Identification and Adaptive Control of Some Stochastic Distributed Parameter Systems*

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Abstract

An important class of controlled linear distributed parameter control systems are those with boundary or point control. A survey of some existing adaptive control problems with their solutions for the boundary or the point control of a linear stochastic distributed parameter systems is presented. The distributed parameter system is modeled by an evolution equation with an infinitesimal generator for an analytic semigroup. Since there is boundary or point control, the linear transformation for the control in the state equation is also an unbounded operator. The unknown parameters in the model appear affinely in both the infinitesimal generator of the semigroup and the linear transformation of the control. Strong consistency is verified for a family of least squares estimates of the unknown parameters. For a quadratic cost functional of the state and the control, the certainty equivalence control is self-optimizing, that is the family of average costs converges to the optimal ergodic cost. Another control problem considered here is when the control occurs on the boundary. The "highest-order" operator is assumed to be known but the "lower-order" operators contain unknown parameters. Furthermore, the linear operators of the state and the control on the boundary contain unknown parameters. The noise in the system is a cylindrical white Gaussian noise. The performance measure is an ergodic, quadratic cost functional. This time for the identification of the unknown parameters a diminishing excitation is used that has no effect on the ergodic cost functional but ensures sufficient excitation for strong consistency. The adaptive control is the certainty equivalence control for the ergodic, quadratic cost functional with switchings to the zero control.

1 Introduction

A survey of some existing adaptive control problems with their solutions for the boundary or the point control of a linear stochastic distributed parameter systems is presented.

An important family of controlled linear, distributed parameter systems are those with boundary or point control. Perturbations or inaccuracies in the mathematical model can offer be effectively modeled by white noise. Since in many control situations there are unknown parameters in these linear, stochastic distributed parameter systems, it is necessary to solve a stochastic adaptive control problem. The unknown linear stochastic distributed parameter system is described by an evolution equation where the unknown parameters appear in the infinitesimal generator of an analytic semigroup and the unbounded linear transformation for

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the boundary control. The noise process is a cylindrical, white noise. A family of least squares estimates is constructed from the observations of the unknown stochastic system. This family of estimates is shown to be strongly consistent under verifiable conditions. A stochastic differential equation is given for the family of estimates. The self-tuning and the self-optimizing properties of an adaptive control law are investigated. If an adaptive control is seff-tuning, then it is shown that the system satisfies some stability properties and the adaptive control is self-optimizing. The certainty equivalence adaptive control, that is, using the optimal stationary control with the estimates of the parameters, is shown to be self-optimizing; that is, the optimal ergodic cost is achieved.

An important class of models for linear distributed parameter systems is the family that is described by analytic semigroups. To model some perturbations or inaccuracies in these models, it is often reasonable to consider stochastic, linear distributed systems. In many applications of controlled linear distributed parameter system it is natural to consider that the control occurs on the boundary or at discrete points because it is often unreasonable to expect that the control can be applied throughout the domain.

If there is an ergodic, quadratic cost functional, then under suitable assumptions the optimal control can be obtained from the solution of an algebraic or stationary Riccati equation. Typically the stochastic differential equation model for the stochastic, linear distributed parameter system contains some unknown parameters so there is the problem of stochastic adaptive control. It is assumed that the "highest-order" operator is known but that "lower-order" operators contain unknown parameters. The unknown operators include linear operators on the state on the boundary and on the control on the boundary or at discrete points in the domain. When the "highest-order" linear operator is known the analysis is simplified compared with the case when this linear operator is unknown. In many applications it seems that this form for the unknowns is a reasonable model.

For the identification of the unknown parameters that occur in the linear operator acting on the control, it is necessary to ensure that there is sufficient excitation. This is accomplished by a diminishing excitation which has no effect on the ergodic cost functional but ensures sufficient excitation for strong consistency. This strong consistency is obtained for a family of least-squares estimates. It is assumed that the analytic semigroup is stable. The control at time t is required to be measurable with respect to the past (of the state process) until time $t - \Delta$ where $\Delta > 0$ is arbitrary but fixed. This assumption accounts for some natural delay in processing the information for the construction of the control. No boundedness assumptions are made on the range of the unknown parameters.

The adaptive control is the certainty equivalence control for the ergodic, quadratic cost functional with switchings to the zero control. These switchings are determined to ensure stability of the estimated infinitesimal generator and to satisfy a suitable boundedness for the control.

2 Adaptive Boundary and Point Control of Linear Stochastic Distributed Parameter Systems

The unknown linear stochastic distributed parameter system with boundary or point control is formally described by the following stochastic differential equation

$$\begin{aligned} dX(t;\alpha) &= (A(\alpha)X(t;\alpha) + B(\alpha)U(t))dt + \Phi dW(t), \\ X(0;\alpha) &= X_0 \end{aligned} \tag{2.1}$$

where $X(t;\alpha) \in H$; H is a real, separable, infinite-dimensional Hilbert space; $(W(t),\, t \geq 0)$ is a cylindrical Wiener process on H; $\Phi \in \mathcal{L}(H)$, $\alpha = (\alpha^1, \dots, \alpha^q)$; and $t \geq 0$.

The probability space is denoted $(\Omega, \mathcal{F}, P)$, where P is a probability measure that is induced from the cylindrical Wiener measure and $\mathcal{F}$ is the P-completion of the Borel σ-algebra on Ω. Let $(\mathcal{F}_t,\, t \geq 0)$ be an increasing P-complete family of σ-algebras of $\mathcal{F}$ such that X_t is $\mathcal{F}_t$-measurable for $t \geq 0$ and $(\langle \ell, W(t)\rangle, \mathcal{F}_t,\, t \geq 0)$ is a martingale for each $\ell \in H$. $A(\alpha)$ is the infinitesimal of an analytic semi group on H. For some $\beta \geq 0$, the operator $-A(\alpha) + \beta I$ is strictly positive, so that the fractional powers $(-A(\alpha) + \beta I)^\gamma$ and $(-A(\alpha)^* + \beta I)^\gamma$ and the spaces $D^\gamma_{A(\alpha)} = \mathcal{D}((-A(\alpha) + \beta I)^\gamma)$ and $D^\gamma_{A^*(\alpha)} = \mathcal{D}((-A^*(\alpha) + \beta I)^\gamma)$ with the graph norm topology for $\gamma \in \mathbb{R}$ can be defined, It is assumed that $B(\alpha) \in \mathcal{L}\left(H_1, D^{\varepsilon-1}_{A(\alpha)}\right)$, where H_1 is a real, separable Hilbert space and $\varepsilon \in (0,1)$ (cf. assumption (A4) below). For the solution of (2.1) on $[0,T]$, the control $(U(t),\, t \in [0,T])$ is an element of $M^p_W(0,T,H_1)$, where $M^p_W(0,T,H_1) = \{u : [0,T] \times \Omega \to H_1$, is $(\mathcal{F}_t)$-nonanticipative and $E\int_0^T |u(t)|^p dt < \infty\}$ and $p > \max(2, 1/\varepsilon)$ is fixed.

A selection of the following assumptions are used subsequently.

(A1) The family of unknown parameters are the elements of a compact set $\mathcal{K}$.

(A2) For $\alpha \in \mathcal{K}$, the operator $\Phi^*(-A^*(\alpha)+\beta I)^{-1/2+\delta}$ is Hilbert–Schmidt for some $\delta \in \left(0, \frac{1}{2}\right)$.

(A3) There are real numbers $M > 0$ and $\omega > 0$ such that, for $t > 0$ and $\alpha \in \mathcal{K}$,

$$|S(t;\alpha)|_{\mathcal{L}(H)} \leq Me^{-\omega t}$$

and

$$|A(\alpha)S(t;\alpha)|_{\mathcal{L}(H)} \leq Mt^{-1}e^{-\omega t},$$

where $(S(t;\alpha),\, t \geq 0)$ is the analytic semigroup generated by $A(\alpha)$.

(A4) For all $\alpha_1, \alpha_2 \in \mathcal{K}$, $\mathcal{D}(A(\alpha_1)) = \mathcal{D}(A(\alpha_2))$, $D^\delta_{A(\alpha_1)} = D^\delta_{A(\alpha_2)}$ and $D^\delta_{A^*(\alpha_1)} = D^\delta_{A^*(\alpha_2)}$ for $\delta \in \mathbb{R}$.

(A5) For each $\alpha \in \mathcal{K}$ and $x \in H$, there is a control $u_{\alpha,x} \in L^2(\mathbb{R}_+, H_1)$ such that

$$y(\cdot) = S(\cdot;\alpha)x + \int_0^\cdot S(\cdot - t;\alpha)B(\alpha)u_{\alpha,x}(t)dt \in L^2(\mathbb{R}_+, H).$$

(A6) The operator $A(\alpha)$ has the form

$$A(\alpha) = F_0 + \sum_{i=1}^q \alpha^i F_i,$$

where F_i is a linear, densely defined operator on H for $i = 0, 1, \dots, q$ such that $\cap_{i=0}^q \mathcal{D}(F_i^*)$ is dense in H.

It is well known that the strong solution of (2.1) may not exist, so usually the mild solution of (2.1) is used, that is,

$$X(t;\alpha) = S(t;\alpha)X_0 + \int_0^t S(t-r;\alpha)B(\alpha)U(r)dr + \int_0^t S(t-r;\alpha)\Phi dW(r), \tag{2.2}$$

where $S(t;\alpha) = e^{tA(\alpha)}$. The mild solution is equivalent to the following inner product equation: For each $y \in \mathcal{D}(A^*(\alpha))$,

$$\langle y, X(T;\alpha)\rangle = \langle y, X(0)\rangle + \int_0^t \langle A^*(\alpha)y, X(s;\alpha)\rangle ds + \int_0^t \langle \Phi(\alpha)y, U(s)\rangle ds \langle \Phi^* y, W(t)\rangle, \tag{2.3}$$

where $\Phi(\alpha) = B^*(\alpha) \in \mathcal{L}\left(D^{1-\varepsilon}_{A^*(\alpha)}, H_1\right)$. $(X(t;\alpha), t \in [0,T])$ is a well-defined process in $M^p_W(0,T,H)$ (see (Duncan *et al.*, 1994)).

Consider the quadratic cost functional

$$J(X_0, U, \alpha, T) = \int_0^T [\langle QX(s), X(s)\rangle + \langle PU(s), U(s)\rangle] ds, \tag{2.4}$$

where $T \in (0,\infty]$, $X(0) = X_0$, $Q \in \mathcal{L}(H)$, $P \in \mathcal{L}(H_1)$ are selfadjoint operators satisfying

$$\langle Qx, x\rangle \geq r_1|x|^2, \tag{2.5}$$
$$\langle Py, y\rangle \geq r_2|y|^2 \tag{2.6}$$

for $x \in H$, $y \in H_1$ and constants $r_1 > 0$ and $r_2 > 0$.

For adaptive control, the control policies $(U(t), t \geq 0)$ that are considered are linear feedback controls, that is,

$$U(t) = K(t)X(t), \tag{2.7}$$

where $(K(t), t \geq 0)$ is an $\mathcal{L}(H, H_1)$-valued process that is uniformly bounded almost surely by a constant $R > 0$. Let $\Delta > 0$ be fixed. It is assumed that the $\mathcal{L}(H, H_1)$-valued process $(K(t), t \geq 0)$ has the property that $K(t)$ is adapted to $\sigma(X(u), u \leq t - \Delta)$ for each $t \geq \Delta$ and it is assumed that $(K(t), t \in [0, \Delta])$ is a deterministic, operator-valued function. For such an admissible adaptive control there is a unique solution of (2.1) with $K(t) = \tilde{K}(X(s), 0 \leq s \leq t - \Delta)$. If $\Delta = 0$ then (2.1) may not have a unique solution. Furthermore, the delay $\Delta > 0$ accounts for some time that is required to compute the adaptive feedback control law from the observation of the solution of (2.1).

Two more assumptions, (A7) and (A8) are given that are used for the verification of the strong consistency of a family of least squares estimates of the unknown parameter vector α. Define $\mathbb{K} \subset \mathcal{L}(H, H_1)$ as

$$\mathbb{K} = \{K \in L(H, H_1) : |K|_{\mathcal{L}(H,H_1)} \leq R\}$$

where R is given above.

Assume that $B(\alpha)$ is either independent of $\alpha \in \mathcal{K}$ or has the form

$$B(\alpha) = \Phi^*(\alpha) \tag{2.8}$$

where $\Phi(\alpha) = \hat{B}^* A^*(\alpha) \in \mathcal{L}(D^{1-\varepsilon}_{A^*(\alpha)}, H_1)$ and the operator $\hat{B} \in \mathcal{L}(H_1, D^{\varepsilon}_{A(\alpha)})$ is given.

(A7) There is a finite dimensional projection $\tilde{P}$ on H with range in $\cap_{i=1}^q \mathcal{D}(F_i^*)$ such that $i_{\tilde{P}} \Phi\Phi^* i^*_{\tilde{P}} > 0$ where $i_{\tilde{P}} : H \to \tilde{P}(H)$ is the projection map and $B(\alpha)$ is either independent of α or has the form (2.8). In the latter case there is a finite dimensional projection $\tilde{P}$ on H and a constant $c > 0$ such that

$$|\hat{P}(I + K^*\hat{B}^*)F^*\tilde{P}|_{\mathcal{L}(H)} > c$$

is satisfied for all $F \in \{F_1, \ldots, F_q\}$ and $K \in \mathbb{K}$.

It is easy to verify that if H is infinite dimensional, $\hat{B} \in \mathcal{L}(H_1, H)$ is compact and $(F_i^*)^{-1} \in \mathcal{L}(H)$ for $i = 1, 2, \ldots, q$ then (A7) is satisfied.

Let $(U(t), t \geq 0)$ be an admissible control, denoted generically as $U(t) = K(t)X(t)$, where $(X(t), t \geq 0)$ is the (unique) mild solution of (2.1) using the above admissible control. Let

$$\mathcal{A}(t) = (a_{ij}(t)) \tag{2.9}$$

and

$$\tilde{\mathcal{A}}(t) = (\tilde{a}_{ij}(t)) \tag{2.10}$$

where

$$a_{ij}(t) = \int_0^t \langle \tilde{P}F_iX(s), \tilde{P}F_jX(s)\rangle ds \tag{2.11a}$$

if B does not depend on α or

$$a_{ij}(t) = \int_0^t \langle \tilde{P}(F_i + F_i\hat{B}K(s))X(s), \tilde{P}(F_j + F_j\hat{B}K(s))X(s)\rangle ds \tag{2.11b}$$

if $B(\alpha)$ has the form (2.8) and

$$\tilde{a}_{ij}(t) = \frac{a_{ij}(t)}{a_{ii}(t)}. \tag{2.12}$$

It is easy to verify that the integrations in (2.11a) and (2.11b) are well defined.

For the verification of the strong consistency of a family of least squares estimates of the unknown parameter vector, the following assumption is used.

(A8) For each admissible adaptive control law, $(\tilde{\mathcal{A}}(t),\, t \geq 0)$ satisfies

$$\liminf_{t\to\infty} |\det\tilde{\mathcal{A}}(t)| > 0 \quad \text{a.s.}$$

3 Parameter Identification

For the identification of the unknown parameters in the linear stochastic distributed parameter system (2.1), a family of least squares estimates are formed. In this section it is assumed that $\beta = 0$, that is, $-A(\alpha)$ is strictly positive. Let $\tilde{P}$ be the projection given in (A7). The estimate of the unknown parameter vector at time t, $\hat{\alpha}(t)$, is the minimizer of the quadratic functional of α, $L(t;\alpha)$, given by

$$\begin{aligned} L(t;\alpha) = &-\int_0^t \langle \tilde{P}(A(\alpha) + B(\alpha)K(s))X(s), d\tilde{P}X(s)\rangle \\ &+ \frac{1}{2}\int_0^t |\tilde{P}(A(\alpha) + B(\alpha)K(s))X(s)|^2 ds, \end{aligned} \tag{3.1}$$

where $U(s) = K(s)X(s)$ is an admissible adaptive control.

Theorem 3.1 *Let $(K(t),\, t \geq 0)$ be an admissible feedback control law. Assume that* (A2), (A6)–(A8) *are satisfied and $\alpha_0 \in \mathcal{K}^0$. Then the family of least squares estimates $(\hat{\alpha}(t),\, t > 0)$, where $\hat{\alpha}(t)$ is the minimizer of (3.1), is strongly consistent, that is,*

$$P_{\alpha_0}\left(\lim_{t\to\infty} \hat{\alpha}(t) = a_0\right) = 1, \tag{3.2}$$

where α_0 is the true parameter vector.

Proof. See (Duncan *et al.*, 1994).

For the applications of identification and adaptive control it is important to have recursive estimators of the unknown parameters. Let $\langle \tilde{F}(s)x, y\rangle$ be the vector whose ith component is $\langle \tilde{P}F_i(I + \hat{B}K(s))x, y\rangle$. Using (2.1) we have

$$\hat{\alpha}(t) = \mathcal{A}^{-1}(t)\int_0^t \langle \tilde{F}(s)X(s), d\tilde{P}X(s) - \tilde{P}F_0X(s)ds\rangle. \tag{3.3}$$

Since $\mathcal{A}^{-1}(t)$ satisfies the differential equation

$$d\mathcal{A}^{-1}(t) = -\mathcal{A}^{-1}(t)d\mathcal{A}(t)\mathcal{A}^{-1}(t),$$

the differential of (3.3) satisfies

$$d\hat{\alpha}(t) = \mathcal{A}^{-1}(t)\langle \tilde{F}(t)X(t), d\tilde{P}X(s) - \tilde{P}A(\hat{\alpha}(t))(I + \hat{B}K(t))X(t)dt\rangle. \tag{3.4}$$

4 Optimality for an Adaptive Control

The certainty equivalence, optimal ergodic control law is self-tuning and self-optimizing. The self-tuning property is obtained by using the continuity of the solution of a stationary Riccati equation with respect to parameters in the topology induced by a suitable operator norm. Since the unbounded operator $B(\alpha)$ appears in the linear transformation of the control in (2.1), this operator topology is more restrictive than for bounded linear transformations on the Hilbert space. This continuity property is also used to show that the certainty equivalence control stabilizes the unknown system in a suitable sense.

The self-optimizing property is formulated for a self-tuning adaptive control in the following.

Theorem 4.1 *Assume that* (A1)–(A4), (A6)–(A8) *are satisfied. Let* $(\hat{\alpha}(t),\, t \geq 0)$ *be the family of least squares estimates where* $\hat{\alpha}(t)$ *is the minimizer of* (3.1). *Let* $(K(t),\, t \geq 0)$ *be an admissible adaptive control law such that*

$$K(t) = -P^{-1}\Psi(\hat{\alpha}(t-\Delta))V(\hat{\alpha}(t-\Delta)) \tag{4.1}$$

where $\Psi(\alpha) = B^*(\alpha)$ *and* $V(\alpha)$ *is the solution of the formal stationary Riccati equation (see (Duncan* et al.*, 1994) for details) for* $\alpha \in \mathcal{K}$. *Then the family of estimates* $(\hat{\alpha}(t),\, t \geq 0)$ *is strongly consistent,*

$$\lim_{t\to\infty} K(t) = k_0 \quad \text{a.s.} \tag{4.2}$$

in $\mathcal{L}(H, H_1)$ *where* $k_0 = -P^{-1}\Psi(\alpha_0)V(\alpha_0)$ *and*

$$\lim_{T\to\infty} \frac{1}{T} J(X_0, U, \alpha_0, T) = \mathrm{Tr}\Pi(V(\alpha_0)) \quad \text{a.s.} \tag{4.3}$$

where $U(t) = K(t)X(t)$ *and* $\Pi(V)$ *is given in the following corollary.*

Proof. See (Duncan *et al.*, 1994).

Corollary. *Let* $V \in \mathcal{L}(H)$ *be self adjoint such that* $V \in \mathcal{L}(H, D_{A^*}^{1-\varepsilon})$ *and* $|\langle Vx, Ax\rangle| \leq k|x|^2$ *for* $x \in \mathcal{D}(A)$, *where* $k > 0$. *Assume that one of the following conditions is satisfied:*

(i) F *is Hilbert-Schmidt*

(ii) V *is nuclear*

(iii) $V \in \mathcal{L}(D_A^{\delta-(1/2)}, D_{A^*}^{(1/2)-\delta})$.

Then for all $0 \leq \tau \leq t \leq T$,

$$\begin{aligned}\langle VX(t), X(t)\rangle - \langle VX(t), X(t)\rangle = & \int_\tau^t [h(X(s)) + 2\langle U(s), \Phi VX(s)\rangle + \Pi(V)]ds \\ & + 2\int_\tau^t \langle \Phi^* VX(s), dW(s)\rangle \quad \text{a.s.}\end{aligned}$$

where h *is the continuous extension of* $2\langle Vx, Ax\rangle$ *on* H, *and for* (i) *and* (ii) $\Pi(V) = \mathrm{Tr}V\Phi\Phi^*$ *and, for* (iii) $\Pi(V) = \mathrm{Tr}(R^*(\beta))^{\delta-(1/2)}V\Phi\Phi^*(R^*(\beta))^{(1/2)-\delta}$.

5 Adaptive Boundary Control of Linear Stochastic Distributed Parameter Systems Described by Analytic Semigroups

The stochastic system is described by the stochastic evolution equation

$$\begin{aligned} dX(t;a) &= [A_0 + A_1(\alpha) + A_0 BC(a)]X(t;\alpha)dt \\ &\quad + A_0 BD(\alpha)U(t)dt + GdW(t) \\ X(0;\alpha) &= x \end{aligned} \tag{5.1}$$

in a separable Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ where A_0 is the infinitesimal generator of an exponentially stable analytic semigroup $(S_0(t),\, t \geq 0)$ on H, $A_0 = A_0^*$, $\alpha \in \mathcal{K} \subset \mathbb{R}^q$. Let D_A^γ for $\gamma \in \mathbb{R}$ be the domain of the fractional power $(-A_0)^\gamma$ with the $(-A_0)^\gamma$ graph norm. Let $B \in \mathcal{L}(H_1, D_A^\varepsilon)$ for some $\varepsilon \in (0,1)$, $A_1^*(\alpha) \in \mathcal{L}(D_A^\eta, H)$ for some $\eta \in [0,1)$, $C(\alpha) \in \mathcal{L}(H, H_1)$ and $D(\alpha) \in \mathcal{L}(H_2, H_1)$ for each value of $\alpha \in \mathcal{K}$ where H_1 and H_2 are separable Hilbert spaces. The formal process $(W(t),\, t \geq 0)$ is a cylindrical Wiener process with the incremental covariance the identity, $I \in \mathcal{L}(H)$, that is defined on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t,\, t \geq 0)$. For $p \geq 2$ let $M_W^p(H2) = \cap_{T>0} M_W^p(0,T,H_2)$ where

$$M_W^p(0,T,H_2) = \Bigg\{ U | U : [0,T] \times \Omega \to H_2,\ (U(t),\, t \geq 0) \\ \text{is } (\mathcal{F}_t) \text{ adapted and } E\int_0^T |U(t)|^p dt < \infty \Bigg\}. \tag{5.2}$$

The control process $(U(t),\, t \geq 0)$ in (5.1) is assumed to belong to the space $M_W^p(H_2)$ for some fixed $p > \max((1/\varepsilon), 1/(1-\eta))$ and $p \geq 2$.

For the control problem, the following ergodic, quadratic cost functional is used

$$J(\alpha, U) = \limsup_{t \to \infty} \frac{1}{t} J(t, x, \alpha, U) \tag{5.3}$$

where

$$J(t,x,\alpha,U) = \int_0^t [\langle Q_1 X(s,\alpha), X(s;\alpha)\rangle + \langle Q_2 U(s), U(s)\rangle] ds$$

and $Q_1 = D_1^* \in \mathcal{L}(H)$, $Q_1 \geq 0$, $Q_2 = D_2^* \in \mathcal{L}(H_2)$ and $Q_2 \geq cI$, $c > 0$.

For the remainder of this section the dependence of $A_1(\cdot)$, $C(\cdot)$ and $D(\cdot)$ on the parameter α is suppressed because only general properties of the solution of (5.1) are investigated that are valid for each fixed value of α. A solution of (5.1) is understood as a mild solution, that is, an H-valued process $(X(t),\, t \geq 0)$ that satisfies (almost surely)

$$\begin{aligned} X(t) = S_0(t)x &+ \int_0^t S_0(t-r)A_1 X(r)dr + \int_0^t A_0 S_0(t-r)BDU(r)dr \\ &+ \int_0^t A_0 S_0(t-r)BCX(r)dr + Z(t) \end{aligned} \tag{5.4}$$

where

$$Z(t) = \int_0^t S_0(t-r)GdW(t). \tag{5.5}$$

The operator $S_0(t-r)A_1$ is identified with its (unique) extension as an element of $\mathcal{L}(H)$ which exists because

$$|S_0(t-r)A_1x| \leq \frac{c}{(t-r)^\eta}|x| \tag{5.6}$$

for $x \in \mathcal{D}(A_0)$, $0 \leq r < t \leq T$ and some $c > 0$ where $A_1^*(a) \in \mathcal{L}(D_A^\eta, H)$ and the analyticity of $S_0(\cdot)$ are used. To ensure that the stochastic integral (5.5) is a "nice" process it is assumed that the following condition is satisfied

(C1) $(-A_0)^{-\delta}G$ is Hilbert-Schmidt for some $0 \leq d < \frac{1}{2}$.

If A_0 has a compact resolvent then it is easy to verify that (C1) is equivalent to the following condition

(C1′) $\int_0^T t^{2\delta-1}|S_0(t)G|_{\mathrm{HS}}^2 dt < \infty$ for each $T > 0$ where $|\cdot|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm on $\mathcal{L}(H)$. There is a unique mild solution of (5.1) with continuous sample paths.

In a similar way we can obtain the existence and the uniqueness for the solution of (5.1) with a feedback control $U(t) = K(t)X(t)$ where $K(t) = \tilde{K}(t, X(u),\, u \leq t-\Delta)$, $\Delta > 0$ is fixed, $K(\cdot)$ is deterministic on $[0,\Delta]$ and $K(\cdot) : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, H_2)$ is uniformly bounded and measurable and adapted to $(\mathcal{F}_{t-\Delta},\, t \in \mathbb{R}_+)$. The equation

$$\begin{aligned} X(t) = S_0(t)x + \int_0^t S_0(t-r)A_1X(r)dr + \int_0^t A_0S_0(t-r)BCX(r)dr \\ + \int_0^t A_0S_0(t-r)BDK(r)X(r)dr + Z(t) \end{aligned} \tag{5.7}$$

can be treated similarly to (5.4). The feedback control is an element of $M_W^p(H_2)$.

6 Parameter Estimation

To estimate the parameters of the unknown system (5.1) a family of least squares estimates is given that is shown to be strongly consistent. Some additional conditions are introduced.

(C5) The semigroup generated by $A_0 + \mathcal{C}(\alpha)$ is stable for each a where $\mathcal{C}(\alpha) = A_1(\alpha) + [C^*(\alpha)\Psi]^*$ and $\Psi \in \mathcal{L}(D_A^{1-\varepsilon}, H_1)$ is the extension of B^*A_0.

(C6) The linear operators $A_1(\alpha)$, $C(\alpha)$ and $D(\alpha)$ have the following form:

$$A_1(\alpha) = A_{10} + \sum_{i=1}^{q_1} \alpha_i A_{1i},$$

$$C(\alpha) = C_0 + \sum_{i=1}^{q_1} \alpha_i C_i,$$

$$D(\alpha) = D_0 + \sum_{i=1}^{q} \alpha_i D_i,$$

where $A_{1i}^* \in \mathcal{L}(D_A^\eta, H)$, $C_i \in \mathcal{L}(H, H_1)$ for $i = 0, \ldots, q_1$ and $D_i \in \mathcal{L}(H_2, H_1)$ for $i = 0, q_1 + 1, \ldots, q$. Define the linear operators $\mathcal{C}_i$ and $\mathcal{B}_i$ as follows $\mathcal{C}_i = A_{1i} + [C_i^*\Psi]^*$ for $i = 0, \ldots, q_1$ and $\mathcal{B}_i = [D_i^*\Psi]^*$ for $i = 0, q_1+1, \ldots, q$. Clearly $\mathcal{C}_i \in \mathcal{L}(H, D_A^{-\gamma})$ for $i = 0, \ldots, q_1$, where $\gamma = \max(1-\varepsilon, \eta)$ and $\mathcal{B}_i \in \mathcal{L}(H_2, D_A^{\varepsilon-1})$.

(C7) There is a finite dimensional projection $\tilde{P} : D_A^{-1} \to \tilde{P}(D_A^{-1}) \subset H$ and $(\tilde{P}\mathcal{B}_i,\ i = q_1 + 1, \ldots q)$ are linearly independent and for each nonzero $\mathcal{B} \in \mathbb{R}^{q_1}$,

$$\mathrm{tr} \sum_{i=1}^{q_1} \beta_i(\tilde{P}(\mathcal{C}_i)) \int_0^\Delta S(r;\alpha_0)GG^*S^*(r;\alpha_0)dr \sum_{i=1}^{q_1} \beta_i(\tilde{P}(\mathcal{C}_i))^* > 0,$$

where $(S(t;\alpha_0),\, t \geq 0)$ is the C_0-semigroup with the infinitesimal generator $A_0 + \mathcal{C}(\alpha_0)$.

Let $(\Omega, \mathcal{F}, P)$ denote a probability space for (5.1) where P includes a measure induced from the cylindrical Wiener process and a family of independent random variables for a diminishingly excited control introduced subsequently. $\mathcal{F}$ is the P-completion of an appropriate σ-algebra on ω and $(\mathcal{F}_t,\, t \geq 0)$ is a filtration so that the cylindrical Wiener process $(W(t),\, t \geq 0)$, the solution $(X(t),\, t \geq 0)$ of (5.1) and the diminishingly excited control are adapted to $(\mathcal{F}_t,\, t \geq 0)$.

For the adaptive control problem it is convenient to enlarge the class of controls to $\tilde{M}^p_W(H_2) = \cap_{T>0}\tilde{M}^p_W(0,T,H_2)$ where

$$\tilde{M}^p_W(0,T,H_2) = \Big\{ U | U : [0,T] \times \Omega \to H_2,$$

$$(U(t),\, t \geq 0) \text{ is } (\mathcal{F}_t)\text{-adapted and } \int_0^T |U(s)|^p ds < \infty \text{ a.s.} \Big\}.$$

It is elementary to verify that the regularity properties of the sample paths of the solution of (5.1) with $U \in M^p_W(H_2)$ carry over to $U \in \tilde{M}^p_W(H_2)$.

A family of least squares estimates $(\hat{\alpha}(t),\, t \geq 0)$ of the true parameter vector α_0 is defined as the solution of the following affine stochastic differential equation

$$\begin{aligned} d\hat{\alpha}(t) &= \Gamma(t)[\varphi(t) \times (d\tilde{P}X(t) - \tilde{P}(A_0 + \mathcal{C}_0)X(t)dt \\ &\quad - \tilde{P}\mathcal{B}_0 U(t)dt - \varphi(t) \cdot \hat{\alpha}(t)dt], \\ \hat{\alpha}(0) &= \alpha(0), \end{aligned} \tag{6.1}$$

where $U \in \tilde{M}^p_W(H_2)$.

Let $\hat{\alpha}(t) = \alpha_0 - \hat{\alpha}(t)$ for $t \geq 0$. The process $(\hat{\alpha}(t),\, t \geq 0)$ satisfies the following stochastic differential equation

$$\begin{aligned} d\hat{\alpha}(t) &= -\Gamma(t)[\varphi(t) \times (\varphi(t) \cdot \hat{\alpha}(t)dt + \tilde{P}GdW(t))], \\ \hat{\alpha}(0) &= \alpha_0 - \alpha(0). \end{aligned} \tag{6.2}$$

Since $d\Gamma/dt = -\Gamma(t)[\varphi(t) \times \varphi(t)]\Gamma(t)$, $\Gamma(0) = \alpha I$ we have

$$\hat{\alpha}(t) = -\Gamma(t)\Gamma^{-1}(0)\hat{\alpha}(0) - \Gamma(t)\int_0^t (\varphi(s) \times \tilde{P}GdW(s)). \tag{6.3}$$

The control is a sum of a desired (adaptive) control and a diminishing excitation control. Let $(Z_n,\, n \in \mathbb{N})$ be a sequence of H_2-valued, independent, identically distributed, random variables that is independent of the cylindrical Wiener process $(W(t),\, t \geq 0)$. It is assumed that $EZ_n = 0$ and the covariance of Z_n is Λ for all n where Λ is positive and nuclear and there is a $\sigma > 0$ such that $|Zn|^p \leq s$ a.s. Choose $\tilde{\varepsilon} \in (0, 1/2)$ and fix it. Define the H_2-valued process $(V(t),\, t \geq 0)$ as

$$V(t) = \sum_{n=0}^{[t/\Delta]} \frac{Z_n}{n^{\tilde{\varepsilon}/2}} 1_{[n\Delta,(n+1)\Delta)}(t). \tag{6.4}$$

Clearly we have that

$$\lim_{t \to \infty} |V(t)| = 0 \quad \text{a.s.} \tag{6.5}$$

and for each $l_1, l_2 \in H_2^* = H_2$,

$$\lim_{t\to\infty} \frac{1}{t^{1-\tilde{\varepsilon}}} \int_0^t \langle l_1, V(s)\rangle\langle l_2, V(s)\rangle ds = \lim_{t\to\infty} \frac{1}{t^{1-\tilde{\varepsilon}}} \sum_{i=1}^{[t/\Delta]} \frac{\langle l_1, Z_i\rangle\langle l_2, Z_i\rangle}{i^{\tilde{\varepsilon}}}\Delta + o(1)$$
$$= \Delta^{\tilde{\varepsilon}}(1-\tilde{\varepsilon})^{-1}\langle \Lambda l_1, l_2\rangle \quad \text{a.s.} \tag{6.6}$$

It is assumed that $Z_n \in \mathcal{F}_{n\Delta}$ and Z_n is independent of $\mathcal{F}_s$ for $s < n\Delta$ for all $n \in \mathbb{N}$.

The diminishingly excited control is

$$U(t) = U^d(t) + V(t) \tag{6.7}$$

for all $t \geq 0$.

7 Adaptive Control

In this section a self-optimizing adaptive control is constructed for the unknown linear stochastic system with the ergodic quadratic cost functional using the family of least squares estimates $(\hat{\alpha}(t),\, t \geq 0)$ that satisfies (6.1).

The family of admissible controls $\mathcal{U}(\Delta)$ is

$$\mathcal{U}(\Delta) = \Big\{ U : U(t) = U^d(t) + U^1(t),\ U^d(t) \in \mathcal{F}((t-\Delta)\vee 0)$$
$$\text{and } U^1(t) \in \sigma(V(s),\ (t-\Delta)\vee 0 \leq s \leq t) \text{ for all } t \geq 0,$$
$$U \in \tilde{M}_W^p(H_2),\ \limsup_{t\to\infty} \frac{|X(t)|^2}{t} = 0 \text{ a.s., and}$$
$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t (|X(s)|^2 + |U(s)|^2)ds < \infty \text{ a.s.} \Big\}. \tag{7.1}$$

Define the H_2-valued (control) process $(U^0(t),\, t \geq \Delta)$ by the equation

$$U^0(t) = -Q_2^{-1}\tilde{B}^*(t-\Delta)P(t-\Delta)$$
$$\times \left(S(\Delta; t-\Delta)X(t-\Delta) + \int_{t-\Delta}^t S(t-s; t-\Delta)\tilde{B}(t-\Delta)Ud(s)ds \right), \tag{7.2}$$

where $\tilde{B}^*(t) = (\mathcal{B}^*(\hat{\alpha}(t)))^*$, $S(\tau; t) = e^{\tau A(t)}$, and $A(t)$ is defined as

$$A(t) = \begin{cases} A_0 + \mathcal{C}(\hat{\alpha}(t)) & \text{if } A_0 + \mathcal{C}(\hat{\alpha}(t)) \text{ is stable,} \\ \tilde{A} & \text{otherwise,} \end{cases} \tag{7.3}$$

and $\tilde{A}$ is a fixed stable infinitesimal generator (that is, the associated semigroup is stable) such that $\tilde{A} = A_0 + \mathcal{C}(\alpha_1)$ for some parameter vector α_1, $P(t)$ is the minimal solution of the Riccati equation using $A(t)$ and $\tilde{B}^*(t)$. It will be clear by the construction of U^d that $U^0 \in \mathcal{U}(\Delta)$.

Define two sequences of stopping times $(\sigma_n,\, n = 0, 1, \dots)$ and $(\tau_n,\, n = 1, 2, \dots)$ as follows:

$$\sigma_0 \approx 0$$
$$\sigma_n = \sup\left\{ t \geq \tau_n : \int_0^s |U^0(r)|^p dr \leq \tau_n^\delta s \text{ for all } s \in [\tau_n, t) \right\} \tag{7.4}$$
$$\tau_n = \inf\left\{ t > \sigma_{n-1} + 1 : \int_0^t |U^0(r)|^p dr \leq t^{1+\delta/2} \text{ and } |X(t-\Delta)|^2 \leq t^{1+\delta/2} \right\}. \tag{7.5}$$

where $\delta > 0$ is fixed and $(1+\delta)/2 < 1-\varepsilon$ and U^0 is given by (6.2). It is clear that $(\tau_n - \sigma_{n-1}) \geq 1$ on $\{\sigma_{n-1} < \infty\}$ for all $n \geq 1$.

Define the adaptive control $(U^*(t),\, t \geq 0)$ by the equation

$$U^*(t) = U^d(t) + V(t) \tag{7.6}$$

for $t \geq 0$ where

$$U^d(t) = \begin{cases} 0 & \text{if } t \in [\sigma_n, \tau_{n+1}) \text{ for some } n \geq 0, \\ U^0(t) & \text{if } t \in [\tau_n, \sigma_n) \text{ for some } n \geq 1 \end{cases}$$

and $U^0(t)$, $V(t)$ satisfy (6.2), (5.6), respectively. It is clear that $U^d \in \tilde{M}^p_W(H_2)$.

Theorem 7.1 *If* (C1)–(C7) *are satisfied then the adaptive control* $(U^*(t),\, t \geq 0)$ *for* (5.1) *given by* (7.6) *is an element of* $\mathcal{U}(\Delta)$ *and is self-optimizing, that is,*

$$\begin{aligned} &\inf_{U \in \mathcal{U}} \limsup_{t\to\infty} \frac{1}{t} J(x, U, \alpha_0, t) \\ &\quad = \lim_{t\to\infty} \frac{1}{t} J(x, U^*, \alpha_0, t) \\ &\quad = \mathrm{Tr}(-A_0)^{\delta} PGG^*(-A_0)^{-\delta} + \mathrm{Tr}\tilde{B}^* PR(\Delta)P\tilde{B}Q_2^{-1} \quad a.s., \end{aligned} \tag{7.7}$$

where J *is given above.*

Proof. See (Duncan *et al.*, 1996).

8 Conclusions

The identification methods presented in Section 7 are very promising in being applied to adaptive control problems described in earlier sections. What makes these methods special is that the method of verification of consistency associates a family of control problems to the identification problem and the asymptotic behavior of the solutions of a family of stationary Riccati equations from the control problem implies a persistent excitation property for the identification problem. It means that a persistent excitation property is proved by control theory methods.

8.1 Future Research: Noise Modelled by the Fractional Brownian Motion

With the very recent development of Stochastic Calculus for Fractional Brownian Motion see (Duncan *et al.*, 1999) it will be natural to consider the identification problem for the stochastic partial differential equation with fractional white noise as the input noise but first the theory of stochastic partial differential equation with fractional Brownian motion needs to be well established.

Since the pioneering work of Hurst (1951, 1956) and Mandelbrot (1983), the fractional Brownian motions have played an increasingly important role in many fields of application such as hydrology, economics and telecommunications.

Let $0 < H < 1$. It is well-known that there is a Gaussian stochastic process $(B^H_t,\, t \geq 0)$ such that

$$\mathbb{E}\left(B^H_t\right) = 0\,, \quad \mathbb{E}(B^H_t B^H_s) = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}$$

for all $s, t \in \mathbb{R}_+$. This process is called the fractional Brownian motion with Hurst parameter H. To simplify the presentation, it is always assumed that the fractional Brownian motion is 0 at $t = 0$.

If $H = 1/2$, then the corresponding fractional Brownian motion is the usual standard Brownian motion. If $H > 1/2$, then the process $(B_t^H,\, t \geq 0)$ exhibits a long-range dependence, that is, if $r(n) = \operatorname{cov}(B_1^H, B_{n+1}^H - B_n^H)$, then $\sum_{n=1}^{\infty} r(n) = \infty$. A fractional Brownian motion is also self-similar, that is, $B_{\alpha t}^H$ has the same probability law as $\alpha^H B_t^H$. A process satisfying this property is called a self-similar process with the Hurst parameter H.

Since in many problems related to network traffic analysis, mathematical finance and many other fields, the processes under study seem empirically to exhibit the self-similar properties and the long-range dependent properties and since the fractional Brownian motions are the simplest processes of this kind, it is important to have a systematic study of these processes and to use them to construct other stochastic processes. One way to approach this is to follow by analogy the methods for Brownian motion. In the stochastic analysis, the Brownian motion can be used as the input (white) noise and many other processes (e.g. general diffusion processes) can be constructed as solutions of stochastic differential equations. One powerful tool for determining the solutions is the Itô formula. This theory of stochastic calculus for Fractional Brownian Motion was developed by Duncan, Hu and Pasik-Duncan (1999).

8.2 Future Research: Generalization to Semilinear Systems

The identification methods described in this paper can be developed for semilinear stochastic distributed parameter systems. In (Duncan *et al.*, 1998) the adaptive control problem is formulated and solved for the following semilinear stochastic distributed parameter system:

Let $(X(t),\, t \geq 0)$ be an H-valued, parameter dependent, controlled process that satisfies the stochastic differential equation

$$\begin{aligned} dX(t) &= (AX(t) + f(\alpha, X(t)) - u(t))dt + Q^{1/2} dW(t) \\ X(0) &= x \end{aligned}$$

where H is a real, separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$, $A : \mathrm{Dom}(A) \to H$ is a densely defined, unbounded linear operator on H, $f(\alpha, \cdot) : H \to H$ for each $\alpha \in \mathcal{A} \subset \mathbb{R}^d$ that is a compact set of parameters, $(W(t),\, t \geq 0)$ is a standard, cylindrical H-valued Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and $Q^{1/2} \in \mathcal{L}(H)$. The family of admissible controls is

$$\mathcal{U} = \{u : \mathbb{R}_+ \times \Omega \to B_R \mid u \text{ is measurable and } (\mathcal{F}_t) \text{ adapted}\}$$

where $B_R = \{y \in H \mid |y| \leq R\}$ and $R > 0$ is fixed. A family of Markov controls, e.g., $u(t) = \tilde{u}(X(t))$, is also considered where $\tilde{u} \in \tilde{\mathcal{U}}$ and

$$\tilde{\mathcal{U}} = \{\tilde{u} : H \to B_R \mid \tilde{u} \text{ is Borel measurable}\}.$$

The cost functionals $J(x, \lambda, u)$ and $\tilde{J}(x, u)$ are given as

$$J(x, \lambda, u) = \mathbb{E}_{x,u} \int_0^{\infty} e^{-\lambda t} (\psi(X(t)) + h(u(t)))dt$$

and

$$\tilde{J}(x, u) = \liminf_{T \to \infty} \mathbb{E}_{x,u} \frac{1}{T} \int_0^T (\psi(X(t)) + h(u(t)))dt$$

where $\lambda > 0$, $h : B_R \to \mathbb{R}_+$ and $\psi : H \to \mathbb{R}$ that describe a discounted and an ergodic control problem, respectively.

It is very important to solve the identification problem for this model. It seems that some identification methods described in this paper could be developed for semilinear stochastic systems with unknown parameters.

References

Duncan, T. E., B. Goldys, and B. Pasik-Duncan (1991). "Adaptive control of linear stochastic evolution systems," *Stochastics and Stochastic Reports*, **36**, pp. 71–90.

Duncan, T. E., Y. Hu, and B. Pasik-Duncan (1999). "Stochastic calculus for fractional Brownian motion," *SIAM J. Control Optim.*, to appear.

Duncan, T. E., B. Maslowski, and B. Pasik-Duncan (1994). "Adaptive boundary control of linear stochastic distributed parameter systems," *SIAM J. Control Optim.*, **32**, pp. 648–672.

Duncan, T. E., B. Maslowski, and B. Pasik-Duncan (1996). "Adaptive boundary control of stochastic linear distributed parameter described by analytic semigroups," *Appl. Math. Optim.*, **33**, pp. 107–138.

Duncan, T. E., B. Maslowski, and B. Pasik-Duncan (1998). "Adaptive control for semilinear stochastic systems," *Proceedings of the 37th CDC, Tampa, Florida.*

Flandoli, F. (1987). "Albebraic Riccati equation arising in boundary control problems," *SIAM J. Control Optim.*, **25**, pp. 612–636.

Hurst, H. E. (1951). "Long-term storage capacity in reservoirs," *Trans. Amer. Soc. Civil Eng.*, **116**, pp. 400–410.

Hurst, H. E. (1956). "Methods of using long-term storage in reservoirs," *Proc. Inst. Civil Engineers*, Part I, Chapter 5, pp. 519–590.

Lasiecka, I. and R. Triggiani (1987). "The regulator problem for parabolic equations with Dirichlet Boundary control. I," *Appl. Math. Optim.*, **16**, pp. 147–168.

Mandelbrot, B. B. (1983). *The Fractal Geometry of Nature.* San Francisco, CA: Freeman.

Pasik-Duncan, B. (1992). "On the consistency of least squares identification procedure in linear evolution systems," *Stochastics and Stochastic Reports*, **39**, pp. 83–94.