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Improving Efficiency in the Computation of Piecewise Quadratic Lyapunov Functions

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Abstract

In a series of papers, the authors have developed a method for analysis of piecewise linear systems. The idea is to use Lyapunov functions that are piecewise quadratic. Such Lyapunov functions can be computed via convex optimization in terms of linear matrix inequalities. This paper presents two approaches for improving the efficiency of these computations. It is shown that by splitting the analysis computations into two distinct steps, one can decrease the computations with roughly 50% *essentially without introducing conservatism. By using ellipsoidal boundings rather than polyhedral descriptions of the operating regimes, it is possible to reduce the computations even further. Combined, the two approaches allow the computation times to be reduced with an order of magnitude compared to previous formulations. However, it is shown that the use of ellipsoidal cell boundings in the S-procedure introduces conservatism in comparison with analysis based on polytopic region descriptions. An explicit formula for the minimal volume ellipsoid containing a simplex is also given, together with a complete proof.*

1 Introduction

Piecewise linear systems have a wide applicability in a range of engineering sciences. Some of the most common nonlinear components encountered in control systems, such as relays and saturations, are piecewise linear. Diodes and transistors, key components in even the simplest electronic circuits, are naturally modeled as piecewise linear. Many advanced controllers, notably gain scheduled control systems, are based on piecewise linear ideas. In a series of papers (Johansson and Rantzer, 1996; Rantzer and Johansson, 1997), the authors have developed an approach for analysis of piecewise linear systems. A key idea is to base the analysis on Lyapunov functions that are continuous and piecewise quadratic. The approach gives a drastic reduction of conservatism compared to approaches based on a single quadratic Lyapunov function (see (Johansson, 1999) for several illuminating examples.) The analysis conditions are formulated as convex optimization problems in terms of linear matrix inequalities (LMIs). This allows complex nonlinear systems to be analyzed using efficient numerical computations. Moreover, the same ideas allow constructive extensions of several aspects of linear systems with quadratic constraints to nonlinear systems and non-quadratic constraints. Important examples are uncertainty modeling, passivity and gain computations, as well as solutions to optimal control problems, see (Johansson and Rantzer, 1996; Rantzer and Johansson, 1997; Johansson, 1999). An attractive feature of this approach is that the results can be packaged into software that can be easily used also by inexperienced users, see (Hedlund and Johansson, 1999).

Piecewise quadratic Lyapunov functions are much more powerful than the globally quadratic Lyapunov functions. Naturally, this additional power comes at a price. System analysis using piecewise quadratic Lyapunov functions is computationally more demanding than analysis based on quadratic Lyapunov functions. Indeed, stability analysis using the conditions given in (Johansson and Rantzer, 1996) can be time consuming for large systems. It is therefore important to look for methods that decrease the computations of the piecewise quadratic analysis without introducing excessive conservatism. This paper presents two such methods. The first idea is to split the stability analysis into two steps. This reduces the number of analysis constraints by roughly 50% and eliminates a large number of search variables, essentially without introducing conservatism. The second idea is to restrict the number of free parameters in the S-procedure by *a priori* fixing an ellipsoidal cell bounding, rather than optimizing over the ellipsoidal cell boundings as was done in the original work. This allows us to reduce the number of search variables even further. However, this approach is not always appropriate as it introduces conservatism in the analysis. We will show that for some important classes of polytopes, it is possible to compute the minimum volume covering ellipsoid explicitly using basic matrix manipulations. To the best of the authors' knowledge, these are completely novel results. They have a direct practical use as they eliminate some intricate computations that would otherwise have been necessary, but they are also of theoretical interest. In this paper, we will use these results to prove that polyhedral relaxation used in our original work (Johansson and Rantzer, 1996) is always less conservative than the ellipsoidal relaxations used in this paper. This is contrary to a statement in (Hassibi and Boyd, 1998). Hence, by using ellipsoidal relaxations one reduces the computational effort at the expense of conservatism in the analysis. The developments also give useful insight in the use of the S-procedure in the analysis computations.

2 Piecewise Linear Systems

We consider piecewise linear systems on the form

$$\begin{cases} \dot{x} = A_i x + a_i + B_i u \\ y = C_i x + c_i + D_i u \end{cases} \qquad \text{for } x \in X_i. \tag{1}$$

Here, $\{X_i\}_{i \in I} \subseteq \mathbf{R}^n$ is a partition of the state space into a number of closed (possibly unbounded) polyhedral cells, see Figure 1, and I is the index set of the cells.

For convenient notation, we introduce

$$\left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array}\right] = \left[\begin{array}{cc|c} A_i & a_i & B_i \\ 0 & 0 & 0 \\ \hline C_i & c_i & D_i \end{array}\right], \qquad \bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}, \tag{2}$$

and write the system dynamics as

$$\begin{cases} \dot{\bar{x}} = \bar{A}_i \bar{x} + \bar{B}_i u \\ y = \bar{C}_i \bar{x} + \bar{D}_i u \end{cases} \qquad \text{for } x \in X_i.$$

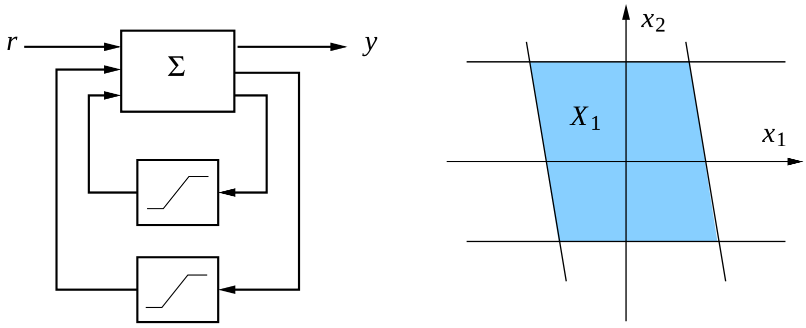

Figure 1: Linear system with saturated feedbacks (left) and the corresponding polyhedral partition of $\mathbf{R}^2$ (right).

The focus of this paper will be to investigate (possibly global) stability properties of equilibria. We therefore let $I_0 \subseteq I$ be the set of indices for the cells that contain the origin and $I_1 \subseteq I$ be the set of indices for cells that do not contain the origin. We will assume that $a_i = 0$, $c_i = 0$ for $i \in I_0$. Further details on piecewise linear modeling are given in (Johansson, 1999).

3 Analysis via Piecewise Quadratic Lyapunov Functions

We will base our analysis on piecewise quadratic Lyapunov functions on the form

$$V(x) = \begin{cases} x^T P_i x & \text{for } x \in X_i, i \in I_0 \\ \begin{bmatrix} x \\ 1 \end{bmatrix}^T \bar{P}_i \begin{bmatrix} x \\ 1 \end{bmatrix} = x^T P_i x + 2q_i^T x + r_i & \text{for } x \in X_i, i \in I_1. \end{cases} \tag{3}$$

In the computations, we need to enforce continuity of the Lyapunov function candidate and exploit the fact that a certain linear system only describes the dynamics in a specific region of the state space. This is done by constructing matrices $\bar{F}_i = \begin{bmatrix} F_i & f_i \end{bmatrix}$ and $\bar{E}_i = \begin{bmatrix} E_i & e_i \end{bmatrix}$ for every $i \in I$ such that

$$\bar{F}_i \bar{x} = \bar{F}_j \bar{x} \qquad \text{for } x \in X_i \cap X_j \tag{4}$$

$$\bar{E}_i \bar{x} \succeq 0 \qquad \text{for } x \in X_i. \tag{5}$$

Here, the vector inequality $z \succeq 0$ means that each entry of the vector z is non-negative. The matrices $\bar{F}_i$ express the continuity condition, while the matrices $\bar{E}_i$ are used to bound the operation regimes X_i. To obtain strict LMI conditions in the analysis, we will have to eliminate affine terms in the regions that contain the origin. This is done by imposing the additional constraints

$$e_i = f_i = 0 \qquad \text{for } i \in I_0. \tag{6}$$

Now, let T be a symmetric matrix of appropriate dimensions. Then,

$$V(x) = \bar{x}^T \bar{F}_i^T T \bar{F}_i \bar{x} := \bar{x}^T \bar{P}_i \bar{x} \qquad \text{for } x \in X_i$$

is a continuous and piecewise quadratic function on the form (3). The following result, first stated in (Johansson and Rantzer, 1996), forms the starting point for the developments in this paper.

Proposition 1 (Piecewise Quadratic Stability) *Consider symmetric matrices T, U_i and W_i, such that U_i and W_i have non-negative entries, while $P_i = F_i^T T F_i$ for $i \in I_0$ and $P_i = F_i^T T F_i$ for $i \in I$ satisfy*

$$\begin{cases} 0 > A_i^T P_i + P_i A_i + E_i^T U_i E_i \\ 0 < P_i - E_i^T W_i E_i \end{cases} \qquad i \in I_0 \tag{7}$$

$$\begin{cases} 0 > \bar{A}_i^T \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{E}_i^T U_i \bar{E}_i \\ 0 < \bar{P}_i - \bar{E}_i^T W_i \bar{E}_i \end{cases} \qquad i \in I_1 \tag{8}$$

Then $x(t)$ tends to zero exponentially for every continuous piecewise C^1 trajectory in $\cup_{i \in I} X_i$ satisfying (1) with $u \equiv 0$ for $t \geq 0$.

Proof: See (Johansson, 1999) for a proof based on Lemma 2 of this paper.

Note that the stability conditions of Proposition 1 are LMI conditions in the variables T, U_i and W_i. In the absence of attractive sliding modes, the above conditions assure that the function (3) is a Lyapunov function for the system (1).

Simple algorithms for computing constraint matrices that satisfy (4), (5) and (6) for a number of important partition types are given in (Johansson, 1999). For the purpose of this paper, it is sufficient to show how one can compute constraint matrices for simplex partitions.

4 Constraint Matrices for Simplex Partitions

Consider a compact domain of the state space $\{X_i\}_{i \in I} \subset \mathbf{R}^n$ partitioned into convex polytopes. A *polytope* is defined as the convex hull of a finite number of affinely independent vertices $v_k \in \mathbf{R}^n$. It is called an *n-dimensional simplex* if the number of vertices equals $n+1$. Note that any polytope which is not a simplex can be partitioned into two polytopes, each with fewer vertices than the original one. Repeating this procedure eventually generates a partition of the original polytope into simplices, see Figure 2.

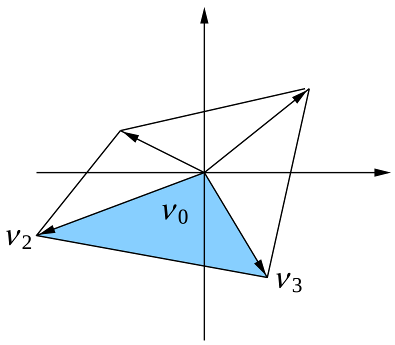

Figure 2: Simplex partition of $\mathbf{R}^2$.

Let $X_i \subset \mathbf{R}^n$ be a simplex. Each $x \in X_i$ has a unique representation as a convex combination of the cell vertices

$$x = \sum_k z_k v_k, \qquad x, v_k \in X_i \tag{9}$$

with $z_k \geq 0$, $\sum_k z_k = 1$. The numbers z_k are sometimes called the barycentric coordinates of x. Since they are non-negative for $x \in X_i$, this indicates that the mapping from x to the barycentric coordinates of X_i would qualify for the cell bounding (5). Moreover, if x lies on the boundary between two cells the only non-zero coordinates will be those z_k that describe this boundary. This observation allows us to construct a mapping on the form (4) from x to the set of all z_k of the partition, see (Rantzer and Johansson, 1997).

To describe the computations, let $v_0, \ldots, v_p$ be the vertices of the partition with $v_0 = 0$. Using the notation $\bar{v}_i^T = \begin{bmatrix} v_i^T & 1 \end{bmatrix}$, construct the matrices

$$V = \begin{bmatrix} v_1 & \ldots & v_p \end{bmatrix}, \qquad \bar{V} = \begin{bmatrix} \bar{v}_1 & \ldots & \bar{v}_p \end{bmatrix}. \tag{10}$$

Finally, we need the following definition.

Definition 1 (Extraction Matrix) *For each simplex X_i, define an extraction matrix $\mathcal{E}_i$ as follows: the kth row of $\mathcal{E}_i$ is zero for all $k \in \{1..p\}$ such that $v_k \notin X_i$ and the non-zero rows of $\mathcal{E}_i$ are equal to the rows of an identity matrix.*

Note that since k ranges from 1 to p, $\mathcal{E}_i \in \mathbf{R}^{n \times p}$ for $i \in I_0$ and $\mathcal{E}_i \in \mathbf{R}^{(n+1) \times p}$ when $i \in I_1$. Now, let $\mathcal{E}_i$ be the extraction matrix for cell X_i and compute the constraint matrices as

$$E_i = (V\mathcal{E}_i)^{-1}, \quad F_i = \mathcal{E}_i E_i \qquad \text{for } i \in I_0, \tag{11}$$

$$\bar{E}_i = (\bar{V}\mathcal{E}_i)^{-1}, \quad \bar{F}_i = \mathcal{E}_i \bar{E}_i \qquad \text{for } i \in I_1. \tag{12}$$

The following result can be established, cf. (Johansson and Rantzer, 1997).

Lemma 1 (Simplicial Domain Description) *The matrices $\bar{E}_i$, $\bar{F}_i$, E_i and F_i describing a simplicial partition $\cup X_i \subseteq \mathbf{R}^n$ as constructed in (10),(11), (12) satisfy the conditions (4),(5) and (6).*

5 Improved Efficiency via Two-Step Analysis

The computations resulting from a straightforward implementation of the LMI conditions of Proposition 1 are often time consuming. This is particularly true if the state space partitioning is performed in high-dimensional space. It is therefore of interest to look for methods that decrease the computational burden without introducing excessive conservatism. Essentially, such savings can be done either by reducing the number of search variables (the entries of T, U_i and W_i), or by reducing the number of constraints that have to be satisfied.

Our first suggestion is to initially ignore the positivity condition in the Lyapunov function search. A solution to the remaining inequalities guarantees that the function (3) is decreasing along all system trajectories. Once such a function is found, we proceed to check if it has the desired positivity properties. This separation is justified by the following result.

Lemma 2 *Let* $x(t) : [0, \infty) \longrightarrow \mathbf{R}^n$ *and let and let* $V(t) : [0, \infty) \longrightarrow \mathbf{R}$ *be a non-increasing and piecewise* C^1 *function satisfying*

$$\frac{d}{dt} V(t) \leq -\gamma \|x(t)\|^p \tag{13}$$

for some $\gamma > 0$ *and some* $p > 0$*, almost everywhere on* $[0, \infty)$*.*
If there exists $\alpha > 0$ *such that*

$$\alpha \|x(t)\|^p \leq V(t) \leq \beta \|x(t)\|^p \tag{14}$$

then $\|x(t)\|$ *tends to zero exponentially. If the maximal* α *that satisfies (14) is negative, then* $\|x(t)\| \to \infty$ *as* $t \to \infty$*.*

Proof: See (Johansson, 1999).

Lemma 2 suggests that if we can find a function which is decreasing along all system trajectories, it contains all information about the stability properties of the system. If the function can be shown to be positive on the partition, stability follows analogously to Proposition 1. If we find some point where the computed function is negative, then no trajectory in the partition starting at this point can approach the origin as $t \to \infty$.

The computational consequence of Lemma 2 is that the LMI problem in Proposition 1 can be split into several smaller problems. In order to find $V(x)$ with the desired decreasing properties, one only needs to solve the decreasing conditions in Proposition 1 (the upper LMIs in (7) and (8)). This reduces the number of LMI conditions by roughly 50% and gives a large decrease in the number of search variables. Hence, this problem can be solved in a fraction of the time needed to solve the original problem. Moreover, if this LMI problem has no solution, then neither has the original formulation in Proposition 1.

In most cases, a reduction of the number of constraints in an optimization problem can only be done at the cost of increased conservatism in the solution. The remarkable consequence of Lemma 2 is that for the suggested separation, the situation is completely opposite! By first looking for a function with the desired decreasing properties and then checking positivity, one can not only assess exponential stability, but may also detect instabilities. This is illustrated by the following example.

Example 1 (Detection of Instabilities) *Consider the following piecewise linear system*

$$\dot{x} = Ax - B\,max(Cx, 0)$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 \\ 0 \end{bmatrix}^T.$$

The LMI conditions of Proposition 1 have no solution. Hence, no conclusion can be drawn regarding system stability. By disregarding the positivity constraints, however, it is possible to find a continuous piecewise quadratic function that decreases along all system trajectories. Verifying positivity of the computed function fails for the region $Cx \leq 0$*. By plotting the level set* $\{x \mid V(x) < 0\}$ *and invoking Lemma 2, we conclude that the computations* ***prove*** *state divergence for all initial value within this set, see Figure 3.*

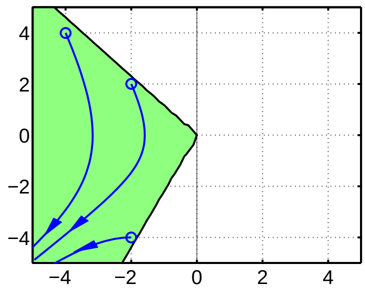

Figure 3: State divergence is proved for all trajectories starting in the shaded region.

Positivity can, for example, be checked as in Proposition 1. Checking positivity then amounts to solving a number of small independent problems (one for each region). Since the Lyapunov function candidate obtained from the first step is now fixed, these problems consist of only one constraint and few free parameters (the entries of the matrix W_i).

6 Improved Efficiency via Ellipsoidal Cell Boundings

In many cases it is the number free parameters of the relaxation terms that add the most parameters to the Lyapunov function search. A second way to reduce the computational burden is therefore to try to minimize the number of free parameters in the S-procedure. Returning to Proposition 1 we see that, for a given $\bar{P}_i$, a solution to the inequality

$$\bar{P}_i - \bar{E}_i^T U_i \bar{E}_i > 0$$

does not only imply that $V(x) = \bar{x}^T \bar{P}_i \bar{x} > 0$ for all $x \in X_i$, but that $V(x)$ is positive for all x in the quadratic set

$$\mathcal{E}_i = \{x \mid \bar{x}^T \bar{E}_i^T U_i \bar{E}_i \bar{x} \geq 0\}$$

We may view the term $\bar{S}_i = \bar{E}_i^T U_i \bar{E}_i$ as a description of a quadratic set derived from its polyhedral representation. From this perspective, the free parameters in U_i are used to adjust the quadratic set so as to verify the desired inequality, see Figure 4.

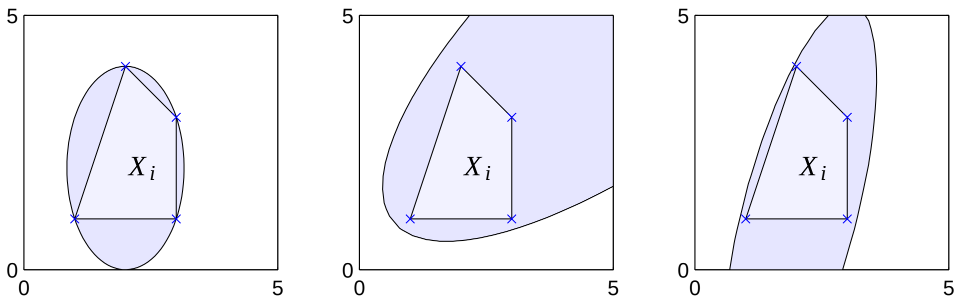

Figure 4: Several quadratic boundings $\mathcal{E}_i(\bar{E}_i)$ (dark) of the cell X_i (light) can be derived by optimizing the free parameters in the matrix U_i.

One way of reducing the number of search variables would be to simply fix the matrix $\bar{S}_i$ before the Lyapunov function search. This is equivalent to specifying a quadratic set

$$\mathcal{E}_i(\bar{S}_i) = \{x \mid \bar{x}^T \bar{S}_i \bar{x} \geq 0\}$$

that contains the cell X_i, *i.e.*, $\mathcal{E}_i \supseteq X_i$. Now, rather than using the polyhedral relaxation term $\bar{E}_i^T U_i \bar{E}_i$ with $U_i \succeq 0$ in Proposition 1, we use the ellipsoidal relaxation $u_i \bar{S}_i$ with $u_i \geq 0$.

Clearly, this approach admits large savings in the number of search variables. More precisely, if $\bar{E}_i \in R^{p\times(n+1)}$, the polyhedral relaxation has $(p-1)(p-2)/2$ free parameters, while the ellipsoidal relaxation uses just one free parameter.

The drawback of this approach is that the quadratic set has to be fixed *before* the optimization. On the contrary, the polyhedral relaxation has a lot of freedom in adjusting a quadratic superset of the region *during* the Lyapunov function search. This freedom may be critical, as will be shown later in this paper.

We also note that ellipsoidal cell boundings may sometimes be attractive for other reasons that computational efficiency. In (Pettersson, 1996), ellipsoidal cell descriptions are used to guarantee robustness of hybrid systems, while in (Hassibi and Boyd, 1998), ellipsoidal cell boundings are used to allow the S-procedure in quadratic stabilizability computations for piecewise linear systems.

6.1 Computing Minimal Volume Ellipsoids

A natural candidate for quadratic approximation of a polyhedral cell is to compute the ellipsoid with minimum volume that contains the cell (Hassibi and Boyd, 1998). The covering ellipsoid of minimal volume can be obtained by solving the following convex optimization problem (Vandenberghe *et al.*, 1998).

Proposition 2 (Minimal Volume Ellipsoids) *Let X be a convex polytope with vertices v_i,*

$$X = conv(v_1, \ldots, v_p)$$

The ellipsoid

$$E = \{x \mid ||Px + b|| \leq 1\}$$

of minimum volume that contains X is given by the solution to the convex optimization problem

$$\begin{aligned} \min_{P,b} \quad & \ln\det P^{-1} \\ s.t. \quad & P = P^T > 0 \\ & \begin{bmatrix} I & Pv_i + b \\ (Pv_i + b)^T & 1 \end{bmatrix} \geq 0 \textit{ for } i = 1, \ldots, p \end{aligned}$$

The matrices $\bar{S}_i$ can be computed from the solution P, b of the optimization problem in Proposition 2 as

$$\bar{S}_i = \begin{bmatrix} -P^T P & -P^T b \\ -b^T P & 1 - b^T b \end{bmatrix}.$$

Note that in order to compute the minimum volume ellipsoid, we need to compute all the vertices of the cell. This computation is called a vertex enumeration, and it is known that solving the vertex enumeration problem can be computationally intensive (Avis *et al.*, 1997). Once the vertices are found, Proposition 2 can be invoked to compute the optimal bounding ellipsoid. In the light of these computations, the actual savings in using ellipsoidal cell boundings are not always obvious.

7 A Comparative Example

To give a flavor of the benefits and limitations of the different formulations of piecewise quadratic stability we consider analysis of the system shown in Figure 5(left). The system dynamics is given by

$$\dot{x} = Ax + b_1 f(x_1) + b_2 f(x_2)$$

where $A \in \mathbf{R}^{2\times 2}$, $b_1, b_2 \in \mathbf{R}^{2\times 1}$ and $f(z) = \arctan(z)$. We will present results for both piecewise linear approximations and piecewise linear sector bounds on the nonlinearities, see Figure reffig:compex (right). The analysis conditions for the different approaches are given explicitly in (Johansson, 1999).

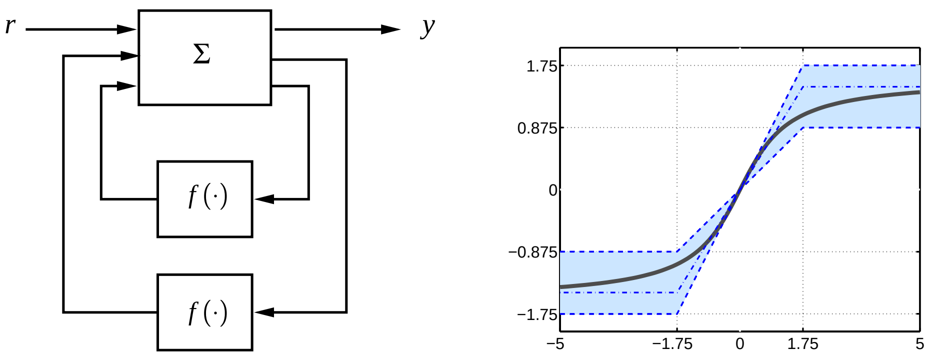

Figure 5: The system used as comparative example (left). The nonlinearity $f_i(x_i)$ is shown in full lines in the right figure. The dash-dotted line illustrates a piecewise linear approximation and the dashed lines show piecewise linear sector bounds.

In all cases, the nonlinearity descriptions induce a partition of the domain $[-5, 5] \times [-5, 5]$ into nine regions. If piecewise linear approximations are used, the resulting dynamics is piecewise linear. If piecewise linear sector bounds are used, the dynamics in each region is specified by a linear differential inclusion with four extreme dynamics.

First, we let

$$A = \begin{bmatrix} -3 & 2 \\ 1 & -3 \end{bmatrix}, \quad b_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

and use the piecewise linear approximation of $f_i(x_i)$. In this case, all approaches verify stability. The different computational requirements are shown in Table 1. The computations were performed on a SUN Ultra 10 computer using the LMI software (Gahinet *et al.*, 1995). In the table the acronym P refers to the use of polyhedral cell description in the S-procedure

Approach	Time (s)	#Variables	#Constraints
P-1	1.04	117	114
P-2	0.41	69	57
Q-1	0.23	37	34
Q-2	0.11	29	17

Table 1: First set-up. All approaches verify stability. Large savings in computations are obtained from the alternative formulations (P-2,Q-1,Q-2).

terms while Q indicates the use of quadratic cell boundings. The number 1 means that the analysis was performed in a single step (enforcing both positivity and decreasing conditions simultaneously) while 2 means that the analysis was performed in two steps (enforcing the decreasing condition in the Lyapunov function search and then verifying positivity). As seen in Table 1, the two-step procedure (P-2) results in a large reduction in computation time compared with the computations required by Proposition 1 (P-1). The computational savings when using quadratic cell boundings (Q-1) are even greater. In this case, the quadratic cell boundings are taken as the minimal volume ellipsoids that cover each region. By combining the two-step analysis procedure with quadratic cell boundings (Q-2), the computational time is reduced to around than 10% of what was required by the original formulation.

Using the same matrices A, b_1 and b_2, we now consider the case when the nonlinearities are described by piecewise linear sector bounds. This approach allows rigorous stability analysis of the original system, but requires more computations than an approach based on piecewise linear approximations. In each region, the system is now described by a differential inclusion with four extreme dynamics. Stability is assured by searching for a Lyapunov function that is decreasing with respect to each extreme dynamics, see (Johansson, 1999). As the main burden in analysis of such systems is verification of the multiple decreasing conditions, the savings of the two step analysis procedure gets somewhat lower, see Table 2.

Approach	Time (s)	#Variables	#Constraints
P-1	3.79	261	285
P-2	2.17	213	228
Q-1	0.80	61	85
Q-2	0.45	53	58

Table 2: Second set-up. The use of piecewise linear sector bounds decreases the benefits of the two-set analysis procedure, but good savings are still obtained.

The problem with ellipsoidal cell boundings is that there is very little freedom in adjusting the S-procedure terms during the Lyapunov function search. This introduces some conservatism as can be seen by letting

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -2 \end{bmatrix}, \quad b_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

and using piecewise linear sector bounds on the nonlinearities. The computational results are shown in Table 3. Stability can no longer be verified using ellipsoidal cell boundings, while the computational savings in the use of the two-step procedure (P-2) remain the same.

Approach	Time (s)	#Variables	#Constraints
P-1	4.15	261	285
P-2	2.62	213	228
Q-1	fails	–	–
Q-2	fails	–	–

Table 3: Final set-up. Quadratic cell boundings fail to verify stability.

8 Explicit Formulas for Optimal Covering Ellipsoids

The computations outlined above are unnecessarily complicated if we have cells with special structure. In this section, we will derive explicit expressions for the minimal volume ellipsoids containing a simplex. An explicit formula for the minimal volume ellipsoid that contains a parallelepiped can be found in (Johansson, 1999). This makes ellipsoidal boundings computationally efficient and easily applicable for these type of cells. Moreover, these results are useful for the theoretical study of piecewise quadratic analysis. Amazingly enough, the problem of finding the minimum volume ellipsoid that contains a simplex seems to be if not unknown, then very poorly documented in the mathematical literature. For some related results, see (John, 1948; Ball, 1992).

Appendix A contains a full proof of the fact that the minimal volume ellipsoid containing the standard simplex is a sphere.

Theorem 1 *The minimum volume ellipsoid containing the standard simplex,*

$$X_i = \{x \in \mathbf{R}^{n+1} \mid x_k \geq 0, \sum_{k=1}^{n+1} x_k = 1\}$$

is the ball

$$\mathcal{E}^* = \{x \mid x^T x \leq 1, \sum_{k=1}^{n+1} x_k = 1\} \tag{15}$$

Proof: See Appendix A.

The following result for simplex boundings now follows.

Corollary 1 (Simplex Bounding) *Let X_i be a non-degenerate simplex, and let $\bar{E}_i$ be the corresponding cell identification matrix, as computed in (12). Then, the ellipsoid of minimum volume that contains X_i is given by*

$$\bar{x}^T \bar{E}_i^T \bar{E}_i \bar{x} \leq 1. \tag{16}$$

Proof: The map $\lambda = \bar{E}_i \bar{x}$ is a bijective affine map that transforms a non-degenerate simplex into a standard simplex in the constraint hyperplane $\sum_i \lambda_i = 1$. In these coordinates (16) describes the ball (15), and the desired result follows.

9 Polytopic Relaxation is Stronger than Ellipsoidal

The use of ellipsoidal descriptions in the S-procedure allows significant reductions in the number of free variables of the optimization problem compared to the use of polytopic descriptions. As indicated by the example in the previous section, this reduction in number of search parameters comes at the price of increased conservatism in the analysis.

Next, we will show that the use of quadratic cell boundings always introduces some conservatism compared to the use of polytopic relaxation terms. More precisely, we will show that if the piecewise quadratic computations with ellipsoidal cell boundings have a solution, the so has the computations in Proposition 1, while the opposite is not always true. This is contrary to a statement in (Hassibi and Boyd, 1998)(Section 4.1) where it was conjectured that computations using ellipsoidal cell boundings would be less conservative than those using polyhedral relaxations since the S-procedure may be lossy when several quadratic terms are used.

Proposition 3 *For simplex cells, the polytopic S-procedure relaxation $\bar{E}^T U \bar{E}$ is stronger than the S-procedure using optimal outer ellipsoids, $u\bar{S}$.*

More precisely, let X be a simplex, $\bar{E}$ be the associated cell bounding satisfying (10) and (12), and let $\bar{S}$ be the minimal volume ellipsoid that contains X. Then, if the LMI

$$\bar{P} - u\bar{S} > 0 \tag{17}$$

has a solution, then so has the LMI

$$\bar{P} - \bar{E}^T U \bar{E} > 0, \tag{18}$$

but there exists cases when (18) admits a solution while (17) does not.

Proof: According to Lemma 1, the outer ellipsoidal approximation is given by

$$\bar{S} = \begin{bmatrix} 0_{n\times n} & 0 \\ 0^T & 1 \end{bmatrix} - \bar{E}\bar{E}$$

Let $\bar{0} = \begin{bmatrix} 0_{1\times n} & 1 \end{bmatrix}$ and $\bar{1} = \begin{bmatrix} 1_{1\times n} & 1 \end{bmatrix}$. From the computation of $\bar{E}$, Equations (10), (12), we have

$$\bar{1}^T \bar{E} = \bar{0}^T$$

and thus

$$u\bar{S} = u\bar{E}^T(\bar{1}\bar{1}^T - I)\bar{E}$$

This expression is on the form $\bar{E}^T U \bar{E}$ with $u_{ij} = u$ if $i \neq j$ and 0 otherwise. This concludes the first part of the proof.

For the second part of the proof, consider the simplex

$$X = \text{conv}(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix})$$

for which we have

$$\bar{E} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \qquad \bar{S} = \begin{bmatrix} 2 & 1 & -4 \\ 1 & 2 & -4 \\ -4 & -4 & 10 \end{bmatrix}.$$

Let

$$\bar{P} = \begin{bmatrix} 20 & 0 & 5 \\ 0 & 1 & 0 \\ 5 & 0 & -25 \end{bmatrix}$$

Pre- and postmultiplying the LMI (17) by $\bar{z} = \begin{bmatrix} 3 & 6 & 4 \end{bmatrix}^T$ we obtain $-64 - 2u$, which is negative for all admissible values of u ($u \geq 0$). Hence, there is no solution to this LMI. However, for the formulation (18), it is straightforward to verify that the choice

$$U = \begin{bmatrix} 0 & 20 & 5 \\ 20 & 0 & 20 \\ 5 & 20 & 0 \end{bmatrix}$$

solves the LMI.

A similar result can be established also for hyper-rectangular cells.

To understand the use of the S-procedure in the piecewise quadratic analysis, it is fruitful to consider the problem of verifying the constraint

$$\bar{x}^T \bar{P}_i \bar{x} > 0 \qquad\qquad x \in X_i$$

using LMI computations. In this case, the role of the S-procedure is to separate the set $V_i^- = \{x \mid \bar{x}^T \bar{P}_i \bar{x} < 0\}$ from the set X_i. The volume of the covering ellipsoid may have very little to do with this separation. This is illustrated in Figure 6. The minimum volume ellipsoid of X_i intersects the set V_i^-, hence it cannot be used to verify the desired inequality. By using the polyhedral relaxation, there is a lot of freedom in optimizing over the quadratic bounding and separation can easily be accomplished, see Figure 6(right).

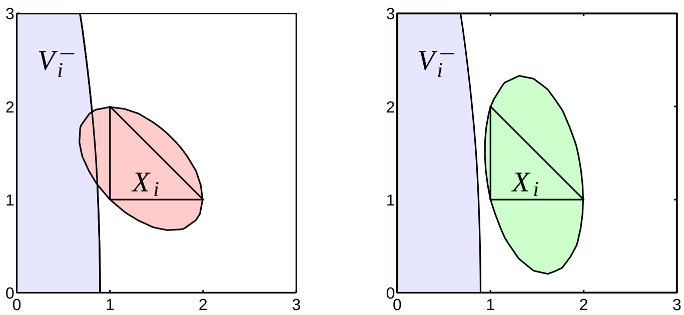

Figure 6: The counter example in Proposition 3. The minimal volume ellipsoid fails to separate X_i from V_i^- (left), while optimizing over the covering ellipsoids using the polytopic formulation easily finds a separating supset.

Another point is that although the S-procedure is only sufficient when there is more than one quadratic constraint, adding new constraints can never make the inequalities harder to satisfy since the associated multipliers can always be set to zero. On the contrary, adding new terms may allow separations that would otherwise not be possible.

10 Conclusions

This paper presented two methods for improving efficiency in the computations of piecewise quadratic Lyapunov functions. The first idea was to split the analysis problem into two distinct steps. It was shown that once a function with the desired decreasing properties has been found, it contains all information about the stability properties of the system. Thus, by *not* imposing positivity constraints in the Lyapunov function search one does not only decrease the computational burden by 50%, but one also obtains more information from the computations than in the original formulation. This additional information can be used to detect instabilities.

The second idea was to use ellipsoidal cell boundings. This approach gives a large decrease in the number of search variables, but comes at the cost of increased conservatism. More precisely, it was shown that for simplex partitions the use of ellipsoidal cell boundings in the S-procedure is always more conservative than the use of polyhedral region descriptions. An explicit formula for the minimal volume ellipsoid that contains a regular simplex was given, together with a complete proof.

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A Auxiliary Results on Ellipsoids

A.1 Existence and uniqueness of the minimal volume ellipsoid containing a compact convex set

Lemma 3 *Given two matrices $P = P^* > 0$ and $Q = Q^*$ there exists a matrix S, $\det(S) = 1$, such that S^*PS and S^*QS are both diagonal.*

Proof: There exists an orthogonal matrix U such that $D = U^*PU$ is diagonal. The matrix $R = D^{-1/2}U^*QUD^{-1/2}$ is Hermitian and then, similarly, there exists an orthogonal matrix V such that V^*RV is diagonal. The matrix $S = UD^{-1/2}VD^{1/2}$ satisfies the conditions above.

Denote $\mathcal{P} \subset \mathbf{R}^{n\times n}$ the set of all positive definite (Hermitian) matrices.

Lemma 4 *The function $f(P) = \ln\det(P)$ is strictly concave on $\mathcal{P}$.*

Proof: Let P_1, $P_2 \in \mathcal{P}$, $P_1 \neq P_2$, λ_1, $\lambda_2 > 0$ and $\lambda_1 + \lambda_2 = 1$. We are to prove that

$$f(\lambda_1 P_1 + \lambda_2 P_2) > \lambda_1 f(P_1) + \lambda_2 f(P_2).$$

By Lemma 3 there exists a matrix S with $\det S = 1$ such that S^*P_1S and S^*P_2S are both diagonal.

$$\begin{aligned} S^*P_1S &= \operatorname{diag}(d_{11}, d_{12}, \ldots, d_{1n}), \\ S^*P_2S &= \operatorname{diag}(d_{21}, d_{22}, \ldots, d_{2n}). \end{aligned}$$

Then $\det(\lambda_1 P_1 + \lambda_2 P_2) = \prod_{i=1}^{n}(\lambda_1 d_{1i} + \lambda_2 d_{2i})$ and the fact that ln is strictly concave yields

$$\begin{aligned} f(\lambda_1 P_1 + \lambda_2 P_2) = \sum_{i=1}^{n} \ln(\lambda_1 d_{1i} + \lambda_2 d_{2i}) &> \\ > \ & \lambda_1 \sum_{i=1}^{n} \ln(d_{1i}) + \lambda_2 \sum_{i=1}^{n} \ln(d_{2i}) = \\ = \ & \lambda_1 \ln\det(S^*P_1S) + \lambda_2 \ln\det(S^*P_2S) = \\ = \ & \lambda_1 f(P_1) + \lambda_2 f(P_2). \end{aligned}$$

Theorem 2 *The minimal volume ellipsoid containing a compact convex set $C \in \mathbf{R}^n$ exists and is unique.*

Proof: There is a bijective map between the set of all ellipsoids in $\mathbf{R}^n$ and the space $\mathcal{P} \times \mathbf{R}^n$. This map is given by

$$(P, a) \quad \leftrightarrow \quad E(P, a) = \{\, x \in \mathbf{R}^n \ : \ \|Px + a\| \leq 1 \,\}.$$

Let $\{\, x_i \,\}_{i=1}^{p}$ be the extremal points of C. The set

$$\begin{aligned} \mathcal{E} &= \{\, (P, a) \in \mathcal{P} \times \mathbf{R}^n \ : \ E(P, a) \supseteq C \,\} = \\ &= \bigcap_{i=1}^{p} \{\, (P, a) \in \mathcal{P} \times \mathbf{R}^n \ : \ x_i \in E(P, a) \,\} \end{aligned}$$

is convex as the intersection of the convex sets. The minimal volume ellipsoid $E(P_*, a_*)$ is a solution of the following optimization problem

$$\begin{aligned} (P_*, a_*) &= \arg\min_{\mathcal{E}} (-\ln\det(P)) = \\ &= \arg\min_{\mathcal{E} \cap \mathcal{D}} (-\ln\det(P)) \end{aligned}$$

where

$$\mathcal{D} = \{\, (P, a) \ : \ \sigma(P)| \leq \mathrm{diam}(C) \,\}$$

and σ denotes the matrix spectrum. By Lemma 4 this is the minimization of the strictly convex continuous function over the compact convex set. Hence there exists a unique solution.

A.2 The minimal volume ellipsoid containing the standard simplex is a ball

Lemma 5 *A hyperplane $L \subset \mathbf{R}^n$ is the hyperplane of symmetry for an ellipsoid E if and only if the center of E belongs to L and the normal vector of L coincides with a semi-axis of the ellipsoid.*

Proof: The **if** implication is trivial.

Only if. Without loss of generality we assume that the center of the ellipsoid is the origin. It evidently belongs to L as an invariant point of symmetry. The ellipsoid and its boundary is given by

$$\begin{aligned} E &= \{\, x \in \mathbf{R}^n \ : \ x^* H x \leq 1 \,\}, \\ \partial E &= \{\, x \in \mathbf{R}^n \ : \ x^* H x = 1 \,\}. \end{aligned}$$

First of all we remark that the hyperplane L is the symmetry hyperplane for all ellipsoids λE, $\lambda \in \mathbf{R}$ if it is at least for one of them. Let $x \in L$ and n be the normal vector of the hyperplane L. The symmetry condition means that the points $x + \gamma n$ and $x - \gamma n$ belong or not belong to $\partial(\lambda E)$ simultaneously. Thus the symmetry condition is equivalent to the equality

$$(x + \gamma n)^* H (x + \gamma n) = (x - \gamma n)^* H (x - \gamma n)$$

for all $x \in L$ and $\gamma \in \mathbf{R}$. This yields

$$x^* H n = 0, \quad \forall x \in L,$$

that is $Hn \perp L$. Since $n \perp L$ we conclude that $Hn \parallel n$, i.e. n is the eigenvector of H.

Lemma 6 *Any hyperplane of symmetry for the standard simplex $K \in \mathbf{R}^n$ is also the hyperplane of symmetry for the minimal volume ellipsoid containing K.*

Proof: Theorem 2 implies the minimal volume ellipsoid $E(K)$ containing the standard simplex K is unique. Then the map $K \mapsto E(K)$ is well defined. Denote $S(F)$ the mirror image of a set F with respect to the hyperplane L. We have $K \mapsto E(K)$. Hence $S(K) \mapsto S(E(K))$. Since $K = S(K)$ we conclude $E(K) = S(E(K))$.

Theorem 3 *The minimal volume ellipsoid containing the standard simplex in $\mathbf{R}^n$ is the ball.*

Proof: For each pair of the standard simplex vertices P_i and P_j, the normal vector $P_i - P_j$ and the point $(P_i + P_j)/2$ define the hyperplane uniquely. It is easy to show that this hyperplane contains the center and all other vertices of the simplex and is the hyperplane of symmetry. By Lemma 6 it is also the hyperplane of symmetry for the minimal volume ellipsoid and by Lemma 5 we conclude that $P_i - P_j$ is the semiaxis the ellipsoid. Thus the ellipsoid in $\mathbf{R}^n$ has at least n semiaxes, $\{ P_{n+1} - P_i \}_{i=1}^n$, each pair of which is nonorthogonal. Hence it is the ball.