Proceedings of the 7th Mediterranean Conference on Control and Automation (MED99) Haifa, Israel - June 28-30, 1999

The Suboptimal Tracking Problem in Linear Systems*

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Abstract

The contribution is focused on control system design for the purposes of suboptimal LQ tracking in continuous-time SISO linear systems. The proposed method is based on the polynomial approach. The presented procedures are proposed for a class of references frequently used in practice. The resulting controller is obtained via the solution of a polynomial Diophantine equation with the right side given by spectral factorization. The theoretical results are tested on an illustrative example.

1 Introduction

Linear quadratic optimal control methodologies have been intensively studied in recent years. Optimal control design, based on the LQ performance criterion, has been developed historically first in terms of the state space approach. By this method we solve differential or algebraic Riccati equations. Progress in polynomial algebra and the polynomial approach to the analysis and synthesis of control systems have offered new tools for tackling the LQ control problem. The procedures, based on the polynomial approach lead to spectral factorizations and other algebraic operations in the polynomial ring. The problems of both deterministic (LQ) and stochastic (LQG) control have been solved by many authors. Some recent results in this field can be found in works of (Kucera and Šebek, 1984; Hunt, Kucera and Šebek, 1992) for discrete-time SISO systems, (Kucera and Šebek, 1985) for continuous-time SISO systems, (Johnson and Grimble, 1987; Hunt and Šebek, 1991) and (Mosca, 1995) for both continuous-time and discrete-time MIMO systems. Some results, obtained for discrete-time and MIMO systems, respectively, can be employed in procedures for continuous-time SISO system control design.

This paper deals with the problem of deterministic LQ tracking. This problem is given by some properties of the control of real technological processes. In most theoretical works the reference signal is assumed to be from a class of stochastic functions. When applied in practice, the references always belong to a class of deterministic functions. It is known (see, e.g. Kucera and Šebek, 1984) that the solution of optimal deterministic tracking problem results in an ill-posed controller whose parameters depend upon the initial conditions of the controlled system. Naturally, this controller cannot be acceptable for control purposes. Here we will propose a well-posed controller which enables the suboptimal tracking for a defined class of references.

* This work was supported by the Grant Agency of the Czech Republic under grant No. 102/99/1292.
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2 Control system description

The feedback control system is depicted in Fig.1. The controlled system is described in the time domain by differential equation

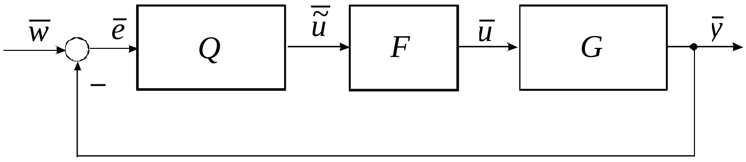

Figure 1. Control system scheme

$$a(\sigma)\,\bar{y}(t) = b(\sigma)\,\bar{u}(t) \tag{1}$$

where t is time, σ is the derivative operator, $\bar{y}$ is the controlled output, $\bar{u}$ is the control input and $\bar{e}$ is the tracking error. Both a and b are polynomials in σ. Generally, nonzero initial conditions for both output and input variables are considered. Using the Laplace transform, the system is represented in the complex domain by

$$y(s) = \frac{b(s)}{a(s)}u(s) + \frac{o_1(s)}{a(s)} \tag{2}$$

where s is the complex argument and o_1 is the transform of initial conditions. Both a and b are now coprime polynomials in s. Assuming deg $b \le$ deg a , transfer function $G(s) = b(s)/a(s)$ is proper. The condition deg $o_1 <$ deg a ensues from Laplace transform properties. Moreover, conditions $a(0) \neq 0$ and $b(0) \neq 0$ are assumed.
The relations between other signals in the control system are described as

$$\tilde{\bar{u}}(t) = f(\sigma)\,\bar{u}(t) \quad \text{and} \quad p(\sigma)\,\tilde{\bar{u}}(t) = q(\sigma)\,\bar{e}(t) \tag{3}$$

in the time domain and

$$\tilde{u}(s) = f(s)\,u(s) - o_2(s) \quad \text{and} \quad p(s)\,\tilde{u}(s) = q(s)\,e(s) + o_3(s) \tag{4}$$

in the complex domain, where o_2 and o_3 are transforms of generally nonzero initial conditions.
The feedback controller Q is represented by its transfer function

$$Q(s) = \frac{q(s)}{p(s)} \tag{5}$$

where q and p are coprime polynomials and F is a pre-compensator with the transfer function

$$F(s) = \frac{1}{f(s)}\,. \tag{6}$$

Evidently, the pre-compensator is only a component of the feedback controller. In some of the following procedures the pre-compensator may be formally separated from the controller. In this case, polynomial p in (3) fulfills condition $p(0) \neq 0$. Further we consider the reference from a class of step or exponential functions frequently used in practice with the transform in the form of polynomial fraction

$$w(s) = \frac{h_w(s)}{\tilde{f}_w(s)} = \frac{h_w(s)}{s\,f_w(s)} \tag{7}$$

so that $\deg h_w \leq \deg f_w$.
After some algebraic manipulation the transform of signals in the control scheme can be obtained in the form (to simplify writing, polynomial arguments *s* are omitted)

$$y(s) = \frac{1}{d}\left\{bq\,w(s) + p(f\,o_1 + bo_2) + bo_3\right\} \tag{8}$$

$$u(s) = \frac{1}{d}\left\{q\left[a\,w(s) - o_1\right] + a\,po_2 + ao_3\right\} \tag{9}$$

$$e(s) = \frac{1}{d}\left\{p\left[a\,f\,w(s) - f\,o_1 - bo_2\right] - bo_3\right\} \tag{10}$$

$$\tilde{u}(s) = \frac{1}{d}\left\{q\left[a\,f\,w(s) - f\,o_1 - bo_2\right] + a\,f\,o_3\right\} \tag{11}$$

where $d = a\,f\,p + bq$.
The basic properties required on the control system are formulated as
- internal properness and stability of the control system
- asymptotic tracking of the reference signal

The condition of control system stability can be found for instance in the work of (Kucera, 1986). We cann express both controlled system and feedback controller transfer functions in the form of rational function fractions

$$G(s) = \frac{B(s)}{A(s)}, \quad Q(s) = \frac{Y(s)}{X(s)} \tag{12}$$

where

$$A(s) = \frac{a(s)}{m_1(s)}, \quad B(s) = \frac{b(s)}{m_1(s)} \tag{13}$$

$$X(s) = \frac{\tilde{p}(s)}{m_2(s)}, \quad Y(s) = \frac{q(s)}{m_2(s)} \tag{14}$$

with stable polynomials m_1 and m_2 so that deg m_1 = deg a and deg m_2 = deg $\tilde{p}$. Then *A*, *B*, *X* and *Y* belong to a ring of stable and proper rational functions and stabilizing controllers are derived from the solution of a Diophantine equation in the ring of rational functions

$$AX + BY = 1. \tag{15}$$

Substituting Eqs. (13) and (14) into (15), the condition of stability in a polynomial ring takes the form

$$a\tilde{p} + bq = m_1 m_2\,. \tag{16}$$

Since m_1, m_2 are stable polynomials, their product *d* is also a stable polynomial. Taking into account the pre-compensator relation $\tilde{p} = f\,p$ holds and the stabilizing feedback controller is derived from the solution of polynomial Diophantine equation

$$a\,f\,p + bq = d \tag{17}$$

with a stable polynomial *d* on the right side.
The control system satisfies the condition of internal properness only when the transfer functions of all its components are proper. The degrees of polynomials of the controller transfer function (inclusive of the pre-compensator) must then fulfill the inequality

$$\deg q \leq \deg p + \deg f\,. \tag{18}$$

From analysis of the solvability of Eq. (17) and taking into account condition (18), the degree of q is given as

$$\deg q = \deg a + \deg f - 1 . \tag{19}$$

The asymptotic tracking of reference $\overline{w}$ is only ensured for $f(s)$ in Eq. (10) divisible by s in the denominator of $w(s)$ in Eq. (7). This claim will be always fulfilled for $f(s) = s$ and the pre-compensator in Fig. 1 is then an integrator. Now, substituting this relation into Eqs. (10) and (11), respectively, the relevant signals take the form

$$e(s) = \frac{1}{f_{\mathrm{w}}\, d}\left\{ p\, z - b\, f_{\mathrm{w}}\, o_3 \right\} \tag{20}$$

$$\tilde{u}(s) = \frac{1}{f_{\mathrm{w}}\, d}\left\{ q\, z + a\, \tilde{f}_{\mathrm{w}}\, o_3 \right\} \tag{21}$$

where

$$z = a\, h_{\mathrm{w}} - \tilde{f}_{\mathrm{w}}\, o_1 - b\, f_{\mathrm{w}}\, o_2 . \tag{22}$$

3 Suboptimal LQ tracking

The idea of *suboptimal* control issues from the *optimal* control theory. The goal of optimal deterministic LQ tracking is to design a feedback controller Q that enables the control system to satisfy the above basic requirements and in addition the control law minimizes the cost function in the complex domain

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left\{ e^{*}(s) \mu e(s) + \tilde{u}^{*}(s) \varphi \tilde{u}(s) \right\} ds \tag{23}$$

where $\mu \geq 0$ and $\varphi > 0$ are weighting coefficients. It is known (see, for example Kuèera and Šebek, 1984) that the solution of this problem results in a controller where the parameters depend upon the initial conditions of the signals in the control system. Clearly, this controller is ill-posed and unacceptable for control purposes.

Note: The resulting controller is given by the solution of a polynomial Diophantine equation. The initial conditions of the control system variables (in our case o_1, o_2 and o_3) enter into spectral factorization of one part of the right side of the polynomial Diophantine equation.

An acceptable controller which ensures *suboptimal* tracking may only be obtained for the above determined class of references with transform in the form (7). The design procedure can be now realized by taking the following steps:

- Calculate stable polynomials g and n as the results of spectral factorizations

$$(a\, f)^{*} \varphi\, a\, f + b^{*} \mu b = g^{*} g \tag{24}$$

$$n^{*} n = a^{*} a \tag{25}$$

  where $f(s) = s$.
- Both polynomials q and p of the controller transfer function are then given by the solution of coupled polynomial equations

$$g^{*} q - v^{*} a\, f = b^{*} \mu\, n \tag{26}$$

$$g^{*} p + v^{*} b = (a\, f)^{*} \varphi\, n \tag{27}$$

  so that $\deg v < \deg g$.
- Eliminating v^{*} results in only one polynomial Diophantine equation

$$a\, f\, p + b\, q = gn \tag{28}$$

As a matter of interest, by expressing both integrand parts in (23) as

$$S_{\mathrm{e}} = e^{*} \mu e = \frac{1}{f_{\mathrm{w}}^{*} d^{*} d\, f_{\mathrm{w}}} \left\{ (pz - b\, f_{\mathrm{w}}\, o_3)^{*}\, \mu\, (pz - b\, f_{\mathrm{w}}\, o_3) \right\} \tag{29}$$

$$S_{\tilde{u}} = \tilde{u}^{*}\, \varphi\, \tilde{u} = \frac{1}{f_{\mathrm{w}}^{*} d^{*} d\, f_{\mathrm{w}}} \left\{ (qz + a\, \tilde{f}_{\mathrm{w}}\, o_3)^{*}\, \varphi\, (qz + a\, \tilde{f}_{\mathrm{w}}\, o_3) \right\} \tag{30}$$

and using some algebraic manipulations, the integrand in (23) takes the form

$$S = S_{\mathrm{e}} + S_{\tilde{u}} = \frac{1}{f_{\mathrm{w}}^{*} g^{*} g\, f_{\mathrm{w}}} \left\{ \mu\, \varphi\, z^{*} z + \frac{\tilde{v}^{*} \tilde{v}}{n^{*} n} \right\} \tag{31}$$

where $\tilde{v} = z\, v^{*} + f_{\mathrm{w}} g^{*} o_3$.

The resulting feedback controller stabilizes the control system. Dividing (28) by its right side and denoting $A = \dfrac{a}{n}$, $B = \dfrac{b}{n}$, $X = \dfrac{f\, p}{g}$, $Y = \dfrac{q}{g}$ where both g and n are stable polynomials, the condition of stability (15) is evidently fulfilled.

The transfer function of the feedback controller (inclusive the pre-compensator) is strictly proper. The degree of the right side of (28) is given by $\deg (g\, n) = 2 \deg a + 1$. Taking into account relations $\deg q = \deg a + \deg f - 1 = \deg a$ and $\deg (f\, p) = \deg (g\, n) - \deg a = \deg a + 1$, respectively, the strict properness of QF is evident.

4 Illustrative example

Suboptimal control was simulated for a second order controlled system represented by the transfer function

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_0}{s^2 + a_1 s + a_0}$$

where $b_0 = 0.02$, $a_0 = 0.0125$ and $a_1 = 0.225$, with zero initial conditions at the beginning of control. The references $w(t) = 1 - \exp(-0.8 \cdot t)$ for $0 \le t < 50$, $w(t) = -1$ for $50 \le t < 100$ and $w(t) = 1$ for $t > 100$ have been chosen. The weighting coefficient μ in the cost function (23) has been chosen equal to one. The influence of the weighting coefficient φ upon the behaviour of the control system has been investigated. Both stable polynomials g and n obtained from spectral factorizations (24), (25) take the form

$$g(s) = g_3 s^3 + g_2 s^2 + g_1 s + g_0 \;; \quad n(s) = s^2 + n_1 s + n_0$$

with coefficients

$$g_0 = \sqrt{\mu b_0^2} \;;\; g_3 = \sqrt{\varphi} \;;\; g_1 = \sqrt{2 g_2 g_0 + \varphi a_0^2} \;;\; g_2 = \sqrt{2 g_3 g_1 + \varphi (a_1^2 - 2 a_0)}$$

$$n_0 = \sqrt{a_0^2} \;;\; n_1 = \sqrt{2 n_0 + a_1^2 - 2 a_0} \,.$$

The resulting degrees of both polynomials of controller transfer function Q are deg q = deg p = 2. Their coefficients have been calculated from polynomial equation (28) by employing the so-called method of uncertain coefficients. The strictly proper transfer function of the feedback controller inclusive of pre-compensator is in the form

$$\tilde{Q}(s) = Q(s)\,F(s) = \frac{q_2 s^2 + q_1 s + q_0}{s\left(p_2 s^2 + p_1 s + p_0\right)}\,.$$

The control input is then computed in the time domain from the differential equation

$$p_2\,\bar{u}^{(3)}(t) + p_1\,\bar{u}^{(2)}(t) + p_0\,\bar{u}^{(1)}(t) = q_2\,\bar{e}^{(2)}(t) + q_1\,\bar{e}^{(1)}(t) + q_0\,\bar{e}(t)$$

The time responses of signals in the control system are illustrated for various φ values in Figs.2-4.

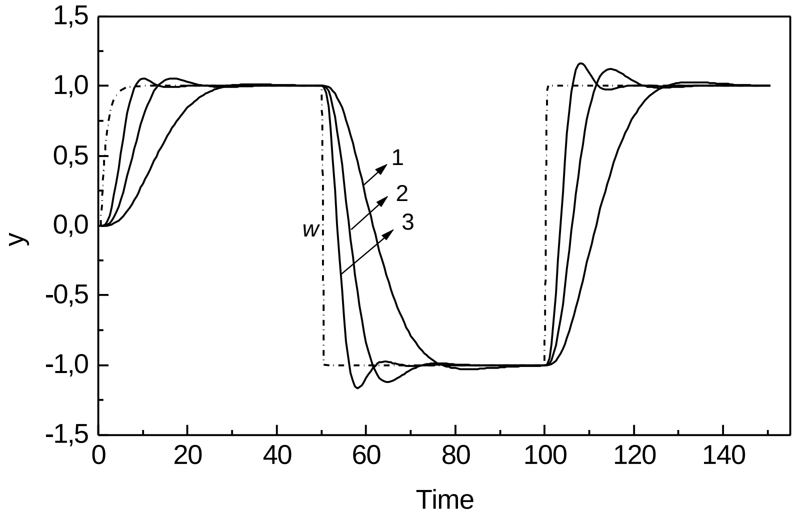

Figure 2. Controlled output time responses for φ = 0.04 (1), 1 (2), 25 (3)

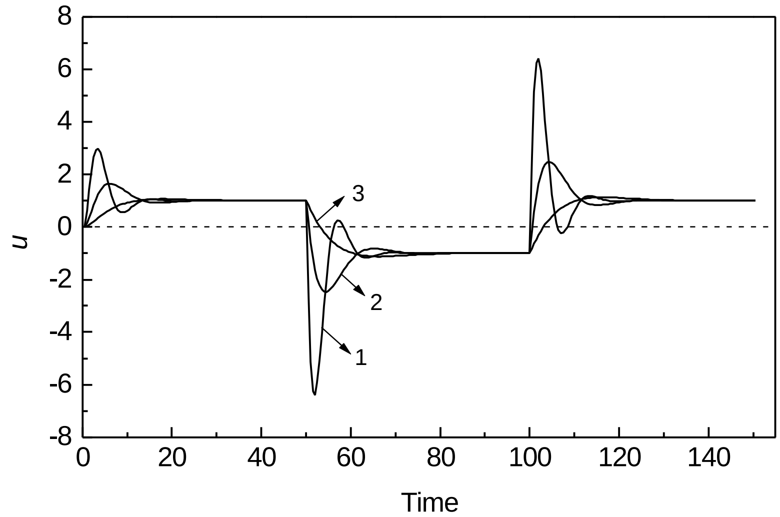

Figure 3. Control input time responses for φ = 0.04 (1), 1 (2), 25 (3)

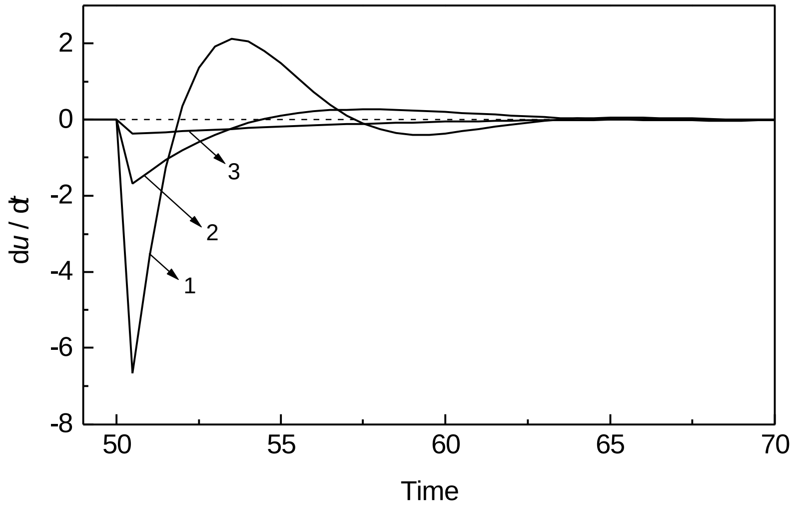

Figure 4. Control input derivative time responses for φ = 0.04 (1), 1 (2), 25 (3)

5 Conclusion

This paper has introduced an suboptimal tracking design procedure for linear continuous-time SISO systems. The resulting control law has been derived for a class of references given by practical needs. The control simulation results clearly demonstrate the influence of weighting coefficients in the quadratic criterion on control behaviour. This fact can be important for the purposes of controlling some technological processes.

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