Proceedings of the 7th Mediterranean Conference on Control and Automation (MED99) Haifa, Israel - June 28-30, 1999

Robust stability condition for the system with feedback connected uncertainty and uncertain number of unstable poles*

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Abstract

In this paper, we consider the robust stabilization problem for single-input/single-output continuous time-invariant systems with feedback connected uncertainty such that the number of poles of the plant in the right half plane is not necessarily equal to that of the nominal plant. First of all, we define a class of uncertainty to be considered. The necessary and sufficient robust stability condition for the system with such class of uncertainty is presented by using a relation between the plant and the nominal plant.

1 Introduction

Notations

R	field of real numbers
$\mathcal{C}$	field of complex numbers
$R(s)$	the space of all real-rational transfer function
$\Re\{\cdot\}$	real part of $\{\cdot\} \in \mathcal{C}$

2 Introduction

In the present paper, we emamin the robust stability problem for single-input and single-output continuous time-invariant systems with an uncertainty. On robust stability problem, several paper has been considered (Doyle and Stein, 1981; Doyle, Wall and Stein, 1982; Chen and Desoer, 1982; Kishore and Pearson, 1992; Glover and Doyle, 1988; Doyle, Glover, Khargonekar and Francis, 1988; Verma, Helton and Jonckheere, 1986; McFarlane and Glover, 1989; Doyle, Francis and Tannenbaum, 1992). Doyle and Stein built the basis for this problem (Doyle and Stein, 1981) under the assumption that the number of unstable poles of the plant is equal to that of the nominal plant. Chen and Deseor gave the complete proof of the result by Doyle and Stein (Chen and Desoer, 1982). Doyle et al. summarized the robust stability condition for several type of uncertaity (Doyle, Francis and Tannenbaum, 1992). Kishore and Pearson clarifid if an class of uncertainty is closed set then the gap between the necessary robust stability condition and

*This research was supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Encouragement of Young Scientists, 10750330 and Saneyoshi Scholarship Foundation.

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sufficient one exist(Kishore and Pearson, 1992). In addition, if an class of uncertainty is open set then the necessary robust stability condition is equal to that of sufficient one. In this way, the robust stability condition for the system with invariant number unstable poles completely clarified.

However, these study on robust control such as (Doyle and Stein, 1981; Doyle, Wall and Stein, 1982; Chen and Desoer, 1982; Kishore and Pearson, 1992; Glover and Doyle, 1988; Doyle, Glover, Khargonekar and Francis, 1988; McFarlane and Glover, 1989; Doyle, Francis and Tannenbaum, 1992) can not be applicabele for the system having uncertain number of poles in the closed right half plane. There exist applications such that the number of unstable poles changes. For example, the number of right half plane poles of a large flexible spacecraft changes when the configuration of the spacecraft is changed (Vidyasagar and Kimura, 1986). The problem of obtaining the robust stability condition for the system having a varying number of the closed right half plane poles is difficult because the problem does not reduce to the small gain theorem. To this problem, Verma et al. gave a solution ascribed to Nevanlinna-Pick theory (Verma, Helton and Jonckheere, 1986). Verma et al. showed that it is necessary to restrict the plants to have no more unstable poles than the nominal plant and under this assumption the necessary and sufficient robust stability condition is that Nevanlinna-Pick matrix related to the set of plant is nonnegative difinite(Verma, Helton and Jonckheere, 1986). Actually speaking , study on this problem has never been completed.

In the present paper, we consider the robust stability plebem for the system having uncertain number of poles in the closed right half plane using the relation between the plant and the nominal plant. First of all, we define the class of uncertainty to be considered. This uncertainty is conected to the plant such that the plant has the uncertainty with feedback loop. The condition that the set of the plant and the nominal plant is contained in such class, is made it clear. By using this relation, the necessary and sufficient robust stability condition for the system having uncertainty number of unstable poles is shown.

3 Problem Formulation

Let us consider the control system as below.

$$\begin{cases} y(s) &= G(s)u(s) \\ u(s) &= C(s)\left(r(s) - y(s)\right) \end{cases} \tag{1}$$

Here $G(s) \in R(s)$ is the strictly proper plant with single-input and single-output. $C(s)$ is the controller, r is the reference input, y is the output. The nominal plant of $G(s)$ denotes $G_m(s) \in R(s)$. $G(s)$ is assumed to have same number of right half plane zeros of $G_m(s)$.

The purpose of this paper is to obtain robust stability condition for the system included in following class

$$G(s) = \frac{G_m(s)}{1 - G_m(s)\Delta(s)} \tag{2}$$

$$|\Delta(j\omega)| < |W(j\omega)| \quad \forall\omega \in R. \tag{3}$$

Here Ω denotes the set of plant with included in above class of uncertainty. That is,

$$\Omega = \left\{ G(s) | G(s) = \frac{G_m(s)}{1 - G_m(s)\Delta(s)}, |\Delta(j\omega)| < |W(j\omega)| \ \forall\omega \in R \right\} \tag{4}$$

Note that this uncertainty permit that the number of unstable poles of $G(s)$ is not always equal to that of $G_m(s)$. The connection of the nominal plant $G_m(s)$ and the uncertainty $\Delta(s)$ in (2) is shown in Fig. 1 . Therefore we call the plant (2) the plant with feedback connected uncertainty.

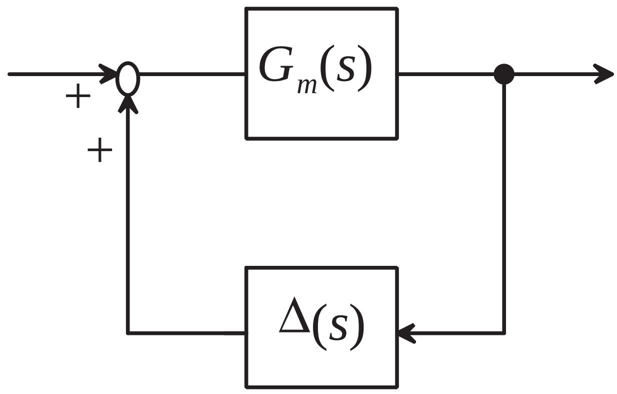

Figure 1: Connection of $G_m(s)$ and $\Delta(s)$

Before considering the robust stability condition for the above class of uncertainty, the robust stability condition under general assumption is presented. That is the robust stability condition under the assumption that the number of unstable poles of $G(s)$ is equal to that of $G_m(s)$ is summarized as follows.

Theorem 1 *Let $C(s)$ stabilize the nominal plant $G_m(s)$. It is assumed that the number of the poles of the plant $G(s)$ is equal to that of the nominal plant $G_m(s)$. $C(s)$ is the robustly stabilizer for all $\Delta(s)$ satisfying*

$$|\Delta(j\omega)| < |W(j\omega)| \tag{5}$$

if and only if

$$\left\| \frac{G_m(s)W(s)}{1+G_m(s)C(s)} \right\|_\infty \leq 1 \tag{6}$$

holds. □

(Proof is omitted.)

4 Relation between nominal plant and plant

In this section, we describe the relation between the nominal plant $G_m(s)$ and the plant $G(s)$ that there exist controller $C(s)$ satisfying Theorem 1.

From internally stability condition, the controller must be proper. When the controller $C(s)$ is proper, we have

$$\lim_{\omega\to\infty} \left| \frac{1}{1+G_m(s)C(s)} \right| = 1 \tag{7}$$

since the nominal plant is assumed to be strictly proper.

From (7), we have following theorem.

Theorem 2 *Necessary condition that there exists robust stabilizer $C(s)$ is that the relative degree of $G(s)$ is equal to that of $G_m(s)$.* □

Proof: *The proof is to show if the relative degree of $G(s)$ is not equal to that of $G_m(s)$, then (6) of Theorem 1 can not be satisfied.*

To satisfy Theorem 1, from (2),

$$\begin{aligned}\lim_{\omega\to\infty}&\left|\frac{G_m(j\omega)\Delta(j\omega)}{1+G_m(j\omega)C(j\omega)}\right| \\ &< \lim_{\omega\to\infty}\left|\frac{G_m(j\omega)W(j\omega)}{1+G_m(j\omega)C(j\omega)}\right| \\ &\leq 1 \end{aligned} \tag{8}$$

must be hold for all $\Delta(s)$ satisfying (3). From (2) and (3), we have

$$\begin{aligned}\lim_{\omega\to\infty}\left|\frac{G_m(j\omega)\Delta(j\omega)}{1+G_m(j\omega)C(j\omega)}\right| &= \lim_{\omega\to\infty}\left|\frac{G_m(j\omega)}{1+G_m(j\omega)C(j\omega)}\frac{G(j\omega)-G_m(j\omega)}{G_m(j\omega)G(j\omega)}\right| \\ &= \lim_{\omega\to\infty}\left|\frac{1}{1+G_m(j\omega)C(j\omega)}\frac{G(j\omega)-G_m(j\omega)}{G(j\omega)}\right| \\ &= \lim_{\omega\to\infty}\left|\frac{1}{1+G_m(j\omega)C(j\omega)}\right|\lim_{\omega\to\infty}\left|\frac{G(j\omega)-G_m(j\omega)}{G(j\omega)}\right| \\ &= \lim_{\omega\to\infty}\left|\frac{G(j\omega)-G_m(j\omega)}{G(j\omega)}\right|. \end{aligned} \tag{9}$$

If the relative degree of $G_m(s)$ is greater than that of $G(s)$, we have

$$\begin{aligned}\lim_{\omega\to\infty}\left|\frac{G_m(j\omega)\Delta(j\omega)}{1+G_m(j\omega)C(j\omega)}\right| &= \lim_{\omega\to\infty}\left|\frac{G(j\omega)-G_m(j\omega)}{G(j\omega)}\right| \\ &= 1 \end{aligned} \tag{10}$$

In this case, (8) can not be satisfied.

Conversely if the relative degree of $G_m(s)$ is smaller than that of $G(s)$, then

$$\begin{aligned}\lim_{\omega\to\infty}\left|\frac{G_m(j\omega)\Delta(j\omega)}{1+G_m(j\omega)C(j\omega)}\right| &= \lim_{\omega\to\infty}\left|\frac{G(j\omega)-G_m(j\omega)}{G(j\omega)}\right| \\ &= \infty. \end{aligned} \tag{11}$$

This case can not satisfy (8), too.

We have thus completed the proof of this theorem. ■

From the proof of Theorem 2, we have following corollary.

Corollary 1 *Necessary condition that there exists robust stabilizer $C(s)$ is*

$$\lim_{\omega\to\infty}|G_m(j\omega)W(j\omega)| \leq 1 \tag{12}$$

□

Proof: *Proof is obvious from (8) and (9).* ■

5 Robust stability condition

In this section, we consider the robust stability condition for the number of poles of the nominal plant $G_m(s)$ in the closed right half plane is not equal to that of $G(s)$.

The robust stability condition for the class of the system with uncertain number of poles in the right half plane is summarized as following theorem.

Theorem 3 *Let $C(s)$ stabilize $G_m(s)$. $C(s)$ is the robust stabilizer for Ω if and only if*

$$\left\| \frac{G_m(s)W(s)}{1+G_m(s)C(s)} \right\|_\infty \quad \leq \quad 1. \tag{13}$$

□

The proof of Theorem 3 requires the following lemma.

Lemma 1 *Let $W(s)$ satisfies (12). It is assumed that $G_m(s)$ has q-th number of zeros in the closed right half plane and the p_m-th number of poles in the closed right half plane, and $G(s)$ has q-th number of zeroes in the closed right half plane and the p-th number of poles in the closed right half plane. Then the Nyquist plot of $1/(1-G_m(s)\Delta(s))$ encircles the origin $(0,0)$ $p-p_m$ times in the counter-clockwise direction.* □

Proof: *From the assumption that $W(s)$ satisfies (12) and Theorem 2, the relative degree of $G(s)$ is equivalent to that of $G_m(s)$. Therefore $G(s)$ and $G_m(s)$ can be drown by*

$$G(s) \quad = \quad \frac{k \prod_{i=1}^{n-\beta-q} (s+\mu_i) \prod_{i=n-\beta-q+1}^{n-\beta} (s-\mu_i)}{\prod_{i=1}^{p} (s-\gamma_i) \prod_{i=p+1}^{n} (s+\gamma_i)} \tag{14}$$

$$\begin{aligned} & G_m(s) \\ & = \frac{k_m \prod_{i=1}^{n_m-\beta-q} (s+\mu_{mi}) \prod_{i=n_m-\beta-q+1}^{n_m-\beta} (s-\mu_{mi})}{\prod_{i=1}^{p_m} (s-\gamma_{mi}) \prod_{i=p_m+1}^{n_m} (s+\gamma_{mi})} \end{aligned} \tag{15}$$

respectively. Where β is the relative degree of $G(s)$ and $G_m(s)$. Without loss of generality, let us assume to satisfy $n_m-\beta-q \geq 0$, $n-\beta-q \geq 0$, $\Re\{\gamma_i\} > 0 (i=1,\ldots,n)$, $\Re\{\gamma_{mi}\} > 0 (i=1,\ldots,n_m)$, $\Re\{\mu_i\} > 0 (i=1,\ldots,n-\beta)$, $\Re\{\mu_{mi}\} > 0 (i=1,\ldots,n_m-\beta)$ From (2),(14) and (15), $1/(1-G_m(s)\Delta(s))$ is written by

$$\begin{aligned} & \frac{1}{1-G_m(s)\Delta(s)} \\ & = \frac{G(s)}{G_m(s)} \\ & = \frac{k \prod_{i=1}^{n-\beta-q} (s+\mu_i) \prod_{i=n-\beta-q+1}^{n-\beta} (s-\mu_i)}{\prod_{i=1}^{p} (s-\gamma_i) \prod_{i=p+1}^{n} (s+\gamma_i)} \end{aligned}$$

$$\frac{\prod_{i=1}^{p_m}(s-\gamma_{mi})\prod_{i=p_m+1}^{n_m}(s+\gamma_{mi})}{k_m\prod_{i=1}^{n_m-\beta-q}(s+\mu_{mi})\prod_{i=n_m-\beta-q+1}^{n_m-\beta}(s-\mu_{mi})} \tag{16}$$

From above equation and argument principle, the Nyquist plot of $1/(1-G_m(s)\Delta(s))$ *encircles the origin* $(0,0)$ $p-p_m$ *times in the counter-clockwise direction.*

In this way, we have thus completed proof of this theorem. ■

Theorem 3 is proven using the above lemma.

Proof: If the Nyquist plot of the characteristics polynomial of (1) for all $G(s)$ included in Ω encircles the origin $p-p_m$ times in the counter-clockwise direction, then (1) is robustly stable. The characteristics polynomial of (1) is given by

$$\begin{aligned} 1+G(s)C(s) &= 1+\frac{G_m(s)C(s)}{1-G_m(s)\Delta(s)} \\ &= (1+G_m(s)C(s))\frac{1}{1-G_m(s)\Delta(s)}\left(1-\frac{G_m(s)}{1+G_m(s)C(s)}\Delta(s)\right). \end{aligned} \tag{17}$$

From the assumption that $C(s)$ stabilize $G_m(s)$, the Nyquist plot of $1+G_m(s)C(s)$ encircles the origin p_m+p_c times in the counter-clockwise direction. Here p_c means the number of zeroes of the controller $C(s)$ in the closed right half plane. In addition, from Lemma 1, the Nyquist plot of $1/(1-G_m(s)\Delta(s))$ encircles the origin $p-p_m$ times. Therefore the necessary and sufficient condition that $C(s)$ is the robust stabilizer for Ω is equivalent to the condition that the Nyquist plot of

$$1-\frac{G_m(s)}{1+G_m(s)C(s)}\Delta(s)$$

does not encircle the origin any times.

The remaining problem is to prove the necessary and sufficient condition that the Nyquist plot of

$$1-\frac{G_m(s)}{1+G_m(s)C(s)}\Delta(s)$$

does not encircle the origin any times, is same to (13).

Sufficient part of the proof is as follows. Assume that

$$\left\|\frac{G_m(s)W(s)}{1+G_m(s)C(s)}\right\|_\infty \leq 1. \tag{18}$$

It is clear that the Nyquist plot of

$$1-\frac{G_m(s)}{1+G_m(s)C(s)}\Delta(s)$$

can encircle the origin no time for arbitraly $\Delta(s)$ satisfying (3).

Necessary part is to show if

$$\left\| \frac{G_m(s)W(s)}{1+G_m(s)C(s)} \right\|_\infty > 1$$

, then $\Delta(s) \in \Omega$ exists to let the Nyquist plot of

$$1 - \frac{G_m(s)}{1+G_m(s)C(s)}\Delta(s)$$

encircle the origin. If

$$\left\| \frac{G_m(s)W(s)}{1+G_m(s)C(s)} \right\|_\infty > 1$$

, then some ω exists satisfying

$$\frac{G_m(j\omega)W(j\omega)}{1+G_m(j\omega)C(j\omega)} = \epsilon \ \ (|\epsilon| > 1)$$

If

$$\Delta(j\omega) = \frac{W(j\omega)}{\epsilon}, \tag{19}$$

$\Delta(s)$ satisfies (3) on ω because of

$$\begin{aligned} |\Delta(j\omega)| &= \frac{|W(j\omega)|}{|\epsilon|} \\ &< |W(j\omega)|. \end{aligned} \tag{20}$$

We have

$$1 - \frac{G_m(j\omega)\Delta(j\omega)}{1+G_m(j\omega)C(j\omega)} = 0. \tag{21}$$

This means the Nyquist plot of (1) pass on the origin and (1) is unstable.

From above discussion, the proof of this theorem is shown. ■

6 Control system design

The problem to construct the controller to satisfy (13) is precisely H_∞ control problems. Therefore we can obtain the controller to hold (13) by using the method the references (Glover and Doyle, 1988; Doyle, Glover, Khargonekar and Francis, 1988) or LMI base H_∞ control design method (Iwasaki and Skelton, 1994; Gahinet and Apkarian, 1994).

7 Conclusion

In this paper, we considered the robust stability for single input and single output continuous time invariant systems with feedback connected type of uncertainty. The necessary and sufficient robust stability condition for the system having uncertainty number of unstable poles was shown. This result is very important to give a design procedure for the system having a uncertain number of unstable poles.

Acknowledgements

The author would like to express his gratitude to Prof. K. Watanabe of Yamagata University, Prof. M. Ikeda of Osaka University, Associate Prof. Y. Ohta of Osaka University and Prof. Y. Nasu of Yamagata University for their helpful discussions.

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