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Adaptive Sliding Backstepping Control of Nonlinear Semi-Strict Feedback Form Systems

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Abstract

This paper considers the application of a combined adaptive backstepping sliding mode control (SMC) algorithm to a class of nonlinear continuous uncertain processes which can be converted to a semi-parametric strict form. The algorithm follows a systematic procedure for the design of dynamical adaptive SMC laws for the output regulation of observable minimum phase nonlinear systems.

1 Introduction

The backstepping design procedure is a systematic design technique for globally stable and asymptotically tracking adaptive controllers for a class of nonlinear systems. In fact, adaptive backstepping algorithm has been its applicability to systems which can be transformed to a triangular form, in particular, the parametric pure feedback (PPF) form and the parametric static feedback (PSF) form (Kanellakopoulos *et al.*, 1991). This method has widely been studied in the last few years (Kanellakopoulos *et al.*, 1991, 1992; Rios-Bolívar and Zinober, 1994, 1997a,b). When plants include uncertainty with lack of information about the bounds of unknown parameters, adaptive control is more convenient; whilst, if some information about uncertainty (e.g. bounds) is available, robust control is usually employed. Sliding mode control (SMC) is a robust control method, and backstepping can be considered a method of adaptive control. The combination of these methods yields benefits from both methods. A systematic design procedure has been proposed to combine adaptive control and SMC for the nonlinear systems with relative degree one (Yao and Tomizuka, 1994). The sliding mode backstepping approach has been considered for some classes of nonlinear systems which need not be in the PPF or PSF forms (Rios-Bolívar and Zinober, 1994, 1997a,b; Rios-Bolívar, 1997). A symbolic algebra toolbox allows striagthforward design (Rios-Bolívar and Zinober, 1998).

If a plant contains unmatched uncertainty, the system may be stabilized via state feedback control (Corless and Leitmann, 1996). Some techniques have been proposed for the case of plant containing unmatched uncertainty (Freeman and Kokotović, 1996). In this paper we consider nonlinear systems which can be converted to a particular form, the so called semi-parametric strict form (SPSF). In this form the plant contains unmodeled terms and unmeasurable external disturbance which are bounded by known functions. We extend the classical backstepping

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method to this class of systems in Section 2 to achieve the output tracking of a dynamical reference signal. Sliding mode control design is studied in Section 3. An example illustrating the results is presented in Section 4 with some conclusions in Section 5.

2 Adaptive Robust Control

Consider the semi-strict feedback form

$$\begin{aligned}
\dot{x}_1 &= x_2 + \varphi_1^T(x_1)\theta + \eta_1(x,w,t) \\
\dot{x}_2 &= x_3 + \varphi_2^T(x_1,x_2)\theta + \eta_2(x,w,t) \\
\dot{x}_3 &= x_4 + \varphi_3^T(x_1,x_2,x_3)\theta + \eta_3(x,w,t) \\
&\vdots \\
\dot{x}_{n-1} &= x_n + \varphi_{n-1}^T(x_1,x_2,\ldots,x_{n-1})\theta + \eta_{n-1}(x,w,t) \\
\dot{x}_n &= f(x) + g(x)u + \varphi_n^T(x)\theta + \eta_n(x,w,t) \\
y &= x_1
\end{aligned} \tag{1}$$

where $x = [x_1, x_2, \ldots, x_n]$ is the state, y the output, u the control and $\varphi_i(x_1,\ldots,x_i) \in \mathbb{R}^\rho$, $i = 1,\ldots,n$, are known functions which are assumed to be sufficiently smooth. $\theta \in \mathbb{R}^\rho$ is the vector of constant unknown parameters and $\eta_i(x,w,t)$, $i = 1,\ldots,n$, are the unknown nonlinear scalar functions including all the disturbances. w is an uncertain time-varying parameter.

Assumption. The functions $\eta_i(x,w,t)$, $i = 1,\ldots,n$ are bounded by known functions $h_i(x_1,\ldots x_i)$ $\in \mathbb{R}^\rho$, i.e.

$$|\eta_i(x,w,t)| \le h_i(x_1,\ldots x_i), \qquad i = 1,\ldots,n \tag{2}$$

Suppose $y_r(t)$ is the bounded reference signal with bounded n-th order derivative.

We now follow the backstepping approach.

Step 1. Define the error variable $z_1 = x_1 - y_r$ then

$$\dot{z}_1 = x_2 + \varphi_1^T(x_1)\theta + \eta_1(x,w,t) - \dot{y}_r \tag{3}$$

From (3)

$$\dot{z}_1 = x_2 + \omega_1^T\hat{\theta} + \eta_1(x,w,t) - \dot{y}_r + \omega_1^T\tilde{\theta} \tag{4}$$

with $\omega_1(x_1) = \varphi_1(x_1)$ and $\tilde{\theta} = \theta - \hat{\theta}$ where $\hat{\theta}(t)$ is an estimate of the unknown parameter θ. It is desired that $\hat{\theta} \to \theta$ when $t \to \infty$. The subsystem (3) can be stabilized with respect to the Lyapunov function

$$V_1(z_1,\hat{\theta}) = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \tag{5}$$

where Γ is a positive definite matrix. The derivative V_1 is

$$\dot{V}_1(z_1,\hat{\theta}) = z_1\left(x_2 + \omega_1^T\hat{\theta} + \eta_1(x,w,t) + \omega_1 T\tilde{\theta} - \dot{y}_r\right) + \tilde{\theta}^T\Gamma^{-1}\left(\Gamma\omega_1 z_1 - \dot{\hat{\theta}}\right) \tag{6}$$

Define $\tau_1 = \Gamma\omega_1 z_1$. If $\dot{\hat{\theta}} = \tau_1$, the estimate error $\tilde{\theta}$ can be eliminated from $\dot{V}_1$, i.e. the second term of $\dot{V}_1$ is zero. Let

$$\beta_1 = \alpha_1(x_1, \hat{\theta}, t) + \dot{y}_r = -\omega_1^T\hat{\theta} - h_1(x_1)\mathrm{sgn}(z_1) + \dot{y}_r - c_1 z_1 \tag{7}$$

with c_1 is a positive number. In subsystem (3), x_2 acts a virtual control. So with the control law $x_2 = \beta_1(x_1, \hat{\theta})$, $\dot{V}_1(z_1, \hat{\theta}) = -c_1 z_1^2 + \eta_1(x, w, t)z_1 - h_1(x_1)|z_1|$. However, x_2 is not the actual control and therefore $x_2 \neq \beta_1(x_1, \hat{\theta})$. Define the error variable as

$$z_2 = x_2 - \alpha_1(x_1, \hat{\theta}, t) - \dot{y}_r = x_2 + \omega_1^T\hat{\theta} + h_1(x_1)\mathrm{sgn}(z_1) - \dot{y}_r + c_1 z_1 \tag{8}$$

Then

$$\dot{z}_1 = -c_1 z_1 + z_2 + \omega_1^T\tilde{\theta} + \eta_1(x, w, t) - h_1(x_1)\mathrm{sgn}(z_1) \tag{9}$$

and $\dot{V}_1$ is now converted to

$$\dot{V}_1(z_1, \hat{\theta}) = -c_1 z_1^2 + z_1 z_2 + \eta_1(x, w, t)z_1 - h_1(x_1)|z_1| + \tilde{\theta}^T\Gamma^{-1}\left(\tau_1 - \dot{\hat{\theta}}\right)$$

Step 2. Consider the second Lyapunov function

$$V_2(z_1, z_2, \hat{\theta}) = V_1 + \frac{1}{2}z_2^2 \tag{10}$$

Then

$$\begin{aligned}\dot{V}_2 \;=\;& -c_1 z_1^2 + \eta_1(x, w, t)z_1 - h_1(x_1)|z_1| + z_2\left[z_1 + x_3 + \omega_2^T\hat{\theta} + \left(\eta_2(x, w, t) - \frac{\partial\alpha_1}{\partial x_1}\eta_1(x, w, t)\right)\right.\\ &\left. -\frac{\partial\alpha_1}{\partial t} - \frac{\partial\alpha_1}{\partial x_1}x_2 - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} - \ddot{y}_r(t)\right] + \tilde{\theta}^T\Gamma^{-1}\left(\tau_2 - \dot{\hat{\theta}}\right)\end{aligned} \tag{11}$$

where $\omega_2 = \varphi_2^T(x_1, x_2) - \dfrac{\partial\alpha_1}{\partial x_1}\varphi_1^T(x_1)$ and $\tau_2 = \tau_1 + \Gamma\omega_2 z_2 = \Gamma(\omega_1 z_1 + \omega_2 z_2)$.
Let

$$\begin{aligned}\beta_2(x_1, x_2, \hat{\theta}, t) = \alpha_2(x_1, x_2, \hat{\theta}, t) + \ddot{y}_r \;=\;& -z_1 - c_2 z_2 - \omega_2^T\hat{\theta} - \left(h_2(x_1, x_2) + \left|\frac{\partial\alpha_1}{\partial x_1}\right| h_1(x_1)\right) \times\\ & \mathrm{sgn}(z_2) + \frac{\partial\alpha_1}{\partial x_1}x_2 + \frac{\partial\alpha_1}{\partial\hat{\theta}}\tau_2 + \frac{\partial\alpha_1}{\partial t} + \ddot{y}_r(t)\end{aligned} \tag{12}$$

and $z_3 = x_3 - \beta_2 = x_3 - \alpha_2 - \ddot{y}_r$. If $x_3 = \beta_2$ and $\tau_2 = \dot{\hat{\theta}}$, then $\dot{V}_2 < 0$. But $x_3 \neq \beta_2$, and so

$$\begin{aligned}\dot{z}_2 \;=\;& -z_1 - c_2 z_2 + z_3 + \omega_2^T\tilde{\theta} + \left(\eta_2(x, w, t) - \frac{\partial\alpha_1}{\partial x_1}\eta_1(x, w, t)\right)\\ & - \left(h_2(x_1, x_2) + \left|\frac{\partial\alpha_1}{\partial x_1}\right| h_1(x_1)\mathrm{sgn}(z_2)\right) + \frac{\partial\alpha_1}{\partial\hat{\theta}}\left(\tau_2 - \dot{\hat{\theta}}\right)\end{aligned} \tag{13}$$

Step k **(**$1 \leq k \leq n-1$**)**. The time derivative of the error variable z_k is

$$\begin{aligned}\dot{z}_k \;=\;& x_{k+1} + \omega_k^T\hat{\theta} - \sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial x_i}x_{i+1} - \frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} + \xi_k - y_r^{(k)}(t) +\\ & \left(\sum_{i=1}^{k-2} z_{i+1}\frac{\partial\alpha_i}{\partial\hat{\theta}}\Gamma w_k\right) + \omega_k^T\tilde{\theta} - \frac{\partial\alpha_{k-1}}{\partial t}\end{aligned} \tag{14}$$

where

$$\begin{aligned}
\omega_k &= \varphi_k(x_1, \dots, x_k) - \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial x_i} \varphi_i(x_1, \dots, x_i) \\
\zeta_k &= h_k(x_1, \dots, x_k) + \sum_{i=1}^{k-1} \left| \frac{\partial \alpha_{k-1}}{\partial x_i} \right| h_i(x_1, \dots, x_i) \\
\xi_k &= \eta_k - \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial x_i} \eta_i
\end{aligned} \tag{15}$$

Since $x_{k+1} = z_{k+1} + \beta_k = z_{k+1} + \alpha_k + y_r^{(k)}$,

$$\begin{aligned}
\alpha_k(x_1, x_2, \dots, x_k, \hat{\alpha}, t) &= -z_{k+1} - c_k z_k - \omega_k^T \hat{\theta} + \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial x_i} x_{i+1} + \frac{\partial \alpha_{k-1}}{\partial t} - \zeta_k \mathrm{sgn}(z_k) \\
&\quad + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \tau_k + \left(\sum_{i=1}^{k-2} z_{i+1} \frac{\partial \alpha_i}{\partial \hat{\theta}} \right) \Gamma w_k
\end{aligned} \tag{16}$$

Then the time derivative of the error variable z_k is

$$\dot{z}_k = -z_{k-1} - c_k z_k + z_{k+1} + \omega_k^T \tilde{\theta} + \xi_k - \zeta_k \mathrm{sgn}(z_k) - \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \left(\dot{\hat{\theta}} - \tau_k \right) + \left(\sum_{i=1}^{k-2} z_{i+1} \frac{\partial \alpha_i}{\partial \hat{\theta}} \right) \Gamma w_k \tag{17}$$

The time derivative of V_k is

$$\begin{aligned}
\dot{V}_k &= -\sum_{i=1}^{k-1} c_i z_i^2 + z_k z_{k+1} + \sum_{i=1}^{k-1} \left(\xi_i - \zeta_i \mathrm{sgn}(z_i) \right) z_i - \left(\sum_{i=1}^{k-2} \frac{\partial \alpha_i}{\partial \hat{\theta}} z_{i+1} \right) \left(\tau_k - \dot{\hat{\theta}} \right) \\
&\quad + \tilde{\theta}^T \Gamma^{-1} \left(\tau_k - \dot{\hat{\theta}} \right)
\end{aligned} \tag{18}$$

since

$$\tau_k = \tau_{k-1} + \Gamma \omega_k z_k = \Gamma \sum_{i=1}^{k} \omega_i z_i \tag{19}$$

Step n. Define

$$z_n = x_n - \beta_{n-1} = x_n - \alpha_{n-1} - y_r^{(n)}$$

with α_{n-1} obtained from (16) for $k = n$. Then the time derivative of the error variable z_n is

$$\begin{aligned}
\dot{z}_n &= f(x) + g(x)u + \eta_n(x, w, t) + \omega_n^T(x, t)\hat{\theta} - \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_i} x_{i+1} - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \frac{\partial \alpha_{n-1}}{\partial t} \\
&\quad + \omega_n^T(x, t)\tilde{\theta} - \xi_n - y_r^{(n)} + \left(\sum_{i=1}^{n-2} z_{i+1} \frac{\partial \alpha_i}{\partial \hat{\theta}} \right) \Gamma w_n
\end{aligned} \tag{20}$$

where $\omega_n(x, \hat{\theta})$ is defined in (15) for $k = n$. Consider

$$V_n = V_{n-1} + \frac{1}{2} z_n^2 \tag{21}$$

The time derivative of V_n is now

$$\begin{aligned}\dot{V}_n &= -\sum_{i=1}^{n-1} c_i z_i^2 + \sum_{i=1}^{n-1}(\xi_i - \zeta_i \mathrm{sgn}(z_i))\, z_i + z_n \Big[z_{n-1} + f(x) + g(x)u + \omega_n^T \hat{\theta} - \\ &\quad \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial x_i} x_{i+1} - \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - \frac{\partial \alpha_{k-1}}{\partial t} + \xi_n + \left(\sum_{i=1}^{n-2} z_{i+1} \frac{\partial \alpha_i}{\partial \hat{\theta}} \Gamma w_n \right) \\ &\quad - y_r^{(n)} - \frac{\partial \alpha_{n-1}}{\partial t} \Big] + \tilde{\theta}^T \Gamma^{-1} \left(\tau_n - \dot{\hat{\theta}} \right)\end{aligned} \tag{22}$$

with $\tau_n = \tau_{n-1} + \Gamma \omega_k^T z_n = \Gamma \sum_{i=1}^{n} \omega_i^T z_i$ and ζ_n (15). The control law

$$\begin{aligned}u &= \frac{1}{g(x)} \Big[-f(x) - z_{n-1} - c_n z_n - \omega_n^T \hat{\theta} + \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial x_i} x_{i+1} + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \tau_n + \frac{\partial \alpha_{k-1}}{\partial t} \\ &\quad - \left(\sum_{i=1}^{n-2} z_{i+1} \frac{\partial \alpha_i}{\partial \hat{\theta}} \right) \Gamma w_n + y_r^{(n)} - \zeta_n \mathrm{sgn}(z_n) \Big]\end{aligned} \tag{23}$$

and the final tuning function

$$\dot{\hat{\theta}} = \tau_n \tag{24}$$

guarantee the last term of (23) to be zero and, then $\dot{V}_n < 0$, i.e. the global stability of the error system is achieved.

3 Sliding Mode Control Design

Sliding mode techniques yield robust control, and adaptive control techniques are popular when there is uncertainty in the plant. The combination of these methods has been studied in recent years (Rios-Bolívar and Zinober, 1994, 1997a,b). In general, at each step of the backstepping method, the new update tuning function and the defined error variables (and virtual control law) take the system to the equilibrium position. At the final step, the system is stabilized via the control. With sliding mode control we desire the trajectory to tend to the equilibrium point along the sliding hyperplane. If the sliding surface is given by the final variable z_n, i.e. $z_n = 0$, the sliding mode condition holds for the control design (23) (Rios-Bolívar and Zinober, 1994). However, if the sliding hyperplane is a given hyperplane, $\sigma \neq z_n$, some additional conditions on the sliding gain matrix and the sliding equation are needed (Rios-Bolívar and Zinober, 1997a,b).

Consider the sliding surface $\sigma = z_n = 0$. The sufficient condition for existence of the sliding mode is $\dot{\sigma}\sigma < 0$. Consider the Lyapunov function (21) and the control law

$$\begin{aligned}u &= \frac{1}{g(x)} \Big[-f(x) - z_{n-1} - c_n z_n - \omega_n^T \hat{\theta} + \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial x_i} x_{i+1} + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \tau_2 + \frac{\partial \alpha_{k-1}}{\partial t} - \\ &\quad \left(\sum_{i=1}^{n-2} z_{i+1} \frac{\partial \alpha_i}{\partial \hat{\theta}} \right) \Gamma w_n + y_r^{(n)} - (\zeta_n + K)\, \mathrm{sgn}(z_n) \Big]\end{aligned} \tag{25}$$

where K is a non-negative real number and is a design parameter. The control law (25) guarantees that the condition $\dot{V} < 0$ is satisfied. Both the control laws (23) and (25) assure system stability and guarantee that the trajectories tend to the equilibrium point along the surface $\sigma = z_n = 0$. However, the control (25) contains a gain parameter K which can be changed by designer to yield additional design freedom.

4 Example

Consider the second order system in semi-PSF form

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1\theta + ax_1^2\cos(bx_1x_2) \\ \dot{x}_2 &= u \end{aligned}$$

where a and b are unknown but it is known that $|a| \leq 2$ and $|b| \leq 3$. We have

$$\begin{aligned} h_1 &= 2x_1^2 \\ z_1 &= x_1 - y_r \\ z_2 &= x_2 + x_1\hat{\theta} + 2x_1^2\mathrm{sgn}\,(x_1 - y_r) + c_1\,(x_1 - y_r) - \dot{y}_r \\ \alpha_1 &= -x_1\hat{\theta} - 2x_1^2\mathrm{sgn}(z_1) - c_1z_1 \\ \omega_1 &= x_1 \\ \omega_2 &= -\frac{\partial\alpha_1}{\partial x_1}x_1 \\ \tau_2 &= \Gamma\,(x_1z_1 + \omega_2z_2) \\ \zeta_2 &= \left|\frac{\partial\alpha_1}{\partial x_1}\right|2x_1^2 \end{aligned}$$

Then the control law (25) becomes

$$u \quad = \quad -z_1 - c_2z_2 - \omega_2^T\hat{\theta} + \frac{\partial\alpha_1}{\partial x_1}x_2 + \frac{\partial\alpha_1}{\partial\hat{\theta}}\tau_2 + \frac{\partial\alpha_1}{\partial t} + y_r^{(2)} - (\zeta_2 + K)\,\mathrm{sgn}(z_2) \tag{26}$$

For $y_r = 0$, simulation results are shown in Figs. 1 and 2 for different values of K. Fig. 3 shows the simulation results when $K = 0$ and $y_r = 0.05\sin(0.5\pi t)$.

5 Conclusions

Backstepping technique is a systematic method to design a control so that the systems is stabilized via the control. The sliding mode control is a robust control method design and adaptive backstepping is an adaptive control design method. In this paper the method of design has benefited both design methods. So the design method has the advantages of both methods. The method was employed for a class of nonlinear systems which can be converted to the semi-parametric strict form. The considered plant may have a unmodeled or external disturbance in the system equations. The discontinuous control may contain a gain design so that the designer can select to change the velocity of the convergence to impose the trajectories to the sliding hyperplane.

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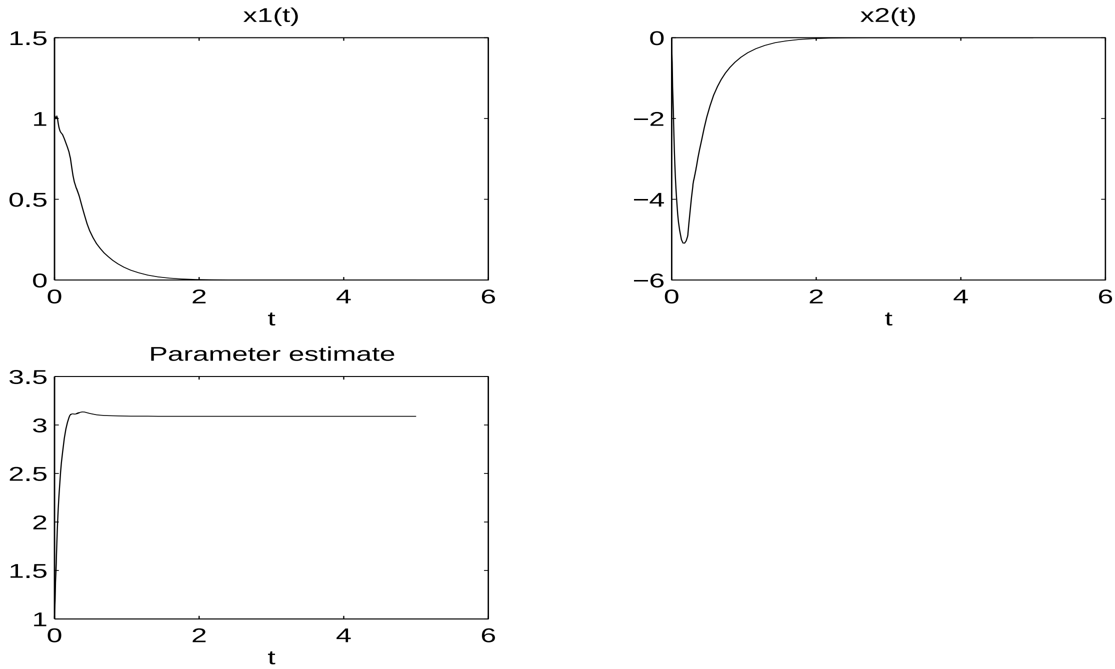

Figure 1: Responses the example with $K = 0$

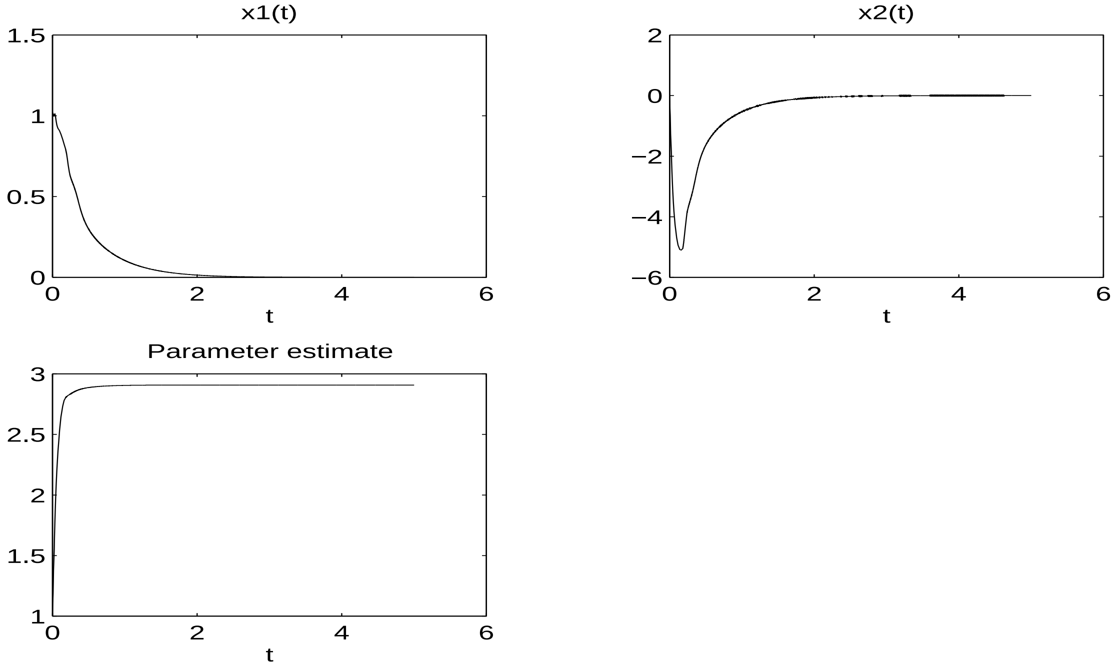

Figure 2: Responses the example with $K = 4$

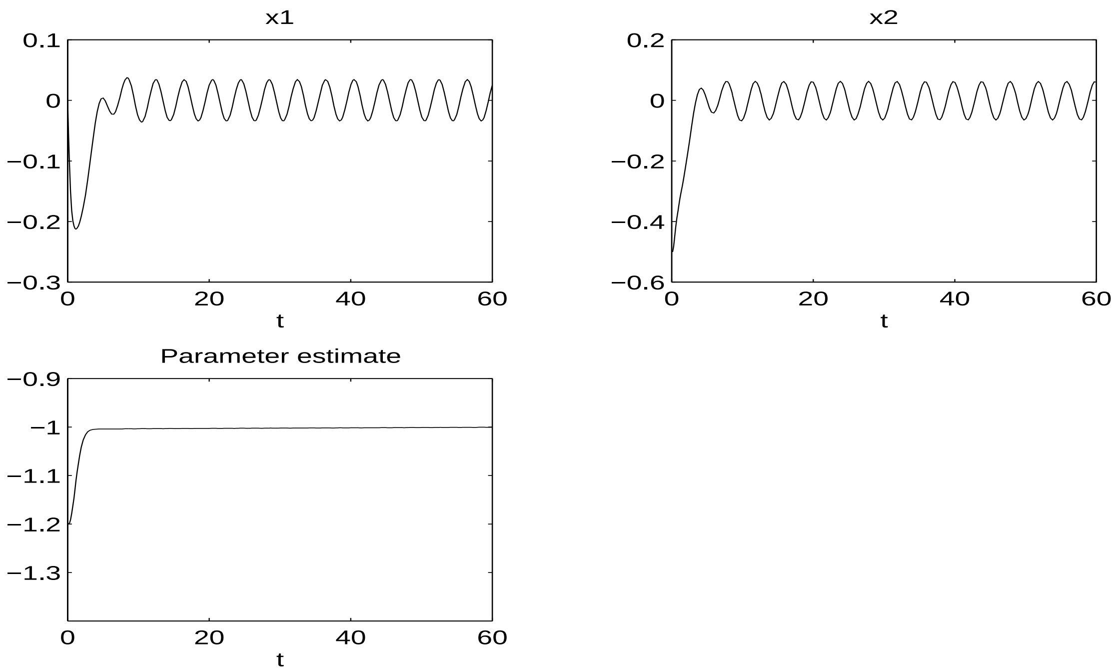

Figure 3: Responses the example with $K = 0$ and $y_r = 0.05 \sin(0.5\pi t)$