Reduction of singular 2D models to equivalent standard models

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Abstract. A new extended Roesser type model is introduced. It is shown that:

1. Any singular 2D general model (1) with $E \neq 0$ can be reduced to the model (6) (or (6`)),
2. Regular singular 2D model (9) can be reduced to standard extended Roesser type model (11),

Sufficient conditions are established under which a singular 2D general model (1) can be reduced to standard models of the form (28) or (35).

Key Words: Extended Roesser model, singular 2D model, reduction, sufficient conditions

1. Introduction

The most popular two-dimensional (2D) models are the discrete models proposed by Roesser [18] and by Fornasini and Marchesini [2,3] and Kurek [15]. Extensions of these models from 2D to nD has been suggested in [13,12,19]. Polynomial and algebraic approachs for transformations and recasting of different 2D and nD models have been studied by Galkowski [5-7]. Singular 2D and nD general models have been introduced in [9-12]. Recently the relationship between the general nD Roesser model and Fornasini-Marchesini models has been given by Miri and Applevich in [17].
In [1,8,12] the 2D shuffle algorithm has been used for checking of the regularity of singular 2D model and its reduction to standard form. It is known that in some cases the algorithm may stop without reaching a definite conclusion about the regularity of the singular 2D model and its reduction to the standard form [1,8,12]. The results of this paper clarify at least partly why it takes place.
In this paper a new extended Roesser type model will be introduced and some new procedures for reduction of singular 2D general to standard models will be proposed.

3. Models of 2D systems.

Let $R^{n\times m}$ be set of $m\times n$ real matrices and $R^n := R^{n\times 1}$. The set of nonnegative integers will be denoted by Z_+.
Consider a 2-D system described by the equations [9,10]

(1a) $$\mathrm{E}x_{i+1,j+1} = A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + Bu_{ij}$$

(1b) $$y_{ij} = Cx_{ij} + Du_{ij} \qquad i, j \in Z_+$$

where $x_{ij} \in R^n$ is the emistate vector at the point (i, j), $u_{ij} \in R^m$ is the input vector, $y_{ij} \in R^p$ is the output vector and $E \in R^{n\times n}$, $A_k \in R^{n\times n}, k = 0,1,2, B \in R^{n\times m}, C \in R^{p\times n}, D \in R^{p\times m}$
Boundary conditions for (1a) are given by

(2) x_{i0} for $i \in Z_+$ and x_{0j} for $j \in Z_+$

The model (system) (1) is called standard if $E = I_n$ (the identity matrix) and it is called singular if $\det E = 0$

If

(3) $\det[Ez_1z_2 - A_0 - A_1 - A_2] \neq 0$ for some $z_1z_2 \in \mathbf{C}$ (the field of complex numbers)

the model (system) (1) is called regular.

We shall also considered an extended 2D Roesser type model of the form

(4a) $$E\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}\begin{bmatrix} x^h_{i,j+1} \\ x^v_{i+1,j} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij}$$

(4b) $$y_{ij} = [C_1 \ \ C_2]\begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + Du_{ij} \qquad \text{i, j} \in \mathrm{Z}_+$$

where $x^h_{ij} \in R^{n_1}$ is the horizontal semistate vector, $u_{ij} \in R^m$ is the input vector, $y_{ij} \in R^p$ is the output vector, $A_{11}, F_1 \in R^{n_1 \times n_1}$, $A_{22}, F_2 \in R^{n_2 \times n_2}$, $E \in R^{n \times n}$, $n = n_1 + n_2$, $B_1 \in R^{n_1 \times m}$, $B_2 \in R^{n_2 \times m}$, $C_1 \in R^{p \times n_1}, C_2 \in R^{p \times n_2}, D \in R^{p \times m}$

The extended model (4) is called standard if $E = I_n$ and it is called singular if $\det E = 0$.

If

(5) $$\det\begin{bmatrix} E_{11}z_1 - A_{11} - F_1z_2\,, & E_{12}z_2 - A_{12} \\ E_{21}z_1 - A_{21} & , E_{22}z_2 - A_{22} - F_2z_1 \end{bmatrix} \neq 0 \quad \text{for some} \quad z_1, z_2 \in \mathbf{C}$$

then the model (4) is called regular. For $F_1 = 0$ and $F_2 = 0$ from (4) we obtain the singular 2D Roesser model [11].

Theorem 1. The model (1) can be reduced to the form

(6a) $$\tilde{A}_0\tilde{x}_{ij} + \tilde{A}_1\tilde{x}_{i+1,j} + \tilde{A}_2\tilde{x}_{i,j+1} + \tilde{B}u_{ij} = 0$$

(6b) $$y_{ij} = \tilde{C}\tilde{x}_{ij}$$

or

(6\`a) $$\tilde{A}'_0\tilde{x}'_{ij} + \tilde{A}'_1\tilde{x}'_{i+1,j} + \tilde{A}'_2\tilde{x}'_{i,j+1} + \tilde{B}'u_{ij} = 0$$

(6\`b) $$y_{ij} = \tilde{C}'\tilde{x}'_{ij}$$

where

$$\tilde{x}_{ij} := \begin{bmatrix} x_{i+1,j} \\ x_{ij} \end{bmatrix},\ \tilde{A}_0 := \begin{bmatrix} A_1 & A_0 \\ I_n & 0 \end{bmatrix}, \tilde{A}_1 := \begin{bmatrix} 0 & 0 \\ 0 & -I_n \end{bmatrix}, \tilde{A}_2 := \begin{bmatrix} -E & A_2 \\ 0 & 0 \end{bmatrix}, \tilde{B} := \begin{bmatrix} B \\ 0 \end{bmatrix}, \tilde{C} := [0 \quad C]$$

$$\tilde{x}'_{ij} := \begin{bmatrix} x_{i,j+1} \\ x_{ij} \end{bmatrix},\ \tilde{A}'_0 := \begin{bmatrix} A_2 & A_0 \\ I_n & 0 \end{bmatrix}, \tilde{A}'_1 := \begin{bmatrix} -E & A_1 \\ 0 & 0 \end{bmatrix}, \tilde{A}'_2 := \begin{bmatrix} 0 & 0 \\ 0 & -I_n \end{bmatrix},, \tilde{B}' := \begin{bmatrix} B \\ 0 \end{bmatrix}, \tilde{C}' := [0 \quad C]$$

Proof. The equations (1) can be written in the form

(7) $$[-E,\ A_2]\begin{bmatrix} x_{i+1,j+1} \\ x_{i,j+1} \end{bmatrix} + [A_1,\ A_0]\begin{bmatrix} x_{i+1,j} \\ x_{ij} \end{bmatrix} + Bu_{ij} = 0$$

and

(8) $$y_{ij} = [0\ \ C]\begin{bmatrix} x_{i+1,j} \\ x_{ij} \end{bmatrix}$$

Using (7) we can write the equation

$$\begin{bmatrix} A_1 & A_0 \\ I_n & 0 \end{bmatrix}\begin{bmatrix} x_{i+1,j} \\ x_{ij} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -I_n \end{bmatrix}\begin{bmatrix} x_{i+2,j} \\ x_{i+1,j} \end{bmatrix} + \begin{bmatrix} -E & A_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{i+1,j+1} \\ x_{i,j+1} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_{ij} = 0$$

which is equivalent to (6a). The proof of the dual equations (6`) is similar.

4. **Reduction of singular models.**

Now first we shall consider a particular case of (1a) for $E = 0$, i.e.

(9) $$A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + Bu_{ij} = 0$$

under the assumption that

(10) $$\det[A_1 z + A_2] \neq 0 \quad \text{for some } s \in \mathbf{C}$$

Theorem 2. If the condition (10) is satisfied then the equation (9) can be reduced to the form

(11) $$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}\begin{bmatrix} x^h_{i,j+1} \\ x^v_{i+1,j} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij}, \qquad \text{i, j} \in Z_+$$

where $x^h_{ij}, x^v_{ij}, u_{ij}$ and the submatrices A_{ij}, B_i, F_i $(i, j = 1,2)$ are defined in the same way as for (4a)

Proof. It is well-known [12] that if (10) holds then there exists a pair of nonsingular matrices $P, Q \in R^{n\times n}$ such that

(12) $$PA_1Q = \begin{bmatrix} I_{n_1} & 0 \\ 0 & -F_2 \end{bmatrix},\ PA_2Q = \begin{bmatrix} -F_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}$$

where n_1 is equal to the degree of the polynomial $\det[A_1 z + A_2]$, $F_1 \in R^{n_1\times n_1}$, $F_2 \in R^{n_2\times n_2}$ is a nilpotent matrix.

Premultiplying (9) by the matrix P and introducing the new subvectors $x^h_{ij} \in R^{n_1}, x^v_{ij} \in R^{n_2}$ defined by

(13) $$\begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} = Q^{-1} x_{ij}$$

and using (12) we obtain

(14) $$-\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + \begin{bmatrix} I_{n_1} & 0 \\ 0 & -F_{21} \end{bmatrix}\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i+1,j} \end{bmatrix} + \begin{bmatrix} -F_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}\begin{bmatrix} x^h_{i,j+1} \\ x^v_{i,j+1} \end{bmatrix} - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij} = 0$$

where

(15) $$PA_0Q = -\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad PB = -\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

It is easy to see that the equation (14) can be rewritten in the desired form (11).
From theorem 2 it follows that if (10) holds then the singular model (9) with $\det A_1 = 0$ (or and $\det A_2 = 0$) can be reduced to standard extended Roesser type model (11).

To find the solution $\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix}$ of (11) it is enough to know the boundary conditions

(16) $$x_{0j}^h \quad \text{for} \quad j \in Z_+ \quad \text{and} \quad x_{i0}^v \quad \text{for} \quad i \in Z_+$$

and u_{ij} for $i, j \in Z_+$. Knowing (2) the boundary conditions (16) may be computed the from equation (13).

Theorem 3. If $\det F_1 \neq 0$ then the model (11) can be transformed to the standard model

(17) $$\begin{bmatrix} x_{i,j+1}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} -F_1^{-1}A_{11}, & -F_1^{-1}A_{12} \\ A_{21}, & A_{22} \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + \begin{bmatrix} -F_1^{-1}B_1 \\ B_2 \end{bmatrix} u_{ij} + \begin{bmatrix} F_1^{-1} & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} x_{i+1,j}^h \\ x_{i+1,j}^v \end{bmatrix}$$

and if $\det F_2 \neq 0$ then the model (11) can be transformed to the standard model

(18) $$\begin{bmatrix} x_{i+1,j}^h \\ x_{i+1,j}^v \end{bmatrix} = \begin{bmatrix} A_{11}, & A_{12} \\ -F_2^{-1}A_{21}, & -F_2^{-1}A_{22} \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ -F_2^{-1}B_2 \end{bmatrix} u_{ij} + \begin{bmatrix} F_1 & 0 \\ 0 & F_2^{-1} \end{bmatrix} \begin{bmatrix} x_{i,j+1}^h \\ x_{i,j+1}^v \end{bmatrix}$$

Proof. The model (11) can rewritten in the form

(19) $$\begin{bmatrix} I & 0 \\ 0 & -F_2 \end{bmatrix} \begin{bmatrix} x_{i+1,j}^h \\ x_{i+1,j}^v \end{bmatrix} + \begin{bmatrix} -F_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_{i,j+1}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij}$$

If the $\det F_1 \neq 0$ then premultiplying (19) by $\begin{bmatrix} -F_1^{-1} & 0 \\ 0 & I \end{bmatrix}$ we obtain (17) and if $\det F_2 \neq 0$ then premultiplying (19) by $\begin{bmatrix} I & 0 \\ 0 & -F_2^{-1} \end{bmatrix}$ we obtain (18).

In general case of (1) it is assumed that $|\det A_1| + \det|A_2| \neq 0$.

It is well-known [12] that if $rank\, E = r < n$ then there exist nonsingular matrices $T_1, T_2 \in R^{n\times n}$ such that

(20) $$T_1 E T_2 = \begin{bmatrix} 0 & I_r \\ 0 & 0 \end{bmatrix}$$

Premultiplying (1a) by T_1 and defining the new subvectors $x_{ij}^1 \in R^{n-r}, x_{ij}^2 \in R^r$ by $\begin{bmatrix} x_{ij}^1 \\ x_{ij}^2 \end{bmatrix} = T_2^{-1} x_{ij}$

we obtain

(22a) $$x_{i+1,j+1}^2 = A_{11}^0 x_{ij}^1 + A_{12}^0 x_{ij}^2 + A_{11}^1 x_{i+1,j}^1 + A_{12}^1 x_{i+1,j}^2 + A_{11}^2 x_{i,j+1}^1 + A_{12}^2 x_{i,j+1}^2 + B_1 u_{ij}$$

(22b) $$0 = A_{21}^0 x_{ij}^1 + A_{22}^0 x_{ij}^2 + A_{21}^1 x_{i+1,j}^1 + A_{22}^1 x_{i+1,j}^2 + A_{21}^2 x_{i,j+1}^1 + A_{22}^2 x_{i,j+1}^2 + B_2 u_{ij}$$

where

(22c) $\overline{A}_k := T_1 A_k T_2 = \begin{bmatrix} A_{11}^k & A_{12}^k \\ A_{21}^k & A_{22}^k \end{bmatrix}, k = 0,1,2,\ T_1 B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \begin{matrix} A_{11}^k \in R^{r\times(n-r)},\ B_1 \in R^{r\times m} \\ A_{22}^k \in R^{n-r\times r},\ B_2 \in R^{n-r\times m} \end{matrix}$

Note that $\det A_1 \neq 0$ implies the full row rank of the matrix

(23) $$\tilde{A}_1 = \left[A_{21}^1, A_{22}^1\right]$$

and there exists its right inverse $\left(\tilde{A}_1 \tilde{A}_1^r = I_{n-r}\right)$ of the form

(24) $$\tilde{A}_1^r = \tilde{A}_1^T \left[\tilde{A}_1 \tilde{A}_1^T\right]^{-1}$$

where T denotes the transposition.
Using (24) from (22b) we have

(25) $$x_{i+1,j}^2 = -\tilde{A}_{21}\left(A_{21}^0 x_{ij}^1 + A_{22}^0 x_{ij}^2 + A_{21}^2 x_{i,j+1}^1 + A_{22}^2 x_{i,j+1}^2 + B_2 u_{ij}\right)$$

where $\tilde{A}_{21} := \tilde{A}_{22}^{1T}\left[\tilde{A}_1 \tilde{A}_1^T\right]^{-1}$
From (22a) and (25) we obtain

(26) $$\begin{aligned} 0 = {} & A_{11}^0 x_{ij}^1 + A_{12}^0 x_{ij}^2 + A_{11}^1 x_{i+1,j}^1 + A_{12}^1 x_{i+1,j}^2 + \left(A_{11}^2 + \tilde{A}_{21} A_{21}^0\right) x_{i,j+1}^1 + \left(A_{12}^2 + \tilde{A}_{21} A_{22}^0\right) x_{i,j+1}^2 + \\ & + \tilde{A}_{21}\left(A_{21}^2 x_{i,j+2}^1 + A_{22}^2 x_{i,j+2}^2 + B_2 u_{i,j+1}\right) + B_1 u_{ij} \end{aligned}$$

The equations (22b) and (26) can be written in the form

(27) $$\overline{A}_0 x_{ij} + \overline{A}_1 x_{i+1,j} + \overline{A}_2' x_{i,j+1} + \overline{A}_3' x_{i,j+2} + \overline{B}_0 u_{ij} + \overline{B}_1 u_{i,j+1} = 0$$

where $\overline{A}_0$ and $\overline{A}_1$ are defined by (22c) and

$$\overline{A}_2' = \begin{bmatrix} A_{11}^2 + \tilde{A}_{21} A_{21}^0, & A_{12}^2 + \tilde{A}_{21} A_{22}^0 \\ A_{21}^2, & A_{22}^2 \end{bmatrix}, \overline{A}_3' = \begin{bmatrix} \tilde{A}_{21} A_{21}^2, & \tilde{A}_{21} A_{22}^2 \\ 0, & 0 \end{bmatrix}, \overline{B}_0 := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \overline{B}_1 := \begin{bmatrix} \tilde{A}_{21} B_2 \\ 0 \end{bmatrix}, x_{ij} = \begin{bmatrix} x_{ij}^1 \\ x_{ij}^2 \end{bmatrix}$$

From (22c) it follows that $\det A_1 \neq 0$ implies $\det \overline{A}_1 \neq 0$ and from (27) we obtain

(28) $$x_{i+1,j} = A_0 x_{ij} + A_2 x_{i,j+1} + A_3 x_{i,j+2} + B_0 u_{ij} + B_1 u_{i,j+1}$$

where

(29) $$A_0 := -\overline{A}_1^{-1} \overline{A}_0,\ A_2 := -\overline{A}_1^{-1} \overline{A}_2',\ A_3 := -\overline{A}_1^{-1} \overline{A}_3',\ B_0 := -\overline{A}_1^{-1} \overline{B}_0,\ B_1 := -\overline{A}_1^{-1} \overline{B}_1$$

If $\det A_2 \neq 0$ then the matrix

(30) $$\tilde{A}_2 := \left[A_{21}^2 \ \ A_{22}^2\right]$$

has full row rank and there exists its right inverse

(31) $$\tilde{A}_2^r = \tilde{A}_2^T \left[\tilde{A}_2 \tilde{A}_2^T\right]^{-1}$$

Using (31) from (22b) we have

(32) $$x_{i,j+1}^2 = -\tilde{A}_{22}\left(A_{21}^0 x_{ij}^1 + A_{22}^0 x_{ij}^2 + A_{21}^1 x_{i+1,j}^1 + A_{22}^1 x_{i+1,j}^2 + B_2 u_{ij}\right)$$

where $\tilde{A}_{22} := \tilde{A}_{22}^{2T}\left[\tilde{A}_2 \tilde{A}_2^T\right]^{-1}$
From (22a) and (32) we obtain

(33) $$\begin{aligned} 0 = {} & A_{11}^{00} x_{ij}^1 + A_{12}^0 x_{ij}^2 + \left(A_{11}^1 + \tilde{A}_{22} A_{21}^0\right) x_{i+1,j}^1 + \left(A_{12}^1 + \tilde{A}_{22} A_{22}^0\right) x_{i+1,j}^2 + A_{11}^2 x_{i,j+1}^1 + A_{12}^2 x_{i,j+1}^2 + \\ & + \tilde{A}_{22} A_{21}^1 x_{i+2,j}^1 + \tilde{A}_{22} A_{22}^1 x_{i+2,j}^2 + B_1 u_{ij} + \tilde{A}_{22} B_2 u_{i+1,j} \end{aligned}$$

The equations (22b) and (33) can be written in the form

(34) $$\overline{A}_0 x_{ij} + \overline{A}_1' \, x_{i+1,j} + \overline{A}_2 \, x_{i,j+1} + A_3' \, x_{i+2,j} + \overline{B}_0' \, u_{ij} + \overline{B}_1' \, u_{i+1,j} = 0$$

where $\overline{A}_0$ and $\overline{A}_2$ are defined by (22c) and

$$\overline{A}_1' := \begin{bmatrix} A_{11}^1 + \tilde{A}_{22} A_{21}^0 \ , \ A_{12}^1 + \tilde{A}_{22} A_{22}^0 \\ A_{21}^1 \quad , \quad A_{22}^1 \end{bmatrix}, \overline{A}_3' = \begin{bmatrix} \tilde{A}_{22} A_{21}^1 \ , \ \tilde{A}_{22} A_{22}^1 \\ 0 \quad , \quad 0 \end{bmatrix}, \overline{B}_0' := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \overline{B}_1' := \begin{bmatrix} \tilde{A}_{22} B_2 \\ 0 \end{bmatrix}, x_{ij} = \begin{bmatrix} x_{ij}^1 \\ x_{ij}^2 \end{bmatrix}$$

From (22c) it follows that $\det A_2 \neq 0$ implies $\det \overline{A}_2 \neq 0$ and from (34) we obtain

(35) $$x_{i,j+1} = A_0' \, x_{ij} + A_1' \, x_{i+1,j} + A_3' \, x_{i+2,j} + B_0' \, u_{ij} + B_1' \, u_{i+1,j}$$

where

$$A_0' = -\overline{A}_2^{-1} \overline{A}_0 \, , A_1' = -\overline{A}_2^{-1} \overline{A}_1' , A_3' = -\overline{A}_2^{-1} \tilde{A}_3' , B_0' := -\overline{A}_2^{-1} \overline{B}_0' , \, B_1' = -\overline{A}_2^{-1} \overline{B}_1'$$

Therefore, we have proved the following theorem

Theorem 4. The singular model (1a) can be reduced to the standard model (28) if $\det A_1 \neq 0$ and to the standard model (35) if $\det A_2 \neq 0$.

Note that to find the solution x_{ij} for $i, j \in Z_+$ of the equation (28) we need to know only x_{0j} for $j \in Z_+$ and u_{ij} for $i, j \in Z_+$.

Similarily to find the solution x_{ij} for $i, j \in Z_+$ of the equation (35) we need to know only x_{0j} for $i \in Z_+$ and u_{ij} for $i, j \in Z_+$.

Example 1. Consider the model (1a) with

(36) $$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

In this case $n = 3, r = 1$, matrix A_1 is invertible and A_2 singular.

By theorem 4 we may reduced the singular model (1a) with (36) to the standard model (28).
Taking into account that in this case

$$\overline{A}_0 = A_0 = \left[\begin{array}{cc:c} 1 & 0 & 0 \\ \hdashline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \overline{A}_1 = A_1 = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ \hdashline 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right], \overline{A}_2 = A_2 = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ \hdashline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right],$$

$$\tilde{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tilde{A}_1^r = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tilde{A}_{21} = [0 \ \ 1]$$

$$\overline{A}_2' = \begin{bmatrix} A_{11}^2 + \tilde{A}_{21} A_{21}^0 \ , \ A_{12}^2 + \tilde{A}_{21} A_{22}^0 \\ A_{21}^2 \quad , \quad A_{22}^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\overline{A}_3' = \begin{bmatrix} \tilde{A}_{21} A_{21}^2 \ , \ \tilde{A}_{21} A_{22}^2 \\ 0 \quad , \quad 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \overline{B}_0 = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \overline{B}_1 = \begin{bmatrix} \tilde{A}_{21} B_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and using (29) we obtain the model (28) with the matrices

$$A_0 = -\overline{A}_1^{-1}\overline{A}_0 = -\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ A_2 = -\overline{A}_1^{-1}\overline{A}_2' = -\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},\ A_3 = -\overline{A}_1^{-1}\overline{A}_3' = -\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_0 = -\overline{A}_1^{-1}\overline{B}_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, B_1 = -\overline{A}_1^{-1}\overline{B}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Note that in this case A_2 is singular but $\overline{A}_2'$ is invertible. Therefore the singular model (1a) with (36) may be also reduced to the standard model (35).
Note that using the 2D shuffle algorithm [1,8,12] we can not reduce the model (1a) with (36) to the standard 2D general model since both matrices $\left[A_{21}^1 \ \ A_{22}^1\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left[A_{21}^2 \ \ A_{22}^2\right] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ of (22) are nonzero.

Theorem 5. The singular model (1a) can be reduced to the standard model (28) with

(37) $\quad A_0 := -\overline{A}_1'^{-1}\overline{A}_0 , A_2 := -\overline{A}_1'^{-1}\overline{A}_2 , A_3 := -\overline{A}_1'^{-1}\tilde{A}_3', B_0 := -\overline{A}_1'^{-1}\overline{B}_0',\ B_1 := -\overline{A}_1'^{-1}\overline{B}_1'$

if the matrix (30) has full row rank and $\det \overline{A}_1' \neq 0$ and to the standard model (35) with

(38) $\quad A_0' := -\overline{A}_2'^{-1}\overline{A}_0 , A_1' := -\overline{A}_2'^{-1}\overline{A}_1 , A_3' := -\overline{A}_2'^{-1}\overline{A}_3', B_0' := -\overline{A}_2'^{-1}\overline{B}_0 ,\ B_1' := -\overline{A}_2'^{-1}\overline{B}_1$

if the matrix (23) has full row rank and $\det \overline{A}_2' \neq 0$.

Proof. If the matrix (30) has full row rank then we can find the equation (34) in the same way as in the proof of theorem 4. If $\det \overline{A}_1' \neq 0$ then solving the equation (34) with respect to $x_{i+1,j}$ we obtain (28) with (37). If the matrix (23) has full row rank then we can find the equation (27) in the same way as in the proof of theorem 4. If $\det \overline{A}_2' \neq 0$ then solving the equation (27) with respect to $x_{i,j+1}$ we obtain (35) with (38).

Example 2. Consider the model (1a) with

(39) $$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

In this case $n = 3,\ r = 1$, the matrices A_1 and A_2 are singular but the matrix

(40) $$\tilde{A}_1 = \left[A_{21}^1 \ \ A_{22}^1\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has full row rank. To check if $\det \overline{A}_2' \neq 0$ we compute

$$\tilde{A}_{21} = \tilde{A}_{22}^{1T}\left[\tilde{A}_1 \ \tilde{A}_1^T\right]^{-1} = [0 \ \ 1]\,, \quad \overline{A}_3' = \begin{bmatrix} \tilde{A}_{21}A_{21}^2 \ , \ \tilde{A}_{21}A_{22}^2 \\ 0 \ \ , \ \ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

(41)
$$\overline{A}_2' = \begin{bmatrix} A_{11}^2 + \tilde{A}_{21}A_{21}^0 \ , \ A_{12}^2 + \tilde{A}_{21}A_{22}^0 \\ A_{21}^2 \quad , \quad A_{22}^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix (40) has full row rank and the matrix (41) is invertible. Therefore the conditions of theorem 5 are satisfied and the singular model (1a) with (39) can be reduced to the standard model (35) with

$$A_0' = -\overline{A}_2'^{-1}\overline{A}_0 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, A_1' = -\overline{A}_2'^{-1}\overline{A}_1 = -\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, A_3' = -\overline{A}_2'^{-1}\overline{A}_3' = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_0' = -\overline{A}_2' \ \overline{B}_0 = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}, B_1' = -\overline{A}_2' \ \overline{B}_1 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

It is easy to check that in this case the $\det \overline{A}_1' = 0$.

4. Concluding remarks.

It has been shown that the singular 2D general model (1) can be always reduced to the singular model (6) or (6`) (theorem 1). If the condition (1) is satisfied, then the equation (9) can be reduced to the standard form (11) (theorem 2). If $\det A_1 \neq 0$ ($\det A_2 \neq 0$) then the singular model (1a) can be reduced to the standard model (28) ((35)) (theorem 4). If the matrix (30) ((23)) has full row rank and $\det \overline{A}_1' = 0$ ($\det \overline{A}_2' = 0$) then the singular model (1a) can be reduced to the standard model (28) ((35)) (theorem 5). The above considerations can be extended for singular nD (*n*>2) models. Note that by theorem 4 the singular model (1) can not be reduced to the standard 2D general model but it can be reduced to the standard model (28) or (35). Futher investigations are needed to established necessary and sufficient conditions under which the singular 2D general model (1) can be reduced to the standard 2D general model.

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