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BIBO Stability of NARX Models*

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Abstract

This paper explains an approach to BIBO stability of NARX control systems. The approach is based on difference inequalities and assumes availability of an approximate NARX model and the system order. Sufficient conditions for modelling error are derived ensuring boundedness of the error between model's and plant's outputs for the same inputs. For this class of bounded inputs sufficient conditions for BIBO stability are given and shown practicable. They also allow designing a controller using the model, leading to BIBO stable closed-loop system.

1 Introduction

In this paper we consider single-input, single-output, deterministic, discrete-time systems in the NARX (Nonlinear Autoregressive Moving Average) form (Leontaritis and Billings, 1985; Chen and Billings, 1989):

$$\begin{aligned} z(k+1) &= g(z(k),\dots,z(k-n+1),u(k),\dots,u(k-m+1)),\\ z(k_0+i) &= z_i, \qquad i=0,1,\dots,n-1. \end{aligned} \tag{1}$$

We assume that g in (1) is unknown, but an approximate model is available:

$$\begin{aligned} y(k+1) &= f(y(k),\dots,y(k-n+1),u(k),\dots,u(k-m+1)),\\ y(k_0+i) &= y_i, \qquad i=0,1,\dots,n-1, \end{aligned} \tag{2}$$

where f is known, so that z is the true and y predicted output. The following *a priori* knowledge is required: (i) values of m and n, (ii) an estimate of modelling error, i.e., the discrepancy between f and g over admissible inputs of interest. The latter is made precise in Lemma 1 below, equation (12).

Availability of an approximate model (2) of the underlying NARX description (1) is of particular interest in neural modelling (Chen *et al.*, 1990b,a; Narendra and Parthasarathy, 1991; Żbikowski and Dzieliński, 1996; Dzieliński and Żbikowski, 1996). By necessity, system analysis and controller design must be done for (2), but we want it to be applicable to (1). In this paper we tackle two problems: (a) influence of discrepancy between f and g on difference between y and z under action of the same control signal u; (b) BIBO stability of (2) and (1) both in the open- and closed-loop settings. In the context the BIBO stability is meant according to the following definition

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Definition 1 A NARX system is said to be bounded-input, bounded-output (BIBO) stable if, and only if, any admissible, bounded input results in a bounded output. $\square$

where by boundedness we shall mean uniform boundedness, as formalised by the following definition.

Definition 2 *A function $x\colon I_{k_0} \to \mathcal{R}$ is said to be bounded if, and only if, there exists $M \in \mathcal{R}$, such that $\|x\| \leq M$.* $\square$

If modelling error results in bounded error between y and z for admissible inputs of interest, then all boundedness results for y in (2) also apply to z in (1). Thus, bounded-input, bounded-output (BIBO) stability enters naturally and is necessary anyway, because of the input-output setting. Engineering constraints usually restrict the class of inputs, e.g., imposing limitations on their growth. In this paper we deal with a subset of bounded control signals, technically specified in the theorems below, for which we show boundedness of outputs.

BIBO stability for the restricted class $\mathcal{U}$ of bounded inputs is considered both in the open- and closed-loop settings. The latter leads to *BIBO redesign*, the design of a controller using approximate model (2) and an estimate of modelling error. It is required that the controller's output is always a bounded signal $u \in \mathcal{U}$, leading to a bounded z of the plant (1) for a given reference ξ. Thus, the closed-loop system arises by applying to the real plant (1) the controller designed for the approximate model (2). Its BIBO stability is a guarantee that $u \in \mathcal{U}$ and z is bounded.

The problems outlined are approached in a novel way, using recent techniques of difference inequalities. This is entirely different from the functional analytic approach (Desoer and Vidyasagar, 1975; Vidyasagar, 1992), traditionally used for the input-output stability analysis.

The main contributions of the paper are: (i) the difference inequalities approach to BIBO stability of NARX systems; (ii) sufficient conditions for quantifying influence of modelling error between (1) and (2) on the difference between z and y (Lemma 1); (iii) use and analysis of Agarwal's boundedness criterion (Theorem 1); (iv) BIBO redesign (Theorem 3); (v) BIBO stability analysis (Theorem 2 plus Agarwal's criterion).

2 Applicability of Approximate NARX Models of a NARX Plant

In this section we discuss adequacy of an approximate (due to modelling errors) NARX model (2) of (1) for control purposes. In other words, we want to know if an inaccurate NARX representation (2) of the real NARX system (1) would reflect well the system's behaviour, when influenced by the same control signal.

Consider a function constituting a bound on the norm of the modelling error, i.e., the difference between f in (2) and g in (1). This function should bound the norm uniformly in u (for all admissible control signals u). The question is what this tells us about the error between y and z. If the error is small, then applying a control signal to the approximate model would cause similar behaviour of the real system. This also applies to BIBO stability analysis, because if we prove stability for the model (2), then it will hold for the real plant (1), provided the difference between y and z is bounded.

In this paper we approach the problem using finite difference inequalities. Recall that finite differences are defined as

$$\Delta^{(0)}y(k) \quad = \quad y(k),$$

$$\begin{aligned} \Delta^{(1)}y(k) &= \Delta y(k) = y(k+1) - y(k), \\ \Delta^{(n)}y(k) &= \Delta(\Delta^{(n-1)}y(k)) \quad \text{for } n \geq 1 \end{aligned}$$

for functions $y\colon I_{k_0} \to \mathcal{R}$, where $I_{k_0} = \{k_0, k_0+1, \dots\} \in \mathcal{Z}_+$.

Proposition 1 Controlled difference equation (2) is equivalent to the following n-th order controlled finite difference equation

$$\Delta^{(n)}y(k) = \bar{f}(Y(k), U(k)), \tag{3}$$

where

$$\begin{aligned} Y(k) &= (\Delta^{(n-1)}y(k), \dots, \Delta y(k), y(k)), \\ U(k) &= (\Delta^{(n-1)}u(k), \dots, \Delta u(k), u(k)). \quad \square \end{aligned} \tag{4}$$

Proof: We have (Lakshmikantham and Trigiante, 1988, p. 3)

$$x(k+l) = \sum_{i=0}^{l} \binom{l}{i} \Delta^i x(k). \tag{5}$$

Renumbering the indices in (2) we obtain

$$y(k+n) = f(y(k+n-1), \dots, y(k), u(k+n-1), \dots, u(k+n-m)). \tag{6}$$

Introducing (5) into (6) and appropriately rearranging the terms we obtain a finite difference equation in the form (3). $\square$

From now on $\bar{f}$ denotes the right-hand side (RHS) of the finite difference equation corresponding to (2) and, similarly, $\bar{g}$ for (1).

By Proposition 1, we may consider the model-plant correspondence and BIBO stability in the framework of (controlled) finite difference equations.

Lemma 1 Let the function $W\colon \mathcal{R}^n \times I_{k_0} \to \mathcal{R}$ be continuous, non-negative, monotonically increasing on $\mathcal{R}^n$ for each $k \in I_{k_0}$ and let $r\colon I_{k_0} \to \mathcal{R}$ be the solution of

$$\begin{aligned} \Delta^{(n)}r(k) &= W(R(k), k), \\ \Delta^{(i)}r(k_0) &= \bar{r}_i \quad \text{for} \quad i = 0, 1, \dots, n-1, \end{aligned} \tag{7}$$

where

$$R(k) = (\Delta^{(n-1)}r(k), \dots, \Delta r(k), r(k)). \tag{8}$$

With the notation as in (4), consider the two n-th order finite difference equations:

$$\begin{aligned} \Delta^{(n)}z(k) &= \bar{g}(Z(k), U(k)) \\ \Delta^{(i)}z(k_0) &= \bar{z}_i \quad \text{for} \quad i = 0, 1, \dots, n-1, \end{aligned} \tag{9}$$

for the true NARX description (1) of the plant, where

$$Z(k) = (\Delta^{(n-1)}z(k), \dots, \Delta z(k), z(k)), \tag{10}$$

and

$$\Delta^{(n)}y(k) = \bar{f}(Y(k), U(k))$$

$$\Delta^{(i)}y(k_0) \quad = \quad \bar{y}_i \quad \text{for} \quad i = 0, 1, \ldots, n-1, \tag{11}$$

being the approximate NARX model (2) of the plant. Here $\bar{f}\colon \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}$ and $\bar{g}\colon \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}$ are assumed to be continuous and satisfy:

$$\begin{aligned} &\|\bar{f}(Y(k), U(k)) - \bar{g}(Z(k), U(k))\| \leq \\ &\leq W(\|\Delta^{(n-1)}y(k) - \Delta^{(n-1)}z(k)\|, \ldots, \|y(k) - z(k)\|, k) \end{aligned} \tag{12}$$

uniformly with respect to $u \in \mathcal{U}$ for all $k \geq k_0$, where $\mathcal{U}$ is the set of admissible control signals, defined on I_{k_0}. Finally, let $y\colon I_{k_0} \to \mathcal{R}$ and $z\colon I_{k_0} \to \mathcal{R}$ be any solutions of (11) and (9), respectively, such that

$$\|\bar{y}_i - \bar{z}_i\| \leq \bar{r}_i \quad \text{for} \quad i = 0, 1, \ldots, n-1. \tag{13}$$

Then

$$\|y(k) - z(k)\| \leq r(k) \quad \text{for all } k \geq k_0. \quad \square \tag{14}$$

Proof: Proof is a straightforward application of Theorem 4 in (Pachpatte, 1970) by noting that for a given $u\colon I_{k_0} \to \mathcal{R}$, $u \in \mathcal{U}$, say $u(k) \equiv \vartheta(k)$, (11) can be rewritten as

$$\Delta^{(n)}y(k) = \hat{f}(Y(k), k) = \bar{f}(Y(k), U(k))|_{u(k)\equiv\vartheta(k)}$$

and, similarly, (9)

$$\Delta^{(n)}z(k) = \hat{g}(Z(k), k) = \bar{g}(Y(k), U(k))|_{u(k)\equiv\vartheta(k)}. \quad \square$$

Remark 1 Continuity of $W: \mathcal{R}^n \times I_{k_0} \to \mathcal{R}$ is to be understood by interpreting W as a restriction of a function continuous on $\mathcal{R}^n \times \mathcal{R}$. Alternatively, we may require that W is continuous on $\mathcal{R}^n$ for each fixed $k \in I_{k_0}$. $\square$

Remark 2 Note that we assume the same order, n, of the model (11) and the plant (9). $\square$

Remark 3 For BIBO stability considerations the set of admissible control signals $\mathcal{U}$ is the set of bounded functions $u\colon I_{k_0} \to \mathcal{R}$ (cf. Definition 2). Because of the form of (11), $\mathcal{U}$ describes not only the constraints on u, but also on its finite differences $\Delta^{(n-1)}u(k), \ldots, \Delta u(k)$. $\square$

Lemma 1 has important consequences for modelling and control of NARX systems. Solutions of (7) determine the discrepancy (see (13) and (14)) between the model (11) and the real plant (9) under the action of the same control signal. Thus, if we can find W satisfying (12) with $\bar{r}_i$, $i = 0, 1, \ldots, n-1$, satisfying (13) and such that solutions of (7) are bounded, then the discrepancy is also bounded.

This is of particular interest for neural modelling of NARX systems (Żbikowski and Dzieliński, 1996; Dzieliński and Żbikowski, 1996), where (11) is a neural approximation of the real plant. If the discrepancy is small, then the controller designed for the *approximate* NARX model should perform well for the *real* NARX plant. In practice, this equivalence may be provable only for a subset of admissible controls $\mathcal{U}$, because Lemma 1 gives only sufficient conditions. These ideas are fully developed in section 3.1.

3 BIBO Stability of the Control System Based on a NARX Model

In this section the BIBO stability of NARX model is analysed (2) using some results on asymptotic behaviour of solutions of n-th order finite difference equations. This is done by exploiting difference inequalities.

As noted in the proof of Lemma 1, for a given control signal $u(k) \equiv \vartheta(k)$ the controlled finite difference equation (3) becomes the time-varying finite difference equation

$$\begin{aligned} \Delta^{(n)}y(k) &= \hat{f}(Y(k), k) = \bar{f}(Y(k), U(k))|_{u(k)\equiv\vartheta(k)} \\ \Delta^{(i)}y(k_0) &= \bar{y}_i \quad \text{for} \quad i = 0, 1, \dots, n-1. \end{aligned} \tag{15}$$

Thus, proving BIBO stability of (3) reduces to showing boundedness of solutions of (15) for all admissible u. In fact, it suffices to prove *asymptotic* boundedness because of the following result.

Proposition 2 If a solution of equation (15) is asymptotically bounded, then it is bounded. □

Proof: Recall (Lakshmikantham and Trigiante, 1988, page 11) that for *any* right hand side of (15) there exists a unique solution of the equation. Hence $y(k)$ is a finite number for each $k \geq k_0$. Therefore, if (by hypothesis) $\lim_{k\to\infty} |y(k)| < \infty$, then there exists $M = \sup\{y(k)|\ k \geq k_0\} < \infty$, such that $y(k) \leq M$ for all $k \geq k_0$. □

This is in a marked contrast with the initial value problem for ordinary differential equations (ODEs); there, even for locally Lipschitz equations, the finite escape time phenomenon may occur. Thus, from asymptotic boundedness of a solution of an ODE we cannot infer this property for all earlier times.

A criterion for asymptotic boundedness of solutions of (15) is a result due to Agarwal (Agarwal, 1985, Theorem 3.1) (see also (Agarwal, 1992, Theorem 6.17.1)). Before formulating the theorem, recall (Lakshmikantham and Trigiante, 1988, page 5) that the nth factorial power of $l \in \mathcal{Z}_+$ is defined as

$$l^{(n)} = l(l-1) \cdot \ldots \cdot (l-n+1) = \prod_{i=0}^{n-1}(l-i)$$

with $l^{(0)} = 1$.

Theorem 1 (Agarwal) Let the function $\hat{f} \colon \mathcal{R}^n \times I_{k_0} \to \mathcal{R}$ satisfy

$$|\hat{f}(x_{n-1}, \dots, x_1, x_0, k)| \leq \sum_{i=0}^{n-1} p_i(k)|x_i|, \tag{16}$$

for all $(x_{n-1}, \dots, x_1, x_0, k) \in \mathcal{R}^n \times I_{k_0}$, where $p_i \colon I_{k_0} \to \mathcal{R}$, $i = 0, 1, \dots, n-1$, are nonnegative functions and

$$\prod_{l=k_0}^{\infty}\left[1 + \sum_{i=0}^{n-1} l^{(n-i-1)}p_i(l)\right] < \infty, \tag{17}$$

where $l^{(n-i-1)}$ is the factorial expression. Then the finite difference equation (15) has nonoscillatory[1] solutions for which $\lim_{k\to\infty} |y(k)| < \infty$. □

[1] A solution of (15) is called nonoscillatory if, and only if, it is eventually of fixed sign, i.e., $\exists k_1 \in I_{k_0}\ \forall k > k_1\quad y(k)y(k+1) > 0$ (Agarwal, 1992, p. 322).

Relation between notations of (16) and (15) is $x_i \leftrightarrow \Delta^i y(k)$, $i = 0, 1, \ldots, n-1$.

In order to use Theorem 1 effectively, we must first of all establish the classes of functions p_i, $i = 0, 1, \ldots, n-1$, satisfying (17). It follows from the theory of infinite products (Knopp, 1990, Chapter VII) that a product $\Pi(1 + a_l)$ with positive terms a_l is convergent if, and only if, the series Σa_l converges. Since the terms of product (17) are positive, it suffices to prove convergence of the series

$$\sum_{l=k_0}^{\infty} \left[\sum_{i=0}^{n-1} l^{(n-i-1)} p_i(l) \right] = \sum_{i=0}^{n-1} \left[\sum_{l=k_0}^{\infty} l^{(n-i-1)} p_i(l) \right], \tag{18}$$

which is the same as showing convergence of the n series

$$\sum_{l=k_0}^{\infty} l^{(n-i-1)} p_i(l), \quad i = 0, 1, \ldots, n-1. \tag{19}$$

The terms of the series are functions p_i premultiplied by polynomials of degrees from 0 (for p_{n-1}) to $n-1$ (for p_0), implying $p_i(l) = O(1/l^{n-i+\epsilon})$, $\epsilon > 0$, as a necessary condition for convergence of (19). This makes the right-hand side of (16), $P(x, k) = \sum_{i=0}^{n-1} p_i(k)|x_i|$, a function with $\lim_{k\to\infty} P(x, k) = 0$ for each $x \in \mathcal{R}^n$, restricting in this way dependence of $\hat{f}$ on k in (16).

Functions p_i ensuring convergence of (19) are plentiful (Knopp, 1990, Chapters III, IX).

3.1 BIBO Redesign

In this section we apply Lemma 1 and the developments of the previous section to BIBO stability analysis of (1) in the closed-loop context. Recall that we don't know g of (1), or—equivalently—$\bar{g}$ of (9), but have f of (2), or—equivalently—$\bar{f}$ of (11). Thus, design of a control law must be based on $\bar{f}$, but the control signal will be applied to the real plant (9). The fundamental requirement is that this approach will lead to a BIBO stable closed-loop system. The closed-loop system is the real plant (9) with a controller designed for its approximate model (11). Stability is the BIBO stability of Definition 1, meaning that the controller generates bounded inputs resulting in bounded outputs of (9). In this sense we can talk about closed-loop BIBO stability.

The proof of the main result of this section is based on one of the comparison theorems given by Pachpatte (see (Pachpatte, 1970, Theorem 5)).

Theorem 2 Let the functions $\hat{f}_1 \colon \mathcal{R}^n \times I_{k_0} \to \mathcal{R}$ and $\hat{f}_2 \colon \mathcal{R}^n \times I_{k_0} \to \mathcal{R}$ be continuous, non-negative and monotonically increasing on $\mathcal{R}^n$ for each $k \in I_{k_0}$. Moreover, let $\hat{f}_1$ and $\hat{f}_2$ satisfy the inequalities

$$\hat{f}_1(Y(k), k) \leq \Delta^{(n)} y(k) \leq \hat{f}_2(Y(k), k) \tag{20}$$

for all $k \geq k_0$, where $\Delta^{(n)} y(k)$ is as in (15). Let $v(k)$ and $w(k)$ be the solutions of

$$\begin{aligned} \Delta^{(n)} v(k) &= \hat{f}_1(\Delta^{(n-1)} v(k), \ldots, v(k), k), \\ \Delta^{(i)} v(k_0) &= v_0^i, \quad \text{for} \quad i = 0, 1, \ldots, n-1, \end{aligned} \tag{21}$$

$$\begin{aligned} \Delta^{(n)} w(k) &= \hat{f}_2(\Delta^{(n-1)} w(k), \ldots, w(k), k), \\ \Delta^{(i)} w(k_0) &= w_0^i, \quad \text{for} \quad i = 0, 1, \ldots, n-1, \end{aligned} \tag{22}$$

respectively, such that

$$v_0^i \leq \Delta^{(i)} y(k_0) \leq w_0^i, \quad \text{for} \quad i = 0, 1, \dots, n-1. \tag{23}$$

Then

$$v(k) \leq y(k) \leq w(k) \tag{24}$$

for all $k \geq k_0$. $\square$

Proof: The proof follows by a direct application of Theorem 1 and Theorem 5 in (Pachpatte, 1970). $\square$

Thus, if we are able to find two functions $\hat{f}_1$ and $\hat{f}_2$ such that (20) holds, it means (by (15)) that

$$\hat{f}_1(Y(k), k) \leq \hat{f}(Y(k), k) \leq \hat{f}_2(Y(k), k) \tag{25}$$

for all $k \geq k_0$ and for a given $u \in \mathcal{U}$, say $u(k) \equiv \vartheta(k)$. Additionally, let $\hat{f}_1$ and $\hat{f}_2$ be such that equations (21) and (22) have bounded solutions (e.g., by Theorem 1) and u be bounded. Then, from (24), the solution of (3) corresponding to $u(k) \equiv \vartheta(k)$ is also bounded. If this can be shown for all $u \in \mathcal{U}$, then the system described by (3) is BIBO stable in accordance with Definition 1 and, by Proposition 1, so is (2).

These considerations can be applied in the closed-loop context, where u is generated by a control law. However, it should be borne in mind that the controller design is possible only on the basis of $\bar{f}$ in (11), while u will be applied to (9).

In order to obtain a stable closed-loop system the following procedure of *BIBO redesign* can be devised. First, the model-plant equivalence must be established, i.e., a set $\mathcal{U}$ of admissible, bounded inputs must be found for which Lemma 1 holds. Thus, based on *a priori* estimate of the modelling error, a function W satisfying (12) should be constructed, so that (14) is satisfied with r bounded. The set $\mathcal{U}$ for which these hold is then the starting point for the second step of BIBO redesign, for it will ensure that bounded $\|\bar{f}(Y(k), U(k)) - \bar{g}(Z(k), U(k))\|$ results in bounded $\|y(k) - z(k)\|$ for all $k \geq k_0$. Now, given a reference signal $\xi\colon I_{k_0} \to \mathcal{R}$, a control law ϕ, $\phi(Y(k), \xi(k)) = u(k)$ with $u \in \mathcal{U}$, must be designed, so that (25) holds with $\hat{f}_1$, $\hat{f}_2$ of Theorem 2 satisfying Theorem 1.

This reasoning is made precise by the following theorem.

Theorem 3 (BIBO redesign) Let $\mathcal{U}$ be a set of admissible, bounded inputs for which (14) of Lemma 1 holds with r bounded. Suppose further that $\xi\colon I_{k_0} \to \mathcal{R}$ is a desired reference signal for (1). Finally, let a control law ϕ, $\phi(Y(k), \xi(k)) = u(k)$ with $u \in \mathcal{U}$, giving rise to $\Phi(k) = (\Delta^{(n-1)} u(k), \dots,$
$\Delta u(k), u(k))$, be such that (25) is satisfied with $\hat{f}(Y(k), k) = \bar{f}(Y(k), \Phi(k))$, where $\bar{f}$ is as in (11), and $\hat{f}_1$, $\hat{f}_2$ satisfy (21), (22) and Theorem 1. Then the closed-loop system

$$\Delta^{(n)} z(k) = \hat{g}(Z(k), k) = \bar{g}(Z(k), \Phi(k)), \tag{26}$$

with $\bar{g}$ as in (9), is BIBO stable with respect to $\mathcal{U}$. $\square$

Proof: The set $\mathcal{U}$ of the hypothesis specifies admissible, bounded inputs required by Definition 1. The inputs are defined by the control law ϕ, which—by assumption—results in $u \in \mathcal{U}$. Thus, the task is to show that for all $u \in \mathcal{U}$ the corresponding solutions of (26) are bounded.

It follows that $\mathcal{U}$ ensures (by Lemma 1 and boundedness of r) that for all $u \in \mathcal{U}$ the difference $\|y(k) - z(k)\|$ is bounded for all $k \geq k_0$. Hence, if u applied to (11) results in bounded y, then so is z of (26). The postulated construction of ϕ guarantees (by application of Theorems 1 and 2) that u will generate such y and the result follows. $\square$

4 Conclusions

We have considered BIBO stability of approximate NARX models in the framework of difference inequalities. Two main questions were: (a) influence of modelling error on the error between model's and plant's output and (b) BIBO stability of NARX systems in the open- and closed-loop settings. An answer to (a) were sufficient conditions on modelling error ensuring boundedness of the error between the outputs. This allowed concentrating on BIBO stability of the model, leading to a solution to (b), also based on difference inequalities. In particular, a controller can be designed using the approximation, but applied to the real NARX plant, resulting in a BIBO stable closed-loop system—the new methodology of BIBO redesign.

An enabling result is the criterion (due to Agarwal) of boundedness of solutions of finite difference equations and we showed its applicability. However, an important aspect is that all results put restrictions on the class of inputs. Not only must they be bounded, but they should also be asymptotically decreasing.

The approach presented is novel and entirely different from the traditional functional analytic techniques. As shown in examples, the derived sufficient conditions rely on (i) convergence of series with positive terms and (ii) inequalities. There exists vast literature on both topics with many concrete techniques and examples. Thus, within the limitations imposed by sufficiency of the BIBO stability conditions, there is a considerable scope for ingenious design.

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