ON ALGORITHMS FOR ATTITUDE ESTIMATION USING GPS

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Abstract

This paper discusses algorithms for attitude determination using GPS differential phase measurements, assuming that the cycle integer ambiguities are known. The problem of attitude determination is posed as a parameter optimization problem. One proposed set of optimal solutions, which includes solutions of Wahba`s problem, is based on least squares fit of some attitude parameters to a set of *vector* measurements. The use of these algorithms requires the conversion of the basic GPS scalar phase measurements into unit vectors.

Another possible approach is based on a least squares fit of the attitude quaternion to the GPS phase measurements themselves. The cost function of the fit is given in the literature in the most straightforward formulation as a function of the attitude matrix. The paper presents the conversion of the matrix-based cost function to a quaternion-based cost function, which corresponds to the cost function minimized by QUEST. However, unlike the QUEST cost function, the converted cost function is not a simple quadratic form, therefore the simple QUEST solution is not applicable in this case. Three iterative solutions for finding the optimal quaternion are derived. The first algorithm is a linearly convergent one whose convergence rate is slow. The other two converge very fast.

The algorithms presented in this paper can handle cases of planar antenna arrays and thus cover a deficiency in earlier algorithms. The efficiency of the new algorithms is demonstrated through numerical examples.

I. Introduction

Attitude determination using GPS carrier signals has been given a considerable attention in the last decade (Cohen, 1992; Lightsey et al., 1994). Much attention was given to concept, hardware, and algorithm development as well as to testing. Algorithms for GPS attitude determination given differential phase measurements can be broken into integer resolution and attitude calculations. Several methods for integer resolution were presented in the literature (see e.g. Cohen, 1992; Conway et al., 1996). In this work we assume that the integer ambiguity is solved and we are concerned only with attitude calculation. The problem of attitude determination can be expressed as the problem of minimizing the cost function

$$\rho\left(D_a^e\right) = \frac{1}{4}\sum_i^n p_i \sum_j^2 \left| B_{ji} - \mathbf{a}_j^T D_a^e \mathbf{s}_j \right|^2 \qquad (1)$$

with respect to D_a^e where $\mathbf{s}_i$ be a unit vector in the direction of an observed GPS satellite which is designated as satellite number i, *e* is the reference (earth) coordinate system in which $\mathbf{s}_i$ is resolved, *a* is the body coordinate system, $\mathbf{a}_j$ is the $j^{\underline{th}}$ axis of the latter system, D_a^e is the transformation matrix from *e* to *a*, and B_{ji} is the processed phase measurement. The transpose is denoted by T and p_i is a weight given to the measurement related to the $i^{\underline{th}}$ satellite. It is assumed that the components of $\mathbf{s}_i$ in *e* are known. Note that we are considering a planar problem; that is, only two components of B are

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measured for each satellite. The components are projections on the body coordinate axes of processed phase measurements.

The purpose of this work is to find algorithms, which yield the attitude matrix (or other equivalent attitude parameterizations) that, minimizes r.

II. Attitude Determination using GPS Vectorized Observations

Several efficient algorithms for attitude determination based on a least squares fit of the attitude to *vector* measurements were introduced in the past. To make use of these algorithms, the phase measurements have to be converted into vector measurements in the body coordinate system as follows.

$$\mathbf{s}_{ia} = \begin{bmatrix} B_{1i} \\ B_{2i} \\ \left(1 - B_{1i}^2 - B_{2i}^2\right)^{1/2} \end{bmatrix} \tag{2}$$

Note that the third component of $\mathbf{s}_{ia}$ is chosen to be the positive root of the expression in parentheses. This was done since only the signals of those GPS satellites which are above the antenna plane, and thus in the positive direction of the $\mathbf{a}_3$ axis, are received by the antennae. The vector $\mathbf{s}_{ia}$, resolved in earth reference coordinates, is denoted by $\mathbf{s}_{ie}$. The latter is easily computed since both the satellite and the vehicle positions are known in earth coordinates. With the pairs $\mathbf{s}_{ia}$, $\mathbf{s}_{ie}$ on hand, $i = 1,2\ldots,n$, one can replace Eq. (1) by the following cost function introduced by Wahba[4]

$$\rho'(D_a^e) = \frac{1}{4}\sum_i^n p_i \left|\mathbf{s}_{ia} - D_a^e \mathbf{s}_{ie}\right|^2 \tag{3}$$

and use QUEST (Shuster and Oh, 1981), or other similar algorithms, to obtain a weighted least squares attitude quaternion fit which minimizes ρ'. For the sake of comparison between QUEST, which operates on measured vectors, and the algorithms that will be developed later, which operate on phase measurements, a short description of QUEST is given next.

Since D_a^e is a known function of the attitude quaternion (Wertz, 1984), $\mathbf{q}$, then $\rho\left(D_a^e\right)$ can be replaced by w($\mathbf{q}$) where

$$w(\mathbf{q}) = \frac{1}{4}\sum_i^n p_i \left|\mathbf{s}_{ia} - D_a^e(\mathbf{q})\mathbf{s}_{ie}\right|^2 \tag{4}$$

It can be shown that $\mathbf{q}$*, the $\mathbf{q}$ which minimizes w($\mathbf{q}$), is the same $\mathbf{q}$ which maximizes the cost function

$$\eta(\mathbf{q}) = \frac{1}{2}\mathbf{q}^T K \mathbf{q} \tag{5}$$

where

$$K = \left[\begin{array}{c|c} S - \sigma I & \mathbf{z} \\ \hline \mathbf{z}^T & \sigma \end{array}\right] \tag{6}$$

and where

$$m_n = \sum_{i=1}^n p_i \quad (7.a) \qquad\qquad \sigma = \frac{1}{m_n}\sum_{i=1}^n p_i \mathbf{s}_{ia}^T \mathbf{s}_{ie} \quad (7.b)$$

$$B = \frac{1}{m_n}\sum_{i=1}^{n} p_i \mathbf{s}_{ia}\mathbf{s}_{ie}^T \qquad (7.c) \qquad\qquad S = B + B^T \qquad (7.d)$$

$$\mathbf{z} = \frac{1}{m_n}\sum_{i=1}^{n} p_i(\mathbf{s}_{ia} \times \mathbf{s}_{ie}) \qquad (7.e)$$

The matrix I is the third order identity matrix. It turns out that **q*** is the eigenvector which corresponds to the largest eigenvalue of K. QUEST[5] is an algorithm which yields this **q***.

III. Attitude Determination Using GPS Phase Measurements Directly

III.a: Cost function conversion to quadratic forms

Recall Eq. (1)

$$\rho(D_a^e) = \frac{1}{4}\sum_{i}^{n} p_i \sum_{j}^{2} \left|B_{ji} - \mathbf{a}_j^T D_a^e \mathbf{s}_i\right|^2 \qquad (8)$$

We wish to find D_a^e which minimizes $\rho(D_a^e)$. Since, as mentioned earlier, D_a^e is a known function of the attitude quarternion, **q**, then $\rho(D_a^e)$ can be replaced by J(**q**) where

$$J(\mathbf{q}) = \frac{1}{4}\sum_{i}^{n} p_i \sum_{j}^{2} \left|B_{ji} - \mathbf{a}_j^T D_a^e(\mathbf{q})\mathbf{s}_i\right|^2 \qquad (9)$$

In order to facilitate the search for the quarternion **q*** which minimizes J(**q**) the latter is now converted into a function of matrix quadratic forms. To meet this end define

$$C_{ji} = \mathbf{s}_i \mathbf{a}_j^T \qquad (10.a) \qquad\qquad E_{ji} = C_{ji} + C_{ji}^T \qquad (10.b)$$

$$\mathbf{p}_{ji} = \mathbf{a}_j \times \mathbf{s}_i \qquad (10.c) \qquad\qquad \mu_{ji} = \mathbf{a}_{ji}^T \mathbf{s}_i \qquad (10.d)$$

and then define

$$L_{ji} = \left[\begin{array}{c|c} E_{ji} - \mu_{ji} I & \mathbf{p}_{ji} \\ \hline \mathbf{p}_{ji}^T & \mu_{ji} \end{array}\right] \qquad (11)$$

It can be shown that (Bar-Itzhack et al., 1998)

$$\mathbf{a}_j^T D_a^e(\mathbf{q})\mathbf{s}_i = \mathbf{q}^T L_{ji}\mathbf{q} \qquad (12)$$

Substitution of Eq. (12) into Eq. (9) yields

$$J(\mathbf{q}) = \frac{1}{4}\sum_{i}^{n} p_i \sum_{j}^{2} \left|B_{ji} - \mathbf{q}^T L_{ji}\mathbf{q}\right|^2 \qquad (13)$$

Define

$$\Phi_{ji} = B_{ji} I \qquad (14)$$

then since $\mathbf{q}^T\mathbf{q} = 1$, one can write

$$B_{ji} = \mathbf{q}^T \Phi_{ji} \mathbf{q} \tag{15}$$

therefore

$$B_{ji} - \mathbf{q}^T L_{ji}\, \mathbf{q} = \mathbf{q}^T [\Phi_{ji} - L_{ji}]\, \mathbf{q} \tag{16}$$

Let

$$M_{ji} = p_i (\Phi_{ji} - L_{ji}) \tag{17}$$

then using Eqs. (16) and (17) in Eq. (13) the following is obtained

$$J(\mathbf{q}) = \frac{1}{4} \sum_{i}^{n} \sum_{j}^{2} \left| \mathbf{q}^T M_{ji}\, \mathbf{q} \right|^2 \tag{18}$$

or

$$J(\mathbf{q}) = \frac{1}{4} \mathbf{q}^T \left[\sum_{i}^{n} \sum_{j}^{2} M_{ji}\, \mathbf{q}\mathbf{q}^T M_{ji} \right] \mathbf{q} \tag{19}$$

III.b: The first algorithm for finding the optimal q (Steepest Descent)

The problem of finding the matrix D_a^e which minimizes $\rho(D_a^e)$, defined in Eq. (1), has been transformed into finding **q** that minimizes J(**q**) of either Eq. (18) or Eq. (19). Unfortunately J(**q**), is quartic in **q** whereas the cost function which has to be optimized when solving Wahba`s problem is only quadratic in **q**. For this reason the QUEST solution *is not suitable* in the present case. One needs to use some other methods for minimizing J(**q**). An iterative solution is suggested here which is based on the *gradient projection technique*[8].

Consider the cost function of Eq. (19). Suppose that as a result of the iterative technique to reduce J, $\mathbf{q}_k$ was computed at the k^th iteration. Perform now the k + 1^st iteration by changing **q** as follows

$$\mathbf{q}_{k+1} = \mathbf{q}_k + \varepsilon \mathbf{h} \tag{20}$$

where **h** is a four element column matrix which determines the direction one moves from $\mathbf{q}_k$ to $\mathbf{q}_{k+1}$ in $\mathbf{R}^4$ and ε is the distance one moves in this direction. Substituting Eq. (20) into Eq. (19) yields the following cost function at $\mathbf{q}_{k+1}$

$$J(\varepsilon, \mathbf{h}) = \frac{1}{4} (\mathbf{q}_k + \varepsilon \mathbf{h})^T \left[\sum_{i}^{n} \sum_{j}^{2} M_{ji} (\mathbf{q}_k + \varepsilon \mathbf{h}) (\mathbf{q}_k + \varepsilon \mathbf{h})^T M_{ji} \right] (\mathbf{q}_k + \varepsilon \mathbf{h}) \tag{21}$$

For a given direction, **h**, one wishes to move at a distance ε which will minimize J; that is, one wants to move from $J_k = J(\mathbf{q}_k)$ to $J_{k+1} = J(\mathbf{q}_{k+1})$ at the steepest descent route. The rate of descent of J from point $\mathbf{q}_k$ to point $\mathbf{q}_{k+1}$ is the derivative of J with respect to ε at the point $\varepsilon = 0$. When performing this operation on J, given in Eq. (21), the following is obtained

$$\left. \frac{\partial J}{\partial \varepsilon} \right|_{\varepsilon=0} = \mathbf{q}_k^T \left[\sum_{i}^{n} \sum_{j}^{2} M_{ji} \mathbf{q}_k \mathbf{q}_k^T M_{ji} \right] \mathbf{h} \tag{22}$$

Note that the last equation yields the rate of descent but not necessarily the steepest one. While moving from $\mathbf{q}_k$ to $\mathbf{q}_{k+1}$ the following normality constraint of the quaternion of rotation has to be satisfied.

$$g(\mathbf{q}) \equiv \mathbf{q}^T\mathbf{q} - 1 = 0 \tag{23}$$

Satisfying the constraint implies that when moving an incremental distance, g has to stay zero. In other words, the rate of change of g has to be zero at the point $\mathbf{q}_k$. Put in analytic terms and using Eqs. (23) and (20) it means that

$$\left.\frac{dg}{d\varepsilon}\right|_{\varepsilon=0} = \left.\frac{d}{d\varepsilon}[(\mathbf{q}_k + \varepsilon\mathbf{h})^T(\mathbf{q}_k + \varepsilon\mathbf{h}) - 1]\right|_{\varepsilon=0} = 0 \tag{24}$$

which yields

$$\mathbf{q}_k^T\mathbf{h} = 0 \tag{25}$$

As for **h**, since it is a direction, it is actually a unit vector, therefore its length has to be 1; that is,

$$\mathbf{h}^T\mathbf{h} - 1 = 0 \tag{26}$$

Everything is ready now for the computation of the direction of the steepest descent. The direction of the steepest descent is determined by **h**. The column matrix **h** is sought which yields the smallest (the most negative) value of the derivative expressed by Eq. (23). Define this derivative as $\Psi(\mathbf{h})$; that is

$$\Psi(\mathbf{h}) = \mathbf{q}_k^T\left[\sum_i^n\sum_j^2 M_{ji}\mathbf{q}_k\mathbf{q}_k^T M_{ji}\right]\mathbf{h} \tag{27}$$

which is the function to be minimized with respect to **h** subject to the constraints expressed by Eqs. (25) and (26). As usual, this is being done by adding the constraints to $\Psi(\mathbf{h})$ using the scalar Langrange Multipliers, λ_1 and λ_2, and minimizing the new function. Thus a new merit function, $\mu(\mathbf{h})$, is defined as follows

$$\begin{aligned}\mu(\mathbf{h}) = {} & \mathbf{q}_k^T(\sum_i^n\sum_j^2 M_{ji}\mathbf{q}_k\mathbf{q}_k^T M_{ji})\mathbf{h} \\ & + \lambda_1\mathbf{q}_k^T\mathbf{h} + \lambda_2(\mathbf{h}^T\mathbf{h} - 1)\end{aligned} \tag{28}$$

This is the function which is to be minimized with respect to **h**. The concept of *directional derivative* is used to accomplish that. Accordingly, denote the direction of the steepest descent by $\mathbf{h}_o$ then any other direction, **h**, can be expressed as

$$\mathbf{h} = \mathbf{h}_o + \nu\mathbf{d} \tag{29}$$

where **d** is the direction from $\mathbf{h}_o$ to **h**, and ν is the distance one has to move in this direction in order to reach **h** starting at $\mathbf{h}_o$. Substitution of the latter expression for **h** into Eq. (28) yields

$$\begin{aligned}\mu(\mathbf{h}_o + \nu\mathbf{d}) = {} & \mathbf{q}_k^T\left[\sum_i^n\sum_j^2 M_{ji}\mathbf{q}_k\mathbf{q}_k^T M_{ji} + \lambda_1 I\right](\mathbf{h}_o + \nu\mathbf{d}) \\ & + \lambda_2[(\mathbf{h}_o + \nu\mathbf{d})^T(\mathbf{h}_o + \nu\mathbf{d}) - 1]\end{aligned} \tag{30}$$

A necessary condition for $\mathbf{h}_o$ to be a stationary point is

$$\left.\frac{d\mu}{d\nu}\right|_{\nu=0} = 0 \qquad \forall\, \mathbf{d} \tag{31}$$

Applying this condition to $\mu(\mathbf{h}_o + \nu\mathbf{d})$ of Eq. (30) yields

$$\left\{ q_k^T \left[\sum_i^n \sum_j^2 M_{ji} q_k q_k^T M_{ji} + \lambda_1 I \right] + 2\lambda_2 h_o^T \right\} d = 0 \qquad \forall\, d \tag{32}$$

This condition can hold only if

$$\mathbf{q}_k^T \left[\sum_i^n \sum_j^2 M_{ji} \mathbf{q}_k \mathbf{q}_k^T M_{ji} + \lambda_1 I \right] + 2\lambda_2 \mathbf{h}_o^T = 0 \tag{33}$$

which yields

$$\mathbf{h}_o = -\frac{1}{2\lambda_2} \left[\sum_j^n \sum_j^2 M_{ji} \mathbf{q}_k \mathbf{q}_k^T M_{ji} + \lambda_1 I \right] \mathbf{q}_k \tag{34}$$

Since $\mathbf{h}_o$ has to satisfy the condition of Eq. (25), then from the last equation

$$-\mathbf{q}_k^T \frac{1}{2\lambda_2} \left[\sum_i^n \sum_j^2 M_{ji} \mathbf{q}_k \mathbf{q}_k^T M_{ij} + \lambda_1 I \right] \mathbf{q}_k = 0 \tag{35}$$

which yields

$$\frac{1}{2\lambda_2} \left[\sum_i^n \sum_j^2 \mathbf{q}_k^T M_{ji} \mathbf{q}_k \mathbf{q}_k^T M_{ji} \mathbf{q}_k + \lambda_1 \mathbf{q}_k^T \mathbf{q}_k I \right] = 0 \tag{36}$$

Since $\mathbf{q}_k^T \mathbf{q}_k = 1$ the last equation implies that

$$\lambda_1 = -\sum_i^n \sum_j^2 \mathbf{q}_k^T M_{ji} \mathbf{q}_k \mathbf{q}_k^T M_{ji} \mathbf{q}_k \tag{37}$$

A comparison between the last equation and Eq. (19) indicates that

$$\lambda_1 = -4J(\mathbf{q}_k) \tag{38}$$

Substitution of this result into Eq. (34) yields

$$\mathbf{h}_o = -\frac{1}{2\lambda_2} \left[\sum_i^n \sum_j^2 M_{ji} \mathbf{q}_k \mathbf{q}_k^T M_{ji} - 4J(\mathbf{q}_k) I \right] \mathbf{q}_k \tag{39}$$

In order to find λ_2 substitute the last expression for $\mathbf{h}_o$ into the constraint expressed in Eq. (26) to obtain

$$\left(\frac{1}{2\lambda_2}\right)^2 \mathbf{q}_k^T \left[\sum_i^n \sum_j^2 M_{ji}\mathbf{q}_k\mathbf{q}_k^T M_{ji} - 4J(\mathbf{q}_k)I\right] \bullet \left[\sum_i^n \sum_j^2 M_{ji}\mathbf{q}_k\mathbf{q}_k^T M_{ji} - 4J(\mathbf{q}_k)I\right]\mathbf{q}_k = 1 \tag{40}$$

Define

$$C_k = \sum_i^n \sum_j^2 M_{ji}\mathbf{q}_k\mathbf{q}_k^T M_{ji} \tag{41}$$

then using this definition, Eq. (40) can be written as

$$\lambda_2^2 = \frac{1}{4}\mathbf{q}_k^T[C_k - 4J(\mathbf{q}_k)I][C_k - 4J(\mathbf{q}_k)I]\mathbf{q}_k \tag{42}$$

which yields

$$\lambda_2^2 = \frac{1}{4}\mathbf{q}_k^T C_k C_k \mathbf{q}_k - 2\mathbf{q}_k^T J(\mathbf{q}_k) C_k \mathbf{q}_k + 4\mathbf{q}_k^T J(\mathbf{q}_k)^2 \mathbf{q}_k \tag{43}$$

The last equation can be written as

$$\lambda_2^2 = \frac{1}{4}\mathbf{q}_k^T C_k C_k \mathbf{q}_k - 2J(\mathbf{q}_k)\mathbf{q}_k^T C_k \mathbf{q}_k + 4J(\mathbf{q}_k)^2 \mathbf{q}_k^T \mathbf{q}_k \tag{44}$$

which after noting that $\mathbf{q}_k^T C_k \mathbf{q}_k = 4J(\mathbf{q}_k)$, can be written as

$$\lambda_2^2 = \frac{1}{4}\mathbf{q}_k^T C_k C_k \mathbf{q}_k - 8J(\mathbf{q}_k)^2 + 4J(\mathbf{q}_k)^2 \tag{45}$$

Let

$$\mathbf{v}_k = C_k \mathbf{q}_k \tag{46}$$

then from Eq. (45) one obtains

$$\lambda_2 = \pm\left[\frac{1}{4}\mathbf{v}_k^T \mathbf{v}_k - 4J(\mathbf{q}_k)^2\right]^{1/2} \tag{47}$$

In order to choose the appropriate sign for λ_2 it is necessary to examine the second derivative of $\mu(\mathbf{h}_o + \nu\mathbf{d})$ with respect to ν evaluated at $\nu = 0$. Using Eq. (30) it is evident that

$$\left.\frac{d^2}{d\nu^2}\mu(\mathbf{h}_o + \nu\mathbf{d})\right|_{\nu = o} = 2\lambda_2 \mathbf{d}^T\mathbf{d} \tag{48}$$

For $\mathbf{h}_o$ to be a minimum point, the second derivative has to be positive, therefore the positive sign has to be chosen in the computation of λ_2 in Eq. (47). With this in mind and using the definition of C_k, Eq. (39) is written as

$$\mathbf{h}_o = -\frac{1}{2|\lambda_2|}[C_k - 4J(\mathbf{q}_k)I]\mathbf{q}_k \tag{49}$$

Recall that $\mathbf{h}_o$ is the direction of the steepest descent. Substituting the last expression for $\mathbf{h}_o$ into Eq. (20) yields

$$\mathbf{q}_{k+1} = \left\{ I - \frac{\varepsilon}{2|\lambda_2|}\left[C_k - 4J(\mathbf{q}_k)I\right]\right\}\mathbf{q}_k \tag{50}$$

Finally, let

$$W_k = \frac{1}{2|\lambda_2|}\left[C_k - 4J(\mathbf{q}_k)I\right] \tag{51}$$

then Eq. (50) can be written as

$$\mathbf{q}_{k+1} = (I - \varepsilon W_k)\mathbf{q}_k \tag{52}$$

A value should be selected for ε. In principle this can be done by substituting $\mathbf{q}_{k+1}$ of the last equation into J(ε,**h**) given in Eq. (21) and then minimizing the result with respect to ε. This, however, yields a complicated third order algebraic equation in ε whose solution has to be obtained at every time step. Another possible approach is of finding empirically a suitable value for ε. In summary the recursive algorithm for minimizing the cost function of Eq. (19) is as follows:

1. Determine $\mathbf{q}_1$, the initial guess of **q,** and set k = 1.
2. Compute $C_k = \sum_i^n \sum_j^2 M_{ji}\mathbf{q}_k\mathbf{q}_k^T M_{ji}$.
3. Compute $J(\mathbf{q}_k) = \mathbf{q}_k^T C_k \mathbf{q}_k$.
4. Compute $W_k = \frac{1}{2|\lambda_2|}\left[C_k - 4J(\mathbf{q}_k)I\right]$.
5. Compute $\mathbf{q}_{k+1} = (I - \varepsilon\, W_k)\mathbf{q}_k$.
6. If $|\mathbf{q}_{k+1} - \mathbf{q}_k| \le \delta$ where δ is a pre-determined constant, then stop. Otherwise increase the argument by 1 and go back to step 2.

III.c: The second algorithm for finding the optimal q (Eigen Problem)

Using the definition of $C(\mathbf{q})$ in Eq. (41), the merit function of Eq. (19) can be written as

$$J(\mathbf{q}) = \mathbf{q}^T C(\mathbf{q})\mathbf{q} \tag{53}$$

now removed.

Recall that we wish to minimize J with respect to **q** where the latter has to satisfy the normality constraint of Eq. (23). To accomplish this, define the following Lagrange function

$$L(\mathbf{q},\lambda) = J(\mathbf{q}) - 2\lambda(\mathbf{q}^T\mathbf{q} - 1) = \mathbf{q}^T C(\mathbf{q})\mathbf{q} - 2\lambda(\mathbf{q}^T\mathbf{q} - 1) \tag{54}$$

That **q** which minimizes J subject to the constraint on **q**, is a stationary point of L. Moreover, λ too is a stationary point of L. That is

$$\frac{\partial L}{\partial \mathbf{q}} = 0 \qquad (55.a) \qquad\qquad \frac{\partial L}{\partial \lambda} = 0 \qquad (55.b)$$

When performing the differentiations and after some elaborate manipulations one obtains the following corresponding equations[9]

$$\left.\begin{aligned} C(\mathbf{q})\mathbf{q} &= \lambda\mathbf{q} \\ \mathbf{q}^T\mathbf{q} &= 1 \end{aligned}\right\} \quad \mathbf{q} \in \Re^4; \qquad \lambda \in \Re \tag{56}$$

Eq. (56) is a set of 5 nonlinear algebraic equations. Their solution yields the sought **q**. (It is still necessary of course to show that the found **q** is a minimum point but this is quite easy). It was shown (Nadler, 1998) that these equations have no analytic solution, which is similar to that, which gave rise to QUEST. In the absence of an analytic solution one naturally seeks an iterative solution. One such solution that immediately comes to mind is the following. Guess a starting **q**, use it to compute C in Eqs. (56), and then find the eigenvalues and eiqenvectors of C. It was shown in (Nadler, 1998) that J is minimal when **q** is the unit eigenvector which corresponds to the smallest eigenvalue of C. Therefore use this eigenvector as **q** for the following iteration. This algorithm, however, is problematic. Experiments showed that its convergence was slow and near the end it alternated between two values, none of which is the correct solution (Nadler, 1998). It was observed, however, that the two values were almost symmetric about the correct solution. Therefore the algorithm was modified in the following way. The solutions obtained from two consecutive iterations were averaged. The average solution was then fed into the iterative algorithm, which was run twice again. The results of these two iterations was averaged again and so on. A step by step description of the algorithm is as follows.

1. Guess $\mathbf{q}_k$, and set $k = 0$.
2. Compute $C_k(\mathbf{q}_k)$.
3. Find the eigenvalues and eigenvectors of $C_k(\mathbf{q}_k)$.
4. Set $\mathbf{q}_{k+1}$ to the eigenvector corresponding to the smallest eigenvalue of $C_k(\mathbf{q}_k)$.
5. Go once more to Step 2, repeat Steps 3 and 4.
6. Replace $\mathbf{q}_{k+1}$ by the average of the last two $\mathbf{q}_{k+1}$`s.
7. If $|\mathbf{q}_{k+1} - \mathbf{q}_k| \le \delta$ where δ is a pre-determined constant, then stop. Otherwise increase the argument by 1 and go back to step 2.

III.d: The third algorithm for finding the optimal q (Newton-Raphson)

The classic approach to the solution of Eqs. (56) is the Newton-Raphson approach. Its steps are as follows. Define

$$\mathbf{f}(\mathbf{x}) = \left[\frac{C(\mathbf{q})\mathbf{q} - \lambda\mathbf{q}}{\mathbf{q}^T\mathbf{q} - 1}\right] \tag{57}$$

where $\mathbf{x}^T = [\mathbf{q}^T|\lambda]$. To solve the equation $\mathbf{f}(\mathbf{x}) = 0$ iteratively we compute first the Jacobian J where

$$J = \frac{\partial\mathbf{f}(\mathbf{x})}{\partial\mathbf{x}} \tag{58}$$

For the **f** defined in Eqs. (57) J takes the form (Nadler, 1998)

$$J = \left[\begin{array}{c|c} 2C(\mathbf{q}) + D(\mathbf{q}) - \lambda I & -\mathbf{q} \\ \hline 2\mathbf{q}^T & 0 \end{array}\right] \tag{59}$$

where

$$D(\mathbf{q}) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{q}^T M_{ji}\mathbf{q} M_{ji} \tag{60}$$

Then iterate as follows

$$\mathbf{x}_{k+1} = \mathbf{x}_k - J^{-1}(\mathbf{x}_k)\cdot \mathbf{f}(\mathbf{x}_k) \tag{61}$$

When convergence occurs the first four elements of **x** constitute the sought **q**.

IV. Examples

In this example there were five GPS satellites, where $\mathbf{s}_i$, $i = 1,2,\ldots,5$ were the unit vectors to the five satellites. They were given the following values

$$s_1 = \begin{bmatrix} 0.953 \\ 0.095 \\ 0.288 \end{bmatrix} \quad s_2 = \begin{bmatrix} -0.195 \\ 0.976 \\ 0.097 \end{bmatrix} \quad \mathbf{s}_3 = \begin{bmatrix} -0.432 \\ -0.259 \\ 0.864 \end{bmatrix} \quad s_4 = \begin{bmatrix} -0.316 \\ 0.632 \\ 0.708 \end{bmatrix} \quad \mathbf{s}_5 = \begin{bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{bmatrix}$$

The vectors $\mathbf{s}_i$ were expressed in the reference coordinate system. The vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ were the x and y coordinate axes of the body system, *a*, in which we performed the computations. Thus

$$\mathbf{a}_1^T = [1 \quad 0 \quad 0] \qquad \mathbf{a}_2^T = [0 \quad 1 \quad 0]$$

The transformation matrix, D_a^e, from the reference to the body coordinates and its corresponding quarternion were

$$D_a^e = \begin{bmatrix} 0.713 & 0.659 & 0.241 \\ -0.579 & 0.359 & 0.732 \\ 0.396 & -0.661 & 0.638 \end{bmatrix} \qquad \mathbf{q}^T = [0.423, 0.047, 0.376, 0.823]$$

The corresponding Euler angles of this attitude were

$$\begin{aligned} \psi &= 42.7530^\circ \\ \theta &= -13.9390^\circ \\ \phi &= 48.9390^\circ \end{aligned}$$

For this geometry the *nominal* phase measurements were

$$\begin{array}{lllll} B_{11} = 0.811 & B_{12} = 0.527 & B_{13} = -0.270 & B_{14} = 0.362 & B_{15} = 0.931 \\ B_{21} = -0.307 & B_{22} = 0.534 & B_{23} = 0.790 & B_{24} = 0.928 & B_{25} = 0.295 \end{array}$$

IV.a: Vectorized phase measurements.

When vectorizing the phase measurements according to Eq. (2), and using QUEST to compute **q***, the achieved accuracy is

$$|\mathbf{q} - \mathbf{q}^*| < 1\cdot 10^{-15}$$

Next, zero-mean random measurement error was added to the GPS phase measurements. The respective standard deviation of the error for each of the five satellites were

$$\sigma_1 = 0.01 \quad \sigma_2 = 0.05 \quad \sigma_3 = 0.03 \quad \sigma_4 = 0.02 \quad \sigma_5 = 0.02$$

The errors themselves were errors in B_{ji}; that is, in the projection of $\mathbf{s}_i$, the unit vector to satellite i, on the body coordinate system axis j. When again vectorizing the phase measurements according to Eq. (17), and using QUEST to compute the optimal attitude quaternion, **q***, the following quaternion error was achieved

$$|q - q^*| < 0.0131$$

corresponding errors in yaw, pitch, and roll were obtained

$$\begin{aligned} \delta\psi &= 1.145° \\ \delta\theta &= 1.011° \\ \delta\phi &= -0.227° \end{aligned}$$

IV.b: Direct phase measurements.

Instead of using the vectorized measurements and consequently the QUEST algorithm, here we find the attitude quaternion using the iterative algorithms which we developed for finding **q*** directly from the phase measurements themselves. In other words, we use the three iterative algorithm to find the quaternion which minimizes J of Eq. (19).

IV.b.1: Using algorithm 1 (Steepest Descent)

Using the preceding data we started the iteration with the arbitrarily chosen initial quaternion

$$\hat{\mathbf{q}}^T = [0.451,\ 0.107,\ 0.215,\ 0.859]$$

which corresponds to the following initial attitude estimation error expressed in terms of Euler angles

$$\begin{aligned} \delta\psi &= 14.05° \\ \delta\theta &= -14.78° \\ \delta\phi &= 0.802° \end{aligned}$$

The optimal iteration step size was found empirically to be ε = 2.75. The iterative solution settles on

$$\hat{\mathbf{q}}^T = [0.42363, \quad 0.04862, \quad 0.37751, \quad 0.82198]$$

and the final attitude estimation error in terms of yaw, pitch and roll was

$$\begin{aligned} \delta\psi &= -0.251° \\ \delta\theta &= -0.110° \\ \delta\phi &= -0.035° \end{aligned}$$

Observing the convergence rate of the recurrent solutions to the correct one reveals that the convergence rate was only linear; however, the final accuracy of the iterative solution was better than that of the

QUEST solution when applied to the vectorized phase measurements. An error analysis reveals that the error associated with the QUEST solution contains a term that does not exist when using the cost function J(**q**) of Eq. (19) to find **q**. The term is a function of the GPS satellite elevations with respect to the antenna coordinates z-axis. This term diminishes when the elevations are high. This is the reason why we obtained better results when using the iterative solution which minimizes J(**q**). The two solutions are, of course, identical when they process ideal measurements.

IV.b.2: Using algorithm 2 (Eigen Problem)

The same data that was used in the previous example, was used here too only that he iteration started with $\hat{\mathbf{q}}_0 = [0.5,\ 0.5,\ 0.5,\ 0.5]$ and $\hat{\lambda}_0 = 0$. This choice of initial quaternion corresponds to the following initial angular errors:

$$\delta\psi = -47.2470°$$
$$\delta\theta = -13.9390°$$
$$\delta\phi = -41.0610°$$

After 13 iterations the solution settled on

$$\mathbf{q}_{13}^{\mathrm{T}} = [0.42099,\ \ 0.04887,\ \ 0.37599,\ \ 0.82401]$$

The corresponding final attitude estimation error in terms of yaw, pitch and roll was

$$\delta\psi = -0.092°$$
$$\delta\theta = -0.286°$$
$$\delta\phi = \ \ 0.192°$$

and the absolute value of the difference between the final quaternion and the true quaternion was

$$|\mathbf{q} - \mathbf{q}_{13}| = 0.0030$$

IV.b.3: Using algorithm 3 (Newton-Raphson)

The exact same data that was used in the previous example was used here too. After only 9 iterations the solution settled on practically the same final quaternion as did the previous algorithm. Differences in the final Euler angles between the two algorithms were of the order of 10^{-12} degrees.

V. Conclusions

This paper presented algorithms for attitude determination using phase difference between GPS signals arriving at different antennae. Since the number of measurements is greater than the number of the unknown attitude parameters, and since the phase measurements are corrupted by noise, it is advantageous to find the attitude as a least squares fit. The cost function to be minimized in the fitting process is that given in Eq. (1).

It was shown that the phase measurements can be easily converted into vector measurements. Then the cost function becomes the one given in Eq. (3), and the least squares fit can be found using one of the available algorithms like QUEST.

The paper treats another possible approach, which is based on a least squares fit of the attitude quaternion to the basic GPS phase measurements. In the literature the cost function of the attitude fit is given as a function of the attitude matrix; that is, in the form of Eq. (3). It is, however, desirable to express the cost function as a function of the attitude quaternion. This stems from the success attained in quaternion fitting to vector measurements, which was achieved using QUEST. The paper presents the conversion of the matrix-based cost function to a quaternion-based cost function, which is given in Eqs. (18) and (19). The latter corresponds to Eq. (5), the cost function minimized by QUEST. A comparison between the latter function, and that of Eq. (18) reveals that unlike the case of vector measurements, where the cost function reduces to a quadratic form of a symmetric matrix, in the case of phase measurements, the cost function is a sum of squares of quadratic forms, therefore a simple QUEST-like solution is not applicable in this case.

A possible solution to the problem of finding $\mathbf{q}^*$ which minimizes $J(\mathbf{q})$ of Eq. (18) is an iterative one. Indeed, the paper presents such a solution. It is based on the gradient projection technique, which was used to develop a steepest descent search for the local minimum of the cost function. It was found though that the iteration process converged slowly. Therefore a faster converging algorithm was sought.

To meet this end, a Lagrange function was defined that included the quaternion unity-constraint. The conditions required for the Lagrange function to be stationary yielded five nonlinear algebraic equations with five unknowns, four of which were the components of the optimal quaternion. Four of the equations had the form of eigen-value/eigen-vector problem, therefore a corresponding iterative algorithm was developed. Another algorithm, which was developed for solving the set of five nonlinear algebraic equations, was a Newton-Raphson algorithm. It was shown that not only did the latter two algorithms reached the solution very fast, their accuracy was better than that of QUEST, particularly when the elevation of the GPS satellites was low. This was so because the measurement errors were amplified by low elevation when the phase measurements were converted to vector measurements, a step always needed when QUEST is used.

Finally, it should be noted that the algorithms presented in this work cover a deficiency in earlier work in that they are also applicable to attitude determination systems employing planar antenna arrays.

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