Generalized versions of Bode's Theorem

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Abstract

The classical theorem of Bode related to the sensitivity integral is generalized. A version with respect to weighted sensitivity integrals is presented. The theorem is also extended to include transfer functions with time delay. Some examples are provided to show its utility.

1. Introduction

The fundamental theorem of Bode (1945) states that the integral over all frequencies of the natural log of the magnitude of the sensitivity function $\ln|S(j\omega)|$ vanishes. This was seen to be true for an open-loop stable transfer function with the difference between the degrees of the numerator and denominator at least 2 . Consequently, it is not possible to have $|S(j\omega)| < 1$ over all frequencies. The extension of Bode's theorem to unstable open-loop systems has been carried out by (Freudenberg and Looze, 1985) . They have proved that the integral of the log of the sensitivity function is proportional to the sum of the unstable open-loop poles. The extension of the theorem to eliminate the limitation that the difference between numerator and denumerator must be greater than 1, has been performed by (Kwaakernak and Sivan, 1972). Their analysis was restricted to asymptotically stable open-loop systems. Wu and Jonekheere (1992) presented a simplified approach to prove the revised generalized Bode's theorem, which combines the work of Freudenberg *et al.* (1985) and of Kwaakernak *et al.* (1972). They also introduced a discrete generalized Bode's theorem . Chen (1997) has provided alternative proofs for those theorems using some properties of Laplace and Z transforms.

A generalized version of Bode's theorem to include weighted sensitivity integrals has been presented by (Skogestad and Postlethwaite, 1997) and by (Chen, 1997). In the following, Bode's theorem will be generalized for a new family of weighted sensitivity integrals.

The theorem has been restricted to rational open-loop transfer functions. Its extended version to include systems with time delay will also be presented.

2. Weighted sensitivity integral

Before introducing a version of Bode's theorem for weighted sensitivity, the following Lemmas are necessary.[1]

Lemma 1 : If 'a' and 'b' are real numbers, then for any t,

$$\int_{-\infty}^{\infty} \ln \left|(j\omega-a)/(j\omega-b)\right|^2 e^{jt\omega}d\omega = 2\pi/t \,\{[e^{-bt}\, u(b) + e^{bt}\, u(-b) - e^{-at}\, u(a) - e^{at}\, u(-a)]\, u(t) -[e^{-bt}\, u(-b) - e^{-at}\, u(-a) + e^{bt}\, u(b) - e^{at}\, u(a)]\, u(-t)\} \tag{2-1}$$

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u(t) is the unit step function.

Proof : By integration by parts:

$$u = \ln(j\omega + a) \quad \text{and} \quad dv = e^{jt\omega}\, d\omega \;\Rightarrow\; v = e^{jt\omega}/(jt)\,,$$

we have

$$\int_{-\sigma}^{\sigma} \ln(j\omega + a)\, e^{jt\omega}\, d\omega = 1/(jt)\,[\ln(j\sigma + a)\, e^{jt\sigma}(- \ln(-j\sigma + a)\, e^{-jt\sigma}] - 1/t \int_{-\sigma}^{\sigma} e^{jt\omega}/(j\omega+a)\, d\omega \tag{2-2}$$

The following cases must be considered.

I) $a\neq 0$, $b\neq 0$: It is known that for $a > 0$ and for any t :

$$1/(2\pi) \int_{-\infty}^{\infty} e^{jt\omega}/(j\omega+a)\, d\omega = e^{-at}\, u(t) \tag{2-3}$$

and after changing variables ($t'=-t$, $\omega'=-\omega$) ,

$$1/(2\pi) \int_{-\infty}^{\infty} e^{jt\omega}/(j\omega-a)\, d\omega = -e^{at}\, u(-t) \tag{2-4}$$

Therefore, for any $a \neq 0$,

$$\int_{-\infty}^{\infty} e^{jt\omega}/(j\omega+a)\, d\omega = 2\pi\, e^{-at}\, [u(t)\, u(a) - u(-t)\, u(-a)] \tag{2-5}$$

Referring to (2-2) with $a,b \neq 0$, and $\sigma \rightarrow 1$:

$$\int_{-\infty}^{\infty} \ln((j(\omega+a)/(j\omega+b))\, e^{jt\omega}\, d\omega =$$

$$= 2(\pi/t\, \{e^{-bt}\, [u(t)\, u(b) - u(-t)\, u(-b)] - e^{-at}\, [u(t)\, u(a) - u(-t)\, u(-a)]\} \tag{2-6}$$

and the expression in (2-1) follows thereafter .

II) If $a=0$ or $b=0$ (or both) :

Although the limit of the unit step function $u(x)$ as $x \rightarrow 0$ doesn't exist, it seems appropriate to introduce the generalized limit :

$$u(x) \rightarrow 0.5\, [u(0+) + u(0-)] = 0.5 \quad \text{as } x \rightarrow 0 \tag{2-7}$$

Accordingly, the validity of (2-1) may be observed. :

Conclusion : For $t \rightarrow 0$, it may be observed that

$$\int_{-\infty}^{\infty} \ln \left|(j\omega-a)/(j\omega-b)\right|^2 d\omega = 2\pi\, (\left|a\right| - \left|b\right|) \tag{2-8}$$

which coincides with the result of (Wu and Jonekheere, 1992).

Removing the constraint that 'a' and 'b' should be real, Lemma 1 is extended as follows :

Lemma 2 : $$\int_{-\infty}^{\infty} \ln \left| (j\omega-z)/(j\omega-p) \right|^2 e^{jt\omega}\, d\omega =$$

$$= 2\pi/t \,\{ e^{pt}\,[u(t)\, u(-Re(p)) - u(-t)\; u(Re(p))] \\ -e^{zt}\,[u(t)\, u(-Re(z)) - u(-t)\, u(Re(z))] \\ +e^{-p^*t}\,[u(t)\, u(Re(p)) - u(-t)\, u(-Re(p))] \\ -e^{-z^*t}\,[u(t)\, u(Re(z)) - u(-t)\, u(-Re(z))]\} \qquad (2\text{-}9)$$

where 'p' and 'z' are complex numbers (p*, z* their conjugates).
The proof is a simple generalization of the proof of Lemma 1.

Conclusion : For $t \to 0$,

$$\int_{-\infty}^{\infty} \ln \left| (j\omega-z)/(j\omega-p) \right|^2 d\omega = 2\pi \left(\left| Re(z) \right| - \left| Re(p) \right| \right) \qquad (2\text{-}10)$$

Consider a single input single output(SISO), linear, time invariant system with open-loop transfer function L(s) given by

$$L(s) = K \prod_{i=1}^{m} (s-z_i) \,/ \prod_{j=1}^{n} (s-p_j) \qquad (2\text{-}11)$$

where $K \neq 0$ and it is chosen so that the closed-loop system is asymptotically stable.
'n_1' of the open-loop poles are stable.
Let :

$$\nu = n - m$$

The sensitivity function is :

$$S(s) = 1/(1+L(s)) \qquad (2\text{-}12)$$

Theorem 1 (Revised Generalized Bode's Theorem for Weighted Sensitivity) :
For any $t > 0$,

$$\int_{-\infty}^{\infty} \ln \left| S(j\omega) \right| e^{jt\omega}\, d\omega = \begin{cases} 2\pi/t \left[\sum_{i=1}^{n} \exp(r_i t) - \sum_{j=1}^{n_1} \exp(p_{Sj}\, t) - \sum_{j=n_1+1}^{n} \exp(-p^*_{Uj}\, t) \right] & \nu \geq 1 \\ -4\pi \ln \left| K+1 \right| \delta(t) + \\ \quad + 2\pi/t \left[\sum_{i=1}^{n} \exp(r_i t) - \sum_{j=1}^{n_1} \exp(-p_{Sj}\, t) - \sum_{j=n_1+1}^{n} \exp(-p^*_{Uj}\, t) \right] & \nu = 0 \end{cases} \qquad (2\text{-}13)$$

where r_i , p_{Ui} , p_{Si} denote the i-th closed-loop pole, the i-th unstable open-loop pole, and the i-th stable closed-loop pole, respectively.

If $n_1=0$, then $\overset{n_1}{\underset{j=1}{}}\ 0$.

Proof : (1) If $\nu \geq 1$, following the proof of theorem 1 of (Wu *et al., 1992)*, it may be shown that

$$S(s) = \prod_{i=1}^{n} (s-p_i) / \prod_{i=1}^{n} (s-r_i) \tag{2-14}$$

where r_i and p_i denote the i-th closed-loop and open-loop poles, respectively. Owing to the closed-loop stability, $\mathrm{Re}\,(r_i) < 0$ $i = 1,...,n$. The result is then achieved using (2-9) (Lemma 2) , for any fixed $t > 0$.

(2) If $\nu = 0$ (n=m), then

$$S(s) = M \prod_{i=1}^{n} (s-p_i) / \prod_{i=1}^{n} (s-r_i) \tag{2-15}$$

for which $M=1/(K+1)$, $K\neq-1$. The case of $K=-1$ may be handled in a different way.
Assume any fixed $t > 0$. Referring to (2-10) (Lemma 2) , the result is obtained.
Comment : The results of Wu *et al., (1992)* may be derived as a limit case of this general theorem, using a generalized limit value of u(t) like in (2-7).

3. Systems with time delay

A different extension of Bode's theorem will now be presented for open-loop transfer functions that include a time delay factor :

$$L\,(s) = L_0\,(s)\,e^{-sT} \tag{3-1}$$

where T is the time delay duration.
It seems worthy to combine the work of Wu *et al., (1992)* together with a rational approximation of the exponential factor. The following is used :

$$e^{-sT} = \lim\,[1 / (1+sT/q)^q] \ \text{ as } \ q \to \infty \tag{3-2}$$

For any integer value of 'q', the additional poles due to the inclusion of the e^{-sT} term are all LHP poles of the open-loop transfer function. Like before, let

$$L_0(s) = K \prod_{i=1}^{m} (s-z_i) / \prod_{j=1}^{n} (s-p_j) \tag{3-3}$$

and 'K' is chosen so that the closed-loop system is asymptotically stable.

Theorem 2 : For $L(s) = L_0(s)\, e^{-sT}$,

$$\int_{-\infty}^{\infty} \ln |S(j\omega)|\, d\omega = 2\pi \sum_i \mathrm{Re}(p_{Ui}) \tag{3-4}$$

where p_{Ui} are the unstable open-loop poles of L_0.

Proof : For any integer ‘q’, approximation (3-2) of e^{-sT} introduces only LHP poles into the new $L(s)$ function, and $\nu = n\text{-}m \rightarrow \infty$ as $q \rightarrow \infty$. Relation (3-4) comes out as a result of theorem 1 in [6]. A different proof has been given by Freudenberg and Looze using complex variable theory (Freudenberg *et al.*, 1985).

Another way of analysis may be carried out through the series expansion :

$$\ln [1+K\, L(j\omega)] = \sum_{p=1}^{\infty} (-1)^{p+1} (K^p / p)\, L^p(j\omega) \tag{3-5}$$

which is valid for any ω for which

$$|K\, L(j\omega)| < 1 \tag{3-6}$$

Therefore, if

$$|K| < \min (1 / |L(j\omega)|) \tag{3-7}$$

then

$$\int_{-\infty}^{\infty} \ln [1+K\, L(j\omega)]\, d\omega = \sum_{p=1}^{\infty} (-1)^{p+1} (K^p / p) \int_{-\infty}^{\infty} L^p(j\omega)\, d\omega \tag{3-8}$$

In case of systems with a time delay : $L = L_0\, e^{-j\omega T}$, the integrals on the right-hand side are Fourier transforms of the powers of the nominal open-loop transfer function.

Example : $L(s) = e^{-sT} / (s+a)$ $a > 0$

It is known that for any natural ‘p’ :

$$1/(2\pi) \int_{-\infty}^{\infty} e^{j\omega pT} / (j\omega+a)^p\, d\omega = (pT)^{p-1} / (p-1)!\; e^{-apT}\, u(T) \tag{3-9}$$

so that

$$\int_{-\infty}^{\infty} e^{-j\omega pT} / (j\omega+a)^p\, d\omega = 2\pi\, (-pT)^{p-1} / (p-1)!\; e^{apT}\, u(-T) \tag{3-10}$$

Consequently, from (3-8),

$$\int_{-\infty}^{\infty} \ln [1+K e^{-j\omega T} / (j\omega+a)]\, d\omega = 0 \tag{3-11}$$

for $T > 0$, $|K| < a$, which is in correlation with the result of theorem 2 (3-4).

4. Numerical Examples

Calculations have been performed using trapezoidal or Simpson quadrature. The following nominal (without time delay) open-loop transfer functions were chosen :

(1) $L_0(s) = 1/(s+1)^2$

(2) $L_0(s) = 1/((s+1)(s-0.5))$

(3) $L_0(s) = 2/(s-1)$

(4) $L_0(s) = (s+1)/(s+2)$

A slight time delay was taken : $T = 0.1$. It was checked that the closed-loop is asymptotically stable for each example. The value 'n' is the order of approximation of e^{-sT} according to (3-2). Identical integration bounds and steps were used for purpose of comparison. The following results were calculated :

Example no.	Exact value of integral (theorem 2)	Numerical quadrature	Approximation order (n)	Integration of approximate term (3-2)
1	0	$2\cdot10^{-7}$	1	$-8\cdot10^{-8}$
2	π	3.1416	1	3.1416
3	2π	6.2809	1	6.2832
4	0	*	2	10^{-8}

Table 4-1 : Comparison between theoretical results, numerical integration and approximate calculation

It is observed that the first order approximation :

$$e^{-sT} \approx 1 - sT \tag{4-1}$$

is practically sufficient for the analysis in case of a slight time delay $T = 0.1$. Obviously, higher order approximations may be needed for larger values of T.

Example no. 4 exhibits the situation for which the regular numerical quadrature does not converge due to the oscillatory nature of the integrand, whereas integration over the approximate term converges right on. This indicates the superiority of the later for the case of $\nu = n - m = 0$.

5. Conclusions

Some extensions of the revised generalized Bode thoerem have been introduced. The theorem is applicable to weighted sensitivity integrals. It is also valid for systems with time delay. The assumption ,that the only way to prove Bode's theorem for nonrational transfer functions is by complex variable theory, was contradicted. The same approach may be carried over to the discrete version. Bode's theorem is thus observed to be much wider and more abundant in its applicability than its original version.

6. References

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