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An Algorithm for Control System Loop Gain Identification

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Abstract

The identification of a linear discrete-time control system's loop gain is addressed. The classical Kalman filter theory for linear control systems is extended and the control system's state and loop gain are jointly estimated. Explicit formulae for the loop gain's (unbiased) estimate and estimation error covariance are derived.

1 Introduction

Feedback, and the reliance on high gain action, are used to mitigate the ill effects of the unstructured environment where the controlled plant is operating. At the same time, the benefits of feedback control, and, in particular, high gain action, are severely circumscribed by sensor noise (Houpis and Pachter, 1997). System identification entails the estimation of a control system's parameters from measurements on the system's inputs and outputs (Ljung, 1987); as such, system identification lends itsef well to integration into modern feedback control synthesis, because no additional hardware, i.e., sensors or actuators, above and beyond the components used in conventional feedback control, are required. The incorporation of system identification into feedback control law synthesis calls for additional signal processing, however, a reduction in plant uncertainty is achieved. Therefore, lower gains in the feedback control law are possible. Hence, there is a strong incentive for the incorporation of system identification into control law synthesis and the employment of indirect adaptive control. Unfortunately, system identification, which entails the estimation of all the plant's parameters, resides in the realm of nonlinear filtering. It is however recognized that of paramount importance in control law design is accurate information on the control matrix parameters, e.g., in flight control one then refers to the "control derivatives" (Chandler *et al.*, 1998). Now, in linear control systems, and provided that the dynamics matrix is known, the exclusive estimation of the parameters of the control matrix only is reducible to a problem in linear regression, and therefore, is amenable to analysis using linear mathematics. Hence, a rigorous, i.e., an unbiased, estimate of the parameters of the control matrix can be obtained. In this paper a simplified version of this problem is addressed and an algorithm for the estimation of a control system's critical loop gain parameter is developed. The inclusion of a "forgetting factor" into this basic algorithm will afford on-line operation. Thus, a mechanization of an indirect adaptive control system which incorporats the loop gain identification algorithm developed in this paper is possible.

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1.1 Problem Statement

The linear discrete-time control system is considered,

$$x_{k+1} = Ax_k + Kbu_k + \Gamma w_k, \quad k = 0, 1, ..., N-1, \quad E(w_k w_k^T) = Q \tag{1}$$
$$x_o = N(\overline{x}_o, P_{ox}) \tag{2}$$
$$K = N(K_o, P_{o_K}) \tag{3}$$
$$y_{k+1} = Cx_{k+1} \tag{4}$$
$$z_{k+1} = y_{k+1} + \upsilon_{k+1}, \quad E(\upsilon_{k+1}\upsilon_{k+1}^T) = R \tag{5}$$

In the special case of a single output, the measurement equation (5) is

$$z_{k+1} = y_{k+1} + \upsilon_{k+1}, \qquad \upsilon_{k+1} = N(0, \sigma^2) \tag{6}$$

The control system's state $x_k \in \Re^n$. The dynamics matrix A, the control matrix b, the observation matrix C and the vector Γ are known. The respective process and sensor noise intensities, Q and R (or σ) are also known. In addition, the prior information specified in eqs. (2) and (3) is provided. It is required to identify the scalar loop gain K, given the input sequence $u_0, u_1, ..., u_{N-1}$ and the measurements record $z_1, z_2, ..., z_N$. An stimate of the control system's state is also obtained.

Here, the loop gain parameter K is treated as an unknown and an algorithm for the identification of K is developed. Hence, e.g., in the flight control application, one can now handle control surface failure: Obviously, for an unfailed plant (aircraft) the loop gain $K = 1$ (by definition), viz., K is unity, until a failure at time t_f forces $K < 1$.

In this paper the identification of the control system's loop gain K is undertaken. The classical Kalman filter theory for linear control systems (Maybeck, 1982) is extended and the control system's state and loop gain are jointly estimated. Explicit formulae for the loop gain's (unbiased) estimate and estimation error covariance are derived. The state estimate and the covariance of the state estimation error are also obtained. The main develoment is undertaken in Section 2 and the results are summarized and discussed in Section 3. Concluding remarks are made in Section 4.

2 Recursive System Identification Algorithm

Since the unknown loop gain K is a constant, we augment the dynamics as follows.

$$K_{k+1} = K_k \tag{7}$$

Hence, the augmented state dynamics evolve in $\Re^{n+1}$ and are

$$\begin{pmatrix} x_{k+1} \\ K_{k+1} \end{pmatrix} = \begin{pmatrix} A & u_k b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_k \\ K_k \end{pmatrix} + \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} w_k \tag{8}$$

and the measurement equation is

$$z_{k+1} = \begin{pmatrix} C & \vdots & 0 \end{pmatrix} \begin{pmatrix} x_{k+1} \\ K_{k+1} \end{pmatrix} + \upsilon_{k+1} \tag{9}$$

The prior information at time k is

$$\begin{pmatrix} x_k \\ K_k \end{pmatrix} = N\left(\begin{pmatrix} \hat{x}_k \\ \hat{K}_k \end{pmatrix}, \; P_{k_{(x,K)}} \right) \tag{10}$$

where

$$P_{k_{(x,K)}} = \begin{pmatrix} P_{k_{xx}} & p_{k_{xK}} \\ p_{k_{xK}}^T & p_{k_{KK}} \end{pmatrix} \tag{11}$$

is the estimation error covariance matrix.

Note: The estimation error covariance matrix is partitioned as follows:

$$P_{k_{xx}} \in \Re^{n\times n}, \quad p_{k_{xK}} \in \Re^{n}, \quad p_{k_{KK}} \in \Re^{1}.$$

Hence, before the z_{k+1} measurement is recorded, the augmented state

$$\begin{aligned}
\begin{pmatrix} x_{k+1} \\ K_{k+1} \end{pmatrix} &= N\left(\begin{pmatrix} A & u_k b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{x}_k \\ \hat{K}_k \end{pmatrix}, \begin{pmatrix} A & u_k b \\ 0 & 1 \end{pmatrix} P_{k_{(x,K)}} \begin{pmatrix} A^T & 0 \\ u_k b^T & 1 \end{pmatrix} + \begin{pmatrix} \Gamma Q \Gamma^T & 0 \\ 0 & 0 \end{pmatrix}\right) \\
&= N\left(\begin{pmatrix} A\hat{x}_k + \hat{K}_k b u_k \\ \hat{K}_k \end{pmatrix}, \right. \\
&\qquad \left. \left(\begin{array}{c:c} \begin{array}{c} AP_{k_{xx}}A^T + u_k(AP_{k_{xK}}b^T + \\ bp_{k_{xK}}^T A^T) + u_k^2 p_{k_{KK}} bb^T + \Gamma Q \Gamma^T \end{array} & Ap_{k_{xK}} + u_k p_{k_{KK}} b \\ \hdashline p_{k_{xK}}^T A^T + u_k p_{k_{KK}} b^T & p_{k_{KK}} \end{array}\right)\right)
\end{aligned}$$

Next, apply the Bayesian estimation formula

$$\hat{x}^+ = \hat{x}^- + \mathcal{K}\left[z - H\hat{x}\right], \tag{12}$$

viz.,

$$\begin{aligned}
\begin{pmatrix} \hat{x}_{k+1} \\ \hat{K}_{k+1} \end{pmatrix} &= \begin{pmatrix} A\hat{x}_k + \hat{K}_k b u_k \\ \hat{K}_k \end{pmatrix} + \mathcal{K}\left(z_{k+1} - \begin{pmatrix} C & \vdots & 0 \end{pmatrix}\begin{pmatrix} A\hat{x}_k + \hat{K}_k b u_k \\ \hat{K}_k \end{pmatrix}\right) \\
&= \begin{pmatrix} A\hat{x}_k + \hat{K}_k b u_k \\ \hat{K}_k \end{pmatrix} + \mathcal{K}\left(z_{k+1} - CA\hat{x}_k - u_k \hat{K}_k C b\right)
\end{aligned}$$

where the Kalman gain

$$\mathcal{K} = \left(\begin{array}{ccc} \begin{array}{c} AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T + bp_{k_{xK}}^T A^T) \\ +u_k^2 p_{k_{KK}} bb^T + \Gamma Q \Gamma^T \end{array} & \vdots & Ap_{k_{xK}} + u_k p_{k_{KK}} b \\ \cdots\cdots\cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ p_{k_{xK}}^T A^T + u_k p_{k_{KK}} b^T & \vdots & p_{k_{KK}} \end{array} \right) \times$$

$$\left(\begin{array}{c} C^T \\ 0 \end{array} \right) \times \{CAP_{k_{xx}}A^T C^T + u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T]$$
$$+ u_k^2 p_{k_{KK}}(Cb)(Cb)^T + C\Gamma Q \Gamma^T C^T + R\}^{-1}$$

$$= \left(\begin{array}{c} AP_{k_{xx}}A^T C^T + u_k[Ap_{k_{xK}}(Cb)^T + b(CAp_{k_{xK}})^T] \\ +u_k^2 p_{k_{KK}} b(Cb)^T + \Gamma Q \Gamma^T C^T \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (CAp_{k_{xK}})^T + u_k p_{k_{KK}}(Cb)^T \end{array} \right) \times$$
$$\{CAP_{k_{xx}}A^T C^T + u_k[CAp_{k_{xK}}(Cb)^T + Cb(CAp_{k_{xK}})^T]$$
$$+ u_k^2 p_{k_{KK}}(Cb)(Cb)^T + C\Gamma Q \Gamma^T C^T + R\}^{-1} \tag{13}$$

Finally,

$$P_{k+1_{(x,K)}} = P_{k_{(x,K)}} - \mathcal{K} H P_{k_{(x,K)}} \tag{14}$$

Hence, we calculate

$$P_{k+1_{(x,K)}} = \left(\begin{array}{ccc} \begin{array}{c} AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T + \\ bp_{k_{xK}}^T A^T) + u_k^2 p_{k_{KK}} bb^T + \Gamma Q \Gamma^T \end{array} & \vdots & Ap_{k_{xK}} + u_k p_{k_{KK}} b \\ \cdots\cdots\cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ p_{k_{xK}}^T A^T + u_k p_{k_{KK}} b^T & \vdots & p_{k_{KK}} \end{array} \right) -$$

$$\left(\begin{array}{cc} \begin{array}{c} \{AP_{k_{xx}}A^T C^T + u_k[Ap_{k_{xK}}(Cb)^T + b(CAp_{k_{xK}})^T] \\ +u_k^2 p_{k_{KK}} b(Cb)^T + \Gamma Q \Gamma^T C^T\} \times \{CAP_{k_{xx}}A^T C^T \\ +u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] + u_k^2 p_{k_{KK}}(Cb)(Cb)^T \\ +C\Gamma Q \Gamma^T C^T + R\}^{-1} \times [CAP_{k_{xx}}A^T + u_k(CAp_{k_{xK}}b^T \\ +Cbp_{k_{xK}}^T A^T) + u_k^2 p_{k_{KK}} Cbb^T + C\Gamma Q \Gamma^T] \end{array} & \vdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \vdots \\ \begin{array}{c} [(CAp_{k_{xK}})^T + u_k p_{k_{KK}}(Cb)^T] \times \{CAP_{k_{xx}}A^T C^T \\ +u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] + u_k^2 p_{k_{KK}}(Cb)(Cb)^T \\ +C\Gamma Q \Gamma^T C^T + R\}^{-1} \times [CAP_{k_{xx}}A^T + u_k(CAp_{k_{xK}}b^T \\ +Cbp_{k_{xK}}^T A^T) + u_k^2 p_{k_{KK}} Cbb^T + C\Gamma Q \Gamma^T] \end{array} & \vdots \end{array} \right.$$

$$
\left.\begin{array}{c|c}
\vdots & \begin{array}{c} \{AP_{k_{xx}}A^TC^T + u_k[Ap_{k_{xK}}(Cb)^T + b(CAp_{k_{xK}})^T] \\ +u_k^2 p_{k_{KK}} b(Cb)^T + \Gamma Q \Gamma^T C^T\} \times \{CAP_{k_{xx}}A^TC^T \\ +u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] + u_k^2 p_{k_{KK}}(Cb)(Cb)^T \\ +C\Gamma Q \Gamma^T C^T + R\}^{-1} \times (CAp_{k_{xK}} + u_k p_{k_{KK}}(Cb)) \end{array} \\
\vdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\vdots & \begin{array}{c} [(CAp_{k_{xK}})^T + u_k p_{k_{KK}}(Cb)^T] \times \{CAP_{k_{xx}}A^TC^T \\ +u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] + u_k^2 p_{k_{KK}}(Cb)(Cb)^T \\ +C\Gamma Q \Gamma^T C^T + R\}^{-1} \times (CAp_{k_{xK}} + u_k p_{k_{KK}}(Cb)) \end{array}
\end{array}\right) \qquad (15)
$$

Thus,

$$
\begin{aligned}
P_{k+1_{xx}} = {} & [AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T + bp_{k_{xK}}^T A^T) + u_k^2 p_{k_{KK}} bb^T \\
& +\Gamma Q \Gamma^T]\{[AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T + bp_{k_{xK}}^T A^T) \\
& + u_k^2 p_{k_{KK}} bb^T + \Gamma Q \Gamma^T]^{-1} - C^T\{CAP_{k_{xx}}A^TC^T \\
& +u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] + u_k^2 p_{k_{KK}}(Cb)(Cb)^T \\
& +C\Gamma Q \Gamma^T C^T + R\}^{-1}C\}[AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T + bp_{k_{xK}}^T A^T) \\
& +u_k^2 p_{k_{KK}} bb^T + \Gamma Q \Gamma^T]
\end{aligned} \qquad (16)
$$

Next, recall the complete Matrix Inversion Lemma (MIL):

Lemma 1 *Assume relevant matrices are compatible and invertible. Then*

$$
(A_1 - A_2 A_4^{-1} A_3)^{-1} = A_1^{-1} + A_1^{-1} A_2 (A_4 - A_3 A_1^{-1} A_2)^{-1} A_3 A_1^{-1} \qquad (17)
$$

Applying the MIL to the expression in the outer curly brackets from eq. (16), viz,

$$
\begin{aligned}
& \{[AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T + bp_{k_{xK}}^T A^T) + u_k^2 p_{k_{KK}} bb^T + \Gamma Q \Gamma^T]^{-1} \\
& - C^T\{CAP_{k_{xx}}A^TC^T + u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] \\
& + u_k^2 p_{k_{KK}}(Cb)(Cb)^T + C\Gamma Q \Gamma^T C^T + R\}^{-1}C\}^{-1}
\end{aligned}
$$

where we set

$$
\begin{aligned}
& A_1 = [AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T + bp_{k_{xK}}^T A^T) + u_k^2 p_{k_{KK}} bb^T + \Gamma Q \Gamma^T]^{-1} \\
& A_2 = C^T \\
& A_3 = C \\
& A_4 = \{CAP_{k_{xx}}A^TC^T + u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] \\
& \qquad + u_k^2 p_{k_{KK}}(Cb)(Cb)^T + C\Gamma Q \Gamma^T C^T + R\}\ ,
\end{aligned}
$$

we obtain

$\{[AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T + bp_{k_{xK}}^T A^T) + u_k^2 p_{k_{KK}} bb^T + \Gamma Q \Gamma^T]^{-1} - C^T\{CAP_{k_{xx}}A^T C^T$
$+u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] + u_k^2 p_{k_{KK}}(Cb)(Cb)^T + C\Gamma Q\Gamma^T C^T + R\}^{-1}C\}^{-1}$

$= [AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T + bp_{k_{xK}}^T A^T) + u_k^2 p_{k_{KK}} bb^T + \Gamma Q\Gamma^T] + [AP_{k_{xx}}A^T$
$+u_k(Ap_{k_{xK}}b^T + bp_{k_{xK}}^T A^T) + u_k^2 p_{k_{KK}} bb^T + \Gamma Q\Gamma^T]C^T\{\{CAP_{k_{xx}}A^T C^T + u_k[CAp_{k_{xK}}(Cb)^T$
$+(Cb)(CAp_{k_{xK}})^T] + u_k^2 p_{k_{KK}}(Cb)(Cb)^T + C\Gamma Q\Gamma^T C^T + R\} - C[AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T$
$+bp_{k_{xK}}^T A^T) + u_k^2 p_{k_{KK}} bb^T + \Gamma Q\Gamma^T]C^T\}^{-1}C[AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T + bp_{k_{xK}}^T A^T)$
$+u_k^2 p_{k_{KK}} bb^T + \Gamma Q\Gamma^T]$

Reducing the above gives

$[AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T + bp_{k_{xK}}^T A^T) + u_k^2 p_{k_{KK}} bb^T + \Gamma Q\Gamma^T]\{[AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T$
$+bp_{k_{xK}}^T A^T) + u_k^2 p_{k_{KK}} bb^T + \Gamma Q\Gamma^T]^{-1} + C^T R^{-1} C\}[AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T + bp_{k_{xK}}^T A^T)$
$+u_k^2 p_{k_{KK}} bb^T + \Gamma Q\Gamma^T]$

Hence, eq. (16) can now be reduced to:

$$\begin{aligned} P_{k+1_{xx}} = \{[&AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T + bp_{k_{xK}}^T A^T) \\ &+ u_k^2 p_{k_{KK}} bb^T + \Gamma Q\Gamma^T]^{-1} + C^T R^{-1} C\}^{-1} \end{aligned} \tag{18}$$

In addition,

$$\begin{aligned} p_{k+1_{KK}} = p_{k_{KK}} &- [(CAp_{k_{xK}})^T + u_k p_{k_{KK}}(Cb)^T]\{CAP_{k_{xx}}A^T C^T \\ &+ u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] + u_k^2 p_{k_{KK}}(Cb)(Cb)^T \\ &+ C\Gamma Q\Gamma^T C^T + R\}^{-1}(CAp_{k_{xK}} + u_k p_{k_{KK}}(Cb)) \end{aligned} \tag{19}$$

and

$$\begin{aligned} p_{k+1_{xK}} = Ap_{k_{xK}} &+ u_k p_{k_{KK}} b - \{AP_{k_{xx}}A^T C^T + u_k[Ap_{k_{xK}}(Cb)^T \\ &+ b(CAP_{k_{xK}})^T] + u_k^2 p_{k_{KK}} b(Cb)^T + \Gamma Q\Gamma^T C^T\} \times \\ &\{CAP_{k_{xx}}A^T C^T + u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] \\ &+ u_k^2 p_{k_{KK}}(Cb)(Cb)^T + C\Gamma Q\Gamma^T C^T + R\}^{-1}(CAp_{k_{xK}} \\ &+ u_k p_{k_{KK}}(Cb)) \end{aligned} \tag{20}$$

We also partition the Kalman gain vector as follows

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_x \\ \mathcal{K}_K \end{pmatrix} \tag{21}$$

where

$$\begin{aligned} \mathcal{K}_x = \{&AP_{k_{xx}}A^T C^T + u_k[Ap_{k_{xK}}(Cb)^T + b(CAp_{k_{xK}})^T] \\ &+ u_k^2 p_{k_{KK}} b(Cb)^T + \Gamma Q\Gamma^T C^T\} \times CAP_{k_{xx}}A^T C^T \\ &+ u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] + u_k^2 p_{k_{KK}}(Cb)(Cb)^T \\ &+ C\Gamma Q\Gamma^T C^T + R\}^{-1} \end{aligned} \tag{22}$$

and

$$\begin{aligned}\mathcal{K}_K = {} & [(CAp_{k_{xK}})^T + u_k p_{k_{KK}}(Cb)^T]\{CAP_{k_{xx}}A^T C^T \\ & + u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] + u_k^2 p_{k_{KK}}(Cb)(Cb)^T \\ & + C\Gamma Q\Gamma^T C^T + R\}^{-1} \end{aligned} \tag{23}$$

Hence,

$$\hat{x}_{k+1} = A\hat{x}_k + \hat{K}_k b u_k + \mathcal{K}_x(z_{k+1} - CA\hat{x}_k - \hat{K}_k C b u_k) \tag{24}$$
$$\hat{K}_{k+1} = \hat{K}_k + \mathcal{K}_K(z_{k+1} - CA\hat{x}_k - \hat{K}_k C b u_k) \tag{25}$$

3 Results

The above derivations are summarized in the following.

Theorem 2 *Consider the following linear estimation problem: The linear dynamical system is*

$$x_{k+1} = Ax_k + Kbu_k + \Gamma w_k, \quad k = 0, 1, ..., N-1, \quad E(w_k w_k^T) = Q \tag{26}$$

The prior information is

$$x_o = N(\bar{x}_o, P_{o_x}) \tag{27}$$
$$K = N(K_o, P_{o_K}) \tag{28}$$

The output signal

$$y_{k+1} = Cx_{k+1} \tag{29}$$

and the observation equation is

$$z_{k+1} = y_{k+1} + v_{k+1}, \quad E(v_{k+1} v_{k+1}^T) = R. \tag{30}$$

The martices A, b, C and Γ are known. The respective process noise and measurement noise covariance matrices, Q and R, are also known.

Denote by $\hat{x}_k$ and $\hat{K}_k$ the respective estimates of the state at time k, x_k, and the loop gain, K, given the measurements record $z_1, ..., z_k$, the input sequence $u_0, ..., u_{k-1}$, and the prior information on x_o and K. The covariance of the estimation error of the $\begin{pmatrix} x_k \\ K \end{pmatrix}$ vector is denoted by the partitioned matrix $P_k = \begin{pmatrix} P_{k_{xx}} & p_{k_{xK}} \\ p_{k_{xK}}^T & p_{k_{KK}} \end{pmatrix}$

Initially, set

$$\hat{x}_o = \bar{x}_o, \qquad \hat{K}_o = K_o, \qquad P_{o_{xx}} = P_{o_x}, \qquad p_{o_{KK}} = P_{o_K}, \qquad p_{o_{xK}} = 0.$$

Then for $k = 0, 1, ..., N-1$, the state and gain estimates are

$$\hat{x}_{k+1} = A\hat{x}_k + \hat{K}_k b u_k + \mathcal{K}_x(z_{k+1} - CA\hat{x}_k - \hat{K}_k C b u_k) \tag{31}$$
$$\hat{K}_{k+1} = \hat{K}_k + \mathcal{K}_K(z_{k+1} - CA\hat{x}_k - \hat{K}_k C b u_k) \tag{32}$$

where the Kalman gains

$$\begin{aligned}\mathcal{K}_x = \{&AP_{k_{xx}}A^TC^T + u_k[Ap_{k_{xK}}(Cb)^T + b(CAp_{k_{xK}})^T] \\ &+ u_k^2 p_{k_{KK}} b(Cb)^T + \Gamma Q \Gamma^T C^T\} \times \{CAP_{k_{xx}}A^TC^T \\ &+ u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] \\ &+ u_k^2 p_{k_{KK}}(Cb)(Cb)^T + C\Gamma Q \Gamma^T C^T + R\}^{-1}\end{aligned} \tag{33}$$

and

$$\begin{aligned}\mathcal{K}_K = [&(CAp_{k_{xK}})^T + u_k p_{k_{KK}}(Cb)^T] \times \{CAP_{k_{xx}}A^TC^T \\ &+ u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] \\ &+ u_k^2 p_{k_{KK}}(Cb)(Cb)^T + C\Gamma Q \Gamma^T C^T + R\}^{-1}\end{aligned} \tag{34}$$

Furthermore, the estimation error covariances are

$$\begin{aligned}P_{k+1_{xx}} = \{&[AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T + bp_{k_{xK}}^T A^T) \\ &+ u_k^2 p_{k_{KK}} bb^T + \Gamma Q \Gamma^T]^{-1} + C^T R^{-1} C\}^{-1}\end{aligned} \tag{35}$$

$$\begin{aligned}p_{k+1_{KK}} = p_{k_{KK}} - [&(CAp_{k_{xK}})^T + u_k p_{k_{KK}}(Cb)^T]\{CAP_{k_{xx}}A^TC^T \\ &+ u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] + u_k^2 p_{k_{KK}}(Cb)(Cb)^T \\ &+ C\Gamma Q \Gamma^T C^T + R\}^{-1}(CAp_{k_{xK}} + u_k p_{k_{KK}}(Cb))\end{aligned} \tag{36}$$

and

$$\begin{aligned}p_{k+1_{xK}} = &Ap_{k_{xK}} + u_k p_{k_{KK}} b - \{AP_{k_{xx}}A^TC^T + u_k[Ap_{k_{xK}}(Cb)^T \\ &+ b(CAp_{k_{xK}})^T] + u_k^2 p_{k_{KK}} b(Cb)^T + \Gamma Q \Gamma^T C^T\} \times \\ &\{CAP_{k_{xx}}A^TC^T + u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] \\ &+ u_k^2 p_{k_{KK}}(Cb)(Cb)^T + C\Gamma Q \Gamma^T C^T + R\}^{-1} \times \\ &(CAp_{k_{xK}} + u_k p_{k_{KK}}(Cb))\end{aligned} \tag{37}$$

Remark 3 *An application of the MIL will reduce the number of matrix inversions such that only the low-order matrix*

$$CAP_{k_{xx}}A^TC^T + u_k[CAp_{k_{xK}}(Cb)^T + (Cb)(CAp_{k_{xK}})^T] + u_k^2 p_{k_{KK}}(Cb)(Cb)^T + C\Gamma Q \Gamma^T C^T + R$$

needs to be inverted.

It is important to realize that the absence of complete plant information, viz., the uncertainty in the loop gain parameter K, causes both the loop gain and the state estimation error covariances to be dependent on the input signal - see, e.g., eqs. (35)-(37). This is a major departure from the classical state estimation paradigm in linear control theory. Thus, the loop gain estimate $\hat{K}$ (and alo the loop gain estimation error covariance) are now time depedent; obviously, the best loop gain estmate is obtained at the end of the estimation interval, at time N. In addition, the algorithm - provided loop gain and state estimates are correlated. Furthermore, the loop gain and state estimates' dependence on the input signal is nonlinear. The input signal dependence of the loop gain and state estimation error covariances, is a unique manifestation of the *dual control* effect.

Corollary 4 *Consider the classical Kalman filter paradigm where K is known, i.e., K=1. In this special case*

$$p_{o_{KK}} = 0, \qquad p_{o_{xK}} = 0, \qquad p_{k_{KK}} = 0, \qquad p_{k_{xK}} = 0 \qquad \text{for all } k = 1, 2, ...$$

and it follows that

$$P_k = P_{k_{xx}} \tag{38}$$
$$\mathcal{K}_k = 0 \tag{39}$$
$$\mathcal{K}_x = (AP_{k_{xx}}A^T + \Gamma Q \Gamma^T)C^T\{CAP_{k_{xx}}A^TC^T + C\Gamma Q \Gamma^T C^T + R\}^{-1} \tag{40}$$
$$P_{k+1_{xx}} = [(AP_{k_{xx}}A^T + \Gamma Q \Gamma^T)^{-1} + C^T R^{-1} C]^{-1} \tag{41}$$

Thus, the classical Kalman filter formulae are recovered.

Remark 5 *If x_o is known, viz., $x_o = N(\bar{x}_o, 0)$, i.e, $P_{o_x} = 0$, and only the loop gain parameter K is not known, i.e., $P_{o_{xx}} = 0$, $p_{o_{xK}} = 0$, one nevertheless has to deal with an uncertain x at time k (even if $\Gamma = 0$ and if there is no process noise), and one must propagate $\begin{pmatrix} \hat{x}_k \\ \hat{K}_k \end{pmatrix}$ and $P_{k_{(n+1)\times(n+1)}}$.*

Corollary 6 *Special case: C is a row vector ($\equiv$ scalar measurement).*
Then the estimation algorithm is

$$\hat{x}_{k+1} = A\hat{x}_k + \hat{K}_k b u_k + \mathcal{K}_x(z_{k+1} - CA\hat{x}_k - \hat{K}_k(Cb)u_k) \tag{42}$$
$$\hat{K}_{k+1} = \hat{K}_k + \mathcal{K}_k(z_{k+1} - CA\hat{x}_k - \hat{K}_k(Cb)u_k) \tag{43}$$

where the Kalman gain for state estimation

$$\mathcal{K}_x = \frac{1}{X}\{AP_{k_{xx}}A^TC^T + u_k[(Cb)Ap_{k_{xK}} + (CAp_{k_{xK}})b] + u_k^2(Cb)p_{k_{KK}}b + \Gamma Q \Gamma^T C^T\} \tag{44}$$

and where the scalar X,

$$X = CAP_{k_{xx}}A^TC^T + 2u_k(Cb)CAp_{k_{xK}} + u_k^2(Cb)^2 p_{k_{KK}} + C\Gamma Q \Gamma^T C^T + R \tag{45}$$

The Kalman gain for loop gain estimation is

$$\mathcal{K}_K = \frac{[CAp_{k_{xK}} + u_k(Cb)p_{k_{KK}}]}{X} \tag{46}$$

Finally, the estimation error covariances are

$$P_{k+1_{xx}} = \{[AP_{k_{xx}}A^T + u_k(Ap_{k_{xK}}b^T + bp_{k_{xK}}^T A^T) + u_k^2 p_{k_{KK}} bb^T + \Gamma Q \Gamma^T]^{-1} + \frac{1}{R}C^TC\}^{-1} \tag{47}$$
$$p_{k+1_{KK}} = p_{k_{KK}} - \frac{[CAp_{k_{xK}} + u_k(Cb)p_{k_{KK}}]^2}{X} \tag{48}$$
$$p_{k+1_{xK}} = Ap_{k_{xK}} + u_k p_{k_{KK}} b - \frac{CAp_{k_{xK}} + u_k(Cb)p_{k_{KK}}}{X} \{AP_{k_{xx}}A^TC^T + u_k[(Cb)Ap_{k_{xK}} + (CAp_{k_{xK}})b] + u_k^2(Cb)p_{k_{KK}}b + \Gamma Q \Gamma^T C^T\} \tag{49}$$

4 Conclusions

An algorithm for the identification of a control system's loop gain is presented. The derived algorithm yields joint estimates of both the system's state and the loop gain. The loop gain/state estimation algorithm is being referred to as a system identification algorithm, and rightfully so. Our assertion is not just contingent on the obvious, viz., the (crucial) loop gain plant parameter is now estimated - this, vis a vis the classical Kalman filtering paradigm where the estimation of the state of a completely known linear system is exclusively addressed. Indeed, our estimation algorithm, while superficially similar to the classical linear Kalman filtering algorithm, entails time-varying dynamics. Moreover, in contrast to the classical linear Kalman filtering algorithm, in our algorithm 1) The loop gain and state estimates and the respective covariances of the estimation errors are dependent on the input signal, and 2) The input signal is processed in a nonlinear fashion. Thus, a unique instance, where the dual effects of control are at work, is analyzed. At the same time, and most importantly, unbiased estimates of the loop gain and of the system's state, and the respective estimation errors' covariances, are obtained. This lays the foundation for on-line control system loop gain identification and indirect adaptive and reconfigurable control.

References

Chandler P., J. Buffington, and M. Pachter (1998). "Integration of On-Line System Identification and Optimization-Based Control," *AIAA Guidance, Navigation and Control Conference*, Boston, MA, August 10–12, 1998, AIAA paper No 98-4487.

Houpis C. H. and M. Pachter (1997). "Application of QFT to Control System Design - An Outline for Engineers," *Int. J. of Robust and Nonlinear Control*, **7**, no. 6, pp. 515–531.

L. Ljung (1987). *System Identification for the User*, Prentice Hall.

P. Maybeck (1982). *Stochastic Models, Estimation and Control*, Academic Press.