Estimation theory for graph linear systems: applications to 1D and multi-D filtering and smoothing

Extended Abstract

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1 Introduction

In this paper we consider the estimation problem for a set of vectors $X = \{x_i\}$, $x_i \in \mathrm{R}^{n_i}$, based on all or part of observations:

$$o_k : \quad z_k = \mathcal{L}_k(X) + G_k u_k \tag{1}$$

where u_k's are zero-mean, independent Gaussian vector with covariance I and $\mathcal{L}_k$'s linear "local" operators, i.e.,

$$\mathcal{L}_k(X) = \sum_{j=1}^{J_k} A_{ki} x_{i_{kj}}. \tag{2}$$

Unknown vectors $x_{i_{jk}}$, $1 \leq j \leq J$, are in some sense "close". This structure is best illustrated by a graph which we shall call an xo-graph. An xo-graph is a graph with x's and o's as nodes and relations (1) as arcs. For each k, (1) is coded in the xo-graph by J_k arcs connecting z_k to the $x_{i_{jk}}$'s. An obvious property of the xo-graph is that to go from an x to another x, we must go through an o and vice versa. Graphs having this property are called bipartite.

Let us illustrate all of this with an example:

Example 1 Consider the following estimation problem:

$$X = \{x_1, x_2, x_3, x_4, x_5\}$$

$$O = \{o_1, o_2, o_3, o_4\}$$

with

$$o_1 \qquad z_1 = A_{11}x_1 + A_{12}x_2 + G_1 u_1 \tag{3}$$
$$o_2 \qquad z_2 = A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + G_2 u_2 \tag{4}$$
$$o_3 \qquad z_3 = A_{33}x_3 + A_{34}x_4 + G_3 u_3 \tag{5}$$
$$o_4 \qquad z_4 = A_{44}x_4 + A_{45}x_5 + G_4 u_4 \tag{6}$$

The xo-graph associated with this problem is illustrated in Figure 1.

Note that by putting on each arc of the xo-graph the associated A matrix, and in each node o, the value of the corresponding observation z and the corresponding G matrix, we can completely code the estimation problem in the xo-graph. In this way, we establish a complete equivalence between the estimation problem and the xo-graph.

Example 2 The Fornasini-Marchesini model [1] is a first order model for 2D systems. The estimation problem for this 2D model can also be put into an xo-graph.

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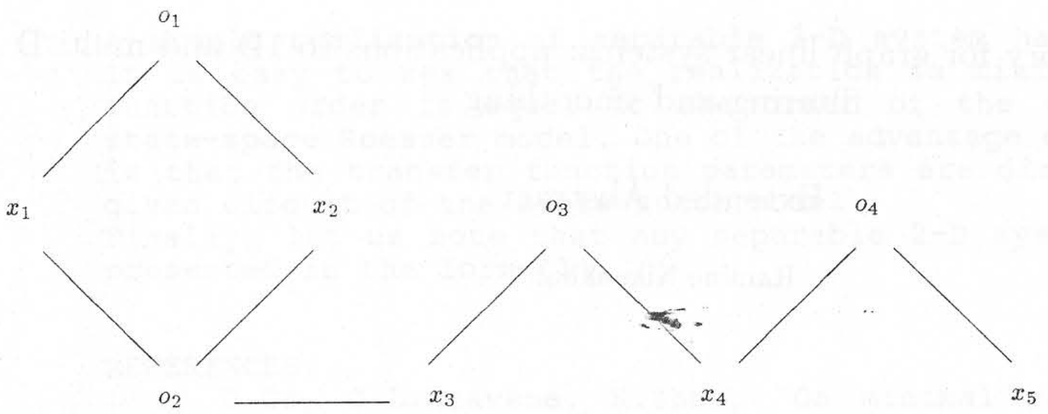

Figure 1:

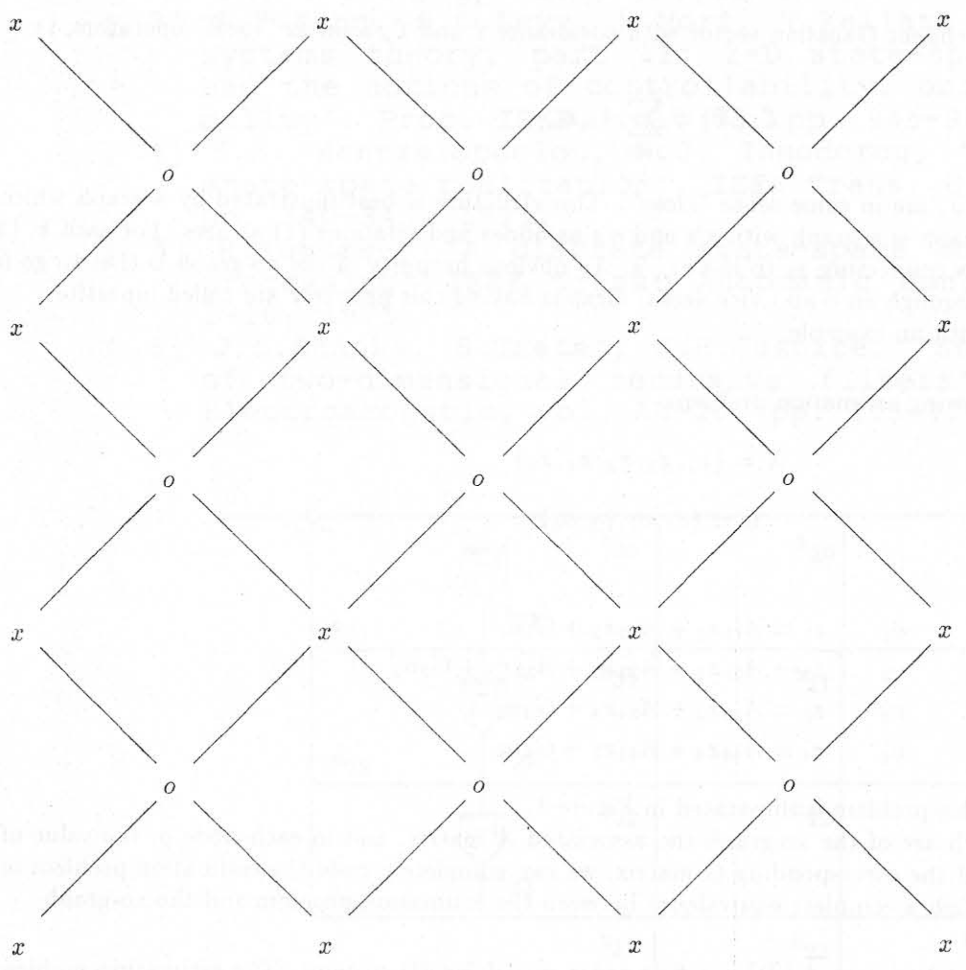

Figure 2:

2 xo-graph

There are two basic operations on the xo-graph which allow us to reduce it to an acyclic xo-graph.

2.1 x-aggregation

This operation consists in putting together x's to form larger x's. The effect on the A matrices is straightforward. Let us examine it on Example 1 by aggregating x_1, x_2 and x_3; let

$$x_{123} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Then the observations become

$$\begin{aligned}
o_1 \quad & z_1 = [A_{11}\ A_{12}\ 0]x_{123} + G_1 u_1 && (7)\\
o_2 \quad & z_2 = [A_{21}\ A_{22}\ A_{23}]x_{123} + G_2 u_2 && (8)\\
o_3 \quad & z_3 = [0\ 0\ A_{33}]x_{123} + A_{34}x_4 + G_3 u_3 && (9)\\
o_4 \quad & z_4 = A_{44}x_4 + A_{45}x_5 + G_4 u_4 && (10)
\end{aligned}$$

and the new xo-graph is:

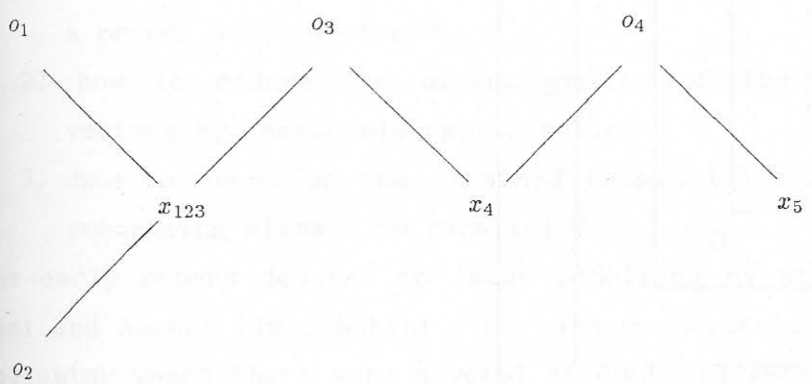

Figure 3:

2.2 o-aggregation

This operation is similar to x-aggregation except that in this case we put together observations o's to form one observation; this is done by stacking up the associated z vectors. Consider again the xo-graph in Figure 1 and let us examine aggregating o_1 and o_2. By defining

$$z_{12} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

in the problem in Figure 1, we replace o_1 and o_2 by o_{12}:

$$o_{12}: \ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} A_{11} \\ A_{31} \end{pmatrix} x_1 + \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ A_{23} \end{pmatrix} x_3 + \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \qquad (11)$$

2.3 xo-graph reduction

With the aggregation operations introduced in the previous sections, it is possible to reduce any xo-graph into an acyclic xo-graph. The obvious way is to aggregate all x's and all o's. This solution is of course not very interesting because it destroys all structure. The idea here is to do the reduction by doing the least number of aggregation operations.

For the problem in Example 2, a method for aggregation is illustrated in Figure 4:

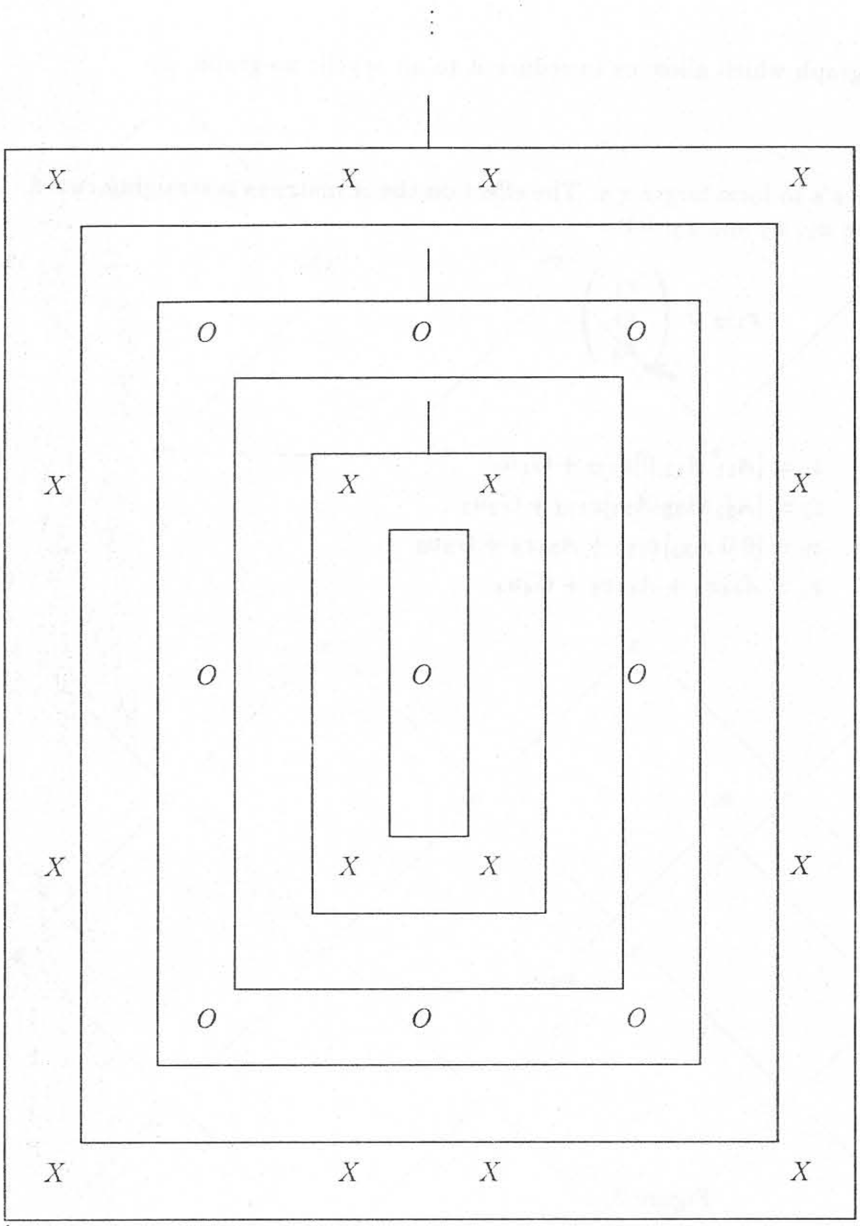

Figure 4:

3 Acyclic xo-graph

We have seen that every xo-graph can be reduced to an acyclic graph. That is why in this paper we concentrate on estimation of acyclic xo-graphs.

References

[1] E. Fornasini and G. Marchesini, "State-space realization theory of two-dimensional filters", *IEEE Trans. Automat. Control*, vol. 21, pp. 484-491, 1976.

[2] R. Nikoukhah, A.S. Willsky, and B.C. Levy, "Kalman filtering and Riccati equations for descriptor systems", *IEEE Trans. Automat. Control*, vol. 37, Sept. 1992.

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[4] L. Chisci and G. Zappa, "Square-root Kalman filtering of descriptor systems", preprint, 1992.