Proceedings of the 7th Mediterranean Conference on Control and Automation (MED99) Haifa, Israel - June 28-30, 1999

Control Systems with Actuator Saturation and Bifurcations at Infinity*

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Abstract

It is known that unstable open-loop plants can be stabilized under constrained controls only locally. To understand this fact, it is shown how bifurcations at infinity are always involved in the stabilizing process. These bifurcations are easily detected by studying the Nyquist plots. The approach is illustrated with a concrete example of a anti-windup scheme taken from the literature.

1 Introduction

The class of linear systems with saturating actuators has deserved the attention of researchers in the last years as most practical systems have bounded control, see for instance (Bernstein and Michel, 1995). In this context, the specific problem of stabilizing open-loop unstable plants has been addressed in many recent papers, see in particular the theoretical results in (Zhao and Jayasuriya, 1995).

However, it is our feeling that in order to achieve a deeper insight in the behavior of physical dynamic systems, which are subject to hard (nonlinear) constraints (as is the case of control systems, where actuator saturations cannot be avoided) a more frequent resort to bifurcation theory must be done.

No doubt, in the particular case of control systems with saturation, the situation is changing, since control engineers are more aware of the benefits that the tools of qualitative theory of dynamical systems and especially bifurcation theory can report (Abed *et al.*, 1996). Nonetheless, it seems still relevant to pay attention to some phenomena not yet well known by the majority of practitioners. It should be mentioned that saturations can be responsible not only for local bifurcations but for global ones (Aracil *et al.*, 1998).

*This work has been partially supported by the Spanish Ministry of Education and Science under grant CICYT TAP 97-0553 and by the Brazilian Ministry of Education (CAPES) under grant BEX0439/95-6.

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In this paper, we address a particular topic related to bifurcation theory, which is responsible for certain global problems in control system with saturated inputs, namely the generic appearance of bifurcations at infinity. This phenomenon has been shown to be in the core of windup problems (Aracil *et al.*, 1997) as well as in robustness issues of the system (Ponce *et al.*, 1996). Also, it has important implications when a time-delay is present (Pagano *et al.*, 1997). These concepts will be illustrated through an example (Middleton, 1996) from Chapter 20 of the IEEE Control Handbook (Levine, 1996), devoted to systems with actuator saturation.

2 Preliminary results: Equilibria and limit cycles of nonlinear SISO systems

In this section, some preliminary results about equilibria and limit cycles of nonlinear SISO systems will be reviewed. More details about the methodology used can be found in (Tesi *et al.*, 1997). Consider the following nonlinear control system in $\mathbf{R}^n$, composed by a linear plant

$$\dot{\mathbf{x}} = \mathbf{A}_\mu \mathbf{x} + \mathbf{b}_\mu u, \tag{1}$$

subject to a nonlinear feedback defined by a memoryless function $u = -\phi(y)$, where as usual the output is a linear combination of the states, that is $y = \mathbf{c}_\mu \mathbf{x}$. Equivalently,

$$\dot{\mathbf{x}} = \mathbf{A}_\mu \mathbf{x} - \mathbf{b}_\mu \phi(\mathbf{c}_\mu \mathbf{x}). \tag{2}$$

Here $\mu \in \mathbf{R}^m$ denotes a vector of parameters. The linear part can be described by the transfer function $G_\mu(s) = \mathbf{c}_\mu(s\mathbf{I} - \mathbf{A}_\mu)^{-1}\mathbf{b}_\mu$ so that, assuming that $\mathbf{A}_\mu$ is invertible, $G_\mu(0) = -\mathbf{c}_\mu \mathbf{A}_\mu^{-1}\mathbf{b}_\mu$ is the open-loop static gain. Alternatively, if D denotes the differential operator with respect to time, the dynamical system can be represented by the scalar differential equation

$$q_\mu(D)y(t) + p_\mu(D)\phi[y(t)] = 0, \tag{3}$$

where q_μ and p_μ are polynomials of degree n and $n-1$ respectively, so that $G_\mu(s) = p_\mu(s)/q_\mu(s)$.

As is well known, the expression (3) can be used to determine the equilibrium points of the system. If y^e is the output value corresponding to an equilibrium steady state then $Dy^e = 0$ and $D\phi(y^e) = 0$, and so

$$q_\mu(0)y^e + p_\mu(0)\phi(y^e) = 0. \tag{4}$$

If $\mathbf{A}_\mu$ is invertible then the above equation is equivalent to

$$y^e = -G_\mu(0)\phi(y^e). \tag{5}$$

Otherwise, when $\det \mathbf{A}_\mu = 0$, then $q_\mu(0) = 0$ and Eq. (4) becomes

$$\phi(y^e) = 0, \tag{6}$$

where it has been assumed that $p_\mu(0) \neq 0$.

Suppose that y^e is a solution of Eq. (5) or a solution of Eq. (6) when $q_\mu(0) = 0$. By linearizing $\phi(y)$ around y^e, the stability character of the corresponding equilibrium point can be obtained by considering the roots of the characteristic polynomial of the perturbation equation

$$[q_\mu(D) + p_\mu(D)\phi'(y^e)]\bar{y}(t) = 0, \tag{7}$$

where the apostrophe ($'$) denotes the derivative of the function with respect to its argument. Thus, if $\phi'(y^e) \neq 0$, it suffices to study the polynomial

$$q_\mu(\lambda) + p_\mu(\lambda)\phi'(y^e) = 0, \tag{8}$$

and that can be graphically made by plotting $G_\mu(j\omega)$ and using the Nyquist criterion about the point $-1/\phi'(y^e)$ in the complex plane. Note also that for $\phi'(y^e) = 0$ the characteristic polynomial of (8) reduces to $q_\mu(\lambda)$ and so the stability of the equilibrium point coincides with the stability character of the open-loop system.

So equilibrium points are related to the solutions of (5)-(6). Due to parameter modifications, they can undergo bifurcations when one or more of their eigenvalues cross the imaginary axis of the complex plane. The phenomenon can be visualized by observing the polar plot. Taking for instance a solution of (5), when a parameter movement makes to change the relative position of the plot of the transfer function with respect to the point $-1/\phi'(y^e)$ a bifurcation phenomenon must be involved.

Apart from equilibria, other important invariant sets that organize the dynamics of the system are limit cycles, i.e. isolated periodic orbits. To determine approximately periodic orbits and their stability, a describing function approach can be used. Taking only in consideration memoryless nonlinearities, a first harmonic analysis by means of the determining equation

$$1 + N(A)G_\mu(j\omega) = 0, \tag{9}$$

where $N(A)$ is the corresponding describing function, can be done by looking at the intersections of the graphics $G_\mu(j\omega)$ and $-1/N(A)$. The stability of predicted limit cycles can be deduced from Loeb criterion, see (Moiola and Chen, 1983) but also (Llibre and Ponce, 1996).

3 Generic bifurcations for systems with constrained controls

After some rescaling if needed and without loss of generality, suppose that the nonlinearity is the normalized saturation function $\phi(y) = \text{sgn}(y)\min\{|y|, 1\}$, as shown in Fig. 1. From (5) the equilibrium points can be determined by solving the equation

$$-\frac{1}{G_\mu(0)} y^e = \phi(y^e). \tag{10}$$

Note first the degeneration corresponding to $G_\mu(0) = -1$, where there appears an infinity of nonisolated equilibrium points. If $G_\mu(0) < -1$, the system has three equilibrium points, the origin and the points corresponding to $y^e = \pm G_\mu(0)$, and only one equilibrium point (at the origin) otherwise, see Fig. 1.

From Eq. (8) and using that $\phi'(0) = 1$, the stability of the origin (or closed-loop stability) is governed by the characteristic polynomial

$$Q_\mu(\lambda) = q_\mu(\lambda) + p_\mu(\lambda). \tag{11}$$

If the origin is stable one must have as a necessary condition that

$$Q_\mu(0) = q_\mu(0) + p_\mu(0) > 0. \tag{12}$$

Analogously, the stability of the symmetric pair of equilibria that exist for $1 + G_\mu(0) < 0$ is given by $q_\mu(\lambda)$ (the open-loop characteristic polynomial), since then $\phi'(y^e) = 0$ in Eq. (8). Note that when $q_\mu(0) \neq 0$ and the origin is stable, from (12) one has that

$$Q_\mu(0) = q_\mu(0)[1 + G_\mu(0)] \tag{13}$$

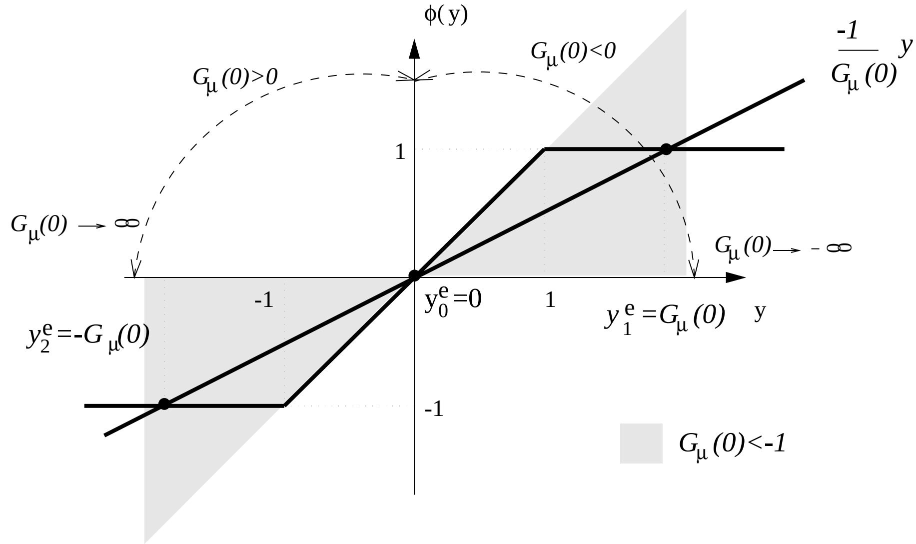

Figure 1: Equilibrium points of the system according to $G_\mu(0)$.

is positive and so, if there are other equilibria, they are unstable since $q_\mu(0) < 0$. Reciprocally, if they are stable then $q_\mu(0) > 0$ and from (13) the origin must be unstable.

Therefore two different bifurcations related to a change in the number of equilibrium points are possible. First, a degenerate *pitchfork bifurcation* (to be denoted by P_0) occurs at $G_\mu(0) = -1$, giving rise to two new equilibrium points. Note that from (13) an eigenvalue associated with the equilibrium at the origin vanishes when $1+G_\mu(0) = 0$. Second, for $q_\mu(0) = 0$ we have another pitchfork bifurcation (denoted by P_∞) where the two additional equilibrium points disappear (appear) going to (coming from) infinity. We will say that P_∞ represents a *pitchfork bifurcation at infinity.*

Considering only the equilibrium at the origin, other bifurcations (not giving rise to new equilibria) can take place when a change of its stability is produced by moving parameters, so that one or a pair of eigenvalues crosses the imaginary axis. From (11) and (12), one eigenvalue changes its sign if

$$q_\mu(0) \equiv 0, \text{and} \quad p_\mu(0) = 0 \tag{14}$$

to become negative, and this phenomenon does not imply any change in the number of equilibria. This represents a *saddle-node bifurcation* of the origin and will be denoted by SN_0. Again from (11), a crossing of a pair of eigenvalues occurs for the values of μ such that

$$q_\mu(j\omega) + p_\mu(j\omega) = q_\mu(j\omega)[1 + G_\mu(j\omega)] = 0, \tag{15}$$

where it is assumed $q_\mu(j\omega) \neq 0$, that is, the open-loop dynamics has no poles in the imaginary axis. The phenomenon generically corresponds with the so called *Hopf bifurcation* and is associated to the birth of a limit cycle along with the change of the origin stability. It will be denoted by H_0 and from (15) it corresponds with parameter values in the set

$$H_0 = \{\mu \in \mathbf{R}^m : \exists\omega > 0 \ \text{ with } \ G_\mu(j\omega) = -1\}. \tag{16}$$

This bifurcation can be interpreted in terms of the describing function analysis and visualized by means of the Nyquist plot evolution as parameters change. As is well known, the describing function of the saturation in (9) verifies $0 < N(A) \leq 1$ for every value of $A > 0$, and then

$-1/N(A) \in (-\infty, -1]$. Therefore, from (9), if the polar plot of $G_\mu(j\omega)$ does not intercept the interval $(-\infty, -1]$ then we have no periodic orbits. Assuming that this is the case, a parameter change such that $G_\mu(j\omega)$ begins to cut $-1/N(A)$ implies that the system undergoes a bifurcation giving rise to a limit cycle.

Consequently, a passing of the graph of $G_\mu(j\omega)$ through -1 should correspond, recalling Eq. (16), with a Hopf bifurcation at the origin H_0, and is associated to the birth of a limit cycle of small amplitude. As it has been already noted, this bifurcation implies a stability change of the origin, what is graphically confirmed, since $-1 = -1/\phi'(0)$ and the relative position of this point with respect to the plot of the transfer function $G_\mu(j\omega)$ changes.

But a second possibility for the graph of $G_\mu(j\omega)$ is to begin to cut the interval $(-\infty, -1]$ *from the left*, that is in the large amplitude points of the graph of $-1/N(A)$. The corresponding bifurcation will be denoted by H_∞, and we will say that the system undergoes a *Hopf bifurcation at infinity*, see (Llibre and Ponce, 1997) for a more mathematical treatment in the planar case. A necessary condition for this phenomenon can be derived as follows. Let $\omega_c > 0$ denote a value of ω such that $\mathrm{Im}G_\mu(j\omega_c) = 0$. Then, assuming that this condition defines implicitly ω_c as a function of μ in some open set $M \subset \mathbf{R}^m$, the parameter values $\mu^* \in M$ corresponding to H_∞ will satisfy $\lim_{\mu\to\mu^*} G_\mu(j\omega_c(\mu)) = -\infty$, so that

$$H_\infty \subseteq \{\mu \in \mathbf{R}^m : \exists\omega \geq 0 \ \text{ with } \ q_\mu(j\omega) = 0\}. \tag{17}$$

It should be noticed that in the H_∞-case no changes of the local stability of the origin are required.

Of course, other more complicated situations are also possible, but here we want only to remark the different character between local bifurcations (P_0, SN_0, H_0) and global ones (P_∞, H_∞).

4 Application to windup problems

Consider the following example (Middleton, 1996) shown in Fig. 2, where an unstable open loop plant is controled with a PI controller with saturation. This example is used by the author to illustrate the undesirable effects of the actuator saturation.

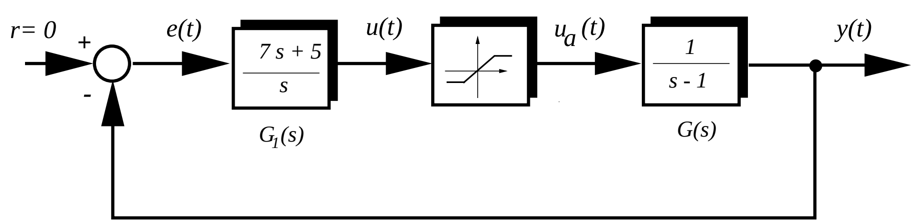

Figure 2: Open-loop unstable system with actuator saturation.

In (Middleton, 1996) is shown that a step change of 0.8 units in the system reference leads it out of control. No explanation is given about what actually happens. The fact is that the system has an unstable limit cycle that encircles the origin, as is easily checked using the describing function method (Fig. 3*a*) or simply by constructing the corresponding state portrait (Fig. 3*b*). Probably that is implicit for the author, but we think that this is quite relevant as the appearance of the limit cycle is a global phenomenon due to the nonlinearity.

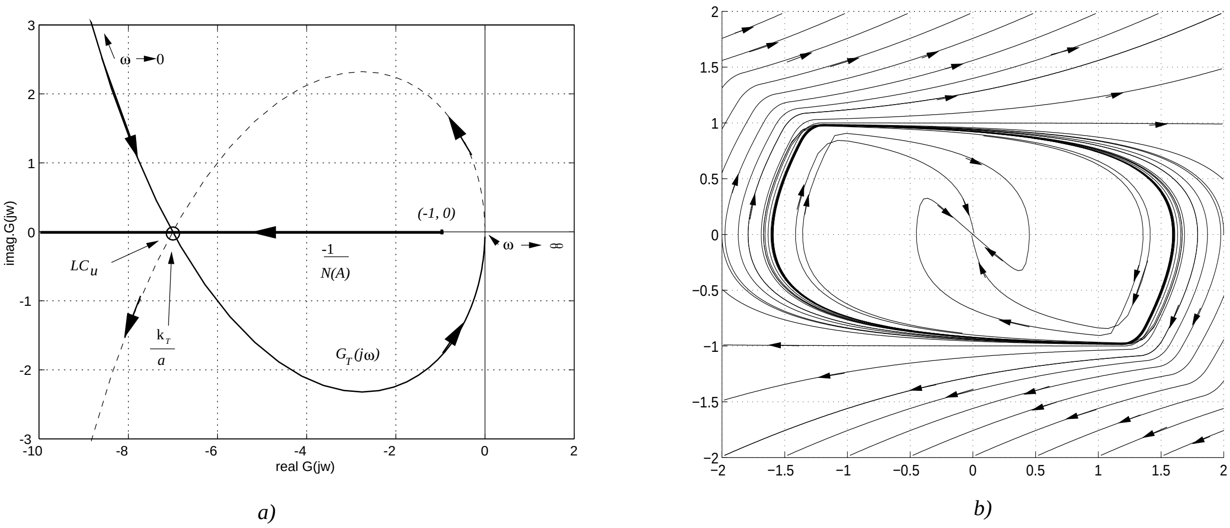

Figure 3: System without anti-windup for $k_T = 7$ and $a = -1$: a) polar plot of $G_T(j\omega)$; b) state portrait (x_1, x_2).

The system in Fig. 2 is equivalent to the one in Fig. 4. This last figure shows the conventional separation of the linear and nonlinear part, which gives rise to a representation amenable to describing function method. The linear part is such that

$$G_T = G_1 G = \frac{k_1(s + k_2)}{s} \frac{k}{s + a},$$

and the nonlinearity $\phi(u)$ is a normalized saturation.

The parameters in the concrete case of (Middleton, 1996) take the values $k_1 = 7$, $k_1 k_2 = 5$, $k = 1$ and $a = -1$. However, a generic case will be analysed here fixing only $k_2 = 5/7$, in order to get universality and take advantage of the bifurcation point of view.

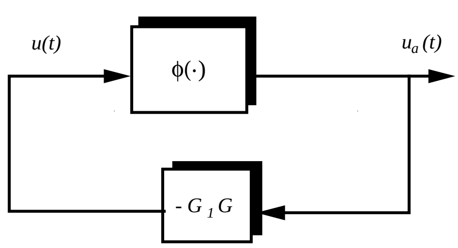

Figure 4: Equivalent scheme of the system without anti-windup.

Defining $k_T = kk_1$ and $\mu = (a, k_T)$, we write $q_\mu(s) = s(s + a)$, $p_\mu(s) = k_T(s + k_2)$, so that

$$Q_\mu(s) = s^2 + (a + k_T)s + k_T k_2.$$

Therefore, the system is closed loop stable if $k_T > 0$ and $a + k_T > 0$. Applying the concepts of previous section, we see from (14) that a saddle-node bifurcation SN_0 appears for $k_T = 0$, and then for $k_T < 0$ the system becomes unstable. Also, from (17) we detect a Hopf bifurcation at infinity H_∞ for $a = 0$ that gives rise to one unstable limit cycle of great amplitude for $k_T < a < 0$, limiting the attraction basin of the origin. This region of local stability, see Fig. 5, ends to the left in the line $0 < k_T = -a$ where, recalling (15)-(16), a Hopf bifurcation (denoted by H_0^u) is produced. This bifurcation makes the operating point unstable, giving rise to the disappearance of the unstable limit cycle. It should be remarked that the stability is global for $a > 0$ and $k_T > 0$, but only local for $a < 0$ and $a + k_T > 0$. In Fig. 6 the bifurcations diagram of the system for a positive value of k_T is displayed. This diagram shows the loci of equilibria and limit cycles as parameter a varies. It is clear that the Hopf bifurcation at infinity H_∞^u, which gives rise to the unstable limit cycle and thus a limited attraction basin, is responsible of the global troubles with the actuator saturation for this open-loop unstable plant ($a < 0$).

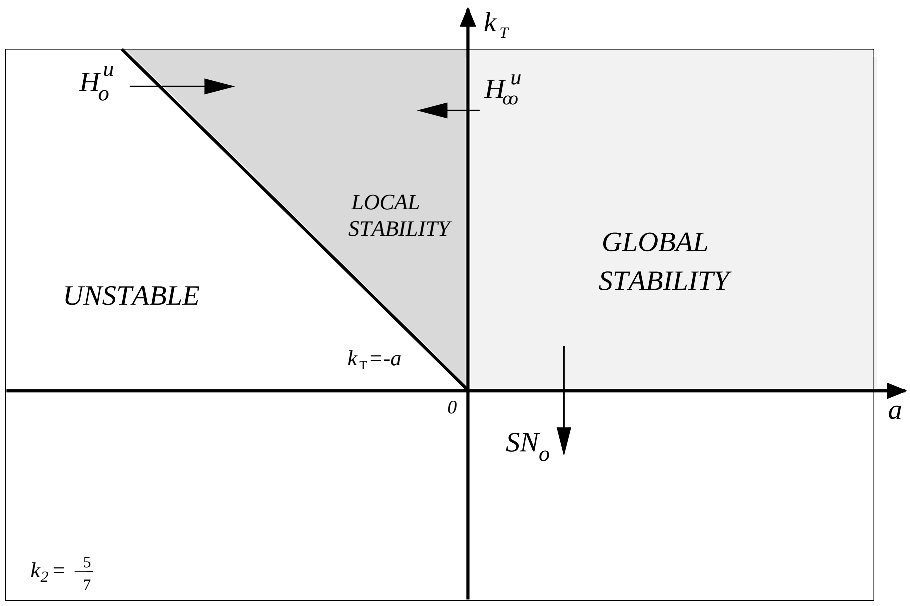

Figure 5: Bifurcation set in the parameter plane (a, k_T) for the system without anti-windup.

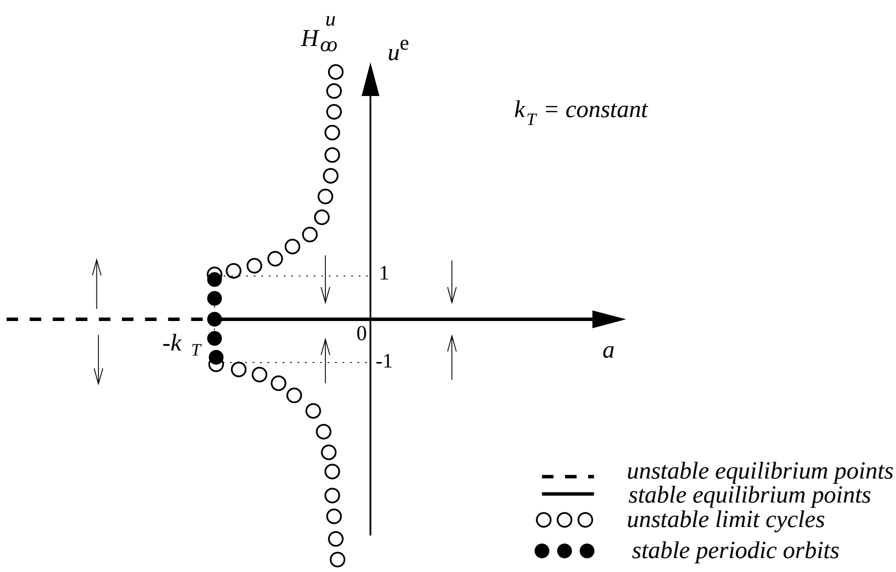

Figure 6: Bifurcations diagram of the system without anti-windup, for $k_T > 0$.

The diference between local and global phenomena is more crucial when analysing the solution given in Fig. 7, where it is proposed an anti-integral windup scheme (Middleton, 1996). This scheme corrects only the local effects and if, as will be seen, the unstable limit cycle disappears, new equilibria are born and the global phenomena remain.

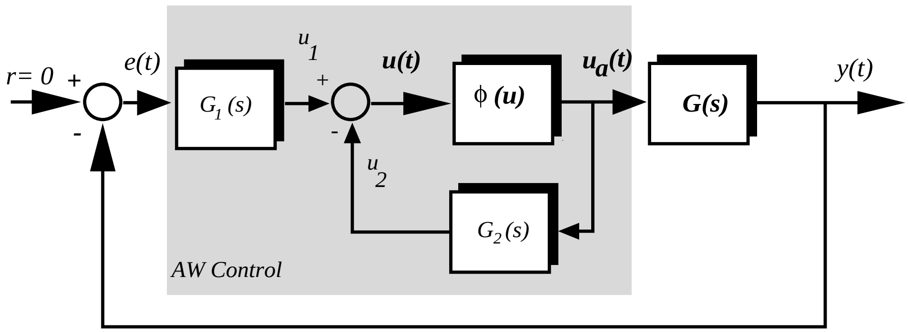

Figure 7: Diagram of the system with anti-windup.

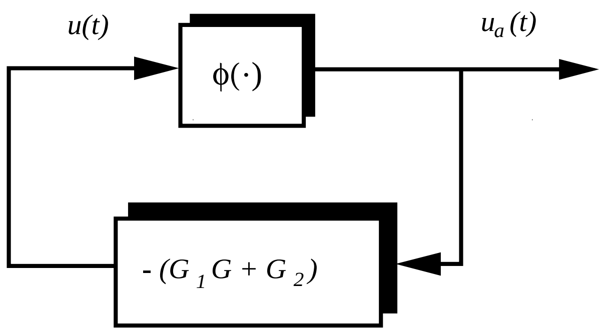

Figure 8: Equivalent scheme of the system with anti-windup.

The anti-windup scheme proposed in (Middleton, 1996) is such that $G_1(s) = (7s+5)/(s+5)$, $G_2(s) = -5/(s+5)$ and $G(s) = 1/(s-1)$. If the system reference is considered equal to zero, then this scheme is equivalent to the one shown in Fig. 8. To verify this equivalence it should be considered that the constraints on the control variable are only included in the control law and not in the plant because of redundancy. It should be remarked that with the change of scheme the type of the linear system has varied from type 1 to type 0. This change of type is deeply related with the global changes, as will be seen in what follows.

The linear part now results to be

$$G_T = G_1G + G_2 = \frac{k_1(s+k_2)}{s+b}\frac{k}{s+a} - \frac{b}{s+b},$$

and then

$$G_T(0) = \frac{k_T k_2}{ab} - 1,$$

being $b = 5$ and the other parameters as above.

In the case considered, where $a = -1$ and all the other parameters b, k, k_1, and k_2 are positive, it is clear that $G_T(0) < -1$. So the shape of the polar plot of $G_T(j\omega)$ and the state portrait of the system are the ones shown in Fig. 9. This figure has been plotted with the same scale of Fig. 3 to remark the effects of the anti-windup on the system gain. Recalling the results

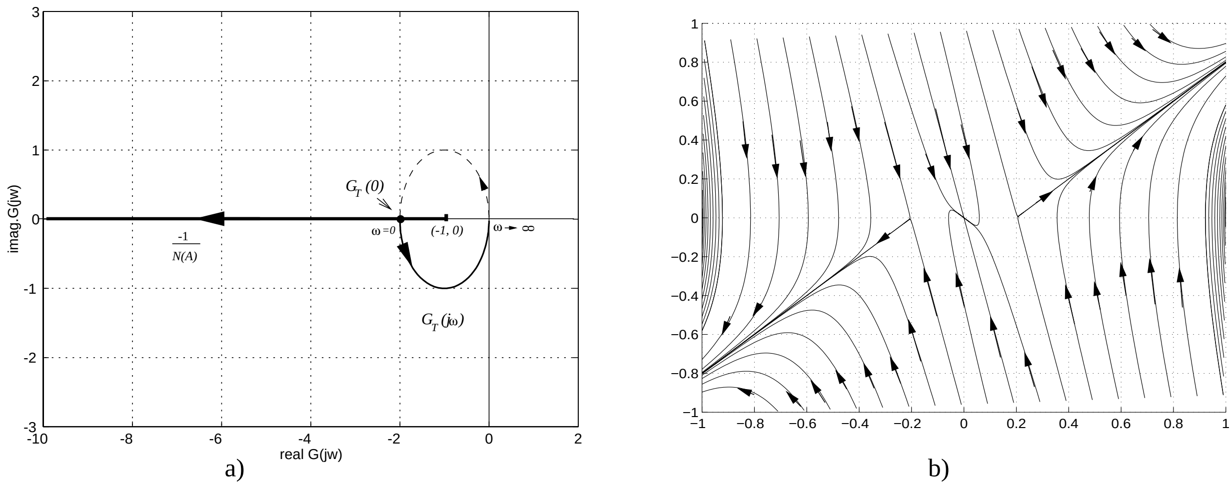

Figure 9: System with anti-windup: a) polar plot of $G_T(j\omega)$; b) state portrait (x_1, x_2).

in Sec. 3, it is deduced that now the closed loop system has in the state space two saddles, further than the equilibrium point at the origin.

To gain a wider perspective on what actually happens it is worthy to analyse Fig. 10 where the different behavior regions of system behavior are displayed in the (a, k_T) plane. In this figure it is shown that again for $a > 0$ and $k_T > 0$ the system is globally stable. However, for $a < 0$ it is only locally stable due to a pitchfork bifurcation at infinity P_∞. This bifurcation gives rise to two equilibria (saddles) whose stable manifolds limit again the attraction basin of the origin.

Using the ideas previously developed, other bifurcation lines limit the local stability parameter region. At $a = -k_T$ the system undergoes a Hopf bifurcation at the origin H_0. The dashed curve of Fig. 10 corresponds to *double saddle connection bifurcation* points (DSC), and intersects in the point T the line of Hopf bifurcation points H_0, separating supercritical Hopf bifurcation points H_0^s from subcritical ones H_0^u. The DSC bifurcation is a more complex phenomenon that is far from the scope of this paper, see (Guckenheimer and Holmes, 1983). For more details, see (Aracil *et al.*, 1997).

In this way it is concluded that the anti-windup scheme (even improving the local behavior at the equilibrium point) does not solve at all the global problems. The unstable limit cycle has disappeared but instead two saddles have arised giving rise to new global difficulties. For instance in (Middleton, 1996) it is said that for a step change of 1.0 units the system recovers the operating point. That is true, but for 1.2 units it gets again out of control. So the advantage in a global setting is more limited than thought. The qualitative analysis proposed here could help to better understand the global problems involved in the design and how to improve it.

5 Conclusions

In this paper, bifurcations at infinity have been shown to be in the core of global state-space problems raised by the nonlinear structure of systems with constrained controls. Some insight has been gained regarding the behavior problems in systems with actuator saturations, and in particular for an anti-windup scheme.

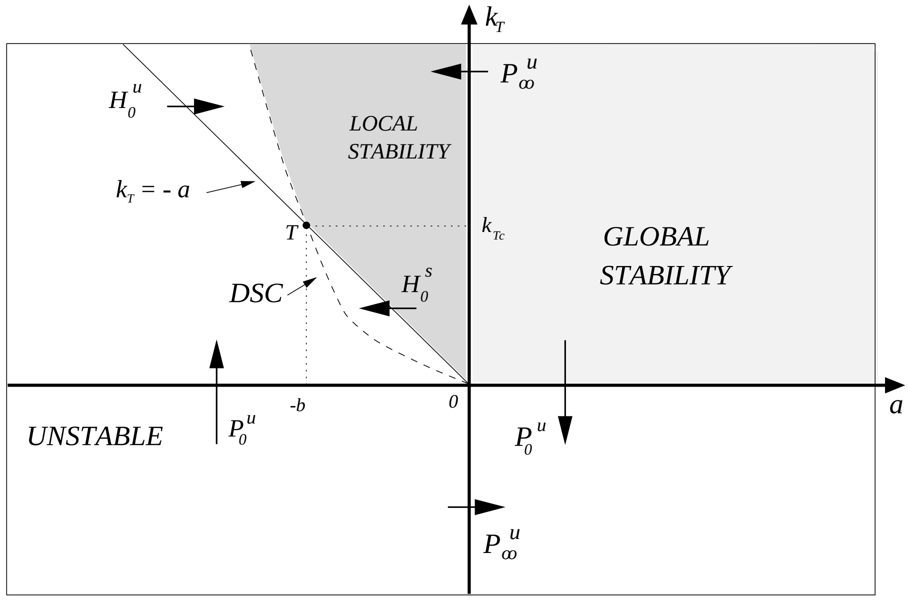

Figure 10: Bifurcation set in the parameter plane (a, k_T) for the system with anti-windup.

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