FEED-BACK CONTROL FOR DESCRIPTOR SYSTEMS*

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Abstract

The control in the feed-back form is obtained for five linear-quadratic optimal control problems in a Hilbert space with the state equation unresolved with respect to the derivative, namely, the problem with the quadratic or linear-quadratic performance index, the problem with fixed points, the periodic problem, the regulation problem on an infinite interval.

For that purpose the operator which is the solution of the operator Riccati equation is used , this operator acts in all state space, unlike in a subspace in the case of a singular operator before the derivative, as it was in the papers of other authors.

In contrast to previous works of other authors the regularity of the pencil of the operators from the state equation is not required.

1 Introduction

For the last twenty years many works devoted to the study of optimal control problems by systems, the state equations of which are not resolved with respect to the derivative have been published (see, for example, reviews in (Lewis, 1986; Kurina, 1992)). In the literature such systems are frequently named as descriptor, implicit, singular, differential–algebraic, etc.

In the present paper, the control in the feed-back form is obtained for five following linear-quadratic optimal control problems for descriptor systems in a Hilbert space: the problem with the quadratic or linear-quadratic performance index, the problem with fixed points, the periodic problem, the regulation problem on an infinite interval.

For that purpose it is not necessary to select from the state equation an equation resolved with respect to the derivative as it was made in numerous works devoted to linear-quadratic control problems for descriptor systems. Besides, the form of relations, defining the control in the feed-back form, is identical both for a singular operator, standing before the derivative, and for a nonsingular operator, that is very convenient in a research of singularly perturbed control problems. For the entry of the control in the feed-back form the operator is used which is the solution of the operator Riccati equation and it acts in all state space, unlike in a subspace in the case of a singular operator before the derivative, as it was in previous works of other authors (see, for example, in (Bender and Laub, 1987; Cobb, 1983; Yue-Yun Wang et al., 1987)).

In contrast to previous works of other authors (Bender and Laub, 1987; Cobb, 1983; Lewis, 1985; Pandolfi, 1981; Yue-Yun Wang et al., 1987) the regularity of the pencil of the operators from the state equation is not recuired.

2 Quadratic performance index

Let's consider the problem of the minimization of the quadratic functional

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$$J(u)=\frac{1}{2}<x(T),Vx(T)>+\frac{1}{2}\int_0^T(<x(t),W(t)x(t)>+2<x(t),S(t)u(t)>+<u(t),R(t)u(t)>)dt \quad (1)$$

on trajectories of a linear system unresolved with respect to the derivative

$$\frac{d}{dt}(Ax(t))=C(t)x(t)+B(t)u(t), \quad (2)$$

$$Ax(0)=x_0. \quad (3)$$

Here $t \in [0, T]$, $T>0$ is fixed, $x(t) \in X$, $u(t) \in U$; X, Y, U are real Hilbert spaces, $<\cdot,\cdot>$ means a scalar product in appropriate spaces, $V, W(t) \in L(X)$, $S(t) \in L(U, X)$, $R(t) \in L(U)$; $A, C(t) \in L(X, Y)$, $B(t) \in L(U, Y)$, $W(t) = W'(t)$, $R(t) = R'(t)$ for all $t \in [0, T]$, $V = V'$, the prime with a notation of an operator denotes the conjugate operator, the operators $W(t), S(t), R(t), C(t), B(t)$ are continuous with respect to t.

Admissible controls are continuous functions with values in U for which there is a solution of the problem (2), (3).

Remark 1. *From the relation (4) it follows that $x_0 \in ImA$, that is for some $\tilde{x}_0 \in X$ the equality $x_0 = A\tilde{x}_0$ should take place. Therefore initial condition (3) for descriptor systems is frequently denoted as $Ax(0) = A\tilde{x}_0$ (see, for example, in (Bender and Laub, 1987)).*

Theorem 1. *If for every $t \in [0, T]$ the invertible operator $R(t)$ is positive, the operator $K(t) \in L(X,Y)$ is a solution of the differential operator equation*

$$\frac{d}{dt}(A'K(t))=-K'(t)C(t)-C'(t)K(t)+(K'(t)B(t)+S(t))R^{-1}(t)(K'(t)B(t)+$$
$$+S(t))'-W(t) \quad (4)$$

with the condition

$A'K(T) = V$ *(5)*

and $x^*(t)$ is a solution of the problem (2), (3) with the control, defined by the formula

$u^*(t)=-R^{-1}(t)(B'(t)K(t)+S'(t))x^*(t),$ *(6)*

then the equality (6) defines an optimal control for the problem (1)-(3) in the feed-back form, the minimum value of the performance index (1) is

$J(u^*) = \frac{1}{2}< x(0), A'K(0)x(0)>.$ *(7)*

Remark 2. *From (4), (5) and the symmetry of the operators $R(t)$, $W(t)$, V it follows that the operator $A'K(t)$ is symmetric for every $t \in [0, T]$, that is*

$A'K(t)=K'(t)A.$ *(8)*

Proof of theorem 1. Let's find the derivative of the function $<x(t), AK'(t)x(t)>$ in view of the expressions (2), (4), (8). Fulfilling the simple transformations we have

$$\frac{d}{dt}<x(t),A'K(t)x(t)>=<\frac{d(Ax(t))}{dt},K(t)x(t)>+<x(t),\frac{d(A'K(t))}{dt}x(t)>+$$
$$+<x(t),K'(t)\frac{d(A(t))}{dt}>=-<x(t),W(t)x(t)>-2<x(t),S(t)u(t)>-<u(t),R(t)u(t)>+ \quad (9)$$
$$+<u(t)+R^{-1}(t)(B'(t)K(t)+S'(t))x(t),R(t)(u(t)+R^{-1}(t)(B'(t)K(t)+S'(t))x(t))>.$$

Integrating the last equality on the segment $[0, T]$, by virtue of the relations (1), (5) we obtain

$$J(u)=\frac{1}{2}<x(0),A'K(0)x(0)>+\frac{1}{2}\int_0^T<u(t)+R^{-1}(t)(B'(t)K(t)+S'(t))x(t),R(t)(u(t)+$$
$$+R^{-1}(t)(B'(t)K(t)+S'(t))x(t)>dt. \quad (10)$$

Let's show, that the magnitude $<x(0), A'K(0)x(0)>$ does not depend on a control $u(t)$. By virtue of the remark 1 and the relation (8) we really have

$$<x(0), A' K(0)x(0)> = <A\tilde{x}_0, K(0)x(0)> = <\tilde{x}_0, K'(0)x_0>.$$

The choice of an optimal control is obvious if we use the obtained expression for *J(u)*. That is as the operator *R(t)* is positive from (10) it follows that the minimum of the functional *J(u)* is reached when $u(t)=u^*(t)$ (see (6)) and this minimum is calculated by the formula (7).

3 Linear-quadratic performance index

Let's receive an optimal control in a feed-back form for the problem of the minimization of the linear-quadratic functional

$$J(u) = <d, x(T)> + \frac{1}{2}\int_0^T (<x(t), W(t)x(t)> + 2<x(t), S(t)u(t)> + <u(t), R(t)u(t)>)dt \qquad (11)$$

on trajectories of the system (2), (3). Here $d \in X$ is fixed.

Theorem 2. *If for every* $t \in [0, T]$ *the invertible operator R (t) is positive, the operator K (t) is a solution of the differential operator equation (4) with the condition*

$$A'K(T) = 0, \qquad (12)$$

h (t) is a solution of the problem

$$\frac{d}{dt}(A'h(t)) = -(C'(t) - (K'(t)B(t) + S(t))R^{-1}(t)B'(t))h(t), \qquad (13)$$

$$A'h(T) = d \qquad (14)$$

and $x^*(t)$ *is a solution of the problem (2), (3) with the control, defined by the formula*

$$u^*(t) = -R^{-1}(t)((B'(t)K(t) + S'(t))x^*(t) + B'(t)h(t)), \qquad (15)$$

then the equality (15) defines an optimal control for the problem (11), (2), (3) in the feed-back form, the minimum value of the performance index (11) is

$$J(u^*) = <x(0), A'(\frac{1}{2}K(0)x(0) + h(0))> - \frac{1}{2}\int_0^T <h(t), B(t)R^{-1}(t)B'(t)h(t)> dt.$$

The proof of this theorem is similar to the proof of the theorem 1. Namely, it is necessary to find the expression for

$$\int_0^T \frac{d}{dt} <x(t), A'(h(t) + \frac{1}{2}K(t)x(t))> dt,$$

using the relations (4), (12), (13), (14) for *K(t)* and *h(t).*

4 Problem with fixed points

Now we deal with the fixed points problem, which is the minimization of the functional

$$J(u) = \frac{1}{2}\int_0^T (<x(t), W(t)x(t)> + 2<x(t), S(t)u(t)> + <u(t), R(t)u(t)>)dt \qquad (16)$$

on trajectories of the system (2) with the boundary values

$$Ax(0)=x_0,\ Ax(T)=x_T. \qquad (17)$$

Here admissible controls are continuous functions with values un *U* for which there is a solution of the problem (2), (17).

Theorem 3. *If for every* $t \in [0, T]$ *the invertible operator R (t) is positive, the operator* $K(t) \in L(X,Y)$ *is a solution of the differential operator equation (4) under the symmetry condition (8) and* $x^*(t)$ *is a solution of the problem (2), (17) with the control, defined by the formula (6), then the equality (6)defines an optimal* *control for the problem (16), (2), (17) in the feed-back form, the minimum value of the performance index (16) is*

$J(u^*)=\frac{1}{2}(<x(0),A'K(0)x(0)>-<x(T),A'K(T)x(T)>).$ *(18)*

Proof. Integrating the equality (9) on the segment *[0,T]*, by virtue of the relations (16), (17) we obtain

$$J(u)=\frac{1}{2}(<x(0),A'K(0)x(0)>-<x(T),A'K(T)x(T)>)+$$

$$+\frac{1}{2}\int_0^T<u(t)+R^{-1}(t)(B'(t)K(t)+S'(t))x(t),R(t)(u(t)+R^{-1}(t)(B'(t)K(t)+S'(t))x(t))>dt. \quad (19)$$

We further establish that the term outside the integral on the right side of (19) is equal to

$$\frac{1}{2}<\tilde{x}_0,K'(0)x_0>-<\tilde{x}_T,K'(T)x_T>,$$

where $x_T=A\tilde{x}_T$, that is it does not depend on a control *u(t)*, it depends on boundary values (17) only.

As the operator *R(t)* is positive definite from (19) it follows that the minimum value of the functional (16) is reached when *u(t)* is defined by the formula (6) and this minimum is calculated by the formula (18).

5 Periodic problem

Now let's consider the periodic problem of the minimization of the functional (16) on trajectories of the system

$$\frac{d}{dt}(Ax(t))=C(t)x(t)+B(t)u(t)+g(t), \quad (20)$$

$$x(0)=x(T). \quad (21)$$

Here in addition to the previous conditions we assume that all operators and $g(t)\in Y$ are *T*-periodic in t functions.

Theorem 4. *If for every $t\in[0, T]$ the invertible operator R (t) is positive, $K(t)\in L(X,Y)$ is a solution of the differential operator equation (4) under the conditions (8) and*

$K(0) = K(T),$ *(22)*

$\varphi(t)$ is a solution of the problem

$$\frac{d}{dt}(A'\varphi(t))=-(C'(t)-(K'(t)B(t)+S(t))R^{-1}(t)B'(t))\varphi(t)-K'(t)g(t), \quad (23)$$

$$\varphi(0)=\varphi(T), \quad (24)$$

and $x^*(t)$ is a solution of the problem (20), (21) with the control, defined by the formula

$u^*(t) = -R^{-1}(t)((B'(t)K(t) + S'(t))x^*(t) + B'(t)\varphi(t)),$

then $u^*(t)$ is an optimal control for the problem (16), (20), (21), the minimum value of the functional (16) is

$$J(u^*)=\frac{1}{2}\int_0^T<\varphi(t),2g(t)-B(t)R^{-1}(t)B'(t)\varphi(t)>dt.$$

For the proof of the theorem 4 it is necessary to find the expression for

$$\int_0^T\frac{d}{dt}<x(t),A'(\varphi(t)+\frac{1}{2}K(t)x(t))>dt,$$

using the relations (16), (20), (21), (4), (8), (22)-(24).

6 Regulation problem on infinite interval

Let's consider the problem of the minimization of the functional

$$J(u) = \frac{1}{2}\int_0^{+\infty}(< x(t), Wx(t) > +2 < x(t), Su(t) > + < u(t), Ru(t) >)dt \tag{25}$$

on trajectories of the system

$$\frac{d}{dt}(Ax(t)) = Cx + Bu, \tag{26}$$

$$Ax(0) = x_0. \tag{27}$$

Admissible controls are controls *u(t)* ensuring a finite value of the functional (25) and the relation

$Ax(+\infty) = 0,$ (28)

where x(t) is a solution of the problem (26), (27) corresponding to the control *u(t).*

Theorem 5. *If the invertible operator R is symmetric positive, the operator* $K \in L(X,Y)$ *is a solution of the algebraic operator Riccati equation*

$K'C + C'K - (K'B + S) R^{-1} (K'B + S)' + W = 0$ (29)

under the condition

$A'K = K'A,$ (30)

$x^*(t)$ *is a solution of the problem (26), (27) with the control, defined by the formula*

$u^*(t) = -R^{-1}(t) (B'K + S') x^*(t),$ (31)

and this control is admissible, then $u^*(t)$ *is an optimal control for the problem (25)-(27) and the minimum value of the performance index (25) is*

$J(u^*) = ½ < x(0), A'Kx(0) >.$ (32)

Proof. Let *u(t)* is an arbitrary admissible control and *x(t)* is a corresponding trajectory (the solution of the problem (26), (27)). Let's find the derivative of the function $<x(t), A'Kx(t)>$ taking into account the

$$\frac{d}{dt} < x(t), A'Kx(t) >=< \frac{d(Ax(t))}{dt}, Kx(t) > + < x(t), K'\frac{d(Ax(t))}{dt} >=$$
$$= - < x(t), Wx(t) > -2 < x(t), Su(t) > - < u(t), Ru(t) > +$$
$$+ < u(t) + R^{-1}(B'K + S')x(t), R(u(t) + R^{-1}(B'K + S')x(t)) > .$$

expressions (26), (29), (30)). After simple transformations we have

Integrating the last equality on the interval $[0, +\infty]$, by virtue of (25), (28) we obtain

$$J(u) = \frac{1}{2}\int_0^{+\infty} < u(t) + R^{-1}(B'K + S')x(t), R(u(t) + R^{-1}(B'K + S')x(t)) > dt +$$
$$+ \frac{1}{2} < x(0), A'Kx(0) > .$$

As $<x(0), A'Kx(0)>$ does not depend on control *u(t)* and *R* is positive, from the last equality it follows, that the minimum of the functional (25) is reached when $u(t)=u^*(t)$ (see 31) and this minimum is calculated by the formula (32).

Some from the above cosidered theorems were established in the finite-dimensional case when *X=Y, S = 0* in (Kurina, 1984; 1992; 1993).

As the equation (29) is equivalent to the equation

$K'(C - BR^{-1}S') + (C - BR^{-1}S')'K - K'BR^{-1}B'K + W - SR^{-1}S' = 0,$

it is possible to apply the results from (Kurina, 1993) for the analysis of the solvability of the equation (29) under the condition (30) in the finite-dimensional case when *X=Y.*

Remark 3. *For the entry of the control in the feed-back form in the finite-dimensional case when X=Y the authors in the pareps (Lewis, 1985; 1986; Mehrmann, 1991) use the operators, which are the solutions of the special operator Riccati equations and they act in all state space.*

We write these equations when the operator R is invertible and S=0:

$$A\ '\frac{dZ(t)}{dt}A = -A\ 'Z(t)C - C\ 'Z(t)A + A\ 'Z(t)BR^{-1}B\ 'Z(t)A - W, \qquad (33)$$

$$A\ 'ZC + C\ 'ZA - A\ 'ZBR^{-1}B\ 'ZA + W = 0. \qquad (34)$$

But there are simple examples (see, for example, in (Kurina, 1992; 1993)) showing that in the case of a noninvertible operator A these operator Riccati equations are unsolvable although the corresponding optimal control problems have the unique solutions.

Example 1. *Let us consider the following values of the parameters*

$$V = S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \quad R = 1, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then the problems (1)-(3) and (25)-(27) have the unique solutions.

For this example the equation (33), (34) have no solutions. We can see this by comparing the elements in the lower right corners of the matrices on the two sides of equations (33), (34).

However, the problems (1)-(3) and (25)-(27) are easily solved using the solutions of the problems (4), (5) and (29), (30) respectively. We can take

$$K(t) = K = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}.$$

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