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On the feasibility and convergence of H_∞ multistep predictors*

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Abstract

An H_∞ multistep predictor is designed so as to guarantee a prescribed level of energy attenuation from the disturbances to the prediction error. It is shown in this paper that, for a given value of the attenuation, an admissible predictor exists over a finite horizon if and only if the solution of a suitable difference Riccati equation lies uniformly above a computable lower threshold, which depends on the prediction look-ahead horizon (feasibility condition). Moreover, sufficient conditions on the initial state uncertainty are worked out, which ensure the existence of the predictor over an arbitrarily long time interval and its convergence to steady-state.

1 Introduction

The H_∞ approach to the problem of state estimation of dynamic systems has been demonstrated to be effective in achieving robustness in the face of uncertainty on the spectral characterization of the disturbances (see e.g. (Basar and Bernhard, 1991), (Green and Limebeer, 1995), (Hassibi *et al.*, 1996), (Nagpal and Khargonekar, 1991), (Shaked and Theodor, 1992)). The objective of this methodology is to design an estimator able to guarantee a prescribed level γ of energy attenuation from the exogenous disturbances to the estimation error.

In discrete-time, both the design of an H_∞ filter and the design of an H_∞ one-step ahead predictor call for the solution of a suitable difference Riccati equation, subject to some "feasibility" constraints, whose inizialization depends on the initial state uncertainty ((Yaesh and Shaked, 1991)). Differently from the Kalman estimator, that can be extended over an interval of arbitrary length and, under mild assumptions, asymptotically converges to steady-state, the H_∞ estimator may blow up in finite time for some initial conditions. Given a certain γ, it is thus of interest to compute regions for the initial uncertainty such that the existence of a solution to the H_∞ problem is ensured. Moreover, it is often important to determine whether the filter designed over a finite-horizon converges to a stationary filter when the width of the interval tends to infinity. These issues have been studied in (Bolzern *et al.*, 1997), (Bolzern and Maroni, 1999), with reference to the problems of filtering and one-step prediction.

The more general issue of l-step ahead prediction (with $l > 1$) could be tackled by introducing delayed measurements and applying regular H_∞ filtering methods to an augmented system.

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However, this would result in an undesired increase in the dimension of the state and, in turn, of the underlying Riccati equation. Alternatively, the H_∞ multistep prediction problem has been recently addressed in (Hassibi, 1996) under a different viewpoint. It has been shown that the existence of a predictor hinges on the same Riccati equation as in the filtering problem (hereafter called main equation) and a second recursion with a further feasibility condition, that takes into account the absence of measurements along the prediction horizon. This recursion has the form of a Bounded Real Lemma Riccati equation and must be solved at any time instant for l steps with initial conditions that are provided by the solution of the main equation.

The first purpose of the present paper is to revisit the results of (Hassibi, 1996), by using a new formulation based on the "information" matrix ((Bolzern *et al.*, 1997), (Bolzern and Maroni, 1999)). In this way, the feasibility conditions can be expressed as simple constraints on the solution of the main Riccati equation, so preserving strict analogy with previous results on filtering and one-step prediction.

Secondly, the transient and asymptotic behaviour of the multistep predictor is analyzed. More precisely, sufficient conditions on the initial state uncertainty are worked out, which guarantee the existence of the predictor over an arbitrarily long time interval and its convergence to steady-state.

The paper is organized as follows. In Section 2, the results on multistep H_∞ prediction are summarized. Section 3 deals with the computation of a threshold for the initial condition of the main Riccati equation ensuring convergence, whereas in Section 4 an illustrative example is discussed. The paper ends with some concluding remarks.

2 Design of H_∞ predictors

Consider the following discrete-time linear system:

$$x_{k+1} = Ax_k + Bw_k \tag{1}$$
$$y_k = Cx_k + Dw_k \tag{2}$$
$$z_k = Lx_k \tag{3}$$

with initial condition x_0, and $k = 0, 1, 2....., N$, where $x_k \in \Re^n$ is the state vector, $y_k \in \Re^p$ is the measured output vector, $w_k \in \Re^s$ is a noise vector and $z_k \in \Re^m$ is the linear combination of the state to be estimated.

It will be tacitly assumed that $DB' = 0$ and $DD' = I$, which correspond to the usual hypothesis of uncorrelated process and measurement noises and normalized measurement noise. Furthermore the case of reversible systems is handled, namely it is assumed that det $A \neq 0$.

As for notation, $\|\cdot\|$ denotes the Euclidean vector norm. The symbols $[X]_+$, $[X]_-$ represent the positive and negative part of a symmetric matrix X, respectively. Namely, $X = [X]_+ + [X]_-$ where $[X]_+ \geq 0$, $[X]_- \leq 0$ and the positive eigenvalues of X coincide with the non-null eigenvalues of $[X]_+$.

Consider the finite time horizon $[0, N]$ and the cost function

$$J_N = \sum_{k=0}^{N} \left\| z_k - \widehat{z}_{k|k-l} \right\|^2 - \gamma^2 \left(\sum_{k=0}^{N} \|w_k\|^2 + x_0{}' \Pi_0 x_0 \right) \tag{4}$$

where $\widehat{z}_{k|k-l}$ is an estimate of z_k based on the output observations $\{y_0, y_1, ..., y_{k-l}\}$, namely $\widehat{z}_{k|k-l}$ represents an l-step ahead prediction, with $l \geq 1$. In (4), the scalar $\gamma > 0$ is the prescribed level of disturbance attenuation, and $\Pi_0 = \Pi_0' > 0$ is a given weighting matrix reflecting the uncertainty on the initial state x_0. The expression (4) can be viewed as the cost function of a 2-player game. The first player (i.e. the estimator) tries to minimize J_N whereas the second player (say nature) aims at maximizing it by properly selecting x_0 and w_k. When $J_N < 0$, the ratio of the prediction error energy to the energy of the disturbances (including the initial state x_0) is less than γ^2. It will be said that a predictor guarantees a prescribed attenuation level γ if it ensures that $J_N < 0$ for each finite nonzero $(x_0, \{w_0, w_1, ...w_N\})$.

This problem was first treated in (Yaesh and Shaked, 1991) for the case $l = 1$, while a general theory on multistep predictors has been addressed in (Hassibi, 1996) and (Maroni and Bolzern, 1998). The main result is summarized in the following theorem, whose proof can be found in (Maroni and Bolzern, 1998).

Theorem 1 *An l-step predictor guaranteeing a prescribed attenuation level γ exists if and only if there exist two sequences of matrices $\{S_k\}_{k=0}^{N-l}$ and $\{Q_m\}_{m=0}^{l-1}$ satisfying the recursions*

$$\begin{aligned} S_{k+1} &= \left(AS_k^{-1}A' + BB'\right)^{-1} + C'C - L'L\gamma^{-2} && (5)\\ S_0 &= \Pi_0^{-1} + C'C - L'L\gamma^{-2} \\ Q_{m-1} &= A'\Big[(L'L\gamma^{-2}+Q_m) \\ &\quad +(L'L\gamma^{-2}+Q_m)B\big(I-B'(L'L\gamma^{-2}+Q_m)B\big)^{-1}B'(L'L\gamma^{-2}+Q_m)\Big]A && (6)\\ Q_{l-1} &= 0 && (7)\end{aligned}$$

and the conditions

$$\begin{aligned} S_k &> Q_0 + C'C \quad , \quad k = 0, 1...N-l && (8)\\ 0 &< I - B'(L'L\gamma^{-2} + Q_m)B \quad , \quad m = 0, 1, ..., l-1 && (9)\end{aligned}$$

In that case, an admissible predictor is given by

$$\begin{aligned} \widehat{x}_{k+1} &= A\widehat{x}_k + K_k \begin{bmatrix} y_k - C\widehat{x}_k \\ \widehat{z}_{k|k-l} - L\widehat{x}_k \end{bmatrix} \quad , \quad \widehat{x}_0 = 0 && (10)\\ \widehat{x}^k_{k+m+1} &= A\widehat{x}^k_{k+m} + K^k_{k+m}(\widehat{z}_{k+m|k+m-l} - L\widehat{x}^k_{k+m}) \quad , \quad \widehat{x}^k_k = \widehat{x}_k && (11)\\ \widehat{z}_{k|k-l} &= \begin{cases} L\widehat{x}^k_{k+l-1} \quad , & k \geq l \\ 0 \quad , & k < l \end{cases} && (12)\end{aligned}$$

where

$$\begin{aligned} K_k &= AP_k \begin{bmatrix} C' & L' \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C \\ L \end{bmatrix} P_k \begin{bmatrix} C' & L' \end{bmatrix} \right)^{-1} && (13)\\ P_k &= (S_k - C'C + L'L\gamma^{-2})^{-1} && (14)\\ K^k_{k+m} &= A(S^k_{k+m} + L'L\gamma^{-2})^{-1}L'\left(-\gamma^2 I + L(S^k_{k+m} + L'L\gamma^{-2})^{-1}L'\right)^{-1} && (15)\\ S^k_{k+m+1} &= \left(A(S^k_{k+m})^{-1}A' + BB'\right)^{-1} - L'L\gamma^{-2} \quad , \quad S^k_k = S_k - C'C && (16)\\ m &= 0, 1, ..., l-1 && (17)\end{aligned}$$

■

Note that the gains K_k and K_{k+m}^k used in the predictor depend on the solutions of two coupled Riccati equations. The main eq. (5) is the standard Riccati equation arising in the H_∞ filtering problem (see (Yaesh and Shaked, 1991)), where S_k is the so-called "information matrix". Eq. (16) has the form of a Bounded Real Lemma equation and must be solved at any time instant for $l-1$ steps with an initial condition that is dictated by the solution S_k of the main equation (5).

According to (Hassibi, 1996), the existence of an admissible predictor is equivalent to the positiveness of the sequence $\{S_{k+m}^k\}_{k=0,1,\dots,N-l}^{m=0,1,\dots,l-1}$. Actually, it is proven in (Maroni and Bolzern, 1998) that, under (9), the sequence $\{Q_m\}_{m=0}^{l-1}$ evaluated recursively from (6), (7) is positive semidefinite, and the feasibility condition (8) entails that the solution S_{k+m}^k of (16) remains positive up to $m = l-1$. It should be also remarked that the check on (9) and the computation of Q_0 through (6), (7) can be performed in advance, so that the existence of an H_∞ predictor guaranteeing the prescribed attenuation level is eventually related to the fact that the solution S_k of the main equation (5) lies uniformly above a lower threshold given by $Q_0 + C'C \geq 0$.

When one-step-ahead prediction is concerned, i.e. $l = 1$, it turns out that $Q_0 = 0$ and the feasibility condition (8) reduces to $S_k > C'C$, which is the already known condition studied in (Yaesh and Shaked, 1991) and (Bolzern and Maroni, 1999). As the prediction horizon l increases, the feasibility condition becomes more and more stringent since Q_0 monotonically increases with l. This is in accordance with intuition: the larger the prediction horizon is, the more difficult is to maintain the same disturbance attenuation level.

3 Sufficient conditions for convergence

The predictors discussed in the previous section are in general time-varying, even when the original system (1)-(3) is stationary, because they are designed on the basis of a finite-horizon criterion. Regarding the asymptotic properties of such estimators, two main questions are interesting:

1. given a look-ahead interval l and an attenuation level γ, does the predictor exist over an arbitrarily long time horizon?
2. in the affirmative, as the horizon width tends to infinity, does the predictor converge to a steady-state configuration?

In order to answer these questions, it is fundamental to analyze the behaviour of the solution of the Riccati equation (5) as a function of γ and the initial weighting matrix Π_0. It is well recognized (see e.g. (Bolzern *et al.*, 1997)) that H_∞-type Riccati equations may exhibit properties that are completely different from those of the standard Riccati equations of Kalman filtering. For instance, the solution may lose positiveness at a finite instant even though the initial condition is positive definite.

The tool used in the present paper to address the issues posed above is a technical result recently derived in (Bolzern and Maroni, 1999), which states sufficient conditions guaranteeing that the solution of an H_∞-type difference Riccati equation is always above a given threshold. In fact, it has been observed in Section 2 that the feasibility condition of H_∞ multistep predictors can be expressed in terms of a lower bound on the solution S_k of the Riccati equation (5).

Before formulating the main result, consider the algebraic Riccati equation associated with (5), i.e.

$$S = \left(AS^{-1}A' + BB'\right)^{-1} + C'C - L'L\gamma^{-2} \tag{18}$$

and assume that it admits a solution S_S which is stabilizing, namely it is such that all the eigenvalues of $\widehat{A} = (A^{-1})' \left(I + S_S A^{-1} B B' (A^{-1})'\right)^{-1}$ lie inside the unit circle. Moreover, suppose that $S_S > Q_0 + C'C$, where Q_0 has been defined in Section 2 through (6), (7). In this case, S_S will be called "feasible".

Furthermore, introduce the following notation:

$$\begin{aligned}
\Psi &= A^{-1}B\left(I + B'(A^{-1})'S_S A^{-1}B\right)^{-1} B'(A^{-1})' \\
\Theta &= (\widehat{A}^{-1})'\left(\left(S_S - Q_0 - C'C\right)^{-1} - \Psi\right)\widehat{A}^{-1} - \left(S_S - Q_0 - C'C\right)^{-1}
\end{aligned}$$

Finally, consider the Lyapunov equation

$$X = \widehat{A}'X\widehat{A} + [\Theta]_-$$

and let

$$\overline{S_0} = S_S - \left((S_S - Q_0 - C'C)^{-1} - \widehat{A}'X\widehat{A}\right)^{-1} \tag{19}$$

Then, a sufficient condition for the existence and convergence of the l-step ahead predictor is illustrated in the following theorem.

Theorem 2 *Suppose that the pair (A, B) is reachable, and the stabilizing solution of (18) is feasible. Then, the condition*

$$S_0 > \overline{S_0}$$

implies that

(i) a predictor guaranteeing a prescribed attenuation level γ exists over an interval of arbitrary length N, and one possible predictor is that given by (10)-(17);

(ii) as $N \to \infty$, the time-varying predictor (10)-(17) tends to the stationary one described by

$$\begin{aligned}
\widehat{x}_{k+1} &= A\widehat{x}_k + \overline{K}\begin{bmatrix} y_k - C\widehat{x}_k \\ \widehat{z}_{k|k-l} - L\widehat{x}_k \end{bmatrix} \qquad , \qquad \widehat{x}_0 = 0 \\
\widehat{x}^k_{k+m+1} &= A\widehat{x}^k_{k+m} + \overline{K}_m(\widehat{z}_{k+m|k+m-l} - L\widehat{x}^k_{k+m}) \qquad , \qquad \widehat{x}^k_k = \widehat{x}_k \\
\widehat{z}_{k|k-l} &= L\widehat{x}^k_{k+l-1}
\end{aligned}$$

where

$$\begin{aligned}
\overline{K} &= AP_S\begin{bmatrix} C' & L' \end{bmatrix}\left(\begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C \\ L \end{bmatrix} P_S \begin{bmatrix} C' & L' \end{bmatrix}\right)^{-1} \\
P_S &= (S_S - C'C + L'L\gamma^{-2})^{-1} \\
\overline{K}_m &= A(\overline{S}_m + L'L\gamma^{-2})^{-1}L'\left(-\gamma^2 I + L(\overline{S}_m + L'L\gamma^{-2})^{-1}L'\right)^{-1} \\
\overline{S}_{m+1} &= \left(A(\overline{S}_m)^{-1}A' + BB'\right)^{-1} - L'L\gamma^{-2} \qquad , \qquad \overline{S}_0 = S_S - C'C \\
m &= 0, 1, ..., l-1
\end{aligned}$$

■

The proof of this result easily follows from the application of Lemma 2 of (Bolzern and Maroni, 1999). The assumption on the reachability of (A, B) is of rather technical nature and entails that S_S is positive definite and eq. (18) admits also an antistabilizing solution. Relaxation of such a hypothesis appears difficult at the moment, since most available results on H_∞-type Riccati equations require the existence of both the stabilizing and antistabilizing solutions. The same applies to the assumption on the nonsingularity of A.

It is worth noticing that the condition expressed in Theorem 2 is in general only sufficient, but it becomes also necessary in three particular cases, corresponding to the Kalman prediction problem ($\gamma \to \infty$), open-loop estimation ($C = 0$) and first order systems ($n = 1$). Moreover, in the case $l = 1$, the condition of (Bolzern and Maroni, 1999) is immediately recovered.

As a final interpretation, observe that the condition $S_0 > \overline{S_0}$ of Theorem 2 requires that the initial "information matrix" S_0 should not be "too small" (or equivalently, the initial uncertainty measured by Π_0 should not be "too large"). In particular, it can be shown that $\overline{S_0} < S_S$. Thus, convergence may be reached even for some initializations of the Riccati equation (5) which are below the stabilizing solution S_S.

4 An illustrative example

Consider the system (1)-(3) characterized by the matrices

$$\begin{aligned} A &= \begin{bmatrix} 1.5 & -0.5 \\ 1 & 0 \end{bmatrix} \quad , \quad B = \begin{bmatrix} -0.4 & 0 \\ 0.6 & 0 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ D &= \begin{bmatrix} 0 & 1 \end{bmatrix} \quad , \quad L = \begin{bmatrix} 1 & 1 \end{bmatrix} \end{aligned}$$

The minimum levels of attenuation $\gamma_{\min}$ guaranteeing the existence of a stabilizing solution $S_S > Q_0 + C'C$ to the algebraic Riccati equation (18) are reported in Table 1 for different values of the prediction horizon l.

l	1	2	3	4	5	6
$\gamma_{\min}$	2.12	3.16	4.59	6.20	7.89	9.59

Table 1

Now, consider $l = 6$ and $\gamma = 10$. By computing the matrix Q_0 defined in (6), (7) and evaluating (19), one obtains

$$Q_0 + C'C = \begin{bmatrix} 1.8346 & -0.3673 \\ -0.3673 & 0.1664 \end{bmatrix} \quad , \quad \overline{S_0} = \begin{bmatrix} 1.8444 & -0.4308 \\ -0.4308 & 0.6148 \end{bmatrix}$$

Moreover, the conditions (9) are satisfied. Then, according to Theorem 2, the existence of the H_∞ predictor along any time interval is ensured whenever $S_0 > \overline{S_0}$. Figures 1, 2 show the evolution of the diagonal elements and the determinant of $S_k - C'C$ and S^k_{k+m} when $S_0 = \overline{S_0} + 0.1I$. It is apparent that feasibility is preserved and convergence to steady-state is eventually reached. Conversely, Figures 3, 4 display the same quantities when

$$S_0 = \begin{bmatrix} 2.3310 & -0.3410 \\ -0.3410 & 0.4750 \end{bmatrix} \not> \overline{S_0}$$

Note that, in this case, although $S_0 > Q_0 + C'C$, $\det(S^1_6)$ is negative, namely S^1_6 is not positive semidefinite. So, there is no predictor guaranteeing the prescribed attenuation level over an arbitrarily long time interval.

5 Concluding remarks

In this paper, the problem of discrete-time H_∞ multistep prediction has been analyzed. A first result has concerned the reformulation of necessary and sufficient conditions for the existence of an l-step-ahead predictor guaranteeing a prescribed attenuation level, in terms of a lower bound on the solution of a suitable Riccati equation. Secondly, a method to find an admissible region for the initial state uncertainty has been developed. If the initial condition of the relevant Riccati equation lies into the given region, then the solution of the finite-horizon prediction problem exists for any width of the horizon and, as this width tends to infinity, the time-varying predictor converges to a stationary one. Further study is needed to clarify how far from necessity these sufficient conditions are, and to derive similar results in a more general setting (including nonreversible systems). Finally, it does not appear easy to compare the bounds $\overline{S_0}$ obtained for different values of the look-ahead horizon l. Although common-sense suggests that they are increasing with l, no explicit proof is so far available.

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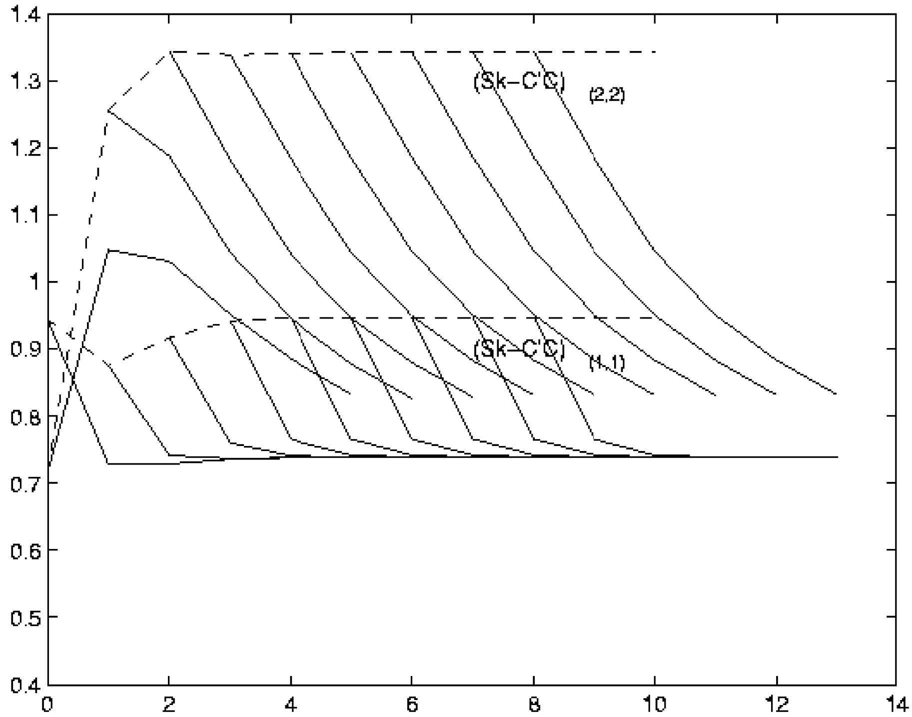

Figure 1: Time evolution of $(S_k - C'C)_{(i,i)}$ (dashed) and $(S^k_{k+m})_{(i,i)}$ (solid) for $S_0 > \overline{S_0}$. $i = 1, 2$

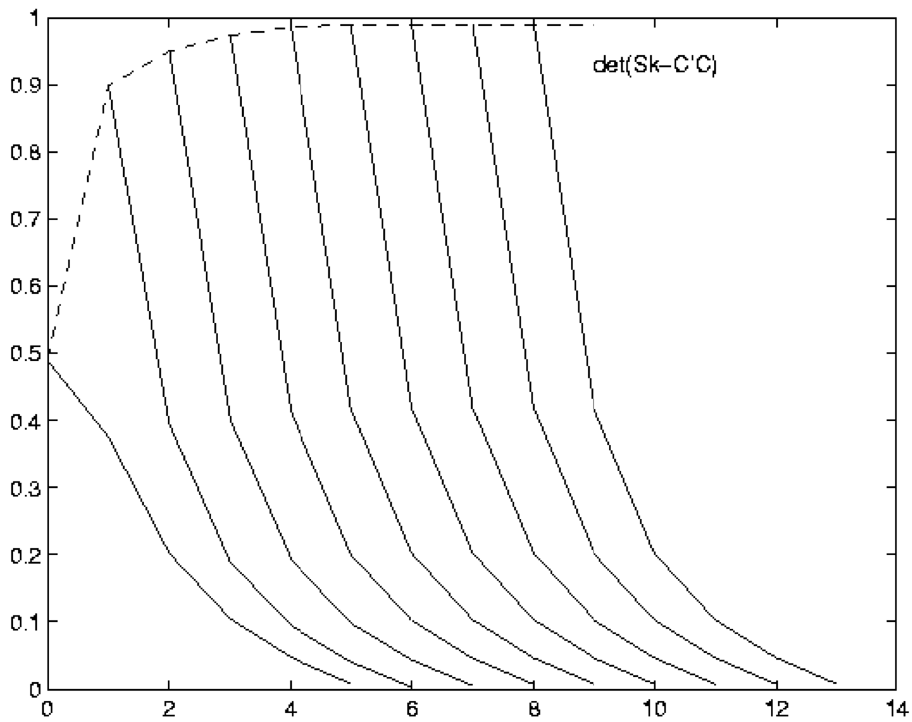

Figure 2: Time evolution of $\det(S_k - C'C)$ (dashed) and $\det(S^k_{k+m})$ (solid) for $S_0 > \overline{S_0}$. $i = 1, 2$

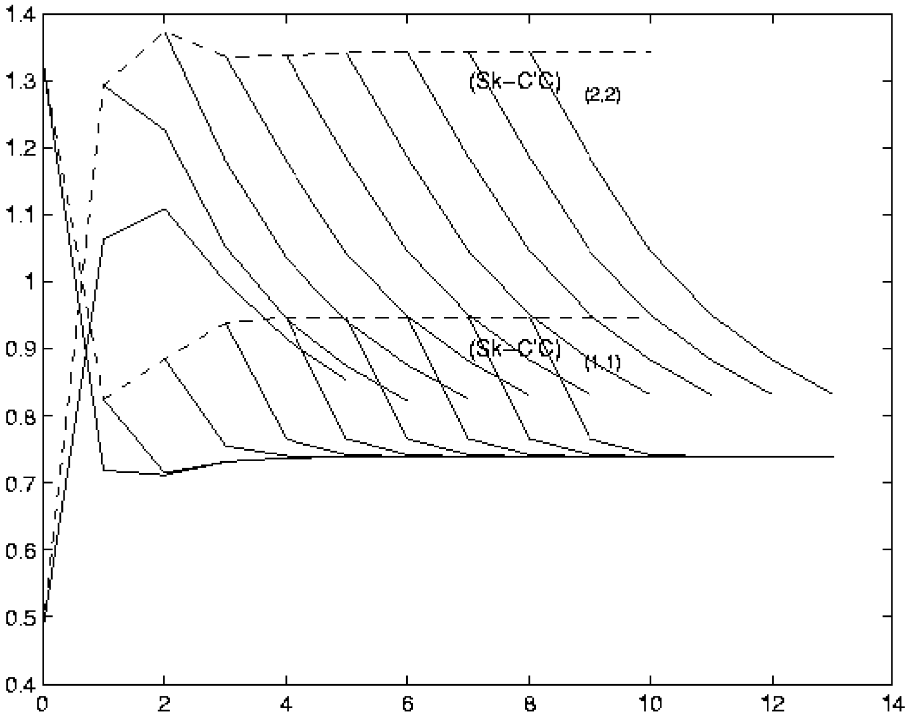

Figure 3: Time evolution of $(S_k - C'C)_{(i,i)}$ (dashed) and $(S^k_{k+m})_{(i,i)}$ (solid) for $S_0 \not> \overline{S_0}$ $i = 1, 2$

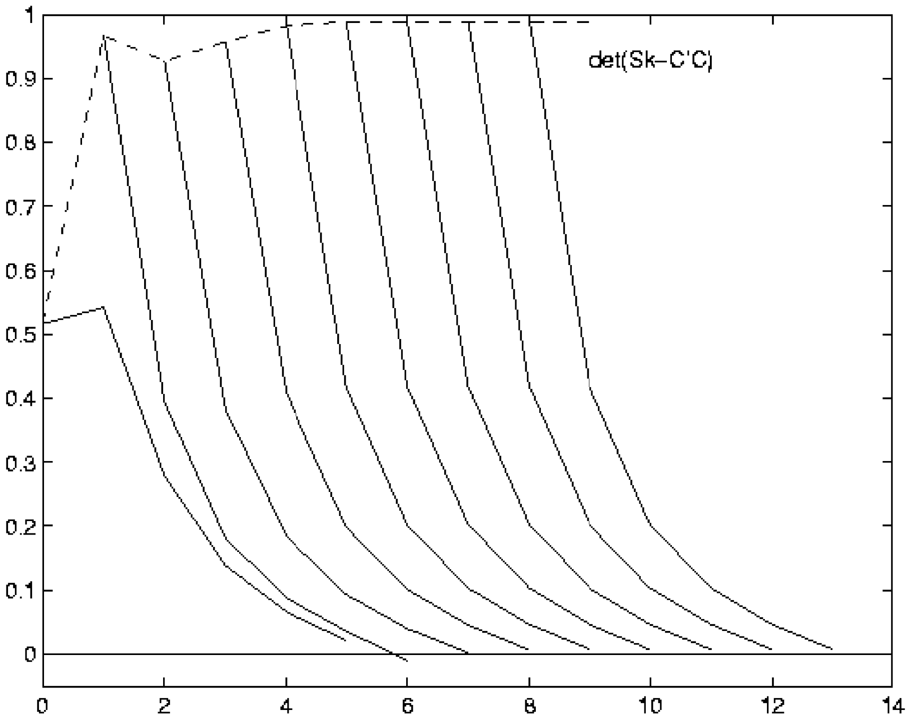

Figure 4: Time evolution of $\det(S_k - C'C)$ (dashed) and $\det(S^k_{k+m})$ (solid) for $S_0 \not> \overline{S_0}$. $i = 1, 2$