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Robust Control for a Class of Linear Infinite Dimensional Systems with Multiplicative Disturbances

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Abstract

In this paper the problem of robust control for a class of linear infinite dimensional systems under mixed disturbances of the multiplicative type is addressed. The Lyapunov function approach is used for proving that there is a controller that stabilizes this class of systems under the presence of uncertainties and perturbations, and guarantees some tolerance level for the joint cost functional. A comment is added to the Riccati operator equation's solution for this problem.

1 Introduction

The affect of uncertainty on the behavior of linear control systems has been for many years a subject of great interest to researches in systems and control engineering. Various ways of modeling uncertainty have been proposed, and it is arguably the case that no single model is best for all applications. It is perhaps most common to model uncertainty about a control system in terms of *imprecisely known "plant parameters"* (suitable defined) or in terms of *disturbances* (either deterministic or stochastic) which corrupt the system's input and output.

There has been a great deal of research effort regarding the modelling and control of infinite dimensional systems (e.g., see, among others, (Bensoussan *et al.*, 1992; Van Keulen, 1993; Curtain and Pritchard, 1978; Curtain and Zwart, 1995; Hinrichsen and Pritchard, 1994; Mordukhovich and Zhang, 1994; Pritchard and Salamon, 1987) and the references there in).

The problem of robust control has been briefly considered in (Curtain and Zwart, 1995) using the semigroup approach, and also in (Bensoussan *et al.*, 1992) through the Method of transposition in the Variational Theory setting. These references propose an optimal program control strategy for the case when all the state is available and do not consider the presence of any type of uncertainties involved in the description of the class of systems they deal with. Recently, however, Curtain *et al.* (1997, 1998a,b), based on Lyapunov redesign, present an adaptive observer along with a parameter adaptive law that, under certain conditions imposed on the plant, achieve state error convergence for a class of infinite dimensional systems with unknown time varying perturbation in the input term.

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Based on our previous works (Poznyak and Rodríguez, 1995, 1999) which deal with full state feedback control and robust boundary control, here we are concerned with the robust control problem for the same class of linear infinite dimensional systems but under mixed disturbances of the multiplicative type, which comprises uncertainties (bounded time-varying unmodeled operators descriptions) as well as perturbations. We assumed that the perturbations are smooth enough such that the Cauchy problem has a unique solution. We used the Lyapunov approach for proving the existence of a controller that robustly stabilizes the already mentioned class of systems together with the disturbance structure described in (Poznyak and Rodríguez, 1995, 1999).

Section two deals with the class of systems and uncertainty. In section three we describe the disturbance structure we are interested in. The problem is posed in section four together with the reference model for robust tracking. The main result of the paper where it is shown that the above mentioned controller exists, is presented in section five. Section six is devoted to the proof of main theorem. Operator Riccati equation and the factorization of all state feedback controllers is given in section seven. And, finally, in the last section we present some brief conclusions.

2 System Description

Let Ω be an open set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $T > 0$, $T \in \mathbb{R}$. Introduce the cylinders

$$Q = \Omega \times (0,T), \qquad \Xi = \partial\Omega \times (0,T).$$

Let U, W, $\mathcal{Z}$, $\mathcal{X}$, $\mathcal{V}$ be Hilbert spaces with their associated scalar product $< \cdot, \cdot >_{\mathcal{X}}$, satisfying $\mathcal{Z} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}$ with continuous dense injections, that is, $\overline{i(\mathcal{Z})} = \mathcal{X}$ and $\overline{i(\mathcal{X})} = \mathcal{V}$. We mention here that $\mathcal{Z}$ is a dense subspace of $\mathcal{X}$ with respect to the norm $\| \cdot \|_{\mathcal{X}}$ induced by the inner product defined on $\mathcal{X}$. Similarly for $\mathcal{X}$ and $\mathcal{V}$.

Following (Poznyak and Rodríguez, 1995, 1999), let us consider the class of linear infinite dimensional dynamical systems, that we denoted by Σ, described as follows:

For all $t \in \mathbb{R}_+$,

$$\frac{\partial x(z,t)}{\partial t} = \mathcal{A}(t)x(z,t) + B_u(t)u(z,t) + B_w(t)w(z,t), \quad z \in \Omega,\ x \in \mathcal{X}, \tag{1}$$

where for all $z \in \Omega$, $t \geq 0$, the densely defined[1] linear operators $\mathcal{A}(t) : \mathcal{D}(\mathcal{A}(t)) \subset \mathcal{X} \to \mathcal{X}$ are given by

$$\mathcal{A}(t)x(z,t) = \alpha\ \nabla \cdot [A(z,t)\nabla^{\mathsf{T}} x(z,t)] + \beta \int_0^t \nabla \cdot [A(z,\tau)\nabla^{\mathsf{T}} x(z,\tau)]d\tau \tag{2}$$

where $\alpha \in \left\{0, 1, \sqrt{-1}\right\}$ and $\beta \in \{0,1\}$, with appropriate boundary and initial conditions which will depend upon the case.

The entries of the operator matrix $A(\cdot,\cdot)$ belong to $C^1(\overline{\Omega} \times \mathbb{R}_+) \cap L^\infty(\overline{\Omega} \times \mathbb{R}_+)$, which can be seen as a nonstationary and spatial variable coefficient. $B_u(t) \in \mathcal{L}(U,\mathcal{V})$, $B_w(t) \in \mathcal{L}(W,\mathcal{V})$, with $u(\cdot,\cdot) \in L^2(\overline{\Omega} \times \mathbb{R}_+; U)$, and $w(\cdot,\cdot) \in L^2(\overline{\Omega} \times \mathbb{R}_+; W)$.

[1] An operator T acting on a Hilbert space $\mathcal{H}$ is *densely defined* in $\mathcal{H}$ if $\mathcal{D}(T)$ is dense in $\mathcal{H}$.

If the operator matrix $A(\cdot,\cdot)$ in (2) is symmetric and positive definite in $\overline{\Omega}\times\mathbb{R}_+$, then the operator

$$\nabla\cdot[A(z,t)\nabla^{\mathsf{T}}x(z,t)]$$

is uniformly elliptic in $\overline{\Omega}\times\mathbb{R}_+$.

The following diagram pictures the scenario for each $t\geq 0$,

$$\begin{array}{ccccc} & & \mathcal{X} & & \\ & & \downarrow \mathcal{A} & & \\ \mathcal{Z} & \hookrightarrow & \mathcal{X} & \hookrightarrow & \mathcal{V} \\ B_w\uparrow & & & & \uparrow B_u \\ W & & & & U \end{array}$$

Since in general, (1) involve unbounded operators we assume that they hold in a dense subspace which is to be determined for each system.

Remark. As has been shown in (Poznyak and Rodríguez, 1995, 1999), this class of linear infinite dimensional systems contains the three known families of PDEs which are used to model many physical systems.

Once defined this class of systems we will turn our attention to the disturbance structure and the disturbed evolution equation.

The source of *internal uncertainty* comes from the term $A(\cdot,\cdot)$. We are going to consider the multiplicative disturbance structure so that

$$A(\cdot,\cdot)=A_0\left[I+\overline{\Delta A}(\cdot,\cdot)\right]\quad\text{or}\quad A(\cdot,\cdot)=A_0+\Delta A(\cdot,\cdot) \tag{3}$$

where A_0 is some central linear operator, which is known and can be consider as representing the *stationary nominal coefficient* of system (1), and $\Delta A(\cdot,\cdot)\doteq A_0\overline{\Delta A}(\cdot,\cdot)$ represents the uncertainty, where the operator $\overline{\Delta A}(\cdot,\cdot)$ is unknown but bounded under an appropriate norn[2]. Therefore, $\Delta A(\cdot,\cdot)$ is unkown but bounded. Hence,

$$\mathcal{A}(t)=\mathcal{A}_0+\Delta\mathcal{A}(t)$$

where for all $z\in\Omega,\ \ t\geq 0$, the uncertainty $\Delta\mathcal{A}(t)$ in operator $\mathcal{A}$, which describes the dynamics of system (1), is given by

$$\Delta\mathcal{A}(t)(\cdot)=\alpha\ \nabla\cdot[\Delta A(z,t)\nabla^{\mathsf{T}}x(z,t)]+\beta\int_0^t\nabla\cdot[\Delta A(z,\tau)\nabla^{\mathsf{T}}x(z,\tau)]\,d\tau \tag{4}$$

and

$$\mathcal{A}_0x(z,t)=\alpha\nabla\cdot[A_0\nabla^{\mathsf{T}}x(z,t)]+\beta\int_0^t\nabla\cdot[A_0\nabla^{\mathsf{T}}x(z,\tau)]d\tau \tag{5}$$

In general, we have that the disturbed system is giving by the disturbed evolution equation

$$\frac{\partial x(z,t)}{\partial t}=[\mathcal{A}_0+\Delta\mathcal{A}(t)]x(z,t)+[B_u^0+\Delta B_u(t)]u(z,t)+[B_w^0+\Delta B_w(t)]w(z,t) \tag{6}$$

[2]See next Section.

where B_u^0, B_w^0 are also central operators, and $\Delta B_u(t)$, $\Delta B_w(t)$ are the perturbation operators associated with B_u and B_w, respectively, given by

$$\Delta B_u(t) \doteq B_u^0 \overline{\Delta B_u}(t), \qquad \Delta B_u(t) \doteq B_w^0\, \overline{\Delta B_w}(t) \tag{7}$$

for suitable operators $\overline{\Delta B_u}(t)$, and $\overline{\Delta B_w}(t)$, unknown but bounded. We will assume that these perturbation operators satisfy certain assumption introduced in next section.

This way, we have included the uncertainty in the system dynamics and the perturbations due to external sources of disturbances, both of the multiplicative type.

3 Disturbance Structure

Uncertainty is present in every control problem and is caused by either a lack of precise knowledge of the models describing the underlying physical system, or the deliberate simplification of the mathematical models for analysis and design convenience. Thus two types of uncertainty models are used in building mathematical models of physical systems for control purposes. A first class includes the so called *unstructured norm bounded perturbations* and is representative of unmodeled or difficult to model system dynamics. The second class includes *structured perturbations*, reflecting uncertainty distributed at several places of the control loop, e.g. plant inputs or outputs, actuator inputs, and sensor outputs. These uncertainties are represented by several 'structured' perturbation blocks. Parametric uncertainty, which is included in the second class, represents the highest level of structure present in plant perturbations.

A more realistic problem is when different types of uncertainties affect the control loop. Here we present a measure of the a priori information regarding uncertainties and perturbations.

Definition 1. (Poznyak and Rodríguez, 1995) Let $\mathcal{H}$ be a Hilbert space, and $\gamma > 0$ a real number. For linear operators H, F, and Λ on $\mathcal{H}$, with $\mathcal{R}(H) \cap \mathcal{R}(F) \subset \mathcal{D}(\Lambda)$, and $\Lambda = \Lambda^* > 0$ is bounded, we define the following seminorm

$$||H||_{F|\Lambda,\gamma} \doteq \left[\sup_z \frac{||Hz||_\Lambda^2}{||Fz||_\Lambda^2 + \gamma} \right]^{1/2} \tag{8}$$

where the sup is taken over all $z \in \mathcal{D}(H) \cap \mathcal{D}(F)$ such that $\mathcal{R}(H) \cap \mathcal{R}(F) \subset \mathcal{D}(\Lambda)$, and

$$||Hz||_\Lambda^2 = < Hz, \Lambda Hz >_{\mathcal{H}} = z^* H^* \Lambda H z. \tag{9}$$

Remark. In definition (8) we do not assume that operators H and F are necessarily bounded. So, this expression can be consider as an extension of the classical operator H^∞-norm commonly used for the class of bounded operators ($F = 0$) to the class of unbounded operators (one can also consider the case when the ratio of two unbounded functions $||Hx||_\Lambda^2$ and $||Fx||_\Lambda^2$ remains bounded over all space). The parameter γ can be seen as a regularizing coefficient to avoid the indeterminacy of 0/0-type in (8).

Remark. If it happens that just F is bounded, then the seminorm is a norm. Certainly it is a norm if both operators are bounded.

Remark. Needless to say that there must be other types of norms that can be proved to be useful, but it is not the intention of this paper to do research on this issue.

Concerning the measurement of perturbations and uncertainties we have the following assumption:

Assumption 1. ***(Uncertainty and Perturbation Measurement)***
Let c, c_u, c_w, γ, γ_u, and γ_w be real positive numbers. Let us take $\mathcal{X}$, $L^2(\overline{\Omega} \times \mathbb{R}_+; U)$, and $L^2(\overline{\Omega} \times \mathbb{R}_+; W)$ as before. Regarding the uncertainty in $\mathcal{A}$, B_u, and B_w we assume that

$$\sup_{t \geq t_0} ||\Delta\mathcal{A}(t)||_{\mathcal{A}_0|\Gamma,\gamma} \leq c, \tag{10}$$

$$\sup_{t \geq t_0} ||\Delta B_u(t)||_{B_u^0|\Gamma_u,\gamma_u} \leq c_u, \tag{11}$$

$$\sup_{t \geq t_0} ||\Delta B_w(t)||_{B_w^0|\Gamma_w,\gamma_w} \leq c_w, \tag{12}$$

where $\Gamma : \mathcal{X} \to \mathcal{X}$, $\Gamma_u : \mathcal{X} \to L^2(\overline{\Omega} \times \mathbb{R}_+; U)$, and $\Gamma_w : \mathcal{X} \to L^2(\overline{\Omega} \times \mathbb{R}_+; W)$ are linear operators such that $\Gamma = \Gamma^* > 0$, $\Gamma_u = \Gamma_u^* > 0$, and $\Gamma_w = \Gamma_w^* > 0$. t, $t_0 \in \mathbb{R}_+$.

Remark. Here we are considering the same disturbance structure for $\Delta\mathcal{A}(t)$ as well as for $\Delta B_u(t)$ and $\Delta B_w(t)$. We use this assumption to deal with uncertainties and perturbations in our systems class, which will be reflected in the following proposition. However, this not need be the case in general. A study of how rich can be made the class of disturbances that can be allowed by fixing bounds on them by other appropriate norms or seminorms. is out of the scope of this paper. We recall that unstructured perturbations are usually modelled as norm bounded perturbations. We will comment on this after the next proposition.

¿From **Assumption 1** we have the following

Proposition 1. *(Poznyak and Rodríguez, 1995)(**Uncertainty Inequality**). $\forall t \geq t_0$; t, $t_0 \in \mathbb{R}_+$, $\forall x \in \mathcal{X}$ it follows from (10) that*

$$< \Delta\mathcal{A}(t)x(z,t), \Gamma\Delta\mathcal{A}(t)x(z,t) > \leq c^2[\gamma + < \mathcal{A}_0 x(z,t), \Gamma\mathcal{A}_0 x(z,t) >] \tag{13}$$

Moreover, similar expressions can be obtained for ΔB_u and ΔB_w. □

Remark. It is important to notice that the constant c^2 describes the margin level of **the relative structured uncertainty** with respect to the nominal model and the constant $c^2\gamma$ describes the allowable tolerance level of **the additive unstructured uncertainty**. The constant c^2 must be small enough as to preserve the properties of the operator $\mathcal{A}$.

This way we have defined the disturbance structure we are going to deal with for purposes of designing the robust controller for the class of systems previously defined. Those are the allowed disturbances for the robust control problem we will pose in next section.

4 Problem Statement

The general goal is to design a robust stabilizing controller for a class of linear infinite dimensional systems in the presence of uncertainties and perturbations of the multiplicative type.

We are going to focus on the robust tracking problem. Consider a system in the class Σ of systems for which the disturbance structure of Assumption 1 holds. Such a system is described by (1) which we rewrite here again

$$\frac{\partial x(z,t)}{\partial t} = \mathcal{A}(t)x(z,t) + B_u(t)u(z,t) + B_w(t)w(z,t), \quad z \in \Omega,\ t \geq 0, \quad x \in \mathcal{X}, \tag{14}$$

where $\mathcal{X} = L^2(\Omega \times \mathbb{R}_+)$.

Remark. Since in general, (14) involve unbounded operators we assume that they hold in a dense subspace which is to be determined for each system (Adams, 1975).

Let us assume that we have a reference system in the class Σ which is given by

$$\frac{\partial x_r(z,t)}{\partial t} = \mathcal{A}_r x_r(z,t) + B_u^r(t) u_r(z,t), \quad z \in \Omega, \quad t \geq 0, \tag{15}$$

with appropriate boundary and initial conditions for known operators $\mathcal{A}_r$, B_u^r, and some control action $u_r \in L^2(\overline{\Omega} \times \mathbb{R}_+; U)$, where x_r is the mild (strong) solution of this reference system (Poznyak and Rodríguez, 1995).

For a to be controlled system (14) in class Σ, whose solution is denoted by $x(\cdot,\cdot)$, we define the error tracking function e as follows

$$e(z,t) \doteq x(z,t) - x_r(z,t) \tag{16}$$

in order to consider a Lyapunov-like approach with Lyapunov function $V(e)$ defined by

$$V(e) \doteq < e, Pe >, \quad P = P^* \geq 0, \tag{17}$$

where P is a positive, self-adjoint operator.

Remark. We recall that for the ordinary algebraic operations with unbounded operators S and T, defined on a specific functional space, say $\mathcal{X}$, the natural definitions for the domains of sums and products are:

$$\begin{aligned} \mathcal{D}(S+T) = & \ \mathcal{D}(S) \cap \mathcal{D}(T), \\ \mathcal{D}(ST) = & \ \{x \in \mathcal{D}(T) \mid Tx \in \mathcal{D}(S)\}. \end{aligned} \tag{18}$$

Remark. In (15) we allow the term $B_u^r(t)u_r(z,t)$ for having a richer class of reference models.

In the following two definitions we are going to introduce the notions of *admissible robust controller* and *stability in average* which we will use as a criterion of stability for the class Σ.

Definition 2. (Poznyak and Rodríguez, 1999) We call a dynamic system **an admissible robust controller** if, for any system belonging to class Σ, it generates the control action $u(\cdot,\cdot)$ such that the control problem

$$\begin{aligned} \frac{\partial x(z,t)}{\partial t} = & \ \mathcal{A}(t)x(z,t) + B_u(t)u(z,t) + B_w(t)w(z,t), \quad z \in \Omega, \ x \in \mathcal{X}, \\ x(z,0) = & \ x_0(z), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z \in \Omega, \end{aligned} \tag{19}$$

for any disturbance satisfying Assumption 1, is well posed.

Definition 3. (Poznyak and Rodríguez, 1999) We say that an admissible robust controller **stabilizes "in average"** the class Σ if for any external perturbation w of "bounded power"[3]

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T < w, w >_{(\cdot)} dt \quad < \infty \tag{20}$$

the corresponding tracking process x as well as the generated control action u have bounded power too, i.e.,

$$\sup_{\Sigma} \overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T \Big(||x||^2_{(\cdot)} + < u, u >_{(\cdot)} \Big) dt \quad < \infty \tag{21}$$

[3]Here, $(\cdot)$ denotes the appropriate space which we are working with.

Remark. Physically, stability in average means that the power (not the energy) of the process is bounded.

Next, we pose the problem of this paper.

Problem Statement: *Given the systems class described by (1) and (2), with the uncertainty and perturbations as in (3) to (7), and the specified disturbance structure of Assumption 1, the problem is to design a robust controller that stabilizes this class of systems in the presence of both uncertainties and perturbations, and to give a tolerance level for which this robust stabilization can be guaranteed.*

5 Result in Robust Controller Structure

In this section we present one of the main results of this work. We must first prove that there is a robust controller, and then we synthesize this control in terms of the solution of a corresponding Riccati Operator Equation which will be the content of the next section. Keeping in mind what we have discussed from (15) to (17), we state the following

Theorem 1. *(**Robust Controller Structure**). Consider the class Σ of systems defined by (1) - (2) for which the disturbance structure of* Assumption 1 *holds. Let us assume that*

1. *the following Riccati operator equation has a positive, possibly self-adjoint, solution P*

$$[\mathcal{A}_0^* P + P\mathcal{A}_0 + P\mathcal{R}P + \mathcal{Q}]e = 0, \quad \forall e \in \mathcal{X} \tag{22}$$

   where

$$\begin{aligned} \mathcal{R} &\doteq \ \Gamma^{-1} + \Psi + \Gamma_w^{-1}, \quad \mathcal{R} > 0 \\ \mathcal{Q} &\doteq \ c^2 \mathcal{A}_0^* \Gamma \mathcal{A}_0 + I, \end{aligned}$$

   with

$$\Psi \doteq \Gamma_u^{-1} - B_u^0 \Lambda_0^{-1} (B_u^0)^*,$$

$$\Psi = \Psi^* > 0, \ \Gamma = \Gamma^* > 0, \ \Gamma_u = \Gamma_u^* > 0, \ \Gamma_w = \Gamma_w^* > 0, \ \Lambda_0 = \Lambda_0^* > 0;$$

2. *there exists a control u that is given by*

$$\begin{aligned} u &= u_0 + \bar{u} \\ \bar{u} &= -\Lambda_0^{-1} (B_u^0)^* P e \end{aligned} \tag{23}$$

   where u_0 is the compensating part of the control action and satisfies the following relationship[4]:

$$B_u^0 u_0 = (\mathcal{A}_r - \mathcal{A}_0) x_r + B_u^r u_r \tag{24}$$

[4]The initial condition $u(z, 0) = n_0(z)$ can be selected based on a priori information.

Then, $\forall z \in \Omega$ and the Lyapunov function (17) it holds that

$$\frac{dV(e)}{dt} \leq -<\bar{u}, \Theta \bar{u}> - <e, e> + <w, \eta w> + \tau \tag{25}$$

where

$$\begin{array}{rll} \Theta \doteq & \Lambda_0 - c_u^2 (B_u^0)^* \Gamma_u B_u^0, & \Theta = \Theta^* > 0, \\ \eta \doteq & (B_w^0)^* \Pi B_w^0, & \eta = \eta^* > 0, \\ \Pi \doteq & \Sigma[(1 + c_w^2) I + \Gamma_0^{-1} \Sigma] + c_7^2 \Gamma_0, & \Pi = \Pi^* > 0, \\ \Sigma \doteq & \Gamma_w [I + (\Lambda_1^{-1} + \Lambda_3^{-1}) \Gamma_w], & \Sigma = \Sigma^* > 0, \\ \Sigma_1 \doteq & (c_1^2 + c_6^2 \Gamma_w \Lambda_2^{-1}) \Gamma_w + c_3^2 \Lambda_1, & \Sigma_1 = \Sigma_1^* > 0, \\ \Sigma_2 \doteq & c_2^2 \Gamma_w + c_4^2 \Lambda_2 + c_5^2 \Lambda_3, & \Sigma_2 = \Sigma_2^* > 0, \\ \tau \doteq & \|\mathcal{A}_0 x_r\|_{\Sigma_2}^2 + \|B_u^0 u_0\|_{\Sigma_1}^2 + \upsilon & \end{array}$$

with

$$\begin{array}{ll} \Lambda_i = \Lambda_i^* > 0, & i = 1, 2, 3; \\ \upsilon \doteq h + h_u + \rho, & \end{array}$$

$$h \doteq c^2 \gamma, \quad h_u \doteq c_u^2 \gamma_u, \quad \rho \doteq h_w + \vartheta, \quad h_w \doteq c_w^2 \gamma_w + c_7^2 \gamma_7,$$

$$\vartheta = \sum_{i=1}^{6} c_i^2 \gamma_i$$

and c, c_u, c_w, γ, γ_u, γ_w; c_i, γ_i, with $i = 1, ..., 7$, are positive constants.

Remark. Expression (24) restricts the allowable reference system models we can deal with, which in turn imposes a restriction on the trajectories we can track.

Remark. The control is given by $u = u_0 - \Lambda_0^{-1} (B_u^0)^* Pe$, where P is a solution of (22). When it happens that $\mathcal{A}_r - \mathcal{A}_0 \equiv 0$, we can select $u_r = 0$, then $u_0 = 0$ also, and $u = -\Lambda_0^{-1} (B_u^0)^* Pe$ is a state feedback type controller without compensating part.

Remark. In expression (25) we do not mean to attain asymptotic stability, i.e., we do not claim that $e \to 0$ as $t \to \infty$. But that the error function e remains bounded, which means that the solution x remains "close" to that of the reference system, x_r, and the power of the process x is bounded

Remark. We cannot cancel the term τ through the control action. Moreover, there is the presence of the perturbation signal w which we consider bounded, and the operator η that is also bounded. So the quantity $<w, \eta w> + \tau$ is known to be bounded as well.

Here we include a Lemma about an inequality between operators which will be useful for proving the main result.

Lemma 1. *(Poznyak and Rodríguez, 1995) (**Hermitian Operator Inequality**). Let X, Y, and Γ be linear operators on a Hilbert space $\mathcal{H}$. Assume that the operator Γ is self-adjoint and strictly positive: $\Gamma = \Gamma^* > 0$. Then the linear operator $\mathcal{F}$, defined by*

$$\mathcal{F} \doteq X^* \Gamma X + Y^* \Gamma^{-1} Y - X^* Y - Y^* X, \tag{26}$$

satisfies $\mathcal{F} \geq 0 \quad \forall w \in \mathcal{H}, \quad w \neq 0$. Where the domains for the algebraic operations of these operators are taken as in (18), and are considered dense in $\mathcal{H}$. □

6 Proof of Theorem

We present the proof of this theorem in sections to make it easier to follow.

6.1 Lyapunov Like Analysis

Proof. ¿From the definition of the error function $e(z,t)$, $u = \bar{u} + u_0$, and (1) we get (omitting arguments for the sake of simplicity)

$$\frac{\partial e}{\partial t} = \mathcal{A}e + (\mathcal{A} - \mathcal{A}_r)x_r + B_u\bar{u} + B_u u_0 - B_u^r u_r + B_w w.$$

Then, with $\mathcal{A} = \mathcal{A}_0 + \Delta\mathcal{A}$, and $B_u = B_u^0 + \Delta B_u$ we have

$$\frac{\partial e}{\partial t} = \mathcal{A}e + B_u\bar{u} + [(\mathcal{A}_0 - \mathcal{A}_r)x_r + B_u^0 u_0 - B_u^r u_r] +$$

$$+[\Delta B_u u_0 + \Delta\mathcal{A}x_r + B_w w].$$

¿From hypothesis, there exists $u_0 \in L^2(\overline{\Omega} \times \mathbb{R}_+; U)$ such that $B_u^0 u_0 = (\mathcal{A}_r - \mathcal{A}_0)x_r + B_u^r u_r$. Hence,

$$\frac{\partial e}{\partial t} = \mathcal{A}e + B_u\bar{u} + \bar{w} \tag{27}$$

where $\bar{w} \doteq \Delta B_u u_0 + \Delta\mathcal{A}x_r + B_w w$.

Taking time derivative of the Lyapunov function candidate (17) along the trajectories of (27) with zero initial conditions and z fixed, we get

$$\frac{dV(e)}{dt} = 2 < \frac{\partial e}{\partial t}, Pe > . \tag{28}$$

Substituting (27) into (28) yields (omitting the arguments for making the expressions simpler),

$$\frac{dV}{dt} = 2 < \mathcal{A}e + B_u\bar{u} + \bar{w}, Pe > . \tag{29}$$

6.2 Estimation of Terms

Here we are going to work with each term of the inner product (29) separately.

i) $(2 < \mathcal{A}e, Pe >)$. From the previous section we have that $\mathcal{A} = \mathcal{A}_0 + \Delta\mathcal{A}$, hence

$$\begin{aligned} 2 < \mathcal{A}e, Pe > \quad &= < \mathcal{A}_0 e, Pe > + < Pe, \mathcal{A}_0 e > + \\ &+ \ < \Delta\mathcal{A}e, Pe > + < Pe, \Delta\mathcal{A}e > . \end{aligned} \tag{30}$$

Now using Lemma 1 for the last two terms of (30) where $X \doteq \Delta\mathcal{A}e$ and $Y \doteq Pe$, with $\Gamma = \Gamma^* > 0$, then

$$\begin{aligned} 2 < \mathcal{A}e, Pe > \quad &\leq \ < \mathcal{A}_0 e, Pe > + < Pe, \mathcal{A}_0 e > + \\ &+ \ < \Delta\mathcal{A}e, \Gamma\Delta\mathcal{A}e > + < Pe, \Gamma^{-1}Pe > . \end{aligned} \tag{31}$$

Using Proposition 1, for $\Delta\mathcal{A}$, we have

$$< \Delta\mathcal{A}e, \Gamma\Delta\mathcal{A}e > \quad \leq \quad c^2[\gamma + < \mathcal{A}_0 e, \Gamma\mathcal{A}_0 e >]$$

hence

$$
\begin{aligned}
2 < \mathcal{A}e, Pe > \;&\le\; < \mathcal{A}_0 e, Pe > + < Pe, \mathcal{A}_0 e > + \\
&+ c^2 < \mathcal{A}_0 e, \Gamma \mathcal{A}_0 e > + < Pe, \Gamma^{-1} Pe > + \; h.
\end{aligned}
\tag{32}
$$

where $h \doteq c^2\gamma$, with c, and γ positive constants.

We are going to do the same for the other two terms.

ii) $(2 < B_u\bar{u}, Pe >)$. Here $B_u = B_u^0 + \Delta B_u$, and using Lemma 1 where $X \doteq \Delta B_u \bar{u}$ and $Y \doteq Pe$, with $\Gamma_u = \Gamma_u^* > 0$, we have

$$
\begin{aligned}
2 < B_u\bar{u}, Pe > \;&\le < B_u^0\bar{u}, Pe > + < Pe, B_u^0\bar{u} > + \\
&+ \; < \Delta B_u \bar{u}, \Gamma_u \Delta B_u \bar{u} > + < Pe, \Gamma^{-1} Pe > .
\end{aligned}
$$

Using Proposition 1 for ΔB_u and rearranging terms we get

$$
\begin{aligned}
2 < B_u\bar{u}, Pe > \;&\le < B_u^0\bar{u}, Pe > + < Pe, B_u^0\bar{u} > + \\
&+ c_u^2 < B_u^0\bar{u}, \Gamma_u B_u^0 \bar{u} > + < Pe, \Gamma_u^{-1} Pe > + \; h_u
\end{aligned}
\tag{33}
$$

where $h_u \doteq c_u^2\gamma_u$, with c_u, and γ_u positive constants.

With $\Lambda_0 = \Lambda_0^* > 0$ and $\Lambda_0 = \Lambda_0^{1/2}\Lambda_0^{1/2}$, the first two terms of the right hand side of (33), for all $e \in \mathcal{X}$, can be written as follows

$$
\begin{aligned}
2 < B_u^0\bar{u}, Pe > \;&= 2 < B_u^0\bar{u}, Pe > \\
&= ||\Lambda_0^{1/2}\bar{u} + \Lambda_0^{-1/2}(B_u^0)^* Pe||^2 - < \bar{u}, \Lambda_0 \bar{u} > - \\
&- \; < (B_u^0)^* Pe, \Lambda_0^{-1}(B_u^0)^* Pe > .
\end{aligned}
\tag{34}
$$

Substituting (34) in (33) yields

$$
\begin{aligned}
2 < B_u\bar{u}, Pe > \;&\le\; ||\Lambda_0^{1/2}\bar{u} + \Lambda_0^{-1/2}(B_u^0)^* Pe||^2 - \\
&- \; < \bar{u}, \Theta \bar{u} > + < Pe, \Psi Pe > + \; h_u
\end{aligned}
\tag{35}
$$

where

$$
\Theta \doteq \Lambda_0 - c_u^2 (B_u^0)^* \Gamma_u B_u^0 \tag{36}
$$

and

$$
\Psi \doteq \Gamma_u^{-1} - B_u^0 \Lambda_0^{-1} (B_u^0)^*.
$$

Besides, $\Theta = \Theta^* > 0$, and $\Psi = \Psi^* > 0$.

¿From hypothesis on $\overline{u}$ we have

$$
2 < B_u\overline{u}, Pe > \le - < \overline{u}, \Theta\overline{u} > + < Pe, \Psi Pe > + \; h_u \tag{37}
$$

iii) $(2 < \bar{w}, Pe >)$. Here $B_w = B_w^0 + \Delta B_w$, and using Lemma 1 where $X \doteq \overline{w}$ and $Y \doteq Pe$, with $\Gamma_w = \Gamma_w^* > 0$, we have

$$
2 < \bar{w}, Pe > \le < \bar{w}, \Gamma_w \bar{w} > + < Pe, \Gamma_w^{-1} Pe > .
$$

¿From the definition of $\overline{w}$ and applying Lemma 1 and Proposition 1 conveniently when needed, we get

$$
< \bar{w}, \Gamma_w \bar{w} > \le \; \|B_u^0 u_0\|_{\Sigma_1}^2 \; + \; \|\mathcal{A}_0 x_r\|_{\Sigma_2}^2 \; + \; < B_w w, \Sigma B_w w > + \; \vartheta \tag{38}
$$

where

$$\begin{array}{rll} \Sigma_1 \doteq & c_1^2\Gamma_w + c_3^2\Lambda_1 + c_6^2\Gamma_w\Lambda_2^{-1}\Gamma_w, & \Sigma_1 = \Sigma_1^* > 0, \\ \Sigma_2 \doteq & c_2^2\Gamma_w + c_4^2\Lambda_2 + c_5^2\Lambda_3, & \Sigma_2 = \Sigma_2^* > 0, \\ \Sigma \doteq & \Gamma_w[I + (\Lambda_1^{-1} + \Lambda_3^{-1})\Gamma_w], & \Sigma = \Sigma^* > 0, \end{array}$$

$$\vartheta \doteq \sum_{i=1}^{6} c_i^2\gamma_i, \quad c_i > 0, \gamma_i > 0, \quad i = 1, ..., 6. \quad \Lambda_i = \Lambda_i^* > 0, \quad i = 1, 2, 3.$$

Knowing that $B_w w = B_w^0 + \Delta B w$ and applying Lemma 1 and Proposition 1 when needed, we obtain

$$< B_w w, \Sigma B_w w > \leq \ \|B_w^0 w\|_\Pi^2 \ + \ h_w \tag{39}$$

where

$$\Pi \doteq \Sigma[(1 + c_w^2)I + \Gamma_0^{-1}\Sigma] \ + \ c_7^2\Gamma_0, \quad \Pi = \Pi^* > 0,$$

with

$$h_w \doteq c_w^2\gamma_w + c_7^2\gamma_7, \tag{40}$$

and c_w, c_7, γ_w, and γ_7 positive constants. Hence,

$$2 < \overline{w}, Pe > \leq < Pe, \Gamma_w^{-1} Pe > \ + \|B_u^0 u_0\|_{\Sigma_1}^2 \ + \ \|\mathcal{A}_0 x_r\|_{\Sigma_2}^2 \ + \|B_w^0 w\|_\Pi^2 \ + \ \rho \tag{41}$$

with

$$\rho \doteq h_w + \vartheta.$$

6.3 Final Estimation of the Time Derivative of the Lyapunov Function

Substituting (32), (37), and (41) into (29) we obtain, for all $e \in \mathcal{X}$, after rearranging terms that

$$\begin{array}{rl} \dfrac{dV}{dt} & \leq < e, \{P\mathcal{A}_0 + \mathcal{A}_0^* P + c^2\mathcal{A}_0^*\Gamma\mathcal{A}_0 + P[\Gamma^{-1} + \Psi + \Gamma_w^{-1}]P + I\}e > - \\ & - \ < \bar{u}, \Theta\bar{u} > - < e, e > + \|B_u^0 u_0\|_{\Sigma_1}^2 \ + \|\mathcal{A}_0 x_r\|_{\Sigma_2}^2 \ + \|B_w^0 w\|_\Pi^2 \ + \upsilon \end{array} \tag{42}$$

where $\upsilon \doteq h + h_u + \rho$.

¿From hypotheses of the theorem we have

$$\frac{dV}{dt} \ \leq \ - \ < \bar{u}, \Theta\bar{u} > - < e, e > + < w, \eta w > + \ \tau \tag{43}$$

where

$$\begin{array}{rll} \eta \doteq & (B_w^0)^* \Pi B_w^0, & \eta = \eta^* > 0, \\ \tau \doteq & \|B_u^0 u_0\|_{\Sigma_1}^2 \ + \ \|\mathcal{A}_0 x_r\|_{\Sigma_2}^2 + \upsilon & \end{array} \tag{44}$$

□

6.4 Estimation of the Joint Cost Functional

¿From the previous theorem, it follows the next result concerning the corresponding *tolerance level* for the suggested robust control.

Corollary 1. *For every object from the class Σ, and under Assumption 1, we have that*

$$J \overset{\Delta}{=} \sup_{\Sigma} \overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T [\ ||e||^2 + \ < \bar{u}, \Theta \bar{u} >]\ dt \ \leq \ \overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T < w, \eta w > dt \ + \tau. \tag{45}$$

where Θ, η, and τ are defined by expressions (36), and (44), respectively.

Proof. It follows directly from (43) if we take into account that

$$-\frac{1}{T} \int_0^T dV(t) = \frac{V(0) - V(T)}{T} \leq \frac{V(0)}{T} \underset{T\to\infty}{\to} 0$$

□

Remark. In the ideal situation when both operators $\mathcal{A}_r(t)$ and $\mathcal{A}_0$ coincide for all $z \in \Omega$ and $t \geq 0$, and there are no external perturbations, then from Corollary 1, we can conclude that the tracking error function e as well as the control action $\overline{u}$ are asymptotically stable, i.e.,

$$||e\,(t)\,||^2 + \ < \bar{u}\,(t)\,, \Theta \bar{u}\,(t) > \ \lim_{t\to\infty} \ \to 0.$$

7 Riccati Operator Equation

One of the difficulties regarding Lyapunov approach is faced when we want to solve the associated Riccati equation involved. For infinite dimensional systems is even more complicated because the equation can have no solution at all or can be rarely solved exactly. Moreover, in infinite dimensions, the Riccati differential equation is not always well posed. There are some results concerning the approximation theory of solutions to operator Riccati equations (*e.g.*, see (Curtain, 1990; Kappel and Salamon, 1990; Lasiecka, 1992; Oostveen and Curtain, 1997; Weiss, 1997) and the references therein).

Here we propose some way to deal with the Riccati Operator Equation of Theorem 1. We will assume that all the operators involved here are densely defined on their appropriate domains.

Lemma 2. *(Poznyak and Rodríguez, 1999) Let us consider the following Riccati Operator Equation*

$$\mathcal{F}(P) \doteq PA + A^*P + PRP + Q = 0 \tag{46}$$

with $R = R^* > 0$ and $Q = Q^*$, that is, R and Q are self-adjoint and, in addition, R is positive. Then, Equation (46) has a solution if, and only if

$$G \doteq A^*R^{-1}A - Q \geq 0 \tag{47}$$

and all the self-adjoint solutions have the following parameterization:

$$P = \frac{1}{2}[R^{-1/2}UG^{1/2} + G^{1/2}U^*R^{-1/2} - R^{-1}A - A^*R^{-1}], \quad \forall U : U^*U = I \tag{48}$$

where the unitary operator U satisfies the linear equation

$$R^{-1/2}UG^{1/2} - G^{1/2}U^*R^{-1/2} - R^{-1}A + A^*R^{-1} = 0 \tag{49}$$

Remark. We take the domains and ranges of R and A so that they satisfy (18) and are such that equation (46) makes sense for the ordinary algebraic operations of the operators involved: $\mathcal{D}(RP) = \{x \in \mathcal{D}(P) \mid \ Px \in \mathcal{D}(R)\}$. And for the sums we take the intersection of the operators' domains.

Remark. Notice also, that with the assumptions impose on the operator R, the operators R^{-1}, $R^{1/2}$, and $R^{-1/2}$ are well defined.

Remark. Assuming that one can solve (49), then one is face with the problem to solve $U^*U = I$, which in general may be not solvable or very difficult to solve. However, for the one dimensional case this is not that complicated.

Positivity[5] of the solution is not so trivial though. Conditions under which this will be the case for the maximal solution are, for instance (in the matrix case), (A, B) controllable and $Q \leq 0$, or (A, B) stabilizable, (Q, A) detectable and $Q \leq 0$. Weaker conditions are possible. Anyway, some condition like $Q \leq 0$ should be always there.

For the case of unbounded operators, $Q < 0$ plus some condition that guarantees the existence of a stabilizing solution will probably work as well. Moreover, from the semigroup approach, one can work this equation using the Popov function approach for the Pritchard-Salamon systems (see for instance (Van Keulen, 1993; Oostveen and Curtain, 1997; Weiss, 1997)). However, we do not pursue this study here. This will be another research topic for future.

Remark. $G \geq 0$ is a necessary and sufficient condition.

Remark. For any bounded linear operator T on $\mathcal{H}$, $T_1 = [T + T^*]$ is self-adjoint[6].

Remark. In the one-dimensional case, $U \equiv 1$, and these formulae (48), and (49) are constructive. For more dimensions we need apply numerical methods.

7.1 Fractional Power Representation for the Solution of the Riccati Operator Equation

Notice that we have here $A = \mathcal{A}_0$, $R = \mathcal{R}$, and $Q = \mathcal{Q}$. And for the terms $R^{-1/2}$, $G^{1/2}$ and R^{-1} we use, for all $\alpha > 0$, the fractional power representation given by

$$\mathcal{T}^{-\alpha} = \frac{1}{2\pi i} \sum_{j=0}^{\infty} \int_{\Upsilon} \lambda^{-(j+1+\alpha)} d\lambda \mathcal{T}^j.$$

and

$$\mathcal{T}^{\alpha} = \frac{\sin(\pi\alpha)}{\pi} \sum_{j=0}^{\infty} (-1)^j \int_{\lambda=0}^{\infty} \lambda^{-(j+2-\alpha)} d\lambda \mathcal{T}^{j+1}$$

(see (Balakrishnan, 1960; Kato, 1976, 1961, 1962, 1960; Pazy, 1983; Poznyak and Rodríguez, 1999; Tanabe, 1979; Yosida, 1996)).

In our case, $\alpha = 1/2$, and 1. From the practical point of view, one can truncate the sum up to the number of terms that may be relevant for computation.

[5]For an unbounded self-adjoint linear operator $T : \mathcal{H} \to \mathcal{H}$: *(a)* $< Tx, x > \geq 0$, $\forall x \in \mathcal{D}(T) \subset \mathcal{H}$ (briefly: $T \geq 0$) iff $\sigma(T) \subset [0, \infty)$. *(b)* If $T \geq 0$, then there exists a unique self-adjoint $B \geq 0$ such that $B^2 = T$ (see (Rudin, 1991)).

[6]See, for instance, (Rudin, 1991)

8 Conclusions

In this paper we have presented a class of linear infinite dimensional systems, and a disturbance structure. We show in the main theorem that, under certain conditions, there exists a state-feedback controller that stabilizes the class Σ with mixed disturbances and guarantees some tolerance level in a general tracking problem in an infinite dimensional space. Much of this work is under research, and other results will appear elsewhere.

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