SIMPLE, GROUP AND APPROXIMATE FACTORIZATION OF MULTIDIMENSIONAL POLYNOMIALS

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Abstract In this paper an overview and some new results on simple and group m-D (multidimensional) polynomials factorization are presented. Furthermore, the optimal approximation of a non-factorizable m-D polynomial by an m-D factorizable polynomial is examined. Two numerical examples illustrate the proposed techniques.

I. INTRODUCTION

Multidimensional (m-D) systems theory has received during the recent years much attention by researchers and practitioners working in the areas of digital filtering, seismic data processing, network analysis and feedback control. A multidimensional (m-D) system is described by a transfer function which is a ratio of two m-D polynomials. Transfer function factorization, that is numerator and denominator factorization, enables a cascade realization of the corresponding m-D system. As the stability tests are in the form: if $f(z_1,...,z_m)=0$ (in appropriate regions of $z_1,...,z_m$), it is important to factorize $f(z_1,...,z_m)=0$ because, in this case, the stability test is separated into simpler ones.

Let the m-D polynomial

$$f(z_1,...,z_m) = \sum_{i_1=0}^{N_1} ... \sum_{i_m=0}^{N_m} a(i_1,...,i_m) z_1^{i_1} ... z_m^{i_m} \tag{1}$$

By the term "simple factorization", it is meant that:

$$f(z_1,\ldots,z_m) = f_1(z_1)\ldots f_m(z_m) \tag{2}$$

where $f_1(z_1),\ldots,f_m(z_m)$ are 1-D polynomials

while by the term "group factorization", it is meant that:

$$f(z_1,\ldots,z_m) = f_1(\bar{z}_1)\ldots f_k(\bar{z}_k) \tag{3}$$

where $\bar{z}_1,\ldots,\bar{z}_k$ are (mutually disjoint) groups of independent variables, taken from the set $\{z_1,\ldots,z_m\}$ and $f_1(\bar{z}_1),\ldots,f_k(\bar{z}_k)$ are polynomials.

The above definitions (2), (3) have been given in [2]-[3]. The related factorizability conditions have also been given in [3]. In the present paper simpler factorizability conditions are used, based on [7], but proved differently. Furthermore, a nonfactorizable polynomial is approximated by a factorizable one, but the approach followed differs from [7]. The present approach has the advantage to be applied for every nonfactorizable polynomial. It should be noted that, up to now, the general factorization problem, i.e. the factorization of each factorizable polynomial, has not been solved. However, some special cases have been studied [3]-[8].

The paper is organized as follows. In Section II, some simpler factorizability conditions than in [3] are presented. In Section III, a non-factorizable polynomial (as in (2) or in (3)) is optimally approximated by a factorizable polynomial (as in (2) or in (3)). Finally, in IV, the conclusions are presented.

II. SIMPLE AND GROUP FACTORIZABILITY: OVERVIEW AND NEW RESULTS

Let $f(z_1,z_2)$ be a 2-D polynomial:

$$f(z_1,z_2) = \sum_{i_1=0}^{N_1} \sum_{i_2=0}^{N_2} a(i_1,i_2) z_1^{i_1} z_2^{i_2} \tag{4}$$

This can be written as:

$$f(z_1,z_2) = [1 \;\; z_1 \;\; \ldots \;\; z_1^{N_1}] \cdot \begin{bmatrix} a(0,0) & a(0,1) & \ldots & a(0,N_2) \\ a(1,0) & a(1,1) & \ldots & a(1,N_2) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a(N_1,0) & a(N_1,1) & \ldots & a(N_1,N_2) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_2^{N_2} \end{bmatrix} \tag{5}$$

or

$$f(z_1,z_2) = \langle z_1 \rangle \cdot F_{\langle z_1 | z_2 \rangle} \cdot \langle z_2 \rangle^T , \tag{6}$$

where

$$\langle z_i \rangle = [1 \;\; z_i \; \dots \; z_i^{N_i}], \quad i=1,2 \tag{7}$$

and

$$F_{\langle z_1 | z_2 \rangle} = [a(i,j)], \quad i=0,\dots,N_1, \;\; j=0,\dots,N_2 \tag{8}$$

is an $(N_1+1)\times(N_2+1)$ matrix.

The following theorem is stated in [7] and proved in a different way.

Theorem 1 A 2-D polynomial $f(z_1,z_2)$ is simple factorizable ((2)), iff (if and only if)

$$\text{rank } F_{\langle z_1 | z_2 \rangle} = 1 \tag{9}$$

The following theorem provides a generalization of the results of theorem 1 ([7]).

Theorem 2 The m-D polynomial $f(z_1,\dots,z_m)$ is group factorizable into two factors $f_1(\bar{z}_1), f_2(\bar{z}_1^c)$, iff

$$\text{rank } F_{\langle \bar{z}_1 | \bar{z}_1^c \rangle} = 1 \tag{10}$$

where $\bar{z}_1^c$ includes some of the variables $z_1,\dots,z_m$, which are not included in $\bar{z}_1$.

The following theorem holds.

Theorem 3 The m-D polynomial $f(z_1,\dots,z_m)$ is group factorizable as in (3), iff

$$\text{rank } F_{\langle \bar{z}_i | \bar{z}_i^c \rangle} = 1 \quad \forall i = 1,\dots,k \tag{11}$$

Simple factorization, as in (3), is clearly a special case of the above theorem.

III APPROXIMATE FACTORIZATION

In this section, it is supposed that the given m-D polynomial is not factorized as in (2) or (3). This implies: rank $F>1$, when a certain type of factorization is considered. Then, its optimal approximation by a factorizable polynomial is attempted. In other words, we seek for a matrix F', such that: rank $F'=1$ and simultaneously the quantity of $\|F-F'\|^2$ is

The problem is reduced to find $b_1(i), b_2(j)$ or $\left(b_1(\bar{u}_i), b_2(\bar{u}_j)\right)$. These can be found by minimizing $\|F-F'\|^2$. Here it is noted that the utilized norm is

$$\|A\| = \left(\sum_{i,j} a_{ij}^2\right)^{1/2}, \quad \text{where} \quad A = \left[a_{ij}\right]$$

In order to minimize $\|F-F'\|^2$ or equivalently minimize $\sum_{i,j}\left(a(i,j)-b_1(i)b_2(j)\right)^2$ or $\sum_{i,j}\left(a(i,j)-b_1(\bar{u}_i)b_2(\bar{u}_j)\right)^2$, the Levenberg-Marquardt routine is proposed. The problem, for this routine, is stated as follows: minimize over $\underline{x}$ $\left\{f_1^2(\underline{x})+\dots+f_M^2(\underline{x})\right\}$ where $\underline{x}=(x_1,\dots,x_N)$. A sequence of approximation to the minimum point is generated by

$$\underline{x}^{n+1} = \underline{x}^n - \left[a_n D_n + J_n^T J_n\right]^{-1} . J_n^T . \underline{f}(\underline{x}^n) \tag{12}$$

where J_n is the numerical Jacobian matrix evaluated at $\underline{x}^n$, D_n is a diagonal matrix equal to the diagonal of $J_n^T J_n$, a_n is a positive scaling constant and $\underline{f}=[f_1,\dots,f_M]$.

IV. CONCLUSION

An m-D polynomial may be factorized into m polynomial factors. Each factor contains only one variable (exclusively) in the case of simple factorization. In the case of group factorization the m-D polynomial is factorized into k-groups of mutually disjoint variables. In both cases, the necessary and sufficient condition for factorization is: the rank of an appropriate matrix is 1. If this rank is >1, the optimal approximation of this matrix (equivalently of the given polynomial) is attempted by another matrix having rank 1. This approximation results by minimizing the square of the norm of the matrix difference.

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