Proceedings of the 7th Mediterranean Conference on Control and Automation (MED99) Haifa, Israel - June 28-30, 1999

H^∞ Design of Generalized Sampling and Hold Functions with Waveform Constraints

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Abstract

This paper deals with the sampled-data H^∞ control problem where both the discrete-time part of the controller and the A/D (sampler) and D/A (hold) converters are design parameters. It is known that the optimal sampler and hold that solve this problem have continuous (exponential) waveforms and thus are not readily implementable on digital hardware. In this respect, in this paper the problem is treated subject to *waveform constraints* on hold and sampling functions. In partricular, the generalized hold is constrained to be piecewise-constant and the generalized sampler is constrained to have piecewise impulse waveform.

The paper presents complete solution to this problem. A separation between the design of the sampler and the hold is established. Moreover, some interesting interpretations of the resulting sampled-data controller are discussed. In particular, it is shown that the (sub) optimal hold attempts to "reconstruct" the H^∞ state-feedback control law of the single-rate sampled-data control system with faster sampling period.

1 Introduction

Digital controllers connected to continuous-time plants via A/D (sampler) and D/A (hold) converters are widely used in industry owing to their lower price, enhanced reliability and flexibility in comparison with their analog counterparts.

The conventional design of sampled-data control systems uses either a pure discrete-time or a pure continuous-time design approach (Åström and Wittenmark, 1989), due to the difficulties in dealing with the continuous-time behavior of sampled-data systems. The first approach is based on the discretization of both the plant and control goals and the design of the controller is carried out in the discrete-time domain, while in the second one a continuous-time controller is first designed, then discretized and digitally implemented. Both approaches are strongly based on approximations that may be justified only if the sampling rate is "sufficiently fast." This assumption, however, might not be satisfied in practice.

The modern approach to the design of digital controllers for continuous-time plants takes explicitly into consideration the inter-sampling system behavior, by performing the design directly in continuous-time. Such a design, however, is more difficult then the pure continuous- or pure discrete-time design because: *i)* the over-all control system is time-varying even when both the

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plant and the digital controller are time-invariant, and *ii)* the designer has to deal simultaneously with continuous- and discrete-time signals (and, in some cases, also with requirements). Various mathematical tools (like "systems with jumps" and "lifting") were developed and used in order to overcome these difficulties and to solve the H^2 and the H^∞ sampled-data control problems under different assumptions (see (Chen and Francis, 1995) and the references therein). Owing to the more accurate design, the fast sampling requirement is relaxed. Yet, in most of these works, only the digital controller is designed, while the sampler and hold devices are taken to be the "ideal" sampler and the zero-order hold respectively, without taking into consideration the plant dynamics and the control goals. This assumption limits the control system performance when the sampling rate is not "fast" enough with respect to the plant dynamics.

Early attempts to incorporate the sampler and, particulary, the hold in the design process (Chammas and Leondes, 1978; Kabamba, 1987) took into consideration only the discrete-time closed-loop performance. As a result, significant improvements of the discrete-time performance were usually achieved at the expense of a poor inter-sampling behavior and undesirable robustness properties (Feuer and Goodwin, 1994). The design of sampling and/or hold functions on the basis of continuous-time H^2 and H^∞ (sub)optimal performance was treated in (Juan and Kabamba, 1991; Tadmor, 1992; Sun *et al.*, 1993; Mirkin *et al.*, 1997a). On one hand, analysis and simulations of those results show that the "sufficiently fast sampling" assumption can be further relaxed by using the generalized A/D and D/A converters instead of the zero-order hold and the ideal sampler. On the other hand, these generalized sampling and hold functions are not easily implementable, mainly because they all have continuous waveforms.

A possible way to implement the generalized sampling and hold functions designed in (Tadmor, 1992; Mirkin *et al.*, 1997a) is to use custom built analog hardware. Such an approach, however, may diminish the advantages of using a sampled-data controller instead of an analog one. Hence, programmable digital hardware is likely to be more adequate for the realization of those functions. The simplest way to implement them, using existing A/D and D/A converters that can *digitally* modulate the controller and the plant outputs, is to *discretize* the continuous waveform of the generalized hold and sampling functions. That is, to approximate them by piecewise constant and piecewise impulse functions, respectively. This method however may inherit the pitfalls of the approximation based sampled-data control design.

The H^2 optimal design of the hold and the sampling functions with piecewise constant and piecewise impulse waveforms, respectively, was considered by Kahane *et al.* (1999b). Simulations have shown that the resulting A/D and D/A converters have control capabilities comparable with those based on the generalized unrestricted sampling and hold functions, yet they are easy to implement.

Encouraged both by the implementability advantages and by the control capabilities of the H^2 optimal A/D and D/A converters developed in (Kahane *et al.*, 1999b), in this paper, we incorporate waveform constraints into the H^∞ design process of the generalized sampling and hold functions. This paper presents the solution to the H^∞ optimization problem where the sampling and hold functions are *free* design parameters yet *restricted* to have piecewise constant and piecewise impulse waveforms, respectively. Like in (Kahane *et al.*, 1999b), the solution is obtained by transforming the control problem into a pure discrete LTI one using the continuous- and the discrete-time lifting techniques (Chen and Francis, 1995). The computations are carried out using advanced continuous-time lifting techniques (Mirkin and Palmor, 1999) and some recently developed mathematical tools for discrete-time lifting (Kahane *et al.*, 1999a). Consequently, the resulting formulae are explicit (i.e., expressed in terms of the original plant parameters) and provide a meaningful interpretation to the solution.

This paper is organized as follows. Section 2 assembles the mathematical background needed in this paper: Subsection 2.1 briefly reviews the discrete-time lifting technique, while Subsection 2.2 collects some facts concerning a more efficient way of dealing with the parameters of the discrete-lifted plants, based upon dynamical systems operating over a finite time interval. The H^∞ optimization problem of the sampled-data setup based on piecewise impulse/constant sampling and hold functions is formulated in Section 3. In Section 4 this problem is reduced to a standard H^∞ optimization problem in the lifted domain, while in Section 5 the lifted solution is presented and some of its interesting properties are discussed. The lifted solution is "peeled-off" back to the time domain in Section 6, which presents the complete solution and various properties and interpretations of the H^∞ suboptimal piecewise constant hold and piecewise impulse sampler. Some concluding remarks are presented in Section 7.

1.1 Notation

The notation throughout the paper is fairly standard. M' means the transpose of a matrix M, O^* — the adjoint of a Hilbert space operator O, and $O^{1/2}$ — the square root of $O = O^* \geq 0$. As usual, $\mathbb{D}$ denotes the open unit disc. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $L^2_n[0,h]$ denotes the Hilbert space of square integrable $\mathbb{R}^n$-valued functions on the interval $[0,h]$ and $l^2_n[0,\nu-1]$ denotes the Hilbert space of $\mathbb{R}^n$ valued sequences defined over a finite time interval $[0,\nu-1]$. When the dimensions are irrelevant or clear from the context we will write $\mathbb{R}$, $L^2[0,h]$ and $l^2[0,\nu-1]$. The notation L^2 is the shortcut for $L^2[0,\infty]$.

A "bar" above a variable ($\bar{\zeta}$) denotes discrete-time signals in $\mathbb{R}^n$, while "vector" and "breve" ($\vec{\zeta}$ and $\breve{\zeta}$) — denote discrete-time signals in the discrete- and continuous-lifted domains, respectively. Also, we put forward the following operator notation which improves the readability of formulae when both finite and infinite dimensional input/output spaces are involved: a bar (or, in the discrete-lifted domain, a vector) indicates an operator $\bar{O}$ ($\vec{O}$) with both input and output spaces finite dimensional; grave accent — $\grave{O}$, when the input space is finite dimensional and the output infinite dimensional one; acute accent — $\acute{O}$, when the input space is infinite dimensional and the output finite dimensional one; and finally breve — $\breve{O}$, when both input and output spaces are infinite dimensional.

The compact block notation $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ denotes (matrix- or operator-valued) transfer functions either in s or in z domain in terms of their state-space realization. To distinguish linear time-invariant (LTI) systems in the time domain from their corresponding transfer functions, the former are denoted by script capital letters, so $G(s)$ and $\bar{G}(z)$ imply the transfer functions of the LTI systems $\mathcal{G}$ and $\bar{\mathcal{G}}$, respectively. Finally, $\mathrm{Ric}_{\mathbb{D}}$ denotes the discrete-time Riccati function (for definition and properties see (Mirkin *et al.*, 1997b)) defined over a subset called $\mathrm{dom}(\mathrm{Ric}_{\mathbb{D}})$ and which has a one-to-one correspondence with the stabilizing solution of the discrete-time algebraic Riccati equations (DARE).

2 Preliminaries

This section containes some preliminary results, which are required in the sequel. Subsection 2.1 briefly reviews the discrete-time lifting technique and underlines the difficulties arising from its application to the analysis and design of control systems having a multi-rate nature. Subsection 2.2 presents a brief exposition of the results of Kahane *et al.* (1999a) concerning *i)* the representation of the parameters of discrete-lifted plants as discrete-time dynamical systems operating over a finite time interval, and *ii)* some important conections between the discrete-time lifted domain and the discrete time domain.

2.1 The discrete-time lifting technique

The notion of discrete-time lifting consists on establishing a one-to-one correspondence between a discrete-time periodically shift-varying system and a shift-invariant one (but with higher input and output dimensions). Thus it enables the use of the well-established LTI tools for the analysis and design of periodically shift-varying systems, like those arising in many multi-rate sampled-data control problems.

Define the discrete-time lifting operator $\bar{\mathcal{W}}_\nu$ (Chen and Francis, 1995), which transforms the $\mathbb{R}^n$ valued sequences to the $\mathbb{R}^{n\nu}$ valued ones, as follows:

$$\vec{\xi} = \bar{\mathcal{W}}_\nu \bar{\xi} \iff \vec{\xi}[k] = \begin{bmatrix} \bar{\xi}[\nu k] \\ \bar{\xi}[\nu k+1] \\ \vdots \\ \bar{\xi}[\nu k+\nu-1] \end{bmatrix}.$$

The usefulness of this operator follows from the fact that for a ν-periodic system $\bar{\mathcal{G}}$ its lifting $\vec{\mathcal{G}} \doteq \bar{\mathcal{W}}_\nu \bar{\mathcal{G}} \bar{\mathcal{W}}_\nu^{-1}$ is shift-invariant. Also, since $\bar{\mathcal{W}}_\nu$ is an isomorphism, the stability properties are preserved under lifting, and since the restriction of $\bar{\mathcal{W}}_\nu$ to ℓ^p is an isometry, induced norms of the original system are equivalent to norms of the lifted one.

Lifting however, increases the input and output dimensions. For example, let $\bar{\mathcal{G}}$ be a discrete-time LTI system with the following transfer matrix:

$$\bar{G}(z) \doteq \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$. Its lifting, $\vec{\mathcal{G}}$, is also shift-invariant (Chen and Francis, 1995), and:

$$\vec{G}(z) \doteq \left[\begin{array}{c|c} \vec{A} & \vec{B} \\ \hline \vec{C} & \vec{D} \end{array}\right] = \left[\begin{array}{c|cccc} A^\nu & A^{\nu-1}B & A^{\nu-2}B & \dots & B \\ \hline C & D & 0 & \dots & 0 \\ CA & CB & D & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{\nu-1} & CA^{\nu-2}B & CA^{\nu-3}B & \dots & D \end{array}\right]. \tag{2}$$

In principle, (2) describes a standard discrete-time system. Hence, dealing with $\vec{\mathcal{G}}$ is conceptually not more complicated than with $\bar{\mathcal{G}}$. Moreover, the state dimensions in (1) and (2) are equal. Yet the input and output dimensions of $\vec{\mathcal{G}}$ increase by the factor ν with respect to those of $\bar{\mathcal{G}}$. That, in turn, "blows up" the matrices $\vec{B}$, $\vec{C}$, $\vec{D}$. Consequently, *numerical* difficulties associated with lifted solutions increase rapidly as ν grows. This fact reduces the effectiveness of lifting, especially for large lifting frames ν. Moreover, when $\bar{\mathcal{G}}$ is not finite dimensional, but rather the result of a continuous-time lifting, the direct treatment of $\vec{B}$, $\vec{C}$, and $\vec{D}$ as block-matrices does not appear to be helpful.

2.2 A representation of the lifted parameters using dynamical systems

As follows from the discussion at the end of the previous section, treating the lifted parameters as block-matrices limits the efficiency of the discrete-time lifting.

Similar problems arise when the so-called continuous-time lifting (Bamieh *et al.*, 1991; Chen and Francis, 1995) is applied. In that case the parameters of the lifted plants become *infinite-dimensional* operators. Thus, a direct manipulation over those parameters is considerably more difficult than over the parameters in (2). It was shown in (Mirkin and Palmor, 1999), however, that those difficulties can be overcome by representing the parameters of continuous lifted plants as continuous-time dynamical systems with two-point boundary conditions (STPBC) of the form

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \quad t \in [0,\ h],\\
y(t) &= Cx(t) + Du(t),\\
\Omega x(0) &+ \Upsilon x(h) = 0\ ,
\end{aligned}$$

and by replacing the manipulations over the infinite-dimensional operators with operations over STPBC. By a subsequent extension of the results of Gohberg and Kaashoek (1984), the latter manipulations can easily be performed in the state-space, in terms of the matrix parameters of the original plants.

This fact suggests that the manipulations over the parameters of $\vec{\mathcal{G}}$ in (2) can also be simplified by treating them not as unstructured matrices, but rather as a representation of discrete dynamical systems.

A straightforward extension of the results of Mirkin and Palmor (1999) to the case of the discrete-time lifting can be constructed using three components: the class of *discrete-time* STPBC, also defined by Gohberg and Kaashoek (1984), the discrete impulse operator $\bar{\mathcal{I}}_\theta$, and the discrete-time sampling operator $\bar{\mathcal{I}}_\theta^*$.

i) The discrete-time STPBC are linear operators $\vec{\mathcal{O}} : l^2[0, \nu-1] \mapsto l^2[0, \nu-1]$, described by the state equations

$$\vec{\mathcal{O}} : \begin{cases} x[k+1] = Ax[k] + Bu[k], \quad k = 0, \dots, \nu-1, \\ y[k] = Cx[k] + Du[k], \\ \Omega x[0] + \Upsilon x[\nu] = 0, \end{cases}$$

where the square matrices Ω and Υ shape the boundary condition of the state vector x. The boundary conditions are said to be well-posed if $\det(\Omega + A^\nu \Upsilon) \neq 0$ and, in this case, the map $y = \vec{\mathcal{O}} u$ is well defined $\forall u \in l^2[0, \nu-1]$.

For the purpose of this paper only the particular case of the causal discrete-time STPBC (where $\Omega = I$ and $\Upsilon = 0$) is required. We denote these systems using the compact block notation

$$\vec{\mathcal{O}} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_0^{\nu-1},$$

and present here only their solution:

$$y[k] = C \sum_{j=0}^{k-1} A^{k-j-1} Bu[j] + Du[k]\ , \quad k = 0, \dots, \nu-1.$$

Note that a causal discrete-time STPBC is always well-posed.

ii) The discrete impulse operator $\bar{\mathcal{I}}_\theta : \mathbb{R}^n \mapsto l^2[0, \nu-1]$, $\theta = 0, \dots, \nu-1$ is defined in the following manner:

$$\zeta = \bar{\mathcal{I}}_\theta \eta \iff \zeta[k] = \begin{cases} \eta, & \text{if } k = \theta \\ 0, & \text{else.} \end{cases}$$

iii) The adjoint $\bar{\mathcal{J}}_\theta^* : l^2[0, \nu-1] \mapsto \mathbb{R}^n$ of the Hilbert space operator $\bar{\mathcal{J}}_\theta$, is given by:

$$\eta = \bar{\mathcal{J}}_\theta^* \zeta[k] \iff \eta = \zeta[\theta],$$

and, in fact, it is the discrete-time sampling operator.

These operators, together with the causal discrete-time STPBC allows one to introduce the following simple representation for the parameters of the discrete-lifted plant $\vec{\mathcal{G}}$ in (2):

$$\begin{bmatrix} \vec{A} & \vec{B} \\ \vec{C} & \vec{D} \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{J}}_{\nu-1}^* & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{c|cc} A & A & B \\ \hline A & 0 & B \\ C & C & D \end{array}\right]_0^{\nu-1} \begin{bmatrix} \bar{\mathcal{J}}_0 & 0 \\ 0 & I \end{bmatrix}. \tag{3}$$

By using this representation, the involved manipulations over the high dimensional matrix parameters of the lifted plant are replaced by operations over STPBC. The latter manipulations can be performed in the state space, in terms of the low-dimensional parameters of the original plant. Thus, the computational efficiency of the discrete-time lifting technique is considerably improved, and the structures of the original problems are preserved.

Unfortunately, not all the operations over the discrete-lifted parameters can be performed using this STPBC-based representation. The reason is that unlike the class of the continuous-time STPBC, the class of the discrete-time STPBC is not closed under the adjoint (when $\det(A) = 0$) and inverse (when $\det(D) = 0$) operations. At the same time, singular A and D appear frequently in the sampled-data control problems (Chen and Francis, 1995).

To overcome this difficulty, Kahane *et al.* (1999a) developed a representation for the discrete-lifted parameters, based on a broader class of discrete-time dynamical systems operating over a finite time interval. To this end they defined the class of the *discrete-time implicit descriptor systems with two-point boundary conditions* (DIDS), that is, systems of the form

$$\begin{aligned} &Ex[k+1] = Fx[k] + Gy[k] + Hu[k], k = 0, \ldots, \nu - 1, \\ &\Omega x[0] + \Upsilon x[\nu] = 0. \end{aligned}$$

The theory of DIDS was developed in (Kahane *et al.*, 1999a), and it was shown that *i)* this class of systems is closed under all the required operations, and *ii)* the manipulations over DIDS can also be performed in the state space using computations over the low-dimensional parameters of the original plant only. Consequently, the DIDS-based representation preserves the advantages of that based on STPBC and, in addition, covers all possible singularities in the description of $\bar{\mathcal{G}}$.

By using the DIDS-based representation for the discrete-lifted parameters, Kahane *et al.* (1999a) were able to establish some important connections between the discrete-time lifted domain and the discrete time domain. One of those, concerning the relations between the stabilizing solution to the DARE associated with the discrete-lifted plant $\vec{\mathcal{G}}$ and that to the DARE associated with the original plant $\bar{\mathcal{G}}$, is presented in the sequel.

Let J be an appropriately dimensioned square matrix and associate with $\bar{\mathcal{G}}$, (1), the equation

$$A'XA - X + C'JC + (A'XB + C'JD)F = 0, \tag{4}$$

where F is the gain matrix associated with (4):

$$F \doteq -(B'XB + D'JD)^{-1}(B'XA + D'JC).$$

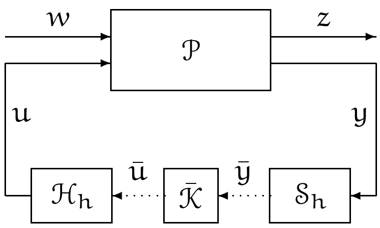

Figure 1: General sampled-data setup in time domain.

Equation (4) is the well known DARE, and finding its stabilizing solution is a crucial step in solving various discrete-time control problems, such as H^2 ($J = I$) and H^∞ ($J = \begin{bmatrix} I & \\ & -I \end{bmatrix}$) optimizations (Chen and Francis, 1995; Zhou *et al.*, 1995).

Similarly, associate with $\vec{\mathcal{G}}$ the equation:

$$\vec{A}^*X\vec{A} - X + \vec{C}^*J\vec{C} + \left(\vec{A}^*X\vec{B} + \vec{C}^*J\vec{D}\right)\vec{F} = 0, \tag{5}$$

where

$$\vec{F} \doteq -\left(\vec{B}^*X\vec{B} + \vec{D}^*J\vec{D}\right)^{-1}\left(\vec{D}^*J\vec{C} + \vec{B}^*X\vec{A}\right).$$

Equation (5) is also a DARE, which arises in many multi-rate sampled-data optimization problems, see, e.g., (Chen and Qiu, 1994).

The next Lemma is important for the reasoning to follow.

Lemma 1 (Kahane *et al.* (1999a)). *A matrix* $X = X'$ *is the stabilizing solution to DARE* (4) *if and only if it is the stabilizing solution to DARE* (5). *Moreover, if* X *is the stabilizing solution to those DARE's, then* $\vec{A} + \vec{B}\vec{F} = (A + BF)^\nu$ *and*

$$\vec{F} = \begin{bmatrix} F \\ F(A+BF) \\ \vdots \\ F(A+BF)^{\nu-1} \end{bmatrix}.$$

3 Problem formulation

The purpose of this section is to formulate the sampled-data H^∞ control problem where the sampling and hold functions are design parameters restricted to have piecewise constant and piecewise impulse waveforms, respectively. This problem will be defined in terms of the feedback setup illustrated in Fig. 1, where $\mathcal{P}$ is a continuous-time generalized plant and w, z, y and u are the continuous-time exogenous input, the regulated output, the measured output and the control signal, respectively. The sampled-data controller consists of three devices: a digital controller $\bar{\mathcal{K}}$, a sampler $\mathcal{S}_h$ and a hold $\mathcal{H}_h$. The three devices are assumed to be synchronized and with a given sampling period h. The generalized plant $\mathcal{P}$,

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{1\bullet} \\ \mathcal{P}_{2\bullet} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{\bullet 1} & \mathcal{P}_{\bullet 2} \end{bmatrix}$$

is assumed to be LTI, with the following state-space representation:

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & 0 & 0 \end{array}\right]. \tag{6}$$

The matrix D_{11} is taken to be zero in order to simplify the derivations and to obtain more transparent results. The conditions $D_{21} = 0$ and $D_{22} = 0$ ensure that the sampler $\mathcal{S}_h$ operates over proper signals (i.e., the measured output is pre-filtered, if necessary, by an anti-aliasing filter before sampling).

As is done commonly in the literature (see Kabamba (1987); Araki (1993) for instance) the generalized (zero-order) hold and the sampler are assumed to act on the output of the digital controller $\bar{u}[k]$ and on the measured output $y(t)$ respectively, to generate:

$$\big(\mathcal{H}_h\bar{u}\big)(kh+\tau) = \phi_H(\tau)\bar{u}[k], \quad \forall\tau \in [0,h) \tag{7a}$$

and

$$\big(\mathcal{S}_h y\big)[k] = \int_0^{h^-} \phi_S(\tau)y(kh^- - \tau)\,d\tau \tag{7b}$$

for some generalized hold and sampling functions ϕ_H and ϕ_S defined on the interval $[0,h)$. During the inter-sample, the hold function ϕ_H shapes the form of the control signal, while the sampling function ϕ_S is used to weight the plant output.

Motivated by the technological requirements, we constrain in this paper the hold and the sampling functions to have the following piecewise constant and piecewise impulse waveforms, respectively:

$$\begin{aligned}
\phi_H(\tau) &= \phi_H[i], \quad \forall\tau \in [h\tfrac{i}{\nu}, h\tfrac{i+1}{\nu}),\ i = 0,\dots,\nu-1\\
\phi_S(\tau) &= \sum_{j=0}^{\mu-1} \phi_S[j]\ \delta(\tau - \tfrac{\mu-1-j}{\mu}h^-),
\end{aligned}$$

where ν and μ are any two natural numbers called, in the sequel, the constrained divisions. Hence, the sampler and the hold assumed throughout this paper act in the following manner:

$$\big(\mathcal{H}_h\bar{u}\big)(kh+\tau) = \phi_H[i]\,\bar{u}[k], \quad \forall\tau \in [h\tfrac{i}{\nu}, h\tfrac{i+1}{\nu}),\ i = 0,\dots(\nu-1) \tag{8a}$$

and

$$\big(\mathcal{S}_h y\big)[k] = \sum_{j=0}^{\mu-1} \phi_S[j]\ y\big((k - \tfrac{\mu-1-j}{\mu})h^-\big). \tag{8b}$$

The operation of the A/D and the D/A converters (8) is as follows. The measured output $y(t)$ is ideally sampled μ times during one sampling period h by the piecewise impulse sampler $\mathcal{S}_h$. The samples are weighted by the gain function $\phi_S[j]$ and then summed to generate $\bar{y}[k]$ — the input to the digital controller at the time instance kh. Since the weighting and summation operations can not be processed instantaneously, the $\bar{y}$ fed into the digital controller at the time instance kh contains information about the measured output prior to kh. The output from the digital controller at the time instance kh is shaped by the gain function $\phi_H[i]$ of the piecewise constant hold $\mathcal{H}_h$ in order to generate the control signal $u(t)$, which changes its value ν times during one sampling period h in a piecewise constant manner. Note that the A/D and D/A converters (8) *digitally* modulate the controller and the plant outputs, hence they are readily implementable by means of digital hardware.

The purpose of this paper is to solve the following H^∞ optimization problem:

OP_{H^∞}: Given the generalized plant $\mathcal{P}$, (6), the sampling period h and the constrained divisions ν and μ, find (if they exist) the discrete part of the controller $\bar{\mathcal{K}}$ and the hold and sampler gain functions $\phi_H[i]$ and $\phi_S[j]$, such that the sampled-data system in Fig. 1 is internally stable and the H^∞ norm of the closed-loop operator from w to z is less than a given $\gamma > 0$.

It is worthwhile noting that under an arbitrary choice of the gain functions $\phi_S[j]$ and $\phi_H[i]$ and any LTI $\bar{\mathcal{K}}$, the system in Fig. 1 is h-periodic in continuous-time. With a slight abuse of notation, we use the term H^∞ norm to denote the L^2 induced norm of the periodic operator from w to z (see Bamieh and Pearson (1992) for a discussion on the extension of the H^∞ system norm notion to periodic systems).

In the sequel, the $\bar{\mathcal{K}}$, $\mathcal{S}_h$ and $\mathcal{H}_h$ solving the OP_{H^∞} are referred to as H^∞ suboptimal digital controller, sampler and hold, respectively. The H^∞ sub-optimality of these devices is understood as the ability to design $\bar{\mathcal{K}}$, $\mathcal{S}_h$ and $\mathcal{H}_h$ so that the overall sampled-data controller $\mathcal{K}_{sd} = \mathcal{H}_h\bar{\mathcal{K}}\mathcal{S}_h$ is γ-suboptimal.

The treatment of the OP_{H^∞} is complicated by the inherent periodicity of the sampled-data system and by its hybrid continuous/discrete nature. To overcome those difficulties, a similar approach to the one used by Kahane *et al.* (1999b), will be followed here. In the next section, the OP_{H^∞} will be lifted by applying the continuous- and discrete-time lifting techniques. Then, it will be reformulated in the lifted domain, where the problem reduces to a rather standard, pure discrete and time-invariant H^∞ optimization. The solution in the lifted domain to this equivalent optimization problem will be presented in Section 5, and the results will be "peeled-off" back to the time domain in Section 6.

4 The lifted problem

The purpose of this section is to reduce the OP_{H^∞} to a standard optimization problem in the lifted domain. Toward this end, observe that

$$\mathcal{H}_h = \mathcal{H}^{\text{ZOH}}_{h_u}\bar{\mathcal{W}}_\nu^{-1}\Phi_H, \quad \Phi_H \doteq \begin{bmatrix} \phi_H[0] \\ \vdots \\ \phi_H[\nu-1] \end{bmatrix}, \quad h_u \doteq h/\nu,$$

and

$$\mathcal{S}_h = \mathcal{U}_h\Phi_S\bar{\mathcal{W}}_\mu\mathcal{S}^{\text{IS}}_{h_y}, \quad \Phi_S \doteq \begin{bmatrix} \phi_S[0] & \dots & \phi_S[\mu-1] \end{bmatrix}, \quad h_y \doteq h/\mu,$$

where

$\mathcal{U}_h$ is the unit time delay operator that provides a time delay of h,

$\mathcal{H}^{\text{ZOH}}_{h_u}$ is the zero order hold: $\big(\mathcal{H}^{\text{ZOH}}_{h_u}\,\bar{u}\big)(ih_u + \tau) = \bar{u}[i]$, $\forall\tau \in [ih_u,\ (i+1)h_u)$,

$\mathcal{S}^{\text{IS}}_{h_y}$ is the ideal predictive sampler: $\big(\mathcal{S}^{\text{IS}}_{h_y}\,y\big)[j] = y\big((j+1)h_y^-\big)$.

Using these relations and the continuous- and discrete-time lifting operations, the sampled-data control setup in Fig. 1 is converted to the equivalent one shown in Fig. 2. The usefulness of this conversion follows from the fact that, after lifting, all the subsystems in Fig. 2 are discrete-time LTI systems. Moreover, all the given information is contained in the lifted plant $\breve{\mathcal{P}}$,

$$\breve{\mathcal{P}} \doteq \begin{bmatrix} \mathcal{W}_h & \\ & \bar{\mathcal{W}}_\mu\mathcal{S}^{\text{IS}}_{h_y} \end{bmatrix}\mathcal{P}\begin{bmatrix} \mathcal{W}_h^{-1} & \\ & \mathcal{H}^{\text{ZOH}}_{h_u}\bar{\mathcal{W}}_\nu^{-1} \end{bmatrix} = \begin{bmatrix} \breve{\mathcal{P}}_{11} & \grave{\mathcal{P}}_{12} \\ \acute{\mathcal{P}}_{21} & \bar{\mathcal{P}}_{22} \end{bmatrix} = \begin{bmatrix} \breve{\mathcal{P}}_{\bullet 1} & \grave{\mathcal{P}}_{\bullet 2} \end{bmatrix} = \begin{bmatrix} \breve{\mathcal{P}}_{1\bullet} \\ \acute{\mathcal{P}}_{2\bullet} \end{bmatrix},$$

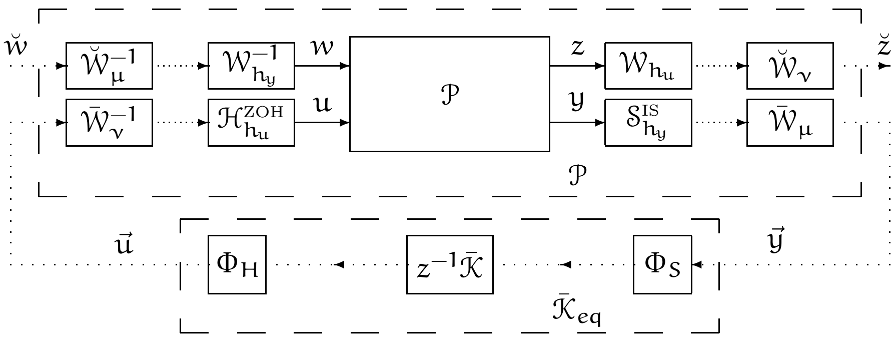

Figure 2: The sampled-data setup in the lifted domain.

while all the design parameters are "absorbed" into the controller $\bar{\mathcal{K}}_{eq}$,

$$\bar{K}_{eq}(z) \doteq z^{-1}\Phi_H \bar{K}(z)\Phi_S \doteq \left[\begin{array}{c|c} A_{k,eq} & B_{k,eq} \\ \hline C_{k,eq} & 0 \end{array}\right]. \tag{9}$$

The backward shift operator z^{-1} was absorbed into the controller in order to preserve the state dimension of $\breve{\mathcal{P}}$.

In the sequel, the LTI plant $\breve{\mathcal{P}}$ is assumed to have the following state-space realization:

$$\breve{P}(z) = \left[\begin{array}{c|cc} \bar{A} & \acute{B}_1 & \bar{B}_2 \\ \hline \grave{C}_1 & \breve{D}_{11} & \grave{D}_{12} \\ \bar{C}_2 & \acute{D}_{21} & \bar{D}_{22} \end{array}\right] = \left[\begin{array}{c|c} \bar{A} & \acute{B} \\ \hline \grave{C}_1 & \grave{D}_{1\bullet} \\ \bar{C}_2 & \acute{D}_{2\bullet} \end{array}\right] = \left[\begin{array}{c|cc} \bar{A} & \acute{B}_1 & \bar{B}_2 \\ \hline \grave{C} & \breve{D}_{\bullet 1} & \grave{D}_{\bullet 2} \end{array}\right]. \tag{10}$$

The OP_{H^∞} reduces then to the following equivalent problem:

$\mathsf{OP}_{H^\infty}^{eq}$: Given the LTI discrete-time generalized plant $\breve{\mathcal{P}}$, (10), find (if such exists) an LTI strictly proper controller $\bar{\mathcal{K}}_{eq}$ that internally stabilizes the plant $\breve{\mathcal{P}}$ and for which the H^∞ norm of the closed-loop operator from $\breve{w}$ to $\breve{z}$ is less than a given $\gamma > 0$.

Note that, disregarding the fact that the lifted plant $\breve{\mathcal{P}}$ has infinite input/output dimensions, the $\mathsf{OP}_{H^\infty}^{eq}$ is a standard discrete-time, H^∞ optimization problem with a strictly proper controller.

Also note that, having the γ-suboptimal solution in the lifted domain, the H^∞ suboptimal digital controller $\bar{\mathcal{K}}$, as well as the H^∞ suboptimal hold and sampler gain matrices Φ_H and Φ_S, can be found by "peeling-off" $\bar{\mathcal{K}}_{eq}$ in the following manner. Find any two matrices Φ_H and C_k such that $C_{k,eq} = \Phi_H C_k$ and find any two matrices Φ_S and B_k such that $B_{k,eq} = B_k\Phi_S$. Then, using (9), the optimal controller $\bar{\mathcal{K}}$ is as follows:

$$\bar{K}(z) = z\left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & 0 \end{array}\right], \quad A_k = A_{k,eq}.$$

5 Solution in the lifted domain

The goal of this section is to solve the $\mathsf{OP}_{H^\infty}^{eq}$. As already explained, this is a rather standard discrete-time LTI H^∞ optimization problem, with the additional constraint that the controller must be strictly proper. This kind of problems was discussed in detail for the case of finite dimensional pure discrete plants in (Mirkin, 1997) and for the case of continuous-lifted plants in (Mirkin *et al.*, 1997a).

Start by imposing the following assumptions on the generalized plant (10):

(A1): The pair $(\bar{A}, \bar{B}_2)$ is stabilizable;

(A2): The pair $(\bar{A}, \bar{C}_2)$ is detectable;

(A3): The operator $\begin{bmatrix} \bar{A} - e^{j\theta}I & \bar{B}_2 \\ \grave{C}_1 & \grave{D}_{12} \end{bmatrix}$ is left invertible $\forall\theta \in [0, 2\pi)$;

(A4): The operator $\begin{bmatrix} \bar{A} - e^{j\theta}I & \acute{B}_1 \\ \bar{C}_2 & \acute{D}_{21} \end{bmatrix}$ is right invertible $\forall\theta \in [0, 2\pi)$.

These assumptions are the counterparts of the standard assumptions imposed on a discrete-time generalized plant in order to guarantee input-output stabilizability and non-singularity of the H^∞ optimization problem.

The solution to the $\mathsf{OP}^{eq}_{H^\infty}$ requires the following two H^∞ DARE's:

$$\bar{X} = \bar{A}'\bar{X}\bar{A} + \grave{C}_1^*\grave{C}_1 - (\acute{B}^*\bar{X}\bar{A} + \breve{D}_{1\bullet}^*\grave{C}_1)^*(\breve{D}_{1\bullet}^*\breve{D}_{1\bullet} - \gamma^2 E_{11} + \acute{B}^*\bar{X}\acute{B})^{-1}(\acute{B}^*\bar{X}\bar{A} + \breve{D}_{1\bullet}^*\grave{C}_1) \quad \text{(11a)}$$

$$\bar{Y} = \bar{A}\bar{Y}\bar{A}' + \acute{B}_1\acute{B}_1^* - (\bar{A}\bar{Y}\grave{C}^* + \acute{B}_1\breve{D}_{\bullet 1}^*)(\breve{D}_{\bullet 1}\breve{D}_{\bullet 1}^* - \gamma^2 E_{11} + \grave{C}\bar{Y}\grave{C}^*)^{-1}(\bar{A}\bar{Y}\grave{C}^* + \acute{B}_1\breve{D}_{\bullet 1}^*)^*, \quad \text{(11b)}$$

where $E_{11} = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

Using (11), necessary and sufficient conditions for the existence of a solution to the $\mathsf{OP}^{eq}_{H^\infty}$, as well as a particular solution can be established:

Theorem 1. *Given plant* (10) *such that the assumptions (A1)–(A4) are satisfied, the following statements are equivalent:*

i) There exists a controller $\bar{\mathcal{K}}_{eq}$ which solves $\mathsf{OP}^{eq}_{H^\infty}$.

ii) The DARE's (11) *have stabilizing solutions $\bar{X} \geq 0$ and $\bar{Y} \geq 0$ such that*

$$\left\| \begin{bmatrix} \bar{X}^{1/2} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A} & \acute{B}_1 \\ \grave{C}_1 & \breve{D}_{11} \end{bmatrix} \begin{bmatrix} \bar{Y}^{1/2} & 0 \\ 0 & I \end{bmatrix} \right\|_2 \leq \gamma. \quad \text{(12)}$$

Moreover, if the conditions of part i) hold, then the matrix $\bar{Z} \doteq (I - \gamma^{-2}\bar{Y}\bar{X})^{-1}$ is well defined and one controller that solves $\mathsf{OP}^{eq}_{H^\infty}$ is

$$\bar{K}_{eq}(z) = \left[\begin{array}{c|c} \bar{A} + \acute{B}\grave{F} + \bar{Z}\bar{L}_2(\bar{C}_2 + \acute{D}_{2\bullet}\grave{F}) & -\bar{Z}\bar{L}_2 \\ \hline \bar{F}_2 & 0 \end{array}\right], \quad \text{(13)}$$

where:

$$\grave{F} \doteq -(\breve{D}_{1\bullet}^*\breve{D}_{1\bullet} - \gamma^2 E_{11} + \acute{B}^*\bar{X}\acute{B})^{-1}(\acute{B}^*\bar{X}\bar{A} + \breve{D}_{1\bullet}^*\grave{C}_1) = \begin{bmatrix} \grave{F}_1 \\ \bar{F}_2 \end{bmatrix}, \quad \text{(14a)}$$

$$\acute{L} \doteq -(\bar{A}\bar{Y}\grave{C}^* + \acute{B}_1\breve{D}_{\bullet 1}^*)(\breve{D}_{\bullet 1}\breve{D}_{\bullet 1}^* - \gamma^2 E_{11} + \grave{C}\bar{Y}\grave{C}^*)^{-1} = \begin{bmatrix} \acute{L}_1 & \bar{L}_2 \end{bmatrix}. \quad \text{(14b)}$$

The solution presented in Theorem 1 is not readily implementable, since it is not clear how to verify assumptions (A1)–(A4) and how to compute the parameters of $\bar{\mathcal{K}}_{eq}$ in (13). Nevertheless, it reveals some interesting properties of the solution to OP_{H^∞}. In particular:

Remark 1. It is clear from (9) that the hold function Φ_H is completely absorbed into the 'C'-part of the sampled-data controller $\bar{\mathcal{K}}_{eq}$, while the sampling function Φ_S is contained in its 'B'-part. Although the 'B'-part of $\bar{\mathcal{K}}_{eq}$ containes the coupling term $\bar{Z}$, the latter can be absorbed into the discrete-time part of the controller. Consequently, by comparing (9) with (13) in Theorem 1, the H^∞ suboptimal hold and sampler are characterized by the operators $\bar{F}_2$ and $\bar{L}_2$, respectively. On the other hand, by inspecting (14) and (11), it is seen that $\bar{F}_2$ (hence, also the optimal generalized piecewise constant hold) depends only on the parameters of $\mathcal{P}_{1\bullet}$ — the subsystem from w and u to z. Similarly, $\bar{L}_2$ (hence, also the optimal generalized piecewise impulse sampler) depends only on the parameters of $\mathcal{P}_{\bullet 1}$ — the subsystem from w to z and y. Hence, there is a separation between the designs of the H^∞ suboptimal $\mathcal{H}_h$ and $\mathcal{S}_h$ in the sense that the hold design does not depend on the measurement $y(t)$, and the sampler design does not depend on the control action $u(t)$. However, both designs are affected by the subsystem from w to z hence the separation is not complete, unlike in the H^2 design case (Kahane *et al.*, 1999b). This is similar to the separation between the designs of the generalized sampler and hold in the unconstrained case (Mirkin *et al.*, 1997a) and in contrast to other works in the literature (Tadmor, 1992; Mirkin and Rotstein, 1997), where the H^∞ suboptimal unconstrained sampler depends on the hold.

Remark 2. Theorem 1 presents a sampled-data controller $\bar{\mathcal{K}}_{eq}$ which solves, in the lifted domain, the OP_{H^∞}. From the discussion at the end of the previous section it is clear, however, that the separation of $\bar{\mathcal{K}}_{eq}$ into $\bar{\mathcal{K}}$, Φ_H and Φ_S is not unique. One possible choice is:

$$\Phi_H = \bar{F}_2, \tag{15a}$$
$$\Phi_S = -\bar{L}_2, \tag{15b}$$

and then

$$\bar{K}(z) = z\left[\begin{array}{c|c} \bar{A} + \acute{B}\grave{F} + \bar{Z}\bar{L}_2(\bar{C}_2 + \acute{D}_{2\bullet}\grave{F}) & \bar{Z} \\ \hline I & 0 \end{array}\right]. \tag{15c}$$

From implementation point of view, however, other separations might be advantageous.

Remark 3. Note that the formal solution to $\mathsf{OP}^{eq}_{H^\infty}$ is exactly the same as the solution to the H^∞ optimization problem defined in the lifted domain by Mirkin *et al.* (1997a) for the case of the unconstrained generalized sampling and hold functions. This is due to the fact that in both cases the same idea was used: to reduce a periodically varying control problem to an equivalent one which is time invariant in the lifted domain. Yet the lifted domains in which those two problems were defined are different: for the H^∞ design problem of the unconstrained generalized sampling and hold functions only the continuous-time lifting was required, while for the OP_{H^∞} the discrete-time lifting had to be used, in addition, due to its muti-rate nature. Consequently, eventhough the lifted problems have the same formal solution, the "peeling-off" process of the solution to $\mathsf{OP}^{eq}_{H^\infty}$ will be different.

6 Main results

This section is devoted to peeling-off the lifted solution given in Theorem 1. This will result in the readily implementable solution to OP_{H^∞} in terms of the original plant parameters. A short discussion on the properties and the interpretations of the H^∞ suboptimal piecewise impulse sampling and piecewise constant hold functions will follow.

Let

$$\Sigma_H \doteq \exp\left(\begin{bmatrix} 0 & -D_{12}'C_1 & -B_2' & -D_{12}'D_{12} \\ 0 & A & \gamma^{-2}B_1B_1' & B_2 \\ 0 & -C_1'C_1 & -A' & -C_1'D_{12} \\ 0 & 0 & 0 & 0 \end{bmatrix} h_u\right) = \begin{bmatrix} I & \Sigma_{H12} & \Sigma_{H13} & \Sigma_{H14} \\ 0 & \Sigma_{H22} & \Sigma_{H23} & \Sigma_{H24} \\ 0 & \Sigma_{H32} & \Sigma_{H33} & \Sigma_{H34} \\ 0 & 0 & 0 & I \end{bmatrix}, \tag{16a}$$

$$\Sigma_S \doteq \exp\left(\begin{bmatrix} A & -B_1B_1' \\ \gamma^{-2}C_1'C_1 & -A' \end{bmatrix} h_y\right) = \begin{bmatrix} \Sigma_{S11} & \Sigma_{S12} \\ \Sigma_{S21} & \Sigma_{S22} \end{bmatrix}, \tag{16b}$$

and

$$\Pi \doteq \begin{bmatrix} \Sigma_{H22} & \Sigma_{H23} \\ \Sigma_{H32} & \Sigma_{H33} \end{bmatrix}^{\nu} = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}, \tag{16c}$$

where: $h_u \doteq h/\nu$ and $h_y \doteq h/\mu$. Define the following two matrix pairs:

$$(\Lambda_\nu, \Delta_\nu) \doteq \left(\begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \begin{bmatrix} \Sigma_{H12} & \Sigma_{H13} & \Sigma_{H14} \\ \Sigma_{H22} & \Sigma_{H23} & \Sigma_{H24} \\ \Sigma_{H32} & \Sigma_{H33} & \Sigma_{H34} \end{bmatrix}\right), \tag{17a}$$

$$(\Lambda_\mu, \Delta_\mu) \doteq \left(\begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \begin{bmatrix} 0 & C_2 & 0 \\ \Sigma_{S11}' & \Sigma_{S21}' & \Sigma_{S11}'C_2' \\ \Sigma_{S12}' & \Sigma_{S22}' & \Sigma_{S12}'C_2' \end{bmatrix}\right). \tag{17b}$$

The usefulness of these matrix pencils follows from the fact that, as shown in (Mirkin *et al.*, 1997b), the solutions to the H^∞ DARE's associated with the continuous-time lifted plants $\breve{\mathcal{P}}_{1\bullet H} \doteq \mathcal{W}_{h_u}\mathcal{P}_{1\bullet}\begin{bmatrix} \mathcal{W}_{h_u}^{-1} & \\ & \mathcal{H}_{h_u}^{\mathrm{ZOH}} \end{bmatrix}$ and $\breve{\mathcal{P}}_{\bullet 1S} \doteq \begin{bmatrix} \mathcal{W}_{h_y} & \\ & \mathcal{S}_{h_y}^{\mathrm{IS}} \end{bmatrix}\mathcal{P}_{\bullet 1}\mathcal{W}_{h_y}^{-1}$ can be computed directly from the deflating subspaces of the extended symplectic pencils $(\Lambda_\nu, \Delta_\nu)$ and $(\Lambda_\mu, \Delta_\mu)$, respectively.

The computation of the $L^2[0,h]$ induced norm of the subsystem from w to z is also required:

$$\gamma_0 \doteq \|\mathcal{P}_{11}\|_{L^2[0,h]}.$$

This quantity can be computed as described in (Chen and Francis, 1995), and it is the lower bound for the H^∞ performance in sampled-data systems under an arbitrary choice of $\mathcal{S}_h$ and $\mathcal{H}_h$. Hence it is natural to consider only the cases where $\gamma > \gamma_0$.

In the sequel we assume that:

(A1′): The pair $(\Sigma_{H22}, \Sigma_{H24})$ is stabilizable for any $\gamma > \gamma_0$;

(A2′): The pair (C_2, Σ_{S11}) is detectable for any $\gamma > \gamma_0$;

(A3′): The matrix $\begin{bmatrix} \Sigma_{H12} & \Sigma_{H14} \\ \Sigma_{H22} - e^{j\theta}I & \Sigma_{H24} \\ \Sigma_{H32} & \Sigma_{H34} \end{bmatrix}$ is left invertible $\forall\theta \in [0, 2\pi)$ and any $\gamma > \gamma_0$;

(A4′): The matrix $\begin{bmatrix} \Sigma_{S11} - e^{j\theta}I & \Sigma_{S12} \\ C_2 & 0 \end{bmatrix}$ is right invertible $\forall\theta \in [0, 2\pi)$ and any $\gamma > \gamma_0$;

Assumptions (A1′) and (A2′) are, in a sense (see (Mirkin *et al.*, 1997b, Subsection 4.2)), the counterparts of the standard assumptions on the stabilizability and detectability of $\mathcal{P}_{22}$. They are necessary for the existence of solutions to OP_{H^∞}. Moreover, as $\gamma \to \infty$ these assumptions become necessary and sufficient for the existence of sampled-data stabilizing controllers for the setup in Fig. 1. Assumptions $(\mathsf{A3}')$ and $(\mathsf{A4}')$ are the counterparts of the standard assumptions on the absence of unit circle (or the imaginary axis, in the continuous-time case) zeros of the subsystems from the control signal to the regulated output and from the exogenous input to the measured output, respectively.

The next Lemma establishes an important relation between assumptions $(\mathsf{A1}')$, $(\mathsf{A3}')$ and $(\mathsf{A1})$, $(\mathsf{A3})$, which were required for the solution to the $\mathsf{OP}^{eq}_{H^\infty}$ in the lifted domain.

Lemma 2. *Whenever $\gamma > \gamma_0$ and OP_{H^∞} is solvable, plant* (6) *satisfies assumptions (A1) and (A3) if and only if it satisfies assumptions (A1′) and (A3′).*

Proof. First, we reformulate assumptions $(\mathsf{A1})$ and $(\mathsf{A3})$ in terms of the parameters of $\breve{\mathcal{P}}_{1\bullet H}$. To this end, note that

$$\breve{\mathcal{P}}_{1\bullet} = \breve{\mathcal{W}}_v \breve{\mathcal{P}}_{1\bullet H} \begin{bmatrix} \breve{\mathcal{W}}_v^{-1} & \\ & \bar{\mathcal{W}}_v^{-1} \end{bmatrix}, \quad \breve{P}_{1\bullet H}(z) = \left[\begin{array}{c|cc} \bar{A}_H & \acute{B}_{1H} & \bar{B}_{2H} \\ \hline \grave{C}_{1H} & \breve{D}_{11H} & \grave{D}_{12H} \end{array}\right].$$

Also note that assumptions $(\mathsf{A1})$ and $(\mathsf{A3})$ are actually (Lancaster and Rodman, 1995) the necessary and sufficient conditions for the existence of the stabilizing solution to the H^2 DARE associated with $\breve{\mathcal{P}}_{12}$. According to Lemma 1, this solution exists if and only if the H^2 DARE associated with $\breve{\mathcal{P}}_{12H}$ possesses a stabilizing solution. This solution, however, exists if and only if the following conditions are satisfied:

$(\mathsf{A1}'')$: The pair $(\bar{A}_H, \bar{B}_{2H})$ is stabilizable;

$(\mathsf{A3}'')$: The operator $\begin{bmatrix} \bar{A}_H - e^{j\theta}I & \bar{B}_{2H} \\ \grave{C}_{1H} & \grave{D}_{12H} \end{bmatrix}$ is left invertible $\forall \theta \in [0, 2\pi)$.

Hence, assumptions $(\mathsf{A1})$ and $(\mathsf{A3})$ both[1] hold true iff so do $(\mathsf{A1}'')$ and $(\mathsf{A3}'')$.

Assume now that plant (6) satisfies assumptions $(\mathsf{A1}')$, $(\mathsf{A3}')$ and that OP_{H^∞} is solvable. Consequently:

i) $(\mathsf{A3}'')$ is satisfied, since it is equivalent to $(\mathsf{A3}')$ (Mirkin *et al.*, 1997b, Lemma 6), and

ii) $(\mathsf{A1})$ holds true, since it is a necessary condition for the existence of stabilizing controllers for plant (6).

Thus, in order to complete the proof to the first part of this Lemma, we only have to show that if assumption $(\mathsf{A1})$ is satisfied, so is $(\mathsf{A1}'')$. We prove that by contradiction. Assume that the pair $(\bar{A}_H, \bar{B}_{2H})$ is not stabilizable. Hence, there exist $|\lambda| \geq 1$ and $\eta \in \mathbb{R}^n$, $\eta \neq 0$ such that $\eta'[\bar{A}_H - \lambda I \;\; \bar{B}_{2H}] = 0$. It means that η' is the left eigenvector of $\bar{A}_H$ associated with the eigenvalue λ. Thus $\eta' \bar{A}_H^k = \lambda^k \eta'$. Consequently, $\eta' \bar{A}_H^k \bar{B}_{2H} = 0$, $\forall k = 0, \ldots, \nu - 1$ and $\eta'(\bar{A}_H^\nu - \lambda^\nu I) = 0$. This, in turn, leads to

$$\eta' [\, \bar{A} - \lambda^\nu I \;\; \bar{B}_2 \,] = \eta' [\, \bar{A}_H^\nu - \lambda^\nu I \;\; \bar{A}_H^{\nu-1} \bar{B}_{2H} \;\; \ldots \;\; \bar{A}_H \bar{B}_{2H} \;\; \bar{B}_{2H} \,] = 0,$$

which means that $(\bar{A}, \bar{B}_2)$ is also not stabilizable, since $|\lambda| \geq 1 \Rightarrow |\lambda^\nu| \geq 1$.

[1] In fact, it is possible to prove that $(\mathsf{A1})$ is equivalent to $(\mathsf{A1}'')$ and $(\mathsf{A3})$ is equivalent to $(\mathsf{A3}'')$. Yet this proof is more involved and not essential for the reasoning to follow.

To prove the second part of this Lemma, we consider the state-feedback single-rate sampled-data H^∞ optimization problem for the plant

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t), \tag{18a}$$
$$z(t) = C_1 x(t) + D_{12} u(t), \tag{18b}$$

where both the zero-order hold and the ideal sampler operate with the sampling period h_u. We claim that, if assumptions $(A1'')$, $(A3'')$ are satisfied and the OP_{H^∞} for plant (18) possesses a γ suboptimal solution, so does this single-rate optimization problem. Then, according to (Mirkin *et al.*, 1997b, Lemma 5), the solution to this problem exists only if assumption $(A1')$ holds true. The fact that $(A3')$ and $(A3'')$ are equivalent, completes the proof.

To prove this claim, let assumptions $(A1'')$ and $(A3'')$ be satisfied and suppose that the OP_{H^∞} for plant (18) possesses a solution. According to Theorem 1, the stabilizing solution $\bar{X}$ to the H^∞ DARE (11a), satisfies

$$\left\| \begin{bmatrix} \bar{X}^{1/2}\acute{B}_1 \\ \breve{D}_{11} \end{bmatrix} \right\|_2 \leq \gamma \Longleftrightarrow \rho\big(\acute{B}_1^*\bar{X}\acute{B}_1 + \breve{D}_{11}^*\breve{D}_{11}\big) \leq \gamma.$$

Note that the hermitian matrix $\acute{B}_1^*\bar{X}\acute{B}_1 + \breve{D}_{11}^*\breve{D}_{11}$ has the form

$$\acute{B}_1^*\bar{X}\acute{B}_1 + \breve{D}_{11}^*\breve{D}_{11} = \begin{bmatrix} ? & ? \\ ? & \acute{B}_{1H}^*\bar{X}\acute{B}_{1H} + \breve{D}_{11H}^*\breve{D}_{11H} \end{bmatrix},$$

where ? denotes irrelevant block terms. According to the Cauchy Theorem of separation (Gantmacher, 1974):

$$\rho\big(\acute{B}_{1H}^*\bar{X}\acute{B}_{1H} + \breve{D}_{11H}^*\breve{D}_{11H}\big) \leq \rho\big(\acute{B}_1^*\bar{X}\acute{B}_1 + \breve{D}_{11}^*\breve{D}_{11}\big) \leq \gamma.$$

Since, as a direct application of Lemma 1, $\bar{X}$ is also the stabilizing solution to the H^∞ DARE associated with the plant $\breve{\mathcal{P}}_{1\bullet H}$, we conclude, according to (Mirkin *et al.*, 1997b, Theorem 2), that the single-rate optimization problem defined for plant (18) possesses a γ suboptimal solution. □

The main result of this paper can now be stated.

Theorem 2. *Suppose plant* (6) *satisfies assumptions* $(A1')$–$(A4')$ *and let* $\nu = \kappa\mu$ *for some* $\kappa \in \mathbb{Z}^+/\{0\}$. *Then, for any* $\gamma > \gamma_0$, *the following statements are equivalent:*

i) There exist $\bar{\mathcal{K}}$, $\phi_H[i]$ *and* $\phi_S[j]$ *which solve* OP_{H^∞}.

ii) $(\Lambda_\nu, \Delta_\nu) \in \mathrm{dom}(\mathrm{Ric}_{\mathbb{D}})$, $(\Lambda_\mu, \Delta_\mu) \in \mathrm{dom}(\mathrm{Ric}_{\mathbb{D}})$ *and the following conditions are satisfied:*

- *(a)* $\bar{X}_\nu \geq 0$ *and* $\rho(\bar{X}_\nu \Pi_{12}\Pi_{22}^{-1}) < 1$;
- *(b)* $\bar{Y}_\mu \geq 0$ *and* $\rho(\Pi_{22}^{-1}\Pi_{21}\bar{Y}_\mu) < \gamma^2$;
- *(c)* $\rho\big(\bar{Y}_\mu(\Pi_{22} + \gamma^{-2}\Pi_{21}\bar{Y}_\mu)^{-1}\bar{X}_\nu(\Pi'_{22} - \Pi'_{12}\bar{X}_\nu)^{-1}\big) < \gamma^2$;

where $(\bar{X}_\nu, \bar{F}_{2\nu}) = \mathrm{Ric}_{\mathbb{D}}(\Lambda_\nu, \Delta_\nu)$ *and* $(\bar{Y}_\mu, \bar{L}'_{2\mu}) = \mathrm{Ric}_{\mathbb{D}}(\Lambda_\mu, \Delta_\mu)$.

Furthermore, if the conditions of part ii) hold, then the matrix $\bar{Z}_{\nu\mu} \doteq (I - \gamma^{-2}\bar{Y}_\mu\bar{X}_\nu)^{-1}$ *is well defined and one possible choice for the sampled-data controller which solves the* OP_{H^∞} *consists of the discrete-time part:*

$$\bar{K}(z) = z\left[\begin{array}{c|c} \bar{Z}_{\nu\mu}\Theta_{12} + \Theta_{22} & \bar{Z}_{\nu\mu} \\ \hline I & 0 \end{array}\right], \tag{19a}$$

the generalized hold of the form (8a) *with*

$$\phi_H[i] = \bar{F}_{2\nu}\big(\Sigma_{H22} + \Sigma_{H24}\bar{F}_{2\nu} + \Sigma_{H23}\bar{X}_\nu\big)^i, \qquad i = 0, \dots, \nu - 1, \tag{19b}$$

and of the generalized sampler of the form (8b) *with*

$$\phi_S[j] = \big(\Sigma_{S11} + \bar{L}_{2\mu}C_2\Sigma_{S11} + \bar{Y}_\mu\Sigma_{S21}\big)^j\,\bar{L}_{2\mu}, \qquad j = 0, \dots, \mu - 1, \tag{19c}$$

where

$$\Theta \doteq \begin{bmatrix} \Sigma_{S11} + \bar{L}_{2\mu}C_2\Sigma_{S11} + \bar{Y}_\mu\Sigma_{S21} & \bar{L}_{2\mu}C_2(\Sigma_{H22} + \Sigma_{H24}\bar{F}_{2\nu} + \Sigma_{H23}\bar{X}_\nu)^\kappa \\ 0 & (\Sigma_{H22} + \Sigma_{H24}\bar{F}_{2\nu} + \Sigma_{H23}\bar{X}_\nu)^\kappa \end{bmatrix}^\mu = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ 0 & \Theta_{22} \end{bmatrix}.$$

Proof. Although assumptions $(\mathsf{A1}')$–$(\mathsf{A4}')$ are, in general, not equivalent to $(\mathsf{A1})$–$(\mathsf{A4})$, according to Lemma 2 and its dual, the replacement of $(\mathsf{A1})$–$(\mathsf{A4})$ with the more readily checkable conditions $(\mathsf{A1}')$–$(\mathsf{A4}')$ does not affect the solution to $\mathsf{OP}^{eq}_{H^\infty}$ presented in Theorem 1. Consequently, it is actually sufficient to prove that the solution presented in this Theorem is equivalent to the one in Theorem 1. To this end, note that

$$\breve{\mathcal{P}}_{\bullet 1} = \begin{bmatrix} \breve{\mathcal{W}}_\mu & \\ & \bar{\mathcal{W}}_\mu \end{bmatrix} \breve{\mathcal{P}}_{\bullet 1S}\breve{\mathcal{W}}_\mu^{-1}, \quad \breve{P}_{\bullet 1S}(z) = \left[\begin{array}{c|c} \bar{A}_S & \acute{B}_{1S} \\ \hline \grave{C}_{1S} & \breve{D}_{11S} \\ \bar{C}_{2S} & \acute{D}_{21S} \end{array}\right].$$

Thus, as a direct application of Lemma 1, $\bar{X}$ and $\bar{Y}$ are the stabilizing solutions to the H^∞ DARE's (11) if and only if they are the stabilizing solutions to the H^∞ DARE's associated with the plants $\breve{\mathcal{P}}_{1\bullet H}$ and $\breve{\mathcal{P}}_{\bullet 1S}$, respectively. Since it was assumed that $\gamma > \gamma_0$ and by a direct application of (Mirkin *et al.*, 1997b, Lemma 7), the latter DARE's have stabilizing solutions if and only if $(\Lambda_\nu, \Delta_\nu) \in \mathrm{dom}(\mathsf{Ric}_{\mathbb{D}})$ and $(\Lambda_\mu, \Delta_\mu) \in \mathrm{dom}(\mathsf{Ric}_{\mathbb{D}})$. Moreover, in this case, $\bar{X} = \bar{X}_\nu$, $\bar{Y} = \bar{Y}_\mu$ and $\bar{Z} = \bar{Z}_{\nu\mu}$. Then, according to (Mirkin *et al.*, 1997b, Lemma 8), the items *(a)–(c)* are equivalent to the coupling condition (12).

Now, consider the separation of $\bar{\mathcal{K}}_{eq}$ into $\bar{\mathcal{K}}$, Φ_H and Φ_S sugested in (15). Denote by $\grave{F}_\nu \doteq \begin{bmatrix} \grave{F}_{1\nu} \\ \bar{F}_{2\nu} \end{bmatrix}$ and $\acute{L}_\mu \doteq \begin{bmatrix} \acute{L}_{1\mu} & \bar{L}_{2\mu} \end{bmatrix}$ the stabilizing gain matrices of the H^∞ DARE's associated with these plants, respectively. By using (Mirkin *et al.*, 1997b, Lemma 7) again, the following relations are established:

$$\begin{aligned} \tilde{A}_H &\doteq \bar{A}_H + \acute{B}_{1H}\grave{F}_{1\nu} + \bar{B}_{2H}\bar{F}_{2\nu} = \Sigma_{H22} + \Sigma_{H24}\bar{F}_{2\nu} + \Sigma_{H23}\bar{X}_\nu, \\ \tilde{A}_S &\doteq \bar{A}_S + \acute{L}_{1\mu}\grave{C}_{1S} + \bar{L}_{2\mu}\bar{C}_{2S} = \Sigma_{S11} + \bar{L}_{2\mu}C_2\Sigma_{S11} + \bar{Y}_\mu\Sigma_{S21}. \end{aligned}$$

On the other hand, it follows from Lemma 1 that

$$\bar{F}_2 = \begin{bmatrix} \bar{F}_{2\nu} \\ \bar{F}_{2\nu}\tilde{A}_H \\ \vdots \\ \bar{F}_{2\nu}\tilde{A}_H^{\nu-1} \end{bmatrix}, \quad \bar{L}_2 = \begin{bmatrix} \bar{L}_{2\mu} & \tilde{A}_S\bar{L}_{2\mu} & \dots & \tilde{A}_S^{\mu-1}\bar{L}_{2\mu} \end{bmatrix},$$

which proves (19b) and (19c).

To complete the proof, the computational formula for the 'A' part of the digital controller, given in (19a), will be derived. To this end, assume that μ is a divisor of ν or, in other words, that

$\nu/\mu = \kappa$, where κ is a natural number. Also note that, in this case, $\bar{\mathcal{P}}_{22} \doteq \bar{\mathcal{W}}_\mu \mathcal{S}^{\text{IS}}_{h_y} \mathcal{P}_{22} \mathcal{H}^{\text{ZOH}}_{h_u} \bar{\mathcal{W}}_\nu^{-1}$ can be written as $\bar{\mathcal{P}}_{22} = \bar{\mathcal{W}}_\mu \bar{\mathcal{P}}_{22S} \bar{\mathcal{W}}_\mu^{-1}$, where $\bar{\mathcal{P}}_{22S} \doteq \mathcal{S}^{\text{IS}}_{h_y} \mathcal{P}_{22} \mathcal{H}^{\text{ZOH}}_{h_u} \bar{\mathcal{W}}_\kappa^{-1}$,

$$\bar{P}_{22S}(z) = \left[\begin{array}{c|c} \bar{A}_S & \bar{B}_{2S} \\ \hline \bar{C}_{2S} & \bar{D}_{22S} \end{array}\right].$$

Denote $\acute{B}_S \doteq \begin{bmatrix} \acute{B}_{1S} & \bar{B}_{2S} \end{bmatrix}$ and $\acute{D}_{2\bullet S} \doteq \begin{bmatrix} \acute{D}_{21S} & \bar{D}_{22S} \end{bmatrix}$. Using the STPBC representation (3) of the discrete-lifted parameters[2], $\bar{C}_2$, $\acute{D}_{2\bullet}$ and $\bar{L}_2$ can be represented as follows:

$$\bar{C}_2 = \left[\begin{array}{c|c} \bar{A}_S & \bar{A}_S \\ \hline \bar{C}_{2S} & \bar{C}_{2S} \end{array}\right]_0^{\mu-1} \bar{\mathcal{I}}_0, \quad \acute{D}_{2\bullet} = \left[\begin{array}{c|c} \bar{A}_S & \acute{B}_S \\ \hline \bar{C}_{2S} & \acute{D}_{2\bullet S} \end{array}\right]_0^{\mu-1}, \quad \bar{L}_2 = \bar{\mathcal{I}}^*_{\mu-1} \left[\begin{array}{c|c} \tilde{A}_S & \bar{L}_{2\nu} \\ \hline \tilde{A}_S & \bar{L}_{2\nu} \end{array}\right]_0^{\mu-1}.$$

Using again the fact that $\nu/\mu = \kappa$, the operator $\grave{F}$ can also be represented as a STPBC operating over the finite time interval $[0, \mu - 1]$:

$$\grave{F} = \left[\begin{array}{c|c} \tilde{A}_H^\kappa & \tilde{A}_H^\kappa \\ \hline \grave{F}_\kappa & \grave{F}_\kappa \end{array}\right]_0^{\mu-1} \bar{\mathcal{I}}_0, \quad \grave{F}_\kappa \doteq \begin{bmatrix} \grave{F}_\nu \\ \grave{F}_\nu \tilde{A}_H \\ \vdots \\ \grave{F}_\nu \tilde{A}_H^{\kappa-1} \end{bmatrix}.$$

Based on the definition of $\bar{\mathcal{P}}_{22S}$, the relations $\bar{C}_{2S} = C_2 \bar{A}_S$, $\acute{D}_{2\bullet S} = C_2 \acute{B}_S$ and $\acute{B}_S \grave{F}_\kappa = \tilde{A}_H^\kappa - \bar{A}_S$ can be derived. Using these relations and the standard formulas for the addition and the multiplication of state-space systems, it is found that

$$\bar{L}_2(\bar{C}_2 + \acute{D}_{2\bullet}\grave{F}) = \bar{\mathcal{I}}^*_{\mu-1} \left[\begin{array}{cc|c} \tilde{A}_S & \bar{L}_{2\mu} C_2 \tilde{A}_H^\kappa & \bar{L}_{2\mu} C_2 \tilde{A}_H^\kappa \\ 0 & \tilde{A}_H^\kappa & \tilde{A}_H^\kappa \\ \hline \tilde{A}_S & \bar{L}_{2\mu} C_2 \tilde{A}_H^\kappa & \bar{L}_{2\mu} C_2 \tilde{A}_H^\kappa \end{array}\right]_0^{\mu-1} \bar{\mathcal{I}}_0,$$

which proves (19a), since $\bar{A} + \acute{B}\grave{F} = \tilde{A}_H^\nu = \Theta_{22}$ (Lemma 1). □

Remark 4. Note that each of the matrices given by (16) plays a different role in the solution process. The matrix exponential Σ_H in (16a) is required only for the computation of the hold gain function $\phi_H[i]$. Similarly, Σ_S in (16b) is required only for the computation of the sampler gain function $\phi_S[j]$, while the matrix Π given by (16c) affects only the conditions *(a)-(c)*.

Remark 5. It is worthwhile noting that the assumption $\nu = \kappa\mu$ is required only to simplify the derivation and the final formula of the main coefficient (the 'A' matrix) of the discrete-time part of the controller $\bar{\mathcal{K}}$. Indeed, this assumption affects neither formulae (19b) and (19c) nor conditions *(a)-(c)*, which still apply in the general case.

Remark 6. Theorem 2 contains the solutions to some known optimization problems already solved in the literature as particular cases. When $\gamma \to \infty$, Theorem 2 actually solves the H^2 design problem of the generalized sampling and hold functions with waveform constraints (Kahane *et al.*, 1999b). When $\nu = \mu = 1$, the sampled-data controller $\mathcal{H}_h \bar{\mathcal{K}} \mathcal{S}_h$ is the solution to the H^∞ suboptimal single-rate sampled-data control problem (Bamieh and Pearson, 1992) based on the zero-order hold and on the ideal sampler converters (Φ_H and Φ_S are absorbed into the digital controller, thus becoming its 'C' and 'B' coefficients, respectively). In the case where

[2]This representation suffices since the derivation of the 'A' part of the digital controller requires only the multiplication and the addition operations.

$\mu = 1$, Theorem 2 actually solves the input multi-rate (Araki, 1993) H^∞ problem, which is a particular case of the general H^∞ multi-rate problem treated in (Chen and Qiu, 1994; Voulgaris and Bamieh, 1993). It is worthwhile noting that Theorem 2 provides a simpler solution in this case both from the computational and conceptual point of view.

The solutions to the piecewise constant hold function in (19b) and the piecewise impulse sampling function in (19c) have interesting properties and interpretations. The remainder of this section is devoted to the discussion of those properties.

The first property is the separation between the designs of the H^∞ suboptimal piecewise impulse sampler and piecewise constant hold in the sense that the hold design does not depend on the measurement $y(t)$, and the sampler design does not depend on the control action $u(t)$. The separation property, which has already been discussed in Remark 1, can be explained by the fact that both the sampler and the hold are, in a sense, open-loop devices.

A nice interpretation for the H^∞ suboptimal piecewise constant hold (19b) can be obtained from the solution to the state-feedback single-rate sampled-data H^∞ optimization problem for the plant (18). Assume that, in this problem, both the zero-order hold and the ideal sampler operate with the sampling period h_u. Hence, the state vector satisfies the following equation:

$$\bar{x}[i+1] = \bar{A}_H\bar{x}[i] + \acute{B}_{1H}\check{w}[i] + \bar{B}_{2H}\bar{u}[i].$$

It was shown by Mirkin *et al.* (1997a) that the solution to this problem is based on the DARE $(\bar{X}_\nu, \bar{F}_{2\nu}) = \mathrm{Ric}_{\mathbb{D}}(\Lambda_\nu, \Delta_\nu)$ (the same as the one used in Theorem 2) and that the resulting state-feedback control law is $\bar{u}[i] = \bar{F}_{2\nu}\bar{x}[i]$. Assume now that the disturbance w is given by $\check{w}[i] = \grave{F}_{1\nu}\bar{x}[i]$. The closed-loop state vector satisfies

$$\bar{x}[i+1] = (\bar{A}_H + \acute{B}_{1H}\grave{F}_{1\nu} + \bar{B}_{2H}\bar{F}_{2\nu})\bar{x}[i] = (\Sigma_{H22} + \Sigma_{H24}\bar{F}_{2\nu} + \Sigma_{H23}\bar{X}_\nu)\bar{x}[i],$$

and, consequently, the control signal $u(t)$ satisfies

$$u(kh + ih_u + \tau) = \bar{F}_{2\nu}(\Sigma_{H22} + \Sigma_{H24}\bar{F}_{2\nu} + \Sigma_{H23}\bar{X}_\nu)^i\bar{x}[k], \quad \forall\tau \in [0, h_u).$$

On the other hand, it follows from (8a) and (19b) that the H^∞ suboptimal piecewise constant hold $\mathcal{H}_h$ produces the control signal

$$u(kh + ih_u + \tau) = \bar{F}_{2\nu}(\Sigma_{H22} + \Sigma_{H24}\bar{F}_{2\nu} + \Sigma_{H23}\bar{X}_\nu)^i\bar{u}[k], \quad \forall\tau \in [0, h_u).$$

The comparison between the latter two expressions yields that the H^∞ suboptimal piecewise constant hold with a sampling period h attempts to "reconstruct" the H^∞ state-feedback control law of the single-rate sampled-data control system with a ν times faster sampling period, assuming that *i)* the digital controller produces at the k-th sampling instance an estimate of the state vector of the plant at $t = kh$; and *ii)* the disturbance $\check{w}[i] = \grave{F}_{1\nu}\bar{x}[i]$ is the worst case one. In other words, the H^∞ suboptimal piecewise constant hold, much alike the H^2 optimal one (Kahane *et al.*, 1999b), tries to compensate for the deterioration in the system performance due to the insufficiently fast sampling rate, by imitating the control law of a faster H^∞ suboptimal sampled-data controller. This is in contrast to an early design (Kabamba, 1987) where the hold device was designed to outperform single-rate sampled-data controllers with faster sampling rates. This property is similar to that of the generalized unconstrained hold developed in (Mirkin *et al.*, 1997a) which tries to imitate the continuous-time H^∞ suboptimal state-feedback control law.

7 Conclusions

In this paper the H^∞ sampled-data control problem has been treated assuming that not only the digital controller but also the sampler and the hold are design parameters. Taking into consideration implementation requirements, the designed sampler and hold have been treated subject to waveform constraints. In particular, the hold has been assumed to belong to the class of piecewise-constant hold functions with a given number ν of intersample corrections of the control signal. The sampler has been assumed to average a given number μ of weighted measurements, equally spread within the intersample (thus, piecewise-impulse waveform of the sampling function). Necessary and sufficient conditions of the existence of a γ-suboptimal sampled-data controller have been obtained and explicit formulae for the suboptimal sampler, hold, and discrete-time part of the controller have been derived. It is believed that these results will be helpful in many applications where the available sampling rate is insufficiently fast.

Note, that the formula for the discrete-time part of the controller was presented only for the case where $\nu = \kappa\mu$ for a positive κ (the existence conditions as well as the formulae for the sampler and hold are valid for arbitrary ν and μ). The reason is that the formula for the general case turns out to be quite complicated. It is believed, however, that a simpler expression exists and its derivation is currently investigated.

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