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Computation in closed form of the equations of motion for a flexible beam with lumped masses and rotational inertias*

Laura Menini†
Dip. Informatica, Sistemi e Produzione, Università di Roma Tor Vergata
Via di Tor Vergata 110, 00133 Roma, Italy.

Antonio Tornambè‡
Dip. Informatica e Automazione, Università di Roma Tre
via della Vasca Navale 79, 00146, Roma, Italy.
Tel. +39 06 5517 3221, Fax +39 06 557 3030.

Luca Zaccarian§
Dip. Informatica, Sistemi e Produzione, Università di Roma Tor Vergata
Via di Tor Vergata 110, 00133 Roma, Italy.

Abstract

This paper is concerned with the problem of modelling flexible structures in which the effects of distributed elasticity and of both distributed and lumped masses are to be taken into account. The eigenvalue-eigenfunction problem, which constitutes the exact model of the free vibrations of the structure, is given for the general case of a flexible beam having lumped masses and rotational inertiae placed along its length. The proposed method is applied to a simple case study: a clamped beam with a rigid body attached to its free end.

1 Introduction

The interest in obtaining accurate dynamic models of system including distributed elasticity has grown up in the past years, due to the great variety of related practical problems, which arise in robotics, space applications and control of large structures.

Todays industrial robots are characterized by an elevate stiffness of the mechanical structure. Such a feature is necessary to achieve the required precision in positioning the end effector of the robots when payloads vary and/or when dynamic loads, due to speed and acceleration, act on the robotic mechanical structure. As a consequence, the ratio of payload to weight of an industrial robot amounts from 1:10 to 1:30 and less. In a number of robotic applications, such as high-speed manipulators and space applications, a demand for lighter robots which operate with the same precisions and speeds can be thoroughly recognized.

These requirements suggest to take into account the dynamic effects of the distributed elasticity in the design of the controller. As a matter of fact, the rigidity assumption, which is

*This research was supported by ASI and MURST.
†Email: *menini@disp.uniroma2.it*.
‡Email: *tornambe@disp.uniroma2.it*, *tornambe@tovvx1.ccd.uniroma2.it*.
§Email: *zack@disp.uniroma2.it*.

the basis of the classical modelling approach for mechanical systems, is valid as long as the control bandwidth remains well below the first elastic frequencies of the system under consideration. In most cases, such an assumption leads to heavy limitations on the maximum speeds and accelerations supported by the systems themselves.

For these reasons, in the last decade, a significant number of works has been developed about modelling and control of structures having flexible parts, *i.e.*, of mechanical structures constituted by deformable bodies.

In this paper, the influence of lumped masses on the models that can be developed for such systems is investigated, with reference to the case in which the deformable body is a flexible beam, characterized by distributed mass and elasticity.

In Section 2 a general approach, based on Lagrangian techniques, for deriving the exact dynamic model of such structures, in the form of a boundary eigenvalue-eigenfunction problem, is presented, whereas, in Section 3, the possibility of obtaining approximate, finite dimensional, models is illustrated. It is stressed that the advantage of the models derived in this framework is that they are constituted by closed form equations, parametric in the order of approximation. This is a great advantage when the purpose of the modelling effort is to design control laws for the system modelled, as a matter of fact, closed-form equations can be useful when proving theoretically the validity of control algorithms. In Section 4, such a general approach is applied to a significant case study in order to compare two different approximate models which can be derived by following the approach described in Section 3.

2 Equations of motion

The purpose of this section is to write an exact model of a mechanical system constituted by a heavy flexible beam, whose mass per unit length and elastic constant are denoted by ρ and k, respectively, along which H heavy rigid bodies are fit, with $H \in \mathbb{Z}^+$. The mass of the i-th rigid body is denoted by M_i, $i = 1, 2, \ldots, H$. It is assumed that the beam is constrained to move on an horizontal plane, so that the effects of gravity can be neglected. Furthermore, it is assumed that a sufficient number of constraints avoid rigid motions of the beam, so that, when undeformed, the beam lies on the x-axis of a suitable inertial, right-handed and orthonormal, reference frame (x, y, z) whose (x, y) plane coincides with the plane of motion.

Under the assumption of small deformations, the Cartesian coordinates of an infinitesimal element of the beam at time $t \in \mathbb{R}$, $t \geq 0$, expressed in the reference frame (x, y, z), are $(\ell, \alpha(t, \ell), 0)$, with $\ell \in [0, L]$, and L being the length of the undeformed beam.

In the following, in order to simplify the notation, the derivative with respect to t will be denoted by $\dot{}$, and the derivative with respect to ℓ will be denoted by the superscript $'$.

In the case considered here, the beam is subject to some physical constraints, which are restricted to be of the following kind:

$$\begin{aligned}
\alpha(t, 0) &= 0, \quad \forall t \geq 0, && \text{(1a)}\\
\alpha'(t, 0) &= 0, \quad \forall t \geq 0, && \text{(1b)}\\
\alpha(t, L) &= 0, \quad \forall t \geq 0, && \text{(1c)}\\
\alpha'(t, L) &= 0, \quad \forall t \geq 0; && \text{(1d)}
\end{aligned}$$

it is easy to see that any set constituted by two or more of the constraints (1), including at least one of the pairs (1a)-(1b), (1a)-(1c), (1c)-(1d), avoids rigid motions of the beam, and, therefore, can be considered in the present setting. An example is a beam subject to constraints (1a)

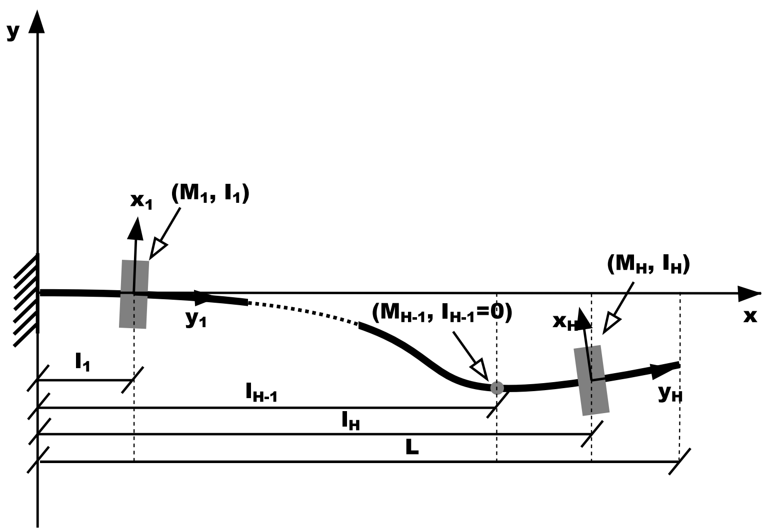

Figure 1: The mechanical system.

and(1b), with the end-point of abscissa $\ell = L$ being free (*i.e.*, (1c) and (1d) do not hold, being $\alpha(t, L)$ and $\dot\alpha(t, L)$ not constrained). Such a case is considered in detail in Section 4.

The H rigid bodies are fixed along the beam in correspondence of the abscissae $\ell_i \in \mathbb{R}$, which are ordered with respect to the subscript so that $0 \le \ell_1 < \ell_2 < \ldots < \ell_H \le L$. The Cartesian coordinates of the center of mass of the i-th rigid body, at time $t \ge 0$, are given by $(\ell_i, \alpha(t, \ell_i), 0)$, $i = 1, 2, \ldots, H$.

For each $i = 1, 2, \ldots, H$, a suitable, right-handed and orthonormal, reference frame (x_i, y_i, z_i) is defined, rigidly connected with the i-th body. Such a reference frame is defined so that, at each time $t \ge 0$ the (x_i, y_i) plane coincides with the plane of motion, the x_i axis coincides with the line tangent to the axis of the deformed beam passing through the point $(\ell_i, \alpha(t, \ell_i), 0)$, and the z_i-axis is parallel to the z-axis. Let I_i be the moment of inertia about the z_i-axis of the i-th rigid body, $i = 1, 2, \ldots, H$. If $I_i > 0$, it is assumed that the i-th rigid body is at rest with respect to the reference frame (x_i, y_i, z_i).

The kinetic energy of the beam is $\frac{\rho}{2}\int_0^L \dot\alpha^2(t, \ell)\mathrm{d}\ell$, the kinetic energy due to the translational motion of the i-th rigid body is $\frac{M_i}{2}\dot\alpha^2(t, \ell_i)$, and (taking into account that, under the assumption of small deformations, $\alpha'(t, \ell_i)$ coincides with the angle from the x-axis to the x_i-axis) the kinetic energy due to the rotation of the i-th rigid body is $\frac{I_i}{2}(\dot\alpha')^2(t, \ell_i)$. Then, the total kinetic energy of the system under consideration can be expressed by the following functional depending on the derivatives of function α:

$$\mathbf{T} := \frac{\rho}{2}\int_0^L \dot\alpha^2(t, \ell)\mathrm{d}\ell + \sum_{i=1}^{H} \frac{M_i}{2}\dot\alpha^2(t, \ell_i) + \sum_{i=1}^{H} \frac{I_i}{2}(\dot\alpha')^2(t, \ell_i),$$

whereas the potential energy is expressed by

$$\mathbf{U} := \frac{k}{2}\int_0^L \left(\alpha''(t, \ell)\right)^2 \mathrm{d}\ell.$$

The dynamic model of the mechanical system under consideration will be obtained by means of the Hamilton principle: the actual path of motion, from time t_1 to time t_2, with initial configuration $\alpha(t_1, \ell) = \alpha_1(\ell)$, for all $\ell \in [0, \ell]$, and final configuration $\alpha(t_2, \ell) = \alpha_2(\ell)$, for all $\ell \in [0, \ell]$, with $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ being fixed, is such that the action integral $\mathcal{A}$, defined by:

$$\mathcal{A} := \int_{t_1}^{t_2} \mathbf{L} \mathrm{d}t,$$

with $\mathbf{L} := \mathbf{T} - \mathbf{U}$ being the lagrangian function, has a stationary value. A sufficiently rich set of admissible functions $\alpha(\cdot, \cdot)$, is the set of continuous functions of both arguments, having continuous second order derivative with respect to the first argument, the time t, continuous first order derivative with respect to the second argument, the abscissa ℓ, and piece-wise continuous (and bounded) second, third and fourth order derivatives with respect to ℓ, with $\alpha''(t, \ell)$, $\alpha'''(t, \ell)$ and $\alpha'^{\nu}(t, \ell)$ continuous for each $t \in [t_1, t_2]$, for each $\ell \in [0, L]$, $\ell \neq \ell_i$, $i = 1, 2, \ldots, H$.

In the remainder of the paper, when confusion cannot arise, if a function is dependent on (t, ℓ), then its arguments are omitted (*i.e.*, symbol α is used instead of $\alpha(t, \ell)$).

Let the lagrangian density function $\mathcal{L}(\dot{\alpha}, \alpha'')$ be defined as follows:

$$\mathcal{L}(\dot{\alpha}, \alpha'') := \frac{1}{2}\rho\, \dot{\alpha}^2 - \frac{1}{2}k\left(\alpha''\right)^2. \tag{2}$$

As $\mathcal{L}(\dot{\alpha}, \alpha'')$ does not depend on α, if $\delta\alpha(t, \ell)$ denotes the variation of $\alpha(t, \ell)$, the first variation of $\mathcal{A}$ can be written as follows:

$$\begin{aligned}\delta\mathcal{A} = &\int_{t_1}^{t_2}\int_0^L \left(\frac{\partial \mathcal{L}(\dot{\alpha}, \alpha'')}{\partial \dot{\alpha}}\delta\dot{\alpha} + \frac{\partial \mathcal{L}(\dot{\alpha}, \alpha'')}{\partial \alpha''}\delta\alpha''\right)\mathrm{d}\ell\,\mathrm{d}t + \\ &\int_{t_1}^{t_2}\sum_{i=1}^{H}\Big(M_i\dot{\alpha}(t, \ell_i)\delta\dot{\alpha}(t, \ell_i) + I_i\dot{\alpha}'(t, \ell_i)\delta\dot{\alpha}'(t, \ell_i)\Big)\mathrm{d}t.\end{aligned} \tag{3}$$

By taking into account the continuity assumptions made about function $\alpha(\cdot, \cdot)$, and the fact that (as $\alpha(t_1, \cdot)$ and $\alpha(t_2, \cdot)$ are fixed) the admissible variations $\delta\alpha(\cdot, \cdot)$ satisfy the following relationships for all $\ell \in [0, L]$, whence also for $\ell = \ell_i$, $i = 1, 2, \ldots, H$:

$$\begin{aligned}\delta\alpha(t_1, \ell) &= 0,\\ \delta\alpha(t_2, \ell) &= 0,\\ \delta\alpha'(t_1, \ell) &= 0,\\ \delta\alpha'(t_2, \ell) &= 0,\end{aligned}$$

the following relationships can be obtained by means of integration by parts:

$$\int_{t_1}^{t_2}\dot{\alpha}(t, \ell_i)\delta\dot{\alpha}(t, \ell_i)\mathrm{d}t = -\int_{t_1}^{t_2}\ddot{\alpha}(t, \ell_i)\delta\alpha(t, \ell_i)\mathrm{d}t, \tag{4a}$$

$$\int_{t_1}^{t_2}\dot{\alpha}'(t, \ell_i)\delta\dot{\alpha}'(t, \ , \ell_i)\mathrm{d}t = -\int_{t_1}^{t_2}\ddot{\alpha}'(t, \ell_i)\delta\alpha'(t, \ell_i)\mathrm{d}t, \tag{4b}$$

$$\int_{t_1}^{t_2}\int_0^L \frac{\partial \mathcal{L}(\dot{\alpha}, \alpha'')}{\partial \dot{\alpha}}\delta\dot{\alpha}\,\mathrm{d}\ell\,\mathrm{d}t = -\int_{t_1}^{t_2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}(\dot{\alpha}, \alpha'')}{\partial \dot{\alpha}}\delta\alpha\,\mathrm{d}\ell\,\mathrm{d}t, \tag{4c}$$

$$\begin{aligned}\int_0^L \frac{\partial \mathcal{L}(\dot{\alpha}, \alpha'')}{\partial \alpha''}\delta\alpha''\,\mathrm{d}\ell = &\sum_{i=0}^{H}\left[\frac{\partial \mathcal{L}(\dot{\alpha}, \alpha'')}{\partial \alpha''}\delta\alpha'\right]_{\ell=\ell_i^+}^{\ell=\ell_{i+1}^-} - \\ &\sum_{i=0}^{H}\left[\frac{\mathrm{d}}{\mathrm{d}\ell}\frac{\partial \mathcal{L}(\dot{\alpha}, \alpha'')}{\partial \alpha''}\delta\alpha\right]_{\ell=\ell_i^+}^{\ell=\ell_{i+1}^-} + \int_0^L \frac{\mathrm{d}^2}{\mathrm{d}\ell^2}\frac{\partial \mathcal{L}(\dot{\alpha}, \alpha''())}{\partial \alpha''}\delta\alpha\mathrm{d}\ell,\end{aligned} \tag{4d}$$

where $\ell_0 := 0$, $\ell_{H+1} := L$, and the following notation has been used for an arbitrary function $f(\cdot)$:

$$
\begin{aligned}
f(T^+) &:= \lim_{\tau \to T^+} f(\tau), \\
f(T^-) &:= \lim_{\tau \to T^-} f(\tau).
\end{aligned}
$$

By means of (2), (4a), (4b), (4c) and (4d), the first variation (3) can be rewritten as follows:

$$
\begin{aligned}
\delta\mathcal{A} = -&\int_{t_1}^{t_2}\int_0^L (\rho\ddot{\alpha} + k\alpha'^{\nu})\delta\alpha \, \mathrm{d}\ell \, \mathrm{d}t + \int_{t_1}^{t_2} \Big(- k\alpha''(t,\, L)\delta\alpha'(t,\, L) + \\
&\sum_{i=1}^{H} \delta\alpha'(t,\, \ell_i) \left(k\,\alpha''(t,\, \ell_i^+) - I_i\, \ddot{\alpha}'(t,\, \ell_i) - k\,\alpha''(t,\, \ell_i^-)\right) + k\alpha''(t,\, 0)\delta\alpha'(t,\, 0) + \\
&k\alpha'''(t,\, L)\delta\alpha(t,\, L) + \sum_{i=1}^{H} \delta\alpha(t,\, \ell_i) \left(-k\,\alpha'''(t,\, \ell_i^+) - M_i\, \ddot{\alpha}(t,\, \ell_i) + k\,\alpha'''(t,\, \ell_i^-)\right) - \\
&k\alpha'''(t,\, 0)\delta\alpha(t,\, 0)\Big)\mathrm{d}t,
\end{aligned}
$$

where, for convenience, the following notation has been used: $\alpha''(t,\, 0^-) := \alpha''(t,\, 0^+)$, $\alpha''(t,\, L^+) := \alpha''(t,\, L^-)$, $\alpha'''(t,\, 0^-) := \alpha'''(t,\, 0^+)$ and $\alpha'''(t,\, L^+) := \alpha'''(t,\, L^-)$, for all $t \in [t_1,\, t_2]$.

By requiring that $\delta\mathcal{A} = 0$ for each admissible variation $\delta\alpha(\cdot,\, \cdot)$ consistent with the subset of the constraints (1a)-(1d) that function $\alpha(\cdot,\, \cdot)$ is assumed to satisfy, the following equations are obtained:

$$\rho\,\ddot{\alpha}(t,\, \ell) + k\,\alpha'^{\nu}(t,\, \ell) = 0, \quad \forall t \in [t_1,\, t_2], \quad \forall \ell \in [0,\, L], \ell \neq \ell_i,\, i = 1,\, 2,\, \ldots,\, H, \tag{5a}$$

$$\alpha'''(t,\, 0) = 0, \quad \forall t \in [t_1,\, t_2], \quad \text{if (1a) does not hold and } \ell_1 \neq 0, \tag{5b}$$

$$k\,\alpha'''(t,\, 0) + M_1\, \ddot{\alpha}(t,\, 0) = 0, \quad \forall t \in [t_1,\, t_2], \quad \text{if (1a) does not hold and } \ell_1 = 0, \tag{5c}$$

$$\alpha''(t,\, 0) = 0, \quad \forall t \in [t_1,\, t_2], \quad \text{if (1b) does not hold and } \ell_1 \neq 0, \tag{5d}$$

$$k\,\alpha''(t,\, 0) - I_1\, \ddot{\alpha}'(t,\, 0) = 0, \quad \forall t \in [t_1,\, t_2], \quad \text{if (1b) does not hold and } \ell_1 = 0, \tag{5e}$$

$$\alpha'''(t,\, L) = 0, \quad \forall t \in [t_1,\, t_2], \quad \text{if (1c) does not hold and } \ell_H \neq L, \tag{5f}$$

$$k\,\alpha'''(t,\, L) - M_H\, \ddot{\alpha}(t,\, L) = 0, \quad \forall t \in [t_1,\, t_2], \quad \text{if (1c) does not hold and } \ell_H = L, \tag{5g}$$

$$\alpha''(t,\, L) = 0, \quad \forall t \in [t_1,\, t_2], \quad \text{if (1d) does not hold and } \ell_H \neq L, \tag{5h}$$

$$k\,\alpha''(t,\, L) + I_H\, \ddot{\alpha}'(t,\, L) = 0, \quad \forall t \in [t_1,\, t_2], \quad \text{if (1d) does not hold and } \ell_H = L, \tag{5i}$$

$$\begin{aligned} k\,\alpha'''(t,\, \ell_i^+) = k\,\alpha'''(t,\, \ell_i^-) - M_i\, \ddot{\alpha}(t,\, \ell_i), \quad \forall t \in [t_1,\, t_2], \\ \forall i = 1,\, 2,\, \ldots,\, H : \ell_i \neq 0 \text{ and } \ell_i \neq L, \end{aligned} \tag{5j}$$

$$\begin{aligned} k\,\alpha''(t,\, \ell_i^+) = k\,\alpha''(t,\, \ell_i^-) + I_i\, \ddot{\alpha}'(t,\, \ell_i), \quad \forall t \in [t_1,\, t_2], \\ \forall i = 1,\, 2,\, \ldots,\, H : \ell_i \neq 0 \text{ and } \ell_i \neq L. \end{aligned} \tag{5k}$$

Equations (5a)-(5k), together with the specified set of constraints chosen among (1a)-(1d) and the initial and final configurations $\alpha(t_1, \cdot)$ and $\alpha(t_2, \cdot)$, at times $t = t_1$, and $t = t_2$, respectively, constitute the exact dynamical behaviour of the mechanical system under consideration. Such a behaviour has been derived under the assumption that $\alpha(t_1, \cdot) = \alpha_1(\cdot)$ and $\alpha(t_2, \cdot) = \alpha_2(\cdot)$, with $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ being fixed; now, it will be shown that the same dynamic behaviour can be obtained by means of equations (5), together with the specified set of constraints chosen among (1a)-(1d) and the initial conditions $\alpha(t_1, \cdot)$ and $\dot{\alpha}(t_1, \cdot)$ at time t_1. This can be done by virtue of the fact that the differential equation (5a), admits an unique solution $\overline{\alpha}(\cdot, \cdot)$ for each $t \in [t_1, t_2]$, for each $\ell \in [0, L]$, from the initial conditions $\overline{\alpha}(t_1, \cdot) = \alpha(t_1, \cdot)$ and $\dot{\overline{\alpha}}(t_1, \cdot) = \dot{\alpha}_1(\cdot)$, whence such a function is certainly a solution of the same equation (5a) with $\alpha(t_1, \cdot) = \alpha_1(\cdot)$ and $\alpha(t_2, \cdot) = \overline{\alpha}(t_2, \cdot)$. However, the two problems are not equivalent, since, in general, more than one solution of equation (5a) with $\alpha(t_1, \cdot) = \alpha_1(\cdot)$ and $\alpha(t_2, \cdot) = \overline{\alpha}(t_2, \cdot)$ may exist. This means that the solution considered here, is only one of the possibly multiple stationarity points of $\mathcal{A}$ with fixed initial and final positions.

Now, solutions of equations (5) satisfying the specified set of constraints chosen among (1a)-(1d) will be sought of the form:

$$\alpha(t, \ell) = \gamma(t)\, \sigma(\ell), \quad \forall t \in [t_1, t_2],\ \forall \ell \in [0, L], \tag{6}$$

where $\gamma(\cdot) : [t_1, t_2] \to \mathbb{R}$ has continuous second order derivative, whereas $\sigma(\cdot) : [0, L] \to \mathbb{R}$ has continuous first order derivative, with $\sigma''(\cdot)$, $\sigma'''(\cdot)$ and $\sigma'^{\nu}(\cdot)$ being continuous for each $\ell \neq \ell_i$, $i = 1, 2, \ldots, H$. With this choice, the partial differential equation (5a) can be recast as follows:

$$\rho\, \frac{\ddot{\gamma}(t)}{\gamma(t)} = -k\, \frac{\sigma'^{\nu}(\ell)}{\sigma(\ell)}, \quad \forall t \in [t_1, t_2], \forall \ell \in [0, L],\ \ell \neq \ell_i,\ i = 1, 2, \ldots, H.$$

Hence, a solution is found if two functions $\gamma(\cdot)$ and $\sigma(\cdot)$, satisfying the above mentioned continuity requirements, are determined so that the following two relationships are satisfied for some real constant c:

$$\rho\, \ddot{\gamma}(t) - c\, \gamma(t) = 0, \quad \forall t \in [t_1, t_2], \tag{7a}$$

$$k\, \sigma'^{\nu}(\ell) + c\, \sigma(\ell) = 0, \quad \forall \ell \in [0, L],\ \ell \neq \ell_i,\ i = 1, 2, \ldots, H, \tag{7b}$$

and the corresponding function $\alpha(\cdot, \cdot)$ given by (6) satisfies relationships (5b)-(5k) and the specified set of constraints chosen among (1a)-(1d).

Notice that, for each $\omega > 0$, the function

$$\sigma(\ell) = a_s \sin(\omega\, \ell) + a_{sh} \sinh(\omega\, \ell) + a_c \cos(\omega\, \ell) + a_{ch} \sin(\omega\, \ell), \quad \ell \in [0, L],$$

where a_s, a_{sh}, a_c, $a_{ch} \in \mathbb{R}$, is solution of (7b) on any of the intervals (ℓ_i, ℓ_{i+1}), $i = 0, 1, \ldots, H$, for $c = -k\, \omega^4$; for such a value of c, the solution of (7a) is

$$\gamma(t) = b_s \sin(\Omega(t - t_1)) + b_c \cos(\Omega(t - t_1)), \quad \forall t \in [t_1, t_2], \tag{8}$$

with $\Omega := \sqrt{\frac{k}{\rho}} \omega^2$ and b_s, $b_c \in \mathbb{R}$.

With the aforementioned definition of the constant c, by means of (6) and (7a), it is possible to recast (7b) and (5b)-(5k) as the following eigenvalue problem:

$$\sigma'^{\nu}(\ell) = \omega^4 \sigma(\ell), \quad \forall \ell \in [0, L],\ \ell \neq \ell_i,\ i = 1, 2, \ldots, H, \tag{9a}$$

$$\sigma(0) = 0, \quad \text{if (1a) holds,} \tag{9b}$$
$$\sigma'''(0) = 0, \quad \text{if (1a) does not hold and } \ell_1 \neq 0, \tag{9c}$$
$$\sigma'''(0) = \frac{M_1}{\rho}\omega^4\sigma(0), \quad \text{if (1a) does not hold and } \ell_1 = 0, \tag{9d}$$
$$\sigma'(0) = 0, \quad \text{if (1b) holds,} \tag{9e}$$
$$\sigma''(0) = 0, \quad \text{if (1b) does not hold and } \ell_1 \neq 0, \tag{9f}$$
$$\sigma''(0) = -\frac{I_1}{\rho}\omega^4\sigma'(0), \quad \text{if (1b) does not hold and } \ell_1 = 0, \tag{9g}$$
$$\sigma(L) = 0, \quad \text{if (1c) holds,} \tag{9h}$$
$$\sigma'''(L) = 0, \quad \text{if (1c) does not hold and } \ell_H \neq L, \tag{9i}$$
$$\sigma'''(L) = -\frac{M_H}{\rho}\omega^4\sigma(L), \quad \text{if (1c) does not hold and } \ell_H = L, \tag{9j}$$
$$\sigma'(L) = 0, \quad \text{if (1d) holds,} \tag{9k}$$
$$\sigma''(L) = 0, \quad \text{if (1d) does not hold and } \ell_H \neq L, \tag{9l}$$
$$\sigma''(L) = \frac{I_H}{\rho}\omega^4\sigma'(L), \quad \text{if (1d) does not hold and } \ell_H = L, \tag{9m}$$
$$\sigma'''(\ell_i^+) = \sigma'''(\ell_i^-) + \frac{M_i}{\rho}\omega^4\sigma(\ell_i), \quad \forall i = 1,\, 2,\, \ldots,\, H:\ \ell_i \neq 0 \text{ and } \ell_i \neq L, \tag{9n}$$
$$\sigma''(\ell_i^+) = \sigma''(\ell_i^-) - \frac{I_i}{\rho}\omega^4\sigma'(\ell_i), \quad \forall i = 1,\, 2,\, \ldots,\, H:\ \ell_i \neq 0 \text{ and } \ell_i \neq L. \tag{9o}$$

Let the integers δ_1, δ_H be defined as follows:

$$\delta_1 = \begin{cases} 1 & \text{if } \ell_1 = 0, \\ 0 & \text{if } \ell_1 > 0, \end{cases}$$
$$\delta_H = \begin{cases} 1 & \text{if } \ell_H = L, \\ 0 & \text{if } \ell_H < L. \end{cases}$$

Now, the function $\sigma(\cdot)$ can be chosen of the following form:

$$\sigma(\ell) = \sum_{i=\delta_1}^{H-\delta_H} \left(a_{i,s}\sin(\omega\,\ell) + a_{i,sh}\sinh(\omega\,\ell) + a_{i,c}\cos(\omega\,\ell) + a_{i,ch}\cosh(\omega\,\ell)\right)\chi_{\ell_i,\ell_{i+1}}(\ell), \quad \forall \ell \in (0,\, L], \tag{10a}$$
$$\sigma(0) = \sigma(0^+), \tag{10b}$$

where, $a_{i,s}$, $a_{i,sh}$, $a_{i,c}$ and $a_{i,ch}$, $i = \delta_1,\, \delta_1+1,\, \ldots,\, H-\delta_H$, are suitable real constants, to be determined, and, for any $a,\, b \in \mathbb{R}$, $a < b$, $\chi_{a,b}(\cdot) : \mathbb{R} \to \mathbb{R}$ is the characteristic function of the interval $(a,\, b]$, which is defined as follows:

$$\chi_{a,b}(\ell) := \begin{cases} 1 & \text{if } \quad a < \ell \leq b, \\ 0 & \text{if } \quad \ell \leq a \text{ or } \ell > b. \end{cases}$$

Continuity of $\sigma(\cdot)$ and $\sigma'(\cdot)$, on the interval $[0,\, L]$, yields the following constraints:

$$\sigma(\ell_i^-) = \sigma(\ell_i^+), \quad \forall i = 1,\, 2,\, \ldots,\, H:\ \ell_i \neq 0,\, \ell_i \neq L, \tag{11a}$$
$$\sigma'(\ell_i^-) = \sigma'(\ell_i^+), \quad \forall i = 1,\, 2,\, \ldots,\, H:\ \ell_i \neq 0,\, \ell_i \neq L. \tag{11b}$$

The derivatives of function $\sigma(\cdot)$ in (10) with respect to ℓ satisfy the following relationships:

$$\begin{aligned}
\sigma'(\ell) &= \omega \sum_{i=\delta_1}^{H-\delta_H} \left(a_{i,\,s}\cos(\omega\,\ell) + a_{i,\,sh}\cosh(\omega\,\ell) - a_{i,\,c}\sin(\omega\,\ell) + a_{i,\,ch}\sinh(\omega\,\ell)\right)\chi_{\ell_i,\,\ell_{i+1}}(\ell), \\
& \qquad \forall \ell \in (0,\,L], \\
\sigma'(0) &= \sigma'(0^+), \\
\sigma''(\ell) &= \omega^2 \sum_{i=0}^{H} \left(-a_{i,\,s}\sin(\omega\,\ell) + a_{i,\,sh}\sinh(\omega\,\ell) - a_{i,\,c}\cos(\omega\,\ell) + a_{i,\,ch}\cosh(\omega\,\ell)\right)\chi_{\ell_i,\,\ell_{i+1}}(\ell), \\
& \qquad \forall \ell \in (0,\,L], \\
\sigma''(0) &:= \sigma''(0^+), \\
\sigma'''(\ell) &= \omega^3 \sum_{i=0}^{H} \left(-a_{i,\,s}\cos(\omega\,\ell) + a_{i,\,sh}\cosh(\omega\,\ell) + a_{i,\,c}\sin(\omega\,\ell) + a_{i,\,ch}\sinh(\omega\,\ell)\right)\chi_{\ell_i,\,\ell_{i+1}}(\ell), \\
& \qquad \forall \ell \in (0,\,L], \\
\sigma'''(0) &:= \sigma'''(0^+),
\end{aligned}$$

which allow to rewrite relationships (9b)-(9o) as follows:

$$a_{\delta_1,\,c} + a_{\delta_1,\,ch} = 0, \quad \text{if (1a) holds,} \tag{12a}$$

$$-a_{0,\,s} + a_{0,\,sh} = 0, \quad \text{if (1a) does not hold and } \ell_1 \neq 0, \tag{12b}$$

$$-a_{1,\,s} + a_{1,\,sh} = \frac{M_1}{\rho}\omega\left(a_{1,\,c} + a_{1,\,ch}\right), \quad \text{if (1a) does not hold and } \ell_1 = 0, \tag{12c}$$

$$a_{\delta_1,\,s} + a_{\delta_1,\,sh} = 0, \quad \text{if (1b) holds,} \tag{12d}$$

$$-a_{0,\,c} + a_{0,\,ch} = 0, \quad \text{if (1b) does not hold and } \ell_1 \neq 0, \tag{12e}$$

$$-a_{1,\,c} + a_{1,\,ch} = -\frac{I_1}{\rho}\omega^3\left(a_{1,\,s} + a_{1,\,sh}\right), \quad \text{if (1b) does not hold and } \ell_1 = 0, \tag{12f}$$

$$a_{H-\delta_H,\,s}\sin(\omega\,L) + a_{H-\delta_H,\,sh}\sinh(\omega\,L) + a_{H-\delta_H,\,c}\cos(\omega\,L) + a_{H-\delta_H,\,ch}\cosh(\omega\,L) = 0, \quad \text{if (1c) holds,} \tag{12g}$$

$$-a_{H,\,s}\cos(\omega\,L) + a_{H,\,sh}\cosh(\omega\,L) + a_{H,\,c}\sin(\omega\,L) + a_{H,\,ch}\sinh(\omega\,L) = 0, \quad \text{if (1c) does not hold and } \ell_H \neq L, \tag{12h}$$

$$\begin{aligned}
&-a_{H-1,\,s}\cos(\omega\,L) + a_{H-1,\,sh}\cosh(\omega\,L) + a_{H-1,\,c}\sin(\omega\,L) + a_{H-1,\,ch}\sinh(\omega\,L) = \\
&\quad -\frac{M_H\omega}{\rho}\left(a_{H-1,\,s}\sin(\omega\,L) + a_{H-1,\,sh}\sinh(\omega\,L) + a_{H-1,\,c}\cos(\omega\,L) + a_{H-1,\,ch}\cosh(\omega\,L)\right), \\
&\qquad \text{if (1c) does not hold and } \ell_H = L,
\end{aligned} \tag{12i}$$

$$a_{H-\delta_H,\,s}\cos(\omega\,L) + a_{H-\delta_H,\,sh}\cosh(\omega\,L) - a_{H-\delta_H,\,c}\sin(\omega\,L) + a_{H-\delta_H,\,ch}\sinh(\omega\,L) = 0, \quad \text{if (1d) holds,} \tag{12j}$$

$$-a_{H,\,s}\sin(\omega\,L) + a_{H,\,sh}\sinh(\omega\,L) - a_{H,\,c}\cos(\omega\,L) + a_{H,\,ch}\cosh(\omega\,L) = 0, \quad \text{if (1d) does not hold and } \ell_H \neq L, \tag{12k}$$

$$\begin{aligned}
&-a_{H-1,\,s}\sin(\omega\,L) + a_{H-1,\,sh}\sinh(\omega\,L) - a_{H-1,\,c}\cos(\omega\,L) + a_{H-1,\,ch}\cosh(\omega\,L) = \\
&\quad \frac{I_H\omega^3}{\rho}\left(a_{H-1,\,s}\cos(\omega\,L) + a_{H-1,\,sh}\cosh(\omega\,L) - a_{H-1,\,c}\sin(\omega\,L) + a_{H-1,\,ch}\sinh(\omega\,L)\right), \\
&\qquad \text{if (1d) does not hold and } \ell_H = L,
\end{aligned} \tag{12l}$$

$$-a_{i,\,s}\cos(\omega\,\ell_i) + a_{i,\,sh}\cosh(\omega\,\ell_i) + a_{i,\,c}\sin(\omega\,\ell_i) + a_{i,\,ch}\sinh(\omega\,\ell_i) =$$

$$-a_{i-1,\,s}\cos(\omega\,\ell_i) + a_{i-1,\,sh}\cosh(\omega\,\ell_i) + a_{i-1,\,c}\sin(\omega\,\ell_i) + a_{i-1,\,ch}\sinh(\omega\,\ell_i) + \frac{M_i\omega}{\rho}\left(a_{i,\,s}\sin(\omega\,\ell_i) + a_{i,\,sh}\sinh(\omega\,\ell_i) + a_{i,\,c}\cos(\omega\,\ell_i) + a_{i,\,ch}\cosh(\omega\,\ell_i)\right), \quad \forall i = 1,\,2,\,\ldots,\,H:\ \ell_i \neq 0,\,\ell_i \neq L, \tag{12m}$$

$$-a_{i,\,s}\sin(\omega\,\ell_i) + a_{i,\,sh}\sinh(\omega\,\ell_i) - a_{i,\,c}\cos(\omega\,\ell_i) + a_{i,\,ch}\cosh(\omega\,\ell_i) = -a_{i-1,\,s}\sin(\omega\,\ell_i) + a_{i-1,\,sh}\sinh(\omega\,\ell_i) - a_{i-1,\,c}\cos(\omega\,\ell_i) + a_{i-1,\,ch}\cosh(\omega\,\ell_i) - \frac{I_i\omega^3}{\rho}\left(a_{i,\,s}\cos(\omega\,\ell_i) + a_{i,\,sh}\cosh(\omega\,\ell_i) - a_{i,\,c}\sin(\omega\,\ell_i) + a_{i,\,ch}\sinh(\omega\,\ell_i)\right), \quad \forall i = 1,\,2,\,\ldots,\,H:\ \ell_i \neq 0,\,\ell_i \neq L, \tag{12n}$$

and relationships (11) as follows:

$$a_{i-1,\,s}\sin(\omega\,\ell_i) + a_{i-1,\,sh}\sinh(\omega\,\ell_i) + a_{i-1,\,c}\cos(\omega\,\ell_i) + a_{i-1,\,ch}\cosh(\omega\,\ell_i) = a_{i,\,s}\sin(\omega\,\ell_i) + a_{i,\,sh}\sinh(\omega\,\ell_i) + a_{i,\,c}\cos(\omega\,\ell_i) + a_{i,\,ch}\cosh(\omega\,\ell_i), \quad \forall i = 1,\,2,\,\ldots,\,H:\ \ell_i \neq 0,\,\ell_i \neq L, \tag{13a}$$

$$a_{i-1,\,s}\cos(\omega\,\ell_i) + a_{i-1,\,sh}\cosh(\omega\,\ell_i) - a_{i-1,\,c}\sin(\omega\,\ell_i) + a_{i-1,\,ch}\sinh(\omega\,\ell_i) = a_{i,\,s}\cos(\omega\,\ell_i) + a_{i,\,sh}\cosh(\omega\,\ell_i) - a_{i,\,c}\sin(\omega\,\ell_i) + a_{i,\,ch}\sinh(\omega\,\ell_i), \quad \forall i = 1,\,2,\,\ldots,\,H:\ \ell_i \neq 0,\,\ell_i \neq L. \tag{13b}$$

Equations (12) and (13) constitute a set of $4(H+1-\delta_1-\delta_H)$ homogeneous linear algebraic equations in the $4(H+1-\delta_1-\delta_H)$ unknowns $a_{i,\,s},\,a_{i,\,sh},\,a_{i,\,c},\,a_{i,\,ch},\ i=\delta_1,\,1,\,\ldots,\,H-\delta_H$, whose coefficients are functions of ω; such equations can obviously be recast as a single matrix equation

$$A(\omega)\,a = 0, \tag{14}$$

where $a \in \mathbb{R}^{4(H+1-\delta_1-\delta_H)}$is given by

$$a := \left[a_{\delta_1,\,s}\ \ a_{\delta_1,\,sh}\ \ a_{\delta_1,\,c}\ \ a_{\delta_1,\,ch}\ \ \ldots\ \ a_{H-\delta_H,\,s}\ \ a_{H-\delta_H,\,sh}\ \ a_{H-\delta_H,\,c}\ \ a_{H-\delta_H,\,ch}\right]^T,$$

and $A(\omega)$ is a square $4(H+1)$ dimensional matrix whose expression is omitted for the sake of brevity. Non null solutions, of the form (10) exist only in correspondence of the values of ω such that matrix $A(\omega)$ is singular; for each $\overline{\omega} \in \mathbb{R}^+$ such that $A(\overline{\omega})$ is singular the number $\overline{\omega}^4$ is called **eigenvalue** and the corresponding function $\sigma(\cdot)$ given by (10) is called **eigenfunction**.

The explicit computation of the eigenvalues and eigenfunctions will be carried out with reference to a significant case in Section 4.

Let $\sigma_1(\cdot),\ \sigma_2(\cdot)$: $[0,\,L] \to \mathbb{R}$ be any two functions satisfying relationships (9b)-(9o). The following relationship can be easily proven by means of integration by parts:

$$\int_0^L \sigma_1(\ell)\,\sigma_2^{\prime\nu}(\ell)\,\mathrm{d}\ell + \sum_{i=1}^{H}\sigma_1(\ell_i)\left(\sigma_2'''(\ell_i^+) - \sigma_2'''(\ell_i^-)\right) - \sum_{i=1}^{H}\sigma_1'(\ell_i)\left(\sigma_2''(\ell_i^+) - \sigma_2''(\ell_i^-)\right) = \int_0^L \sigma_2(\ell)\,\sigma_1^{\prime\nu}(\ell)\,\mathrm{d}\ell + \sum_{i=1}^{H}\sigma_2(\ell_i)\left(\sigma_1'''(\ell_i^+) - \sigma_1'''(\ell_i^-)\right) - \sum_{i=1}^{H}\sigma_2'(\ell_i)\left(\sigma_1''(\ell_i^+) - \sigma_1''(\ell_i^-)\right), \tag{15}$$

with $\sigma_i''(0^-) := 0,\ \sigma_i''(L^+) := 0,\ \sigma_i'''(0^-) := 0,\ \sigma_i'''(L^+) := 0$, for $i = 1,\,2$.

Now, let ω_1^4 and ω_2^4 be two distinct eigenvalues for the eigenvalue problem ((9), (13)) and let $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ be the corresponding eigenfunctions. Equation (9a) yields:

$$\begin{aligned} \sigma_1^{\prime\nu}(\ell) &= \omega_1^4\, \sigma_1(\ell), \quad \forall \ell \in [0,\, L],\, \ell \neq \ell_i,\, i = 1,\, 2,\, \ldots,\, H, &(16)\\ \sigma_2^{\prime\nu}(\ell) &= \omega_2^4\, \sigma_2(\ell), \quad \forall \ell \in [0,\, L],\, \ell \neq \ell_i,\, i = 1,\, 2,\, \ldots,\, H; &(17)\end{aligned}$$

if both sides of equation (16) are multiplied by $\sigma_2(\ell)$ and both sides of equation (17) are multiplied by $\sigma_1(\ell)$, then, by subtracting on both sides and integrating over the interval $[0,\, L]$, the following relationship is obtained:

$$\int_0^L \left(\sigma_2(\ell)\, \sigma_1^{\prime\nu}(\ell) - \sigma_1(\ell)\, \sigma_2^{\prime\nu}(\ell)\right) \mathrm{d}\ell = \left(\omega_1^4 - \omega_2^4\right) \int_0^L \sigma_1(\ell)\, \sigma_2(\ell)\, \mathrm{d}\ell. \tag{18}$$

From (18) and (15), by taking into account (9b)-(9o), the following generalized orthogonality property is obtained, regardless of the set of constraints chosen among (1a)-(1d):

$$\int_0^L \sigma_1(\ell)\, \sigma_2(\ell)\, \mathrm{d}\ell + \sum_{i=1}^{H} \frac{M_i}{\rho}\, \sigma_1(\ell_i)\, \sigma_2(\ell_i) + \sum_{i=1}^{H} \frac{I_i}{\rho}\, \sigma_1'(\ell_i)\, \sigma_2'(\ell_i) = 0, \tag{19}$$

which holds for any pair of eigenfunctions $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ relative to different eigenvalues. Such a relationship is of fundamental importance in the following computations, which will result further simplified if the following normalization condition is imposed:

$$\int_0^L \sigma^2(\ell) \mathrm{d}\ell + \sum_{i=1}^{H} \frac{M_i}{\rho}\, \sigma^2(\ell_i) + \sum_{i=1}^{H} \frac{I_i}{\rho}\, \left(\sigma'(\ell_i)\right)^2 = 1. \tag{20}$$

The possibility of satisfying (20) follows from the properties of the linear system (14); from now on, it will be assumed that, for each eigenvalue ω^4, the choice of the vector a, solution of system (14), is made in order to satisfy (20).

If ω_1^4 and ω_2^4 are two eigenvalues for the eigenvalue problem (9), (11) and $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ are the corresponding eigenfunctions, by virtue of equations (19) and (20), by means of integration by parts it is possible to prove that

$$\int_0^L \sigma_i''(\ell)\, \sigma_j''(\ell)\, \mathrm{d}\ell = \begin{cases} \omega_i^4 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{21}$$

Now, assume that the positive real numbers ω_h, $h \in \mathbb{N}$, ordered with respect to the subscript, such that $A(\omega_h)$ is singular (*i.e.*, such that a corresponding eigenfunction $\sigma_h(\cdot)$ of the form (10) exists) constitute a countable subset of $\mathbb{R}^+$, with $\lim_{h \to +\infty} \omega_h = +\infty$, and that the corresponding set of eigenfunctions $\sigma_h(\cdot)$ is complete in the set of functions $\sigma(\cdot) : [0,\, L] \to \mathbb{R}$ with continuous first order derivatives, piece-wise and bounded second, third and fourth order derivatives, and such that $\sigma''(\cdot)$, $\sigma'''(\cdot)$ and $\sigma^{\prime\nu}(\cdot)$ are continuous for each $\ell \neq \ell_i$, $i = 1,\, 2,\, \ldots,\, H$. Then, for each $t \in [t_1,\, t_2]$, each admissible function $\alpha(t,\, \ell)$ can be expanded into the following absolutely and uniformly convergent series of eigenfunctions:

$$\alpha(t,\, \ell) = \sum_{h=1}^{+\infty} \gamma_h(t)\, \sigma_h(\ell), \quad \forall \ell \in [0,\, L],\, \ell \neq \ell_i,\, i = 1,\, 2,\, \ldots,\, H, \tag{22}$$

so that the infinite dimensional vector $\gamma(t) := [\gamma_1(t)\ \gamma_2(t)\ \ldots]^T$ can be seen as the vector of the components of function $\alpha(t,\, \ell)$ with respect to the basis of eigenfunctions $\sigma_h(\cdot)$, $h \in \mathbb{N}$.

With these positions, by virtue of (19), (20) and (21), for each $t \in [t_1,\, t_2]$, the kinetic energy $\mathbf{T}$ and the potential energy $\mathbf{U}$ of the system under consideration can be rewritten as

$$\begin{aligned} \mathbf{T} &= \frac{\rho}{2} \sum_{h=1}^{+\infty} \dot{\gamma}_h^2(t), \\ \mathbf{U} &= \frac{k}{2} \sum_{h=1}^{+\infty} \omega_h^4 \gamma_h^2(t). \end{aligned}$$

Therefore, the exact dynamical model of the system under consideration can be easily rewritten in terms of vector $\gamma(\cdot)$ by means of the usual Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathbf{L}}{\partial \dot{\gamma}_h} - \frac{\partial \mathbf{L}}{\partial \gamma_h} = 0, \quad h \in \mathbb{N},$$

obtaining the following set of equations

$$\rho\, \ddot{\gamma}_h(t) + k\, \omega_h^4\, \gamma_h(t) = 0, \quad h \in \mathbb{N}, \forall t \in [t_1,\, t_2], \tag{23}$$

which obviously coincide with (7a) rewritten with the values of the constant c corresponding to the eigenvalues ω_h^4, $h \in \mathbb{N}$.

3 Approximate models

The use of reduced order models is a common practice in the design of control laws for high order systems; this approach is much more motivated when dealing with infinite dimensional systems, which do not benefit of the large variety of control design techniques that are available for finite dimensional ones. This motivates the study of finite dimensional models in order to approximate the exact behaviour of infinite dimensional systems, as are the ones considered in this paper.

In order to derive an N-th order, approximate, finite dimensional model of the system under consideration, with N being an arbitrary positive integer, consider the N-th order approximation $\alpha_N(t,\, \ell)$ of the function $\alpha(t,\, \ell)$, obtained by truncating the series in (22):

$$\alpha_N(t,\, \ell) = \sum_{h=1}^{N} \gamma_h(t)\, \sigma_h(\ell), \quad \forall \ell \in [0,\, L],\, \ell \neq \ell_i,\, i = 1,\, 2,\, \ldots,\, H. \tag{24}$$

By virtue of the fact that equations (23) are decoupled (*i.e.*, the time behaviour of each coefficient $\gamma_h(\cdot)$ is not influenced by the one of the other coefficients $\gamma_i(\cdot)$, $i \neq h$), it is clear that the model derived by truncating the series (22) to the first N terms consists of the first N equations in (23):

$$\rho\, \ddot{\gamma}_h(t) + k\, \omega_h^4\, \gamma_h(t) = 0, \quad h = 1,\, 2,\, \ldots,\, N, \forall t \in [t_1,\, t_2], \tag{25}$$

and the coefficients of the truncated series (24) are solutions of the approximate model (25). Then, it follows that the error arising from considering the series (24) instead of (22) can be estimated if the initial conditions $\alpha(t_1,\, \ell)$ and $\dot{\alpha}(t_1,\, \ell)$ are known. As a matter of fact, such functions can be expanded in series as follows:

$$\alpha(t_1,\, \ell) =: \sum_{h=1}^{+\infty} \gamma_{p,\,h} \sigma_h(\ell), \quad \forall \ell \in [0,\, L], \tag{26}$$

$$\dot{\alpha}(t_1,\, \ell) =: \sum_{h=1}^{+\infty} \gamma_{v,\,h} \sigma_h(\ell), \quad \forall \ell \in [0,\, L], \tag{27}$$

and, for each $h \in \mathbb{N}$, the time behaviour of each coefficient $\gamma_h(t)$ depends only on the values of $\gamma_{p,\,h}$ and $\gamma_{v,\,h}$. Moreover, letting $\Omega_h := \sqrt{\frac{k}{\rho}\omega_h^2}$, it can be easily seen that, for each $h \in \mathbb{N}$, the quantity $\dot{\gamma}_h^2(t) + \Omega_h^2\gamma_h^2(t)$ is constant when $t \in [t_1,\, t_2]$, whence it follows that

$$|\gamma_h(t)| \leq \frac{1}{\Omega_h}\sqrt{\gamma_{v,\,h}^2 + \Omega_h^2\gamma_{p,\,h}^2}, \quad \forall t \in [t_1,\, t_2].$$

Now, define the following norm for functions $f(\cdot) : [0,\, L] \to \mathbb{R}$ having continuous first order derivatives:

$$\|f(\cdot)\| := \sqrt{\int_0^L f^2(\ell)\mathrm{d}\ell + \sum_{i=1}^{H} \frac{M_i}{\rho}\, f^2(\ell_i) + \sum_{i=1}^{H} \frac{I_i}{\rho}\, (f'(\ell_i))^2}.$$

By using such a norm, the following relationships are easily proven

$$\begin{aligned} \|\alpha(t,\,\ell) - \alpha_N(t,\,\ell)\| &= \sqrt{\sum_{h=N+1}^{+\infty} \gamma_h^2(t)} \\ &\leq \sqrt{\sum_{h=N+1}^{+\infty} \left(\frac{\gamma_{v,\,h}^2}{\Omega_h^2} + \gamma_{p,\,h}^2(t)\right)}, \quad \forall t \in [t_1,\, t_2], \end{aligned} \tag{28}$$

where the convergence of the series under square root is implied by the absolute convergence of the series (26) and (27) and the fact that $\lim_{h\to+\infty} \Omega_h = +\infty$. The absolute and uniform convergence of the series (26) and (27), and the fact that $\lim_{h\to+\infty} \Omega_h = +\infty$, imply that, for each $\epsilon > 0$, it is possible to determine an integer $\overline{N}$ such that the following inequalities hold:

$$\begin{aligned} \|\alpha(t_1,\,\ell) - \alpha_{\overline{N}}(t_1,\,\ell)\| &= \sqrt{\sum_{h=\overline{N}+1}^{+\infty} \gamma_{p,\,h}^2} \\ &< \frac{\epsilon}{\sqrt{2}}, \\ \|\dot{\alpha}(t_1,\,\ell) - \dot{\alpha}_{\overline{N}}(t_1,\,\ell)\| &= \sqrt{\sum_{h=\overline{N}+1}^{+\infty} \gamma_{v,\,h}^2} \\ &< \frac{\epsilon}{\sqrt{2}}, \\ \Omega_{\overline{N}} &> 1, \end{aligned}$$

thus implying, by virtue of (28), that

$$\|\alpha(t,\,\ell) - \alpha_N(t,\,\ell)\| < \epsilon, \quad \forall N \geq \overline{N}.$$

This means that, for fixed $\alpha(t_1,\,\ell)$ and $\dot{\alpha}(t_1,\,\ell)$, by properly choosing the approximation order N, it is possible to reduce arbitrarily the approximation error, regardless of the length $t_2 - t_1$ of the considered time interval $[t_1,\, t_2]$.

It is easy to see that the above considerations highly depend on the fact that the equations of motion (23) are decoupled; this property has been obtained by considering the series expansion of function $\alpha(t,\, \ell)$ with respect of the set of eigenfunctions $\sigma_h(\cdot)$ for the problem of interest. Since the computation of the eigenvalues and eigenfunctions is not an easy task, in general, one may think about using a different set of functions $\tilde{\sigma}_h(\cdot)$, $h \in \mathbb{N}$, in the series expansion (22) instead of functions $\sigma_h(\cdot)$. Provided that the set $\tilde{\sigma}_h(\cdot)$, $h \in \mathbb{N}$ is complete in the space of admissible functions, for each $t \in [t_1,\, t_2]$, the series defined by

$$\alpha(t,\, \ell) =: \sum_{h=1}^{+\infty} \tilde{\gamma}_h(t)\, \tilde{\sigma}_h(\ell), \quad \forall \ell \in [0,\, L],\, \ell \neq \ell_i,\, i = 1,\, 2,\, \ldots,\, H, \tag{29}$$

is absolutely and uniformly convergent in the interval $[0,\, L]$. For example, the eigenfunctions $\overline{\sigma}_h(\cdot)$ of the mechanical system obtained from the given one by neglecting the lumped masses and inertias, *i.e.* considering $M_i = 0$ and $I_i = 0$, for each $i = 1,\, 2,\, \ldots,\, H$, can be used for this purpose, since in such cases it is well known that the set of the eigenvalues $\overline{\omega}_h^4$ is countable and such that $\lim_{h \to +\infty} \overline{\omega}_h = +\infty$, and that the set of the corresponding eigenfunctions $\overline{\sigma}_h(\cdot)$ is complete. Equations (19), (20) and (21), when rewritten for such a set of eigenfunctions, result in the following relationships:

$$\int_0^L \overline{\sigma}_i(\ell)\, \overline{\sigma}_j(\ell)\, \mathrm{d}\ell = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

$$\int_0^L \overline{\sigma}_i''(\ell)\, \overline{\sigma}_j''(\ell)\, \mathrm{d}\ell = \begin{cases} \overline{\omega}_i^4 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence, the kinetic and potential energies of the system under consideration can be rewritten as

$$\begin{aligned} \mathbf{T} &= \frac{\rho}{2} \sum_{h=1}^{+\infty} \dot{\overline{\gamma}}_h^2(t) + \sum_{i=1}^{H} \frac{M_i}{2} \left(\sum_{h=1}^{+\infty} \dot{\overline{\gamma}}_h(t)\, \overline{\sigma}_h(\ell_i) \right)^2 + \sum_{i=1}^{H} \frac{I_i}{2} \left(\sum_{h=1}^{+\infty} \dot{\overline{\gamma}}_h(t)\, \overline{\sigma}_h'(\ell_i) \right)^2 \\ \mathbf{U} &= \frac{k}{2} \sum_{h=1}^{+\infty} \overline{\omega}_h^4 \overline{\gamma}_h^2(t). \end{aligned}$$

By means of the usual Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathbf{L}}{\partial \dot{\overline{\gamma}}_h} - \frac{\partial \mathbf{L}}{\partial \overline{\gamma}_h} = 0,$$

the exact model of the mechanical system under consideration, rewritten in terms of the coefficients $\overline{\gamma}_h(\cdot)$, is given by the following countable set of equations:

$$\rho\, \ddot{\overline{\gamma}}_h(t) + \sum_{i=1}^{H} M_i\, \overline{\sigma}_h(\ell_i) \sum_{j=1}^{+\infty} \ddot{\overline{\gamma}}_j(t) \overline{\sigma}_j(\ell_i) + \sum_{i=1}^{H} I_i\, \overline{\sigma}_h'(\ell_i) \sum_{j=1}^{+\infty} \ddot{\overline{\gamma}}_j(t) \overline{\sigma}_j'(\ell_i) + k \overline{\omega}_h^4 \overline{\gamma}_h(t) = 0,$$
$$h \in \mathbb{N},\ \forall t \in [t_1,\, t_2]. \tag{30}$$

Such a model is analogous to (23), but is constituted by coupled equations, so that it is not easy to compute the time behaviour of the coefficients $\overline{\gamma}_h(\cdot)$ from fixed initial conditions $\overline{\gamma}_{p,\,h}$, $\overline{\gamma}_{v,\,h}$, $h \in \mathbb{N}$, which are defined on the basis of the initial conditions $\alpha(t_1,\, \ell)$, $\dot{\alpha}(t_1,\, \ell)$, as follows:

$$\alpha(t_1,\, \ell) =: \sum_{h=1}^{+\infty} \overline{\gamma}_{p,\,h} \overline{\sigma}_h(\ell), \quad \forall \ell \in [0,\, L], \ell \neq \ell_i,\, i = 1,\, 2,\, \ldots,\, H, \tag{31}$$

$$\dot{\alpha}(t_1,\, \ell) =: \sum_{h=1}^{+\infty} \overline{\gamma}_{v,\,h} \overline{\sigma}_h(\ell), \quad \forall \ell \in [0,\, L], \ell \neq \ell_i,\, i = 1,\, 2,\, \ldots,\, H. \tag{32}$$

Now, in order to derive a N-order approximate finite dimensional model of the system under consideration, based on the set of functions $\overline{\sigma}_h(\cdot)$, consider the function $\overline{\alpha}^N(t,\,\ell)$ defined as follows:

$$\overline{\alpha}^N(t,\,\ell) = \sum_{h=1}^{N} \overline{\gamma}_h^N(t)\,\overline{\sigma}(\ell), \quad \forall t \in [t_1,\,t_2],\, \forall \ell \in [0,\,L],\, \ell \neq \ell_i,\, i = 1,\,2,\,\ldots,\,H, \tag{33}$$

where the coefficients $\overline{\gamma}_h^N(t)$ constitute the unique solution of the following set of equations:

$$\rho\,\ddot{\overline{\gamma}}_h^N(t) + \sum_{i=1}^{H} M_i\,\overline{\sigma}_h(\ell_i) \sum_{j=1}^{N} \ddot{\overline{\gamma}}_j^N(t)\overline{\sigma}_j(\ell_i) + \sum_{i=1}^{H} I_i\,\overline{\sigma}'_h(\ell_i) \sum_{j=1}^{N} \ddot{\overline{\gamma}}_j^N(t)\overline{\sigma}'_j(\ell_i) + k\overline{\omega}_h^4\overline{\gamma}_h^N(t) = 0,$$
$$h = 1,\,2,\,\ldots,\,N,\, \forall t \in [t_1,\,t_2], \tag{34}$$

from the initial conditions $\overline{\gamma}_h^N(t_1) = \overline{\gamma}_{p,\,h}$, $\dot{\overline{\gamma}}_h^N(t_1) = \overline{\gamma}_{v,\,h}$, $h = 1,\,2,\,\ldots,\,N$.

Equations (34) can be derived by means of the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathbf{L}_N}{\partial \dot{\overline{\gamma}}_h^N} - \frac{\partial \mathbf{L}_N}{\partial \overline{\gamma}_h^N} = 0,$$

if the approximate lagrangian function $\mathbf{L}_N$ is defined as $\mathbf{L}_N := \mathbf{T}_N - \mathbf{U}_N$, where:

$$\begin{aligned}
\mathbf{T}_N &:= \frac{\rho}{2}\sum_{h=1}^{N}\left(\dot{\overline{\gamma}}_h^N(t)\right)^2 + \sum_{i=1}^{H}\frac{M_i}{2}\left(\sum_{h=1}^{N}\dot{\overline{\gamma}}_h^N(t)\,\overline{\sigma}_h(\ell_i)\right)^2 + \sum_{i=1}^{H}\frac{I_i}{2}\left(\sum_{h=1}^{N}\dot{\overline{\gamma}}_h^N(t)\,\overline{\sigma}'_h(\ell_i)\right)^2, \\
\mathbf{U}_N &:= \frac{k}{2}\sum_{h=1}^{N}\overline{\omega}_h^4\left(\overline{\gamma}_h^N(t)\right)^2.
\end{aligned}$$

Since equations (30) are coupled, for $h < N$, the coefficients $\overline{\gamma}_h(t)$ and $\overline{\gamma}_h^N(t)$ do not coincide, in general, whence it is of interest to investigate if some property does exist which guarantees that, for $N \to +\infty$, the function $\overline{\alpha}^N(t,\,\ell)$ defined in (33) is a "good" approximation of function $\alpha(t,\,\ell)$.

4 A case study

In this section the general approach developed in Section 3 is detailed with reference to the problem of modelling a clamped beam with a lumped mass at the end point. Such a problem has been chosen because in this case the solution of the related eigenvalue-eigenfunction problem can be found easily, it is therefore possible to compare the approximated N-th order solutions with the true one.

The eigenvalue problem for such a case can be easily derived from the general case dealt with in Section 2, by considering constraints (1a) and (1b) only, and by letting $H = 1$, $\ell_1 = L$, $M_1 = M$ and $I_1 = 0$. The physical parameters of the system are taken as $L = 0.5$, $\rho = 0.775$, $k = 0.0018$ and $M = \rho\,L/12$.

If a solution of the kind (10) is assumed, relationships (9b)-(9o) result in the following linear system in the unknown coefficients $a_{0,\,s}$, $a_{0,\,sh}$, $a_{0,\,c}$ and $a_{0,\,ch}$:

$$a_{0,\,c} + a_{0,\,ch} = 0, \tag{35a}$$
$$a_{0,\,s} + a_{0,\,sh} = 0, \tag{35b}$$

$$-a_{0,\,s}\cos(\omega L) + a_{0,\,sh}\cosh(\omega L) + a_{0,\,c}\sin(\omega L) + a_{0,\,ch}\sinh(\omega L) = -\frac{M\,\omega}{\rho}\left(a_{0,\,s}\sin(\omega L) + a_{0,\,sh}\sinh(\omega L) + a_{0,\,c}\cos(\omega L) + a_{0,\,ch}\cosh(\omega L)\right), \tag{35c}$$

$$-a_{0,\,s}\sin(\omega L) + a_{0,\,sh}\sinh(\omega L) - a_{0,\,c}\cos(\omega L) + a_{0,\,ch}\cosh(\omega L) = 0. \tag{35d}$$

By simple algebraic manipulations, it is easy to see that system (35) admits non-trivial solutions if and only if the following algebraic equation is satisfied:

$$\frac{M\,\omega}{\rho}\Big(\cos(\omega\,L)\,\sinh(\omega\,L) - \sin(\omega\,L)\,\cosh(\omega\,L)\Big) + \cos(\omega\,L)\,\cosh(\omega\,L) + 1 = 0. \tag{36}$$

The values of $\omega \in \mathbb{R}^+$ such that equation (36) is satisfied, allow to compute the corresponding eigenvalues ω^4; it is easy to see that they constitute a countable set $\{\omega_i,\, i \in \mathbb{N}\}$ and, moreover, if the numbers ω_i are ordered with respect to the subscript, one has $\lim\limits_{i\to+\infty} = +\infty$. As a matter of fact, by taking into account that, for $\omega \gg 1$ one has $\cosh(\omega\,L) \approx \sinh(\omega\,L)$, it follows that $\omega_h \to \dfrac{1}{L}\left(\dfrac{\pi}{4} + h\,\pi\right)$ for $h \to +\infty$.

The first six values ω_i (real and positive fourth roots of the first six eigenvalues ω_i^4), $i = 1$, 2, ..., 6, are reported in Figure 2, whereas the graphs of the corresponding eigenfunctions are reported, with a continuous line, in Figure 3.

ω_i	$\overline{\omega}_i$	$\overline{\overline{\omega}}_i$
3.4883	3.4883	3.4897
8.8643	8.8650	8.9936
14.9756	14.9822	15.2701
21.1156	21.1420	21.4543
27.2963	27.3701	29.1003
33.5024	33.6832	43.1148

Figure 2: True and approximated fourth roots of the first six eigenvalues.

The method described at the end of Section 3 has been applied twice in order to approximate the eigenvalues and the eigenfunctions of this case study, in both cases the order of approximation has been chosen as $N = 6$. The first attempt has been performed by choosing as functions $\tilde{\sigma}_i(\cdot)$ the eigenfunctions $\overline{\sigma}_i(\cdot)$ of the similar eigenvalue problem obtained by neglecting the presence of the lumped mass at the free end. The estimates $\overline{\omega_i}$ of the fourth roots of the first six eigenvalues are reported in Figure 2, whereas the graphs of the corresponding estimates of the eigenfunctions are reported in Figure 3, with a dashed line. It is worth noticing that the first 4 estimates are so close to the true eigenfunctions that their graphs practically coincide with the continuous line representing the true eigenfunctions.

The second attempt has been made by using as functions $\tilde{\sigma}_i(\cdot)$ suitable independent polynomials, chosen so to satisfy the boundary conditions of the clamped beam, in which the presence of the lumped mass at the free end is neglected. The estimates $\overline{\overline{\omega}}_i$ of the fourth roots of the first six eigenvalues are reported in Figure 2, whereas the graphs of the corresponding estimates of the eigenfunctions are reported in Figure 3, with a dotted line.

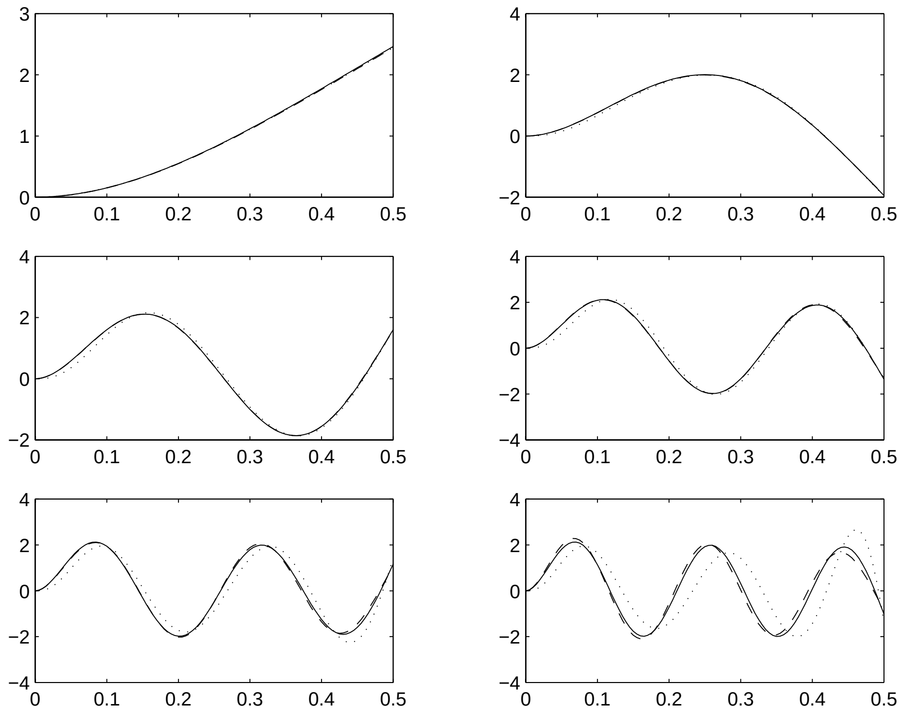

Figure 3: True and approximated eigenfunctions.

5 Concluding remarks

In this paper the problem of modelling flexible structures containing lumped masses and rotational inertiae has been dealt with by means of Lagrangian techniques. It has been shown that, if the eigenvalues and the eigenfunctions of the problem are known, it is possible to approximate the infinite dimensional original system by means of a finite dimensional one, with prescribed degree of accuracy. The approach used in Section 3 to approximate the eigenvalues and the eigenfunctions, leads to the same results as the well known Rayleigh-Ritz method (see (Meirovitch, 1967)). Notice that the convergence of the estimates $\tilde{\omega}_i$ to the values ω_i (the fourth root of the true eigenvalues) can be proven, whereas the convergence of the corresponding estimates of the eigenfunctions is not guaranteed (see (Courant and Hilbert, 1937)).

References

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