Parameter Identification in a Nonautonomous Nonlinear Volterra Integral Equation

Azmy S. Ackleh*
Department of Mathematics
University of Southwestern Louisiana
Lafayette, LA 70504-1010, USA

Sergiu Aizicovici†
Department of Mathematics
Ohio University
Athens, Ohio 45701, USA

Robert R. Ferdinand‡
Department of Mathematics
East Central University
Ada, OK 74820, USA

Simeon Reich§
Department of Mathematics
The Technion-Israel Inst. of Technology
32000 Haifa, Israel

Abstract

We propose a least squares technique for identifying parameters in a nonautonomous nonlinear Volterra integral equation. Numerical results indicating the feasibility of this method are presented.

1 Introduction

The development of theoretical and computational methods for inverse problems involving the identification of linear and nonlinear distributed parameter systems has been the focus of many researchers in the past decade (see, e.g. Ackleh and Fitzpatrick, 1996a; Ackleh and Fitzpatrick 1996b; Ackleh and Reich, 1998; Banks and Kunisch, 1989; Banks *et al.*, 1989; Banks *et al.*, 1990a; Banks *et al.*, 1990b; Banks *et al.*, 1991; Fitzpatrick, 1995). In a recent paper, Aizicovici *et al.* (1993) developed an abstract approximation framework and convergence theory for Galerkin approximations to inverse problems involving autonomous nonlinear Volterra integral equations. Their results guaranteed the convergence of solutions of a sequence of finite dimensional Galerkin approximations to a solution of the original infinite dimensional identification problem. In (Ackleh *et al.*, 1999) we discussed implementation questions involving such approximations to inverse problems and reported on several computational studies and experiments. Both of these papers were concerned with the autonomous case. The goal of this work is the numerical study of an identification problem involving a nonautonomous nonlinear Volterra integral equation.

*E-mail: ackleh@usl.edu. Research supported in part by the Louisiana Education Quality Support Fund under grant #LEQSF(1996-99)-RD-A36.

†E-mail: aizicovi@bing.math.ohiou.edu.

‡E-mail: rferdand@mailclerk.ecok.edu.

§E-mail: sreich@techunix.technion.ac.il. Research was partially supported by the Israel Science Foundation founded by the Israel Academy of Sciences and Humanities, by the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund.

This paper is organized as follows. In Section 2, a least squares technique for numerically identifying parameters in a nonlinear nonautonomous Volterra integral equation is discussed. In Section 3, we present results of parameter estimates obtained using computationally generated data. Finally, in Section 4 we make some concluding remarks.

2 The least squares problem

We consider the following parameter identification problem: Given observations $z(t_i, x)$ at times $\{t_i\}_{i=1}^{K}$, with $0 \leq t_1 < t_2 < \cdots < t_K \leq T$ and a position $x \in (0,1)$, find a parameter $\bar{q} \in Q$ which minimizes the performance index

$$\Phi(u; z) = \sum_{i=1}^{K} \int_0^1 |u(t_i, x, q) - z(t_i, x)|^2 \, dx, \tag{2.1}$$

where for each $q \in Q$, $u(t, x, q)$ is the parameter dependent solution of the following Volterra integral equation

$$\begin{cases} u(q) - b * \{(c(t, u_x(q))\, u_x(q))_x - q(t, u(q))\} = f(t, x) & (t, x) \in [0, T] \times [0, 1] \\ u(t, 0, q) = 0 = u(t, 1, q) & t \in [0, T]. \end{cases} \tag{2.2}$$

Here, we assume that $u \in L^2(0, T; H_0^1(0,1)) \cap C(0, T; L^2(0,1))$, $\dot{u} \in L^2(0, T; H^{-1}(0,1))$, $f \in W^{1,1}(0, T; L^2(0,1))$ and $b \in W^{1,1}(0, T)$. We assume that the function $c(\cdot, \cdot)$ satisfies the following two conditions:

1. The mapping $\theta \to c(t, \theta)$ is C^1 for almost every $t \in [0, T]$;
2. There exists a constant $\delta > 0$ for which

   $$(c(t, \theta)\theta - c(t, \eta)\eta) \cdot (\theta - \eta) \geq \delta |\theta - \eta|^2$$

   for almost every $t \in [0, T]$ and every $\theta,\ \eta \in R$.

To define our admissible parameter set Q we let $D = C_B([0, T] \times \mathrm{R})$, the space of bounded uniformly continuous functions on $[0, T] \times \mathrm{R}$ with the supremum norm, and for fixed values of σ, ρ, and $\bar{u} > 0$, we choose Q to be the D closure of the set

$$\{q \in C_B([0, T] \times \mathbb{R}) \ : \ |q(t, u)| \leq \rho,\ |q_t(t, u)|\,,\ |q_u(t, u)| \leq \sigma,$$

$$q(t, u) = 0 \text{ for } u \leq 0 \text{ and } q(t, u) = q(t), \text{ independent of } u,$$

$$\text{for } u \geq u_q(t) \text{ where } u_q \text{ satisfies } 0 < u_q(t) \leq \bar{u}\}.$$

One can verify that Q is a compact subset of D. We note that other choices for Q are possible. However, the techniques developed here can easily be modified to work for different choices of Q. As a first step towards solving this inverse problem, the solution of equation (2.2) must be approximated. To this end, we use the following Galerkin scheme. Define the approximating solution $u^N(t, x)$ as follows:

$$u^N(t, x) = \sum_{i=0}^{N} \lambda_i^N(t)\, \phi_i^N(x)\,, \tag{2.3}$$

where $\phi_i^N(x)$ represents the i^{th} linear B-spline on the interval $[0,1]$ which is defined by using the uniform mesh $\left\{0, \frac{1}{N}, \frac{2}{N}, \cdots, 1\right\}$ (see, Ackleh *et al.*, 1998; Banks *et al.*, 1989). From the boundary condition in (2.2), it follows that $\lambda_0^N(t) = 0 = \lambda_N^N(t)$. This gives rise to the following finite dimensional problem:

$$\Lambda^N \lambda^N(t) + J^N\left(\lambda^N(t); q\right) = F^N(t), \; t \in [0,T], \tag{2.4}$$

where $\lambda^N(t) = \left(\lambda_1^N, \lambda_2^N, \ldots, \lambda_{N-1}^N\right) \in R^{N-1}$. The matrix Λ^N is an $(N-1)\times(N-1)$ matrix whose $(i,j)^{th}$ element is given by $\left\langle \phi_i^N, \phi_j^N \right\rangle_{i,j=1}^{N-1}$. Using the definition of the linear B-splines ϕ_j^N, $j = 1, \ldots, N-1$, the inner product $\langle a, b\rangle = \int_0^1 a \cdot b \; dx$, $\Delta x = \frac{1}{N}$, and the integral approximation

$$\int_0^1 a(x)\, dx \approx \sum_{i=1}^{N} a\left(\frac{i}{N}\right) \Delta x,$$

we determine $F^N(t)$, an $(N-1)$-dimensional vector, in the following manner:

$$F_i^N(t) = \Delta x f\left(t, \frac{i}{N}\right) \quad i = 1, \ldots, N-1.$$

Similarly, $J^N(\cdot; q) : \mathrm{R}^{N-1} \to \mathrm{R}^{N-1}$ is given by:

$$\begin{aligned} J_1^N(\gamma; q) \quad = \quad & -\int_0^t b(t-s)\left[\frac{\gamma_1(s)}{\Delta x} c\left(s, \frac{\gamma_1(s)}{\Delta x}\right)\right. \\ & \left. + \left(\frac{\gamma_1(s) - \gamma_2(s)}{\Delta x}\right) c\left(s, \frac{\gamma_2(s) - \gamma_1(s)}{\Delta x}\right) - q(s, \gamma_1)\right] ds; \end{aligned}$$

$$\begin{aligned} J_{N-1}^N(\gamma; q) \quad = \quad & -\int_0^t b(t-s)\left[\frac{\gamma_{N-1}(s)}{\Delta x} c\left(s, -\frac{\gamma_{N-1}(s)}{\Delta x}\right)\right. \\ & \left. + \left(\frac{\gamma_{N-1}(s) - \gamma_{N-2}(s)}{\Delta x}\right) c\left(s, \frac{\gamma_{N-1}(s) - \gamma_{N-2}(s)}{\Delta x}\right) - q(s, \gamma_{N-1})\right] ds; \end{aligned}$$

and for $i = 2, \ldots, N-2$

$$\begin{aligned} J_i^N(\gamma; q) = & -\int_0^t b(t-s)\left[\left(\frac{\gamma_i(s) - \gamma_{i-1}(s)}{\Delta x}\right) c\left(s, \frac{\gamma_i(s) - \gamma_{i-1}(s)}{\Delta x}\right)\right. \\ & \left. + \left(\frac{\gamma_i(s) - \gamma_{i+1}(s)}{\Delta x}\right) c\left(s, \frac{\gamma_{i+1}(s) - \gamma_i(s)}{\Delta x}\right) - q(s, \gamma_i)\right] ds. \end{aligned}$$

Since our parameter space Q is infinite dimensional, a finite dimensional approximation for this space is needed as well. To this end, we approximate any $q \in Q$ as follows:

$$\left(I_{M_1, M_2} q\right)(t, u) = \sum_{i=0}^{M_2} \sum_{j=1}^{M_1} q\left(\frac{iT}{M_2}, \frac{j u_q(\frac{iT}{M_2})}{M_1}\right) \psi_{M_1}^j\left(u; u_q(\frac{iT}{M_2})\right) \lambda_{M_2}^i(t; T),$$

where $\psi^j_{M_1}\left(u; u_q(\frac{iT}{M_2})\right)$, $j = 0, \ldots, M_1$, are the linear B-splines defined by using the uniform partition $\{0, \frac{u_q(\frac{iT}{M_2})}{M_1}, \ldots, u_q(\frac{iT}{M_2})\}$ of the interval $[0, u_q(\frac{iT}{M_2})]$. Similarly $\lambda^j_{M_2}(t;T)$, $j = 0, \ldots, M_2$, represent the linear B-splines defined by using the uniform mesh $\{0, \frac{T}{M_2}, \ldots, T\}$. The function $(I_{M_1,M_2}q)(t,u)$ is extended to a continuous function over the entire real line by setting $(I_{M_1,M_2}q)(t,u) = 0$ for any $u \le 0$, and $\psi^j_{M_1}\left(u; u_q(\frac{iT}{M_2})\right) = \psi^j_{M_1}\left(u_q(\frac{iT}{M_2}); u_q(\frac{iT}{M_2})\right)$ for any $u \ge u_q(\frac{iT}{M_2})$. The Peano Kernel Theorem is used to yield

$$\lim_{M_1,M_2\to\infty} I_{M_1,M_2}q = q \quad \text{in } \mathrm{C_B}\left([0,\mathrm{T}]\times\mathbb{R}\right),$$

uniformly in q, for $q \in Q$ (Schultz, 1973). Hence, if $q_M \in Q_M = I_M(Q)$, $M = (M_1, M_2)$, is given by

$$q_M(t,u) = \sum_{i=0}^{M_2}\sum_{j=1}^{M_1} \eta^{i,j}_{M_1,M_2}\psi^j_{M_1}\left(u; u^i_{M_2}\right)\lambda^i_{M_2}(t;T^*),$$

then the solution of our finite dimensional identification problem involves identifying the $(M_1+1)(M_2+1)$ coefficients $\left\{\eta^{i,j}_{M_1,M_2}, u^i_{M_2}\right\}_{i=0,j=1}^{M_2,M_1}$ from a compact subset of $\mathbb{R}^{M_1 M_2+M_1+M_2+1}$ so as to minimize the least squares cost functional $\Phi((u^N\cdot;q_M);z)$, where $u^N(\cdot;q_M)$ is obtained by solving (2.4) with q_M in place of q.

3 Numerical example

In this section we test the least squares technique discussed in Section 2 using data that we computationally generate as follows: We choose the parameter functions

$$b(t) = 1 + \sin(20t), \quad f(t,x) = 3\exp(-20t),$$

$$c(t,\theta) = 1 - 0.5\exp(-3(t+0.5)\theta^2), \; q = (1+10t^2)\frac{u^2}{r+u^2}$$

and we set $r = 0.003$ and the final time $T = 0.15$. Then we solve the finite dimensional problem (2.4) using these parameters with $N = 9$, and let $z(t_i, x)$ $t_i = 0.005(i-1)$, $i = 1, 2, \ldots, 31$, be equal to the solution of the finite dimensional problem at these points in time.

To identify $q(t,u)$, we apply the technique discussed in Section 2 with the assumption that $u_q(t) = \bar{u}$, and set $M = (4,4)$ and $N = 9$. Hence, our identification problem involves estimating 21 constants $\{\eta^{i,j}, \bar{u}\}_{i=0,j=1}^{4,4}$ from a compact subset of $\mathbb{R}^{21}$. In Figures 1 and 2 we present the exact function $q(t,u)$ and the estimated function $q_M(t,u)$, respectively. Figure 3 gives the difference between the exact and estimated functions. We note that the mean of the difference is approximately zero and the final least squares value for this estimate is of the order 10^{-7}.

4 Concluding remarks

We developed a numerical scheme for identifying parameters in a nonlinear nonautonomous Volterra integral equation. The numerical experiments presented here indicate that this technique is very promising. The focus of this paper was on the numerical implementation. General questions concerning the existence and uniqueness of solutions to nonlinear nonautonomous Volterra integral equations as well as the convergence of parameter estimates are expected to be discussed by the authors in a future work. This theory will cover the model discussed in this paper as a special case.

We also remark that we have performed experiments for identifying the parameter $c(t, u_x)$ from computationally generated data using a slight modification of the method presented in Section 2. The estimates were as good as those obtained for the function $q(t, u)$. Finally, we mention that the experiments presented here are computationally very intensive. For example, the experiment with $M = (4, 4)$ presented in Section 3 lasted approximately 22 hours on an Ultra-Sparc 2000.

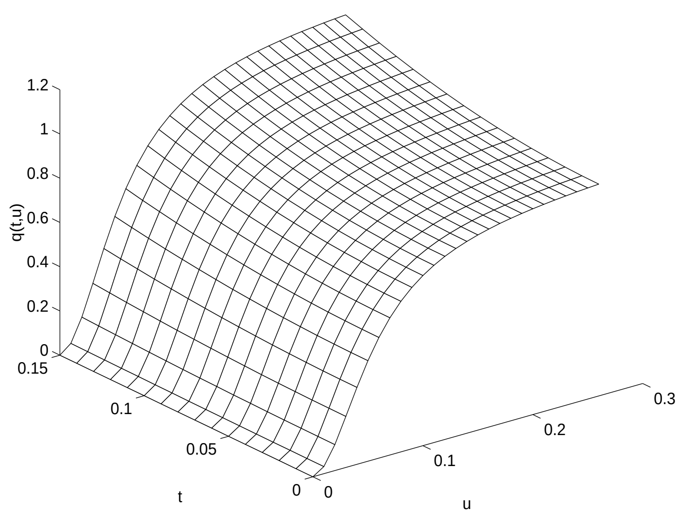

Figure 1: Exact function $q(t, u)$

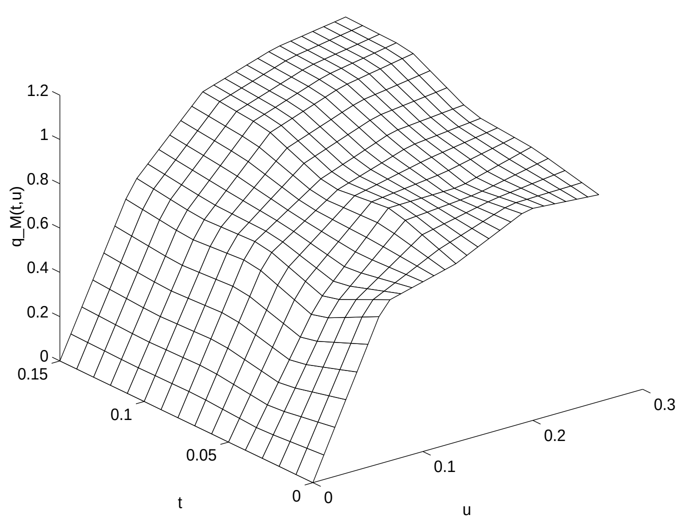

Figure 2: Estimated function $q_M(t, u)$ with $M = (4, 4)$

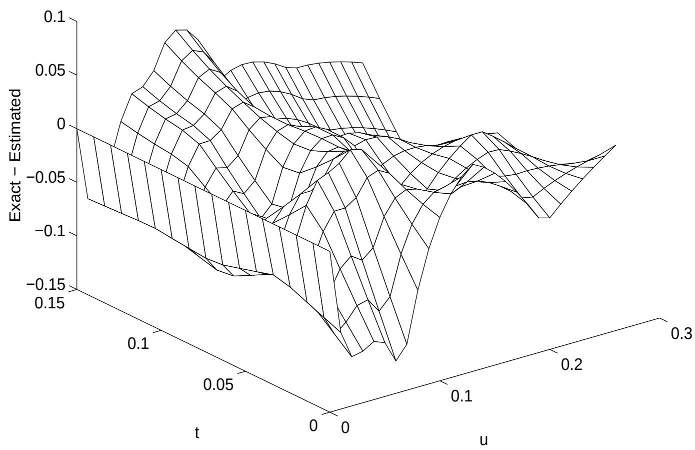

Figure 3: The difference between $q(t,u)$ and $q_M(t,u)$

References

Ackleh, A.S., R. R. Ferdinand, S. Aizicovici and S. Reich (1999). "Numerical Studies of Parameter Estimation Techniques for Nonlinear Volterra Equations", *Theory and Practice of Control and Systems* (A. Tornambè, G. Conte and A.M. Perdon, eds.), World Scientific, Singapore , to appear.

Ackleh, A.S., R.R. Ferdinand, and S. Reich (1998). "Numerical Studies of Parameter Estimation Techniques for Nonlinear Evolution Equations", *Kybernetika*, **6** , pp. 693-712.

Ackleh, A.S. and B. G. Fitzpatrick (1996a). "Estimation of Time Dependent Parameters in General Parabolic Evolution Systems", *Journal of Mathematical Analysis and Applications*, **203**, pp. 464-480.

Ackleh, A.S. and B. G. Fitzpatrick (1996b). "Estimation of Discontinuous Parameters in General Parabolic Systems", *Kybernetika*, **32**, pp. 543-556.

Ackleh, A.S. and S. Reich (1998). "Parameter Estimation in Nonlinear Evolution Equations", *Numerical Functional Analysis and Optimization*, **19**, pp. 933-947.

Aizicovici, S., S. Reich and I.G. Rosen (1993). "An Approximation Theory for the Identification of Nonlinear Volterra Equations", *Numerical Functional Analysis and Optimization,* **14**, pp. 213-227.

Banks, H.T. and K. Kunisch (1989). *Estimation Techniques for Distributed Parameter Systems,* Birkhäuser, Boston.

Banks, H.T., C.K. Lo, S. Reich and I.G. Rosen (1989). "Numerical Studies of Identification in Nonlinear Distributed Parameter Systems", *International Series of Numerical Mathematics*, **91**, pp. 1-20.

Banks, H.T., S. Reich and I. G. Rosen (1990a). "An Approximation Theory for the Identification of Nonlinear Distributed Parameter Systems", *SIAM Journal on Control and Optimization*, **28**, pp. 552-569.

Banks, H.T., S. Reich and I. G. Rosen (1990b). "Estimation of Nonlinear Damping in Second Order Distributed Parameter Systems", *Control Theory and Advanced Technology*, **6**, pp. 395-415.

Banks, H.T., S. Reich and I. G. Rosen (1991). "Galerkin Approximation for Inverse Problems for Nonautonomous Nonlinear Distributed Systems", *Applied Mathematics and Optimization,* **24**, pp. 233-256.

Fitzpatrick, B.G. (1995). "Analysis and Approximation for Inverse Problems in Contaminant Transport and Biodegradation Models", *Numerical Functional Analysis and Optimization*, **16**, pp. 847-866.

Schultz, M.H. (1973). *Spline Analysis*, Prentice-Hall, Englewood Cliffs, N.J.