Robust stability analysis of nonlinear processes using empirical state affine models and LMI's

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Abstract
A novel methodology is proposed for the analysis of robust stability of a nonlinear process under linear control. The analysis is based on state-affine empirical models regressed from input-output data. The state model is represented by a set of polynomial matrices nonlinear with respect to the manipulated variables. This model in combination with a linear PI controller results in a closed loop model that can be shown to lie in a polytope of matrices. This allows for the formulation of a Lyapunov stability test in terms of a simple set of LMI's (Linear Matrix Inequalities). This set of inequalities can be also expanded to account for input saturation. The stability analysis produces regions of stability, in terms of the PI controller parameters, that are significantly larger than the regions previously calculated by a μ (Structured Singular Value) test. The conservativeness of the analysis is assessed by comparison to closed loop simulations of a highly nonlinear CSTR (Continuous Stirred Tank Reactor) under PI control.

Keywords: robust stability, Linear Matrix Inequalities, μ analysis, CSTR reactor.

1-Introduction

There are many options to consider when choosing a control strategy for a process, but, regardless of which control strategy is implemented, it will generally be based on a dynamic model of the process. These models generally have varying degrees of accuracy, although no model is ever perfect. Therefore, controllers designed based on these models have to be robust in the presence of model uncertainty or model inaccuracy.

Accurate models of chemical processes are often difficult to obtain mechanistically since many of the parameters are poorly known. In a previous work, Knapp and Budman (1998) use empirical models identified from experimental input/output data to design robust controllers. The models used by Knapp and Budman are based on the work done by Sontag (1979) and further developed by Diaz and Desrochers (1988). Sontag (1979) studies a general type of input/output relation known as a response map. This map specifies how past values of the input affect the present output of the system. Using the response map description, Sontag develops a very general realization theory for a class of nonlinear systems called state affine system i.e. systems that are affine in the state variables. This system is represented by equations of the form:

$$\begin{aligned} \underline{x}(t+1) &= F(u(t))\underline{x}(t) + G(u(t)) \\ y(t) &= H(u(t))\underline{x}(t) \end{aligned} \tag{1}$$

Where $\underline{x}$ is the state vector and F(·), H(·) and G(·) are polynomial matrices.

Diaz and Desrochers (1988) introduced an algorithm to compute a polynomial state affine model for a discrete-time nonlinear system. The method uses the parameters of a NARMA model to directly compute a Behavior matrix which is then used to find the state affine model. Knapp and Budman (1998) have previously shown the formulation of a nonlinear Structured Singular

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Value problem, using the model given by equation (1.2), to study the robustness of the closed loop nonlinear system.

Other approaches have been proposed in the past to study the robustness of nonlinear systems under linear control. For example, a robust linear controller for an exothermic CSTR was designed by Doyle, Packard and Morari (1989) using a structured singular value approach. In Doyle's work, a first principle model of the process is developed and bounds (conic sectors) on the nonlinearities are found from this first principle model. Assuming that the model uncertainty is entirely due to the nonlinearities of the process, it was possible to use these conic bounds to describe the process with uncertainty in the standard M-Δ structure for robust stability analysis. The identification of the bounds shown by Doyle (1990) is not trivial and requires careful observation of the nonlinearities to be bounded. In addition, for many nonlinear chemical processes, adequate first principle models are not available.

The methodology proposed by Knapp and Budman for the analysis and design of robust controllers has the distinctive advantage that it is based solely on empirical models. In this methodology , a NARMA (Nonlinear Autoregressive Moving Average) model is initially identified from input output data using a conventional least squares fitting procedure. Unfortunately, the NARMA model is not suitable for robust stability analysis due to the dependence of the output on past inputs and outputs raised to different powers and in different product combinations. If all these products and high order term will be assumed to be model uncertainty in a robustness analysis, a very conservative design will surely result. On the other hand, nonlinear discrete state affine models, given in equation 1, are ideally suited for the robust stability test. They also have the distinct advantage that the nonlinear terms, which are assumed in this work to be the source of model mismatch with respect to a nominal linear model, are a function of the current inputs u(t) only. This greatly facilitates the calculation of the uncertainty bounds since the inputs have a priori known limits due to e.g. actuator limits. Then, for the purpose of robustness analysis, a minimal state affine model realization of the identified NARMA model was obtained using the method proposed by Diaz and Desrochers (1988) mentioned above. A minimal realization is sought to avoid conservativeness in the robust stability analysis. The conversion of the NARMA model to the state affine model was done through an intermediate step where a Volterra series model of the process is calculated. Based on the robust stability analysis Knapp and Budman (1998) computed regions in terms of the parameter values of a PI controller chosen to control the system using a Structured Singular Value (SSV) test. Unfortunately, Knapp and Budman previously found it that when the window of operating conditions is too large, regions of robust stability cannot be found. Therefore, there is a motivation to reduce the conservativeness of the analysis.

In this work we propose a new formulation of the robust control problem, based still on the state affine models, but using a LMI (Linear Matrix Inequalities) test instead of the SSV test used in the previous work by Knapp and Budman. It will be shown that the LMI test, being less conservative than the SSV test, results in larger regions of robust stability than the ones resulting from the SSV test. The theoretical results are demonstrated through a CSTR process case study.

2-Minimal State Affine Models: Summary

The objective is to find minimal state affine models of the type described by Equation (1). The first step towards the development of such model is the calculation of a NARMA model. This type of models is the easiest to obtain from input output data and is given by the general form:

$$\begin{aligned} y(t) = \sum_{i=1}^{s} F_i^{y} y(t-i) + \sum_{i=1}^{s} F_i^{u} u(t-i) + \sum_{i=1}^{s}\sum_{j=1}^{s} F_{ij}^{yy} y(t-i) y(t-j) + \\ \sum_{i=1}^{s}\sum_{j=1}^{s} F_{ij}^{yu} y(t-i) u(t-j) + \sum_{i=1}^{s}\sum_{j=1}^{s} F_{ij}^{uu} u(t-i) u(t-j) + ☺ \end{aligned} \tag{2}$$

The parameters F_i^y, F_{ij}^{yu} etc. may be calculated using least squares since y(t) is linear with respect to the parameters in equation (2).

In order to obtain an accurate model of the process by linear regression, persistent excitation in the inputs must be provided. For linear models, Gaussian white noise (GWN) is sufficient to excite the system although for nonlinear models this technique often leads to problems of cross-correlation between unmeasured disturbances and the output, resulting in a poor model (Nowak and Van Veen, 1994). Another commonly used method for persistent excitation is that of a pseudorandom binary sequence (PRBS) in the input. This method entails creating pulses of random time length between two distinct input values. In the case of a nonlinear process, however, PRBS does not provide adequate excitation and will also result in a poor model. Nowak and Van Veen show that the use of a pseudorandom multilevel sequence (PRMS) of pulses in the input will provide adequate excitation to the nonlinear process if N+1 distinct input levels are used to model a process of order N. This excitation signal was adopted in this work.

The basis functions in Equation (2) are selected by testing the crosscorrelation between different possible candidate functions to the output y. The basis functions which result in the highest correlations are selected for use in the NARMA model. Then, following Desrochers and Diaz methodology (1988) a Volterra model is obtained from the NARMA model as an intermediate step towards the calculation of the final minimal state affine model. As explained in the introduction, both the NARMA and Volterra models were found unsuitable for robustness analysis due to the dependency of these models on current and past values of the inputs and outputs which leads to a conservative uncertainty description and consequently to a conservative analysis of system robustness. A Behavior matrix B is constructed from the Volterra series coefficients. Then, the algorithm of Sontag(1979) is used to find a minimal state affine model (Equation (1)) from the calculated B matrix. Following this algorithm an m by m nonsingular submatrix of the matrix B is recursively found.

Let ϕ be a mxm nonsingular submatrix of B(f) and let α_i denote the rows of ϕ and β_i denote the columns of ϕ. A state affine model may then be determined from the realization:

$$F_i = \phi^{-1}\phi_i \quad i = 0,\ldots,n \tag{3}$$

$$G_i = \phi^{-1}\left[a_{i\beta_1},\ldots,a_{i\beta_m}\right]^T \quad i = 1,\ldots,n \tag{4}$$

$$H_i = \left[a_{\alpha_1 i},\ldots,a_{\alpha_m i}\right] \quad i = 0,\ldots,n \tag{5}$$

Where ϕ_i is a submatrix of B(f) with the same rows as ϕ but with columns indexed by $\alpha_1 i,\ldots,\alpha_m i$. The model given by equation (1) with coefficients calculated from (3)-(5) is used in the next section to formulate a robust stability problem using Linear Matrix Inequalities (LMI).

3- Robust Stability Analysis

The uncertainty of the system will be assumed to be the difference between the nonlinear model given by equation (1) and a nominal linear model defined by the linear terms of that equation,i.e. the affine model:

$$\begin{aligned} \underline{x}(t+1) &= F_0\,\underline{x}(t) + G_1 u(t) \\ y(t) &= H_0\,\underline{x}(t) \end{aligned} \tag{6}$$

It is also assumed that all of the uncertainty in the nonlinear state affine model is due to the nonlinearity of the process around this operating point. It is therefore possible to describe the model uncertainty, δ_i, in the form:

$$\delta_i = u(t)^i \quad i = 1, \ldots, n \tag{7}$$

Equation (4.10) is the key advantage of the methodology presented here. In general it is very difficult to quantify the uncertainty, δ (Doyle, 1989). In our case, since δ is equal to the input, it can be easily quantified. For example, each input in a process is known to lie between a lower and an upper limit known during the design stage due to, for example, actuator constraints or economic considerations, thus:

$$u_{min} \leq u \leq u_{max} \tag{8}$$

The input u normalized based on these bounds between –1 and 1, i.e. from (7) $|\delta_i| \leq 1$.

Using Equation (7) and assuming that there is no direct effect of the input u on the output y , i.e. $H_1=H_2=..=0$, the state affine model given by (1) may be rewritten as follows:

$$\begin{aligned} \underline{x}(t+1) &= \left(F_0 + \sum_{i=1}^{n} F_i \delta_i \right) \underline{x}(t) + \left(G_1 + \sum_{i=2}^{n} G_i \delta_i \right) u(t) \\ y(t) &= H_0 \underline{x}(t) \end{aligned} \tag{9}$$

Since closed loop robust stability is desired, a linear controller is added to the process. A convenient form of linear controller to use is a state feedback controller given by:

$$\begin{aligned} \xi(t+1) &= A_c \xi(t) + B_c e(t) \\ u(t) &= C_c \xi(t) + D_c e(t) \end{aligned} \tag{10}$$

Where A_c, B_c, C_c, and D_c are control parameters , ξ(t) is the controller internal state, and e(t) is the tracking error in the closed loop expressed as $e(t) = y_d(t) - y(t)$, $y_d(t)$ being the desired setpoint of the process. In this work, a standard PI controller was desired and therefore the PI control parameters were transformed into state space to find the state controller parameters A_c, B_c, C_c, and D_c by:

$$\begin{aligned} A_c &= 1 & B_c &= 1 \\ C_c &= \frac{K_c}{\tau_I} & D_c &= K_c + \frac{K_c}{\tau_I} \end{aligned} \tag{11}$$

Assuming that this controller is used to stabilize the process, set $y_d(t) = 0$ without loss of generality. Then, after substituting (10) into (9) the following closed loop model is obtained:

$$\begin{aligned} \underline{x}(t+1) &= (F_0 + \sum_i F_i \delta_i) \underline{x}(t) + (G_1 + \sum_i G_i \delta_i)(C_c \xi - D_c H_0 \underline{x}(t)) \\ \xi(t+1) &= \xi(t) - H_0 \underline{x}(t) \\ y(t) &= H_0 \underline{x}(t) \end{aligned} \tag{12}$$

By redefining an augmented state vector: $\underline{\eta}(t) = \left[\underline{x}(t) \quad \underline{\xi}(t) \right]^T$, then from (12):

$$\eta(t+1) = \begin{bmatrix} F_0 + \sum_i F_i\delta_i - (G_1 + \sum_i G_i\delta_i)D_c H_0 & (G_1 + \sum_i G_i\delta_i)D_c H_0 \\ \\ -H_0 & 1 \end{bmatrix} \eta(t+1) \qquad (13)$$

This structure was used previously by Knapp and Budman to analyze robust stability using a μ test. Since this analysis lead to conservative results for some situations, in the current work an alternative robust stability test was sought that will result in a less conservative design. The model given by Equation (13) is ideally suited for the application of an alternative robust stability test using LMI's (Linear Matrix Inequalities, see e.g. Gahinet et al (1994)). Equation (13) may be viewed as a polytopic system for which:

$$\eta(t+1) = A\eta(t) \qquad (14)$$

Where A is a time varying matrix due to the time variation of the input u and consequently of δ according to Equation (7) and may be considered as lying in a polytope of matrices given by:

$$\begin{aligned} &A \in Co(A_0, A_1, \ldots\ldots, A_n) \\ &with \quad A = \alpha_1 A_1 + \alpha_2 A_2 + \ldots\ldots + \alpha_n A_n \\ &and \quad \sum_i \alpha_i = 1 \qquad \alpha_i \geq 0 \end{aligned} \qquad (15)$$

Then, for this polytopic model a Lyapunov stability test can be formulated as follows. Assume a Lyapunov function:

$$V_k = \eta_k{}^T P \eta_k \qquad (16)$$

Where P is a positive definite matrix. Then for stability:

$$V_{k+1} - V_k < 0 \qquad (17)$$

Then, from substitution of (14) into (17):

$$A^T PA - P < 0 \qquad (18)$$

Generally, the solution of Equation (18) is intractable since it poses an infinite number of constraints on P. However, since A lies in a polytope of matrices, described by (15), the problem above may be reduced to the solution of a finite set of n Linear Matrix Inequalities (LMI's, see Gahinet et al, 1994). The set of LMI's given by:

$$A_i{}^T PA_i - P < 0 \qquad i = 1,2,\ldots..n \qquad (19)$$

is a sufficient condition for the stability of the system defined by (13).

The vertices of the polytope defined by (15) correspond to the different combinations of the extreme values of δ_i as defined by Equation (7). In defining δ_i it was assumed that the inputs are between upper and lower limits due to for example actuator constraints. However, the saturation problem occurring when the inputs reached these limits was not accounted for. The model given by Equation (13) can be slightly modified to account for the saturation situation. Assuming that an antiwindup scheme will be implemented with the controller such as the error is reset to zero when the calculated input is larger than the saturation limit. Then, the model given by (13) becomes:

$$\eta(t+1) = \begin{bmatrix} F_0 + \sum_i F_i\delta_i - (G_1 + \sum_i G_i\delta_i)D_c H_0 & (G_1 + \sum_i G_i\delta_i)D_c H_0 \\ 0 & 1 \end{bmatrix} \eta(t+1) \qquad (20)$$

Following this, ξ, the internal state of the controller is kept constant when the input is saturated. Therefore, convergence of the whole state vector η is not possible. Instead, during saturation, convergence of only the state vector $\underline{x}$ may be tested. In order to include this situation in the polytopic model given by Equation (13) a new variable ϕ is introduced into the matrix A such as:

$$\eta(t+1) = \begin{bmatrix} F_0 + \sum_i F_i\delta_i - (G_1 + \sum_i G_i\delta_i)D_c H_0 & (G_1 + \sum_i G_i\delta_i)D_c H_0 \Phi \\ -H_0\Phi & \Phi \end{bmatrix} \eta(t+1) \qquad (21)$$

The variable Φ is defined by:

$$\begin{array}{lll} if & |u(t)| > 1 \quad \Rightarrow & \Phi = 0 \\ otherwise & & \Phi = 1 \end{array} \qquad (22)$$

and $\delta_1 = u(t)$ based on equation (7).

The system of LMI 's given by (19) where the matrices A_i are now defined from Equation (21) may be solved using the function FEASP in the LMI Matlab toolbox (Gahinet et al, 1995). Results of this analysis for a CSTR case study are presented in the following section.

4- Case study: continuous stirred tank reactor (CSTR)

The CSTR process data was produced with the following nondimensional equations representing the component and energy balance for the reactor:

$$\dot{x}_1 = -x_1 + Da(1 - x_1)e^{\frac{x_2}{1 + x_2/\gamma}} \qquad (23)$$

$$\dot{x}_2 = -x_2 + \mathbf{B}Da(1 - x_1)e^{\frac{x_2}{1 + x_2/\gamma}} - \beta(x_2 - x_c) \qquad (24)$$

x_1 and x_2 are the reactor's nondimensional concentration and temperature respectively. x_c is the cooling temperature The process has one stable steady state when the nondimensional groups are Da=0.072, B=1.0, β=0.3, and γ=20.0.

The cooling temperature x_c was selected as the manipulated variable and the concentration x_1 as the controlled variable. The controller was chosen to be a PI+antiwindup. Two ranges of cooling temperatures were investigated:

$$\begin{aligned} &i) \quad 5 \le x_c \le 23 \quad and, \\ &ii) \quad -10 \le x_c \le 40 \end{aligned} \tag{25}$$

These values were assumed to result from saturation limits of the cooling rate. The idea in investigating two different ranges of the cooling temperature, one small and the second one larger, was to assess how the set of controller parameters that provide stability in the range of operation, is reduced as the range of operation becomes larger. This was expected due to the larger amount of nonlinearity in the larger range as compared to the smaller one.

Following the procedure explained in the previous section, NARMA models were initially identified for both ranges defined above using multilevel PRBS signals in x_c. Then, from the NARMA models, state affine models were obtained for each one of the cooling rate ranges considered. The resulting state affine models, using the notation given in Equation (9), are:

For $5 \le x_c \le 23$:

$$F_0 = \begin{bmatrix} 0 & -0.0116 & -0.1167 & 0.0286 \\ 0 & 0.5493 & 0 & 0.3889 \\ 1 & 0.1261 & 0.6132 & -0.3097 \\ 0 & -0.0305 & 0 & 0.0524 \end{bmatrix} \quad F_1 = \begin{bmatrix} 0 & 0 & -0.0152 & 0 \\ 0 & 0 & -0.0363 & 0 \\ 0 & 0 & 0.1645 & 0 \\ 1 & 0 & 0.4703 & 0 \end{bmatrix} \tag{24}$$

$$G_1 = [1 \quad 0 \quad 0 \quad 0]^T \quad G_2 = [0 \quad 1 \quad 0 \quad 0]^T$$
$$H_0 = [0.4427 \quad 0 \quad 0.2715 \quad -0.1003]$$

For $-10 \le x_c \le 40$:

$$F_0 = \begin{bmatrix} 0 & 0.0217 & 0.2124 \\ 0 & -0.4827 & 0.3177 \\ 1 & 0.0158 & -0.0235 \end{bmatrix} \quad F_1 = \begin{bmatrix} -0.3111 & 0 & 0.0919 \\ 3.1386 & 0 & 0.4964 \\ -0.2269 & 0 & 0.0670 \end{bmatrix} \tag{25}$$

$$G_1 = [1 \quad 0 \quad 0]^T \quad G_2 = [0 \quad 1 \quad 0]^T \qquad H_0 = [0.4722 \quad 0.0460 \quad 0.2564]$$

Using these models, Knapp and Budman (1998) have previously shown by a μ-Structured Singular Value analysis that a finite region of stability, defined in terms of the proportional gain and the reset time of the PI controller, may be obtained for the smaller range of cooling temperatures. However, the Structured Singular Value analysis was unable to provide a finite stability region for the larger range of cooling temperatures.

In order to conduct an alternative stability analysis using LMI's as explained in the previous section, the vertices of the polytope defined in (15) are found. Notice that for the models given by (24) and (25), only the power one of the input u appears in the models, i.e. $\delta_l = u$ $(-1 \le \delta_1 \le 1)$ and Φ=0 or 1 according to the definition (22). Then, it is easy to verify that the vertices of the polytope for this example, for both ranges of cooling temperatures, are:

$$A_1 = \begin{bmatrix} F_0 + F_1 - G_2 D_c H_0 - G_1 D_c H_0 & (G_1 C_c + G_2 C_c) \\ & \\ -H_0 & 1 \end{bmatrix} \qquad (u = 1, \Phi = 1)$$

$$A_2 = \begin{bmatrix} F_0 + F_1 + G_2 D_c H_0 - G_1 D_c H_0 & (G_1 C_c - G_2 C_c) \\ & \\ -H_0 & -1 \end{bmatrix} \qquad (u = 1, \Phi = 1)$$

$$A_3 = \begin{bmatrix} F_0 + F_1 & 0 \\ & \\ 0 & 0 \end{bmatrix} \qquad (u = 1, \Phi = 0)$$

$$A_4 = \begin{bmatrix} F_0 - F_1 & 0 \\ & \\ 0 & 0 \end{bmatrix} \qquad (u = -1, \Phi = 0) \qquad (26)$$

The first two matrices in (26) bound the cases when the input u is smaller than or exactly equal to the saturation limits. The last two matrices represent the cases where the calculated input u is larger than the saturation limits and the antiwindup scheme is activated, as explained in the previous section. The region of stability, calculated for the small range of cooling temperatures using μ by Knapp and Budman (1998), are given in Figure 1. Since in that work saturation was not explicitly considered in the calculation, initially a region of stability using LMI was computed for comparison purposes without considering the saturation explicitly. This implies to solve the LMI's corresponding to only matrices A_1 and A_2 in Equation (26). It is clear that the LMI analysis is significantly less conservative than the μ analysis resulting in a larger region of stability.

The regions of stability using the LMI calculation with saturation are shown in Figure 2 for the small range of cooling temperature and in Figure 3 for the larger range of this temperature. As expected, the stability region for the larger cooling temperature range is smaller than the one computed for the smaller range due to the different amount of nonlinearity occurring in the process. The indication that the stability bounds are related to the nonlinearity of the empirical models and no to the stability of the nominal linear system, is obvious from closed loop simulations using these models. For instance, a closed loop simulation using the empirical model given by (24) with K_c=3.1 and τ_I=2 results in instability (see Figure 4) as predicted from the bounds shown in Figure 2 whereas the nominal closed loop linear system is still stable. Although unstable, the controlled variable in Figure 4 evolves between bounds due to the assumed input constraints.

The fact that the quadratic Lyapunov stability analysis is consistently less conservative than the μ analysis is known and it was theoretically proven in a previous work by Zhow et al (1992). On the other hand, the novelty of the current work is that the Lyapunov stability analysis is solely based on a special class of empirical models, which resulted in the formulation, and solution of a relatively simple set of LMI's.

To test the conservativeness of the LMI stability analysis, a large number of closed loop simulations were conducted starting from different initial conditions, for different set point values and using the two ranges of saturation limits on the cooling temperature. The simulations were conducted using both the state affine empirical models and the actual CSTR differential equation. For simplicity proportional control only was implemented, i.e. finite K_c and infinite reset time τ_I. The objective was to find the maximal values of K_c that provided stability and compared them to the limits shown in figures 2 and 3. The results are presented in Table 1 together with the stability limits calculated using both LMI and μ analysis for comparison purposes.

Range of cooling temperature x_c	LMI test	State-affine simulations	Actual CSTR equations
5 - 23	2.75	3.8	9
-10 - 40	0.54	2.3	3.4

Table 1: limiting values of the proportional gain K_c for closed loop stability. The gain values are based on normalized manipulated variable values between –1 and 1.

Although this table indicates that the LMI stability analysis is conservative compared to the simulations, it is still possible that worst cases, which will impose tighter stability limits, were not simulated. It is also clear from these results that conservativeness of the analysis is partly due to the approximation of the actual CSTR equations by the empirical state affine model. This is the price to be paid for the basic assumption made in this work that the first principle model of the process is unavailable *a priori* to the designer.

5- Conclusions

A methodology is proposed for the stability analysis of a nonlinear system under linear closed loop control. The key advantage of the method is that it is based solely on empirical models of the nonlinear system calculated directly from experimental data. The empirical state affine models are shown to lie in a polytope of matrices. This fact allows for the formulation of the stability analysis in the form of Linear Matrix Inequalities. Effect of saturation in the manipulated variable can be explicitly taken into account in this analysis. The stability limits in terms of the tuning parameters of a PI controller were calculated for a chemical reactor and shown to be significantly less conservative than the limits previously computed using a μ analysis. A large number of computer simulations were conducted to assess the conservativeness of the method. The conservativeness varies as a function of the operating limits of the manipulated variable (e.g. cooling temperature) under consideration and it also depends on the accuracy of the approximation of the actual process by the identified empirical models.

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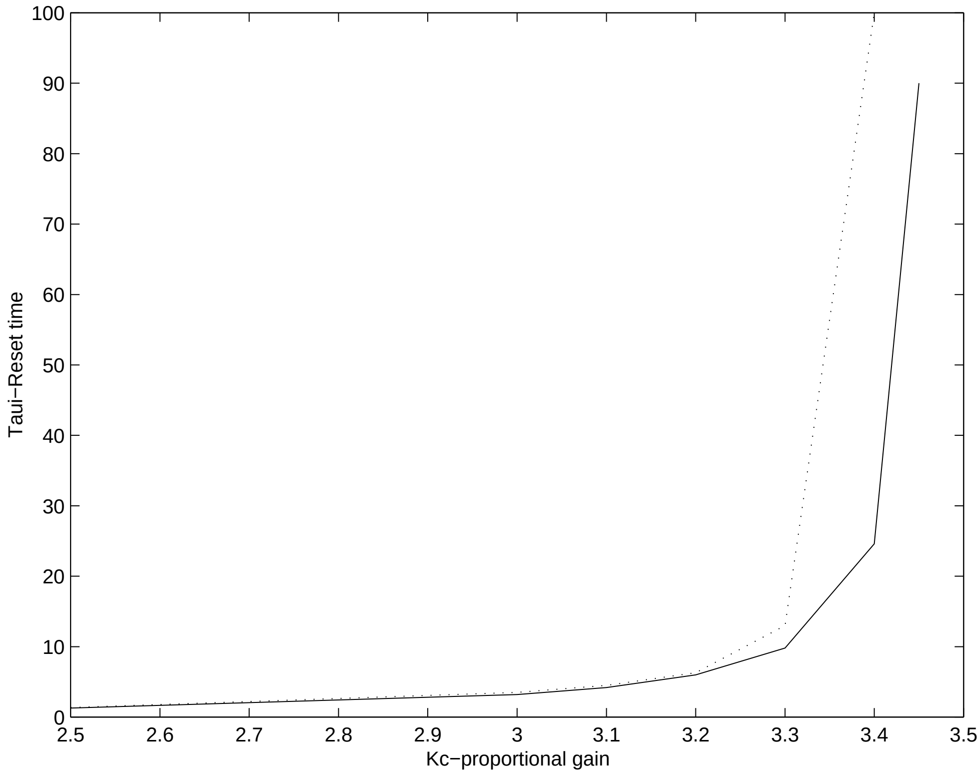

Figure 1- Stability regions (above the lines) based on the μ test (dotted) and the LMI test (solid).

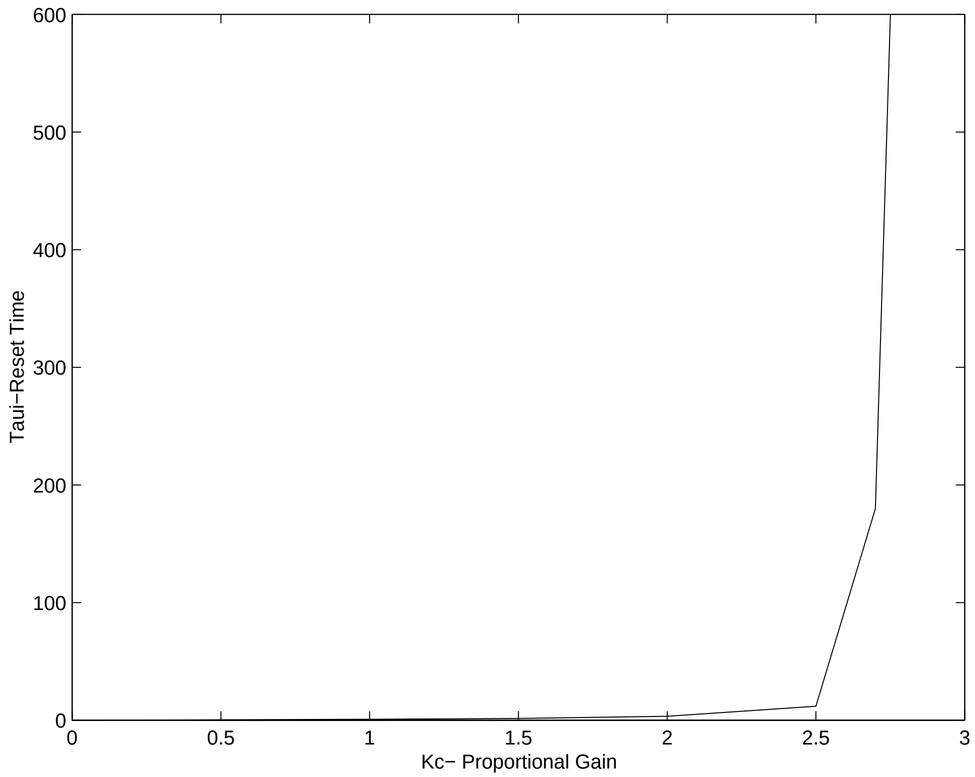

Figure 2- Stability region (above the line) computed with LMI analysis for small x_c range.

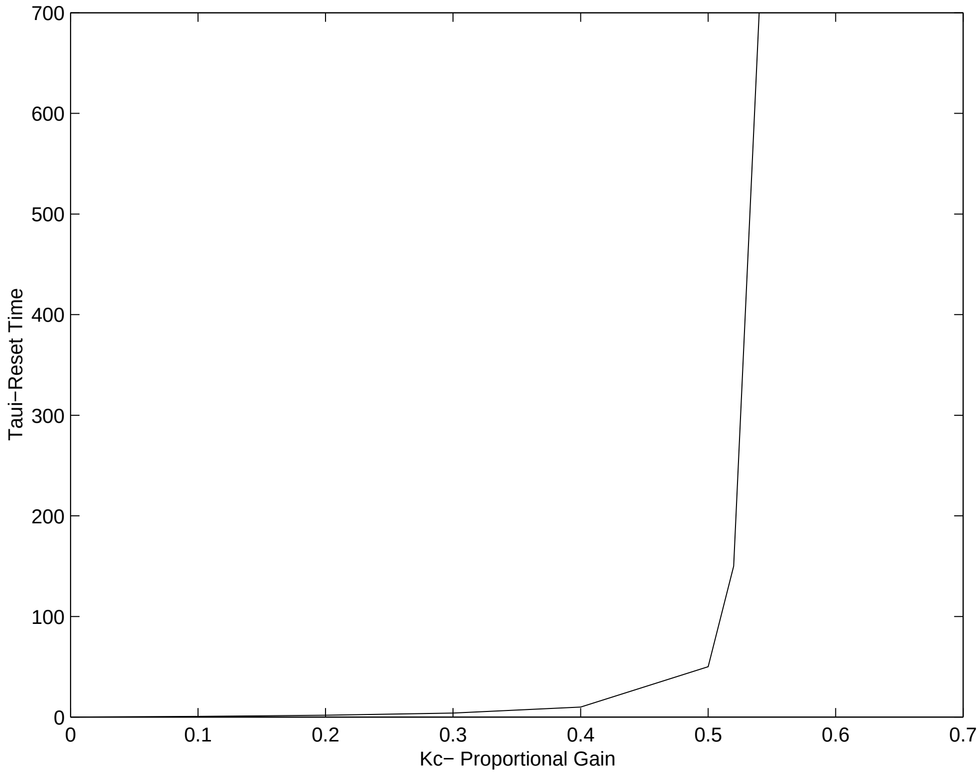

Figure 3- Stability region (above the line) computed from LMI analysis for large x_c range.

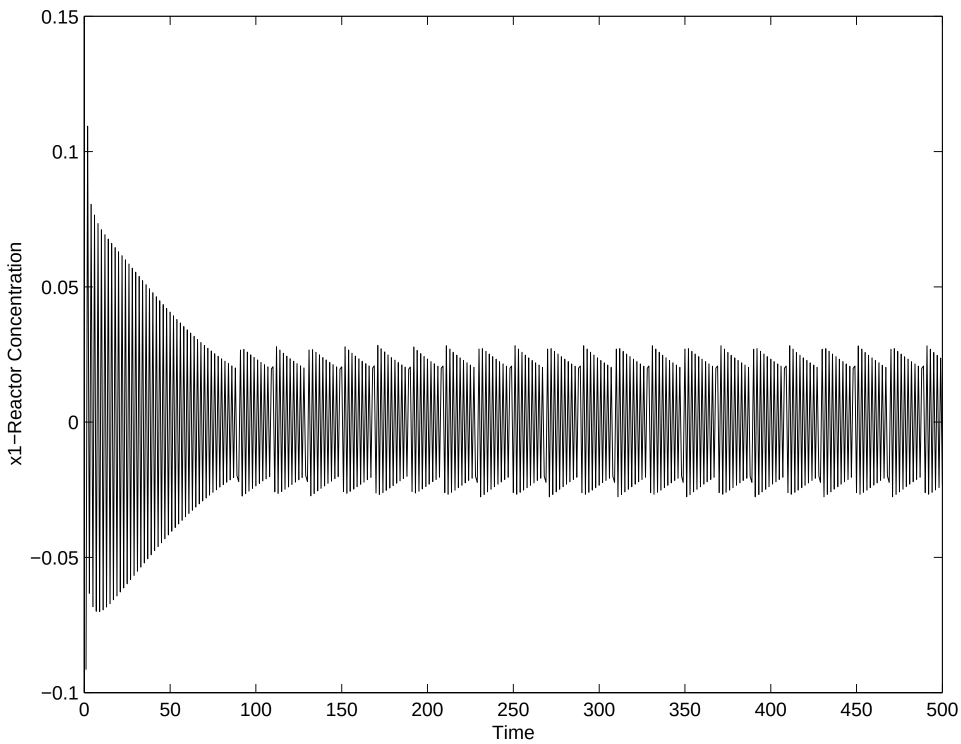

Figure 4- Closed loop response of the concentration for K_c=3.1 and τ_I=2.