

A CLOSED FORMULA FOR THE DETERMINATION OF THE IMPULSIVE SOLUTIONS OF LINEAR HOMOGENEOUS MATRIX DIFFERENTIAL EQUATIONS

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Abstract

In the recent years there has been a growing interest in the system-theoretic problems of generalized state space or singular systems due to the extensive applications of this kind of systems in large-scale, singular perturbation theory, circuits, robotics, economics, demography, control theory and other areas. Generalized state space systems are systems described by:

$$(1) \quad E\dot{x}(t) = Ax(t) + Bu(t)$$

where $E \in \mathbb{R}^{r \times r}$, $\text{rank}_{\mathbb{R}} E \leq r$, $A \in \mathbb{R}^{r \times r}$, $B \in \mathbb{R}^{r \times m}$ whereas (regular) state space systems are the systems described by:

$$(2) \quad \dot{x}(t) = Ax(t) + Bu(t)$$

which represent a particular case of (1) with $E = I_r$, the identity matrix. Regular state space systems were studied explicitly in the past decades [see e.g. [10]]. On the other hand many researchers in the recent years explored special properties of generalized state space systems and found many connections between them and regular. Generalized state space systems were studied in the frequency domain [see [11], [14]–[16]] as well as in time-domain [see [2]–[6], [8]–[9]]. Generalized state space systems as (1) represent a particular case of Generalized Singular Systems (G.S.S.) i.e. physical systems whose dynamics can be described by a linear matrix differential equation having the form:

$$(3) \quad A(\rho)\beta(t) = B(\rho)u(t)$$

where $\rho := d/dt$ the differential operator, $A(\rho) = \sum_{i=0}^q A_i \rho^i \in \mathbb{R}^{r \times r}[\rho]$, $A_i \in \mathbb{R}^{r \times r}$, $i = 0, 1, 2, \dots, q$, $q \geq 1$

with $\text{rank}_{\mathbb{R}} A_{q_1} < r$, $B(\rho) = \sum_{j=0}^{\sigma} B_j \rho^j \in \mathbb{R}^{r \times m}[\rho]$, $B_j \in \mathbb{R}^{r \times m}$, $j = 0, 1, 2, \dots, \sigma$, $\sigma \geq 0$, $\beta(t)$ the pseudo-state

of the system and $u(t)$ the control input to the system. In a recent paper [13] various known results regarding the smooth and impulsive solutions of homogeneous generalized state space systems have been translated to the more general case of G.S.S. A new treatment of G.S.S. using Algebraic methods is given in [12]. Among other topics the author gives a formula for the determination of the impulsive solutions of L.H.M.D.E. in terms of the notions of fast and slow states at $t = 0^-$ which are introduced. Campbell in [2]–[3] has found the solution of systems of the form (1) using the notion of Drazin inverse. Also in [4]–[6] Cobb used a different approach which utilizes the Weierstrass canonical form. Comparing those methods to ours we remark the following: Each method has its advantages and disadvantages for different problems. The main advantage of our method is that it treats systems of the form (3) which seem to be generalizations of system of the form (1). The possible disadvantage is that the proposed solution is associated with the evaluation of several matrices which are derived by means of some previously established algorithms. On the other hand the approach based on the canonical decomposition which separates the system behavior in the finite and infinite frequencies provides deep insight into the structure of singular systems in systems analysis. However, it does not always provide a useful framework for actual computation. The canonical form also use a change of the internal variable which is inconvenient in practical situations since the original variables are chosen to have their own significance. The main drawback of the method proposed by Campbell is the computational complexity of the proposed algorithm. Our approach differs from that given in [12] because we give a formula such that the set of the Dirac impulses and its derivatives which are used in the determination of the impulsive solutions of L.H.M.D.E., as well as, the set of finite and infinite spectral data of the associated polynomial matrix are presented in a — easy to use — closed form.

Specifically consider the linear, homogeneous matrix differential equation:

$$(4) \quad A(\rho)\beta(t) = 0 \quad t \geq 0$$

and let $S_{A(s)}^{\infty}(s)$ be the Smith-McMillan form at $s = \infty$ of $A(s) = A_0 + A_1 s + \dots + A_{q_1} s^{q_1}$:

$$(5) \quad S_{A(s)}^{\infty}(s) = \text{diag}[s^{q_1}, s^{q_2}, \dots, s^{q_k}, \frac{1}{s^{\hat{q}_{k+1}}}, \dots, \frac{1}{s^{\hat{q}_r}}] \in \mathbb{R}^{r \times r}(s)$$

and if $A(s)$ has at least one zero at $s = \infty$ then the Laurent expansion of $A(s)^{-1}$ can be written according to [12]:

$$(6) \quad A(s)^{-1} = H_{\hat{q}_r} s^{\hat{q}_r} + H_{\hat{q}_r-1} s^{\hat{q}_r-1} + \dots + H_1 s + H_0 + H_{-1} s^{-1} + H_{-2} s^{-2} + \dots = H_{\text{pol}}(s) + H_{\text{spr}}(s)$$

where $H_{\text{pol}}(s) \in \mathbb{R}^{r \times r}[s]$ is the polynomial part of $A(s)^{-1}$ and $H_{\text{spr}}(s) \in \mathbb{R}^{r \times r}(s)$ is the strictly proper part of $A(s)^{-1}$. Let (C, J) and (C_{∞}, J_{∞}) be a finite and an infinite Jordan pair respectively of the polynomial matrix $A(s)$. Then we can write (see [7]) $A(s)^{-1} = C_{\infty}[sJ_{\infty} - I_{\mu}]^{-1} B_{\infty} + C[sJ - J]^{-1} B$, where B, B_{∞} are constant matrices with appropriate dimensions. Considering now the Laplace transformed equation (4) we obtain:

$$(7) \quad \hat{\beta}(s) = A(s)^{-1} \hat{a}(s) = \hat{\beta}_{\text{pol}}(s) + \hat{\beta}_{\text{spr}}(s) \in \mathbb{R}^{r \times 1}$$

where $\hat{a}(s) \in \mathbb{R}^{r \times 1}[s]$ is the initial condition vector associated with the initial values of $\beta(t)$ and its (q_1-1) -derivatives at $t = 0^-$ i.e. $\beta(0^-), \beta^{(1)}(0^-), \dots, \beta^{(q_1-1)}(0^-)$ given by [1]:

$$(8) \quad \hat{a}(s) = [s^{q_1-1} I_r, s^{q_1-2} I_r, \dots, s I_r, I_r] \times F$$

$$(9) \quad F = \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0^-) \\ \beta^{(1)}(0^-) \\ \vdots \\ \beta^{(q_1-1)}(0^-) \end{bmatrix}$$

where $\hat{\beta}_{\text{pol}}(s), \hat{\beta}_{\text{spr}}(s)$ are respectively the polynomial and the strictly proper part of $\hat{\beta}(s)$. We examine the form of $\hat{\beta}_{\text{pol}}(s)$. Taking into account the identity $A(s)^{-1} A(s) = I_r$ which can be written also:

$$(10) \quad A(s)^{-1} A(s) = [H_{\hat{q}_r} s^{\hat{q}_r} + \dots + H_1 s + H_0 + H_{-1} s^{-1} + H_{-2} s^{-2} + \dots][A_0 + A_1 s + \dots + A_{q_1} s^{q_1}] = I_r$$

In light of (7) and the relations obtained after multiplying the matrices involved in (10) we obtain:

$$(11) \quad \hat{\beta}_{\text{pol}}(s) = L \Delta(s) F$$

where:

$$(12) \quad L = [C \mid -C_{\infty}] \begin{bmatrix} J_{\infty}^{\hat{q}_r-1} & J_{\infty}^{\hat{q}_r-2} & \dots & J_{\infty} I_{\mu} & \mid & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & J_{\infty} I_{\mu} & \mid & I_n & J & \dots & J^{q_1-2} \end{bmatrix} \times \text{block diag}[B_{\infty}, B_{\infty}, \dots, B_{\infty} \mid B, B, \dots, B]$$

$$(13) \Delta(t) = \begin{bmatrix} 0 & 0 \dots 0 & 0 & \dots & 0 & \delta^{(\hat{q}_r-1)}(t)I_r \\ 0 & 0 \dots 0 & 0 & \dots & \delta^{(\hat{q}_r-1)}(t)I_r & \delta^{(\hat{q}_r-2)}(t)I_r \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 \dots 0 & \delta^{(\hat{q}_r-1)}(t)I_r & \dots & \delta^{(1)}(t)I_r & \delta(t)I_r \\ 0 & \dots & \dots & \dots & \delta(t)I_r & 0 \\ \delta^{(\hat{q}_r-1)}(t)I_r & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \delta(t)I_r & \dots & 0 & 0 \\ \delta(t)I_r & \dots & 0 & \dots & 0 & 0 \end{bmatrix}$$

Thus we have found :

$$(14) \quad \beta_{pol}(s) = L^{-1}[\hat{\beta}_{pol}(s)] = L \Delta(t) F = \beta_0 \delta(t) + \beta_1 \delta^{(1)}(t) + \dots + \beta_{\hat{q}_r-1} \delta^{(\hat{q}_r-1)}(t)$$

where β_i $i = 0, 1, \dots, \hat{q}_r-1$ are $r \times 1$ vectors obtained after some manipulations in the terms of (14). From equation (14) and the definition of F in (9) it follows that if the initial conditions $\beta(0^-), \dots, \beta^{(\hat{q}_r-1)}(0^-)$ are appropriate then $\beta(t)$ has an "impulsive behaviour" at $t = 0$ which consists of a Dirac impulse $\delta(t)$ and its (\hat{q}_r-1) distributional derivatives. In other words when the initial conditions are imposed on $\beta(t)$ and its (\hat{q}_r-1) derivatives at $t = 0^-$, $\beta(t)$ may exhibit an impulsive behaviour at $t = 0$ which is a consequence of the fact that (4) forces $\beta(t)$ and $\beta^{(i)}(t)$, $i = 1, 2, \dots, \hat{q}_r-1$ to satisfy certain constraints at $t = 0^-$. The exact derivation of these constraints and their relation to the structure at $s = \infty$ of $A(s)$ are examined in [13] explicitly. From equation (14) it is also clear that the impulsive solutions of L.H.M.D.E. are closely related — because of the form of L in (12) — to the finite and infinite Jordan pairs (C, J) and (C_∞, J_∞) respectively, of the polynomial matrix $A(s)$.

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