Proceedings of the 7th Mediterranean Conference on Control and Automation (MED99) Haifa, Israel - June 28-30, 1999

(J, J^0)-dissipative matrices and singular H_∞ control*

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Abstract

This paper deals with solving a class of H_∞ control problems where the transfer matrix from the external input to the measured output is invertible at infinity while there is no assumption about the infinite and/or imaginary-axis zeros of the transfer matrix from the control input to the penalized output. Our approach is based on the chain-scattering representation and a newly proposed (J, J^0)-dissipative factorization extending thus the well-known approach of H. Kimura, while preserving its simplicity. We provide also a characterization of the set of controllers solving the given problem.

1 Introduction

This paper presents a new approach to solving a special class of H_∞ control problems based on chain-scattering representation and a newly proposed factorization. The main result is a new parametrization of the solution set.

We are going to assume that the transfer matrix from the external input to the measured output is invertible at infinity—hence, we shall deal with a kind of generalized two-block problem. On the other hand, we lift the assumptions that the transfer matrix from the control input to the penalized output is left-invertible at infinity and it has no imaginary-axis zeros.

Although this problem has been solved by several different methods, see e.g. (Sampei *et al.*, 1990; Stoorvogel, 1992; Gahinet and Apkarian, 1994), very recently also by (Miyazaki and Hosoe, 1997; Xin *et al.*, 1998), we believe that our alternative approach can provide a new insight in the structure of H_∞ control. It is also fairly simple—it generalizes the well known approach of (Kimura, 1997) (its two-block version) which is one of the simplest and most elegant approaches to the standard H_∞ control. We use essentially the same technical tools of chain-scattering and systems factorization. The main point of our approach is introducing the class of (J, J^0)-dissipative matrices which replace the (J, J')-lossless ones. They take over the troublesome

*This research was done when the author was with the University of Tokyo, Department of Mathematical Engineering and Information Physics as a JSPS Fellow. He wishes to acknowledge the Monbusho Grant No 96335.

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infinite and/or imaginary-axis zeros which do not need to be passed to the outer factor. This helps us to avoid some technicalities like descriptor formalism, infinite-zeros compensation etc.

In this paper we reformulate the H_∞ control problem into the factorization one and show how to obtain the set of all controllers solving the given problem. Due to the lack of space, we omit the state-space solution to the factorization problem which is available but will appear elsewhere; it is very similar to the solution of a slightly less general problem in (Baramov, 1998).

2 Problem formulation and review of the standard case

Consider the plant P whose input-output behavior is described as follows:

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, \tag{1}$$

w and u are the external and control inputs, respectively, and z and y are, respectively, the penalized and measured outputs. In what follows we shall assume that $dim(w) = dim(y) = r$, $dim(z) = m$ and $dim(u) = p$. The closed loop transfer matrix from w to z obtained by connecting the plant P and a controller K, which satisfies $u(s) = K(s)y(s)$, can be expressed as

$$T(s) = LFT(P(s), K(s)) = P_{11}(s) + P_{12}(s)K(s)(I + P_{22}(s)K(s))^{-1}P_{21}(s)$$

LFT stands here for the *linear Fractional Transformation.* If $P_{21}(\infty)$ is invertible, we can obtain the *chain-scattering representation* of the plant (see (Kimura, 1997)) as

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} := \begin{bmatrix} P_{12}(s) - P_{11}(s)P_{21}(s)^{-1}P_{22}(s) & P_{11}(s)P_{21}(s)^{-1} \\ -P_{21}(s)^{-1}P_{22}(s) & P_{21}(s)^{-1} \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} P_{12}(s) & P_{11}(s) \\ 0 & I_r \end{bmatrix} \begin{bmatrix} I_p & 0 \\ P_{22}(s) & P_{21}(s) \end{bmatrix}^{-1} \tag{3}$$

We also write $G(s) = CHAIN(P(s))$. $G(s)$ assigns z and w to a pair of u and y. The original feedback form is obtained from the chain scattering representation via the the inverse mapping to $CHAIN$ which is called $SCAT$.

$$P(s) = \begin{bmatrix} G_{12}(s)G_{22}(s)^{-1} & G_{11}(s) - G_{12}(s)G_{22}(s)^{-1}G_{21}(s) \\ G_{22}(s)^{-1} & -G_{22}(s)^{-1}G_{21}(s) \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} G_{11}(s) & G_{12} \\ 0 & I_r \end{bmatrix} \begin{bmatrix} G_{21}(s) & G_{22}(s) \\ I_p & 0 \end{bmatrix}^{-1} \tag{5}$$

The chain-scattering representation simplifies notations significantly compared to the feedback form (1). The closed loop transfer matrix can be expressed in terms of $G(s)$ as

$$T(s) = [G_{11}(s)K(s) + G_{12}(s)][G_{21}(s)K(s) + G_{22}(s)]^{-1} \tag{6}$$

The mapping assigning to $G(s)$ and $K(s)$ the closed loop transfer matrix from w to z is called *Homographic Transformation* and denoted by $T(s) = HMT(G(s), K(s))$.

The H_∞ control problem is to find a controller K such that the closed loop is internally stable (in the usual sense, see e.g. (Kimura, 1997)) and $\|T(s)\|_\infty < 1$, where $\|\cdot\|_\infty$ denotes the standard L_∞-norm of a transfer matrix, analytic on the $j\omega$-axis.

This problem has been solved under various assumptions; the standard ones are as follows:

Assumption 1 *The transfer matrix $G(s)$ given by (2) is left-invertible at ∞ and has neither poles nor zeros with zero real parts.*

Notice that this assumption means that neither $P_{12}(s)$ nor $P_{21}(s)$ have imaginary axis or infinite zeros. We give the definition of the (J, J')-lossless transfer matrices which play the key role in solving the H_∞ problem under these standard assumptions. In the following, the symbols I_p and 0_{mn} stand for the identity matrix of dimension p and the zero matrix of dimension $m \times n$, respectively. The symbol J_{pq} denotes the signature matrix $diag(I_p, -I_q)$. The subscripts can be dropped, if these dimensions are clear from the context. Further, for a transfer matrix $G(s)$, $G^{\sim}(s)$ and $G^*(s)$ denote $G^T(-s)$ and $G^T(\bar{s})$, respectively.

Definition 1 *A transfer matrix $\Theta(s)$ is said to be (J_{mr}, J_{pr})-lossless, if $\Theta^{\sim}(j\omega)J_{mr}\Theta(j\omega) = J_{pr}$ for all ω and $\Theta^*(s)J_{mr}\Theta(s) \le J_{pr}$ for all s with positive real part.*

Instead of (J_{mr}, J_{pr}) we may write just (J, J')-lossless. In the case of $r = 0$, we deal with (I, I')-lossless matrices which are called simply *lossless* or *inner*. Necessary and sufficient conditions for solvability of the standard H_∞ problem are as follows:

Theorem 1 *Let $G(s)$ be the chain-scattering representation of a plant P which satisfies Assumption 1. Then, there exist a controller which stabilizes the closed loop $T(s) = HMT(G(s), K(s))$ and renders $\|T(s)\|_\infty < 1$ iff $G(s) = \Theta(s)\Pi(s)$ where $\Theta(s)$ is a (J_{mr}, J_{pr})-lossless matrix and $\Pi(s)$ is a stable matrix with stable and proper inverse. Then, any such controller can be expressed as $K(s) = HMT(\Pi(s)^{-1}, Q(s))$, where $Q(s)$ is a stable transfer matrix of $\|Q(s)\|_\infty < 1$.*

In this paper we shall deal with the H_∞ problem under the following relaxed conditions:

Assumption 2 *The transfer matrix $G(s)$ given by (2) has no pole with zero real part.*

In other words, we shall lift all assumptions about imaginary-axis/infinite zeros of $P_{12}(s)$. Under this assumption, $G(s)$ may not be factorizable into the product of a (J, J')-lossless matrix (which is of full rank at $s = j\omega$) and $\Pi(s)$, which is invertible for all s with nonnegative real part.

3 (J, J^0)-dissipative transfer matrices

First, let us denote by I^0_{pq} and J^0_{pqr} the singular matrices $diag(I_p, 0_{qq})$ and $diag(I^0_{pq}, -I_r)$, respectively. We shall write J^0 and I^0 instead of J^0_{pqr} and I^0_{pq} if it causes no confusion. Then we define a new class of (J, J^0)-dissipative transfer matrices which take over the role of the (J, J')-lossless ones in singular H_∞ control.

Definition 2 *An $(m+r) \times (p+q+r)$ transfer matrix $\Theta(s)$ is said to be (J_{mr}, J^0_{pqr})-dissipative, if $\Theta^T(\infty)J_{mr}\Theta(\infty) = J^0_{pqr}$ and $\Theta^*(s)J_{mr}\Theta(s) \le J_{(p+q)r}$ for all s of the closed right halfplane.*

The following lemma shows that (J, J^0)-dissipative matrices are mapped into (I, I^0)-dissipative ones and vice versa by the $SCAT$ and $CHAIN$ transformations, respectively.

Lemma 1 *If $\Theta(s)$ is (J_{mr}, J^0_{pqr})-dissipative then $\Sigma(s) = SCAT(\Theta(s))$ exists and is $(I_{m+r}, I^0_{(p+r)q})$-dissipative. Conversely, if $\Sigma(s)$ is $(I_{m+r}, I^0_{(p+r)q})$-dissipative and $\Theta(s) = CHAIN(\Sigma(s))$ is well defined then $\Theta(s)$ is (J_{mr}, J^0_{pqr})-dissipative.*

PROOF: The proof is a straightforward generalization of the analogous lemma on relation between (J, J')-lossless and inner matrices, see (Kimura, 1997). The equation $\Theta^T(\infty) J_{mr} \Theta(\infty) = J^0_{pqr}$ implies that $\Theta_{22}(\infty)^T \Theta_{22}(\infty) = I_r + \Theta_{12}(\infty)^T \Theta_{12}(\infty)$, hence $\Theta_{22}(\infty)$ is nonsingular and $SCAT(\Theta(s))$ exists. Further, consider the representation of $\Theta(s)$ as in (3) where $\Sigma_{ij}(s)$ is substituted for $P_{ij}(s)$. Then, the inequality $J' - \Theta^*(s) J \Theta(s) \geq 0$ for all $Re[s] \geq 0$ implies that

$$
\begin{aligned}
0 &\leq \begin{bmatrix} I_{p+q} & -\Sigma^*_{22}(s) \\ 0 & -\Sigma^*_{21}(s) \end{bmatrix} \begin{bmatrix} I_{p+q} & 0 \\ \Sigma_{22}(s) & \Sigma_{21}(s) \end{bmatrix} - \begin{bmatrix} \Sigma^*_{12}(s) & 0 \\ \Sigma^*_{11}(s) & -I_r \end{bmatrix} \begin{bmatrix} \Sigma_{12}(s) & \Sigma_{11}(s) \\ 0 & I_r \end{bmatrix} \\
0 &\leq \begin{bmatrix} I_{p+q} & 0 \\ 0 & I_r \end{bmatrix} - \begin{bmatrix} \Sigma^*_{12}(s) & \Sigma^*_{22}(s) \\ \Sigma^*_{11}(s) & \Sigma^*_{21}(s) \end{bmatrix} \begin{bmatrix} \Sigma_{12}(s) & \Sigma_{11}(s) \\ \Sigma_{22}(s) & \Sigma_{21}(s) \end{bmatrix}
\end{aligned}
$$

From there it follows that $I - \Sigma^*(s)\Sigma(s) \geq 0$ for all s of the closed right half-plane. As for proving $\Sigma(\infty)^T \Sigma(\infty) = I^0_{(p+r)q}$, we can use essentially the same procedure as above, setting $s := \infty$, and replacing I_{p+q} by I^0_{pq} and the inequality $\leq$ by $=$. The second statement is proven similarly, using the right-fractional representation of $\Sigma(s)$ as in (5). ∎

We have to point out that the (I, I^0)-dissipative matrices are bounded-real (see (Anderson and Vongranitlert, 1973)) with a special property at infinity. The structure of (J, J^0)-dissipative matrices (including the (I, I^0)-dissipative ones as a special case) at infinity is made more explicit in the next lemma.

Lemma 2 *Let $\Theta(s)$ be (J_{mr}, J^0_{pqr})-dissipative. Then, $\Theta(\infty) = \begin{bmatrix} D_{\theta 1} & 0_{(m+r)q} & D_{\theta 3} \end{bmatrix}$, and $D_{\theta 0} := \begin{bmatrix} D_{\theta 1} & D_{\theta 3} \end{bmatrix}$ is (J_{mr}, J_{pq})-unitary, i.e. it satisfies $D^T_{\theta 0} J_{mr} D_{\theta 0} = J_{pq}$.*

PROOF: First, we prove this lemma for (I_s, I^0_{tq})-dissipative matrices $\Sigma(s)$. Let $\Sigma(\infty) := \begin{bmatrix} D_{\Sigma 1} & D_{\Sigma 2} \end{bmatrix}$. As $\Sigma(\infty)^T \Sigma(\infty) = I^0_{tq}$, then $D^T_{\Sigma 1} D_{\Sigma 1} = I_t$ and $D^T_{\Sigma 2} D_{\Sigma 2} = 0$. Hence, $D_{\Sigma 2} = 0$ which proves the assertion for (I, I^0)-dissipative matrices. For a general (J_{mr}, J^0_{pq})-dissipative matrix $\Theta(s)$, $\Sigma(s) := SCAT(\Theta(s)$ is $(I_{m+r}, I^0_{(p+r)q})$-dissipative owing to Lemma 1. Hence, $\Sigma(\infty) = \begin{bmatrix} D_{\Sigma 1} & 0_{(m+r)q} \end{bmatrix}$. The structure of $\Theta(\infty)$ then follows from (2). ∎

Now we give a state-space characterization of (J_{mr}, J^0_{pqr})-dissipative transfer matrices. We shall use the usual notation for a state-space form of $\Theta(s)$:

$$
\Theta(s) = C_\theta (sI - A_\theta)^{-1} B_\theta + D_\theta =: \left[\begin{array}{c|c} A_\theta & B_\theta \\ \hline C_\theta & D_\theta \end{array} \right] = \left[\begin{array}{c|cc} A_\theta & B_{\theta 1} & B_{\theta 2} \\ \hline C_{\theta 1} & D_{\theta 11} & D_{\theta 12} \\ C_{\theta 2} & D_{\theta 21} & D_{\theta 22} \end{array} \right] \tag{7}
$$

where the latter partitioning is compatible with J_{mr} and $J_{(p+q)r}$.

Theorem 2 *A transfer function $\Theta(s)$ given by (7) is (J_{mr}, J^0_{pqr})-dissipative if and only if $D_\theta^T J_{mr} D_\theta = J^0_{pqr}$ and there exist matrices $P \geq 0$ and $S \geq 0$ such that*

$$\begin{aligned} PA_\theta + A_\theta^T P + C_\theta^T J_{mr} C_\theta + PB_\theta O^I_{pqr} B_\theta^T P + S &= 0 && (8)\\ D_\theta^T J_{mr} C_\theta + (I - O^I_{pqr}) B_\theta^T P &= 0 && (9) \end{aligned}$$

where $O^I_{pqr} := J_{q+p,r} - J^0_{pqr} = diag(0_{pp}, I_q, 0_{rr})$.

Proof: To prove this theorem we shall use the well-known properties of bounded-real matrices. Owing to Lemma 1 (J, J^0)-dissipativeness of $\Theta(s)$ is equivalent to the (I, I^0)-dissipativeness of $\Sigma(s) := SCAT(\Theta(s))$. A state-space form of $\Sigma(s)$ is obtained from that of $\Theta(s)$ as

$$\Sigma(s) = \left[\begin{array}{c|c} A_\Sigma & B_\Sigma \\ \hline C_\Sigma & D_\Sigma \end{array}\right], \tag{10}$$

$$A_\Sigma := A_\theta - B_{\theta 2} D_{\theta 22}^{-1} C_{\theta 2}, \qquad B_\Sigma := B_\theta \begin{bmatrix} D_{\theta 21} & D_{\theta 22} \\ I & 0 \end{bmatrix}^{-1}$$

$$C_\Sigma := \begin{bmatrix} I & 0 \\ D_{\theta 12} & D_{\theta 22} \end{bmatrix}^{-1} C_\theta, \quad D_\Sigma := \begin{bmatrix} D_{\theta 11} & D_{\theta 12} \\ 0 & I \end{bmatrix} \begin{bmatrix} D_{\theta 21} & D_{\theta 22} \\ I & 0 \end{bmatrix}^{-1}$$

$\Sigma(s)$ is (I, I^0)-dissipative if and only if it is bounded-real and $D_\Sigma^T D_\Sigma = I^0$. The latter property was shown to be equivalent to $D_\theta^T J D_\theta = J^0$. As was proven already in (Anderson and Vongrantitlert, 1973), $\Sigma(s)$ is bounded-real iff there exist an $n \times n$ symmetric matrix $P \geq 0$ and a $k \times n$ matrix H, $k \geq q$, satisfying

$$\begin{aligned} PA_\Sigma + A_\Sigma^T P + C_\Sigma^T C_\Sigma + H^T H &= 0 && (11)\\ PB_\Sigma + C_\Sigma^T D_\Sigma + H^T E &= 0 && (12) \end{aligned}$$

where E satisfies $E^T E = O^I_{(p+r)q0}$. With no loss of generality we can consider $E = \text{diag}(0_{(k-q)(p+r)}, I_q)$. Let $H = \begin{bmatrix} H_1^T & H_2^T \end{bmatrix}^T$ where H_2 is a matrix of q rows. As follows from (12) and from the structure of D_Σ proven in Lemma 2, $H_2 = -\begin{bmatrix} 0 & I_q \end{bmatrix} B_\Sigma^T P$. By eliminating H_2 from and substituting $H_1^T H_1 =: S$ in (11) we get

$$PA_\Sigma + A_\Sigma^T P + C_\Sigma^T C_\Sigma + PB_\Sigma O^I_{(p+r)q0} B_\Sigma^T P + S = 0 \tag{13}$$

Further, post-multiplying (12) by $I^0_{(p+r)q}$ yields (taking into account the special structure of D_Σ)

$$PB_\Sigma I^0_{(p+r)q} + C_\Sigma^T D_\Sigma = 0 \tag{14}$$

Equations (13) and (14) are already consistent with (8) and (9) if the latter is specialized to the (I, I^0)-dissipative case, i.e. $r = 0$. Otherwise, substituting for A_Σ, B_Σ, C_Σ and D_Σ in (13) and (14) yields, after some manipulations (and taking into account the properties of D_θ), exactly equations (8) and (9). ∎

An important role is played by the so-called (J, J')-lossless complement of a (J, J^0)-lossless matrix $\Theta(s)$ in the subsequent development. It is defined as a matrix $\Theta^{\perp}(s)$ such that

$$\Theta'(s) := \begin{bmatrix} \Theta^{\perp}(s) \\ \Theta(s) \end{bmatrix} \tag{15}$$

is a (J, J')-lossless matrix. The following lemma shows that such a complement always exists, is not unique and that we can always find a stable one.

Lemma 3 *Let $\Theta(s)$ be a (J_{mr}, J^0_{pqr})-dissipative matrix satisfying Assumption 2. Then there always exists a set of $(J_{(m+k+q)r}, J_{(p+k+q)r})$-lossless complements. Moreover, we can always find among them a stable complement $\Theta^{\perp}_s(s)$.*

PROOF: Let us transfer $\Theta(s)$ to the feedback form to obtain a (I, I^0)-dissipative matrix $\Sigma(s) = SCAT(\Theta(s))$. The existence of a lossless complement $\Sigma^{\perp}(s)$ of $\Sigma(s)$ follows from the existence of a spectral factor of $I - \Sigma^{\sim}(s)\Sigma(s)$. $\Theta^{\perp}(s)$ is then obtained via the chain-scattering formula (3):

$$\Theta^{\perp}(s) = \Sigma^{\perp}(s) \begin{bmatrix} I & 0 \\ \Sigma_{22}(s) & \Sigma_{21}(s) \end{bmatrix}^{-1}. \tag{16}$$

Further, for any inner matrix $\Psi(s)$ the product $\Psi(s)\Theta^{\perp}(s)$ is a (J, J')-lossless complement as well:

$$J' \geq \Theta'^{*}(s) J \Theta'(s) \geq \Theta'^{*}(s) \text{ diag}[\Psi^{*}(s), I] \; J \text{ diag}[\Psi(s), I] \; \Theta'(s) \tag{17}$$

for all $Re(s) > 0$ and the equality relations are attained for all $s = j\omega$. It also takes a routine procedure to prove that for any $\Theta^{\perp}(s)$ there exists an inner $\Psi(s)$ such that $\Psi(s)\Theta(s)$ is stable. ∎

It is easy to obtain a state-space form for one of the (J, J')-lossless complements of $\Theta(s)$.

Lemma 4 *Assume that $\Theta(s)$ given by (7) is (J_{mr}, J^0_{pqr})-dissipative. Then*

$$\Theta^{\perp}(s) = \left[\begin{array}{c|c} A_\theta & B_\theta \\ \hline H_1 & 0_{k(p+q+r)} \\ -E_2 B_\theta^T P & E_2 \end{array}\right] \tag{18}$$

where $H_1^T H_1 = S$ and $E_2 := \begin{bmatrix} 0_{qp} & I_q & 0_{qr} \end{bmatrix}$ is its $(J_{(m+k+q)r}, J_{(p+k+q)r})$-lossless complement.

In the case when $\Theta^{\perp}(s)$ has exactly q rows (or $S = 0$ in (8)) then $\Theta(s)$ is said to be (J, J^0)-*semilossless*. This sub-class of (J, J^0)-dissipative matrices was introduced in (Baramov, 1998); it can be used for solving H^{∞} problems where $G(s)$ has no imaginary-axis zero, but need not be left-invertible at infinity.

The following is a converse to Lemma 3, which states that a (J, J^0)-dissipative matrix can be augmented to get a (J, J')-lossless one. Naturally, a (J, J')-lossless matrix can be truncated to get a (J, J^0)-dissipative one, provided it possesses a suitable structure at infinity.

Lemma 5 *Let $\Theta'(s)$ be a $(J_{mr}, J_p q)$-lossless matrix. Assume that there are integers k and q such that $D := \begin{bmatrix} 0 & I_{m+r-k-q} \end{bmatrix} \Theta'(\infty)$ satisfies $D^T J_{(m-k-q)r} D = J^0_{(p-q)qr}$. Then, the matrix $\Theta(s) := \begin{bmatrix} 0 & I_{m+r-k-q} \end{bmatrix} \Theta'(s)$ is $(J_{(m-k-q)r}, J^0_{(p-q)qr})$-dissipative.*

The following two lemmas give some important properties of (J, J^0)-dissipative matrices. Their proofs are obtained (via augmentation by (J, J')-lossless complements) from the known results on (J, J')-lossless systems.

Lemma 6 *Let $\Theta_1(s)$ be a (J_{mr}, J^0_{pqr})-dissipative and $\Theta_2(s)$ a $J_{(p+q)r}$-lossless transfer matrix such that $\Theta_2(\infty) = I_{p+q+r}$. Then, $\Theta_1(s)\Theta_2(s)$ is (J_{mr}, J^0_{pqr})-semilossless.*

Lemma 7 *Any (J_{mr}, J^0_{pqr})-dissipative matrix $\Theta(s)$ can be represented as $\Theta_1(s)\Theta_2(s)$, where $\Theta_1(s)$ is $(J_{mr}, J^0_{pqr})-$dissipative and stable, and $\Theta_2(s)$ is an anti-stable $J_{(p+q)r}$-lossless matrix.*

4 Solutions of the H_∞ problem for (J, J^0)-dissipative plants

(J, J^0)-dissipative matrices are a class of plants which satisfy Assumption 2 (but not necessarily Assumption 1) for which the complete solution of the H_∞ problem is presented first. It is trivial to find *a* solution of the H_∞ problem for them, as in the case of standard (J, J')-lossless plants. However, unlike the standard case, where the set of *all* H_∞ controllers can be easily described (stable matrices $Q(s)$ such that $\|Q(s)\|_\infty < 1$), characterizing all H_∞ controllers is more complicated now. The key role is played by the (J, J')-lossless complement $\Theta^\perp(s)$. Let $\Theta_1^\perp(s)$ and $\Theta_2^\perp(s)$ denote the parts of $\Theta^\perp$ corresponding to u and y, respectively. Denote

$$T_{\Theta Q}(s) := HMT(\Theta(s), Q(s)), \quad \text{and} \quad \Gamma(s) = \begin{bmatrix} \Gamma_1(s) \\ \Gamma_2(s) \end{bmatrix} := \begin{bmatrix} I_p & 0 \\ \Theta_1^\perp(s) & \Theta_2^\perp(s) \end{bmatrix} \begin{bmatrix} Q(s) \\ I_r \end{bmatrix} \tag{19}$$

Theorem 3 *Let $\Theta(s)$ be a (J, J^0)-dissipative system and let $\Theta^\perp(s)$ be its (J, J')-lossless complement. The controller $Q(s)$ renders $T_{\Theta Q}{}^\sim(j\omega)T_{\Theta Q}(j\omega) < I_r$ for all ω iff*

$$\Gamma^\sim(j\omega)J_{(p+q)r}\Gamma(j\omega) < I_r \qquad \forall\omega \tag{20}$$

Proof: Consider the (J, J')-lossless system $\Theta'(s)$ obtained by augmenting $\Theta(s)$ by $\Theta^\perp(s)$. For this system the following power balance holds:

$$\|z(j\omega)\|^2 + \|z^\perp(j\omega)\|^2 - \|w(j\omega)\|^2 - \|u(j\omega)\|^2 + \|y(j\omega)\|^2 = 0 \qquad \forall\omega \tag{21}$$

where $z^\perp(j\omega) := \Theta_1^\perp(j\omega)u(j\omega) + \Theta_2^\perp(j\omega)y(j\omega)$ Then, for all ω, there holds $\|z(j\omega)\|^2 - \|w(j\omega)\|^2 < 0$ if and only if $\|z^\perp(j\omega)\|^2 - \|u(j\omega)\|^2 + \|y(j\omega)\|^2 > 0$ which is equivalent to (20). ∎

Equation (20) has a clear state-space characterization. Let

$$\Gamma(s) = \left[\begin{array}{c|c} A_\Gamma & B_\Gamma \\ \hline C_\Gamma & D_\Gamma \end{array}\right], \qquad T_{\Theta Q}(s) := \left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array}\right]. \tag{22}$$

As is well known, the condition (20) is equivalent to

$$R := I - D_\Gamma^T J D_\Gamma = I - D_{K1}^T D_{K1} > 0. \tag{23}$$

and the existence of a solution to the equation

$$X(A_\Gamma+B_\Gamma R^{-1}D_\Gamma^T JC_\Gamma)+(A_\Gamma^T+C_\Gamma^T JD_\Gamma R^{-1}B_\Gamma^T)X+XB_\Gamma R^{-1}B_\Gamma^T X+C_\Gamma^T(J+JD_\Gamma R^{-1}D_\Gamma^T J)C_\Gamma=0 \tag{24}$$

This is true for $\Gamma(s)$ based on any (J,J')-lossless complement $\Theta^\perp(s)$. The following theorem states the necessary and sufficient conditions for $Q(s)$ to satisfy the condition of Theorem 3 and to internally stabilize the closed loop $HMT(\Theta(s),Q(s))$. Without loss of generality, we will choose the complement given by (18). A state-space form of $\Gamma(s)$ is then given by:

$$A_\Gamma := \begin{bmatrix} A_\theta & B_{\theta 1}C_Q \\ 0 & A_Q \end{bmatrix} \; B_\Gamma := \begin{bmatrix} B_{\theta 2} \\ B_Q \end{bmatrix} \; C_\Gamma := \begin{bmatrix} 0 & C_Q \\ 0 & H_1 \\ -E_2 B_\theta^T P & E_{21}C_Q \end{bmatrix} \; D_\Gamma := \begin{bmatrix} D_Q \\ 0 \\ E_{21}D_Q \end{bmatrix} \tag{25}$$

where $E_{21} = \begin{bmatrix} 0 & I_q \end{bmatrix}$.

Theorem 4 *Let $\Theta(s)$ be a $(J,J^0)-$semilossless matrix given by (7) and let $\Gamma(s)$ be given by (25). A controller $Q(s)$ solves the H_∞ problem for the plant $\Theta(s)$, if (23) holds and there exists a stabilizing solution X to (24) satisfying $X+\mathrm{diag}(P,0)\geq 0$ where $P\geq 0$ satisfies (8).*

PROOF: According to the Bounded Real Lemma there exists a matrix $Y\geq 0$ satisfying the equation

$$Y(A_T+B_T R^{-1}D_T^T C_T)+(A_T^T+C_T^T D_T R^{-1}B_T^T)Y+YB_T R^{-1}B_T^T Y+C_T^T(I+D_T R^{-1}D_T^T)C_T=0 \tag{26}$$

such that $A_T+B_T R^{-1}(D_T^T C_T+B_T^T Y)$ is a stable matrix. Now, substitute $Y:=X+\mathrm{diag}(P,0)$ into (26). After some routine algebraic manipulations using (8)(9), which are omitted here for space considerations, we arrive to the identity (24). This proves the necessity part. Sufficiency is proven by making these manipulations in the reversed order. ■

We can get some insight by applying the above theorem to the standard case where $\Theta(s)$ is (J,J')-lossless. Then we can set $E_2=0$, $E_{21}=0$ and $H_1=0$ (i.e. $\Theta^\perp(s)$ is a zero matrix). Then we get $X=diag(0_{nn},X_{22})$ and $Y=diag(P,X_{22})$ where $X_{22}\geq 0$. Hence, $Q(s)$ must be stable and $\|Q(s)\|_\infty<1$ as expected.

We have some necessary and sufficient conditions for a controller to solve the H_∞ control problem for a (J,J^0)-dissipative plant. We can show that, for instance, a stable $Q(s)$ such that $\|Q(s)\|_\infty<1$ satisfies these conditions. However, we still may not have a clear idea about what is the whole set of Q's like. We shall derive alternative and, hopefully, more intuitive conditions. Let us consider a transfer matrix $\Phi(s)$ which satisfies

$$\Phi(s)\begin{bmatrix} I_r \\ \Gamma_2(s) \end{bmatrix} = \Gamma_1(s) \tag{27}$$

This transfer matrix always exists and is not unique; it can take the following state-space form:

$$\Phi(s) = \begin{bmatrix} \Phi_1(s) & \Phi_2(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_\Gamma - LC_{\Gamma 2} & B_\Gamma - LD_{\Gamma 2} & L \\ \hline C_{\Gamma 1} & D_{\Gamma 1} & 0 \end{array}\right] \tag{28}$$

Here, L is a matrix of suitable dimension and $C_{\Gamma 1}$, $C_{\Gamma 2}$, $D_{\Gamma 1}$ and $D_{\Gamma 2}$ are subblocks of C_Γ and D_Γ of (22) corresponding to u and $z^\perp$, respectively. As follows from (19) and (27), we can express the controller $Q(s)$ of the plant $\Theta(s)$ in terms of $\Phi(s)$ as

$$Q(s) = [I - \Phi_2(s)\Theta_1^\perp(s)]^{-1}[\Phi_1(s) + \Phi_2(s)\Theta_2^\perp(s)]. \tag{29}$$

We shall find necessary and sufficient conditions on $\Phi(s)$ so that the above $Q(s)$ is an H_∞ controller of $\Theta(s)$. The following theorem is an analogue to Theorem 3 which uses $\Phi(s)$ instead of $\Gamma(s)$.

Theorem 5 *Let $\Theta(s)$ be (J, J^0)-dissipative and let $\Theta^\perp(s)$ be its (J, J')-lossless complement. The controller $Q(s)$ given by (29) renders $T_{\Theta Q}\tilde{}(j\omega)T_{\Theta Q}(j\omega) < I_r$ $\forall\omega$ iff $\Phi\tilde{}(j\omega)\Phi(j\omega) < I_{p+q}$ $\forall\omega$.*

Proof: Necessity. We need to show that there exist a matrix X satisfying

$$\begin{aligned} X(A_\Gamma - LC_{\Gamma 2} + B_\Gamma R^{-1}D_{\Gamma 1}C_{\Gamma 1}) + (A_\Gamma^T - C_{\Gamma 2}^T L^T + C_{\Gamma 1}^T D_{\Gamma 1}^T R^{-1} B_\Gamma^T)X + X[B_\Gamma R^{-1} B_\Gamma^T - \\ - B_\Gamma R^{-1} D_{\Gamma 2}^T L - LD_{\Gamma 2} B_\Gamma^T + L(I + D_{\Gamma 2} D_{\Gamma 2}^T)L^T]X + C_{\Gamma 1}^T(I + D_{\Gamma 1} R^{-1} D_{\Gamma 1}^T)C_{\Gamma 1} \end{aligned} = \tag{30}$$

Let us take the stabilizing solution to (24) for X. Assume that $\det(X) \neq 0$. Then $L := X^{-1}C_{\Gamma 2}^T$ will do the job. If X is singular, we can find a $X' > X$ which renders the left side of (24) strictly negative. Then for $L := X'^{-1}C_{\Gamma 2}^T$ we can find a matrix $X < X'$ solving (30).

Sufficiency. It follows from our assumption on $\Phi(s)$ that the inequality $\|z^\perp(j\omega)\|^2 + \|y(j\omega)\|^2 > \|u(j\omega)\|^2$ holds for all $y(j\omega)$ and $z^\perp(j\omega)$. Relation (20) is nothing else than this inequality for $z^\perp(j\omega) := \Theta_1^\perp(j\omega)u(j\omega) + \Theta_2^\perp(j\omega)y(j\omega)$. ■

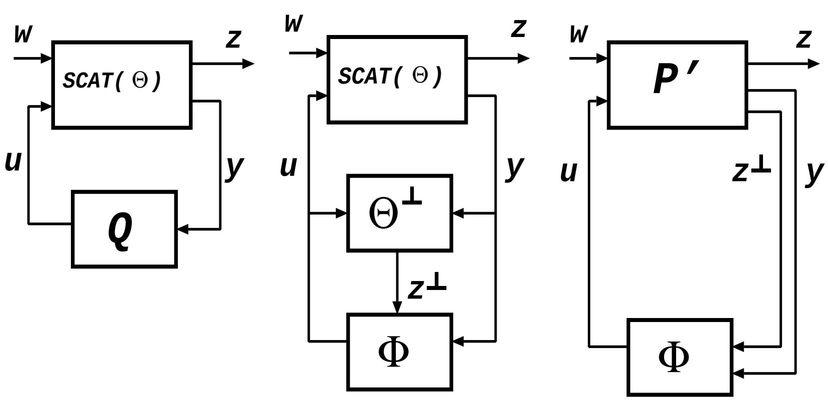

Figure 1: Three equivalent forms of the closed loop transfer matrix

This theorem addresses only the closed loop L_∞ norm and puts aside the stability issue which shall be discussed next. First, we shall consider only stable complements $\Theta^\perp(s)$. Let

$$P'(s) := \begin{bmatrix} 0 & I_{m+r} \\ I_{q+k} & 0 \end{bmatrix} SCAT(\begin{bmatrix} \Theta^\perp(s) \\ \Theta(s) \end{bmatrix})$$

This transfer matrix is, due to Lemma 1, innner. Now, taking into account (29), the following identity holds for the closed loop transfer matrix:

$$HMT(\Theta(s), Q(s)) = LFT(P'(s), \Phi(s)) \tag{31}$$

We note that the output $z^{\perp}$ originally added to Θ as an additional penalized output becomes a measurement output of $P'(s)$ along with y. Hence, $P'(s)$ does not have the chain-scattering representation. Further, it can be proven (for instance by standard state-space transformations) that the reduction in (31) involves no unstable cancellation. This is due to the fact that $\Theta^{\perp}(s)$ is stable. Now, we have an inner (although nonstandard) plant P'; according to Theorem 5, we need to consider only those controllers $\Phi(s)$ for it whose L_∞-norm is less than one. As follows from the *small gain theorem*, if $\Phi(s)$ is stable, then the loop $LFT(P'(s), \Phi(s))$ is also (internally) stable. Thus, we have a sufficient condition for a controller given by (29) to be the desired H_∞ controller for $\Theta(s)$: any stable $\Phi(s)$ satisfying $\|\Phi(s)\|_\infty < 1$ will do the job. This parametrizes most of the H_∞ controllers for $\Theta(s)$. For finding the rest, let us assume that there is a controller given by (29) where $\|\Phi(s)\|_\infty < 1$ but $\Phi(s)$ is not stable. Then, $LFT(P'(s), \Phi(s))$ cannot be internally stable. There is only one way to avoid the contradiction with the assumption that $HMT(\Theta(s), Q(s))$ is internally stable: the unstable cancellations in $LFT(P'(s), \Phi(s))$ are identical to the unstable cancellations in (29) which occur *inside the controller* $Q(s)$ and do not affect internal stability of $HMT(\Theta(s), Q(s))$. These ideas are formulated precisely in the following theorem:

Theorem 6 *Let $\Theta(s)$ be a (J, J^0)-dissipative system and let $\Theta^{\perp}(s)$ be a (J, J')-lossless complement of $\Theta(s)$. The controller $Q(s)$ given by (29) solves the H_∞ problem for $\Theta(s)$ iff $\Phi^{\sim}(j\omega)\Phi(j\omega) < I$ for all ω and makes the transfer matrix*

$$LFT(\begin{bmatrix} P'_a(s) \\ P'(s) \end{bmatrix}, \Phi(s)), \qquad P'_a := \begin{bmatrix} 0 & I_p \\ P'_{21}(s) & P'_{22}(s) \end{bmatrix}, \tag{32}$$

where $P'_{21}(s)$ and $P'_{22}(s)$ are subblocks of $P'(s)$ corresponding to measurement outputs y and $z^{\perp}$ (see Fig 2), (externally) stable.

5 (J, J^0)-dissipative factorization and H_∞ control

In this section we define the (J, J^0)-dissipative factorization and show its relevance for solving nonstandard H_∞ problems.

Definition 3 *A transfer matrix $G(s)$ is said to admit a (J, J^0)-dissipative factorization, if $G(s) = \Theta(s)\Pi(s)$ where $\Theta(s)$ is (J, J^0)-dissipative and $\Pi(s)$ is stable with stable and proper inverse.*

We now show that this factorization is indeed a natural extension of the (J, J')-lossless one in the sense that its existence is necessary and sufficient for solving the H_∞ problem for plants satisfying Assumption 2 (a substantially relaxed compared to Assumption 1).

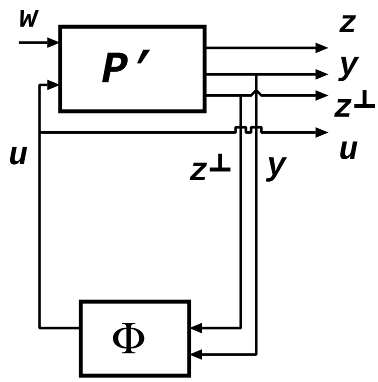

Figure 2: Augmented closed loop

Theorem 7 *Assume that the plant in the chain-scattering representation $G(s)$ given by (2) satisfies Assumption 2. Then there exists a solution of the H_∞ problem for $G(s)$ if and only if $G(s)$ admits a $(J_{mr}, J^0_{(p-q)qr})$-dissipative factorization, where $q = p + r - \mathrm{rank}(G(\infty))$.*

PROOF: Necessity. Let us assume that there exists an H_∞ controller $K(s)$ for the plant $G(s)$. Let us then consider the augmentation

$$G'(s) = \begin{bmatrix} G^\perp(s) \\ G(s) \end{bmatrix}$$

of $G(s)$ by adding s-rows such that *(i)* $G'(s)$ satisfies Assumption 1, *(ii)* $G^\perp(\infty)^T G(\infty) = 0$ and *(iii)* $K(s)$ is an H_∞ controller also for $G'(s)$. An augmentation satisfying conditions *(i)* and *(ii)* can always be found; it requires the number of rows of $G^\perp(s)$ $s \geq k + q$, where k is the number of $j\omega$-axis zeros of $G(s)$. It is clear that this augmentation need not change the closed loop dynamics. Moreover, getting rid of the undesired imaginary axis and infinite zeros can be done with arbitrarily small $G^\perp(s)$ (in the ∞-norm sense); hence, we can always find an $G'(s)$ satisfying all the conditions above. Then, due to the standard result on H_∞, see (Kimura, 1997), the $(J_{(m+s)r}, J_{(p+s)r})$-lossless factorization $G'(s) = \Theta'(s)\Pi(s)$ exists. Due to the orthogonality of $G(s)$ and $G^\perp(s)$, an (J, J')-lossless factor $\Theta(s)$ can be found which has the structure at infinity as required by Lemma 5. Then, truncating the first s rows of $\Theta'(s)$ yields a $(J_{mr}, J^0_{(p-q)qr})$-dissipative matrix $\Theta(s)$. $\Theta(s)\Pi(s)$ is then the desired factorization of $G(s)$. Proving sufficiency is straightforward. ∎

It follows from the proof of the above theorem that the (J, J^0)-dissipative factorization is highly nonunique and we can obtain an arbitrary number of pairs of factors of completely different dynamics. In contrast, the standard (J, J')-lossless factorization is unique up to a constant, J'-unitary factor U, which can be post-multiplied to the (J, J')-lossless factor $\Theta(s)$ and its inverse pre-multiplied to the unimodular factor $\Pi(s)$. One of the main results of this paper is that any fixed (J, J^0)-dissipative factorization yields the whole set of H_∞ controllers of $G(s)$. This result is based on the previous section and standard properties of chain-scattering representation which are taken from (Kimura, 1997) and summarized as follows:

Lemma 8 *Assume that the plant in the chain-scattering representation $G(s)$ given by (2) satisfying Assumption 2 admits a $(J_{mr}, J^0_{(p-q)qr})$-dissipative factorization $\Theta(s)\Pi(s)$. If $Q(s)$ solves the H_∞ problem for the plant $\Theta(s)$, then*

$$K(s) = HMT(\Pi(s)^{-1}, Q(s)) \tag{33}$$

is an H_∞ controller for $G(s)$. Moreover, any H_∞ controller of $G(s)$ has the representation (33) where $Q(s)$ solves the H_∞ problem for $\Theta(s)$.

Now, solving the H_∞ problem under Assumption 2 was shown to be equivalent to the (J, J^0)-dissipative factorization of the plant in the chain-scattering form. A state-space solution to this problem is available, but it is beyond the scope of this paper and will appear elsewhere. It is essentially similar to the result on (J, J^0)-semilossless factorization of (Baramov, 1998). It requires solving two parametrized Riccati equations. For the existence test, however, we need to solve two parameter-free Ricatti equations: one of reduced order, known from singular state-feedback control, and one of the standard H_∞ estimation problem.

6 Conclusions

A class of (J, J^0)-dissipative transfer matrices was introduced. Its relevance for class of singular H_∞ problems was demonstrated—we have obtain a new parametrization of solutions for a class of singular H_∞ control problem which provide a new, and we believe that also a useful insight into the structure of H_∞ control.

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