ON A CONJECTURE AND THE INTERNAL MODEL

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ABSTRACT

In the robust tracking problem with two-output plants it is shown that if the plant is unstable, the numerators corresponding to the two outputs cannot be unrelated. However, if the plant is stable, the two parts of the plant can be unrelated and in fact, the compensator which solves the problem incorporates an "inverse internal model" of the exogenous signal.

1. INTRODUCTION

The linear multivariable robust tracking problem has been addressed for over 25 years, having achieved a degree of maturity in the solution of the main issues. In the mid 70's the problem was studied, among others, in the state-space/matrix formulation by Davison (1976), in the state-space/geometric approach by Francis and Wonham (1975), in the state-space/Laplace transform by Ferreira (1976) and by Desoer and Wang (1980).

In the early / mid 80's the problem in the input–output /Laplace transform was solved by Vidyasagar (1985) for one-output plant, one-degree-of-freedom compensator, by Sugie and Yoshikawa (1986) for one-output plant, two-degree-of-freedom compensator and by Sugie and Vidyasagar (1989) for the general problem, namely, two-output plant, two-degree-of-freedom compensator

Most recently Howze and Bhattacharyya (1997) addressed the issue of the robustness with respect to perturbations of the compensator in the scalar problem with one-output plant, two-degree-of-freedom compensator.

The present paper addresses a conjecture made by Sugie and Vidyasagar (1989), namely, that the numerators of the two-output plant have to be related if the robust tracking problem is to have a solution.

In this paper, after the set-up of the problem in the following section, we show in the third section that indeed the problem has no solution with unstable plant if the numerators are unrelated, but in the fourth section it is shown that if the plant is stable, the problem does have a solution ,which is somehow surprising: the compensator must incorporate an "inverse internal model" of the exogenous signal.

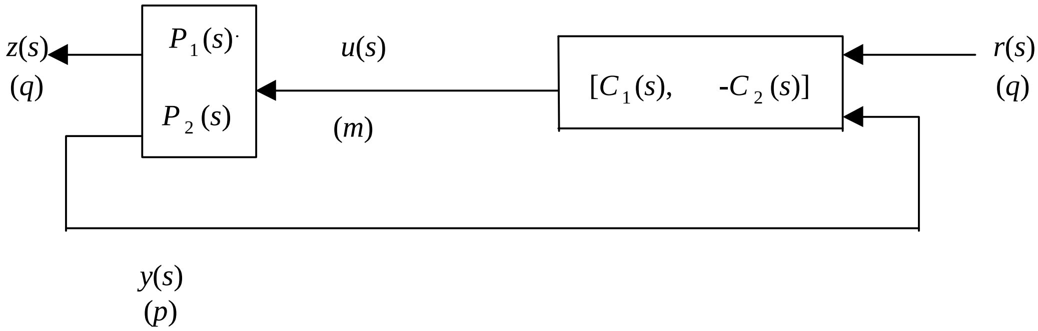
2. SETTING-UP THE PROBLEM

Notation and abbreviations: The set of proper and stable rational functions, a principle ideal domain (Vidyasagar, 1985), is denoted by **S**. The set of matrices with elements in **S** is denoted by **M**(**S**). **R** is the field of real numbers. Left coprime will be abbreviated by *l. c.* , right coprime will be abbreviated by *r. c.*

In the figure $z(s)$ and $r(s)$ are q-valued vectors, $u(s)$ is a m-valued vector and $y(s)$ is a p-valued vector.

$$P(s) = \begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix}$$ represents the given plant.

$C(s) = [\, C_1(s) \quad -C_2(s)]$ is the compensator to be designed.



Omitting henceforth the argument (s) when convenient, we have:

$$z = P_1 u\,, \quad y = P_2 u\,, \quad u = C_1 r - C_2 y\,.$$

P_1, P_2, C_1 and C_2 are proper rational matrices and have the appropriate dimensions. P_2 is assumed strictly proper for convenience in terms of well - posedness and because this is the case in most practical situations. This assumption might be dropped easily. The exogenous signal r is assumed proper. P_1 and P_2 are assumed to have full rank. All the factorizations in the paper are over **S**.

Let $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} D^{-1}$, a $r.\,c.$ factorization.

Let $C = [C_1 \quad -C_2] = \overline{D}_c^{-1} [\overline{N}_{c1} \quad -\overline{N}_{c2}]$, a $l.\,c.$ factorization.

We assume that the exogenous signal r has all its poles in the closed right complex plane; those are the relevant poles, since the modes corresponding to stable poles decay asymptotically to zero. This assumption is standard in the literature of the servo problem, but might be easily dropped.

$r = \overline{D}_r^{-1} \, \overline{N}_r \, r_0$,

where $\overline{D}_r$ is a known matrix, $\overline{N}_r$ need not be known, $\overline{D}_r$ and $\overline{N}_r$ are $l.\,c.$ and r_0 is an arbitrary vector of real numbers.

α_m will denote the largest invariant factor of $\overline{D}_r$.

We use the standard definition of closed loop stability. It is known (Desoer and Gündes, 1988) that if the closed loop is stable, $\overline{D_c}$, $\overline{N_{c2}}$, D and N_2 can be chosen, without loss of generality such that

$$\overline{D_c}\, D + \overline{N_{c2}}\, N_2 = I\,, \tag{1}$$

where I is the identity matrix.

Let H be the transfer function matrix between z and r. Asymptotic tracking is said to take place if and only if the loop is stable and

$$(I - H) r \in \mathbf{M}(\mathbf{S}) \qquad (2)$$

Straightforward calculations give, in view of (1):

$$H = N_1 N_{c1} . \qquad (3)$$

Perturb the plant, $P \longrightarrow P^*$. Let H^* be the resulting transfer matrix between z and r.
We say that C is a <u>robust tracking compensator</u> if and only if the perturbed closed loop is stable and

$$(I - H^*) r \in \mathbf{M}(\mathbf{S}) , \qquad (2^*)$$

whatever be the perturbation in a given set.

<u>Remark</u>: Recall (Vidyasagar, 1985) that if $\|F\|_\infty < 1$, with $F \in \mathbf{M}(\mathbf{S})$, then $I + F$ is unimodular.

We have next a technical result:

<u>Lemma</u>
Let Δ_D, $\Delta_N \in \mathbf{M}(\mathbf{S})$ be such that $I + \overline{D_c}\,\Delta_D + \overline{N}_{c2}\,\Delta_N$ is unimodular. Then there exist e_1, $e_2 \in \mathbf{R}$ and Q_1, $Q_2 \in \mathbf{M}(\mathbf{S})$ such that

$$(I + \overline{D_c}\,\Delta_D + \overline{N}_{c2}\,\Delta_N)^{-1} = I - e_1 \overline{D_c}\, Q_1 - e_2 \overline{N}_{c2} Q_2 \qquad (4)$$

Conversely, let e_1, $e_2 \in \mathbf{R}$ and Q_1, $Q_2 \in \mathbf{M}(\mathbf{S})$ be such that $I - e_1 \overline{D_c}\, Q_1 - e_2 \overline{N}_{c2} Q_2$ is unimodular. Then there exists Δ_D, $\Delta_N \in \mathbf{M}(\mathbf{S})$ such that (4) holds.
Moreover,

$$\Delta_D = e_1 Q_1 (I - e_1 \overline{D_c}\, Q_1 - e_2 \overline{N}_{c2} Q_2)^{-1} , \qquad (5a)$$

$$\Delta_N = e_2 Q_2 (I - e_1 \overline{D_c}\, Q_1 - e_2 \overline{N}_{c2} Q_2)^{-1} \qquad (5b)$$

$$e_1 Q_1 = \Delta_D (I + \overline{D_c}\,\Delta_D + \overline{N}_{c2}\,\Delta_N)^{-1} \qquad (5c)$$

$$e_2 Q_2 = \Delta_N (I + \overline{D_c}\,\Delta_D + \overline{N}_{c2}\,\Delta_N)^{-1} \qquad (5d)$$

Proof:
From (5a) and (5b), we have:

$I + \overline{D_c}\,\Delta_D + \overline{N}_{c2}\,\Delta_N = I + \overline{D_c}\,e_1 Q_1 (I - e_1 \overline{D_c}\,Q_1 - e_2 \overline{N}_{c2} Q_2)^{-1} + \overline{N}_{c2}\,e_2 Q_2 (I - e_1 \overline{D_c}\,Q_1 - e_2 \overline{N}_{c2} Q_2)^{-1} = (I - e_1 \overline{D_c}\,Q_1 - e_2 \overline{N}_{c2} Q_2 + e_1 \overline{D_c}\,Q_1 + e_2 \overline{N}_{c2} Q_2)\,(I - e_1 \overline{D_c}\,Q_1 - e_2 \overline{N}_{c2} Q_2)^{-1} = (I - e_1 \overline{D_c}\,Q_1 - e_2 \overline{N}_{c2} Q_2)^{-1}$,

which is (4).
By the same token, (4) is obtained from (5c) and (5d) also.
∇

Remark:
Q_1 , $Q_2 \in \mathbf{M}(\mathbf{S})$ may be chosen arbitrarily and yet $I - e_1 \overline{D_c}\,Q_1 - e_2 \overline{N}_{c2} Q_2$ will be unimodular, provided $|e_1|$ and $|e_2|$ are sufficiently small. Indeed,

$$\| e_1 \overline{D_c}\,Q_1 + e_2 \overline{N}_{c2} Q_2 \|_\infty \le \| e_1 \overline{D_c}\,Q_1 \|_\infty + \| e_2 \overline{N}_{c2} Q_2 \|_\infty ,$$
$$\|e_1 \overline{D_c}\,Q_1\|_\infty \le |e_1|\,\|\overline{D_c}\|_\infty \|Q_1\|_\infty , \qquad \|e_2 \overline{N_{c2}}\,Q_2\|_\infty \le |e_2|\,\|\overline{N_{c2}}\|_\infty \|Q_2\|_\infty .$$

So, according to the Remark before the Lemma, choose $|e_1|$, $|e_2|$ small enough, establishing the claim.
∇

Let Δ_N and $\Delta_D \in \mathbf{M}(\mathbf{S})$ be the perturbations of N_2 and D , respectively. The admissible Δ_N and Δ_D are such that $\overline{D_c}(D + \Delta_D) + \overline{N}_{c2}(N_2 + \Delta_N)$ is unimodular. It is clear that these perturbations are arbitrary, provide that the norms of Δ_N and Δ_D are sufficiently small.

In their elegant and important paper, Sugie and Vidyasagar (1989) assume that N_1 and N_2 are related by

$$N_1(s) = L(s)\; N_2(s) , \tag{6}$$

where the zeros and poles of L are disjoint from those of $\overline{D_r}$. Notice that L can be improper and unstable (but of course N_1 is proper and stable, by definition). This relationship between N_1 and N_2 is a rather mild one. The authors call it “mode readability”, a weaker condition than “readability” , assumed by Davison (1976) and Francis and Wonham (1975), in which L is constant.
Sugie and Vidyasagar allow perturbations of L even though not arbitrary. We omit here the class of allowed perturbations of L for the sake of brevity, remitting it to that paper.
Sugie and Vidyasagar make the following <u>conjecture</u>: the relationship (6) is necessary for robust tracking.
We show in the next section that if the plant is unstable and the problem is to have a solution, N_1 and N_2 cannot be unrelated, confirming somehow Sugie and Vidyasagar’s conjecture. In the fourth section we show that if the plant is stable, the problem does have a solution when P_1 is

held fixed. In this case, we will see that the compensator must incorporate an inverse internal model of the exogenous signal.

3. PERTURBING THE UNSTABLE PLANT

Theorem 1:
Perturb D and N_2 "arbitrarily" (in the sense defined above), while maintaining N_1 fixed. Then the robust tracking problem has no solution.

Proof:
Perturb $D \rightarrow D + \Delta_D$ and fix N_1 and N_2. Then, it is easy to obtain

$$z = N_1 (I + \overline{D_c} \Delta_D)^{-1} \overline{N}_{c1} r = N_1 (I - e_1 \overline{D_c} Q_1) \overline{N}_{c1} r, \quad \text{in view of (4).}$$

Hence,

$$\begin{aligned} e = r - z &= [I - N_1 (I - e_1 \overline{D_c} Q_1) \overline{N}_{c1}] r \\ &= (I - N_1 \overline{N}_{c1}) r + N_1 e_1 \overline{D_c} Q_1 \overline{N}_{c1} r . \end{aligned}$$

Now, in view of (2) and (3) asymptotic tracking implies

$$N_1 e_1 \overline{D_c} Q_1 \overline{N}_{c1} r \in \mathbf{M}(\mathbf{S}).$$

And from the definition of r we get

$$N_1 \overline{D_c} Q_1 \overline{N}_{c1} \overline{D_r}^{-1} \in \mathbf{M}(\mathbf{S}). \tag{7}$$

Now, in view of (2) and (3) it is clear that $\overline{N}_{c1}$ and $\overline{D_r}$ are $r.\,c.$
Let $\overline{N}_{c1} \overline{D_r}^{-1} =: A^{-1} B$, a $l.\,c.$ factorization. It is clear that A and $\overline{D_r}$ have the same invariant factors.
Then, in view of (7), we have

$$N_1 \overline{D_c} Q_1 A^{-1} \in \mathbf{M}(\mathbf{S}). \tag{8}$$

Let S_A be the Smith form of A and let U and V be unimodular matrices such that $A = U S_A V$. Define $\overline{Q} = Q_1 V^{-1}$. Then from (8)

$$N_1 \overline{D_c} \overline{Q} S_A^{-1} \in \mathbf{M}(\mathbf{S}). \tag{9}$$

Let α_j be the invariant factors of A, $j \in \boldsymbol{m}$, where $\boldsymbol{m} = \{1,2,...m\}$.
Let n_j, $j \in \boldsymbol{p}$, be the columns of $N_1 \overline{D_c}$.
Let q_{ki} be the elements of $\overline{Q}$. Choose $\overline{Q}$ such that

$$q_{jm} = 1, \quad q_{ki} = 0 \quad \forall (k, i) \neq (j, m).$$

Then straightforward calculations from (9) give $n_j \alpha_m^{-1} \in \mathbf{M}(\mathbf{S})$, $\forall j \in \boldsymbol{p}$, or,

$$N_1 \overline{D_c} \alpha_m^{-1} \in \mathbf{M}(\mathbf{S}). \tag{10}$$

Now perturb $N_2 \rightarrow N_2 + \Delta_N$, fixing D and N_1. From the block diagram we obtain

$z = N_1 (I + \overline{N_{c2}} \Delta_N)^{-1} \overline{N}_{c1} r = N_1 (I - e_2 \overline{N_{c2}} Q_2) \overline{N}_{c1} r$,
in view of (4).
Hence,

$$e = (I - N_1 \overline{N}_{c1}) r + N_1 e_2 \overline{N_{c2}} Q_2 \overline{N}_{c1} r.$$

So, robust tracking implies, in view of (2) and (3),

$$N_1 \overline{N_{c2}} Q_2 \overline{N}_{c1} r \in \mathbf{M(S)} \Rightarrow N_1 \overline{N_{c2}} Q_2 \overline{N}_{c1} \overline{D_r}^{-1} \in \mathbf{M(S)}.$$

Defining matrices A, S_A as above (after (8)) and choosing an appropriate matrix in the same way as $\overline{Q}$, we obtain

$$N_1 \overline{N_{c2}} \alpha_m^{-1} \in \mathbf{M(S)} . \tag{11}$$

From (10) and (11), we have

$$\alpha_m^{-1} N_1 [\overline{D_c} , \overline{N_{c2}}] \in \mathbf{M(S)}.$$

But from (1) $\overline{D_c}$ and $\overline{N_{c2}}$ are *l.c.*, hence the last implies $\alpha_m^{-1} N_1 \in \mathbf{M(S)}$.

Hence there exists $N_{11} \in \mathbf{M(S)}$ such that $N_1 = a_m N_{11}$.

But from (2) and (3), there should exist $W \in \mathbf{M(S)}$ such that

$$N_{11} \overline{N}_{c1} a_m + W D_r = I .$$

And it is clear that there is no solution for this equation in $\overline{N}_{c1}$ and W, since $a_m I$ and D_r are not *r. c.*, proving the theorem.
∇

4. SOLUTION OF THE PROBLEM WITH STABLE PLANTS, P_1 AND P_2 UNRELATED

We have now $N_1 = P_1$, $N_2 = P_2$, $D = I$.
Perturb $P_2 \rightarrow P_2 + \Delta$, $\Delta \in \mathbf{M(S)}$, P_1 is fixed.

<u>Theorem 2</u>:
Let P_1 be fixed and P_2 arbitrarily perturbed (in the sense defined above). Then C is a robust tracking compensator if and only if it stabilizes the closed loop system and

a) $(I - P_1 \overline{N}_{c1}) \overline{D_r}^{-1} \in \mathbf{M(S)}$

b) $P_1 \overline{N_{c2}} \alpha_m^{-1} \in \mathbf{M(S)}$.

Proof:
The perturbed transfer matrix between z and r is

$$H^* = P_1 (I + \overline{N_{c2}} \Delta)^{-1} \overline{N}_{c1} = P_1 (I - \varepsilon \overline{N_{c2}} Q) \overline{N}_{c1} ,$$

in view of (4).
Hence, asymptotic tracking takes place if and only if the closed loop is stable and

$$[\, I - P_1 (I - \varepsilon\, \overline{N_{c2}}\; Q)\, \overline{N}_{c1}]\; \overline{D_r}^{\,-1} \in \mathbf{M}(\mathbf{S}) . \tag{12}$$

From (2) and (3) it is clear that condition a) of the theorem is necessary for asymptotic tracking with the nominal plant ($\Delta = 0$). Then from (12) and a) we must have:

$$P_1\, \overline{N_{c2}}\; Q\, \overline{N}_{c1}\; \overline{D_r}^{\,-1} \in \mathbf{M}(\mathbf{S}) .$$

Now proceed as in the development after (8), defining A , S_A and so on. We obtain likewise:

$$P_1\, \overline{N_{c2}}\; \alpha_{\mathrm{m}}^{-1} \in \mathbf{M}(\mathbf{S}) .$$

The sufficiency of the conditions follows at once from (12). ∇

We have next the condition for the solvability of the problem. Recall (Wolovich, 1978) that two matrices A and B are externally skew prime (*e.s.p.*) iff there exist matrices X and Y such that $A X + Y B = I$.

Theorem 3:
Assume that the plant is stable. Fix P_1 and perturb P_2 as in theorem 2. Then there exists a robust tracking compensator C if and only if P_1 and D_{r} are *e.s.p.*

Proof:
The necessity of the condition is a direct consequence of a) of theorem 2. For the sufficiency, find $\overline{N}_{c1}$ such that a) of theorem 1 is satisfied. Next do $\overline{N_{c2}} = \overline{\overline{N_{c2}}}\; \alpha_{\mathrm{m}}$ and find $\overline{\overline{N_{c2}}}$ and $\overline{D_c}$ such that the closed loop is stabilized, which can always be found whatever be P_2 stable, since I and $\alpha_{\mathrm{m}}\; P_2$ are right coprime. ∇

5. CONCLUDING REMARKS

1. The problem handled in the last section has no solution with one-degree-of-freedom feedback compensator. Indeed, conditions a) and b) of theorem 1 contradict each other if $\overline{N}_{c1} = \overline{N_{c2}}$.

2. It is important to note that with our assumptions, the problem of the last section has a simple solution without feedback, namely, $\overline{N_{c2}} = 0$. So, it is clear that it would make sense to implement the feedback solution only if it would be necessary for other reasons. This would be the case if there were disturbances to be rejected affecting, say, the plant, or between the plant and the compensator.

3. The result presented in the last section contrasts evidently with the so called <u>internal model principle</u>: see Francis and Wonham (1975) and Davidson (1976) and, more recently, Sugie and Vidyasagar (1989). According to the internal model principle, in order to obtain robust tracking, the compensator must incorporate a replicated internal model of the exogenous signal.

Now, in condition b) of our theorem 2, we have an “inverse internal model” in the sense that the exogenous poles affect the numerator of the feedback channel of the compensator, not the denominator of it.

The apparent contradiction is solved when we consider the assumptions previous to the results. In our paper we assume that P_1 is fixed, while in the three papers mentioned above a relationship is assumed between P_1 and P_2, namely, $P_1 = L\,P_2$. In the first two papers quoted above, L is a fixed matrix of real numbers and then it is said that z is “readable” from y. In the paper by Sugie and Vidyasagar L is a rational matrix, not necessarily proper, and whose zeros and poles are disjoint from the exogenous modes; besides, L is perturbable in a restricted sense and the authors call z “mode readable” from y. It is clear that mode readability is a weaker condition than readability. Sugie and Vidyasagar believe that mode readability is a <u>necessary</u> condition for robust tracking. We assumed that P_1 is fixed, while P_2 is arbitrarily perturbed, so in our assumption there is no mode readability and a fortiori no readability. And it was proved that neither one is necessary if the plant is stable.

4. It might be added that P_1 and P_2 being not related does not mean necessarily that they refer to two distinct plants. So, for example, if

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix},\quad a_1 \text{ and } a_2 < 0,\quad B = \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix},\quad C = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix},$$

we obtain $$P_1(s) = \begin{bmatrix} \dfrac{b_1\,c_1}{s - a_1}, & \dfrac{b_2\,c_1}{s - a_1} \end{bmatrix},\qquad P_2(s) = \begin{bmatrix} 0, & \dfrac{b_3\,c_2}{s - a_2} \end{bmatrix}.$$

5. A practical situation of theorem 2 might occur, say, when P_1 would refer to the digital part, so in most cases virtually unperturbable, of a plant.

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