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Improved Wiener-Hopf method for $\mathcal{H}_2-$design of sampled-data systems*

B. P. Lampe†
University of Rostock, Dept. Electrical Eng., D-18051 Rostock, Germany

Y. N. Rosenwasser
St. Petersburg State University of Ocean Technology
Dept. Automation, Lotsmanskaya ul. 3, 19800 St. Petersburg, Russia

Abstract

The paper deals with the direct design of sampled-data systems in the $\mathcal{H}_2-$metric by applying the Wiener–Hopf method. The presented technique avoids basic controllers during the design procedure that are normally used to parametrize the set of stabilizing controllers. The suggested method is independent from the pole situation of the transfer matrices of the plant model and is also applicable if poles are placed on the imaginary axis.

Key Words. Sampled-data control, MIMO, Parametric transfer matrix, Wiener-Hopf method, $\mathcal{H}_2$-optimization, Parametrization

1 Notation, Concepts

In the paper two complex variables ζ and s will be used that are connected by the relation

$$\zeta = \mathrm{e}^{-sT} \tag{1}$$

where $T > 0$ is a real number. Functions of the argument s that depend in fact on e^{-sT} will be of main interest. Therefore, introduce the notation

$$f(\zeta) = \tilde{f}(s)\Big|_{\mathrm{e}^{-sT}=\zeta}, \quad \tilde{f}(s) = f(\zeta)|_{\zeta=\mathrm{e}^{-sT}}\,. \tag{2}$$

By construction

$$\tilde{f}(s) = \tilde{f}(s + \mathrm{j}\omega), \quad \mathrm{j} = \sqrt{-1}, \quad \omega = \frac{2\pi}{T}\,. \tag{3}$$

For an arbitrary bilateral Laplace transform $F(s)$ introduce the notation Rosenwasser (1994c); Rosenwasser and Lampe (1997)

$$\varphi_F(T, s, t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F(s + k\mathrm{j}\omega)\mathrm{e}^{k\mathrm{j}\omega t} \tag{4}$$

$$\tilde{D}_F(T, s, t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F(s + k\mathrm{j}\omega)\mathrm{e}^{(s+k\mathrm{j}\omega)t}\,. \tag{5}$$

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†`bernhard.lampe@etechnik.uni-rostock.de`

The series (4) is called the *displaced pulse frequency response* (DPFR) of the transform $F(s)$, and the series (5) its *discrete Laplace transform.* A quadratic polynomial matrix $A(\zeta)$ is said to be *stable*, if it has no eigenvalues inside the unit disc or on its border. A rational matrix is called stable, if it has no poles inside or on the border of the unit disc. From this point of view any polynomial matrix is a stable rational matrix.

2 System description

1. Consider the system of Figure 2 where $F(s)$ is the transfer matrix of the continuous part of

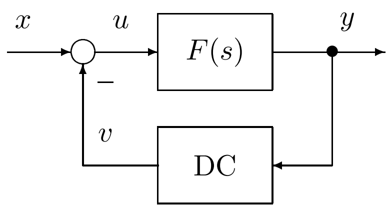

Figure 1: Sampled-data control system

the system, x, y are vectors of the dimensions $m \times 1$, $n \times 1$ respectively, and DC is the MIMO digital computer. The continuous part of the system satisfies the state space equations

$$y = Cx_1\,, \qquad \frac{\mathrm{d}x_1}{\mathrm{d}t} = Ax_1 + Bu \tag{6}$$

with the completely controllable pair (A, B) and the completely observable pair (A, C). Therefore

$$F(s) = \frac{C\,\mathrm{adj}(sI - A)B}{\det(sI - A)} \tag{7}$$

where the fraction on the left side is assumed to be irreducible. In (7) and further I means the identity matrix of appropriate size and $\mathrm{adj}(sI - A)$ stands for the adjoint matrix.

2. The MIMO digital computer DC is given by the structure shown in Figure 2, where ADC

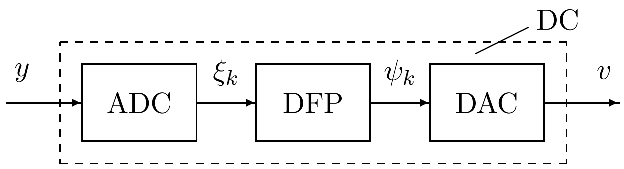

Figure 2: Structure of the digital controller

is the analogue to digital converter, DFP is the digital filter program, and DAC is the digital to analogue converter. The equations of the ADC and the digital filter program take the form

$$\xi_k = y(kT) \tag{8}$$

with the sampling period $T > 0$ and

$$\alpha_0\psi_k + \alpha_1\psi_{k-1} + \ldots + \alpha_p\psi_{k-p} = \beta_0\xi_k + \beta_1\xi_{k-1} + \ldots + \beta_p\xi_{k-p} \tag{9}$$

where α_i, β_i are constant $m \times m$ resp. $m \times n$ matrices. Moreover, the causality condition $\det \alpha_0 \neq 0$ should be satisfied, and at least one of the matrices α_p or β_p should not be a zero matrix. The DAC is described by the equation

$$v(t) = \mu(t - kT)\psi_k\,, \quad kT \le t < (k+1)T \tag{10}$$

where $\mu(t)$ is a scalar function determining the shape of all controlling pulses.

3. The function

$$M(s) = \int_0^T \mu(t)\mathrm{e}^{-st}\,\mathrm{d}t \tag{11}$$

will be called the transfer function of the forming element, and the sum of the series

$$\varphi_{FM}(T,s,t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F(s+k\mathrm{j}\omega)M(s+k\mathrm{j}\omega)\mathrm{e}^{k\mathrm{j}\omega t}\,, \qquad \omega = \frac{2\pi}{T} \tag{12}$$

is referred to as the DPFR of the plant under control.

Denote

$$\tilde{w}_0(s) = \tilde{\varphi}_{FM}(T,s,0) = \tilde{D}_{FM}(T,s,0)\,. \tag{13}$$

Then, the matrix

$$w_0(\zeta) = \left.\tilde{w}_0(s)\right|_{\mathrm{e}^{-sT}=\zeta} \tag{14}$$

is called the *discrete transfer matrix* of the controlled plant.

4. The matrix

$$w_d(\zeta) = \alpha_l^{-1}(\zeta)\beta_l(\zeta) \tag{15}$$

with

$$\begin{aligned} \alpha_l(\zeta) &= \alpha_0 + \alpha_1\zeta + \ldots + \alpha_p\zeta^p \\ \beta_l(\zeta) &= \beta_0 + \beta_1\zeta + \ldots + \beta_p\zeta^p \end{aligned} \tag{16}$$

is named the transfer matrix of the digital controller. Going back to the argument s you find from (15)

$$\tilde{w}_d(s) = \tilde{\alpha}_l^{-1}(s)\tilde{\beta}_l(s) \tag{17}$$

with

$$\tilde{\alpha}_l(s) = \left.\alpha_l(\zeta)\right|_{\zeta=\mathrm{e}^{-sT}}\,, \qquad \tilde{\beta}_l(s) = \left.\beta_l(\zeta)\right|_{\zeta=\mathrm{e}^{-sT}}\,. \tag{18}$$

5. Let us have

$$\det(sI - A) = (s - s_1)^{\mu_1}\cdots(s - s_q)^{\mu_q} \tag{19}$$

and the nonpathological conditions, Rosenwasser (1994c); Rosenwasser and Lampe (1997)

$$\mathrm{e}^{s_iT} \neq \mathrm{e}^{s_\kappa T}, \quad (i \neq \kappa;\, i,\kappa = 1,\ldots,q) \tag{20}$$

$$M(s_i) \neq 0, \quad (i = 1,\ldots,q) \tag{21}$$

may be fulfilled. Then the matrix (14) can be represented in the form, Rosenwasser (1995a,b)

$$w_0(\zeta) = \frac{B(\zeta)}{\Delta(\zeta)} \tag{22}$$

with a polynomial matrix $B(\zeta)$, and

$$\Delta(\zeta) = (1 - \mathrm{e}^{s_1 T}\zeta)^{\mu_1} \cdots (1 - \mathrm{e}^{s_q T}\zeta)^{\mu_q}. \tag{23}$$

In (22) the fraction is not reducible and satisfies the irreducible MFD (matrix fraction description)

$$w_0(\zeta) = a_l^{-1}(\zeta) b_l(\zeta) = -b_r(\zeta) a_r^{-1}(\zeta) \tag{24}$$

with

$$\det a_l(\zeta) \approx \det a_r(\zeta) \approx \Delta(\zeta)\,. \tag{25}$$

In (25) the symbol '$\approx$' stands for the equivalence of scalar polynomials.

6. The lemma of Bézout, Kailath (1980) ensures for any irreducible MFD (24) the existence of matrices $\alpha_{0l}(\zeta)$, $\beta_{0l}(\zeta)$, $\alpha_{0r}(\zeta)$, $\beta_{0r}(\zeta)$ which yield unimodular matrices

$$Q_{0l}(\zeta) = \begin{bmatrix} a_l(\zeta) & -b_l(\zeta) \\ \beta_{0l}(\zeta) & \alpha_{0l}(\zeta) \end{bmatrix}, \quad Q_{0r}(\zeta) = \begin{bmatrix} \alpha_{0r}(\zeta) & -b_r(\zeta) \\ \beta_{0r}(\zeta) & a_r(\zeta) \end{bmatrix} \tag{26}$$

with

$$Q_{0l}(\zeta) Q_{0r}(\zeta) = Q_{0r}(\zeta) Q_{0l}(\zeta) = I\,. \tag{27}$$

The last row of the block matrix $Q_{0l}(\zeta)$ and the first column of the block matrix $Q_{0r}(\zeta)$ are called *basic (left and right) regulators.*

7. As was shown in Rosenwasser (1994a,b), for the stability of the closed loop in the sense of Lyapunov it is necessary and sufficient that the matrices $\alpha_l(\zeta)$, $\beta_l(\zeta)$ of the regulator (15) take the form

$$\begin{aligned} \alpha_l(\zeta) &= D_l(\zeta)\alpha_{0l}(\zeta) - M_l(\zeta) b_l(\zeta) \\ \beta_l(\zeta) &= D_l(\zeta)\beta_{0l}(\zeta) + M_l(\zeta) a_l(\zeta) \end{aligned} \tag{28}$$

where $D_l(\zeta)$, $M_l(\zeta)$ are polynomial matrices, and $M_l(\zeta)$ is arbitrary, but $D_l(\zeta)$ is arbitrary stable. From (28) and (15) follows that the set of transfer matrices of the stabilizing regulators can be parametrized as

$$w_d(\zeta) = \left[\alpha_{0l}(\zeta) - \Psi(\zeta) b_l(\zeta)\right]^{-1} \left[\beta_{0l}(\zeta) + \Psi(\zeta) a_l(\zeta)\right] \tag{29}$$

where

$$\Psi(\zeta) = D_l^{-1}(\zeta) M_l(\zeta) \tag{30}$$

is an arbitrary stable rational matrix.

3 Formulation of the optimization problem

1. Let the closed loop system be stable in the sense of Lyapunov. Then the $\mathcal{H}_2$–norm of the system is determined, Rosenwasser (1995a,b); Rosenwasser and Lampe (1997) by the relation

$$\|H\|_2^2 = \frac{1}{2\pi \mathrm{j}} \int_{-\mathrm{j}\infty}^{\mathrm{j}\infty} \operatorname{tr} B_0(s)\, \mathrm{d}s \tag{31}$$

where 'tr' means the trace of the matrix, and the matrix $B_0(s)$ is given by

$$B_0(s) = \frac{1}{T}\int_0^T \underline{w}'(s,t)w(s,t)\,\mathrm{d}t\,. \tag{32}$$

In (32) is $w(s,t)$ the parametric transfer matrix (PTM) of the system from the input x to the output y, the prime denotes the transpose operator and the underline stands for the substitution of s by $-s$. In Rosenwasser (1997) it was shown that the expression (31), (32) agrees with the formula for the $\mathcal{H}_2$−norm of the standard system derived in Hagiwara and Araki (1995) with the help of the FR-operator theory.

2. In the present case the PTM $w(s,t)$ has the form Rosenwasser (1995a,b); Rosenwasser and Lampe (1997)

$$w(s,t) = F(s) - \varphi_{FM}(T,s,t)\tilde{U}(s)F(s) \tag{33}$$

with

$$\tilde{U}(s) = \tilde{w}_d(s)\left[I + \tilde{w}_0(s)\tilde{w}_d(s)\right]^{-1}\,. \tag{34}$$

Substituting the above relations (32)–(34) into (31) yields a quadratic functional that depends on the matrix $\tilde{U}(s)$ that on its part is a function of the unknown matrix $\tilde{w}_d(s)$. Therefore, the $\mathcal{H}_2$−optimization problem for the system can be formulated as follows:

> **$\mathcal{H}_2$−problem :** For given transfer matrix $F(s)$, function $\mu(t)$, and sampling period T, find a transfer matrix for the discrete-time controller $w_d(\zeta)$ that ensures the stability of the closed-loop system and minimizes the functional (31).

3. In Hagiwara and Araki (1995) a procedure for minimizing the integral (31) is presented that bases on the solution of a certain algebraic matrix Riccati equation. This method requires that the elements of the plant $F(s)$ have no neutral poles, especially, no zero poles. The last assumption is an essential restriction, because in some applications, for instance in dynamic control systems, the plants normally have integral action.

In Lampe and Rosenwasser (1996); Rosenwasser and Lampe (1997) a method for solving the $\mathcal{H}_2$−problem for multi-variable sampled-data systems is given on the basis of the parametrization (28), (29). Following Youla *et al.* (1976); Parks and Bongiorno (1989); Grimble and Kučera (1996) this approach is referred to as Wiener-Hopf method. In a lot of cases the Wiener-Hopf method allows to remove the restrictions for the poles of the matrix $F(s)$. Nevertheless, its practical application leads to serious numerical difficulties, that are caused by the basic regulator as a design parameter. The present paper describes an improved variant of the Wiener-Hopf method that avoids the basic controller during the design process. The method is examplarily illustrated for the system of Figure 2, though it can be extended to more general structures.

4 Main result

Under the above assumptions it can be shown that the $\mathcal{H}_2$−problem is reducible to the minimization problem of a quadratic functional of the form

$$J = \frac{1}{2\pi\mathrm{j}}\oint \mathrm{tr}\left[\Psi(\zeta)G(\zeta)\Psi^*(\zeta)H(\zeta) - \Psi^*(\zeta)C^*(\zeta) - C(\zeta)\Psi(\zeta)\right]\frac{\mathrm{d}\zeta}{\zeta} \tag{35}$$

where the contour integral is taken over the unit circle in positive direction, $\Psi(\zeta)$ is an unknown stable rational matrix, and the asterix denotes the Hermitian conjugate

$$Q^*(\zeta) = Q'(\zeta^{-1})\,. \tag{36}$$

The coefficients of the functional (35) are determined by the relations

$$\begin{aligned}
G(\zeta) &= a_r^*(\zeta) D_{\underline{F'}FM\underline{M}}(T,\zeta,0) a_r(\zeta), \quad G(\zeta) = G^*(\zeta)\\
H(\zeta) &= \frac{1}{T} a_l(\zeta) D_{F\underline{F'}}(T,\zeta,0) a_l^*(\zeta), \quad H(\zeta) = H^*(\zeta)\\
C(\zeta) &= P(\zeta) + H(\zeta)\beta_{0r}^*(\zeta) D_{\underline{F'}FM\underline{M}}(T,\zeta,0) a_r(\zeta)\\
C^*(\zeta) &= P^*(\zeta) + a_r^*(\zeta) D_{\underline{F'}FM\underline{M}}(T,\zeta,0)\beta_{0r}(\zeta) H(\zeta)
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
P(\zeta) &= \frac{1}{T} a_l(\zeta) D_{F\underline{F'}FM}(T,\zeta,0) a_r(\zeta)\\
P^*(\zeta) &= \frac{1}{T} a_r^*(\zeta) D_{\underline{F'}F\underline{F'}\underline{M}}(T,\zeta,0) a_l^*(\zeta)\,.
\end{aligned} \tag{38}$$

If the wanted matrix $\Psi^{\mathrm{opt}}(\zeta)$ is found that minimizes the functional (35) then the transfer matrix of the optimal discrete-time controller $w_d^{\mathrm{opt}}(\zeta)$ is held by (29). On this occasion, because the basic regulator is part of the coefficient $C(\zeta)$ and also of the relation (29), the formal minimalization procedure of the functional (35) and the calculation of $w_d^{\mathrm{opt}}(\zeta)$ depends on the parameters of the basic regulator. Really, it turns out, that using special properties of the matrices (37), (38), the basic controllers can be excluded from the procedure.

Theorem 1 *The matrices (37) permit, independently of the pole properties of the transfer matrix $F(s)$, the representations*

$$\begin{aligned}
G(\zeta) &= \sum_{i=-\nu}^{\nu} G_i\zeta^i\,, \qquad G_i = G'_{-i}\\
H(\zeta) &= \sum_{i=-\kappa}^{\kappa} H_i\zeta^i\,, \qquad H_i = H'_{-i}\\
C(\zeta) &= \sum_{i=-\gamma}^{\delta} C_i\zeta^i\,, \qquad C^*(\zeta) = \sum_{i=-\gamma}^{\delta} C'_i\zeta^{-i}
\end{aligned} \tag{39}$$

where ν, κ, γ, δ are positive integers, and the G_i, H_i, C_i are real constant matrices. ■

The fundamental resultat of the paper consist in the following statement.

Theorem 2 *Suppose the existence of the factorizations*

$$G(\zeta) = \Pi^*(\zeta)\Pi(\zeta), \qquad H(\zeta) = \Gamma(\zeta)\Gamma^*(\zeta) \tag{40}$$

where $\Pi(\zeta)$, $\Gamma(\zeta)$ are stable polynomial matrices that have no eigenvalues of the form $\zeta = \mathrm{e}^{\pm s_i T}$, where s_i $(i = 1, \dots, q)$ are the roots of the polynomial (19). Then, the matrix

$$\Theta(\zeta) = \Pi^{*-1}(\zeta) P^*(\zeta)\Gamma^{*-1}(\zeta) \tag{41}$$

permits a unique separation of the form

$$\Theta(\zeta) = \Theta_1(\zeta) + \Theta_2(\zeta) \tag{42}$$

where $\Theta_2(\zeta)$ *is a strictly proper rational matrix with its poles inside the unit disc and that is analytic in the points* $\zeta_i = \mathrm{e}^{-s_i T}$*, and* $\Theta_1(\zeta)$ *is a rational matrix the poles of which are in the points* ζ_i*. Then, the transfer matrix of the optimal discrete-time controller* $w_d^{\mathrm{opt}}(\zeta)$ *can be found by the formula*

$$w_d^{\mathrm{opt}}(\zeta) = -V_2(\zeta)V_1^{-1}(\zeta) \tag{43}$$

where $V_1(\zeta)$*,* $V_2(\zeta)$ *are rational matrices that can be calculated by the relations*

$$\begin{aligned} V_1(\zeta) &= a_l^{-1}(\zeta) + b_r(\zeta)\Pi^{-1}(\zeta)\Theta_1(\zeta)\Gamma^{-1}(\zeta) \\ V_2(\zeta) &= -a_r^{-1}(\zeta)\Pi^{-1}(\zeta)\Theta_1(\zeta)\Gamma^{-1}(\zeta)\,. \end{aligned} \tag{44}$$

The matrices (44) are stable and analytic in the points $\zeta_i = \mathrm{e}^{-s_i T}$*.* ■

Remark Notice that the matrices $G(\zeta)$, $H(\zeta)$, $P(\zeta)$ do not depend on the parameters of the basic controller. Therefore, the optimization procedure presented in Theorem 2, also does not depend on the parameters of the basic controller.

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