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Lagrange Problem for Non-Standard Nonlinear Singularly Perturbed Systems

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Abstract

This paper considers the infinite horizon optimal control problem for an affine singularly perturbed system which is nonlinear in both, the slow and the fast variables. The relationship between this problem and the analogous one for a descriptor system is investigated. An ϵ-independent composite controller is constructed that solves the problem for the descriptor system and leads the full-order system to the near-optimal performance.

Key Words: Singular perturbations, Nonlinear optimal control, Descriptor systems

1 Introduction

Optimal control of a *standard* singularly perturbed system being nonlinear only on the slow variable has been studied by Chow and Kokotovic (1978, 1981), where a two-stage procedure for design of ϵ-independent composite controller has been suggested. High-order approximations to the optimal controller, optimal trajectory and cost in the case of the standard system have been constructed in (Fridman, 1999). In the *non-standard* case the limit of the value function as $\epsilon \to 0$ has been found by Bensoussan (1988).

A descriptor system approach has been introduced by Wang *et al.* (1988) for the case of LQ problem. It has been shown that the optimal (ϵ-independent) regulator for the descriptor system is a near-optimal regulator for the corresponding singularly perturbed system. In the present paper we extend the latter results to the *non-standard* nonlinear system, which is nonlinear in both, the slow and the fast variables.

Our results are based on the geometric approach of Van der Schaft (1991) and Byrnes(1998) which relates Hamilton-Jacobi equations with special invariant manifolds of Hamiltonian systems. We apply method suggested in (Fridman, 1995) to prove the existence of the solution to Hamilton-Jacobi equation and its asymptotic approximation.

The proofs of the theorems are given in the Appendix.

2 Problem Formulation

Consider the optimal control problem for the system

$$\dot{x}_1 = f_1(x_1, x_2) + B_1(x_1, x_2)u, \quad \epsilon\dot{x}_2 = f_2(x_1, x_2) + B_2(x_1, x_2)u, \tag{1}$$

with respect to the functional

$$J = \int_0^\infty [k'(x_1,x_2)k(x_1,x_2) + u'R(x_1,x_2)u]dt, \tag{2}$$

where $x_1(t) \in \mathbf{R^{n_1}}$ and $x_2(t) \in \mathbf{R^{n_2}}$ are the state vectors, $x = \mathrm{col}\{x_1, x_2\}$, $u(t) \in \mathbf{R^m}$ is the control input, and $z \in \mathbf{R^s}$ is the output to be controlled. The functions f_i, B_iR and k are differentiable with respect to x a sufficient number of times. We assume also that $f_i(0,0) = 0, k(0,0) = 0$ and $R = R' > 0$.

The system (1)-(2) has a non-standard singularly perturbed form in the sense that it is nonlinear in both, the slow variable x_1 and the fast variable x_2. In the standard form the system is linear in x_2 (see e.g. Chow and Kokotovic, 1978).

Denote by $|\cdot|$ the Euclidean norm of a vector. Lagrange problem is to find a nonlinear state-feedback

$$u = \beta(x), \quad \beta(0) = 0, \tag{3}$$

that minimizes the cost (2), where $x(0) = x_0$.

For each $\epsilon > 0$ the control law (3) is locally optimal on $\Omega \subset R^{n_1} \times R^{n_2}$ if there exists Ω_1, $0 \in \Omega \subset \Omega_1$, such that the closed-loop trajectories for initial data in Ω remain in Ω_1 and for any initial condition $x_0 \in \Omega$ and any control $u(t)$ for which

$$(i)\ x(t) \in \Omega_1,\ t \geq 0; \quad (ii)\ J(x_0, u) < \infty; \quad (iii) \lim_{t\to\infty} x(t) = 0$$

we have $J(x_0, u_0) \leq J(x_0, u)$ (Byrnes, 1998).

Consider the Hamiltonian function

$$\mathcal{H}_\gamma(x_1, x_2, p_1, p_2) = p_1' f_1(x_1,x_2) + p_2' f_2(x_1,x_2)$$

$$-\frac{1}{2}(p_1' p_2')\begin{pmatrix} S_{11}(x) & S_{12}(x) \\ S_{21}(x) & S_{22}(x)\end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \frac{1}{2}k'(x_1,x_2)k(x_1,x_2), \tag{4}$$

where prime denotes the transposition of a matrix, p_1 and ϵp_2 play the role of the costate variables and $S_{ij} = B_i R^{-1} B_j'$. The corresponding Hamiltonian system has the form:

$$\dot{x}_1 = f_1(x_1, p_1, x_2, p_2), \quad \dot{p}_1 = f_2(x_1, p_1, x_2, p_2),$$

$$\epsilon\dot{x}_2 = f_3(x_1, p_1, x_2, p_2), \quad \epsilon\dot{p}_2 = f_4(x_1, p_1, x_2, p_2), \tag{5a-d}$$

where $f_1 = \left(\frac{\partial \mathbf{H}}{\partial p_1}\right)'$, $f_2 = -\left(\frac{\partial \mathbf{H}}{\partial x_1}\right)'$, $f_3 = \left(\frac{\partial \mathbf{H}}{\partial p_2}\right)'$, $f_4 = -\left(\frac{\partial \mathbf{H}}{\partial x_2}\right)'$.

For each $\epsilon > 0$ the problem is locally solvable on $\Omega \subset \mathbf{R^{n_1}} \times \mathbf{R^{n_2}}$ if there exists a C^2 nonnegative solution $V : \Omega \to \mathbf{R}$ to the HJ partial differential equation

$$\mathbf{H}_\gamma(\mathbf{x_1}, \mathbf{x_2}, \mathbf{V'_{x_1}}, \epsilon^{-1}\mathbf{V'_{x_2}}) = \mathbf{0}, \quad \mathbf{V(0)} = \mathbf{0}, \tag{6}$$

with the property that the system of (5a) and (5c) with $p_1 = V'_{x_1}, p_2 = \epsilon^{-1}V'_{x_2}$ has an asymptotically stable equilibrium at $x = 0$ (Byrnes, 1998), where (V_{x_1}, V_{x_2}) denotes the Jacobian matrix of V. The latter is equivalent to the existence of the invariant manifold of (5)

$$p_1 = Z_1(x_1, x_2), \quad p_2 = Z_2(x_1, x_2), \tag{7}$$

where

$$V_{x_1} = Z_1', \quad V_{x_2} = \epsilon Z_2', \tag{8}$$

with asymptotically stable flow

$$\dot{x}_1 = f_1(x_1, Z_1, x_2, Z_2)\ , \quad \epsilon\dot{x}_2 = f_3(x_1, Z_1, x_2, Z_2), \tag{9}$$

such that $V \geq 0,\ V(0) = 0$ (that implies $V_x(0) = 0$). The optimal controller that solves the problem is then given by

$$u = -R^{-1}[B_1',\ \epsilon^{-1}B_2']\ V_x' = -R^{-1}B_1'Z_1 - R^{-1}B_2'Z_2. \tag{10}$$

We shall find ϵ-independent controller that near-optimally solves the local Lagrange problem on some ϵ-independent neighborhood Ω for all small enough ϵ.

3 Main results

3.1. Composite controller design. Consider the linearization of (1) at $x = 0$:

$$E_\epsilon \dot{x} = Ax + B_0 u \tag{11}$$

with the quadratic functional

$$J + \int_0^\infty [x'C'Cx + u'R(0)u]dt, \tag{12}$$

where

$$E_\epsilon = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \epsilon I_{n_2} \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{10} \\ B_{20} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

$$A_{ij} = \frac{\partial f_i}{\partial x_j}(0,0), B_{i0} = B_i(0,0),\ C_i = \frac{\partial k}{\partial x_i}(0,0),\ i = 1,2; j = 1,2.$$

Hamiltonian system that corresponds to (11), (12) can be written in the form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{x}_2 \\ \dot{p}_2 \end{bmatrix} = Ham_\gamma \begin{bmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix}, \quad Ham_\gamma = \begin{bmatrix} T_{11} & T_{12} \\ \epsilon^{-1}T_{21} & \epsilon^{-1}T_{22} \end{bmatrix}, \quad T_{ij} = \begin{bmatrix} A_{ij} & -S_{ij}(0) \\ -C_i'C_j & -A_{ji}' \end{bmatrix}. \tag{13a-c}$$

To guarantee that for all small ϵ this LQ problem is solvable we assume (Wang *et al.*, 1988):
A1. The exponential modes of descriptor system (11), where $\epsilon = 0$, are controllable-observable, i.e. both pencils $[sE_0 - A; B]$ and $[sE_0' - A'; C]$ are of full row rank for any finite s.

A2. The triple $\{A_{22},\ B_{20},\ C_2\}$ is controllable-observable.

Under A2 a fast Riccati equation

$$A_{22}'X_f + X_f A_{22} + C_2'C_2 - X_f S_{22}(0) X_f = 0 \tag{14}$$

has a solution $X_f = X_f' \geq 0$, such that the matrix $\Lambda_f = A_{22} - S_{22}(0)X_f$ is Hurwitz.

Under A1 and A2 a slow algebraic Riccati equation

$$X_0A_0 + A_0'X_0 - X_0S_0X_0 + Q_0 = 0, \tag{15}$$

where

$$\begin{bmatrix} A_0 & -S_0 \\ -Q_0 & -A_0' \end{bmatrix} = T_{11} - T_{12}T_{22}^{-1}T_{21} = T_0, \tag{16}$$

has a solution $X_0 = X_0' \geq 0$ such that the matrix $\Lambda_s = A_0 - S_0 X_0$ is Hurwitz.

It is known (Wang *et al.*, 1988) that under A1 and A2 for all small enough ϵ the linear controller

$$u_l = -R^{-1}(0)B_{10}' X_0 x_1 - R^{-1}B_{20}'(X_c x_1 + X_f x_2), \quad X_c = [X_f, \ -I]T_{22}^{-1}T_{21}\begin{bmatrix} I \\ X_0 \end{bmatrix} \tag{17}$$

solves the LQ problem.

Lemma 3.1 *Under A1 and A2*
(i) *The matrix T_{22} has n_2 eigenvalues with negative real parts and n_2 with positive ones.*
(ii) *The matrix T_0 has n_1 eigenvalues with negative real parts and n_1 with positive ones.*
(iii) *In a small enough neighborhood of $\mathbf{R^{n_2}} \times \mathbf{R^{n_2}}$ containing* 0 *the system of equations*

$$f_3(x_1, p_1, x_2, p_2) = 0, \quad f_4(x_1, p_1, x_2, p_2) = 0,$$

has an isolated solution

$$x_2 = \phi(x_1, p_1), \quad p_2 = \psi(x_1, p_1) \tag{18}$$

and the matrix

$$\begin{pmatrix} \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial p_2} \\ \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial p_2} \end{pmatrix}\Big|_{(x_2,p_2)=(\phi(x_1,p_1),\psi(x_1,p_1))}$$

has n_2 stable eigenvalues λ, $\mathrm{Re}\lambda < -\alpha < 0$, and n_2 unstable ones λ, $\mathrm{Re}\lambda > \alpha$.

Proof. Item (i) follows from A1 (Wang *et al.*, 1988). To prove (ii) consider the matrix Ham_γ. It has one group of $2n_1$ small eigenvalues $O(\epsilon)$ close to those of T_0 and another group of $2n_2$ large eigenvalues $O(1)$ close to those of $\epsilon^{-1}T_{22}$ (Kokotovic *et al.*, 1986). Then (ii) follows from the symmetry of the eigenvalues of Ham_γ, of T_{22} and thus of T_0 and from the relation

$$T_0 = \begin{pmatrix} I & 0 \\ X_0 & I \end{pmatrix}\begin{pmatrix} \Lambda_s & -S_0 \\ 0 & -\Lambda_s' \end{pmatrix}\begin{pmatrix} I & 0 \\ -X_0 & I \end{pmatrix}.$$

Item (iii) follows from (i) by the implicit function theorem.

Consider the **reduced** Hamiltonian system

$$\dot{x}_1 = f_1(x_1, p_1, \phi(x_1, p_1), \psi(x_1, p_1)), \quad \dot{p}_1 = f_2(x_1, p_1, \phi(x_1, p_1), \psi(x_1, p_1)). \tag{19a,b}$$

This system results after substitution (18) into (5a,b). From A2, (ii) of Lemma 3.1 and the theory of nonlinear differential equations (Kelley, 1966) it follows that this system has a stable manifold

$$p_1 = N_0(x_1) \tag{20}$$

with asymptotically stable flow:

$$\dot{x}_1 = f_1(x_1, N_0(x_1), \phi(x_1, N_0(x_1)), \psi(x_1, N_0(x_1))) \tag{21}$$

for x_1 from small enough neighborhood of 0. Note that (21) results from substitution of (20) into (19a). Function $N_0 = N_0(x_1)$ satisfies the **slow** partial differential equation (PDE):

$$\frac{\partial N_0}{\partial x_1} f_1(x_1, N_0, \phi(x_1, N_0), \psi(x_1, N_0)) = f_2(x_1, N_0, \phi(x_1, N_0), \psi(x_1, N_0)). \tag{22}$$

This PDE can be derived by differentiating on t of (20),where $p_1 = p_1(t)$, $x_1 = x_1(t)$ and by substituting for $\dot{x}_1$ the right side of (21). The function N_0 can be approximated by

$$N_0(x_1) = X_0 x_1 + O(|x_1|^2). \tag{23}$$

For each x_1 such that (iii) of Lemma 3.1 is valid and (20) exists consider the 'fast' system

$$\dot{x}_2 = \bar{f}_3(x_1, x_2, p_2), \quad \dot{p}_2 = \bar{f}_4(x_1, x_2, p_2), \tag{24}$$

where

$$\bar{f}_i = f_i(x_1, N_0, x_2 + \phi(x_1, N_0), p_2 + \psi(x_1, N_0)) - f_i(x_1, N_0, \phi(x_1, N_0), \psi(x_1, N_0)), \quad i = 3, 4.$$

From A1, (iii) of Lemma 3.1 and the theory of nonlinear differential equations (Kelley, 1966) it follows that this system has a stable manifold $p_2 = M_0(x_1, x_2)$ with asymptotically stable flow

$$\dot{x}_2 = \bar{f}_3(x_1, x_2, M_0(x_1, x_2)) \tag{25}$$

for x_1 and x_2 from small enough neighborhood Ω of $\mathbf{R}^{\mathbf{n_1}} \times \mathbf{R}^{\mathbf{n_2}}$ containing 0. Function $M_0 = M_0(x_1, x_2)$ satisfies the **fast** PDE

$$\frac{\partial M_0}{\partial x_2} \bar{f}_3(x_1, x_2, M_0) = \bar{f}_4(x_1, x_2, M_0). \tag{26}$$

and

$$M_0(x_1, x_2) = X_f x_2 + O\left((|x_1| + |x_2|)|x_2|\right). \tag{27}$$

Define the **composite** controller as follows:

$$u_0 = -R^{-1} B_1' N_0(x_1) - R^{-1} B_2' [\psi(x_1, N_0(x_1)) + M_0(x_1, x_2 - \phi(x_1, N_0(x_1))]. \tag{28}$$

From (17), (23) and (27) it follows that

$$u_0 = u_l + O(|x_1|^2 + |x_2|^2). \tag{29}$$

We shall show that this ϵ-independent controller near-optimally solves Lagrange problem on some ϵ-independent neighborhood for all small enough ϵ. Denote by $\Omega_{m_i} = \{x_i \in \mathbf{R}^{\mathbf{n_i}} : |x_i| < m_i\}$, $i = 1, 2$.

Theorem 3.1 *Under A1 and A2 there exist* $m_1 > 0, m_2 > 0$ *and* $\epsilon_0 > 0$ *such that for all* $\epsilon \in (0, \epsilon_0]$ *the following hold:*
(i) *The* $(2n_1+2n_2)$*-dimensional Hamiltonian system (5) has the invariant on* $\Omega_{m_1} \times \Omega_{m_2}$ *manifold (7) with (9) asymptotically stable.*
(ii) *There exists a* C^2 *function* $V : \Omega_{m_1} \times \Omega_{m_2} \to [0, \infty)$*, satisfying the HJ equation (6) and relations (8). The solution to HJ equation and the optimal controller have the following approximations:*

$$V(x_1, x_2) = V_0(x_1) + O(\epsilon), \quad u(x_1, x_2) = u_0(x_1, x_2) + O(\epsilon), \tag{30}$$

where u_0 *is given by (28) and* $\frac{\partial V_0}{\partial x_1} = N_0(x_1)$*. The composite controller (28) achieves the performance index* $O(\epsilon)$*-close to the optimal one.*

(iii) *The optimal trajectory $x^*(t)$ with initial data $x(0)$ and optimal open-loop control $u^*(t)$ are approximated for $t \in [0, \infty)$ by*

$$\begin{array}{l} x^*(t) = x^{(0)}(t,\tau) + \epsilon r_1(t,\epsilon), \\ x^{(0)}(t,\tau) = col\{\bar{x}_1(t);\ \phi(\bar{x}_1(t), N_0(\bar{x}_1(t)) + \Pi(\tau, \bar{x}_1(t))\}, \quad \tau = \frac{t}{\epsilon}, \\ u^*(t) = -R^{-1}B_1'(x^{(0)})N_0(\bar{x}_1(t)) - R^{-1}B_2'(x^{(0)})[\psi(\bar{x}_1(t), N_0(\bar{x}_1(t))) \\ +M_0(\bar{x}_1(t), \Pi(\tau, \bar{x}_1(t))] + \epsilon r_2(t,\epsilon), \end{array} \tag{31}$$

where $\bar{x}_1(t)$ is a solution to (9) with initial data $x_1(0)$. The boundary layer term $\Pi(\tau, x_1)$, $|\Pi(\tau, x_1)| \leq ce^{-\alpha\tau}$, $\alpha \geq 0$, satisfies for each x_1 the following equations:

$$\frac{\partial \Pi(\tau, x_1)}{\partial \tau} = \bar{f}_3(x_1, \Pi(\tau, x_1), M_0(x_1, \Pi(\tau, x_1))), \quad \Pi(0, x_1) = x_{02} - \phi(x_{10}, N_0(x_{10})). \tag{32}$$

The remainders satisfy inequality $|r_i(t,\epsilon)| \leq ce^{-\alpha t}$, $i = 1,\ 2$.

3.2. Optimal controller for descriptor system approach to the problem. Consider the corresponding to (1) descriptor system

$$E_0\dot{x} = f(x) + B(x)u, \quad z = col\{k(x),\ u\}, \tag{33}$$

where $k(x) = k(x_1, x_2)$ and

$$f = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}, \quad B = \begin{bmatrix} B_1(x_1, x_2) \\ B_2(x_1, x_2) \end{bmatrix}.$$

A controller of the form (10) is called an admissible if the closed-loop system (33), (10) has a unique solution for any initial condition $E_0x(0)$ from small enough neighborhood in R^{n_1} containing 0 as an interior point.

Theorem 3.2 (i) *Let there exists a twice continuously differentiable function $V_d : \Omega_{m_1} \times \{0\} \to R$ such that $V_d(E_0x) \geq 0$,*

$$\frac{\partial V_d(E_0x)}{\partial x} = W(x)E_0, \tag{34}$$

$$2W(x)f(x) - W(x)S(x)W'(x) + k'(x)k(x) = 0, \quad S = BR^{-1}B', \tag{35}$$

with the property that $\tilde{f}_{x_2}(0,0)$ is nonsingular, where $\tilde{f} = f_2 - B_2R^{-1}B_2'W'$. Assume additionally that the system

$$E_0\dot{x} = f(x) - B(x)R^{-1}B'(x)W'(x) \tag{36}$$

is asymptotically stable. Then the controller

$$u_d(x) = -R^{-1}B'(x)W'(x) \tag{37}$$

solves the local Lagrange problem for the descriptor system (33).
(ii) *Under A1 and A2 the composite controller (28) is locally optimal one for (33). The resulting performance index $J_d = V_0(x)$ is $O(\epsilon)$-close to the optimal one for singularly perturbed system (1).*
(iii) *Let $M_0 : \Omega \to R^{n_2}$ be any continuously differentiable function that vanishes at $x_2 = 0$ and such that (25) is exponentially stable uniformly on x_1. Then the controller (28) achieves the performance cost $O(\epsilon)$-close to the optimal one.*

Thus, as in the linear case (Wang *et al.*, 1988), there exist many near-optimal solutions (28) to the problem (1), (2), where the fast gain is any function that exponentially stabilizes (25).

Note that the relation (34) is similar to one in (Xu and Mizukami, 1994).

3.5. Example. Consider the system

$$\dot{x}_1 = -f(x_2) + 2u, \quad \epsilon\dot{x}_2 = f(x_2) - x_1 - u, \quad z = [x_1\ u]', \quad x(0) = [1\ 1]'. \tag{38}$$

Here (14) has the form: $2\dot{f}(0)X_f - X_f^2 = 0$. To guarantee A1 we require $\dot{f}(0) > 0$. We obtain the following Hamiltonian function

$$\mathcal{H} = -p_1 f(x_2) + p_2(f(x_2) - x_1) - 2p_1^2 + 2p_1p_2 - 1/2p_2^2 + 1/2x_1^2$$

and the corresponding Hamiltonian system

$$\begin{aligned} \dot{x}_1 &= -f(x_2) - 4p_1 + 2p_2, \quad \dot{p}_1 = p_2 - x_1, \\ \epsilon\dot{x}_2 &= f(x_2) - x_1 + 2p_1 - p_2, \quad \epsilon\dot{p}_2 = \dot{f}(x_2)(p_1 - p_2). \end{aligned}$$

We find

$$\phi = f^{-1}(x_1 - p_1), \quad \psi = p_1, \quad N_0 = Kx_1, \qquad M_0 = 2f\{x_2 + f^{-1}[(1-K)x_1]\} - 2(1-K)x_1,$$

where f^{-1} is the inverse to f function, $K = -1 + \sqrt{2}$. The composite controller has a form

$$u_0 = (K-2)x_1 + 2f(x_2). \tag{39}$$

Neglecting $O(\epsilon)$-terms, we obtain the following expressions for the optimal trajectory with initial data $x(0)$ and the optimal open-loop control:

$$\begin{aligned} x^*(t) &= col\left\{e^{-(\sqrt{2})t}x_1(0);\ f^{-1}[(2-\sqrt{2})e^{-(\sqrt{2})t}x_1(0)] + \Pi(\tau, e^{-(\sqrt{2})t}x_1(0))\right\}, \\ u^*(t) &= (-3+\sqrt{2})e^{-(\sqrt{2})t}x_1(0) + 2f\left\{f^{-1}[(2-\sqrt{2})e^{-(\sqrt{2})t}x_1(0)] + \Pi(\tau, e^{-(\sqrt{2})t}x_1(0))\right\}, \end{aligned}$$

where $\Pi(\tau, x_1)$ satisfies

$$\frac{\partial \Pi}{\partial \tau} = -f[\Pi + f^{-1}((2-\sqrt{2})x_1)] + (2-\sqrt{2})x_1, \quad \Pi(0, x_1) = x_2(0) - f^{-1}((2-\sqrt{2})x_1(0)).$$

We choose now

$$\bar{M}_0 = 1.5f\{x_2 + f^{-1}[(1-K)x_1]\} - 1.5(1-K)x_1,$$

that stabilizes (25), we obtain (28) given by

$$\bar{u}_0 = (0.5K - 1.5)x_1 + 1.5f(x_2).$$

Applying now u_0 and $\bar{u}_0$ to (38), where $f(x_2) = \arctan x_2$, we find that for $\epsilon = 0.01$ the corresponding values of performance index are $J(x_0, u_0) = 0.4215$ and $J(x_0, \bar{u}_0) = 0.4219$. For $\epsilon = 0.001$ we have $J(x_0, u_0) = J(x_0, \bar{u}_0) = 0.4149$. Thus, for small ϵ both controllers achieve the same values of performance cost that approach to $J_d = 0.4142$ as $\epsilon \to 0$.

4 Conclusions

We have designed ϵ-independent composite controllers for singularly perturbed systems being nonlinear in both, the slow and the fast state variables. We have shown that these controllers are optimal for the corresponding descriptor system and lead the singularly perturbed system to the values of the performance that are $O(\epsilon)$-close to the optimal one. The slow gain of these controllers N_0 is uniquely defined from the slow PDE, while the fast gain M_0 can be found either as a solution to the fast PDE or as a stabilizing gain for the fast system. Even for the case of a standard system this design procedure is a new and more perfect than the two-stage method of Chow and Kokotovic (1981). Moreover, asymptotic approximations of the optimal state-feedback, optimal cost and optimal trajectory have been obtained.

5 Appendix

Proof of Theorem 3.1. (i) follows from (iii) of Lemma 3.1 by using arguments of (Fridman, 1999).

(ii) The invariant manifold (7) with asymptotically stable (9) is Lagrangian (it can be proved as Lemma 1 of Van der Schaft, 1991) and is projectable on the simply connected manifold $\Omega_{m_1} \times \Omega_{m_2}$, that implies the existence of the generating function V, satisfying (8) and (5) (Van der Schaft 1991).

If the closed-loop system of (1) and (10) is asymptotically stable, then $V \geq 0$ (Byrnes, 1998).

Relation (30) follows from the approximations:

$$\frac{\partial V}{\partial x_1} = N_0(x_1) + O(\epsilon). \quad \frac{\partial V}{\partial x_2} = O(\epsilon).$$

(iii) is similar to (Fridman, 1995).

Proof of Theorem 3.2. (i) Let $x(t)$ satisfies (33) and some initial condition $E_0 x(0)$. Applying (34), (33) and (35) we find

$$2\frac{dV_d(E_0 x)}{dt} + z'z = 2W(x)(f(x) + B(x)u) + k'k + u'Ru = |u + R^{-1}B'(x)W'(x)|^2. \tag{40}$$

Integrating (40) on t from 0 to ∞ we find

$$J_d(x_0, u) \geq 2V_d(E_0 x_0) = J_d(x_0, u_d), \tag{41}$$

i.e. u_d is a minimizing controller.

Consider the closed-loop system (33), (37). By the nonsingularity of $\tilde{f}_{x_2}(0,0)$ and the implicit function theorem, the last n_2 algebraic equations of (33) under (37) can be solved with respect to x_2 in a small neighborhood of $x = 0$. Substituting the resulting x_2 into the first n_1 differential equations of (33), (37) we see that the initial condition for x_1 defines the unique solution. Hence, u_d is admissible.

(ii) Choosing

$$V_d(E_0 x) = V_0(x_1), \tag{42}$$

where V_0 satisfies (30), we see that

$$\frac{\partial V_d(E_0 x)}{\partial x} = \begin{bmatrix} \frac{\partial V_0(x_1)}{\partial x_1} & 0 \end{bmatrix} = \begin{bmatrix} N_0'(x_1) & 0 \end{bmatrix} = W(x)E_0,$$

where

$$W'(x) = \left[\begin{array}{cc} N_0'(x_1) & \psi'(x_1, N_0(x_1)) + M'[x_1, x_2 - \phi(x_1, N_0(x_1))] \end{array} \right].$$

Since by Theorem 3.1

$$\left[\begin{array}{cc} \frac{\partial V(x_1,x_2)}{\partial x_1} & \epsilon^{-1}\frac{\partial V(x_1,x_2)}{\partial x_2} \end{array} \right] = W + O(\epsilon),$$

where V is solution to HJ equation (6), then W satisfies (35). Under A1 and A2 the properties of (i) of this theorem are satisfied that implies that (28) is a minimizing controller.

The $O(\epsilon)$-closeness of the performance indexes follows from (41), (42) and (30).

(iii) Consider the closed-loop system (1), (28), where M_0 is any stabilizing function for (25). Compare it with the closed-loop system (1), (28), where M_0 satisfies the fast PDE (24). The reduced problems for these systems are the same. Hence, solutions have the same regular parts in the zero-order approximations. Therefore, the resulting values of J are $O(\epsilon)$-close (boundary layer terms after integrating give $O(\epsilon)$-terms).

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