Adaptive Pole Placement Control of Linear Systems using Periodic Multirate-Input Controllers

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Abstract:
An new indirect adaptive algorithm is derived for pole placement control of linear continuous-time systems with unknown parameters. The control structure proposed relies on a periodic controller, which suitably modulates the sampled output and discrete reference signals by a multirate periodically time-varying function. Such a control strategy, allows us to assign the poles of the sampled closed-loop system to desired prespecified values and does not make assumptions on the plant other than controllability, observability and known order. The proposed indirect adaptive control scheme estimates the unknown plant parameters (and consequently the controller parameters) on-line, from sequential data of the inputs and the outputs of the plant, which are recursively updated within the time limit imposed by a fundamental sampling period T_0. On the basis of the proposed algorithm, the adaptive pole placement problem is reduced to a controller determination based on the well known Ackermanns' formula. Known indirect adaptive pole placement schemes usually resort to the computation of dynamic controllers through the solution of a polynomial Diophantine equation, thus introducing high order exogenous dynamics in the control loop. Moreover, in many cases, the solution of the Diophantine equation for a desired set of closed-loop eigenvalues might yield an unstable controller, and the overall adaptive pole placement scheme is then unstable with unstable compensators because their outputs are unbounded. The proposed control strategy avoids these problems, since here gain controllers are needed to be designed. Moreover, persistency of excitation and, therefore, parameter convergence, of the continuous-time plant is provided without making any assumption either on the existence of specific convex sets in which the estimated parameters belong or on the coprimeness of the polynomials describing the ARMA model, or finally on the richeness of the reference signals, as compared to known adaptive pole placement schemes.

1 Introduction

Periodically varying and/or multirate feedback strategies for continuous-time linear systems have long been the focus of interest by many control designers. Several digital control schemes were proposed in the literature, among them periodically varying gain controllers (Chammas and Leondes, 1978; Greshak and Vergese, 1982; Khargonekar *et al.*, 1985), mutlrate-input controllers (MRICs; Araki and Hagiwara, 1986), intersample-data controllers (Mita *et al.*, 1987), multirate-output controllers (Hagiwara and Araki, 1988; Hagiwara *et al.*, 1990), generalized sampled-data hold fumc-

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tions (GSHF; Kabamba, 1987) and multirate GSHF (Arvanitis, 1995a). These classes of digital controllers have been applied successfully in solving many important control problems (Chammas and Leondes, 1978; Greshak and Vergese, 1982; Khargonekar *et al.*, 1985; Araki and Hagiwara, 1986; Mita *et al.*, 1987; Hagiwara and Araki, 1988; Hagiwara *et al.*, 1990; Kabamba, 1987; Arvanitis, 1995a, 1995b, 1996, 1997; Al-Rahmani and Franklin, 1989; Paraskevopoulos and Arvanitis, 1994; Arvanitis and Paraskevopoulos, 1994, 1995). The increased interest for such a type of feedback strategies is warranted by the new dimensions of flexibility of the design procedure, offered by these control schemes, which also provide, a series of remarkable advantages over ordinary time-invariant feedback strategies, such as state feedback, dynamic compensation or state observers; for an overview of these advantages see Hagiwara and Araki (1988), Kabamba (1987), Arvanitis and Paraskevopoulos (1995), Arvanitis (1995b), (1996).

Araki and Hagiwara (1986), in their inspired work, propose a digital multirate-input controller (MRIC), which suitably modulates the sampled output and discrete reference signals by a multirate periodically varying function, in order to solve the sampled pole placement problem for linear time-invariant continuous-time systems. In Arvanitis and Paraskevopoulos (1994), the MRIC based approach is extended to the solution of the model matching problem. Under certain conditions, the modulating functions can be tailored to a given system in such a way that for the sampled closed loop system a discrete-time transfer function matrix can be arbitrarily assigned. A main feature of the approach reported in Araki and Hagiwara (1986) and Arvanitis and Paraskevopoulos (1994) is that the pole placement or the model matching is obtained without the requirement of pole-zero cancellation.

The purpose of the present paper is to explore the possibility of extending the MRIC based approach presented in Araki and Hagiwara (1986) and subsequently used in Arvanitis and Paraskevopoulos (1994), to the control of linear time-invariant plants with unknown parameters. In particular, we use the certainty equivalence principle to combine the identification method with a control structure derived from the pole placement problem. Adaptive pole placement is of particular interest, since the middle of '70s, for obvious reasons. Several techniques based on either direct or indirect adaptive control schemes were presented to treat the problem and a very large number of papers were reported on the subject (see for example Astrom and Wittenmark (1974; 1980), Wellstead e*t al.* (1979), Egardt (1980), Elliott (1982), Elliott *et al.* (1984; 1985), Anderson and Johnstone (1985), Lozano-Leal and Goodwin (1985), Giri *et al.* (1988; 1989), Mo and Bayoumi (1989), Abramovitch and Franklin (1990), Das and Cristi (1990), Kim *et al.* (1991), and the references therein). The feedback strategies proposed to solve the adaptive pole placement problem, are hitherto based on dynamic output feedback, thus introducing high order exogenous dynamics in the control loop. On the other hand, a common feature of these techniques is that they reduce the solution of the problem to the solution of a polynomial Diophantine equation. This approach, however, does not ensure that the compensators obtained from the solution of the Diophantine equation are necessarily stable. In the case of unstable solutions, the control scheme composed by feedforward and feedback compensators is not stable and thus is not useful. The control signal is calculated from two unbounded signals that are the outputs of the compensators. In a short time the system becomes unstable. It is worth noticed at this point, that unstable solutions of the Diophantine equation, can occur even though, the system under control possesses the parity interlacing property (p.i.p.; is strongly stabilizable; Youla *et al,*, 1974). A plant is said that it possesses the p.i.p. if the number of its real poles between each pair of zeros in the unstable domain is even. In this case, it is possible to obtain a stable controller from these unstable solutions by using the approach presented in Kinaert and Blondel (1992), which is based on an interpolation procedure. Unfortunately, as mentioned above, this approach can be applied only in cases where the system under control is strongly stabilizable. When the system under control contains unknown parameters (as in the case of adaptive pole placement control), this information of crucial importance is not available to the designer. Thus, up to now, the design of a stable and useful adaptive pole placement compensator cannot be guaranteed.

The motivation for studying an adaptive version of the particular controller structure presented in Araki and Hagiwara (1986) and Arvanitis and Paraskevopoulos (1994), is manifold. First, since it does not rely on pole-zero cancellation, it may be readily applicable for solving the adaptive pole placement problem for nonstably invertible plants. Furthermore, the degrees of freedom in the choice of the modulating function, provide a solution to the problem of assuring persistency of excitation of the continuous-time plant under control, without imposing any special assumption either on the existence of special convex sets in which the estimated parameters belong or on the coprimeness of the polynomials describing the ARMA model, as in known techniques, or finally on the richeness of the reference signals (except boundedness), as in known adaptive pole placement techniques. Finally, the MRIC based adaptive pole placers sought are computed here, on the basis of the well known and fairly simple Ackerman's formula. No Diophantine equation is needed to be solved here as compared to known techniques. The designed MRIC based adaptive pole placers are always stable, since gain controllers are needed here, as compared to (possibly unstable) dynamic compensators obtained by known techniques. Therefore, the proposed adaptive scheme is readily applicable to plants which do not possess the p.i.p. As a consequence of this design philosophy, a useful globally stable indirect adaptive control scheme is derived, which estimates the unknown plant parameters (and consequently the controller parameters) on-line, from sequential data of the inputs and the outputs of the plant, which are recursively updated within the time limit imposed by a fundamental sampling period T_0. It is remarked that, the a priori knowledge needed in order to implement the proposed adaptive pole placers, is controllability and observability of the continuous and the discretized plant under control and its order.

2 Preliminaries and Problem Formulation

Consider the continuous-time, linear time-invariant single-input, single-output system of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \;\;,\;\; y(t) = \mathbf{c}^T\mathbf{x}(t) \tag{2.1}$$

where, $\mathbf{x}(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$, are the state, control and output signals, respectively, and **A**, **b**, $\mathbf{c}^T$ are real matrices having appropriate dimensions. With regard to system (2.1), we make the following two assumptions.

Assumption 2.1. System (2.1) is controllable and observable and of known order n.

Assumption 2.2. There is a sampling period $T_0 \in \mathbb{R}^+$, such as the respective discretized system with matrix triplet $\left(\Phi = \exp(\mathbf{A}T_0),\ \tilde{\mathbf{b}} = \int_0^{T_0} \exp(\mathbf{A}\lambda)\mathbf{b}d\lambda,\ \mathbf{c}^T\right)$, is controllable and observable.

Except for this prior information, the matrix triplet (**A**, **b**, $\mathbf{c}^T$) is arbitrary and unknown. In particular, no assumption is made here, on the relative degree of the plant or its stable invertibility.

Now consider applying to system (2.1) the multirate control strategy depicted in Figure 1. With regard to the sampling mechanism, we assume that all samplers start simultaneously at t=0. The sampling period T^* has rational ratio, i.e. $T^* = T_0/N$, where T_0 is the so-called *frame* sampling period and $N \in \mathbb{Z}^+$ is the *input multiplicity* of the sampling. The hold circuits H_N and H_0 are the zero order holds with holding times T^* and T_0, respectively. The signal $w(kT_0) \in \mathbb{R}$, is assumed to be a bounded reference signal. Finally, the compensator f(t) is a periodically time-varying controller with period T_0, i.e.

$$f(t + T_0) = f(t) \;\;,\;\; t \in \left[kT_0,\ (k+1)T_0\right) \tag{2.2}$$

The resulting closed-loop system is described by the following state-space equations

$$\mathbf{x}\left[(k+1)T_0\right] = \left(\boldsymbol{\Phi} - \mathbf{k}_f\mathbf{c}^T\right)\mathbf{x}(kT_0) + \mathbf{k}_f w(kT_0) \;\;,\;\; y(kT_0) = \mathbf{c}^T\mathbf{x}(kT_0) \;\;,\;\; k \geq 0$$

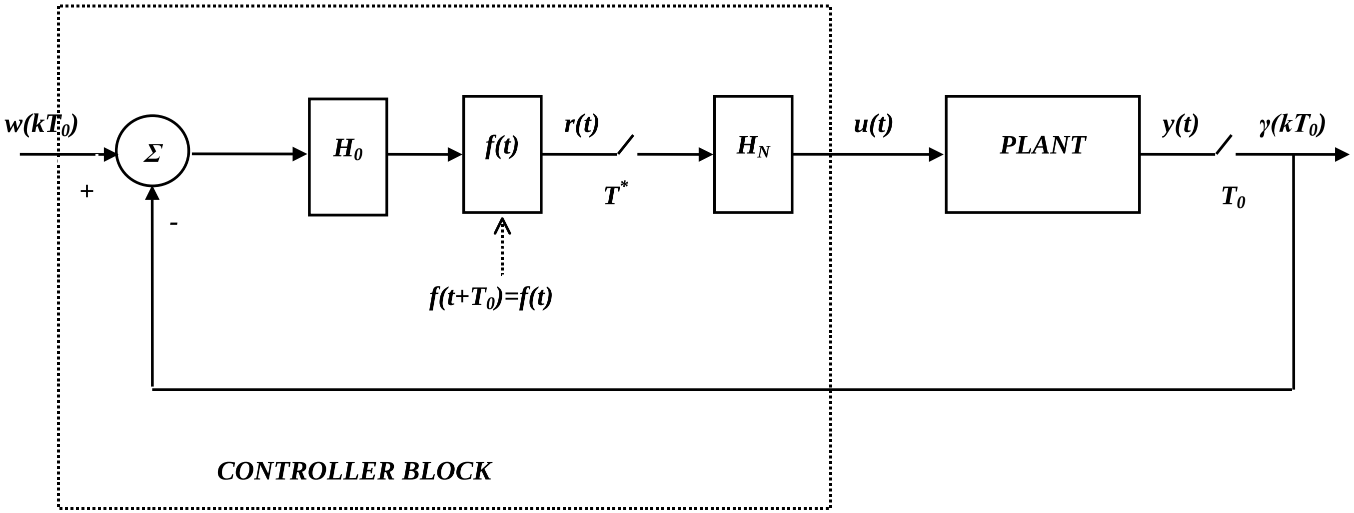

Figure 1. Control strategy in the nonadaptive case

where $\mathbf{x}(kT_0) \in \mathbb{R}^n$ and $y(kT_0) \in \mathbb{R}$ are discrete measurement quantities obtained by sampling $\mathbf{x}(t)$ and y(t), respectively, with sampling period T_0 and the vector $\mathbf{k}_f$ is defined as

$$\mathbf{k}_f = \int_0^{T_0} \exp\left[\mathbf{A}(T_0 - \lambda)\right]\mathbf{b} f(\lambda) d\lambda \tag{2.3}$$

The adaptive pole placement problem treated in the present paper is as follows: Find a periodic controller f(t), which when applied to system (2.1), drives the poles of the resulting closed-loop system to new desired values $\hat{\lambda}_1, \hat{\lambda}_2, \cdots, \hat{\lambda}_n$, where complex poles appear in conjugate pairs.

To solve the above problem, an indirect adaptive control scheme is exhibited in the sequel. In particular, we first solve the pole placement prolem, namely, the assignment of the poles of the sampled system to the given values $\hat{\lambda}_1, \hat{\lambda}_2, \cdots, \hat{\lambda}_n$, using periodic multirate controllers, for known systems. This is done in Section 3. Next, using these results, the pole placement problem is solved for the configuration of Figure 2, wherein the periodic controller f(t) is with prespecified periodic behavior and a persistent excitation signal is introduced in the control loop for future identification purposes. This is done in Section 4. It is remarked that the motivation for modifying the control strategy as in Figure 2, is that it facilitates the derivation of the indirect adaptive control scheme sought, which is presented in Section 5. In Section 5, the global stability of the proposed scheme is also studied.

3 Solution of the Pole Placement Problem for Known Systems

The procedure for stabilization through pole placement using the configuration depicted in Figure 1, consists in finding a periodic controller f(t), such that

$$\det\left(z\mathbf{I} - \mathbf{\Phi} + \mathbf{k}_f \mathbf{c}^T\right) \equiv \hat{p}(z) \tag{3.1a}$$

where

$$\hat{p}(z) = \prod_{i=1}^{n}\left(z - \hat{\lambda}_i\right) \hat{=} z^n + \hat{a}_1 z^{n-1} + \cdots + \hat{a}_{n-1} z + \hat{a}_n \tag{3.1b}$$

Since, $\det\left(z\mathbf{I} - \mathbf{\Phi} + \mathbf{k}_f \mathbf{c}^T\right) \equiv \det\left(z\mathbf{I} - \mathbf{\Phi}^T + \mathbf{c}\mathbf{k}_f^T\right)$, relation (3.1a), is equivalent to the relation

$$\det\left(z\mathbf{I} - \mathbf{\Phi}^T + \mathbf{c}\mathbf{k}_f^T\right) = \hat{p}(z) \tag{3.2}$$

Under Assumption 2.2, the vector $\mathbf{k}_f$ satisfying (3.2) (and consequently (3.1a)) is given by the well known Ackerman's formula, which has the following form

$$\mathbf{k}_f^T = \mathbf{e}^T \mathbf{R} \hat{p}\left(\mathbf{\Phi}^T\right) \tag{3.3}$$

where $\mathbf{e}^T = (0, \cdots, 0, 1)$, $\mathbf{R}$ is the well known observability matrix of the pair $\left(\mathbf{\Phi}, \mathbf{c}^T\right)$ and $\hat{p}\left(\mathbf{\Phi}^T\right)$ is given by

$$\hat{p}\left(\mathbf{\Phi}^T\right) = \left(\mathbf{\Phi}^T\right)^n + \hat{a}_1\left(\mathbf{\Phi}^T\right)^{n-1} + \cdots + \hat{a}_{n-1}\mathbf{\Phi}^T + \hat{a}_n \mathbf{I} \tag{3.4}$$

Using the vector $\mathbf{k}_f$ as specified by (3.3), we can determine the modulating function f(t), by solving (2.3). Under Assumption 2.1, on the controllability of the pair $(\mathbf{A}, \mathbf{b})$, a solution of (2.3) is the following (Kabamba, 1987)

$$f(t) = \mathbf{b}^T \exp\left[\mathbf{A}^T(T_0 - t)\right]\mathbf{W}^{-1}(\mathbf{A}, \mathbf{b}, T_0)\mathbf{k}_f \tag{3.5}$$

where $\mathbf{W}(\mathbf{A}, \mathbf{b}, T_0)$ is the controllability Grammian of the pair $(\mathbf{A}, \mathbf{b})$ on $[0, T_0]$, which has the form

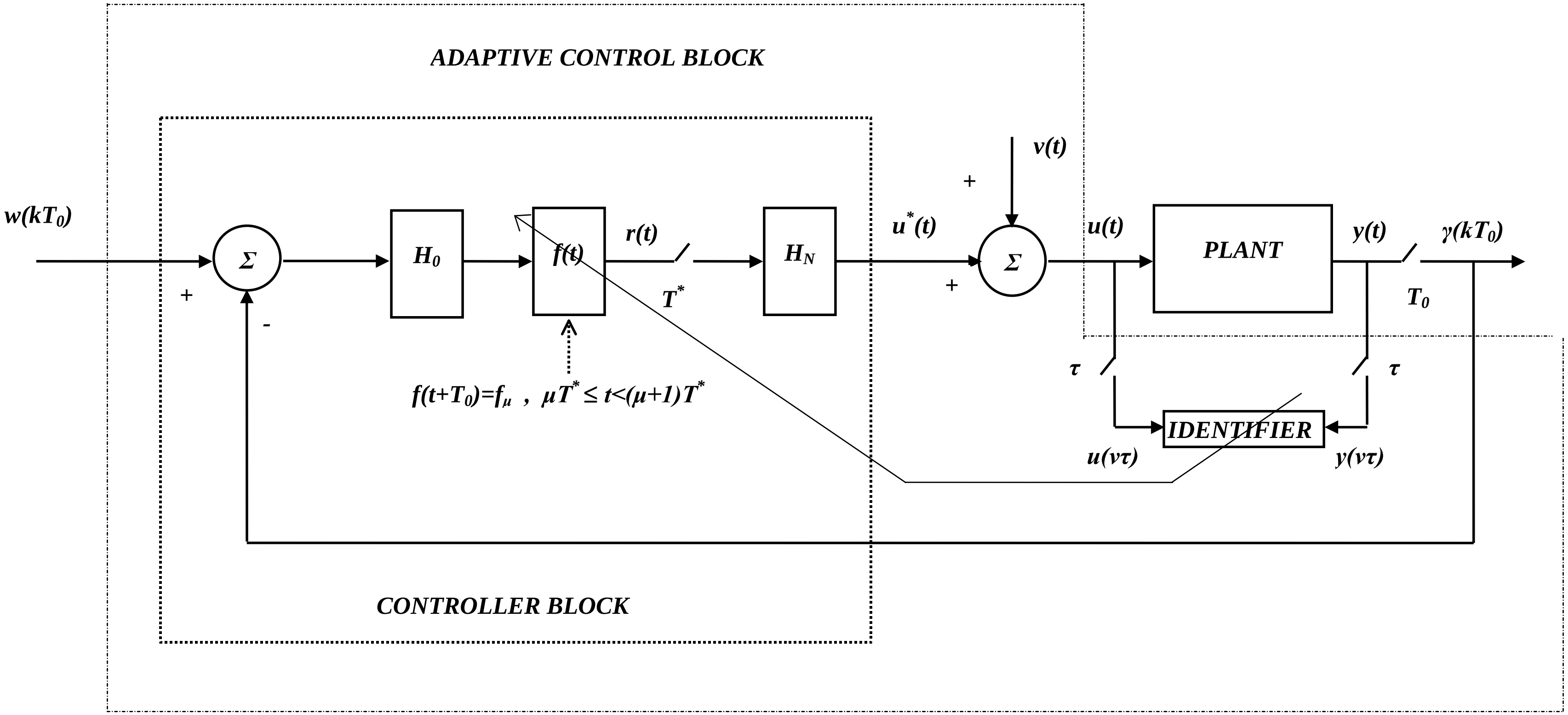

Figure 2. The structure of the adaptive control system

$$\mathbf{W}(\mathbf{A},\mathbf{b},T_0)=\int_0^{T_0}\exp\left[\mathbf{A}(T_0-\lambda)\right]\mathbf{b}\mathbf{b}^T\exp\left[\mathbf{A}^T(T_0-\lambda)\right]d\lambda$$

Note that the controllability Grammian $\mathbf{W}(\mathbf{A},\mathbf{b},T_0)$ is nonsingular and hence a solution of (2.3) of the form (3.5) exists if the pair $(\mathbf{A},\mathbf{b})$ is controllable.

4 Solution of the Problem Appropriate for the Adaptive Case

In order to obtain a solution of the aforementioned pole placement problem which will be more appropriate for application in the case of systems with unknown parameters, we slightly modify in the sequel the control stategy of Figure 1 as it is shown in Figure 2. In particular, we focus our attention to the special class of the time-varying T_0-periodic modulating functions f(t), which are piecewise constant over intervals of length T^*, i.e.

$$f(t)=f_\mu \;,\; \forall t\in\left[\mu T^*,(\mu+1)T^*\right) \;,\; \mu=0,1,\ldots,N-1 \tag{4.1}$$

The persistent excitation signal v(t) is defined as

$$v(t) = \mathbf{q}^T(t)\mathbf{v} \;,\; \mathbf{q}^T(t) = \left[q_0(t),\cdots,q_{N-1}(t)\right]$$

Here, $\mathbf{q}$(t) the T^*-periodic vector function with elements having the form

$$q_i(t) = q_{i,\mu} \;,\; \text{for } t\in\left[\mu T^*,(\mu+1)T^*\right] \;,\; i=0,1,\ldots,N-1 \;,\; \mu=0,1,\ldots,N-1 \tag{4.2}$$

where $q_{i,\mu}$ is constant taking the following values

$$q_{i,\mu} = \begin{cases} 1\,, & \text{for } \mu=i \\ 0\,, & \text{for } \mu\neq i \end{cases} \tag{4.3}$$

It is pointed out that $\mathbf{v}$ is as yet unknown. We remark that the additive term $v(t) = \mathbf{q}^T(t)\mathbf{v}$, in the input of the continuous-time system, is used only for identification purposes and as it will be shown later, it is selected such as it will not influence the pole placement problem. Furthermore define

$$\hat{\mathbf{A}}\triangleq\exp\left(\mathbf{A}T^*\right) \;,\; \hat{\mathbf{b}}\triangleq\int_0^{T^*}\exp(\mathbf{A}\lambda)\mathbf{b}\,d\lambda$$

We are now able to establish the following Theorem.

Theorem 4.1. For the modulating function of the form (4.1), the resulting closed loop system has the form

$$\mathbf{x}\left[(k+1)T_0\right] = \left(\mathbf{\Phi}-\hat{\mathbf{B}}\hat{\mathbf{f}}\mathbf{c}^T\right)\mathbf{x}(kT_0)+\hat{\mathbf{B}}\hat{\mathbf{f}}w(kT_0)+\mathbf{B}^*\mathbf{v} \;,\; y(kT_0) = \mathbf{c}^T\mathbf{x}(kT_0) \;,\; k>0 \tag{4.4}$$

where

$$\hat{\mathbf{B}}=\left[\hat{\mathbf{b}}\quad\hat{\mathbf{A}}\hat{\mathbf{b}}\quad\cdots\quad\hat{\mathbf{A}}^{N-1}\hat{\mathbf{b}}\right]$$

$\hat{\mathbf{f}}$ is the N-dimensional vector of the form

$$\hat{\mathbf{f}}=\begin{bmatrix} f_{N-1}\\ \vdots\\ f_0\end{bmatrix} \tag{4.5}$$

and $\mathbf{B}^*$ is the $n\times N$ matrix having the form

$$\mathbf{B}^* = \left[\hat{\mathbf{A}}^{N-1}\hat{\mathbf{b}}\;\hat{\mathbf{A}}^{N-2}\hat{\mathbf{b}}\;\cdots\;\hat{\mathbf{b}}\right]$$

Proof: To show that the closed-loop system can be written in the form (4.4), we start by discretizing system (2.1) with sampling period T_0, to yield

$$\mathbf{x}\left[(k+1)T_0\right] = \mathbf{\Phi x}\left(kT_0\right) + \int_{kT_0}^{(k+1)T_0} \exp\left\{\mathbf{A}\left[(k+1)T_0 - \lambda\right]\right\}\mathbf{b}u(\lambda)d\lambda \tag{4.6}$$

By observing that $u(t) = r(t) + \mathbf{q}^T(t)\mathbf{v}$ and taking into account the structure of the control scheme of Figure 2, we obtain

$$u(t) = f(t)\varepsilon\left(kT_0\right) + \mathbf{q}^T(t)\mathbf{v} \ , \quad \text{for } t \in \left[\mu T^*, (\mu+1)T^*\right) \tag{4.7}$$

where

$$\varepsilon\left(kT_0\right) = w\left(kT_0\right) - y\left(kT_0\right) = w\left(kT_0\right) - \mathbf{c}^T\mathbf{x}\left(kT_0\right) \tag{4.8}$$

Combining relations (4.6)-(4.8), we obtain the following relationship

$$\mathbf{x}\left[(k+1)T_0\right] = \left(\mathbf{\Phi} - \mathbf{k}_f\mathbf{c}^T\right)\mathbf{x}(kT_0) + \mathbf{k}_f w(kT_0) + \mathbf{\Gamma v} \tag{4.9}$$

where

$$\mathbf{\Gamma} = \int_{kT_0}^{(k+1)T_0} \exp\left\{\mathbf{A}\left[(k+1)T_0 - \lambda\right]\right\}\mathbf{b}\mathbf{q}^T(\lambda)d\lambda$$

Now, partition $\mathbf{\Gamma}$ as follows

$$\mathbf{\Gamma} = \left[\mathbf{\Gamma}_1 \quad \mathbf{\Gamma}_2 \quad \cdots \quad \mathbf{\Gamma}_N\right]$$

Then, the (i+1)-th column of the matrix $\mathbf{\Gamma}$, denoted by $\mathbf{\Gamma}_{i+1}$, for i=0,1,...,N-1, can be expressed as

$$\mathbf{\Gamma}_{i+1} = \int_0^{T_0} \exp\left[\mathbf{A}\left(T_0 - \lambda\right)\right]\mathbf{b}q_i(\lambda)d\lambda \ , \quad \text{for } i = 0,1,\ldots,N-1 \tag{4.10}$$

Introducing (4.2) in (4.10), yields

$$\mathbf{\Gamma}_{i+1} = \sum_{\mu=0}^{N-1} \int_{\mu T^*}^{(\mu+1)T^*} \exp\left[\mathbf{A}\left(T_0 - \lambda\right)\right]\mathbf{b}q_{i,\mu}d\lambda \ , \quad \text{for } q = 0,1,\ldots,N-1 \tag{4.11}$$

Relation (4.11) may further be written as

$$\mathbf{\Gamma}_{i+1} = \sum_{\mu=0}^{N-1} q_{i,\mu} \exp\left[\mathbf{A}\left(N-\mu-1\right)T^*\right]\int_0^{T^*} \exp\left[\mathbf{A}\left(T^* - \lambda\right)\right]\mathbf{b}d\lambda = \left[\sum_{\xi=1}^{N} q_{i,N-\xi}\hat{\mathbf{A}}^{\xi-1}\right]\hat{\mathbf{b}}$$

By making use of relation (4.3), we arrive at the following relationship

$$\mathbf{\Gamma}_{i+1} = \hat{\mathbf{A}}^{N-i-1}\hat{\mathbf{b}}$$

Clearly, $\Gamma \equiv \mathbf{B}^*$. Application of the above algorithm to the first two terms of (4.9) gives $\mathbf{k}_f = \hat{\mathbf{B}}\hat{\mathbf{f}}$ (see Araki and Hagiwara (1986) for the details). This completes the proof of the Theorem □

The following Lemma whose proof is given in Araki and Hagiwara (1986), will be useful in the sequel.

Lemma 4.1. If N is chosen such that N>n, the matrix $\hat{\mathbf{B}}$ has full row rank, for almost every T_0.

Thus far, we have established that the pole placement controller vector $\mathbf{k}_f$ is related to the vector $\hat{\mathbf{f}}$ via the relation $\mathbf{k}_f = \hat{\mathbf{B}}\hat{\mathbf{f}}$. It remains to determine $\hat{\mathbf{f}}$. To this end let $\hat{\mathbf{S}}$ be the following matrix

$$\hat{\mathbf{S}} = \left[\hat{\mathbf{b}} \quad \hat{\mathbf{A}}\hat{\mathbf{b}} \quad \cdots \quad \hat{\mathbf{A}}^{n-1}\hat{\mathbf{b}}\right] \tag{4.12}$$

From Lemma 4.1, it is clear that the matrix $\hat{\mathbf{S}}$ is nonsingular for almost every T_0. Let also $\mathbf{E}$ be the nonsingular permutation matrix with the property $\mathbf{E}^{-1} = \mathbf{E}^T$ and having the form

$$\mathbf{E} = \left[\mathbf{E}_1 \quad \mathbf{E}_2\right]^T$$

where

$$\mathbf{E}_1 = \left[\varepsilon_1 \quad \varepsilon_2 \quad \cdots \quad \varepsilon_n\right] \ , \ \mathbf{E}_2 = \left[\varepsilon_{n+1} \quad \varepsilon_{n+2} \quad \cdots \quad \varepsilon_N\right]$$

where, in general, $\varepsilon_j \in \mathbb{R}^N$ is the column vector whose elements are zeros except to a unity appearing in the jth position. Define

$$\widetilde{\mathbf{B}} \,\hat{=}\, \hat{\mathbf{B}}\mathbf{E}^{-1} \equiv \begin{bmatrix} \hat{\mathbf{S}} & \hat{\mathbf{Q}} \end{bmatrix}$$

where the matrix $\hat{\mathbf{Q}}$ has the form

$$\hat{\mathbf{Q}} = \begin{bmatrix} \hat{\mathbf{A}}^{n}\hat{\mathbf{b}} & \hat{\mathbf{A}}^{n+1}\hat{\mathbf{b}} & \cdots & \hat{\mathbf{A}}^{N-1}\hat{\mathbf{b}} \end{bmatrix}$$

Also, let Δ be the nonsingular permutation matrix with the property $\Delta^{-1} \equiv \Delta^{T}$ and having the form

$$\Delta = \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \end{bmatrix}^{T}$$

where

$$\Delta_1 = \begin{bmatrix} \varepsilon_{N-n+1} & \varepsilon_{N-n+2} & \cdots & \varepsilon_{N} \end{bmatrix} \;,\; \Delta_2 = \varepsilon_{N-n} \;,\; \Delta_3 = \begin{bmatrix} \varepsilon_{N-n-1} & \varepsilon_{N-n-2} & \cdots & \varepsilon_{1} \end{bmatrix}$$

Furthermore, let

$$\widetilde{\mathbf{B}}^{*} \,\hat{=}\, \mathbf{B}^{*}\Delta \equiv \begin{bmatrix} \hat{\mathbf{S}}^{*} & \hat{\mathbf{A}}^{n}\hat{\mathbf{b}} & \hat{\mathbf{Q}}^{*} \end{bmatrix}$$

where

$$\hat{\mathbf{S}}^{*} = \begin{bmatrix} \hat{\mathbf{A}}^{n-1}\hat{\mathbf{b}} & \hat{\mathbf{A}}^{n-2}\hat{\mathbf{b}} & \cdots & \hat{\mathbf{b}} \end{bmatrix} \;,\; \hat{\mathbf{Q}}^{*} = \begin{bmatrix} \hat{\mathbf{A}}^{N-1}\hat{\mathbf{b}} & \hat{\mathbf{A}}^{N-2}\hat{\mathbf{b}} & \cdots & \hat{\mathbf{A}}^{n+1}\hat{\mathbf{b}} \end{bmatrix} \tag{4.13}$$

Using the above definitions, one may determine $\hat{\mathbf{f}}$ by inspection, to have the form

$$\hat{\mathbf{f}} = \mathbf{E}^{T}\begin{bmatrix} \hat{\mathbf{S}}^{-1}\hat{p}(\boldsymbol{\Phi})\mathbf{R}^{T}\mathbf{e} \\ \mathbf{0} \end{bmatrix} \tag{4.14}$$

It only remains to determine the appropriate vector $\mathbf{v}$ which does not influence the pole placement problem. In other words $\mathbf{v} \in \ker \mathbf{B}^{*}$, or $\mathbf{B}^{*}\mathbf{v} = \mathbf{0}$. An obvious selection of such $\mathbf{v}$ obtained also by inspection is the following

$$\mathbf{v} = \Delta^{T}\begin{bmatrix} -\hat{\mathbf{S}}^{*-1}\hat{\mathbf{A}}^{n}\hat{\mathbf{b}} \\ 1 \\ \mathbf{0}_{(N-n-1)\times 1} \end{bmatrix} \tag{4.15}$$

The general form of $\mathbf{v}$ is

$$\mathbf{v} = \mathbf{B}_0^{*}\begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{N-n} \end{bmatrix} \tag{4.16}$$

where $\mathbf{B}_0^{*}$ is the $N \times (N-n)$ matrix whose columns are the linearly indepedent N-dimensional vectors which are orthogonal to the rows of $\mathbf{B}^{*}$ and where $\rho_j,\ j = 1,2,\ldots,N-n$ are arbitrary real parameters.

It is noted that the vector $\mathbf{v}$, eventhough does not affect the pole placement problem, it provides persistent excitation useful for the identification of the system, as it will be shown in the following Section.

Clearly, the modulating function f(t) in the configuration depicted in Figure 2 has the form

$$f(t) = \mathbf{e}_{N-\mu}\hat{\mathbf{f}} \;,\; \forall t \in \left[\mu T^{*}, (\mu+1)T^{*}\right) \;,\; \mu = 0,1,\ldots,N-1 \tag{4.17}$$

where $\mathbf{e}_{N-\mu}$ is the N-dimensional row vector defined as $\mathbf{e}_{N-\mu} = \varepsilon_{N-\mu}^{T}$. Note that, the above periodic function is largely affected upon the multirate mechanism, while the modulating function f(t) of Figure 1, is not. Furthermore, the introduction of the reference signal v(t) in the control loop, greatly facilitates the estimation of the plant parameters in the case of unknown systems. For these reasons, the control strategy of Figure 2 is more appropriate than the control strategy of Figure 1 for the development of the indirect adaptive control scheme that follows.

5 Control Strategy for the Adaptive Case

The control scheme presented in Sections 3 and 4 has a corresponding scheme in the case where the system is unknown. For the case of unknown systems, the control strategy is mainly based on the computation of the vectors $\hat{\mathbf{f}}$ and $\mathbf{v}$ from suitable estimates of the parameters of the plant with updating taking place every kT_0, $k \geq 0$ and results to a globally stable closed-loop system whose poles are located to the prespecified values $\hat{\lambda}_1, \hat{\lambda}_2, \cdots, \hat{\lambda}_n$.

5.1. Identification of the system

System (2.1), when descretized with sampling period $\tau = \dfrac{T^*}{2n+1}$, takes the form

$$\mathbf{x}\big[(\nu+1)\tau\big] = \mathbf{\Phi}_\tau \mathbf{x}(\nu\tau) + \hat{\mathbf{b}}_\tau u(\nu\tau) \ , \ y(\nu\tau) = \mathbf{c}^T\mathbf{x}(\nu\tau) \ , \ \nu \geq 0 \tag{5.1}$$

where

$$\mathbf{\Phi}_\tau = \exp(\mathbf{A}\tau) \ , \ \hat{\mathbf{b}}_\tau = \int_0^\tau \exp\big[\mathbf{A}(\tau-\lambda)\big]\mathbf{b}d\lambda \tag{5.2}$$

Iterating equation (5.1), 2n+1 times and observing that $u(\nu\tau)$ is constant for $\nu\tau \in \big[mT^*, (m+1)T^*\big)$, $m \geq 0$ yields

$$\mathbf{x}\big[(m+1)T^*\big] = \mathbf{\Phi}_{T^*}\mathbf{x}(mT^*) + \hat{\mathbf{b}}_{T^*}u(mT^*) \ , \ y(mT^*) = \mathbf{c}^T\xi(mT^*) \ , \ m \geq 0$$

where

$$\mathbf{\Phi}_{T^*} \equiv \hat{\mathbf{A}} = \mathbf{\Phi}_\tau^{2n+1} \ \text{ and } \ \hat{\mathbf{b}}_{T^*} \equiv \hat{\mathbf{b}} = \sum_{\rho=0}^{2n} \mathbf{\Phi}_\tau^\rho \hat{\mathbf{b}}_\tau \tag{5.3}$$

We also note that matrix $\mathbf{\Phi}$ can be written as

$$\mathbf{\Phi} \equiv \hat{\mathbf{A}}^N = \mathbf{\Phi}_\tau^{(2n+1)N} \tag{5.4}$$

From the above analysis, it is clear that the matrices $\mathbf{\Phi}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{A}}$ (which are the only matrices involved in computing the vectors $\mathbf{k}_f^T$ and $\mathbf{v}$) can be computed on the basis of the pair $\big(\mathbf{\Phi}_\tau \ , \ \hat{\mathbf{b}}_\tau\big)$. Moreover, fixing the coordinate system such that

$$\mathbf{\Phi}_\tau = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_n \\ 1 & 0 & \cdots & 0 & -\alpha_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_1 \end{bmatrix} \ , \ \hat{\mathbf{b}}_\tau = \begin{bmatrix} \beta_n \\ \beta_{n-1} \\ \vdots \\ \beta_1 \end{bmatrix} \ , \ \mathbf{c}^T = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \tag{5.5}$$

only α_i and β_i, $i = 1,2,\ldots,n$ are considered as unknown parameters. Note that relations (5.1) and (5.5), are equivalent to the following difference equation

$$y(\nu\tau) + \sum_{\rho=1}^{n} \alpha_i y(\nu\tau - \rho\tau) \ = \ \sum_{\rho=1}^{n} \beta_i u(\nu\tau - \rho\tau) \ , \ \nu \geq 0 \tag{5.6}$$

Relation (5.6), can now be used for the identification of the parameters of the unknown system. To this end, relation (5.6) can be written in the following linear regression form

$$y(\nu\tau) = \varphi^T(\nu\tau)\theta$$

where

$$\varphi(\nu\tau) = \left[-y(\nu\tau-\tau) \quad \cdots \quad -y(\nu\tau-n\tau) \quad u(\nu\tau-\tau) \quad \cdots \quad u(\nu\tau-n\tau)\right]^T$$

and

$$\theta = \left[\alpha_1 \quad \cdots \quad \alpha_n \quad \beta_1 \quad \cdots \quad \beta_n\right]$$

Next, define

$$\mathbf{Y}(kT_0) = \left[y(kT_0) \quad y(kT_0-\tau) \quad \cdots \quad y\left[(k-1)T_0\right]\right]$$

$$\mathbf{Z}(kT_0) = \left[\varphi(kT_0) \quad \varphi(kT_0-\tau) \quad \cdots \quad \varphi\left[(k-1)T_0\right]\right]$$

$$\hat{\theta}_k = \left[\hat{\alpha}_1(kT_0) \quad \cdots \quad \hat{\alpha}_n(kT_0) \quad \hat{\beta}_1(kT_0) \quad \cdots \quad \hat{\beta}_n(kT_0)\right]$$

Clearly, we have the relation

$$\mathbf{Y}(kT_0) = \mathbf{Z}^T(kT_0)\theta$$

We now choose the recursive algorithm for the estimation of $\hat{\theta}_k$ as

$$\hat{\theta}_{k+1} = \hat{\theta}_k - \left[a\mathbf{I} + \mathbf{Z}^T(kT_0)\mathbf{Z}(kT_0)\right]^{-1}\mathbf{Z}(kT_0)\left[\mathbf{Z}^T(kT_0)\hat{\theta}_k - \mathbf{Y}(kT_0)\right] \tag{5.7}$$

where $a \in \mathbf{R}^+$ is arbitrary and $\hat{\theta}_0$ is arbitrarily specified. It is pointed out that the term a**I** in (5.7), is added in order to avoid numerical ill conditioning, arising in the identification procedure based on the usual least-square algorithm, when the determinant of the matrix $\mathbf{Z}^T(kT_0)\mathbf{Z}(kT_0)$ takes small values.

Clearly, the adaptive law (5.7) describes an on-line estimation procedure which deals with sequential data of the inputs and the outputs of the plant and in which the parameter estimates are recursively updated within the time-limit imposed by the sampling period T_0. The convergence and the boundedness properties of the proposed identification procedure are summarized in the following proposition.

Proposition 5.1. Let $\tilde{\theta}_k$ be the parameter estimation error, defined as

$$\tilde{\theta}_k = \hat{\theta}_k - \theta \tag{5.8}$$

Then, for the parameter estimation algorithm (5.7), the following properties hold

(a) $\left\|\hat{\theta}_k\right\| \le \xi$, for some finite $\xi \in \mathbb{R}^+$.

(b) If $\lim\limits_{k\to\infty}\sum\limits_{\rho=0}^{k}\lambda_{\min}\left(\mathbf{Z}(\rho T_0)\mathbf{Z}^T(\rho T_0)\right) = \infty$ then $\lim\limits_{k\to\infty}\hat{\theta}_k = \theta$

where, $\lambda_{\min}(\bullet)$ denotes the minimum eigenvalue of a matrix.

Proof: (a) Introducing (5.8) in (5.7) and taking into account that $\mathbf{Z}^T(kT_0)\theta - \mathbf{Y}(kT_0) = \mathbf{0}$, we readily obtain

$$\tilde{\theta}_{k+1} = \left\{\mathbf{I} - \left[a\mathbf{I} + \mathbf{Z}(kT_0)\mathbf{Z}^T(kT_0)\right]^{-1}\mathbf{Z}(kT_0)\mathbf{Z}^T(kT_0)\right\}\tilde{\theta}_k \tag{5.9}$$

On the basis of the Matrix Inversion Lemma, relation (5.9) may further be written as

$$\tilde{\theta}_{k+1} = \left\{\mathbf{I} + \frac{1}{a}\mathbf{Z}(kT_0)\mathbf{Z}^T(kT_0)\right\}^{-1}\tilde{\theta}_k \tag{5.10}$$

Therefore,

$$\tilde{\theta}_{k+1}^T\tilde{\theta}_{k+1} = \tilde{\theta}_k^T\left\{\mathbf{I} + \frac{1}{a}\mathbf{Z}(kT_0)\mathbf{Z}^T(kT_0)\right\}^{-2}\tilde{\theta}_k \le \left(1 + \frac{\lambda_{\min}\left(\mathbf{Z}(kT_0)\mathbf{Z}^T(kT_0)\right)}{a}\right)^{-2}\tilde{\theta}_k^T\tilde{\theta}_k \tag{5.11}$$

By repeatedly using the above inequality, we obtain

$$\tilde{\theta}_{k+1}^{T}\tilde{\theta}_{k+1} \le \left\{\prod_{\rho=0}^{k}\left(1+\frac{\lambda_{\min}\left(\mathbf{Z}(\rho T_0)\mathbf{Z}^{T}(\rho T_0)\right)}{a}\right)\right\}^{-2}\tilde{\theta}_0^{T}\tilde{\theta}_0 \le \left\{1+\frac{1}{a}\sum_{\rho=0}^{k}\lambda_{\min}\left(\mathbf{Z}(\rho T_0)\mathbf{Z}^{T}(\rho T_0)\right)\right\}^{-2}\tilde{\theta}_0^{T}\tilde{\theta}_0 \quad (5.12)$$

where, $\tilde{\theta}_0 = \hat{\theta}_0 - \theta$. Hence, $\left\|\tilde{\theta}_k\right\|$ is uniformly bounded by $\left\|\tilde{\theta}_0\right\|$, and since θ is finite, $\hat{\theta}_k$ is also uniformly bounded for some finite $\xi \in \mathbb{R}^+$.

(b) If $\lim\limits_{k\to\infty}\sum\limits_{\rho=0}^{k}\lambda_{\min}\left(\mathbf{Z}(kT_0)\mathbf{Z}^{T}(kT_0)\right) = \infty$, then from (5.11) it follows that $\lim\limits_{k\to\infty}\tilde{\theta}_k = \mathbf{0}$, and therefore, $\lim\limits_{k\to\infty}\hat{\theta}_k = \theta$.

Clearly, Proposition 5.1 states that for the convergence of the plant parameter estimates $\hat{\theta}_k$ to their true values θ, it is sufficient that the regression vector $\mathbf{Z}(kT_0)$ is persistently exciting to the amount that

$$\lim_{k\to\infty}\sum_{\rho=0}^{k}\lambda_{\min}\left(\mathbf{Z}(kT_0)\mathbf{Z}^{T}(kT_0)\right) = \infty$$

Therefore, since adaptation and stability of the adaptive scheme depend on the convergence of the parameter estimates to their true values, it is necessary to prove excitation of $\mathbf{Z}(kT_0)$. This is done in Subsection 5.3, that follows (see Theorem 5.1, therein).

Remark 5.1. It is worth noticed, at this point that the only reason for choosing the sampling period for the identification procedure as $\tau = \frac{T^*}{2n+1}$ is due to the fact that using this period much more information from the plant input and output data is available, for the estimation of the unknown plant parameter vector $\hat{\theta}_k$. Indeed, if we use a sampling period e.g. of the form $\tau = T^*$, we can directly identify matrix $\hat{\mathbf{A}}$ as well as vector $\hat{\mathbf{b}}$ (and hence $\mathbf{\Phi}$ using (5.4)). However, in the later case, only N+1 values of the plant input as well as N+1 values of the plant output are available for the identification procedure, instead of the (2n+1)N+1 values of the input and the (2n+1)N+1 values of the output, available in the case where $\tau = \frac{T^*}{2n+1}$. This fact can deteriorate the results obtained by the identification process, whenever $\tau = T^*$.

5.2. Adaptive controller synthesis algorithm

On the basis of the estimated parameter vector $\hat{\theta}_k$ obtained from (5.7), as well as on the basis of the relations (5.3)-(5.5), one can take the estimates, which are needed for the computation of the matrices $\mathbf{c}^{T}\left(\hat{\theta}_k\right)$, $\hat{\mathbf{A}} \equiv \hat{\mathbf{A}}\left(\hat{\theta}_k\right)$, $\mathbf{\Phi} \equiv \mathbf{\Phi}\left(\hat{\theta}_k\right)$ and $\hat{\mathbf{b}} \equiv \hat{\mathbf{b}}\left(\hat{\theta}_k\right)$, which are involved in the algorithms presented in the previous Sections. Moreover, since the matrices $\mathbf{R}$ and $\hat{p}\left(\mathbf{\Phi}^{T}\right)$ can be constructed on the basis of the matrices $\mathbf{\Phi}\left(\hat{\theta}_k\right)$ and $\mathbf{c}^{T}\left(\hat{\theta}_k\right)$, then provided that the matrix triplet $\left(\mathbf{\Phi}_{T^*}\left(\hat{\theta}_k\right), \hat{\mathbf{b}}_{T^*}\left(\hat{\theta}_k\right), \mathbf{c}^{T}\right)$ is minimal, for all possible values of $\hat{\theta}_k$, we can obtain the following results:

$$\hat{\mathbf{f}} \equiv \hat{\mathbf{f}}\left(\hat{\theta}_k\right) \;,\; \mathbf{v} \equiv \mathbf{v}\left(\hat{\theta}_k\right) \quad (5.13)$$

Overall, the procedure for the synthesis of a periodic multirate adaptive pole placer of the form

(4.1), consists on the main steps given bellow:

Step 1. Choose the sampling period τ such that $\tau = \dfrac{T_0}{(2n+1)N} = \dfrac{T^*}{2n+1}$.

Step 2. Update the estimates using (5.7).

Step 3. Use (5.5) to compute the matrices $\mathbf{\Phi}_\tau$, $\hat{\mathbf{b}}_\tau$ and $\mathbf{c}^T$.

Step 4. Use (5.3) and (5.4) to compute the matrices $\hat{\mathbf{A}}$, $\hat{\mathbf{b}}$ and $\mathbf{\Phi}$.

Step 5. Compute the vector $\mathbf{k}_f$ using relations (3.3) and (3.4).

Step 6. Compute the matrices $\hat{\mathbf{S}}$ and $\hat{\mathbf{S}}^*$ using relations (4.12) and (4.13).

Step 7. Compute the vectors $\hat{\mathbf{f}}$ and $\mathbf{v}$ using relations (4.14) and (4.15) or (4.16).

Step 8. Implement the periodic multirate modulating function f(t) using relation (4.17).

5.3 Stability analysis of the adaptive control scheme

We now investigate the stability of the closed loop system for arbitrary initial conditions on the plant. To this end, the following fundamental result, can be established.

Theorem 5.1. The regressor sequence $\varphi(\nu\tau)$ is persistently exciting, i.e. there is a $\delta > 0$, such that

$$\mathbf{Z}(kT_0)\mathbf{Z}^T(kT_0) = \sum_{\nu=0}^{(2n+1)N} \varphi(kT_0 - \nu\tau)\varphi^T(kT_0 - \nu\tau) \geq \delta\mathbf{I} \tag{5.14}$$

Proof: Let $u(t) = \mathbf{q}^T(t)\mathbf{v}$ and observe that introducing the pseudovariable $\zeta(\nu\tau)$, the plant takes the form

$$\zeta(\nu\tau) + \sum_{i=1}^{n}\alpha_i\zeta(\nu\tau - i\tau) = u(\nu\tau) \ , \ y(\nu\tau) = \sum_{i=1}^{n}\beta_i\zeta(\nu\tau - i\tau) \ , \ \nu \geq 1 \tag{5.15}$$

Defining the following vectors

$$\hat{\varphi}(\nu\tau) = \begin{bmatrix} u(\nu\tau) & \cdots & u(\nu\tau - n\tau) & y(\nu\tau-\tau) & \cdots & y(\nu\tau - n\tau)\end{bmatrix}^T$$

$$\hat{\zeta}(\nu\tau) = \begin{bmatrix}\zeta(\nu\tau) & \cdots & \zeta(\nu\tau - n\tau)\end{bmatrix}^T$$

it is easy to see that

$$\hat{\varphi}(\nu\tau) = \mathbf{P}\hat{\zeta}(\nu\tau) \tag{5.16}$$

where **P** is a Sylvester-matrix which is nonsingular due to Assumption 2.2, and has the following form

$$\mathbf{P} = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 0 & 0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \beta_1 & \cdots & \beta_{n-2} & \beta_{n-1} & \beta_n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_n \end{bmatrix}$$

Observe also that the vectors $\varphi(\nu\tau)$ and $\hat{\varphi}(\nu\tau)$ are interrelated upon the following relation

$$\varphi(\nu\tau) = \mathbf{T}\hat{\varphi}(\nu\tau) \tag{5.17}$$

where $\mathbf{T} \in \mathbb{R}^{2n\times(2n+1)}$ is the full row rank matrix of the form

$$\mathbf{T} = \begin{bmatrix} \mathbf{0}_{2n\times 1} & \begin{matrix} \mathbf{0}_{n\times n} & -\mathbf{I}_{n\times n} \\ \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n}\end{matrix}\end{bmatrix}$$

It is now obvious that excitation of $\hat{\zeta}(\nu\tau)$ implies excitation of $\varphi(\nu\tau)$. Therefore, we next investigate excitation of $\hat{\zeta}(\nu\tau)$. To this end, observe that from relation (5.15), we can write

$$\gamma^{T}\hat{\zeta}(\nu\tau) = u(\nu\tau) \tag{5.18}$$

where $\gamma^{T} \in \mathbb{R}^{2n+1}$ is the following vector

$$\gamma^{T} = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_n & 0 & \cdots & 0 \end{bmatrix}$$

Now, let $\mathbf{X}(\nu\tau) \in \mathbb{R}^{2n\times 2n}$ be the following symmetric matrix

$$\mathbf{X}(\nu\tau) = \begin{bmatrix} \hat{\zeta}(\nu\tau) & \hat{\zeta}(\nu\tau-\tau) & \cdots & \hat{\zeta}(\nu\tau-2n\tau) \end{bmatrix} \tag{5.19}$$

and $\hat{\mathbf{u}}(\nu\tau) \in \mathbb{R}^{2n}$ be the following vector

$$\hat{\mathbf{u}}(\nu\tau) = \begin{bmatrix} u(\nu\tau) & u(\nu\tau-\tau) & \cdots & u(\nu\tau-2n\tau) \end{bmatrix}^{T} \tag{5.20}$$

Combining relations (5.18)-(5.20), we obtain

$$\gamma^{T}\mathbf{X}(\nu\tau) = \hat{\mathbf{u}}^{T}(\nu\tau)$$

Therefore, for every column vector η, with norm equal to unity, we have

$$\left|\eta^{T}\hat{\mathbf{u}}(\nu\tau)\right|^{2} = \left|\eta^{T}\mathbf{X}^{T}(\nu\tau)\gamma\right|^{2} = \left|\gamma^{T}\mathbf{X}(\nu\tau)\eta\right|^{2} \le \|\gamma\|^{2}\left\|\mathbf{X}^{T}(\nu\tau)\eta\right\|^{2}$$

Summing over the interval $\left[kT_0 + (2n+1)\tau\ ,\ kT_0 + (4n+1)\tau\right]$ and observing that

$$\left[\hat{\mathbf{u}}\left(kT_0 + (2n+1)\tau\right) \quad \hat{\mathbf{u}}\left(kT_0 + (2n+2)\tau\right) \quad \cdots \quad \hat{\mathbf{u}}\left(kT_0 + (4n+1)\tau\right)\right] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \hat{\mathbf{U}}(kT_0)$$

we obtain

$$\sum_{\nu=2n+1}^{4n+1}\left|\eta^{T}\hat{\mathbf{u}}\left(kT_0+\nu\tau\right)\right|^{2} = \left\|\hat{\mathbf{U}}(kT_0)\eta\right\|^{2} \le \|\gamma\|^{2}\sum_{\nu=2n+1}^{4n+1}\left\|\mathbf{X}^{T}(\nu\tau)\eta\right\|^{2} \le \|\gamma\|^{2}(2n+1)\sum_{\nu=1}^{4n+1}\left[\hat{\zeta}^{T}\left(kT_0+\nu\tau\right)\eta\right]^{2}$$

Hence,

$$\sum_{\nu=1}^{4n+1}\left[\hat{\zeta}^{T}\left(kT_0+\nu\tau\right)\eta\right]^{2} \ge \left[\|\gamma\|^{2}(2n+1)\right]^{-1}\left\|\hat{\mathbf{U}}(kT_0)\eta\right\|^{2}$$

Since the smallest singular value of $\hat{\mathbf{U}}(kT_0)$ is greater than a constant, there is a constant $\delta > 0$ such that

$$\sum_{\nu=1}^{4n+1}\hat{\zeta}\left(kT_0+\nu\tau\right)\hat{\zeta}^{T}\left(kT_0+\nu\tau\right) \ge \delta$$

Therefore, the vector $\hat{\zeta}(\nu\tau)$ is persistently exciting. According to (5.16) and (5.17), the regressor sequence $\varphi(\nu\tau)$, is also persistently exciting. This completes the proof of the Theorem. □

We are now able to establish the stability of the adaptive control system.

Proposition 5.2. The closed loop adaptive control system is globally stable, i.e. for arbitrary finite initial conditions, all states are uniformly bounded, and as $k \to \infty$, the closed-loop system poles are driven to the prespecified locations $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_n$. Furthermore, the proposed adaptive scheme provides exponential convergence of the estimated parameters.

Proof: Since according to Theorem 5.1, the regressor sequence is persistently exciting, then the difference $\hat{\theta}_k - \theta$ converges to zero. That is, the plant parameter estimates converge to their true values. As a consequence of this and of the fact that $\hat{\theta}_k$ is uniformly bounded, the controller parameter estimates (5.13) also converge to their true values. Therefore, at the sampling instants uniform boundedness of all states and stabilization through pole placement follow on the basis of (4.4). Uniform boundedness of $u(t)$ and $\mathbf{x}(t)$ then follows from (2.1), (4.7), (4.8) and the fact that $w(kT_0)$ is bounded by assumption. Finally, exponential convergence of the plant parameter estimates follows from (5.10), which together with (5.14), ensures that $\hat{\theta}_k \to \theta$ exponentially as $k \to \infty$. □

6 Illustrative Examples and Simulation Study

In this Section, the proposed adaptive pole placement algorithm is tested through two illustrative simulation examples. The simulation results that follows have been obtained by the use of the «ppmric» procedure, which has been programmed for this purpose in MATLAB 4.2c for WINDOWS.

Example 6.1. In this example, we address the same continuous-time plant utilized in Kinaert and Blondel (1992), which is the unstable second order system with transfer function

$$H(s) = \frac{s-10}{s(s-1)}$$

The plant has a non-minimum phase zero at s=10. Note that the plant possesses the p.i.p. and hence it is strongly stabilizable. The plant is discretized by using a zero order hold and a sampling interval of T_0=0.02 secs, that yields the following discrete-time plant

$$H(z) = \frac{0.0182z - 0.0222}{z^2 - 2.0202z + 1.0202}$$

which has a pole at z=1, an unstable pole at z=1.0202 and an unstable zero at z=1.2221. Our aim here is to find an adaptive periodic multirate pole placer of the form (4.1), such that the eigenvalues of the closed loop system to be $\hat{\lambda}_1 = \hat{\lambda}_2 = 0.9231$. This reveals that the closed-loop system model is

$$H_m(z) = \frac{0.0182z - 0.0222}{z^2 - 1.8462z + 0.8521}$$

which represents a second order system with damping ratio coefficient of 1 and a natural frequency of 4 rads/sec.

The above pole placement problem, can be solved by using a dynamic controller, which consists on a precompensator $\frac{T(z)}{R(z)}$ and an output feedback compensator $\frac{S(z)}{R(z)}$, i.e. with a compensator of the form

$$U(z) = \frac{T(z)}{R(z)}W(z) - \frac{S(z)}{R(z)}Y(z) \tag{6.1}$$

where W(z) and Y(z) are the usual $\mathbb{Z}$ -transforms of $w(kT_0)$ and $y(kT_0)$, respectively.

As it can be easily checked, from the solution of the Diophantine equation, the resulting compensators are constituted by the following polynomials

$$T(z) = z \ , \ R(z) = z + 1.2275 \ , \ S(z) = -57.8852z + 56.4102$$

The compensators are unstable and have unbounded outputs. In a short time, the controller produces an non-negligible error in computing the control signal which unables to stabilize the plant. In this case the solution is purely mathematical and useless for applications. Therefore, it is necessary to use other controllers or procedures to obtain solutions with stable compensators.

Let us try to apply the proposed procedure, based on MROCs. By selecting the output multiplicity of the sampling N such that N=4, the sampling period τ has the value τ=1 msec. Application of the results of Sections 3 and 4, result to the following periodic multirate modulating function

$$f(t) = \mathbf{e}_{N-\mu}\begin{bmatrix}890.1630 & -896.0177 & 0 & 0\end{bmatrix}^T \ , \ \mu = 0,1,2,3$$

In the unknown plant case, the simulation has been performed using the proposed modified recursive least square algorithm given in (5.7). The nominal parameter vector θ has the form

$$\theta = [-2.0010 \quad 1.0010 \quad 0.0010 \quad -0.0010]^T$$

The identification algorithm was initialized with the following parameter vector

$$\hat{\theta}_0 = [1 \quad 1 \quad 1 \quad 1]^T$$

and with a=0.2. Simulation results are given in Figures 3-8. To obtain the simulation results both nominal and estimated closed-loop system are excited by a reference signal, which is a unity square wave with period of 12 secs. Similar simulation examples can be obtained in the case where $\hat{\theta}_0$ or a, take other values, e.g. a=0.5 or 0.9 and

$$\hat{\theta}_0 = [4 \quad 4 \quad 3 \quad 3]^T \text{ or } \hat{\theta}_0 = [2 \quad -2 \quad 1 \quad 4]^T$$

Example 6.2. Consider the following unstable third order system with transfer function

$$H(s) = \frac{(s-1)(s+5)}{(s-2)(s+1)(s+3)}$$

Note that the plant has a non-minimum phase blocking zero at s=1 and an unstable pole at s=2. Namely, this plant does not possesses the p.i.p. in the continuous-time sense and hence it is not strongly stabilizable. The plant is discretized by using a zero order hold and a sampling interval of $T_0 = 0.2$ secs, that yields the following discrete-time plant

$$H(z) = \frac{0.2361z^2 - 0.3721z + 0.1025}{z^3 - 2.8594z^2 + 2.4895z - 0.6703}$$

which has two poles at z=0.5488 and z=0.8187, an unstable pole at z=1.4918, a zero at z=0.3557 and a non-minimum phase zero at z=1.2203. The discretized system also does not possesses the p.i.p. Our aim here is to find an adaptive periodic multirate pole placer of the form (4.1), such that the eigenvalues of the closed loop system to be $\hat{\lambda}_1 = 0.9$, $\hat{\lambda}_2 = 0.7$ and $\hat{\lambda}_3 = -0.6$, a fact which reveals that the closed-loop system has the following discrete transfer function

$$H_m(z) = \frac{0.2361z^2 - 0.3721z + 0.1025}{z^3 - z^2 - 0.3301z + 0.3780}$$

The foregoing pole placement problem, can be solved by using a dynamic controller of the form (6.1). As it can be easily checked, from the solution of the Diophantine equation, the resulting compensators are constituted by the following polynomials

$$T(z) = z^2 \quad , \quad R(z) = z^2 - 8.4540z + 2.6579 \quad , \quad S(z) = 43.6825z^2 - 56.7412z + 17.3813$$

The roots of R(z) are z=0.3270 and z=8.1270. Clearly, the compensators are unstable and useless. Therefore, it is necessary to use other controllers or procedures to obtain solutions with stable compensators. However, no such procedure exists hitherto, since, as already mentioned, the given system does not possesses the p.i.p..

Let us try to apply the proposed procedure. By selecting the output multiplicity of the sampling N such that N=5, the sampling period τ has the value τ=5,71 msec. Application of the results of Sections 3 and 4, result to the following periodic multirate modulating function

$$f(t) = \mathbf{e}_{N-\mu}[2196.6 \quad -4756.0 \quad 2570.4 \quad 0 \quad 0]^T \quad , \quad \mu = 0,1,2,3,4$$

In the unknown plant case, the simulation has been performed using the proposed modified recursive least square algorithm given in (5.7). The nominal parameter vector θ has the form

$$\theta = [-2.9888 \quad 2.9774 \quad -0.9886 \quad 0.0057 \quad -0.0114 \quad 0.0056]^T$$

The identification algorithm was initialized with the following parameter vector

$$\hat{\theta}_0 = [-2 \quad 4 \quad 2 \quad 1 \quad 2 \quad 2]^T$$

and with a=1. Simulation results are given in Figures 9-16. To obtain the simulation results both nominal and estimated closed-loop system are excited by a reference signal, which is a unity square wave with period of 12 secs. Similar simulation results can be obtained in the case where $\hat{\theta}_0$ or a, take other values, e.g. a=0.5 or 0.2 and

$$\hat{\theta}_0 = \begin{bmatrix}-5 & 4 & 2 & 1 & 1 & -1\end{bmatrix}^T \text{ or } \hat{\theta}_0 = \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}^T$$

From the two illustrative examples, it is easily seen that the proposed adaptive pole placement algorithm based on MRICs has a good performance even if it is applied to nonminimum-phase plants and to systems which do not possesses the p.i.p.

It is remarked at this point that, the square wave with period of 12 secs used in both examples, as the reference signal, provides sufficient excitation to the plant under control. So in this case the excitation signal v(t) is useless. In the case where v(t) is added in the control loop, simulation results show that, in the case where **v** is evaluated by (4.15), the convergence of the identification algorithm is ameliorated. It is worth noticed that in this case, the excitation v(t) causes to the closed-loop system output, a static steady state error of approximately 15%. However, this static error is eliminated by evaluating **v** through (4.16) and by selecting appropriately the arbitrary parameters ρ_j, j=1,2 (e.g. ρ_1=0.5, ρ_2=-0.45 in the case of Example 6.1).

7 Conclusions

A new periodic multirate sampled-data adaptive pole placer for continuous-time linear systems has been exhibited in the present paper. The proposed control strategy has several advantages over known indirect adaptive pole placement techniques. The main of them are:

(a) It is readily applicable to nostably invertible plants having arbitrary poles and zeros and relative degree. This due to the fact that the approach used to solve the adaptive pole placement problem does not rely on pole-zero cancellations.

(b) Following the proposed technique a gain controller is essentially needed to be designed, as compared to dynamic compensators or state observers needed by known indirect adaptive pole placement schemes. Consequently, the proposed approach avoids the problems of known adaptive pole placement techniques, interwoven with the possibly unstable solutions of the Diophantine equation. Moreover, no exogenous dynamics are introduced in the control loop by the proposed technique, whereas in many known techniques the dynamics introduced are of high order. This fact improves the computational aspect of the problem, since the proposed technique does not require many on-line computations and its practical implementation requires computer memory only for storing the modulating function f(t) over one period of time.

(c) Finally, persistency of excitation of the plant under control and hence parameter convergence, is provided, without making any special richeness assumption on of the reference signals, as compared to known indirect adaptive pole placement control schemes.

The present paper gives some new insights to the adaptive pole placement problem of linear systems. The proposed technique can be easily extended to solve other important problems of the area of adaptive control, such as model reference adaptive control, adaptive LQG regulation, etc. and for other types of systems, such as time-varying periodic and non-periodic linear systems. Adaptive control schemes based on alternative parameter-estimation algorithms or on alternative multirate controllers are currently under investigation.

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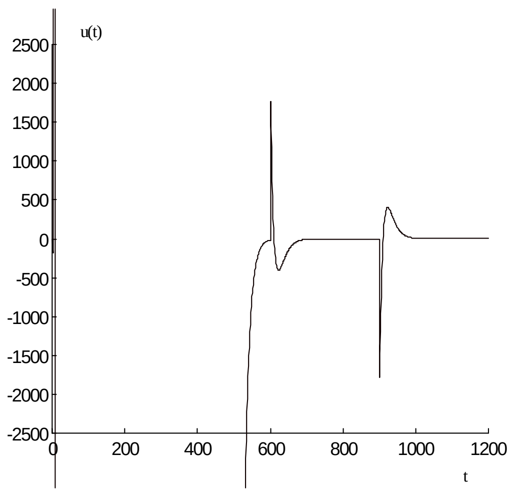

Figure 3. Controller output for Example 6.1.

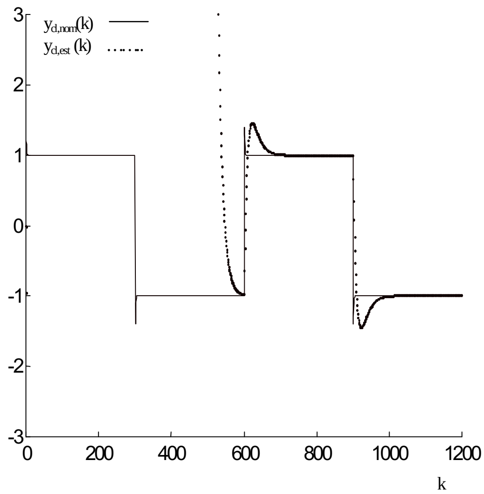

Figure 4. Closed-loop system output under identification process $y_{cl,est}(k)$ versus nominal closed-loop system output $y_{cl,nom}(k)$ for Example 6.1.

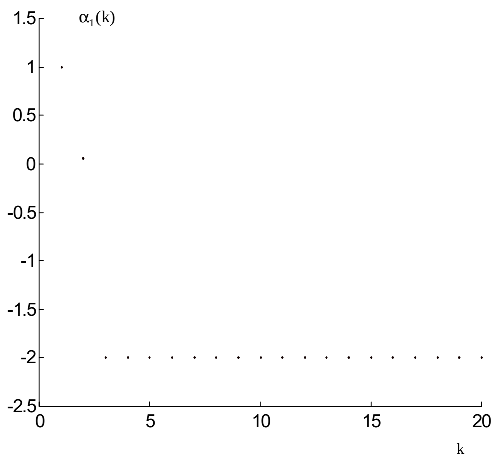

Figure 5. Estimates of $\alpha_1(k)$ for Example 6.1.

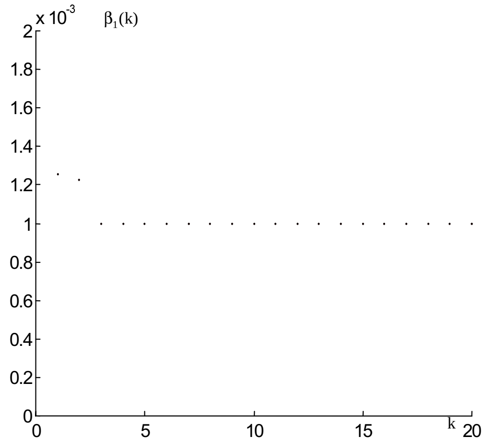

Figure 7. Estimates of $\beta_1(k)$ for Example 6.1.

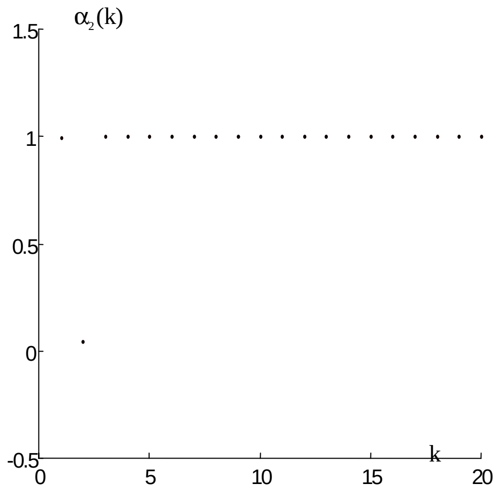

Figure 6. Estimates of $\alpha_2(k)$ for Example 6.1.

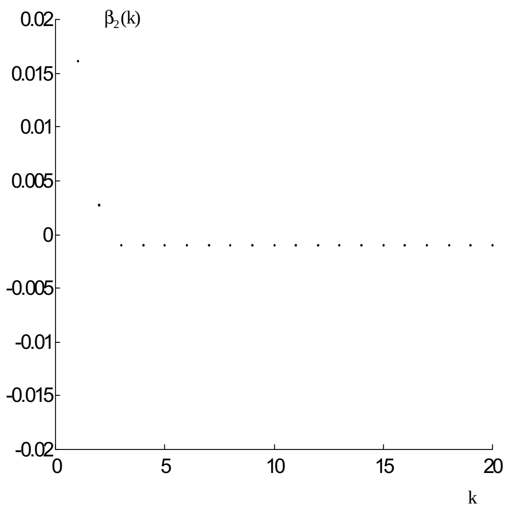

Figure 8. Estimates of $\beta_2(k)$ for Example 6.1.

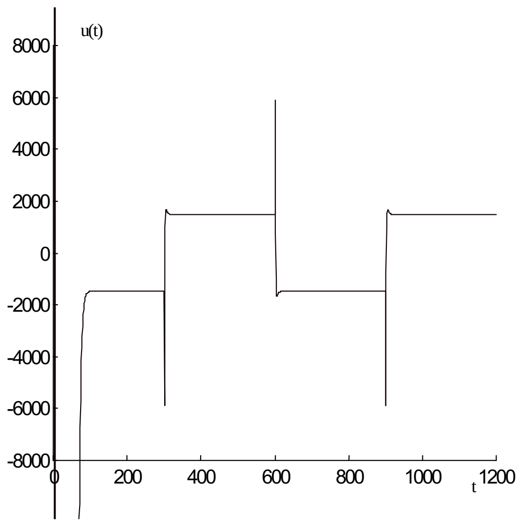

Figure 9. Controller output for Example 6.2.

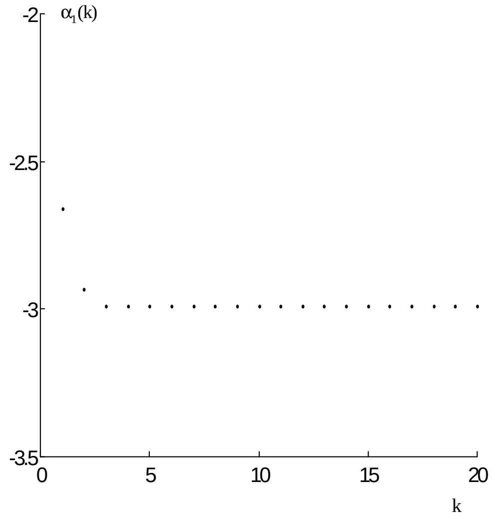

Figure 11. Estimates of $\alpha_1(k)$ for Example 6.2.

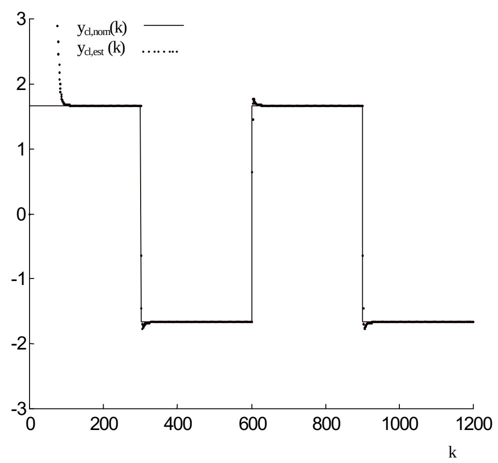

Figure 10. Closed-loop system output under identification process $y_{cl,est}(k)$ versus nominal closed-loop system output $y_{cl,nom}(k)$ for Example 6.2.

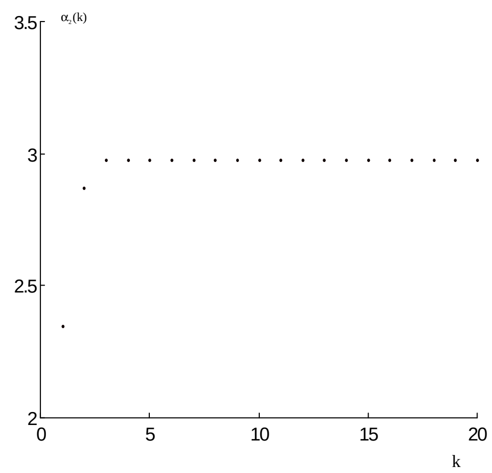

Figure 12. Estimates of $\alpha_2(k)$ for Example 6.2.

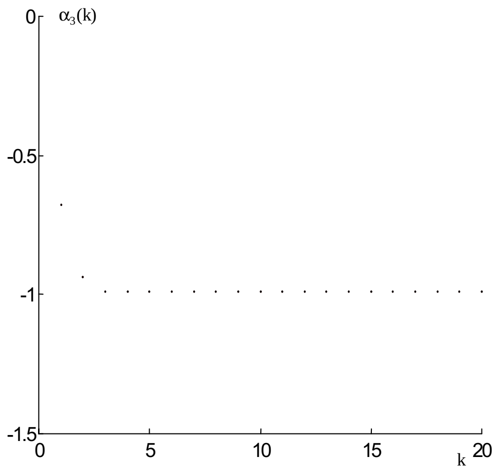

Figure 13. Estimates of α_3(k) for Example 6.2.

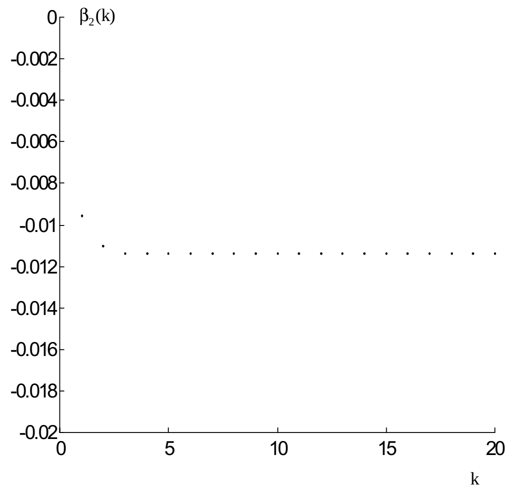

Figure 15. Estimates of β_2(k) for Example 6.2.

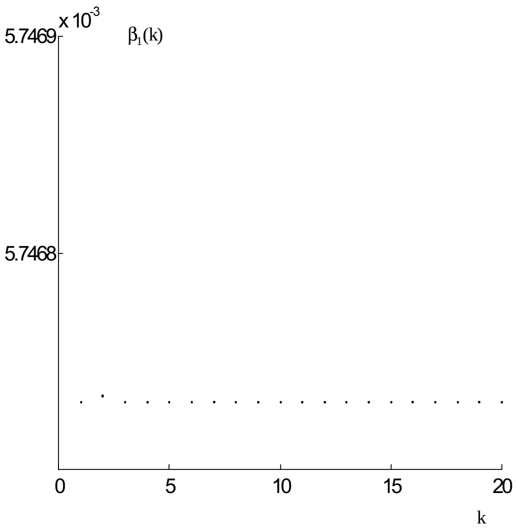

Figure 14. Estimates of β_1(k) for Example 6.2.

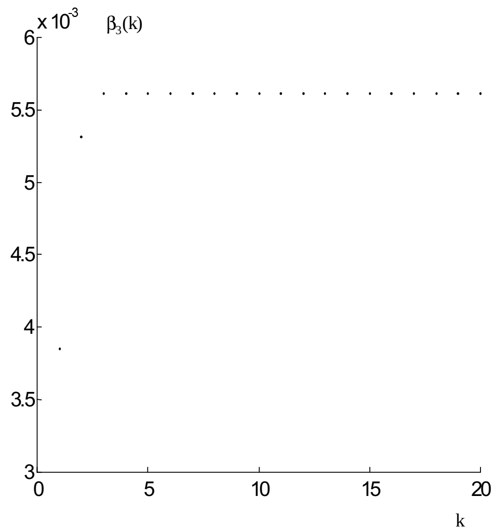

Figure 16. Estimates of β_3(k) for Example 6.2.