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# NONLINEAR TRACKING PROBLEMS BY A SLIDING MANIFOLD APPROACH

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## EXTENDED ABSTRACT


Variable structure control systems are widely used for the control of nonlinear plants. The basic idea is the following. Given a nonlinear system

$$\dot{x}=f(x,u,t),\ x\in\mathbb{R}^n,\ u\in\mathbb{R}^m,\ m\le n. \tag{1}$$

Consider a sliding manifold

$$\mathcal{S}=\left\{x\in\mathbb{R}^n : s(x)=0\right\} \tag{2}$$

where the function $s:\mathbb{R}^n\to\mathbb{R}^m$ is a continuously differentiable function. Given a time interval $[0,T]$ with $0<T\le\infty$, we want to steer and then hold the state vector $x$ on the sliding manifold $\mathcal{S}$ by using feedback control laws $u=u(x,t)\in U\subseteq\mathbb{R}^m$ which are discontinuous along the surfaces

$$s_j(x)=0,\ j=1,2,\dots,m. \tag{3}$$

Such control systems have very good properties; they exhibit stable behaviour, accurate tracking, robust performance, and insensitivity with respect to disturbances and variation of plant parameters which satisfy some matching restrictions. The main drawbacks of these systems are the chattering phenomenon and the necessity of considering a generalized notion of solution of the nonlinear controlled system, usually that due to Filippov instead of the classical solution concept of Caratheodory. Indeed, extra work and extra conditions on $f$ are required when we have to relate the Caratheodory conditions corresponding to the *equivalent control* (which is a Caratheodory function) to the Filippov solutions which lie on the sliding manifold $\mathcal{S}$ and correspond to discontinuous feedback controls.

In this paper we introduce a different approach to the control problem (1)-(2). This approach is based on the consideration of a different class of feedback controls which we will introduce via the theory of singularly perturbed ordinary differential equations. Such controls will depend on a small parameter $\varepsilon>0$ whose corresponding states will not realize, in general, exactly the sliding condition (2). However, for any prescribed neighbourhood of $\mathcal{S}$, we can determine values of the parameter $\varepsilon$ for which the corresponding trajectories of (1) belong to that neighbourhood. Furthermore, by using this class of controls, we eliminate both chattering and the necessity of considering Filippov solutions.

An important class of nonlinear systems for which the control technique proposed in the paper apply is that of nonlinear plants affine in control variables, i.e.

$$\dot{x} = A(x)x + B(x)u \tag{4}$$

and among these the attitude control systems for rigid robotic manipulators or satellite where the input matrix $\mathbf{B}(\mathbf{x})$ can be written as follows

$$\mathbf{B}(\mathbf{x}) = \begin{pmatrix} 0 \\ \\ \overline{\mathbf{B}}(\mathbf{x}) \end{pmatrix} \tag{5}$$

where $\overline{\mathbf{B}}(\mathbf{x})$ is the inverse of the inertia matrix of the mechanical system, and it is positive definite $\forall x \in \mathbb{R}^n$.
Under the hypothesis that the two maps $\mathbf{x} \to \mathbf{A}(\mathbf{x})$ and $\mathbf{x} \to \mathbf{B}(\mathbf{x})$ satisfy a Lipschitz condition on $\mathbb{R}^n$, consider a continuously differentiable map $s: \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^m$, and define the related sliding manifold $\mathcal{S}$ as follows

$$\mathcal{S} = \left\{ (x,t) \in \mathbb{R}^n \times [0,\infty) \ : \ s(x,t) = 0 \right\} . \tag{6}$$

Under weak hypothesis on $\mathcal{S}$, for any $\varepsilon > 0$, consider the following system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u} \tag{7}$$

$$\varepsilon \dot{\mathbf{u}} = \mathbf{g}(\mathbf{x},\mathbf{u},\mathbf{t}) \tag{8}$$

where

$$\mathbf{g}(\mathbf{x},\mathbf{u},\mathbf{t}) = \frac{\partial}{\partial t} \mathbf{s}(\mathbf{x},\mathbf{t}) + \frac{\partial}{\partial \mathbf{x}} \mathbf{s}(\mathbf{x},\mathbf{t}) \Big( \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u} \Big) . \tag{9}$$

**Definition 1.** For any $(\mathbf{x},\mathbf{t}) \in \mathcal{I}$, where $\mathcal{I}$ is a neighbourhood of the manifold $\mathcal{S}$ such that, for any $(\mathbf{x},\mathbf{t}) \in \mathcal{I}$, the map $\mathbf{u} \to \mathbf{g}(\mathbf{x},\mathbf{u},\mathbf{t})$ is one-to-one on $\mathbb{R}^m$ and its range contains zero, the unique solution $\mathbf{u}^*(\mathbf{x},\mathbf{t})$ of the algebraic equation $\mathbf{g}(\mathbf{x},\mathbf{u},\mathbf{t})=0$ is called the *equivalent control* for the problem

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u} \\ \mathbf{s}(\mathbf{x},\mathbf{t}) &= 0 . \end{aligned} \tag{10}$$

If the equilibrium point $\overline{\mathbf{u}}^* = \mathbf{u}^*(\overline{\mathbf{x}},\overline{\mathbf{t}})$ of the equation (8) is asymptotically uniformly stable, for any $(\overline{\mathbf{x}},\overline{\mathbf{t}}) \in \mathcal{I}$. Then the classical result of the singular perturbation theory apply, and if the pair $(\tilde{\mathbf{x}}(\mathbf{t}),\tilde{\mathbf{u}}(\mathbf{t}))$ is the solution of the reduced system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u} \ ; \ \mathbf{x}(0) = \mathbf{x}_0; \\ 0 &= \mathbf{g}(\mathbf{x},\mathbf{u},\mathbf{t}), \end{aligned} \tag{11}$$

then the solution $(\mathbf{x}(\mathbf{t},\varepsilon),\mathbf{u}(\mathbf{t},\varepsilon))$ of the Cauchy problem (7)-(8) with

$x(0)=x_o$ and $u(0)=u_o$ has the following properties:

$$\lim_{\varepsilon\to 0} x(t,\varepsilon) = \tilde{x}(t) \quad \text{uniformly in } [0,\infty)$$
$$\lim_{\varepsilon\to 0} u(t,\varepsilon) = \tilde{u}(t) \quad \text{uniformly in } [t_1,\infty), \tag{12}$$

for any $t_1>0$.

In the paper the theory sketched above will be applied to the tracking problem of a mechanical manipulator. By means of a suitable definition of the sliding manifold for which the equivalent control is well defined, we design a simple PD controller by only using the property of positive definiteness of the inertia matrix and that of the logarithmic norm of a matrix.

The approach is different from that of high gain feedback systems, because fast modes occur only in the control dynamics, and the control signal tends to a neighbourhood of the equivalent control with fast non-oscillating modes and remains within this neighbourhood in the uniform topology for any time. The uniform asymptotic stability of the control scheme will be proved also in the presence of actuator dynamics. The closed-loop system has the approximability property which allows us to state that it is robust with respect to a given class of perturbations on the state variable measurements.