Proceedings of the 7th Mediterranean Conference on Control and Automation (MED99) Haifa, Israel - June 28-30, 1999

Some New Results in Theory of Controllability*

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Abstract

The new necessary and sufficient conditions, formulated in terms of convergence of a certain sequence of operators involving the resolvent of the negative of the controllability operator, are found for deterministic linear stationary control systems to be completely and approximately controllable, respectively. These conditions are applied to study the S_T-*controllability* (that is a property of attaining for the time T an arbitrarily small neighborhood of each point in the state space with a probability arbitrarily near to one) and the C_T-*controllability* (that is the S_T-controllability fortified with some uniformity) of stochastic systems. It is shown that a partially observable linear stationary control system with an additive Gaussian white noise disturbance is S_T-controllable (C_T-controllable) for each $T > 0$ if and only if its deterministic part is approximately (completely) controllable for each time $T > 0$.

1 Introduction

In this paper we present some new results concerning theory of controllability for deterministic as well as for stochastic systems.

Theory of controllability originates from the famous work (Kalman, 1960) where the concept of complete controllability was defined for finite dimensional deterministic linear systems and the rank condition for them was proved. The natural extension of the concept of complete controllability to infinite dimensional systems is too strong for many of them. Therefore, the concept of approximate controllability was defined as a weakened version of the complete controllability. A discussion of the concepts of controllability for deterministic systems the reader can find in (Curtain and Zwart, 1995; Bensoussan *et al.*, 1993; Zabczyk, 1992; Curtain and Pritchard, 1978; Balakrishnan, 1976). Recently, the new necessary and sufficient conditions for the complete and approximate controllabilities were obtained in (Bashirov and Mahmudov, 1998). These conditions are called the resolvent conditions and they are discussed in this paper.

*The results presented in this report are in the main obtained in (Bashirov and Kerimov, 1997; Bashirov and Mahmudov, 1998)

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The natural extension of the complete and approximate controllability concepts to stochastic control systems has no meaning. Therefore, there is a need in further weakening of these concepts in order to extend them to stochastic control systems. Some attempts in this direction are made in (Synahara *et al.*, 1974). In (Bashirov and Hajiyev, 1983 and 1984) an approach based on separation was suggested to define and to study the controllability of stochastic systems and in (Bashirov, 1996; Bashirov and Kerimov, 1997) the concepts of S_T- and C_T-controllability were defined for stochastic systems. Briefly, an S_T-controllable stochastic control system can attain for the time T an arbitrarily small neighborhood of each point in the state space with probability arbitrarily near to one. The C_T-controllability is the S_T-controllability fortified with some uniformity. In this paper we discuss the S_T- and C_T-controllabilities as well.

2 Notation

In this paper X and Y are real separable Hilbert spaces. $\mathbf{R}^k$ denotes the k-dimensional real Euclidean space. As usual, $\mathbf{R}^1 = \mathbf{R}$. The closure of the set D is denoted by $\overline{D}$. The space of all linear bounded operators from X to Y is denoted by $\mathcal{L}(X,Y)$. The brief notation $\mathcal{L}(X) = \mathcal{L}(X,X)$ is used as well. A^* denotes the adjoint of the operator A. The trace of the operator A is denoted by $\operatorname{tr} A$. If $A \in \mathcal{L}(X)$ is self-adjoint and $\langle h, Ah\rangle \geq 0$ (respectively, $\langle h, Ah\rangle \geq c\|h\|^2$, where c =const.> 0) for all $h \in X$, then we write $A \geq 0$ (respectively, $A > 0$), where $\langle\cdot\,,\cdot\rangle$ is an inner product and $\|\cdot\|$ is a norm. For $A \geq 0$, the square root of A is denoted by $A^{1/2}$. The symbol I denotes an identity operator. A zero operator, a zero vector and the number zero are denoted by 0 being clear which is meant from the context.

Always it is supposed that two time moments are given. The initial time moment is identified with zero and it is fixed. The terminal one is denoted by T ($T > 0$) and it is considered as variable. $L_2(0,T;X)$ denotes the space of equivalence classes of all Lebesque measurable and square integrable with respect to the Lebesgue measure functions from $[0,T]$ to X. As usual, we use the brief notation $L_2(0,T) = L_2(0,T;\mathbf{R})$. The notation Δ_T is used for the triangular set $\{(t,s) : 0 \leq s \leq t \leq T\}$. $B_2(\Delta_T, \mathcal{L}(X,Y))$ denotes the space of all $\mathcal{L}(X,Y)$-valued functions on Δ_T that are strongly measurable and square integrable with respect to the Lebesgue measure on Δ_T.

All integrals of vector-valued functions are considered in the Bochner sense. For probability, for expectation and for conditional expectation, the notations $\mathbf{P}$, $\mathbf{E}$ and $\mathbf{E}(\cdot\,|\cdot\,)$ are used, respectively. $\operatorname{cov}(x,y)$ is the covariance operator of the random variables x and y. The brief notation $\operatorname{cov} x$=$\operatorname{cov}(x,x)$ is used as well. The integrals of operator-valued functions (except stochastic integrals) are in the strong Bochner sense.

3 Main definitions

Consider a deterministic or stochastic control system on the finite time interval $[0,T]$ with $T > 0$. Let x_T^u be its (random or not) state value at time T corresponding to the control u taken from the set of admissible controls U_{ad}. If the considered control system is stochastic, then by $\mathcal{F}_T^u$ we denote the smallest σ-algebra generated by the observations on the time interval $[0,T]$ corresponding to the control u. Suppose that X is the state space. Introduce the set

$$D(T) = \{x_T^u : u \in U_{\text{ad}}\}. \tag{1}$$

Definition 1. Given $T > 0$, a deterministic control system will be called

(a) D_T^c-*controllable* if $D(T) = X$;
(b) D_T^a-*controllable* if $\overline{D(T)} = X$.

It is clear that the D_T^c- and D_T^a-controllabilities are the well-known complete and approximate controllabilities for deterministic control systems, respectively. Originally, the D_T^c-controllability was introduced in (Kalman, 1960) as a concept for finite dimensional deterministic control systems. The natural extension of this concept to infinite dimensional control systems is too strong for many of them. Therefore, the D_T^a-controllability was introduced as a weakened version of the D_T^c-controllability.

The natural extension of the complete and approximate controllability concepts to stochastic control systems is meaningless. Therefore, there is a need in further weakening of these concepts in order to extend them to stochastic control systems.

Given $T > 0$, $0 \le \varepsilon < \infty$ and $0 \le p \le 1$, introduce the sets

$$S(T,\varepsilon,p) = \{h \in X : \exists u \in U_{\text{ad}} \;\; \mathbf{P}(\|\mathbf{E}(x_T^u|\mathcal{F}_T^u) - h\|^2 > \varepsilon) \le 1-p\} \tag{2}$$

and

$$C(T,\varepsilon,p) = \{h \in X : \exists u \in U_{\text{ad}} \;\; h = \mathbf{E}x_T^u \text{ and } \mathbf{P}(\|\mathbf{E}(x_T^u|\mathcal{F}_T^u) - h\|^2 > \varepsilon) \le 1-p\}. \tag{3}$$

The following definitions will be used as a step in discussing the main concepts of controllability for stochastic control systems. Given $T > 0$, $\varepsilon \ge 0$ and $0 \le p \le 1$, a stochastic control system will be called

(a) $S_{T,\varepsilon,p}^c$-*controllable* if $S(T,\varepsilon,p) = X$;
(b) $S_{T,\varepsilon,p}^a$-*controllable* if $\overline{S(T,\varepsilon,p)} = X$;
(c) $C_{T,\varepsilon,p}^c$-*controllable* if $C(T,\varepsilon,p) = X$;
(d) $C_{T,\varepsilon,p}^a$-*controllable* if $\overline{C(T,\varepsilon,p)} = X$;
(e) $S_{T,\varepsilon,p}^0$-*controllable* if $0 \in S(T,\varepsilon,p)$.

Geometrically, the $S_{T,\varepsilon,p}^c$-controllability ($S_{T,\varepsilon,p}^a$-controllability) can be interpreted as follows. If a control system with the initial state x_0 is $S_{T,\varepsilon,p}^c$-controllable ($S_{T,\varepsilon,p}^a$-controllable), then with probability not less than p it can pass from x_0 for the time T into the $\sqrt{\varepsilon}$-neighborhood of an arbitrary point in the state space (in a set that is dense in the state space). The interpretation of the $C_{T,\varepsilon,p}^c$- and $C_{T,\varepsilon,p}^a$-controllabilities differs from the same of the $S_{T,\varepsilon,p}^c$- and $S_{T,\varepsilon,p}^a$-controllabilities since among the controls, with the help of which the $\sqrt{\varepsilon}$-neighborhood of any point h is achieved, there exists one with property that the expectation of the state at the time T, corresponding to this control, coincides with h. Obviously, a $C_{T,\varepsilon,p}^c$-controllable ($C_{T,\varepsilon,p}^a$-controllable) control system is $S_{T,\varepsilon,p}^c$-controllable ($S_{T,\varepsilon,p}^a$-controllable), but the converse is not true.

Smaller ε is and larger p is for a control system, better controllable it is, i.e. it is possible to hit into a smaller neighborhood with a higher probability. One can observe that for any $T > 0$, all control systems are $S_{T,\varepsilon,p}^c$-, $S_{T,\varepsilon,p}^a$-, $C_{T,\varepsilon,p}^c$- and $C_{T,\varepsilon,p}^a$-controllable with $\varepsilon \ge 0$ and $p = 0$ or $\varepsilon = \infty$ and $0 \le p \le 1$ if we admit ∞ as a value for ε. At the same time it is clear that a D_T^c-controllable (D_T^a-controllable) deterministic system is $S_{T,0,1}^c$- and $C_{T,0,1}^c$-controllable ($S_{T,0,1}^a$- and $C_{T,0,1}^a$-controllable) with parameters $\varepsilon = 0$ and $p = 1$, since for deterministic systems, $D(T) = S(T,0,1) = C(T,0,1)$. Also, each kind of controllability, mentioned above, with a smaller ε and a greater p implies the same kind of controllability with a greater ε and a smaller p.

Summarizing, we can give the following easy necessary and sufficient conditions for the D_T^c- and D_T^a-controllabilities.

Proposition 2. *Given $T > 0$, for a deterministic control system, the following conditions are equivalent*:

(a) *the D^c_T-controllability;*
(b) *the $S^c_{T,\varepsilon,p}$-controllability for all $\varepsilon \ge 0$ and for all $0 \le p \le 1$;*
(c) *the $C^c_{T,\varepsilon,p}$-controllability for all $\varepsilon \ge 0$ and for all $0 \le p \le 1$.*

Proposition 3. *Given $T > 0$, for a deterministic control system, the following conditions are equivalent:*
(a) *the D^a_T-controllability;*
(b) *the $S^a_{T,\varepsilon,p}$-controllability for all $\varepsilon \ge 0$ and for all $0 \le p \le 1$;*
(c) *the $C^a_{T,\varepsilon,p}$-controllability for all $\varepsilon \ge 0$ and for all $0 \le p \le 1$.*

Excepting the limit values $\varepsilon = 0$ and $p = 1$ from the above mentioned necessary and sufficient conditions of the complete and approximate controllabilities, one can obtain the weakened versions of these concepts. For a moment call a given stochastic control system to be
(a) S^c_T*-controllable* if it is $S^c_{T,\varepsilon,p}$-controllable for all $\varepsilon > 0$ and for all $0 \le p < 1$;
(b) S^a_T*-controllable* if it is $S^a_{T,\varepsilon,p}$-controllable for all $\varepsilon > 0$ and for all $0 \le p < 1$;
(c) C^c_T*-controllable* if it is $C^c_{T,\varepsilon,p}$-controllable for all $\varepsilon > 0$ and for all $0 \le p < 1$;
(d) C^a_T*-controllable* if it is $C^a_{T,\varepsilon,p}$-controllable for all $\varepsilon > 0$ and for all $0 \le p < 1$.

The following proposition shows that the concepts of S^c_T- and S^a_T-controllabilities are equivalent.

Proposition 4. *Given $T > 0$, a stochastic control system is S^a_T-controllable if and only if it is S^c_T-controllable.*

Proof. The sufficiency is obvious. For the necessity, suppose that a given stochastic control system is $S^a_{T,\varepsilon,p}$-controllable for all $\varepsilon > 0$ and $0 \le p < 1$. Let $S(T,\varepsilon,p)$ be the set (2) corresponding to this system. We have $\overline{S(T,\varepsilon,p)} = X$ for all $\varepsilon > 0$ and $0 \le p < 1$, where X is the state space. We have to show that the stronger condition $S(T,\varepsilon,p) = X$ for all $\varepsilon > 0$ and $0 \le p < 1$ holds. Fix arbitrary $\varepsilon_0 > 0$, $0 \le p_0 < 1$ and $h \in X$. Since $\overline{S(T,\varepsilon,p)} = X$ for all $\varepsilon > 0$ and $0 \le p < 1$, there is $h_0 \in S(T,\varepsilon_0/4,p_0)$ such that $\|h_0 - h\|^2 \le \varepsilon_0/4$. At the same time, since $h_0 \in S(T,\varepsilon_0/4,p_0)$, there exists $u \in U_{\mathrm{ad}}$ with

$$\mathbf{P}\{\|\mathbf{E}(x^u_T|\mathcal{F}^u_T) - h_0\|^2 > \varepsilon_0/4\} \le 1 - p_0.$$

Hence, for this $u \in U_{\mathrm{ad}}$, we have

$$\begin{aligned}
\mathbf{P}\{\|\mathbf{E}(x^u_T|\mathcal{F}^u_T)-h\|^2 > \varepsilon_0\} &\le \mathbf{P}\{\|\mathbf{E}(x^u_T|\mathcal{F}^u_T) - h_0\| + \|h_0 - h\| > \sqrt{\varepsilon_0}\} \\
&\le \mathbf{P}\{\|\mathbf{E}(x^u_T|\mathcal{F}^u_T) - h_0\| + \sqrt{\varepsilon_0}/2 > \sqrt{\varepsilon_0}\} \\
&= \mathbf{P}\{\|\mathbf{E}(x^u_T|\mathcal{F}^u_T) - h_0\|^2 > \varepsilon_0/4\} \\
&\le 1 - p_0.
\end{aligned}$$

Thus, $h \in S(T,\varepsilon_0,p_0)$. We obtain $S(T,\varepsilon_0,p_0) = X$ for all $\varepsilon_0 > 0$ and for all $0 \le p_0 < 1$ that proves the proposition.

Also, it will be shown (see Proposition 27) that for partially observable linear stationary control systems with an additive Gaussian white noise disturbance the concept of C^a_T-controllability is equivalent to the concept of S^c_T- or S^a_T-controllability. So, we can define two basic and one additional concepts of controllability for stochastic systems.

Definition 5. Given $T > 0$, a stochastic control system will be called
(a) S_T*-controllable* if it is $S^c_{T,\varepsilon,p}$-controllable (or, equivalently, $S^a_{T,\varepsilon,p}$-controllable) for all $\varepsilon > 0$ and for all $0 \le p < 1$;
(b) C_T*-controllable* if it is $C^c_{T,\varepsilon,p}$-controllable for all $\varepsilon > 0$ and for all $0 \le p < 1$;
(c) S^0_T*-controllable* if it is $S^0_{T,\varepsilon,p}$-controllable for all $\varepsilon > 0$ and for all $0 \le p < 1$.

Geometrically, the S_T-controllability can be interpreted as follows: an S_T-controllable control system can attain for the time T an arbitrarily small neighborhood of each point in the state space with a probability arbitrarily near to one. The C_T-controllability is the S_T-controllability fortified with some uniformity. The S_T^0-controllability is useful in discussing S_T- and C_T-controllabilities.

In order to interprete the S_T-, C_T- and S_T^0-controllabilities in (Mahmudov and Denker, 1999) the sets

$$S(T) = \bigcap S(T, \varepsilon, p) \text{ and } C(T) = \bigcap C(T, \varepsilon, p),$$

where the intersections are taken over all $\varepsilon > 0$ and all $0 \le p < 1$, are introduced. With these sets a stochastic control system is

(a) S_T-controllable if and only if $S(T) = X$;
(b) C_T-controllable if and only if $C(T) = X$;
(c) S_T^0-controllable if and only if $0 \in S(T)$.

Then Proposition 4 easily follows from the fact that for any control system, $S(T)$ is a closed set in X and, hence, the conditions $S(T) = X$ and $\overline{S(T)} = X$ are equivalent. Also, Proposition 2 and Proposition 3 are consequences of the fact that for a deterministic control system

$$D(T) = S(T, 0, 1) = C(T, 0, 1) = \bigcap S(T, \varepsilon, p) = \bigcap C(T, \varepsilon, p),$$

where the intersections are taken over all $\varepsilon \ge 0$ and all $0 \le p \le 1$.

Finally, notice that the abbreviations D, S, C, c and a in the previously introduced controllability concepts mean deterministic, stochastic, combined, complete and approximate, respectively.

4 Description of the system

We will examine the S_T- and C_T-controllabilities of the partially observable linear control system

$$\begin{cases} dx_t^u = (Ax_t^u + Bu_t + f_t)\, dt + d\varphi_t, \ 0 < t \le T, \ x_0^u = x_0, \\ d\xi_t^u = Cx_t^u\, dt + d\psi_t, \ 0 < t \le T, \ \xi_0^u = 0, \end{cases} \tag{4}$$

where x, u and ξ are the state, control and observation processes. Under the set U_{ad} of addmissible controls we consider the set of all controls u in the linear feedback form

$$u_t = \bar{u}_t + \int_0^t K_{t,s}\, d\xi_s^u, \quad 0 \le t \le T, \tag{5}$$

with $\bar{u} \in L_2(0, T; Y)$ and $K \in B_2(\Delta_T, \mathcal{L}(\mathbf{R}^k, Y))$.

Throughout this paper we assume that A is a densely defined on X closed linear operator generating a strongly continuous semigroup $\mathcal{U}$, $B \in \mathcal{L}(Y, X)$, $C \in \mathcal{L}(X, \mathbf{R}^k)$, $f \in L_2(0, T; X)$, x_0 is an X-valued Gaussian random variable, φ and ψ are X- and $\mathbf{R}^k$-valued Wiener processes, respectively, x_0, φ and ψ are independent. We will use the notations

$$\operatorname{cov} x_0 = P_0 \quad \text{and} \quad \operatorname{cov} \varphi_t = Mt,$$

and assume that $\operatorname{cov} \psi_t = It$. If $u \in U_{\text{ad}}$, then under a solution of the equation in (4) it will be meant its mild solution, i.e. the function

$$x_t^u = \mathcal{U}_t x_0 + \int_0^t \mathcal{U}_{t-s}(Bu_s + f_s)\, ds + \int_0^t \mathcal{U}_{t-s}\, d\varphi_s, \ 0 \le t \le T.$$

One can associate two control systems with the system (4). The first of them is the deterministic control system

$$\frac{d}{dt} y_t^v = A y_t^v + B v_t + f_t,\ 0 < t \le T,\ y_0^v = y_0 = \mathbf{E}\, x_0, \tag{6}$$

with the admissible controls v taken from $V_{\text{ad}} = L_2(0,T;Y)$. The second one is the partially observable stochastic control system

$$\begin{cases} dz_t^w = (A z_t^w + B w_t)\, dt + d\varphi_t,\ 0 < t \le T,\ z_0^w = z_0 = x_0 - \mathbf{E}\, x_0, \\ d\eta_t^w = C z_t^w\, dt + d\psi_t,\ 0 < t \le T,\ \eta_0^w = 0, \end{cases} \tag{7}$$

where w is a control from the set of admissible controls W_{ad} consisting of all controls in the form

$$w_t = \int_0^t K_{t,s}\, d\eta_s^w, \quad t \ge 0, \tag{8}$$

with $K \in B_2(\Delta_T, \mathcal{L}(\mathbf{R}^k, Y))$. The same mild sense will be applied to the solutions of the equations in (6) and (7).

5 D_T^c- and D_T^a-controllabilities: the resolvent conditions

In this section the necessary and sufficient conditions in terms of convergence of operators will be obtained for the system (6) on V_{ad} to be D_T^c- and D_T^a-controllable.

With the systems (4), (6) and (7), one can associate the operator-valued function

$$\mathcal{Q}_T = \int_0^T \mathcal{U}_s B B^* \mathcal{U}_s^*\, ds,\ T \ge 0, \tag{9}$$

which is called a *controllability operator*. For $T \ge 0$, the operator $\mathcal{Q}_T$ is nonnegative ($\mathcal{Q}_T \ge 0$) and, hence, $R(\lambda, -\mathcal{Q}_T) = (\lambda I + \mathcal{Q}_T)^{-1}$ is well-defined bounded linear operator for all $\lambda > 0$ and for all $T \ge 0$. If $\mathcal{Q}_T > 0$, then $R(\lambda, -\mathcal{Q}_T)$ is defined for $\lambda = 0$ as well. The operator $R(\lambda, -\mathcal{Q}_T)$ is called the *resolvent* of $-\mathcal{Q}_T$. This resolvent will be used to represent the optimal control in the linear regulator problem of minimizing the functional

$$J(v) = \|y_T^v - h\|^2 + \lambda \int_0^T \|v_t\|^2\, dt, \tag{10}$$

where y^v is a state process defined by (6), v is a control taken from $V_{\text{ad}} = L_2(0,T;Y)$ and $T > 0$, $h \in X$ and $\lambda > 0$ are parameters.

Lemma 6. *Given $T > 0$, $h \in X$ and $\lambda > 0$, there exists a unique optimal control v^λ at which the functional* (10) *takes its minimum value on V_{ad}. Furthermore,*

$$v_t^\lambda = -B^* \mathcal{U}_{T-t}^* R(\lambda, -\mathcal{Q}_T)(\mathcal{U}_T y_0 - h + g),\ a.e.\ t \in [0,T], \tag{11}$$

and

$$y_T^{v^\lambda} - h = \lambda R(\lambda, -\mathcal{Q}_T)(\mathcal{U}_T y_0 - h + g), \tag{12}$$

where $R(\lambda, -\mathcal{Q}_T)$ is the resolvent of $-\mathcal{Q}_T$ and

$$g = \int_0^T \mathcal{U}_{T-t} f_t\, dt.$$

Proof. The existence and uniqueness of the optimal control v^λ follows from the general results about linear regulator problems, see for example (Curtain and Pritchard, 1978). We will prove the formulae (11) and (12). Computing the variation of the functional (10), one can easily obtain

$$v_t^\lambda = -\lambda^{-1} B^* \mathcal{U}_{T-t}^* \left(y_T^{v^\lambda} - h \right), \text{ a.e. } t \in [0, T]. \tag{13}$$

Substituting this in (6) and using (9), we obtain

$$\begin{aligned} y_T^{v^\lambda} &= \mathcal{U}_T y_0 + \int_0^T \mathcal{U}_{T-t}(Bv_t^\lambda + f_t)dt \\ &= \mathcal{U}_T y_0 + g - \lambda^{-1} \int_0^T \mathcal{U}_{T-t} BB^* \mathcal{U}_{T-t}^* \left(y_T^{v^\lambda} - h \right) dt \\ &= \mathcal{U}_T y_0 + g - \lambda^{-1} \mathcal{Q}_T \left(y_T^{v^\lambda} - h \right). \end{aligned}$$

Hence,

$$\lambda y_T^{v^\lambda} = \lambda(\mathcal{U}_T y_0 + g) - \mathcal{Q}_T \left(y_T^{v^\lambda} - h \right),$$

which implies

$$(\lambda I + \mathcal{Q}_T) y_T^{v^\lambda} = \lambda(\mathcal{U}_T y_0 + g) + \mathcal{Q}_T h$$

and, consequently,

$$\begin{aligned} y_T^{v^\lambda} &= \lambda(\lambda I + \mathcal{Q}_T)^{-1}(\mathcal{U}_T y_0 + g) + (\lambda I + \mathcal{Q}_T)^{-1}(\lambda I + \mathcal{Q}_T - \lambda I)h \\ &= \lambda R(\lambda, -\mathcal{Q}_T)(\mathcal{U}_T y_0 + g - h) + h. \end{aligned}$$

Thus, the equality (12) holds. Substituting (12) in (13), we obtain the equality (11). Lemma is proved.

Theorem 7. *Given $T > 0$, the following statements are equivalent:*

(a) *the control system (6) on V_{ad} is D_T^c-controllable;*
(b) $\mathcal{Q}_T > 0$;
(c) *$R(\lambda, -\mathcal{Q}_T)$ converges as $\lambda \to 0$ in uniform operator topology;*
(d) *$R(\lambda, -\mathcal{Q}_T)$ converges as $\lambda \to 0$ in strong operator topology;*
(e) *$R(\lambda, -\mathcal{Q}_T)$ converges as $\lambda \to 0$ in weak operator topology;*
(f) *$\lambda R(\lambda, -\mathcal{Q}_T)$ converges to zero operator as $\lambda \to 0$ in uniform operator topology.*

Proof. The equivalence (a) $\Leftrightarrow$ (b) is well-known. For the implication (b) $\Rightarrow$ (c), let $\mathcal{Q}_T > 0$. Then for all $x \in X$ and for all $\lambda \geq 0$,

$$\langle x, (\lambda I + \mathcal{Q}_T)x \rangle \geq (\lambda + k)\|x\|^2,$$

where $k > 0$ is a constant. Therefore, for all $\lambda \geq 0$,

$$\|R(\lambda, -\mathcal{Q}_T)\| = \|(\lambda I + \mathcal{Q}_T)^{-1}\| \leq \frac{1}{\lambda + k} \leq \frac{1}{k}.$$

We obtain that $\|R(\lambda, -\mathcal{Q}_T)\|$ is bounded with respect to $\lambda \geq 0$. This implies

$$\begin{aligned} \|R(\lambda, -\mathcal{Q}_T) - \mathcal{Q}_T^{-1}\| &= \|(\lambda I + \mathcal{Q}_T)^{-1} - \mathcal{Q}_T^{-1}\| \\ &= \|\mathcal{Q}_T^{-1}(\mathcal{Q}_T - \lambda I - \mathcal{Q}_T)(\lambda I + \mathcal{Q}_T)^{-1}\| \\ &\leq \lambda\|\mathcal{Q}_T^{-1}\| \, \|(\lambda I + \mathcal{Q}_T)^{-1}\| \\ &\leq \lambda k^{-2}. \end{aligned}$$

So, $R(\lambda, -\mathcal{Q}_T)$ converges uniformly to $\mathcal{Q}_T^{-1}$ as $\lambda \to 0$. The implications (c) $\Rightarrow$ (d) $\Rightarrow$ (e) are obvious. The implication (e) $\Rightarrow$ (f) follows from the boundedness of a weakly convergent sequence of operators. For the implication (f) $\Rightarrow$ (b), suppose

$$\lambda\|R(\lambda, -\mathcal{Q}_T)\| = \lambda \left\|(\lambda I + \mathcal{Q}_T)^{-1}\right\| \to 0, \ \lambda \to 0.$$

Then $\lambda^{1/2} \left\|(\lambda I + \mathcal{Q}_T)^{-1/2}\right\| \to 0$ as $\lambda \to 0$. For sufficiently small $\lambda_0 > 0$, we can write

$$\lambda_0^{1/2}\|(\lambda_0 I + \mathcal{Q}_T)^{-1/2}\| \le \frac{1}{\sqrt{2}}.$$

So, for all $x \in X$, we have

$$\begin{aligned} \|x\|^2 &= \|(\lambda_0^{1/2}(\lambda_0 I + \mathcal{Q}_T)^{-1/2})(\lambda_0^{-1/2}(\lambda_0 I + \mathcal{Q}_T)^{1/2})x\|^2 \\ &\le \frac{1}{2}\|\lambda_0^{-1/2}(\lambda_0 I + \mathcal{Q}_T)^{1/2}x\|^2 \\ &= \frac{1}{2}\left\langle \lambda_0^{-1}(\lambda_0 I + \mathcal{Q}_T)x, x\right\rangle, \end{aligned}$$

which implies

$$\left\langle \lambda_0^{-1}(\lambda_0 I + \mathcal{Q}_T)x, x\right\rangle \ge 2\|x\|^2$$

and, consequently,

$$\langle \mathcal{Q}_T x, x\rangle \ge \lambda_0\|x\|^2.$$

Thus, $\mathcal{Q}_T > 0$. The theorem is proved.

Theorem 8. *Given $T > 0$, the following statements are equivalent:*

(a) *the control system (6) on V_{ad} is D_T^a-controllable;*

(b) *if $B^*\mathcal{U}_t^* x = 0$ for all $0 \le t \le T$, then $x = 0$;*

(c) *$\lambda R(\lambda, -\mathcal{Q}_T)$ converges to zero operator as $\lambda \to 0$ in strong operator topology;*

(d) *$\lambda R(\lambda, -\mathcal{Q}_T)$ converges to zero operator as $\lambda \to 0$ in weak operator topology.*

Proof. The equivalence (a) $\Leftrightarrow$ (b) is well-known. For the implication (c) $\Rightarrow$ (a), let $\lambda R(\lambda, -\mathcal{Q}_T)$ be strongly convergent to zero operator as $\lambda \to 0$. Consider an arbitrary $h \in X$ and the functional (10) with this h. By (12), selecting λ sufficiently small, we can make $y_T^{v^\lambda}$ to be close to h. So, the system (6) on V_{ad} is D_T^a-controllable. For the implication (a) $\Rightarrow$ (c), let the control system (6) on V_{ad} be D_T^a-controllable. Then for arbitrary $h \in X$, there exists a sequence $\{\bar{v}^n\}$ in $L_2(0, T; U)$ such that $\|y_T^{\bar{v}^n} - h\| \to 0$ as $n \to \infty$. We have

$$\left\|y_T^{v^\lambda} - h\right\|^2 \le \left\|y_T^{v^\lambda} - h\right\|^2 + \lambda\int_0^T \|v_t^\lambda\|^2 dt \le \left\|y_T^{\bar{v}^n} - h\right\|^2 + \lambda\int_0^T \|\bar{v}_t^n\|^2 dt,$$

where v^λ is the control at which the functional (10) takes on its minimum value. If $\varepsilon > 0$ is given, then we can make $\|y_T^{\bar{v}^n} - h\| < \varepsilon/\sqrt{2}$ for some sufficiently large n and then we can select $\delta > 0$ to be sufficiently small so that for all $0 < \lambda < \delta$,

$$\lambda\int_0^T \|\bar{v}_t^n\|^2 dt < \frac{\varepsilon^2}{2}.$$

Thus, $\|y_T^{v^\lambda} - h\| < \varepsilon$ for all $0 < \lambda < \delta$, i.e. $y_T^{v^\lambda}$ converges to h as $\lambda \to 0$. By (12) and by the arbitrariness of h, this implies the strong convergence of $\lambda R(\lambda, -\mathcal{Q}_T)$ to zero operator as $\lambda \to 0$. Finally, the equivalence (c) $\Leftrightarrow$ (d) is a consequence of $\lambda R(\lambda, -\mathcal{Q}_T) \ge 0$. Theorem is proved.

The conditions (f) in Theorem 7 and (c) in Theorem 8 clearly distinct the D_T^c- and D_T^a-controllabilities of the control system (6) showing that the distinction between them is in a kind of convergence of $\lambda R(\lambda, -\mathcal{Q}_T)$ to zero operator as $\lambda \to 0$. We call these conditions the *resolvent conditions* for the control system (6) to be D_T^c- and D_T^a-controllable, respectively. The conditions (b) in Theorem 7 and (b) in Theorem 8 are the well-known *complete* and *approximate controllability conditions.*

6 Applications of the resolvent conditions

An application of the resolvent conditions to a concrete control system requires a computation of the respective resolvent and then a verification of the respective convergence. These are illustrated below in the examples of controlled one-dimensional heat and wave equations.

Example 9. Consider the controlled one-dimensional heat equation

$$\frac{\partial}{\partial t} y_{t,\theta} = \frac{\partial^2}{\partial \theta^2} y_{t,\theta} + v_{t,\theta},\ 0 \le \theta \le 1,\ 0 < t \le T, \tag{14}$$

with the initial and boundary conditions

$$y_{0,\theta} = f_\theta,\ y_{t,0} = y_{t,1} = 0,\ 0 \le \theta \le 1,\ 0 \le t \le T. \tag{15}$$

Let $X = Y = L_2(0,1)$ and let $f \in X$. In the system (14)-(15), the second order differential operator $d^2/d\theta^2$ stands for the operator A with the domain

$$D(A) = \{h \in X : (d^2/d\theta^2)h \in X,\ h_0 = h_1 = 0\}$$

and it generates the strongly continuous semigroup $\mathcal{U}$ defined by

$$[\mathcal{U}_t h]_\theta = \sum_{i=1}^{\infty} 2e^{-i^2\pi^2 t} \sin(i\pi\theta) \int_0^1 h_\alpha \sin(i\pi\alpha)\, d\alpha,\ 0 \le \theta \le 1,\ t \ge 0,\ h \in X.$$

If v is a control action taken from the set of admissible controls $V_{\mathrm{ad}} = L_2(0,T; L_2(0,1))$, then it is easily seen that $B = B^* = I$ and, since $\mathcal{U}_t$ is self-adjoint,

$$\mathcal{Q}_T = \int_0^T \mathcal{U}_s B B^* \mathcal{U}_s^*\, ds = \int_0^T \mathcal{U}_{2s}\, ds.$$

Therefore, for $h \in X$,

$$\begin{aligned} [\mathcal{Q}_T h]_\theta &= \left[\int_0^T \mathcal{U}_{2s} h\, ds\right]_\theta \\ &= \sum_{i=1}^{\infty} \int_0^T 2e^{-2i^2\pi^2 s} \sin(i\pi\theta) \int_0^1 h_\alpha \sin(i\pi\alpha)\, d\alpha\, ds \\ &= \sum_{i=1}^{\infty} \frac{1 - e^{-2i^2\pi^2 T}}{i^2\pi^2} \sin(i\pi\theta) \int_0^1 h_\alpha \sin(i\pi\alpha)\, d\alpha. \end{aligned}$$

The half-range Fourier sine expansion of $h \in X$ is

$$h_\theta = \sum_{i=1}^{\infty} 2\sin(i\pi\theta) \int_0^1 h_\alpha \sin(i\pi\alpha)\, d\alpha,\ 0 \le \theta \le 1.$$

Using this, we obtain

$$[(\lambda I + \mathcal{Q}_T)h]_\theta = \sum_{i=1}^{\infty} \frac{2i^2\pi^2\lambda + 1 - e^{-2i^2\pi^2 T}}{i^2\pi^2} \sin(i\pi\theta) \int_0^1 h_\alpha \sin(i\pi\alpha)\, d\alpha.$$

Let $(\lambda I + \mathcal{Q}_T)h = g$. If we use the half-range Fourier sine expansion of $g \in X$, then

$$\sum_{i=1}^{\infty} \frac{2i^2\pi^2\lambda + 1 - e^{-2i^2\pi^2 T}}{i^2\pi^2} \sin(i\pi\theta) \int_0^1 h_\alpha \sin(i\pi\alpha)\, d\alpha = \sum_{i=1}^{\infty} 2\sin(i\pi\theta) \int_0^1 g_\alpha \sin(i\pi\alpha)\, d\alpha,$$

which for all $i \in \mathbf{N}$ implies

$$\int_0^1 h_\alpha \sin(i\pi\alpha)\, d\alpha = \frac{2i^2\pi^2}{2i^2\pi^2\lambda + 1 - e^{-2i^2\pi^2 T}} \int_0^1 g_\alpha \sin(i\pi\alpha)\, d\alpha.$$

Therefore,

$$\begin{aligned} h_\theta &= [(\lambda I + \mathcal{Q}_T)^{-1} g]_\theta = [R(\lambda, -\mathcal{Q}_T)g]_\theta \\ &= \sum_{i=1}^{\infty} \frac{4i^2\pi^2}{2i^2\pi^2\lambda + 1 - e^{-2i^2\pi^2 T}} \sin(i\pi\theta) \int_0^1 g_\alpha \sin(i\pi\alpha)\, d\alpha. \end{aligned}$$

If $g_\alpha \equiv 1$, then by Parseval's identity,

$$\begin{aligned} \|R(\lambda, -\mathcal{Q}_T)g\|_X^2 &= \frac{1}{2}\sum_{i=1}^{\infty} \frac{(4i^2\pi^2)^2}{(2i^2\pi^2\lambda + 1 - e^{-2i^2\pi^2 T})^2} \left(\int_0^1 \sin(i\pi\alpha)\, d\alpha\right)^2 \\ &= \sum_{i=1}^{\infty} \frac{8i^2\pi^2(1-(-1)^i)^2}{(2i^2\pi^2\lambda + 1 - e^{-2i^2\pi^2 T})^2} \\ &\geq \sum_{i=1}^{\infty} \frac{8i^2\pi^2(1-(-1)^i)^2}{(2i^2\pi^2\lambda + 1)^2} = \sum_{i=1,3,5,\ldots} \frac{32i^2\pi^2}{(2i^2\pi^2\lambda + 1)^2}. \end{aligned}$$

One can verify that the inequality

$$\frac{i}{2i^2\pi^2\lambda + 1} > \frac{i+1}{2(i+1)^2\pi^2\lambda + 1}$$

holds whenever i is an integer that is greater than the number $1/\sqrt{2\lambda}\pi$. Let N_λ be the smallest odd integer that is greater than $1/\sqrt{2\lambda}\pi$. Then the sequence

$$\{i^2\pi^2/(2i^2\pi^2\lambda + 1)^2\}_{i=1,2,\ldots}$$

is decreasing for $i \geq N_\lambda$. The following limits are obvious:

$$N_\lambda \to \infty \text{ and } \lambda N_\lambda^2 \to \frac{1}{2\pi^2} \text{ as } \lambda \to 0.$$

Using these, for $g_\alpha \equiv 1$, we obtain

$$\begin{aligned} \|R(\lambda, -\mathcal{Q}_T)g\|_X^2 &\geq \sum_{i=N_\lambda}^{\infty} \frac{16i^2\pi^2}{(2i^2\pi^2\lambda + 1)^2} \geq \int_{N_\lambda}^{\infty} \frac{16\pi^2 t^2}{(2\pi^2\lambda t^2 + 1)^2}\, dt \\ &\geq \int_{N_\lambda}^{\infty} \frac{4\pi^2 t}{(2\pi^2\lambda t^2 + 1)^2}\, dt = \frac{1}{\lambda(2\pi^2\lambda N_\lambda^2 + 1)} \to \infty \end{aligned}$$

as $\lambda \to 0$. So, by (a) $\Leftrightarrow$ (d) in Theorem 7, for any $T > 0$, the control system (14)-(15) on $V_{\mathrm{ad}} = L_2(0,T;L_2(0,1))$ is not D_T^c-controllable. At the same time, for all $g \in X$,

$$\|\lambda R(\lambda, -\mathcal{Q}_T)g\|_X^2 = \sum_{i=1}^{\infty} \frac{8i^4\pi^4\lambda^2}{(2i^2\pi^2\lambda + 1 - e^{-2i^2\pi^2 T})^2} \left(\int_0^1 g_\alpha \sin(i\pi\alpha)\, d\alpha \right)^2 \to 0$$

as $\lambda \to 0$ and, hence, by (a) $\Leftrightarrow$ (c) in Theorem 8, for each $T > 0$, the control system (14)-(15) on $V_{\mathrm{ad}} = L_2(0,T;L_2(0,1))$ is D_T^a-controllable.

Example 10. Consider the controlled wave equation

$$\frac{\partial^2}{\partial t^2}\xi_{t,\theta} = \frac{\partial^2}{\partial \theta^2}\xi_{t,\theta} + b_\theta v_t,\ 0 \le \theta \le 1,\ 0 < t \le T, \tag{16}$$

with the initial and boundary conditions

$$\xi_{0,\theta} = f_\theta,\ \frac{\partial}{\partial t}\xi_{t,\theta}\Big|_{t=0} = g_\theta,\ \xi_{t,0} = \xi_{t,1} = 0,\ 0 \le \theta \le 1,\ 0 \le t \le T, \tag{17}$$

where v is a control action taken from the set of admissible controls $V_{\mathrm{ad}} = L_2(0,T)$, i.e. $Y = \mathbf{R}$. We assume that f, g and b are functions in $L_2(0,1)$. For these functions we will use the half-range Fourier sine expansions

$$f_\theta = \sum_{i=1}^{\infty} \alpha_i \sin(i\pi\theta),\ g_\theta = \sum_{i=1}^{\infty} \beta_i \sin(i\pi\theta),\ b_\theta = \sum_{i=1}^{\infty} \gamma_i \sin(i\pi\theta)$$

and suppose that

$$\sum_{i=1}^{\infty} i^2\alpha_i^2 < \infty.$$

Let X be a Hilbert space of all functions

$$h = \begin{bmatrix} f \\ g \end{bmatrix} : [0,1] \to \mathbf{R},$$

where f and g satisfy the above mentioned conditions, endowed with the scalar product

$$\langle h, \tilde{h} \rangle = \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \tilde{f} \\ \tilde{g} \end{bmatrix} \right\rangle = \sum_{i=1}^{\infty} (i^2\pi^2\alpha_i\tilde{\alpha}_i + \beta_i\tilde{\beta}_i),$$

where $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ are the respective Fourier coefficients of $\tilde{f}$ and $\tilde{g}$. In (Curtain and Zwart, 1995; Zabczyk, 1992) this space X is taken as suitable for the problem (16)-(17). For the operator

$$A = \begin{bmatrix} 0 & I \\ d^2/d\theta^2 & 0 \end{bmatrix}, \tag{18}$$

where $d^2/d\theta^2$ has the domain

$$D(d^2/d\theta^2) = \{\eta \in L_2(0,1) : (d^2/d\theta^2)\eta \in L_2(0,1),\ \eta_0 = \eta_1 = 0\},$$

and for $B \in \mathcal{L}(\mathbf{R}, X)$ defined by

$$[Bv]_\theta = \begin{bmatrix} 0 \\ b_\theta v \end{bmatrix},\ 0 \le \theta \le 1,\ v \in \mathbf{R},$$

the problem (16)-(17) can be formulated in the abstract form

$$\frac{d}{dt}y_t = Ay_t + Bv_t,\ 0 < t \leq T, \tag{19}$$

where

$$[y_t]_\theta = \begin{bmatrix} \xi_{t,\theta} \\ (\partial/\partial t)\xi_{t,\theta} \end{bmatrix},\ 0 \leq \theta \leq 1,\ 0 \leq t \leq T;\ y_0 = \begin{bmatrix} f \\ g \end{bmatrix}.$$

In (Curtain and Zwart, 1995; Zabczyk, 1992) it is shown that the operator A defined by (18) generates a continuous group $\mathcal{U}$ as defined by

$$[\mathcal{U}_t h]_\theta = \sum_{i=1}^{\infty} \begin{bmatrix} \cos(i\pi t) & (i\pi)^{-1}\sin(i\pi t) \\ -i\pi\sin(i\pi t) & \cos(i\pi t) \end{bmatrix} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \sin(i\pi\theta),\ 0 \leq \theta \leq 1,\ t \geq 0,$$

where

$$h = \begin{bmatrix} f \\ g \end{bmatrix} \in X$$

and α_i and β_i are Fourier coefficients of f and g, respectively. Since $\mathcal{U}$ is a group, we have $\mathcal{U}_t^* = \mathcal{U}_{-t}$. Therefore, the controllability operator $\mathcal{Q}_T$ of the system (19) is

$$\mathcal{Q}_T h = \int_0^T \mathcal{U}_{T-t}BB^*\mathcal{U}_{T-t}^* h\, dt = \int_0^T \mathcal{U}_t BB^*\mathcal{U}_{-t} h\, dt,\ h \in X.$$

We have

$$[\mathcal{U}_{-t}h]_\theta = \sum_{i=1}^{\infty} \begin{bmatrix} \alpha_i\cos(i\pi t) - \beta_i(i\pi)^{-1}\sin(i\pi t) \\ \alpha_i i\pi\sin(i\pi t) + \beta_i\cos(i\pi t) \end{bmatrix} \sin(i\pi\theta).$$

One can calculate that

$$B^*h = \sum_{i=1}^{\infty} \gamma_i\beta_i,\ h \in X.$$

Hence,

$$B^*\mathcal{U}_{-t}h = \sum_{i=1}^{\infty} \gamma_i(\alpha_i i\pi\sin(i\pi t) + \beta_i\cos(i\pi t))$$

and, consequently,

$$[\mathcal{U}_t BB^*\mathcal{U}_{-t}h]_\theta = \sum_{i=1}^{\infty} \begin{bmatrix} \gamma_i(i\pi)^{-1}\sin(i\pi t) \\ \gamma_i\cos(i\pi t) \end{bmatrix} \sin(i\pi\theta) \times \sum_{j=1}^{\infty} \gamma_j(\alpha_j j\pi\sin(j\pi t) + \beta_j\cos(j\pi t)).$$

Thus, for $T = 2$,

$$[\mathcal{Q}_2 h]_\theta = \int_0^2 [\mathcal{U}_t BB^*\mathcal{U}_{-t}h]_\theta\, dt = \sum_{i=1}^{\infty} \begin{bmatrix} \gamma_i^2\alpha_i \\ \gamma_i^2\beta_i \end{bmatrix} \sin(i\pi\theta).$$

We obtain that

$$[(\lambda I + \mathcal{Q}_2)h]_\theta = \sum_{i=1}^{\infty} (\lambda + \gamma_i^2) \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \sin(i\pi\theta),$$

which implies

$$[R(\lambda, -\mathcal{Q}_2)h]_\theta = [(\lambda I + \mathcal{Q}_2)^{-1}h]_\theta = \sum_{i=1}^{\infty} (\lambda + \gamma_i^2)^{-1} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \sin(i\pi\theta).$$

Finally, for all $h \in X$,

$$\|\lambda R(\lambda, -\mathcal{Q}_2)h\|^2 = \sum_{i=1}^{\infty} \frac{\lambda^2}{(\lambda + \gamma_i^2)^2} (i^2\pi^2\alpha_i^2 + \beta_i^2) \to 0$$

as $\lambda \to 0$ if $\gamma_i \neq 0$ for all $i = 1, 2, \ldots$. Thus, by (a) $\Leftrightarrow$ (c) in Theorem 8, we obtain the following sufficient condition for the approximate controllability of the system (16)-(17) which agrees with Theorem 2.10 in (Zabczyk, 1972): if $T \geq 2$ and b is so that

$$\int_0^1 b_\theta \sin(i\pi\theta)\, d\theta \neq 0, \ i = 1, 2, \ldots,$$

then the control system (16)-(17) on $V_{\mathrm{ad}} = L_2(0,1)$ is D^a-controllable.

7 S_T^0-controllability

In this section the S_T^0-controllability of the control system (7) on W_{ad} is studied. For this, we consider the Riccati equations

$$\frac{d}{dt}Q_t + Q_tA + A^*Q_t - \lambda^{-1}Q_tBB^*Q_t = 0, \ 0 \leq t < T, \ Q_T = I, \ \lambda > 0, \tag{20}$$

and

$$\frac{d}{dt}P_t - AP_t - P_tA^* - M + P_tC^*CP_t = 0, \ 0 < t \leq T, \ P_0 = \operatorname{cov} z_0. \tag{21}$$

Lemma 11. *There exist the unique strongly continuous solutions* (*in scalar product sense*) Q^λ *and* P *of the equations* (20) *and* (21), *respectively, satisfying* $Q_t^\lambda \geq 0$ *and* $P_t \geq 0$ *for all* $0 \leq t \leq T$. *Moreover, the solution of the equation* (20) *has the explicit form*

$$Q_t^\lambda = \lambda \mathcal{U}_{T-t}^* R(\lambda, -\mathcal{Q}_{T-t})\mathcal{U}_{T-t}, \ 0 \leq t \leq T, \ \lambda > 0. \tag{22}$$

Proof. For the existence and for the uniqueness, see (Curtain and Pritchard, 1978). For the representation (22), see (Da Prato and Barbu, 1992; Bashirov and Kerimov, 1997).

Lemma 12. *There exists the finite limit*

$$a_T = \lim_{\lambda \to 0} \operatorname{tr} \int_0^T CP_sQ_s^\lambda P_sC^*\, ds, \tag{23}$$

where Q^λ *and* P *are the solutions of the equations* (20) *and* (21), *respectively.*

Proof. Consider the family of the stochastic optimal control problems on W_{ad} with the state-observation system (7) and the functional

$$J^\lambda(w) = \mathbf{E}\left(\|z_T^w\|^2 + \lambda \int_0^T \|w_t\|^2\, dt \right), \ \lambda > 0, \tag{24}$$

to be minimized. In (Curtain and Pritchard, 1978) it is shown that the functional J^λ takes its minimum value at some control $w^\lambda \in W_{\mathrm{ad}}$ and

$$J^\lambda(w^\lambda) = \operatorname{tr} P_T + \operatorname{tr} \int_0^T CP_sQ_s^\lambda P_sC^*\, ds.$$

Therefore, to prove the lemma it is sufficient to show that the sequence $\{J^\lambda(w^\lambda)\}$ has a finite limit as $\lambda \to 0$. Let $\lambda > \nu > 0$. Then

$$\begin{aligned} J^\nu(w^\nu) &= \mathbf{E}\left(\|z_T^{w^\nu}\|^2 + \nu \int_0^T \|w_t^\nu\|^2\, dt \right) \\ &\le \mathbf{E}\left(\|z_T^{w^\lambda}\|^2 + \nu \int_0^T \|w_t^\lambda\|^2\, dt \right) \\ &\le \mathbf{E}\left(\|z_T^{w^\lambda}\|^2 + \lambda \int_0^T \|w_t^\lambda\|^2\, dt \right) = J^\lambda(w^\lambda). \end{aligned}$$

We conclude that $\{J^\lambda(w^\lambda)\}$ is a nonnegative and nondecreasing function of $\lambda > 0$. Hence, there exists a finite limit of $J^\lambda(w^\lambda)$ as $\lambda \to 0$ proving the lemma.

Lemma 13. *The equality*

$$\inf_{W_{\mathrm{ad}}} \mathbf{E}\, \|\mathbf{E}\,(z_T^w | \mathcal{F}_T^{w,\eta})\|^2 = a_T, \tag{25}$$

holds, where a_T is defined by (23), *Q^λ and P are the solutions of the equations* (20) *and* (21), *respectively.*

Proof. We will compare the functional (24) and

$$\tilde{J}^\lambda(w) = \mathbf{E}\left(\|\mathbf{E}\,(z_T^w | \mathcal{F}_T^{w,\eta})\|^2 + \lambda \int_0^T \|w_t\|^2\, dt \right),$$

where $w \in W_{\mathrm{ad}}$ and z^w is the state of the system (7). Since P_T is the covariance of the error $z_T^w - \mathbf{E}\,(z_T^w | \mathcal{F}_T^{w,\eta})$ independently on $w \in W_{\mathrm{ad}}$, we have

$$\operatorname{tr} P_T = \mathbf{E}\, \|z_T^w - \mathbf{E}\,(z_T^w | \mathcal{F}_T^{w,\eta})\|^2 = \mathbf{E}\, \|z_T^w\|^2 - \mathbf{E}\, \|\mathbf{E}\,(z_T^w | \mathcal{F}_T^{w,\eta})\|^2,$$

and, consequently,

$$\tilde{J}^\lambda(w^\lambda) = J^\lambda(w^\lambda) - \operatorname{tr} P_T = \operatorname{tr} \int_0^T C P_s Q_s^\lambda P_s C^*\, ds.$$

If we denote by $\{\tilde{w}^n\}$ any minimizing sequence of the functional

$$J_0(w) = \mathbf{E}\, \|\mathbf{E}\,(z_T^w | \mathcal{F}_T^{w,\eta})\|^2,$$

then

$$\inf_{W_{\mathrm{ad}}} \mathbf{E}\, \|\mathbf{E}\,(z_T^w | \mathcal{F}_T^{w,\eta})\|^2 \le \tilde{J}^\lambda(w^\lambda) \le \mathbf{E}\left(\left\|\mathbf{E}\left(z_T^{\tilde{w}^n} | \mathcal{F}_T^{\tilde{w}^n,\eta}\right)\right\|^2 + \lambda \int_0^T \|\tilde{w}_t^n\|^2\, dt \right).$$

Consequently, taking the limit as $\lambda \to 0$ and $n \to \infty$, we obtain the statement.

Theorem 14. *Given $T > 0$, $\varepsilon > 0$ and $0 \le p < 1$, the control system* (7) *on W_{ad} is $S^0_{T,\varepsilon,p}$-controllable if*

$$a_T < \varepsilon(1-p), \tag{26}$$

where a_T is defined by (23).

Proof. By Lemma 13, we have

$$\inf_{W_{\mathrm{ad}}} \mathbf{E}\|\mathbf{E}(z_T^w | \mathcal{F}_T^{w,\eta})\|^2 = a_T < \varepsilon(1-p).$$

Therefore, there exists $w^0 \in W_{\rm ad}$ such that

$$\mathbf{E}\left\|\mathbf{E}\left(z_T^{w^0}|\mathcal{F}_T^{w^0,\eta}\right)\right\|^2 < \varepsilon(1-p).$$

Using Chebyshev's inequality, we obtain

$$\mathbf{P}\left(\left\|\mathbf{E}\left(z_T^{w^0}|\mathcal{F}_T^{w^0,\eta}\right)\right\|^2 > \varepsilon\right) \le \frac{1}{\varepsilon}\mathbf{E}\left\|\mathbf{E}\left(z_T^{w^0}|\mathcal{F}_T^{w^0,\eta}\right)\right\|^2 < (1-p),$$

proving the theorem.

It should be noted that the condition (26) being a sufficient condition of $S^0_{T,\varepsilon,p}$-controllability is not necessary in general. In view of this we present the following arguments. For a given system, define the functions

$$\alpha_p = \inf \Phi_p, \quad \Phi_p = \{\varepsilon : \text{the system is } S^0_{T,\varepsilon,p}\text{−controllable}\}, \tag{27}$$

$$\beta_\varepsilon = \sup \Psi_\varepsilon, \quad \Psi_\varepsilon = \{p : \text{the system is } S^0_{T,\varepsilon,p}\text{−controllable}\}. \tag{28}$$

Obviously, α and β are nondecreasing functions with $\alpha_0 = 0$ and $\lim_{\varepsilon\to\infty}\beta_\varepsilon = 1$. It follows from the definitions that the necessary and sufficient condition for the system to be $S^0_{T,\varepsilon,p}$-controllable is

$$\begin{cases} \alpha_p < \varepsilon & \text{if } \inf\Phi_p \text{ is not achieved,} \\ \alpha_p \le \varepsilon & \text{if } \inf\Phi_p \text{ is achieved,} \end{cases} \tag{29}$$

which can also be written in the following equivalent form:

$$\begin{cases} \beta_\varepsilon > p & \text{if } \sup\Psi_\varepsilon \text{ is not achieved,} \\ \beta_\varepsilon \ge p & \text{if } \sup\Psi_\varepsilon \text{ is achieved.} \end{cases} \tag{30}$$

Using (26), define the functions

$$\tilde{\alpha}_p = \begin{cases} a_T(1-p)^{-1}, & 0 \le p < 1, \\ \infty, & p = 1, \end{cases} \qquad \tilde{\beta}_\varepsilon = \begin{cases} 1 - a_T\varepsilon^{-1}, & a_T < \varepsilon < \infty, \\ 0, & 0 \le \varepsilon \le a_T. \end{cases}$$

By (29), (30) and Theorem 14, it follows that

$$\alpha_p \le \tilde{\alpha}_p, \ 0 \le p \le 1, \quad \text{and} \quad \beta_\varepsilon \ge \tilde{\beta}_\varepsilon, \ 0 \le \varepsilon < \infty,$$

i.e. in the case of the control system (7) the functions $\tilde{\alpha}$ and $\tilde{\beta}$, defined with the help of (26), give only approximations of the functions α and β and may not be equal to them. In case $\alpha_p < \tilde{\alpha}_p$ or $\beta_\varepsilon > \tilde{\beta}_\varepsilon$ the condition (26) cannot be a necessary condition of $S^0_{T,\varepsilon,p}$-controllability.

Theorem 15. *Given $T > 0$, the control system* (7) *on $W_{\rm ad}$ is S^0_T-controllable if $a_T = 0$.*

Proof. By Theorem 14, $a_T = 0$ implies that the control system (7) is $S^0_{T,\varepsilon,p}$-controllable for all ε and for all p satisfying $\varepsilon(1-p) > 0$. This condition includes all pairs (ε, p) with $\varepsilon > 0$ and $0 \le p < 1$. So, the system (7) is $S^0_{T,\varepsilon,p}$-controllable for all $\varepsilon > 0$ and for all $0 \le p < 1$ proving the theorem.

Theorem 16. *Given $T > 0$, the control system* (7) *on $W_{\rm ad}$ is S^0_T-controllable if the system* (6) *is D^a_t-controllable for each $0 < t \le T$.*

Proof. From (a) $\Rightarrow$ (c) in Theorem 8, we obtain that $\lambda R(\lambda, -\mathcal{Q}_{T-t})$ strongly converges to zero operator as $\lambda \to 0$ for all $0 \le t < T$. Hence, by Lemma 11, Q^λ_t strongly converges to zero operator as $\lambda \to 0$ for all $0 \le t < T$. Furthermore, substituting $h = \lambda^{1/2}(\lambda I + \mathcal{Q}_{T-t})^{-1/2}x$ in

$$\langle \lambda^{-1}(\lambda I + \mathcal{Q}_{T-t})h, h\rangle \ge \langle h, h\rangle,$$

we obtain

$$\langle \lambda(\lambda I + \mathcal{Q}_{T-t})^{-1}x, x\rangle \leq \|x\|^2.$$

So, $\lambda R(\lambda, -\mathcal{Q}_{T-t}) \leq I$ and by Lemma 11, $Q_t^\lambda \leq \mathcal{U}_{T-t}^* \mathcal{U}_{T-t}$ for all $\lambda > 0$ and for all $0 \leq t \leq T$. Hence, we can change the places of the limit, the integral and the trace in (23) to obtain $a_T = 0$. Thus by Theorem 15, we obtain the S_T^0-controllability of the control system (7) proving the theorem.

Theorem 17. *The control system* (7) *on* W_{ad} *is* S_T^0*-controllable for each* $T > 0$ *if the control system* (6) *on* V_{ad} *is* D_T^a*-controllable for each* $T > 0$.

Proof. This is a direct consequence of Theorem 16.

8 C_T-controllability

In this section the C_T-controllability of the control system (4) on U_{ad} is studied. We use the results about the D_T^c-controllability of the control system (6) on V_{ad} and about the S_T^0-controllability of the control system (7) on W_{ad} from the previous sections.

Lemma 18. $U_{\mathrm{ad}} = V_{\mathrm{ad}} + W_{\mathrm{ad}}$, *where* $+$ *is the sign of the sum of sets.*

Proof. Let $u \in U_{\mathrm{ad}}$ be of the form (5) with $\bar{u} \in L_2(0,T;Y)$ and $K \in B_2(\Delta_T, \mathcal{L}(\mathbf{R}^k, Y))$. Then $\mathbf{E}\, u = \bar{u} \in V_{\mathrm{ad}}$ and, if $w = u - \bar{u}$, then

$$\begin{aligned} w_t &= \int_0^t K_{t,s} C(x_s^u - \mathbf{E}\, x_s^u)\, ds + \int_0^t K_{t,s}\, d\varphi_s \\ &= \int_0^t K_{t,s} C z_s^w\, ds + \int_0^t K_{t,s}\, d\varphi_s = \int_0^t K_{t,s}\, d\eta_s^w. \end{aligned}$$

Thus, $w = u - \bar{u} \in W_{\mathrm{ad}}$ and, consequently, $u \in V_{\mathrm{ad}} + W_{\mathrm{ad}}$. On the other hand, if $v \in V_{\mathrm{ad}}$ and $w \in W_{\mathrm{ad}}$ where w has the form of (8) with $K \in B_2(\Delta_T, \mathcal{L}(\mathbf{R}^k, Y))$, then

$$\begin{aligned} u_t &= v_t + \int_0^t K_{t,s} C z_s^w\, ds + \int_0^t K_{t,s}\, d\varphi_s \\ &= v_t - \int_0^t K_{t,s} C y_s^v\, ds + \int_0^t K_{t,s} C x_s^u\, ds + \int_0^t K_{t,s}\, d\varphi_s \end{aligned}$$

Denote

$$\bar{u}_t = v_t - \int_0^t K_{t,s} C y_s^v\, ds. \tag{31}$$

Then u has the form of (5) with $\bar{u}$ as in (31), i.e. $u \in U_{\mathrm{ad}}$. Thus, $U_{\mathrm{ad}} = V_{\mathrm{ad}} + W_{\mathrm{ad}}$ proving the lemma.

Lemma 19. *If* $u = v + w$ *where* $v \in V_{\mathrm{ad}}$ *and* $w \in W_{\mathrm{ad}}$, *then the* σ*-algebras* $\mathcal{F}_T^{u,\xi}$ *and* $\mathcal{F}_T^{w,\eta}$, *generated by* ξ_s^u, $0 \leq s \leq T$, *and* η_s^w, $0 \leq s \leq T$, *respectively, are equal.*

Proof. It is easy to show that

$$\xi_t^u = \eta_t^w + C \int_0^t y_s^v\, ds, \quad 0 \leq t \leq T. \tag{32}$$

Since the second term in the right-hand side of (32) is nonrandom, we conclude that $\mathcal{F}_T^{u,\xi}$ and $\mathcal{F}_T^{w,\eta}$ are equal.

Lemma 20. *Given* $T > 0$, $\varepsilon > 0$ *and* $0 \leq p < 1$, *the control system* (4) *on* U_{ad} *is* $C_{T,\varepsilon,p}^c$*-controllable if and only if the control system* (6) *on* V_{ad} *is* D_T^c*-controllable and the control system* (7) *on* W_{ad} *is* $S_{T,\varepsilon,p}^0$*-controllable.*

Proof. Let $C(T,\varepsilon,p)$ be the set (3) corresponding to the control system (4). Similarly, let $D(T)$ be the set (1) corresponding to the control system (6). Assume that the system (4) is $C^c_{T,\varepsilon,p}$-controllable. Then from the inclusion $C(T,\varepsilon,p) \subset D(T)$, it follows that the control system (6) is D^c_T-controllable. Let $h \in C(T,\varepsilon,p)$. Then there exists $u \in U_{\mathrm{ad}}$ such that $h = \mathbf{E}\, x^u_T$ and

$$\mathbf{P}\left\{\|\mathbf{E}\,(x^u_T|\mathcal{F}^{u,\xi}_T) - h\|^2 > \varepsilon\right\} \le 1-p.$$

Consider $w = u - \mathbf{E}\, u \in W_{\mathrm{ad}}$. By Lemma 19, $\mathcal{F}^{u,\xi}_T = \mathcal{F}^{w,\eta}_T$. Therefore,

$$\mathbf{P}\left\{\|\mathbf{E}\,(z^w_T|\mathcal{F}^{w,\eta}_T)\|^2 > \varepsilon\right\} = \mathbf{P}\left\{\|\mathbf{E}\,(x^u_T|\mathcal{F}^{u,\xi}_T) - \mathbf{E}\, x^u_T\|^2 > \varepsilon\right\} \le 1-p,$$

i.e. the control system (7) is $S^0_{T,\varepsilon,p}$-controllable. So, the necessity is proved. To prove the sufficiency, let $h \in D(T)$. Then there exists $v \in V_{\mathrm{ad}}$ such that $h = y^v_T$. Also, from the $S^0_{T,\varepsilon,p}$-controllability of the control system (7), we conclude that there exists $w \in W_{\mathrm{ad}}$ with

$$\mathbf{P}\left\{\|\mathbf{E}\,(z^w_T|\mathcal{F}^{w,\eta}_T)\|^2 > \varepsilon\right\} \le 1-p.$$

Consider $u = v + w$. By Lemma 18, $u \in U_{\mathrm{ad}} = V_{\mathrm{ad}} + W_{\mathrm{ad}}$. Moreover,

$$\mathbf{P}\left\{\|\mathbf{E}\,(x^u_T|\mathcal{F}^{u,\xi}_T) - h\|^2 > \varepsilon\right\} = \mathbf{P}\left\{\|\mathbf{E}\,(z^w_T|\mathcal{F}^{w,\eta}_T)\|^2 > \varepsilon\right\} \le 1-p,$$

i.e. $h \in C(T,\varepsilon,p)$. Therefore, $D(T) \subset C(T,\varepsilon,p)$. Since $D(T) = X$, we obtain $C(T,\varepsilon,p) = X$. Thus, the control system (4) is $C^c_{T,\varepsilon,p}$-controllable. Lemma is proved.

Theorem 21. *Given $T > 0$, the control system* (4) *on U_{ad} is C_T-controllable if and only if the control system* (6) *on V_{ad} is D^c_T-controllable and the control system* (7) *on W_{ad} is S^0_T-controllable.*

Proof. This is a direct consequence of Lemma 20.

Theorem 22. *The control system* (4) *on U_{ad} is C_T-controllable for each $T > 0$ if and only if the control system* (6) *on V_{ad} is D^c_T-controllable for each $T > 0$.*

Proof. The necessity follows from Theorem 21. For sufficiency, note that by Theorem 17, the D^c_T-controllability of the control system (6) for each $T > 0$ implies the S^0_T-controllability of the control system (7) for each $T > 0$. Thus, by Theorem 21, the control system (4) is C_T-controllable for each $T > 0$. Theorem is proved.

9 S_T-controllability

In this section the S_T-controllability of the control system (4) on U_{ad} is studied. At first we present the results about C^a_T-controllability which are similar to those of C_T-controllability.

Lemma 23. *Given $T > 0$, $\varepsilon > 0$ and $0 \le p < 1$, the control system* (4) *on U_{ad} is $C^a_{T,\varepsilon,p}$-controllable if and only if the control system* (6) *on V_{ad} is D^a_T-controllable and the control system* (7) *on W_{ad} is $S^0_{T,\varepsilon,p}$-controllable.*

Proof. This can be proved in a similar way as Lemma 20.

Theorem 24. *Given $T > 0$, the control system* (4) *on U_{ad} is C^a_T-controllable if and only if the control system* (6) *on V_{ad} is D^a_T-controllable and the control system* (7) *on W_{ad} is S^0_T-controllable.*

Proof. This is a direct consequence of Lemma 23.

It turns out that Theorem 24 is true if the C^a_T-controllability in it is replaced by the S_T-controllability. To prove this result, we will use the following fact.

Lemma 25. *U_{ad} is a convex set.*

Proof. At first, note that if $w \in W_{\mathrm{ad}}$ is of the form (8) with $K \in B_2(\Delta_T, \mathcal{L}(\mathbf{R}^k, Y))$, then there exists $M \in B_2(\Delta_T, \mathcal{L}(\mathbf{R}^k, Y))$ so that

$$w_t = \int_0^t M_{t,s}\, d\eta_s^0,\ 0 \le t \le T,$$

and vice versa, where η^0 is the observation process of the system (7) corresponding to the zero-control. The proof of this well-known fact one can find in (Curtain and Pritchard, 1978). Therefore, if $u^1, u^2 \in U_{\mathrm{ad}}$, then by Lemma 18,

$$u_t^i = v_t^i + \int_0^t M_{t,s}^i\, d\eta_s^0,\ 0 \le t \le T,\ i = 1, 2,$$

for some $v^1, v^2 \in L_2(0, T; Y)$ and $M^1, M^2 \in B_2(\Delta_T, \mathcal{L}(\mathbf{R}^k, Y))$. Let $\alpha_1 > 0$ and $\alpha_2 > 0$ be so that $\alpha_1 + \alpha_2 = 1$. Then for $v = \alpha_1 v^1 + \alpha_2 v^2$ and for $M = \alpha_1 M^1 + \alpha_2 M^2$, we have

$$u_t = \alpha_1 u_t^1 + \alpha_2 u_t^2 = v_t + \int_0^t M_{t,s}\, d\eta_s^0,\ 0 \le t \le T,$$

with $v \in L_2(0, T; Y)$ and with $M \in B_2(\Delta_T, \mathcal{L}(\mathbf{R}^k, Y))$. Thus, $u \in U_{\mathrm{ad}}$ proving the lemma.

Theorem 26. *Given $T > 0$, the control system (4) on U_{ad} is S_T-controllable if and only if the control system (6) on V_{ad} is D_T^a-controllable and the control system (7) on W_{ad} is S_T^0-controllable.*

Proof. If the control system (6) is D_T^a-controllable and the control system (7) is S_T^0-controllable, then by Theorem 24, the control system (4) is C_T^a-controllable which implies its S_T-controllability since $C(T, \varepsilon, p) \subset S(T, \varepsilon, p)$. Sufficiency is proved. For the necessity, let the control system (4) be S_T-controllable. Take an arbitrary $h \in X$ and consider the sequences $\{\varepsilon_n\}$ and $\{p_n\}$ with $\varepsilon_n > 0$, $0 \le p_n < 1$ and $\varepsilon_n \to 0$, $p_n \to 1$ as $n \to \infty$. From S_{ε_n, p_n}^a-controllability of the system (4), we obtain the existence of the sequence $\{u^n\}$ in U_{ad} such that

$$\mathbf{P}\left\{\|\mathbf{E}\,(x_T^{u^n} | \mathcal{F}_T^{u^n,\xi}) - h\|^2 > \varepsilon_n\right\} \le 1 - p_n.$$

The obtained inequality implies the convergence in probability of $\mathbf{E}\,(x_T^{u^n} | \mathcal{F}_T^{u^n,\xi})$ to h. Indeed, for $\varepsilon > 0$, we can find a number N such that $0 < \varepsilon_n < \varepsilon^2$ for all $n > N$. Therefore, for $n > N$,

$$\mathbf{P}\left\{\|\mathbf{E}\,(x_T^{u^n} | \mathcal{F}_T^{u^n,\xi}) - h\| > \varepsilon\right\} \le \mathbf{P}\left\{\|\mathbf{E}\,(x_T^{u^n} | \mathcal{F}_T^{u^n,\xi}) - h\|^2 > \varepsilon_n\right\} \le 1 - p_n \to 0, \quad n \to \infty.$$

Hence, $\mathbf{E}\,(x_T^{u^n} | \mathcal{F}_T^{u^n,\xi})$ converges to h in probability. Since for all n, $\mathbf{E}\,(x_T^{u^n} | \mathcal{F}_T^{u^n,\xi})$ is a Gaussian random variable, the characteristic function of it has the form

$$\chi_n(x) = \exp\left(i\langle m_n, x\rangle - \frac{1}{2}\langle \Lambda_n x, x\rangle\right), \quad x \in X,$$

where $m_n = \mathbf{E}\,(\mathbf{E}\,(x_T^{u^n} | \mathcal{F}_T^{u^n,\xi})) = \mathbf{E}\, x_T^{u^n}$ and $\Lambda_n = \operatorname{cov} \mathbf{E}\,(x_T^{u^n} | \mathcal{F}_T^{u^n,\xi})$ and i is the imaginary unit. Also, the vector $h \in X$ is considered as a degenerate Gaussian random variable with the characteristic function

$$\chi(x) = \exp\left(i\langle h, x\rangle\right), \quad x \in X.$$

The convergence of $\mathbf{E}\,(x_T^{u^n} | \mathcal{F}_T^{u^n,\xi})$ to h in probability implies $\chi_n(x) \to \chi(x)$ for all $x \in X$. The last convergence is possible when for all $x \in X$,

$$\langle m_n, x\rangle = \langle \mathbf{E}\, x_T^{u^n}, x\rangle \to \langle h, x\rangle \text{ and } \langle \Lambda_n x, x\rangle \to 0 \text{ as } n \to \infty. \tag{33}$$

The first of these convergences means the convergence of $\mathbf{E}\, x_T^{u^n}$ to h in the weak topology of the Hilbert space X. By Mazur's theorem (Balakrishnan, 1976) we can construct the sequence

$$h_n = \sum_{i=1}^{n} c_i^n \mathbf{E} x_T^{u^i}, \quad c_i^n \geq 0, \quad \sum_{i=1}^{n} c_i^n = 1, \quad i = 1, 2, \ldots, n, \quad i = 1, 2, \ldots,$$

of convex combinations of $\mathbf{E}\, x_T^{u^n}$ such that h_n converges to h in the strong topology of X. Denote $\tilde{u}^n = \sum_{i=1}^{n} c_i^n u^i$, $n = 1, 2, \ldots$. By Lemma 25, $\tilde{u}^n \in U_{\text{ad}}$ for all n. Moreover, in view of the affineness of the system (4), $h_n = \mathbf{E}\, x_T^{\tilde{u}^n}$. In terms of the system (6) this means that for the sequence of controls $\tilde{v}^n = \mathbf{E}\, \tilde{u}^n$ in V_{ad}, the sequence of vectors $h_n = \mathbf{E}\, x_T^{\tilde{u}^n} = y_T^{\tilde{v}^n}$ converges to h in the strong topology of X. Since h is an arbitrary point of X, we conclude that the set $D(T)$ defined by (1) for the control system (6) is dense in X, i.e. the control system (6) is D_T^a-controllable. Now consider the second convergence in (33). Let $\{e_i\}$ be a basis in X. We can select a subsequence $\{n_m^1\}$ of $\{n\}$ so that the sequence $\{\langle \Lambda_{n_m^1} e_1, e_1 \rangle\}$ decreases and goes to 0. Then we can select a subsequence $\{n_m^2\}$ of $\{n_m^1\}$ so that the sequence $\{\langle \Lambda_{n_m^2} e_2, e_2 \rangle\}$ decreases and goes to 0. Continuing this procedure for all e_i and taking the diagonal sequence $\{n_m^m\}$, we obtain that for all e_i, the sequence $\{\langle \Lambda_{n_m^m} e_i, e_i \rangle\}$ decreases and goes to 0. Thus, in

$$\lim_{m\to\infty} \operatorname{tr} \Lambda_{n_m^m} = \lim_{m\to\infty} \sum_{i=1}^{\dim X} \langle \Lambda_{n_m^m} e_i, e_i \rangle \tag{34}$$

the series is so that for all m and for all i,

$$\langle \Lambda_{n_m^m} e_i, e_i \rangle \leq \langle \Lambda_{n_1^1} e_i, e_i \rangle.$$

So, we can interchange the places of the limit and the sum in (34) and obtain that $\lim_{m\to\infty} \operatorname{tr} \Lambda_{n_m^m} = 0$. Without loss of generality, assume that $\lim_{n\to\infty} \operatorname{tr} \Lambda_n = 0$. By Lemma 18 and Lemma 19, if $w^n = u^n - \mathbf{E}\, u^n$, then $w^n \in W_{\text{ad}}$ and

$$\Lambda_n = \operatorname{cov} \mathbf{E}\, (x_T^{u^n} | \mathcal{F}_T^{u^n,\xi}) = \operatorname{cov} \mathbf{E}\, (z_T^{w^n} | \mathcal{F}_T^{w^n,\eta}).$$

Therefore,

$$\lim_{n\to\infty} \mathbf{E}\, \| \mathbf{E}\, (z_T^{w^n} | \mathcal{F}_T^{w^n,\eta}) \|^2 = \lim_{n\to\infty} \operatorname{tr} \Lambda_n = 0.$$

By Lemma 13, this implies $a_T = 0$. Finally from Theorem 15, we obtain that the system (7) is S_T^0-controllable. The theorem is proved.

Proposition 27. *Given $T > 0$, the stochastic control system (4) on U_{ad} is S_T-controllable if and only if it is C_T^a-controllable.*

Proof. This follows from Theorem 24 and Theorem 26.

Theorem 28. *The control system (4) on U_{ad} is S_T-controllable for each $T > 0$ if and only if the control system (6) on V_{ad} is D_T^a-controllable for each $T > 0$.*

Proof. The necessity follows from Theorem 26. For sufficiency, note that by Theorem 17, the D_T^a-controllability of the control system (6) for each $T > 0$ implies the S_T^0-controllability of the control system (7) for each $T > 0$. Thus, by Theorem 26, the control system (4) is S_T-controllable for each $T > 0$. Theorem is proved.

Example 29. Consider the control system (4) with the operators A and B as defined in Example 9. It was shown in Example 9 that the deterministic part of this system is D_T^a-controllable for each $T > 0$. Hence by Theorem 28, this system is S_T-controllable for each $T > 0$.

Example 30. Consider the control system (4) with the operators A and B as defined in Example 10. It was shown in Example 10 that the deterministic part of this system is D_T^a-controllable for each $T > 2$ if some additional condition holds. However, Theorem 28 does not guarantee the S_T-controllability of this system for any $T > 0$.

Finally, we present the following theorem about the equivalence of the above discussed concepts of controllability in a finite dimensional case.

Theorem 31. *Given $T > 0$, if $X = \mathbf{R}^n$ and $U = \mathbf{R}^m$, then the following conditions are equivalent*:

(a) *the rank of the matrix $[B, AB, \ldots, A^{n-1}B]$ is n (Kalman's rank condition)*;

(b) *$\mathcal{Q}_T > 0$ (complete controllability condition)*;

(c) *if $B^*\mathcal{U}_t^*x = 0$ for all $0 \le t \le T$, then $x = 0$ (approximate controllability condition)*;

(d) *$\lambda R(\lambda, -\mathcal{Q}_T)$ converges to zero operator as $\lambda \to 0$ in uniform operator topology (resolvent condition of complete controllability)*;

(e) *$\lambda R(\lambda, -\mathcal{Q}_T)$ converges to zero operator as $\lambda \to 0$ in strong operator topology (resolvent condition of approximate controllability)*;

(f) *$\lambda R(\lambda, -\mathcal{Q}_T)$ converges to zero operator as $\lambda \to 0$ in week operator topology*;

(g) *$R(\lambda, -\mathcal{Q}_T)$ converges as $\lambda \to 0$ in uniform operator topology*;

(h) *$R(\lambda, -\mathcal{Q}_T)$ converges as $\lambda \to 0$ in strong operator topology*;

(i) *$R(\lambda, -\mathcal{Q}_T)$ converges as $\lambda \to 0$ in weak operator topology*;

(j) *the control system (4) on U_{ad} is C_T-controllable*;

(k) *the control system (4) on U_{ad} is S_T-controllable*;

(m) *the control system (6) on V_{ad} is D_T^c-controllable*;

(n) *the control system (6) on V_{ad} is D_T^a-controllable.*

Proof. These follow from the results of this paper and from the other well-known results.

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