MINIMAL REALIZATIONS OF THE INVERSE OF A POLYNOMIAL MATRIX USING FINITE AND INFINITE JORDAN PAIRS

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KEYWORDS : Finite and infinite Jordan pairs , minimal realizations , generalized state—space systems, linear multivariable systems , transfer function matrices .

Abstract

It is rather obvious that the connections between control theory and linear algebra are very strong . Several formulas and notions , as well as , known techniques from matrix theory and theory of operators are used efficiently in control theory . The important treatise of [4] gives a nice example of how matrix theory can be applied to the analysis and solution—finding of several difficult problems in control theory . On the other hand Gohberg and other researchers [5] presented their work on operator polynomial and general operator—valued functions , and pointed out the striking similarities among them and formulas and notions in control theory making the observation that "... from the systems theory point of view , we study here systems for which the transfer function matrix is the inverse of a polynomial matrix " [5, page 7] . In this paper we present a simple method which uses the notions of finite and infinite Jordan pairs from the operator theory in such a way to find a minimal realization of the inverse of a polynomial matrix . The notions of finite and infinite Jordan pairs were found originally in [5] and are based on the notions of finite and infinite Jordan chains [4],[5] . Our analysis is based on the theory presented in recent papers [3],[9],[10] , where simple and efficient methods of finding finite and infinite Jordan chains —and as a consequence Jordan pairs— using the notions of finite and infinite elementary divisors , are given . Specifically we prove the following :

Proposition 1 . Let $A(s) \in \mathbb{R}^{r\times r}[s]$ be a polynomial matrix . Let also the finite Jordan pair $C_f \in \mathbb{R}^{r\times n}$, $J_f \in \mathbb{R}^{n\times n}$, and the matrix $B_f \in \mathbb{R}^{n\times r}$ with $n = \deg|A(s)|$,such that $H_{spr}(s) = C_f[sI_n - J_f]^{-1}B_f$ where $H_{spr}(s)$ is the strictly proper part of $A(s)^{-1}$. Then the triple (C_f, J_f, B_f) is a minimal realization for $H_{spr}(s)$. □

Proposition 2 Let $A(s) \in \mathbb{R}^{r\times r}[s]$ be a polynomial matrix . Let also the infinite Jordan pair (C_∞, J_∞) of $A(s)$,with $C_\infty \in \mathbb{R}^{r\times\mu}$, $J_\infty \in \mathbb{R}^{\mu\times\mu}$, where μ is given by $\mu = (r-1)q_1 - \sum_{i=2}^{k} q_i + \sum_{j=k+1}^{r} \hat{q}_j$ where q_i, $i = 1,\dots,k$ and $\hat{q}_j$ $j = k+1,\dots,r$ denote the orders of the poles and zeros at $s = \infty$ of $A(s)$ respectively and $B_\infty \in \mathbb{R}^{\mu\times r}$. Let also $A(s)^{-1} = H_{spr}(s) + H_{pol}(s)$. Then (i) the triple of matrices $(C_\infty, J_\infty, B_\infty)$ is a realization for the polynomial part $H_{pol}(s)$ of $A(s)^{-1}$. (ii) From $(C_\infty, J_\infty, B_\infty)$ we can find a triple of matrices $(\tilde{C}_\infty, \tilde{J}_\infty, \tilde{B}_\infty)$, with $\tilde{C}_\infty \in \mathbb{R}^{r\times\tilde{\mu}}$, $\tilde{J}_\infty \in \mathbb{R}^{\tilde{\mu}\times\tilde{\mu}}$, $\tilde{B}_\infty \in \mathbb{R}^{\tilde{\mu}\times r}$ where $\tilde{\mu}$ is given by $\tilde{\mu} = (r-k) + \sum_{j=k+1}^{r} \hat{q}_j$ Clearly $\tilde{\mu} \leq \mu$ and the triple $(\tilde{C}_\infty, \tilde{J}_\infty, \tilde{B}_\infty)$ constitutes <u>a minimal realization</u> of the polynomial part $H_{pol}(s)$ of $A(s)^{-1}$,i.e. $H_{pol}(s) = \tilde{C}_\infty[s\tilde{J}_\infty - I_{\tilde{\mu}}]^{-1}\tilde{B}_\infty$ and $\mu = \delta_M[1/w\, A(1/w)]$ □

The proposed method can be applied to the so—called realization theory of transfer function matrices of Linear Multivariable Systems [6] , i.e. physical systems of the form $(\Sigma) : A(\rho)\beta(t) = B(\rho)u(t)$ $y(t) = C(\rho)\beta(t)$,where $\rho := d/dt$ is the differential operator , $A(\rho)$, $B(\rho)$, $C(\rho)$ are polynomial matrices and $\beta(t)$, $y(t)$, $u(t)$ are respectively the pseudostate , the output and the input vectors of the system (Σ) . The transfer function matrix of (Σ) is (in frequency—domain) : $G(s) = C(s)A(s)^{-1}B(s)$ which is a

rational matrix(not necessarily proper) in general . It would be interesting to find certain singular systems in generalized state–space form [1],i.e. physical systems of the form (Σ_1) : $E\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ —where E,A,B,C are constant matrices with appropriate dimensions and $x(t),y(t),u(t)$ are respectively the generalized state,the output and the input vectors of the system (Σ_1) — which give rise to the transfer function matrix $G(s)$. In other words the transfer function matrix of system (Σ_1) which is given by : $G_1(s) = C[sE-A]^{-1}B$ satisfies the following condition : $G_1(s) = C[sE-A]^{-1}B = C(s)A(s)^{-1}B(s) = G(s)$.

Definition 1 [2] Assume that $G_1(s) \in \mathbb{R}^{r\times r}(s)$ is a rational matrix . If there exists a quadruple of matrices (E,A,B,C) such that : $G_1(s) = C[sE-A]^{-1}B$ —where $E,A \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $B \in \mathbb{R}^{\hat{n}\times r}$, $C \in \mathbb{R}^{r\times\hat{n}}$ are constant matrices with $\hat{n} \in N - \{0\}$ — , then the generalized state–space system described by (Σ_1) will be called a singular system realization of $G_1(s)$, or simply a realization of $G_1(s)$. Furthermore the system (Σ_1) is called a minimal realization of $G_1(s)$ iff any other realization of $G_1(s)$ has order greater than $\hat{n}$, or equivalently iff the generalized state–space system (Σ_1) has the least number of generalized states $(x(t))$.

Any rational matrix G(s) (not necessarily proper) may be represented as the sum of its strictly proper part $H_{spr}(s)$ and its polynomial part $H_{pol}(s)$, i.e. $G(s) = H_{spr}(s) + H_{pol}(s)$ We know that the inverse of a polynomial matrix $F(s) \in \mathbb{R}^{r\times r}[s]$ is a rational matrix in general . If we now consider the case where $F(s)^{-1} := G(s)$ then the proposed method finds a minimal realization — as this defined in definition 1— of a transfer function matrix (G(s)) of a system (Σ) which has the property its inverse to be a polynomial matrix . In other words we can find a quadruple of matrices [E,A,B,C] which have the following properties : (i) give rise the the generalized state–space system (Σ_1) , and (ii) $G(s) = C[sE-A]^{-1}B$.

To be more precise let a system (Σ) which give rise to a transfer function matrix $G(s) = C(s)A(s)^{-1}B(s) \in \mathbb{R}^{r\times r}(s)$, and assume that G(s) has the following property $G(s)^{-1} := F(s) \in \mathbb{R}^{r\times r}[s]$. Now proposition 1 states that we can find a triple of matrices $C_f \in \mathbb{R}^{r\times n}$, $J_f \in \mathbb{R}^{n\times n}$, $B_f \in \mathbb{R}^{n\times r}$ with $n = \deg|F(s)|$, such that : $H_{spr}(s) = C_f[sI_n - J_f]^{-1}B_f$, where $H_{spr}(s)$ is the strictly proper part of $G(s) = F(s)^{-1}$ and the triple (C_f, J_f, B_f) is a minimal realization of $H_{spr}(s)$. Also proposition 2 states that we can find a triple of matrices $(\tilde{C}_\infty, \tilde{J}_\infty, \tilde{B}_\infty)$, with $\tilde{C}_\infty \in \mathbb{R}^{r\times\tilde{\mu}}$, $\tilde{J}_\infty \in \mathbb{R}^{\tilde{\mu}\times\tilde{\mu}}$, $\tilde{B}_\infty \in \mathbb{R}^{\tilde{\mu}\times r}$ with $\tilde{\mu} = (r-k) + \sum_{j=k+1}^{r} \hat{q}_j$ such that $H_{pol}(s) = \tilde{C}_\infty[s\tilde{J}_\infty - I_{\tilde{\mu}}]^{-1}\tilde{B}_\infty$, where $H_{pol}(s)$ is the polynomial part of $G(s) = F(s)^{-1}$ and the triple $(\tilde{C}_\infty, \tilde{J}_\infty, \tilde{B}_\infty)$ is a minimal realization of $H_{pol}(s)$. Let now define $E := \begin{bmatrix} I_n & 0 \\ 0 & \tilde{J}_\infty \end{bmatrix}$, $A := \begin{bmatrix} J_f & 0 \\ 0 & I_{\tilde{\mu}} \end{bmatrix}$ $B := \begin{bmatrix} B_f \\ \tilde{B}_\infty \end{bmatrix}$, $C := \begin{bmatrix} C_f \ , \ \tilde{C}_\infty \end{bmatrix}$. We can now define the generalized state–space system (Σ_1) with E,A,B,C as above . It is easy to verify that : $G(s) = H_{spr}(s) + H_{pol}(s) = C[sE-A]^{-1}B$

Hence the system (Σ_1) determined by the matrices [E,A,B,C] is a realization of G(s) . Furthermore we can prove easily that the above realization is also a minimal one .

Definition 2 The order $\hat{n}$ of the minimal realization of the transfer function matrix G(s) of (Σ) is called the minimum generalized order of G(s) . Furthermore $\hat{n}$ is the dimension of the generalized state–space system (Σ_1) and is equal to : $\hat{n} = n + \tilde{\mu} = \deg|F(s)| + (r-k) + \sum_{j=k+1}^{r} \hat{q}_j$, where $\hat{q}_j$, $j = k+1,\dots,r$ denote the orders of the zeros at $s = \infty$ of the polynomial matrix F(s) which can be found using the Smith–McMillan form at $s = \infty$ [8] . We can now state the following .

Theorem 1. Let a linear multivariable system (Σ) which give rise to a rational transfer function matrix $G(s) \in \mathbb{R}^{r\times r}(s)$ and has the property to have a polynomial inverse $F(s) \in \mathbb{R}^{r\times r}[s]$. Then we can find a generalized state–space system of the form (Σ_1) and minimum generalized order $\hat{n}$, such that the system (Σ_1) to be a minimal realization of the rational matrix G(s) (according to definition 1) . Furthermore since the two systems (Σ) and (Σ_1) give rise to the same transfer function matrix G(s) they have the same sets of finite and infinite transmission poles and zeros ([6],[7]) . □

We remark here that the problem of transforming a linear multivariable system (Σ) to a generalized state space system (Σ_1) is called linearization and has been considered by many researchers . In our paper we study a special case of linearization ; that is linearization for the class of transfer function matrices G(s) of systems (Σ) with the property of having a polynomial inverse , i.e.:

$$\mathcal{V}=\{G(s) \in \mathbb{R}^{r\times r}(s) \;/\; G(s)^{-1} \in \mathbb{R}^{r\times r}[s]\}$$

REFERENCES

[1] Verghese ,G.,B.C. Levy and T. Kailath , (1981) , "A generalised state–space for singular systems" , IEEE Trans. Autom.Control , Vol.Ac–26, No.4,pp.811–831 .

[2] Dai, L. (1989) . Singular Control Systems , Springer–Verlag , Berlin

[3] Fragulis, G.,F. (1990) , "Analysis of Generalized Singular Systems" , Ph. D. dissertation , Aristotle Univ. of Thessaloniki , Thessaloniki , Greece .

[4] Gantmacher, F.,R., (1959) . Theory of matrices , Chelsea , New York .

[5] Gohberg, I., Lancaster, P., and Rodman, L., (1982). Matrix Polynomials, Academic Press, New York, London.

[6] Rosenbrock, H.H. (1970) . State–space and Multivariable theory . Wiley , New York .

[7] Van der Weiden, A.,J.,J. (1983) "The use of structural properties in linear multivariable control system design" , Ph. D. Thesis,Delf University of Technology , The Netherlands .

[8] Vardulakis, A.I.G. , Limebeer, D. and Karcanias, N. , (1982) , "Structure and Smith–McMillan form of a rational matrix at infinity" , Int. J. Control , Vol. 35 , No. 4 , p. 701 .

[9] Vardulakis, A.I.G. , (1991) . Linear Multivariable Control : Algebraic Analysis and Synthesis Methods , Wiley .

[10] Vardulakis, A.I.G., and Fragulis, G., (1989). "Infinite elementary divisors of polynomial matrices and impulsive solutions of linear homogeneous matrix differential equations", Circuits, Systems & Signal Process. Vol.8, No. 3,357–373.