}(x,y)=0, \; x\in[-b,b] \Rightarrow W_2(x,y)=\frac{\partial W_2}{\partial x}(x,y)=0. \tag{2.21}$$

Consider finally a $C^1$ map $k:\mathbb{R}\to\mathbb{R}^+$ such that $k(x)=1$ for $0\le|x|\le b/2$, $k(x)=0$ for $|x|\ge 3b/4$ and $0<k(x)<1$ otherwise. We are now in a position to prove that the map

$$\Phi_3(x,y)=\frac{1}{m+1}x^{m+1}+k(x)W_1(x,y)+W_2(x,y)$$

is a clf with respect to (1.1). Obviously, $\Phi_3$ is $C^1$ and positive definite on $\mathbb{R}^2$. Next we establish the implication

$$\frac{\partial\Phi_3}{\partial y}(x,y)=0, \; (x,y)\neq 0 \Rightarrow \left(\frac{\partial\Phi_3}{\partial y}f\right)(x,y)<0 \tag{2.22}$$

Indeed, for each nonzero $(x,y)$ with $x\in[-b/2,b/2]$ and $(\partial\Phi_3/\partial y)(x,y)=0$ we have $k(x)=1$, $W_2(x,y)=0$, $(\partial W_1/\partial y)(x,y)=0$, $\Phi_3(x,y)=\Phi_1(x,y)$ and so by (2.16) we obtain $((\partial\Phi_3/\partial y)f)(x,y)<0$. For $(x,y)$ with $|x|>3b/4$ and $(\partial\Phi_3/\partial y)(x,y)=0$ it follows $k(x)=0$, $(\partial W_2/\partial y)(x,y)=0$, $\Phi_3(x,y)=\Phi_2(x,y)$ and so by (2.19) we get $x\in\hat{M}$ and $((\partial\Phi_3/\partial y)f)(x,y)<0$. Finally, for each $(x,y)$ with $x\in I'\div\left(-\frac{b}{2},-\frac{3}{4}b\right]\cup\left[\frac{3}{4}b,\frac{b}{2}\right)$ and

$$\frac{\partial\Phi_3}{\partial y}(x,y)=\frac{\partial W_1}{\partial y}(x,y)k(x)+\frac{\partial W_2}{\partial y}(x,y)=0$$

it follows from (2.17), (2.20) and the fact that $k$ is strictly positive on the region $I'$ that $(\partial W_1/\partial y)(x,y)=(\partial W_2/\partial y)(x,y)=0$ and $y=\sigma(x)$ $(x\neq 0)$. Therefore by (2.13), (2.18) and (2.21) we obtain $W_1(x,\sigma(x))=(\partial W_1/\partial x)(x,\sigma(x))=(\partial W_2/\partial x)(x,\sigma(x))=0$, therefore

$$\left(\frac{\partial\Phi_3}{\partial y}f\right)(x,y)/_{y=\sigma(x)}=x^m f(x,\sigma(x))+\frac{dk}{dx}(x)W_1(x,\sigma(x))$$

$$+\,k(x)\left(\frac{\partial W_1}{\partial x}f\right)(x,\sigma(x))+\left(\frac{\partial W_2}{\partial x}f\right)(x,\sigma(x))=x^m f(x,\sigma(x))<0.$$

hence condition (2.22) is fulfilled and $\Phi_3$ is a clf with respect to (1.1). Moreover $\Phi_3(x_v,y_v)\to+\infty$ for any sequence $\{(x_v,y_v)\in\mathbb{R}^2\}$ with $|x_v|\to+\infty$, however $\Phi_3$ fails in general to be uniformly unbounded on $\mathbb{R}^2$. In order to complete the proof we can construct similar to [3] a $C^1$ nonnegative map $W_3(x,y)$ which vanishes in a closed neighborhood S of the set $\hat{M}$ and satisfies

$$\frac{\partial(\Phi_3+W_3)}{\partial y}(x,y)\neq 0,\ \forall\ (x,y)\in\mathbb{R}^2\setminus S\ ; \tag{2.23}$$

whereas for any sequence $\{(x_v,y_v)\in\mathbb{R}^2\}$, with $x_v$ being bounded and $|y_v|\to+\infty$, it holds that $W_3(x_v,y_v)\to+\infty$. Then we can easily establish that the map

$$\Phi_4(x,y)\doteq\Phi_3(x,y)+W_3(x,y)=\frac{1}{m+1}x^{m+1}+k(x)W_1(x,y)+W_2(x,y)+W_3(x,y)$$

is also a clf with respect to (1.1) (namely (2.22) holds with $\Phi_4$ instead of $\Phi_3$) and further it is uniformly unbounded on $\mathbb{R}^2$. Notice finallly that $\Phi_4(x,y)=\Phi_1(x,y)=(\frac{1}{m+1})x^{m+1}+W_1(x,y)$ near $0\in\mathbb{R}^2$ and using (2.15) and the facts that $\Phi_4$ is a uniformly unbounded clf with respect to (1.1), and $u_0$ is continuous near zero we conclude that the assumptions of Lemma 2.1 are satisfied for the system (1.1); in particular (2.2) and (2.3) hold with $\Phi=\Phi_4$ and $u_0$ as defined in (2.14). Hence (1.1) is G.A.S. by means of a feedback law $u=u(x,y)$ which is $C^0$ on the whole state space $\mathbb{R}^2$. In particular, $u(x,y)=u_0(x,y)$ for $(x,y)$ near zero. ∎

## References


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