About some interconnection between LTR and RPIS

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Abstract

In this paper, the Loop Transfer Recovery design procedure is extended to non stabilizable systems. After a brief description of the systems considered in this paper, we revisit some results concerning the RPIS (Regulator Problem with Internal Stability), and give the structure of the controller. Thereafter we consider the LTR dual approach and stress the particular configuration of the output sensitivity function of the closed-loop system. We show that it is sufficient to recover only a part of the sensitivity function to guarantee the stability robustness of the loop. Finally the adjustment rules which lead to the desired result are described.

1 Introduction

It is well-known that full-state linear-quadratic regulators and Kalman-Bucy filters have attractive robustness properties, but that these properties disappear in the case of observer-based linear control systems (Anderson *et al.*, 1990; Doyle, 1977; Safonov *et al.*, 1977). One of the most popular way of designing robust controllers is then to use the well-known Loop Transfer Recovery techniques. These consist in choosing an appropriate parametrization of the compensator design, so as to recover a robust target loop. A wide range of sophisticated adjustment rules has been proposed in the last two decades, and provides attractive solutions for the control of detectable and stabilizable plants (Doyle *et al.*, 1979;1981; Kwakernaak, 1972; Niemann *et al.*, 1991; Saberi *et al.*, 1993; Stein *et al.*, 1977).
However, there is no so clear theory when the system is not asymptotically stabilizable. In the case of eigenvalues at the origin, for example, there are some techniques which consist in "placing them to the left of it, the distance being much smaller than the required bandwith" (Maciejowski, 1989). The recovery step is then performed on the resulting stable system. This method would give approximate LTR, probably sufficient but not satisfactory for some general setting.
In this paper, systems composed of a stabilizable plant and an non stabilizable exosystem are considered, and we attempt to solve the recovery problem without any approximation of the unstable poles. The class of the systems under consideration is illustrated in figure 1.

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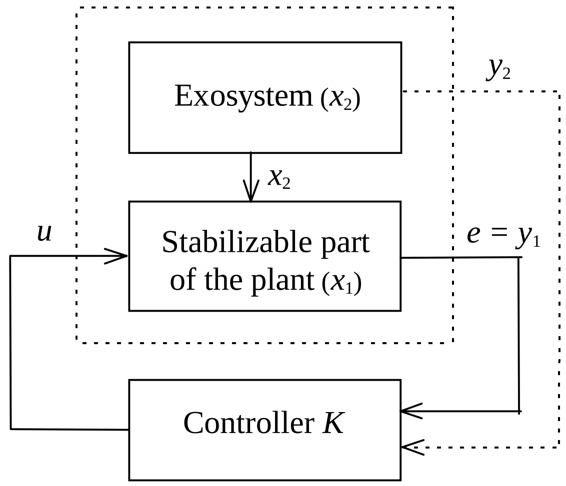

Figure 1 : Configuration of the systems for which RPIS is solvable

Both the system and the controller are finite dimensional LTI systems. The system is composed of a stabilizable part (represented by the state x_1) and a non stabilizable one (state x_2). The measurement vector y contains the quantities to minimize ($y_1 = e$) and possible additional measures (y_2). We stress the fact that the possible additional measures are assumed to concern only the exosystem.

This paper proposes an extent of the LTR techniques to such systems. Following Kwakernaak (Kwakernaak, 1972), we will focus here on the dual LTR (or LTR at the output). In this case (Doyle et al., 1981) one first chooses a observer gain to give some desired properties to a particular target transfer function. In a second step, one calculates a state feedback gain which allows to recover this target transfer function. However, since the exosystem is unstable, direct application of this technique is not possible here. The idea is to decompose the state feedback in two parts : the first one is devoted to solve the disturbance decoupling problem (occultation of the exosystem (Wonham, 1985)) the second one to the stabilisation of the plant and the recovery problem.

The remainder of this paper is thus organized as follows.

We begin in section 2 with an application, to our configuration, of the solution of the RPIS (Regulation Problem with Internal Stability) as it is given in (Wonham, 1985). A simple interpretation of it (de Larminat, 1995) will help us to understand the organization of the closed loop system and the way we tackle the recovery problem.

In Section 3 the dual recovery procedure is discussed in detail. We adapt the concept of sensitivity recovery as developed in (Niemann et al., 1991) to the considered systems. It turns out that there is no need to recover the whole target sensitivity function to guarantee the stability robustness of the loop. We show that partial, but sufficient, recovery is feasible, and we present the appropriate design procedure.

Section 4 is devoted to concluding comments.

2 The regulator Problem with Internal Stability

Consider the LTI system described by the following state-space representation (notations are consistent with figure 1) :

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ e(t) = y_1(t) \end{cases} \tag{2.1}$$

The dimension of the vectors *u* and *e* are equal.
The system is supposed to be detectable, even in the lack of the y_2 measurement, but not stabilizable; the crucial point here is that only (A_{11}, B_1) is assumed to be stabilizable.

Also note that (A_{11}, B_1, C_{11}) is supposed to be of minimum phase and right invertible. These last two hypothesis are needed for the recovery procedure detailled in section 2.3 (Stein *et al.*, 1977). Moreover, for the RPIS to be solvable we need to assume that the following Sylvester system has a solution (Wonham, 1985; de Larminat, 1995) :

$$\begin{cases} A_{11}T_a - T_a A_{22} + B_1 G_a = A_{12} \\ C_{11}T_a = C_{12} \end{cases} \tag{2.2}$$

If such a solution exists, the system described in the previous state space representation can be rewritten in such a way that (see appendix A) :

$$\begin{matrix} A_{12} = B_1 G_a \\ C_{12} = 0 \end{matrix} . \tag{2.3}$$

So, in the re-arranged configuration (see A.7), the non stabilizable part of the system involves a disturbance $d = G_a x_2$, additive to the control input *u* (see figure 2).

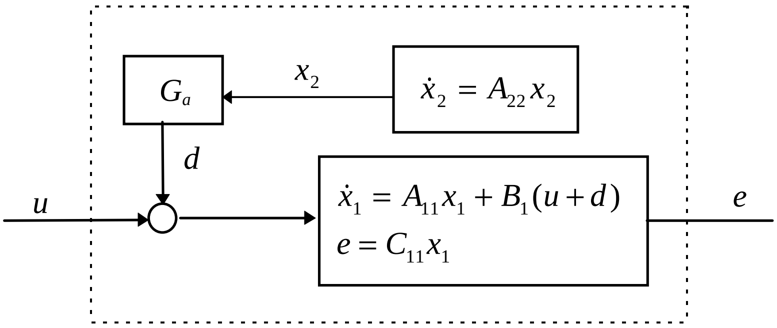

Figure 2 : Equivalence with an additive input disturbance

The full-state feedback law for the regulator problem with internal stability (RPIS) is now straightforward :

$$u = -\begin{bmatrix} K_{c1} & K_{c2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2.4}$$

where K_{c1} stabilizes the subsystem (A_{11}, B_1) and where $K_{c2} = G_a$ rejects the disturbance d. Finally, an observer is included, in order to define the complete output feedback controller :

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_f\,(y(t) - C(t)\hat{x}(t)) \\ u(t) = -K_{c1}\hat{x}_1(t) - G_a\hat{x}_2(t) \end{cases}, \tag{2.5}$$

The observer gain K_f will be calculated through the Kalman-Bucy formalism, and the feedback gain K_{c1} via the Linear-Quadratic optimization.
Our goal is now to check that robust stability of the loop (the exosystem being outside) can be ensured via the LTR design procedure despite the particular configuration of the problem.

3 Sensitivity Recovery Design Techniques

3.1 Introduction

Classically, the first step of the LTR design procedure consists in defining a target loop which has desirable properties in terms of stability margins. Then, the second step lies in minimizing the difference between the loop of the observer-based control system and the target loop.
In this paper we prefer to discuss the sensitivity recovery rather than the loop transfer recovery (Niemann et al., 1991). The reason is that when a plant is unstable, the sensitivity recovery error remains stable, which is not the case for the loop transfer recovery error.
Moreover, we consider the dual LTR where the Kalman-Bucy Filter is defined as a target loop, in place of the full-state Linear Quadratic Control of the LTR primal approach.

3.2 Properties of the target loop sensitivity function

In this section we deal with the first step of the LTR design procedure, which consists in choosing a target loop with good robustness properties. The target loop considered here is a Kalman-Bucy filter loop, the output sensitivity function of which is given below :

$$\bar{S}_o = \left(I + \bar{L}_o\right)^{-1} \tag{3.1}$$

with

$$\bar{L}_o(s) = C(sI - A)^{-1} K_f\,, \tag{3.2}$$

where K_f is the gain of the observer (see eq. 2.5).

As there are two output signals y_1 and y_2, the sensitivity function will be partitioned into

$$\bar{S}_o = \begin{bmatrix} \bar{S}_{o11} & \bar{S}_{o12} \\ \bar{S}_{o21} & \bar{S}_{o22} \end{bmatrix} \tag{3.3}$$

where the block dimensions are fixed by those of y_1 and y_2.

The target sensitivity function $\bar{S}_o$ satisfies the well-known robustness property (Anderson *et al.*, 1990) :

$$\bar{S}_o^T(-j\omega)\,\bar{S}_o(j\omega) \le I \tag{3.4}$$

And a few technical manipulations show that the same property applies to the submatrix $\bar{S}_{o11}$ (see Appendix B).

$$\bar{S}_{o11}^T(-j\omega)\,\bar{S}_{o11}(j\omega) \le I \tag{3.5}$$

3.3 Recovery procedure

Consider the output sensitivity function of the closed-loop system :

$$S_o = \left(I + L_o\right)^{-1} \tag{3.6}$$

where $L_o = GK$ is the loop transfer function of the control system.

The sensitivity function S_o has a particular configuration which derives from the structure of the system (see Appendix C).

$$S_o = \begin{bmatrix} S_{o11} & S_{o12} \\ 0 & I \end{bmatrix} \tag{3.7}$$

One can notice that $S_{o21} \equiv 0$ and $S_{o22} \equiv I$, while $\bar{S}_{o21} \ne 0$ and $\bar{S}_{o21} \ne I$ due to the Kalman-Bucy synthesis of the observer. This implies that the recovery of the complete sensitivity target $\bar{S}_o$ will never be possible.
However, note that that the robust stability of the loop is only linked to the properties of the sensitivity function S_{o11}. It does not depend on S_{o12}, which represents the sensitivity of the output y_1 to the measurement noise through the feedforward branch (remember that y_2 can only be a measure of the exosystem).
As a consequence, it is sufficient that S_{o11} asymptotically recovers the target function $\bar{S}_{o11}$ for the robust stability of the loop to be guaranteed.
Hence, it remains to be shown that S_{o11} asymptotically recovers the target function $\bar{S}_{o11}$ despite the non stabilizable part of the system.

Let the output sensitivity recovery error be defined as the difference between the target loop sensitivity function and the sensitivity function of the observer-based control system.

$$E_{S,o}(s) = \bar{S}_o(s) - S_o(s) \qquad (3.8)$$

Niemann et al. (1991) have shown that the output sensitivity recovery error satisfies

$$E_{S,o}(s) = M_o(s)\,\bar{S}_o(s)\,, \qquad (3.9)$$

where $M_o(s) = C\left(sI - A + BK_c\right)^{-1} K_o$. (3.10)

$M_o(s)$ is called the recovery matrix for the plant output node.
From (2.1), (2.3) and (2.5) ,

$$A = \begin{bmatrix} A_{11} & B_1 G_a \\ 0 & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix}, K_c = \begin{bmatrix} K_{c1} & G_a \end{bmatrix}, K_f = \begin{bmatrix} K_{f11} & K_{f12} \\ K_{f21} & K_{f22} \end{bmatrix}. \qquad (3.11)$$

So $M_o(s)$ can be expanded into

$$M_o(s) = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} sI - A_{11} + B_1 K_{c1} & -B_1 G_a + B_1 G_a \\ 0 & sI - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} K_{f11} & K_{f12} \\ K_{f21} & K_{f22} \end{bmatrix} \qquad (3.12)$$

or

$$M_o(s) = \begin{bmatrix} C_{11}\left(sI - A_{11} + B_1 K_{c1}\right)^{-1} & 0 \\ 0 & C_{22}\left(sI - A_{22}\right)^{-1} \end{bmatrix} \begin{bmatrix} K_{f11} & K_{f12} \\ K_{f21} & K_{f22} \end{bmatrix} \qquad (3.13)$$

The state feedback gain K_{c1} results from a Linear-Quadratic Control minimizing the following criterion :

$$J = \int_{t_o}^{\infty} \left(u(t)^T R_c u(t) + x(t)^T Q_c x(t)\right) dt\,, \qquad (3.14)$$

where R_c is a non-zero matrix, and Q_c is defined as :

$$Q_c = \mu\, C_{11} C_{11}^T. \qquad (3.15)$$

The classical adjustment rule is then applied, assuming that (A_{11}, B_1, C_{11}) is a minimum phase system an right invertible (Stein *et al.*, 1977). It follows that

$$C_{11}\left(sI - A_{11} + B_1 K_{c1}\right)^{-1} \rightarrow 0 \text{ pointwise in } s \text{ as } \mu \rightarrow \infty. \qquad (3.16)$$

As a result, the recovery matrix $M_o(s)$ has the finite limit :

$$\overline{M}_o(s) = \begin{bmatrix} 0 & 0 \\ C_{22}(sI - A_{22})^{-1} K_{f21} & C_{22}(sI - A_{22})^{-1} K_{f22} \end{bmatrix} \tag{3.17}$$

Finally, substituting (3.17) into

$$\begin{bmatrix} E_{S,\,o11}(s) & E_{S,\,o12}(s) \\ E_{S,\,o21}(s) & E_{S,\,o22}(s) \end{bmatrix} = \begin{bmatrix} M_{o11}(s) & M_{o12}(s) \\ M_{o21}(s) & M_{o22}(s) \end{bmatrix} \begin{bmatrix} \overline{S}_{o11}(s) & \overline{S}_{o12}(s) \\ \overline{S}_{o21}(s) & \overline{S}_{o22}(s) \end{bmatrix}. \tag{3.18}$$

yields

$$E_{S,\,o11}(s) \to 0 \text{ as } \mu \to \infty. \tag{3.19}$$

Now since

$$E_{S,\,o11}(j\omega) = S_{o11}(j\omega) - \overline{S}_{o11}(j\omega), \tag{3.20}$$

it follows that asymptotic recovery is obtained for S_{o11}.

4 Conclusion

This paper has attempted to deal with the design of robust controllers for non stabilizable systems. A two-step procedure has been proposed, concerning the systems for which the regulator problem with internal stability is solvable . The first step consists in re-arranging the system via the RPIS solution. The second step is devoted to the output sensitivity recovery, via an appropriate adjustment rule (under the condition of minimum phase and right invertibility of (A_{11}, B_1, C_{11})). A significant point of the proof is that recovery can be performed in the presence of additional measures giving information about the exosystem. These measures tackle the controller as feedforward inputs, and do not affect the stability robustness of the loop.

Appendix A

Consider the LTI system described by the state differential equations :

$$\begin{aligned} \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \end{aligned} \tag{A.1}$$

X_1 represents the stabilizable part of the system, and X_2 the exosystem. The signal vector y is composed of y_1, the measure of the controlled output, and y_2, an additional measure concerning the exosystem.

The state X is transformed into x via

$$x = TX \; , \tag{A.2}$$

where T is a square matrix given by

$$T = \begin{bmatrix} I_{n_{x1}} & T_a \\ 0 & I_{n_{x2}} \end{bmatrix}. \tag{A.3}$$

Considering the transformation (A.2), the state-space representation equivalent to (A.1) is

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & -A_{11}T_a + A_{12} + T_a A_{22} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} v \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} C_{11} & -C_{11}T_a + C_{12} \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{A.4}$$

Define T_a and G_a as a solution of the Sylvester system (assumed to be solvable) (Wonham, 1985 ; de Larminat, 1995) :

$$\begin{cases} -C_{11}T_a + C_{12} = 0 \\ -A_{11}T_a + A_{12} + T_a A_{22} = B_1 G_a \end{cases} \tag{A.5}$$

Substituting (A.5) into (A.4) yields :

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & B_1 G_a \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{A.6}$$

(A.6) may be rewritten as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} (u + G_a x_2) \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \; . \tag{A.7}$$

Hence, the initial system has been reorganized into a new system, with a disturbance $d = G_a x_2$ additive to the input.

Appendix B

Let the matrix M be defined as

$$M = \bar{S}_o^T(-j\omega)\,\bar{S}_o(j\omega) - I \tag{B.1}$$

M expands into :

$$M = \begin{bmatrix} \bar{S}_{o11}^T(-j\omega)\,\bar{S}_{o11}(j\omega) + \bar{S}_{o21}^T(-j\omega)\,\bar{S}_{o21}(j\omega) - I & M_{12} \\ M_{21} & M_{22} - I \end{bmatrix} \tag{B.2}$$

The matrix M is negative definite, which implies

$$\bar{S}_{o11}^T(-j\omega)\,\bar{S}_{o11}(j\omega) + \bar{S}_{o21}^T(-j\omega)\,\bar{S}_{o21}(j\omega) - I \le 0\,. \tag{B.3}$$

(B.3) may be rewritten as

$$\bar{S}_{o11}^T(-j\omega)\,\bar{S}_{o11}(j\omega) \le -\bar{S}_{o21}^T(-j\omega)\,\bar{S}_{o21}(j\omega) + I\,, \tag{B.4}$$

where $\bar{S}_{o21}^T(-j\omega)\,\bar{S}_{o21}(j\omega)$ is positive definite.

It follows that

$$\bar{S}_{o11}^T(-j\omega)\,\bar{S}_{o11}(j\omega) \le I\,. \tag{B.5}$$

Appendix C

Consider the sensitivity function of the system described by (A.6)

$$S = (I + GK)^{-1} \tag{C.1}$$

where G and K are the plant and regulator transfer functions respectively.

G is written in the form :

$$G = C(sI - A)^{-1} B\,, \tag{C.2}$$

which expands into

$$G = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} sI - A_{11} & B_1 G_a \\ 0 & sI - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix}. \tag{C.3}$$

Let

$$M = \begin{bmatrix} M_1 & M_2 \\ 0 & M_3 \end{bmatrix} \tag{C.4}$$

be a square matrix, with M_1 and M_3 non singular.

The formula giving the inverse of M is

$$M^{-1} = \begin{bmatrix} M_1^{-1} & -M_1^{-1} M_2 M_3^{-1} \\ 0 & M_3^{-1} \end{bmatrix} \tag{C.5}$$

Using this formula in (C.3) yields

$$G = \begin{bmatrix} C_{11}(sI - A_{11})^{-1} B_1 \\ 0 \end{bmatrix}. \tag{C.6}$$

The transfer of the regulator is a two-column matrix :

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}. \tag{C.7}$$

Substituting (C.6) and (C.7) into (C.1) yields :

$$S = \begin{bmatrix} I + C_{11}(sI - A_{11})^{-1} B_1 K_1 & C_{11}(sI - A_{11})^{-1} B_1 K_2 \\ 0 & I \end{bmatrix}^{-1} \tag{C.8}$$

Use the formula (C.5) to check that the output sensitivity function has the form

$$S_o = \begin{bmatrix} S_{o11} & S_{o12} \\ 0 & I \end{bmatrix}. \tag{C.9}$$

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