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The Wiener-Hopf Standard Control Problem: A Stable Fractional Approach

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Abstract

In this paper, we present the solution to the standard Wiener-Hopf control problem with the quadratic cost. The solution presented here is based on the stable, rational and proper fractional representation theory and spectral factorizations, and in particular the controller class is proper. Three cases of the external signal are considered. Under a set of assumptions, the minimum and finite costs are given. Meanwhile controllers are also parameterized in terms of an arbitrary stable, real rational, and strictly proper matrix $Z(s)$.

1 Introduction

The standard control problem with a quadratic cost or H_2-norm setting has been widely studied in recent years. For example, the standard H_2 problem has been solved by many researchers using different approaches (such as the state space method together with the Racciti equation by Doyle *et al.* (1989); the polynomial solution based on the polynomial equation by Hunt and Kučera (1992)). For the standard control problem with a quadratic cost, Park and Bongiorno (1989) used the Wiener-Hopf optimization based on the polynomial matrix fraction. Nett (1986) has considered the internal stability and controller parameterization based on the stable fractional representation. With Nett's results, Da Silveira and Correa (1992); Correa and Da Silveira (1995) have extended the Wiener-Hopf optimal design (Youla *et al.*, 1976; Youla and Bongiorno, 1985; Park and Bongiorno, 1989, 1990; Park and Youla, 1992; Bongiorno, 1995) and presented the two-degree-of-freedom Wiener-Hopf optimal design with tracking and disturbance rejection constraints in the stable, proper and rational fractional ring.

In this paper, we will address the Wiener-Hopf design for the standard control system based on Nett's results (1986) and the stable fractional representation. Specifically, formulas are derived under a set of assumptions that give the entire class of proper controllers with which the standard control system is internally stable and a quadratic cost function is finite or minimum. This controller class is also parameterized in terms of an arbitrary stable, real rational, and strictly proper matrix $Z(s)$. The optimal controller corresponds to the choice $Z(s) = 0$. The cost function considered here aims at treating the external signal which is modeled as three cases (a)

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the wide-sense stationary stochastic processes, with zero-mean; (b) the superposition of white-noise and wide-sense stationary stochastic processes with zero-mean; and (c) the superposition of white-noise, deterministic signals, and wide-sense stationary stochastic processes with zero-mean, respectively. All signals are assumed to be independent. Now that different criteria will give different solutions, here we will select the same criteria as Park and Bonogiorno (1989), and not as Correa and Silveria (1992, 1995).

The organization of this paper is as follows. Section 2 summarizes basic definitions and Lemmas. The main results will be stated in Section 3. The concluding remarks are given in Section 4.

Throughout this paper, let $M(s)$ represent the set of all stable, proper and real-rational fractional matrices, and $M(s)^{\perp}$ stand for the set of all proper antistable matrices. The symbols I, trace$(\cdot)$, det$(\cdot)$ and rank$(\cdot)$ are used for the identity matrix, the trace, the determinant, and the rank of matrix, respectively. The matrix $G_*(s)$ is the conjugate transpose of $G(-\overline{s})$ or $G_*(s) = G^*(-\overline{s})$ which, for real rational matrices, reduce to $G_*(s) = G^T(-s)$. A real matrix $G(s)$ is called para-Hermitian when $G = G_*$ Re denotes the real part of complex matrices. In the partial fraction expansion of $G(s)$, the contribution from all poles in $-\infty < Re\ s < 0, 0 \leq Re\ s < \infty$, and at $s = \infty$, are denoted by $(G)_+, (G)_-, (G)_\infty$, respectively. $G(s) \leq o(s^v)$ means that no entry of $G(s)$ grows faster than s^v as $s \to \infty$. Clearly, $(G)_+ \leq 0(s^{-1})$ and $(G)_- \leq 0(s^{-1})$. For the sake of simplification, the argument s of the matrices or vectors is omitted. Assume that all matrices have appropriate dimensions.

2 Problem Formulation and Preliminaries

The standard control system under study is shown in Fig.1. The typical design problem of the control system can be equivalent to the design problem of the standard control system. The subsystem P accounts for all components in the system except the multivariable controller K. The controller has its input the measured variables which are the elements of the vector y. The vector z accounts for the regulated variables, the vector u represents the control input, and the vector e denotes the exogenous inputs such as disturbances, reference inputs, and measurement noise.

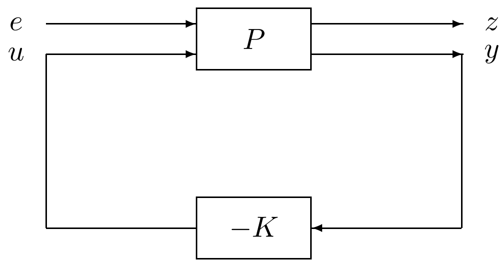

Fig.1 the standard control system

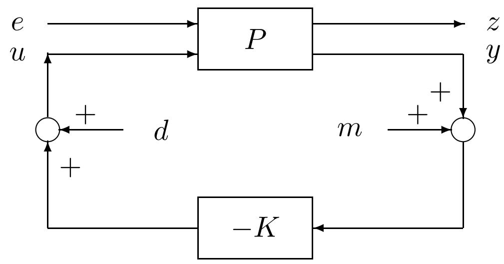

Fig.2 the standard control system for describing internal stability

For the subsystem P, these variables are related by

$$
\begin{aligned}
z(s) &= P_{11}(s)e(s) + P_{12}(s)u(s) \\
y(s) &= P_{21}(s)e(s) + P_{22}(s)u(s) \\
u(s) &= -K(s)y(s)
\end{aligned}
\tag{1}
$$

Furthermore, using a matrix formulation, (1) can be rewritten as

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} e(s) \\ u(s) \end{bmatrix} \tag{2}$$

The Wiener-Hopf standard problem is that: find a real rational proper controller K so that the standard control system is internally stable and the variance of the vector z is finite or minimum.

Fig.2 is used to assess the internal stability of the standard control system, where d, m are introduced to describe the internal stability. The usual internal stability requires one to choose a controller K shown in Fig.2 in such a way as to ensure that the closed loop transfer functions from (e, d, m) to (z, y, u) is rational, proper and stable. The closed loop transfer function matrix is described by

$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} P_{11} - P_{12}K(I+P_{22}K)^{-1}P_{21} & -P_{12}K(I+P_{22}K)^{-1} & P_{12}K(I+KP_{22})^{-1} \\ (I+P_{22}K)^{-1}P_{21} & -(I+P_{22}K)^{-1}P_{22}K & (I+P_{22}K)^{-1}P_{22} \\ -K(I+P_{22}K)^{-1}P_{21} & -K(I+P_{22}K)^{-1} & I - K(I+P_{22}K)^{-1}P_{22} \end{bmatrix} \begin{bmatrix} e \\ m \\ d \end{bmatrix} \tag{3}$$

Let

$$R = K(I+P_{22}K)^{-1} = (I+KP_{22})^{-1}K \tag{4}$$

Substituting (4) into (3) yields

$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} P_{11} - P_{12}RP_{21} & -P_{12}R & P_{12}(I-RP_{22}) \\ P_{21} - P_{22}RP_{21} & -P_{22}R & P_{22}(I-RP_{22}) \\ -RP_{21} & -R & I - RP_{22} \end{bmatrix} \begin{bmatrix} e \\ m \\ d \end{bmatrix} \tag{5}$$

The transformation mapping $K \mapsto R$ is defined for all K such that $\det(I + KP_{22}) \neq 0$ [1]; the inverse mapping, defined for all R such that $(I - RP_{22})$ is biproper, is given by $K = (I - RP_{22})^{-1}R$, and K is a proper rational matrix if and only if R is also a proper rational matrix.

Note that not every P are stabilizable, an obvious non-stabilizable P is $P_{12} = 0$, P_{11} unstable. So we shall only discuss the case when P is stabilizable, while P is called admissible. Let $P_{22} = \widetilde{D}_{22}^{-1}\widetilde{N}_{22} = N_{22}D_{22}^{-1}$ be coprime factorizations in $M(s)$. For such factorizations there are V_{22},U_{22},$\widetilde{V}_{22}$,$\widetilde{U}_{22} \in M(s)$ such that (Vidyasagar, 1985)

$$\begin{bmatrix} V_{22} & U_{22} \\ -\widetilde{N}_{22} & \widetilde{D}_{22} \end{bmatrix} \begin{bmatrix} D_{22} & -\widetilde{U}_{22} \\ N_{22} & \widetilde{V}_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} D_{22} & -\widetilde{U}_{22} \\ N_{22} & \widetilde{V}_{22} \end{bmatrix} \begin{bmatrix} V_{22} & U_{22} \\ -\widetilde{N}_{22} & \widetilde{D}_{22} \end{bmatrix} \tag{6}$$

Lemma 1 (Nett, 1986) Let $Q_a = P_{12}D_{22}$, $Q_b = \widetilde{D}_{22}P_{21}$ and $Q_c = P_{11} - P_{12}D_{22}U_{22}P_{21}$, P is admissible if and only if Q_a, Q_b, $Q_c \in M(s)$.

Assumption 1 Q_a, Q_b, $Q_c \in M(s)$

Under this assumption, the class of all stabilizing controllers is denoted by (Nett 1986)

$$K = (V_{22} + Q\widetilde{N}_{22})^{-1}(U_{22} - Q\widetilde{D}_{22}) \tag{7}$$

for all $Q \in M(s)$ such that $D_{22}(V_{22} + Q\widetilde{N}_{22})$ is biproper. Substituting (7) into (4), we have

$$R = D_{22}(U_{22} - Q\widetilde{D}_{22}) \tag{8}$$

[1]Obviously, the system must be well-defined, namely, $\det(I + KP_{22}) \neq 0$

With (8), the transfer function matrix of the closed loop system is described by

$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} Q_c + Q_a Q Q_b & -Q_a(U_{22} - Q\widetilde{D}_{22}) & Q_a(V_{22} + Q\widetilde{N}_{22}) \\ (\widetilde{V}_{22} + N_{22}Q)Q_b & -N_{22}(U_{22} - Q\widetilde{D}_{22}) & N_{22}(V_{22} + Q\widetilde{N}_{22}) \\ -(\widetilde{U}_{22} - D_{22}Q)Q_b & -D_{22}(U_{22} - Q\widetilde{D}_{22}) & D_{22}(V_{22} + Q\widetilde{N}_{22}) \end{bmatrix} \begin{bmatrix} e \\ m \\ d \end{bmatrix} \tag{9}$$

If the free parameter $Q \in M(s)$ satisfies $D_{22}(V_{22} + Q\widetilde{N}_{22})$ is biproper, then it can be seen from Assumption 1 and Lemma 1 that all entries of the closed loop transfer matrix in (9) must belong to $M(s)$. Namely, the standard control system shown in Fig. 1 is internal stable, and $(I - RP_{22}) = D_{22}(V_{22} + Q\widetilde{N}_{22})$ is biproper. Further, $K = (I - RP_{22})^{-1}R$ is a proper rational matrix, and the standard control system is also well-defined.

In the highlight of analyses and conclusions above, the Wiener-Hopf standard problem is equivalent to find all $Q \in M(s)$ such that the variance of the vector z is finite or minimum and $D_{22}(V_{22} + Q\widetilde{N}_{22})$ is biproper, i.e. $\det(D_{22}(V_{22} + Q\widetilde{N}_{22}))(\infty) \neq 0$. Note that in Fig.2, m, d are fictitious inputs which are taken into account only in the stability condition. In Fig. 1, with (9), we can obtain the transfer function from z to e in terms of the free parameter Q as follows

$$z = (Q_c + Q_a Q Q_b)e \tag{10}$$

where $Q_c + Q_a Q Q_b \in M(s)$.

3 Main Results

3.1 The external signal e is the wide-sense stationary stochastic processes, with zero-mean

Let the spectral density of the vector e be R_e, from the definition of the wide-sense stationary stochastic processes, it is known that R_e has the following properties

(a) R_e is para-Hermitian, i.e. $R_e(s) = R_e^T(-s)$.

(b) R_e is analytic and $R_e(s) \geq 0$ on the $j\omega$-axis, and all entries of R_e have relative degree 2.

In view of the analysis of linear system whose input is the wide-sense stationary stochastic processes, with zero-mean, if the standard control system is internally stable, then the output $\{z(t), t \in T\}$ is also the wide-sense stationary stochastic processes, with zero-mean, and its spectral density R_z can be given by

$$R_z = (Q_c + Q_a Q Q_b)R_e(Q_c + Q_a Q Q_b)_* \quad \text{where} \quad (Q_* = Q^T(-s)) \tag{11}$$

Furthermore, the cost function of the standard control system can be written as

$$\begin{aligned} J = \; & E[z^T(t)z(t)] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{trace}(R_z)ds \\ = \; & \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{trace}(Q_c R_e Q_{c*} + 2Q_{a*}Q_c R_e Q_{b*}Q_* + Q_{a*}Q_a Q Q_b R_e Q_{b*}Q_*)ds \end{aligned} \tag{12}$$

Since some inherent properties of P cannot be changed by the controller K, some assumptions on the general plant P must be made such that the integral is finite, and the optimal solution is unique.

Assumption 2 $Q_{a*}Q_a$,$Q_b R_e Q_{b*}$ are nonsingular on $j\omega$-axis.

Applying the spectral factorization theorem (Youla, 1961), together with the fact that $Q_{a*}Q_a$,$Q_bR_eQ_{b*}$ are para-Hermitian and nonsingular on the $j\omega$-axis (Assumption 2), and Q_a,Q_b,R_e are analytic on the $j\omega$-axis, we have the spectra factorizations of $Q_{a*}Q_a$,$Q_bR_eQ_{b*}$ as follows

$$Q_{a*}Q_a = \Lambda_*\Lambda, \quad Q_bR_eQ_{b*} = \Omega\Omega_* \tag{13}$$

where Λ,Λ^{-1},Ω,Ω^{-1} are analytic in $Re(s) \geq 0$. The following additional assumptions are needed in order for the free parameter Q to be proper, the controller K is also proper, and the standard control system to be well-defined.

Assumption 3 The order relationships

$$(Q_{a*}Q_a)^{-1} \leq o(s^{-2v_1}), \quad (Q_bR_eQ_{b*})^{-1} \leq o(s^{-2v_2}) \text{ and } (\Lambda_*^{-1}A\Omega_*^{-1})_+ \leq o(s^{-(1+v_3)})$$

are satisfied where $A = Q_{a*}Q_cR_eQ_{b*}$.
Assumption 4 $(v_1 + v_2 + v_3) \geq 0$
Assumption 4' $(v_1 + v_2 + v_3 + 1) \geq 0$
Assumption 5 P_{22} is strictly proper.
Assumption 5' P_{22} is proper, V_{22} is biproper.

Either Assumption 4 or 4' must be met, and the same to Assumptions 5 and 5'.

Lemma 3 (a) If Assumption 5 is satisfied, and Q is proper, then the controller K is proper, and the standard control system is well-defined; (b) If Assumption 5' is satisfied, and Q is strictly proper, then the controller K is proper, and the standard control system is well-defined; (c) If Assumptions 5 and 5' are not satisfied, it is necessary to check whether $\det(D_{22}(V_{22}+Q\widetilde{N}_{22}))(\infty)$ is zero or not for determining the properness of the controller K and the well-defined of the standard control system. The proof is trival, and is omitted.

Assumptions 3-5 (or 3, 4', and 5') guarantee that Q is strictly proper (or proper), the controller K is proper, and the system is well-defined. Since the Wiener-Hopf design employs the standard variational method (Youla, *et al.,* 1976; Weston and Bongiorno, 1972), the candidate solution is merely given by this method.

Theorem 1 Suppose that Assumptions 1-3,4-5(or 4'-5') are satisfied. Then the following hold

(a) The set of all acceptable Q and R's that stabilizes the standard control system and yields the finite cost can be written by the formula

$$\begin{aligned} Q &= \Lambda^{-1}(Z - \{\Lambda_*^{-1}A\Omega_*^{-1}\}_+)\Omega^{-1}, \; Z \in M(s) \text{ is strictly proper.} \\ R &= D_{22}U_{22} - D_{22}\Lambda^{-1}(Z - \{\Lambda_*^{-1}A\Omega_*^{-1}\}_+)\Omega^{-1}\widetilde{D}_{22} \end{aligned} \tag{14}$$

(b) The acceptable Q and R which minimize the cost and correspond to the choice $Z = 0$ in (14), are given by

$$\widetilde{Q} = -\Lambda^{-1}\{\Lambda_*^{-1}A\Omega_*^{-1}\}_+\Omega^{-1}, \quad \widetilde{R} = D_{22}U_{22} + D_{22}\Lambda^{-1}\{\Lambda_*^{-1}A\Omega_*^{-1}\}_+\Omega^{-1}\widetilde{D}_{22} \tag{15}$$

(c) Let the minimum cost J be denoted by $\widetilde{J}$, then for any allowed Z (i.e. $Z \in M(s)$ is strictly proper)

$$J = \widetilde{J} + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \text{trace}(ZZ_*)ds \tag{16}$$

$$\widetilde{J} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \text{trace}(Q_cR_eQ_{c*})ds - \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \text{trace}(\{\Lambda_*^{-1}A\Omega_*^{-1}\}_+\{\Lambda_*^{-1}A\Omega_*^{-1}\}_{+*})ds \tag{17}$$

where $A = Q_{a*}Q_cR_eQ_{b*}$.

Proof: First, we shall give the optimal solution, that is, selecting $Q \in M(s)$ to minimize the cost J in (13). Here, the Wiener-Hopf design will employ the standard variational method.

Set

$$Q = \widetilde{Q} + \epsilon\hat{Q} \tag{18}$$

where $\widetilde{Q}$ is the optimal solution, and $\hat{Q}$ is an arbitrary proper, stable rational matrix, i.e. $\hat{Q} \in M(s)$, and ϵ is a scalar. Directly computing $\frac{\partial J}{\partial \epsilon}$, we have

$$\frac{\partial J}{\partial \epsilon} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{trace}(2A\hat{Q}_* + 2\Phi_{rr}\widetilde{Q}B\hat{Q}_* + 2\epsilon\Phi_{rr}\hat{Q}B\hat{Q}_*)ds \tag{19}$$

where $\Phi_{rr} = Q_{a*}Q_a, B = Q_bR_eQ_{b*}$.

Note that the necessary condition that $\hat{Q}$ minimizes the cost J is $\frac{\partial J}{\partial \epsilon}|_{\epsilon=0} = 0$, using this condition to (19), we can obtain

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{trace}(X\hat{Q}_*)ds = 0, \quad X = A + \Phi_{rr}\widetilde{Q}B \tag{20}$$

where $\hat{Q} \in M(s)$, $\hat{Q}_* \in M^{\perp}(s)$. If (20) is identical to zero for an arbitrary $\hat{Q}$, then X must belong to $M^{\perp}(s)$, that is, X is antistable and analytic in $Re(s) < 0$. Moreover, because the integrand in (20) must be finite on $j\omega$-axis, so X is also analytic on $j\omega$-axis. Hence we conclude that X is analytic in $Re(s) \leq 0$. Then we have

$$X = A + Q_{a*}Q_a\widetilde{Q}Q_bR_eQ_{b*} = A + \Lambda_*\Lambda\widetilde{Q}\Omega\Omega_* \tag{21}$$

and

$$\Lambda_*^{-1}X\Omega_*^{-1} = \Lambda_*^{-1}A\Omega_*^{-1} + \Lambda\widetilde{Q}\Omega \tag{22}$$

Further, we can obtain the partial fraction expansion of $\Lambda_*^{-1}A\Omega_*^{-1}$ as follows

$$\Lambda_*^{-1}A\Omega_*^{-1} = (\Lambda_*^{-1}A\Omega_*^{-1})_+ + (\Lambda_*^{-1}A\Omega_*^{-1})_- + (\Lambda_*^{-1}A\Omega_*^{-1})_\infty \tag{23}$$

Since R_e is the spectral density of the wide-sense stationary stochastic process such that $R_e \leq o(s^{-2})$, we have $R_e = J_eJ_{e*}$ and

$$\Lambda_*^{-1}A\Omega_*^{-1} = \Lambda_*^{-1}Q_{a*}Q_cR_eQ_{b*}\Omega_*^{-1} = \Lambda_*^{-1}Q_{a*}Q_cJ_eJ_{e*}Q_{b*}\Omega_*^{-1} \tag{24}$$

From (13), $\Lambda_*^{-1}Q_{a*}$, $J_{e*}Q_{b*}\Omega_*^{-1}$ are finite at $s = \infty$, which means both matrices are proper, i.e. $\Lambda_*^{-1}Q_{a*}$, $J_{e*}Q_{b*}\Omega_*^{-1} \leq o(s^0)$. Further, $Q_c \in M(s)$, $J_e \leq o(s^{-1})$, we have $\Lambda_*^{-1}A\Omega_*^{-1}$ is strictly proper, so $(\Lambda_*^{-1}A\Omega_*^{-1})_\infty \equiv 0$.

Substituting (23) into (22), we can obtain

$$\Lambda_*^{-1}X\Omega_*^{-1} = (\Lambda_*^{-1}A\Omega_*^{-1})_+ + (\Lambda_*^{-1}A\Omega_*^{-1})_- + \Lambda\widetilde{Q}\Omega \tag{25}$$

and further

$$\Lambda_*^{-1}X\Omega_*^{-1} - (\Lambda_*^{-1}A\Omega_*^{-1})_- = (\Lambda_*^{-1}A\Omega_*^{-1})_+ + \Lambda\widetilde{Q}\Omega \tag{26}$$

The left-hand side of (26) is analytic in $Re(s) < 0$, whereas the right-hand side is analytic in $Re(s) \geq 0$. Hence, (26) holds only if both sides of (26) are polynomial matrices, moreover, note that the left-hand side of (26) is also strictly proper, this means that both sides are identically equal to zero, so we have

$$(\Lambda_*^{-1}A\Omega_*^{-1})_+ + \Lambda\widetilde{Q}\Omega = 0 \tag{27}$$

which implies

$$\widetilde{Q} = -\Lambda^{-1}(\Lambda_*^{-1}A\Omega_*^{-1})_+\Omega^{-1} \tag{28}$$

Next, we shall prove $\widetilde{Q}$ in (28) satisfies the sufficient condition under which the cost J is minimum, that is, $\widetilde{Q}$ satisfies the inequality $\dfrac{\partial^2 J}{\partial \epsilon^2} > 0$; together with $\Phi_{rr} = Q_{a*}Q_a$, we have

$$\frac{\partial^2 J}{\partial \epsilon^2} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{trace}(2\Phi_{rr}\hat{Q}B\hat{Q}_*)ds = \frac{1}{\pi}\int_{-j\infty}^{j\infty}\sum_{i,j=1}^{n}|(\Lambda\hat{Q}\Omega)_{ij}|^2(j\omega)d\omega \tag{29}$$

Clearly, $\dfrac{\partial^2 J}{\partial \epsilon^2}$ is nonnegative, and it is equal to zero if and only if $\Lambda\hat{Q}\Omega = 0$, namely, $\hat{Q} = 0$, while the cost J is independent of ϵ; this means the optimization problem is meaningless. Hence we conclude that with (28), we have $\dfrac{\partial^2 J}{\partial \epsilon^2} > 0$. Substituting (28) into (8), we can get (15). Finally, we shall establish (c) in the Theorem. Based on (13), (23) and the residual theorem, the substitution of (28) to (12) yields

$$\begin{aligned} J = {} & \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{trace}(Q_cR_eQ_{c*} + ZZ_* - \{\Lambda_*^{-1}A\Omega_*^{-1}\}_+\{\Lambda_*^{-1}A\Omega_*^{-1}\}_{+*})ds \\ = {} & \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{trace}(Q_cR_eQ_{c*} - \{\Lambda_*^{-1}A\Omega_*^{-1}\}_+\{\Lambda_*^{-1}A\Omega_*^{-1}\}_{+*})ds \\ & + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{trace}(ZZ_*)ds \geq \widetilde{J} \end{aligned} \tag{30}$$

With the equation above, (a) and (c) in Theorem 1 are immediately proven. In view of Lemma 3, under the assumptions, the standard control system is also well-defined. This completes the proof.

3.2 The external signal e is the superposition of white-noise and wide-sense stationary stochastic processes with zero-mean

For the internal stable linear system, when the input is white-noise, the output is also the wide-sense stationary stochastic processes with zero-mean, so the same cost function of the standard control system in (12) can be applied to this case; however, the input vector e is not the wide-sense stationary stochastic processes yet; its spectra density is not strictly proper, but is analytic on $j\omega$-axis. To guarantee the integrand have relative degree of 2, some assumptions are needed to be made such that the variables not adjusted by the controller are acceptable.

Assumption 6 $P_{11}R_eP_{11*} \leq o(s^{-2})$
Assumption 7 $(Q_{a*}Q_a)^{-1} \leq o(s^{-2v_1}),\ (Q_bR_eQ_{b*})^{-1} \leq o(s^{-2v_2})$
Assumption 8 $v_1 + v_2 \geq 0$
Assumption 8' $v_1 + v_2 \geq 1$
Assumption 9 P_{22} is strictly proper.
Assumption 9' P_{22} is proper, and V_{22} is biproper.

Theorem 2 Consider the standard control system shown in Fig.1, the external signal e is the superposition of white-noise and wide-sense stationary stochastic processes with zero-mean. Suppose that Assumptions 1-2, 6-8 and 9 (or 6-8' and 9') are satisfied, then

(a) The set of all acceptable Q and R's that stabilizes the standard control system and yields the finite cost is denoted by the formula

$$\begin{aligned} Q = & \ \Lambda^{-1}(Z - \{\Lambda_*^{-1} A \Omega_*^{-1}\}_+ - \{\Lambda D_{22}^{-1} \widetilde{U}_{22} \Omega\}_\infty)\Omega^{-1}, \quad Z \in M(s) \text{ is strictly proper} \\ R = & \ D_{22}U_{22} - D_{22}\Lambda^{-1}(Z - \{\Lambda_*^{-1} A \Omega_*^{-1}\}_+ - \{\Lambda D_{22}^{-1} \widetilde{U}_{22} \Omega\}_\infty)\Omega^{-1} \widetilde{D}_{22} \end{aligned} \tag{31}$$

(b) The acceptable Q and R which minimize the cost are given by

$$\begin{aligned} \widetilde{Q} = & \ -\Lambda^{-1}(\{\Lambda_*^{-1} A \Omega_*^{-1}\}_+ + \{\Lambda D_{22}^{-1} \widetilde{U}_{22} \Omega\}_\infty)\Omega^{-1} \\ \widetilde{R} = & \ D_{22}U_{22} + D_{22}(\Lambda^{-1}\{\Lambda_*^{-1} A \Omega_*^{-1}\}_+ + \{\Lambda D_{22}^{-1} \widetilde{U}_{22} \Omega\}_\infty)\Omega^{-1} \widetilde{D}_{22} \end{aligned} \tag{32}$$

and correspond to the choice $Z = 0$.

(c) Let the minimum cost J be denoted by $\widetilde{J}$, then for any strictly proper Z (i.e. $Z \in M(s)$)

$$\begin{aligned} J = & \ \widetilde{J} + \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \text{trace}(Z Z_*) ds \geq \widetilde{J} \\ \widetilde{J} = & \ \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \text{trace}(P_{11} R_e P_{11*} - \Gamma \Gamma_* + \{\Lambda_*^{-1} A \Omega_*^{-1}\}_- \{\Lambda_*^{-1} A \Omega_*^{-1}\}_{-*}) ds \end{aligned} \tag{33}$$

where $A = Q_{a*} Q_c R_e Q_{b*}$, $\Gamma = \Lambda_*^{-1} Q_{a*} P_{11} R_e Q_{b*} \Omega_*^{-1}$.

Proof: By Assumptions 2,6 and 7, we have

$$(Q_a \Lambda^{-1})_* (Q_a \Lambda^{-1}) = I, \ (\Omega^{-1} Q_b J_e)(\Omega^{-1} Q_b J_e)_* = I, \ J_e J_{e*} = R_e \tag{34}$$

which means that $\det(Q_a \Lambda^{-1})(\infty)$, and $\det(\Omega^{-1} Q_b J_e)(\infty)$ are finite, so we have $Q_a \Lambda^{-1} \leq o(s^0)$, $\Omega^{-1} Q_b J_e \leq o(s^0)$, thus, $\Gamma_* \leq o(s^{-1})$,$\Gamma \leq o(s^{-1})$. Using the same argument in Theorem 1, it is not difficult to obtain the candidate optimal solution bellow

$$\widetilde{Q} = \Lambda^{-1}(P_1 - \{\Lambda_*^{-1} A \Omega_*^{-1}\}_+)\Omega^{-1} \tag{35}$$

where P_1 is not equal to zero. Note that $\Lambda_*^{-1} A \Omega_*^{-1}$, $\Lambda_*^{-1} X \Omega_*^{-1}$ in (25) are not strictly proper, and $(\Lambda_*^{-1} A \Omega_*^{-1})_\infty$, $(\Lambda_*^{-1} X \Omega_*^{-1})_\infty$ are not identical to zero. Substituting (35) into (12), we have

$$\begin{aligned} J = & \ \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \text{trace}[P_{11} R_e P_{11*} - \Gamma \Gamma_* + (\{\Lambda_*^{-1} A \Omega_*^{-1}\}_- + \{\Lambda_*^{-1} A \Omega_*^{-1}\}_\infty + P_1) \\ & (\{\Lambda_*^{-1} A \Omega_*^{-1}\}_- + \{\Lambda_*^{-1} A \Omega_*^{-1}\}_\infty + P_1)_*] ds \end{aligned} \tag{36}$$

Because R_e,Q_a,Q_b,Q_c are analytic on $j\omega$-axis, P_1 is real, and Λ^{-1},Ω^{-1} are also analytic on $Re(s) \geq 0$, respectively, then $\widetilde{Q}$ is also analytic on $j\omega$-axis. From (10), we conclude that z is analytic on $j\omega$-axis, so is the integrand in (36). Hence we only require that the integrand in (36) has relative degree of 2. By Assumption 6, we have $P_{11} R_e P_{11*} \leq o(s^{-2})$, $\Gamma \Gamma_* \leq o(s^{-2})$. Note that $\{\Lambda_*^{-1} A \Omega_*^{-1}\}_- \leq o(s^{-1})$, hence (36) is integrable only if $P_1 = -\{\Lambda_*^{-1} A \Omega_*^{-1}\}_\infty$. From the analysis above, we can obtain the optimal solution $\widetilde{Q}$ as follows

$$\widetilde{Q} = -\Lambda^{-1}(\{\Lambda_*^{-1} A \Omega_*^{-1}\}_+ + \{\Lambda_*^{-1} A \Omega_*^{-1}\}_\infty)\Omega^{-1} \tag{37}$$

Further, using the identity $\Lambda_*^{-1} A \Omega_*^{-1} = \Gamma - \Lambda D_{22}^{-1} \widetilde{U}_{22} \Omega$, we can obtain

$$\{\Lambda_*^{-1} A \Omega_*^{-1}\}_\infty = \{\Gamma\}_\infty - \{\Lambda D_{22}^{-1} \widetilde{U}_{22} \Omega\}_\infty = -\{\Lambda D_{22}^{-1} \widetilde{U}_{22} \Omega\}_\infty \tag{38}$$

The substitution of (38) into (37) yields (32) in Theorem 2. The same proof can be applied to the properness and stability of Q(or $\widetilde{Q}$), the properness of the controller, and the well-defined of the standard control system. The other equalities in Theorem 2 can be derived from (36). This completes the proof of Theorem 2.

3.3 The external signal e is the superposition of white-noise, deterministic signals, and wide-sense stationary stochastic processes with zero-mean

Due to the existence of the deterministic signal, the input may include persistent signals, while the input is not square-integrable, i.e. is not of finite energy, and the cost function is needed to be defined again. Here, we select the energy of the deterministic output parts as a measure of the system performance. To ensure that the solution of Wiener-Hopf standard problem exists, the system is required not only to be stable but also the output should be square-integrable. Thus, some assumptions are needed to be made.

Assumption 10 $Q_{a*}Q_a$, $Q_bG_eQ_{b*}$ are non-singular on $j\omega$-axis, and $Q_bG_eQ_{b*}$, $Q_cG_eQ_{b*}$ are analytic on $j\omega$-axis (for the definition of G_e see (43)).

Under Assumption 10, there exist the spectra factorizations $Q_{a*}Q = \Lambda_*\Lambda,\ Q_bG_eQ_{b*} = \Omega\Omega_*$, where Λ, Λ^{-1}, Ω, Ω^{-1} are analytic in the $Re(s) \geq 0$.

Assumption 11 $\mathrm{trace}(P_{11}G_eP_{11*} - \Gamma\Gamma_*)$ is analytic on $j\omega$-axis.
Assumption 12 $P_{11}G_eP_{11*} \leq o(s^{-2})$

Definition 1 For the square-integrable deterministic signal, we select the energy as a measure of the system performance. Then we have the following cost function corresponding to the deterministic output

$$J_1 = \int_{-\infty}^{\infty} \overline{z}^T(t)\overline{z}(t)dt < \infty \tag{39}$$

where $\overline{z}(t)$ denotes the deterministic part in the system output.

On the basis of Parseval Theorem, (39) can be rewritten as

$$J_1 = \int_{-\infty}^{\infty} \overline{z}^T(t)\overline{z}(t)dt = \int_{-\infty}^{\infty} \mathrm{trace}[\overline{z}(t)\overline{z}^T(t)]dt = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{trace}[\overline{z}(s)\overline{z}(s)_*]ds \tag{40}$$

For brevity, $[\overline{z}(s)\overline{z}(s)_*]$ is denoted as $< z(s)z(s)_* >$. Hence, together with the performance corresponding to the stochastic part of the system output, the cost function of the standard control system here can be written as

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{trace}[\lambda_1 R_z + \lambda_2 < z(s)z(s)_* >]ds \tag{41}$$

where the first term in the integrand denotes the steady-state performance, and the second denotes the transient performance. λ_1 and λ_2 are nonnegative constants for weighting the relative importance of the deterministic and stochastic components of the regulated variables. Generally, we should select frequency weighting functions, here it is assumed that frequency weighting functions have been absorbed in the general plant P. Substituting (10) into (41), we can obtain

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{trace}[(Q_c + Q_aQQ_b)(\lambda_1 R_e + \lambda_2 < e(s)e(s)_* >)(Q_c + Q_aQQ_b)_*]ds \tag{42}$$

Let

$$G_e = \lambda_1 R_e + \lambda_2 < e(s)e(s)_* > \tag{43}$$

where R_z represents the spectral density matrix for the stochastic component of the output, and the second term on the right-hand side of (43) denotes the equivalent component of the deterministic output. G_e is called the general spectral density. Hence, together with the definition of G_e, we have set the third case of the external signal into the framework of Wiener-Hopf sense. Using (43), (42) can be rewritten as

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \text{trace}[(Q_c + Q_a Q Q_b) G_e (Q_c + Q_a Q Q_b)_*] ds \tag{44}$$

Theorem 3 Consider the standard control system shown in Fig.1, the external signal e is the superposition of white-noise, wide-sense stationary stochastic processes with zero-mean, and deterministic signals. Suppose that Assumptions 10-12 and 7-9 are satisfied, then

(a) The set of all acceptable Q and R's that stabilizes the standard control system and yields the finite cost is denoted by the formula

$$\begin{aligned} Q = &\ \Lambda^{-1}(Z - \{\Lambda_*^{-1} A \Omega_*^{-1}\}_+ - \{\Lambda D_{22}^{-1} \widetilde{U}_{22} \Omega\}_\infty)\Omega^{-1}, \quad Z \in M(s) \text{ is strictly proper} \\ R = &\ D_{22} U_{22} - D_{22} \Lambda^{-1}(Z - \{\Lambda_*^{-1} A \Omega_*^{-1}\}_+ - \{\Lambda D_{22}^{-1} \widetilde{U}_{22} \Omega\}_\infty)\Omega^{-1} \widetilde{D}_{22} \end{aligned} \tag{45}$$

(b) The acceptable Q and R which minimize the cost are given by

$$\begin{aligned} \widetilde{Q} = &\ -\Lambda^{-1}(\{\Lambda_*^{-1} A \Omega_*^{-1}\}_+ + \{\Lambda D_{22}^{-1} \widetilde{U}_{22} \Omega\}_\infty)\Omega^{-1} \\ \widetilde{R} = &\ D_{22} U_{22} + D_{22}(\Lambda^{-1}\{\Lambda_*^{-1} A \Omega_*^{-1}\}_+ + \{\Lambda D_{22}^{-1} \widetilde{U}_{22} \Omega\}_\infty)\Omega^{-1} \widetilde{D}_{22} \end{aligned} \tag{46}$$

and correspond to the choice $Z = 0$.

(c) Let the minimum cost J be denoted by $\widetilde{J}$, then for any allowed Z (i.e. $Z \in M(s)$ is strictly proper),

$$\begin{aligned} J = &\ \widetilde{J} + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \text{trace}(Z Z_*) ds \geq \widetilde{J} \\ \widetilde{J} = &\ \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \text{trace}(P_{11} G_e P_{11*} - \Gamma\Gamma_* + \{\Lambda_*^{-1} A \Omega_*^{-1}\}_- \{\Lambda_*^{-1} A \Omega_*^{-1}\}_{-*}) ds \end{aligned} \tag{47}$$

where $A = Q_{a*} Q_c G_e Q_{b*}$, $\Gamma = \Lambda_*^{-1} Q_{a*} P_{11} G_e Q_{b*} \Omega_*^{-1}$

Proof: Under the assumptions above, G_e is proper, but it may be not analytic on $j\omega$-axis. The Assumptions 10-12 ensure the integral exists, and they also simplify the problem, otherwise it is difficult to solve the problem mathematically. The proof is similar to the proof of Theorem 1, thus it is omitted.

4 Conclusion

The optimal control system design in the Wiener-Hopf sense has been treated for the standard control system in the stable, proper and rational fractional ring. The so-called Wiener-Hopf standard problem is defined. Using the stable fractional representation approach and the general results (Nett, 1986) of the internal stability for the control system with the standard configuration, specifically, formulas are derived under a set of assumptions that give the entire class of proper controllers for which the standard control system is internally stable and a quadratic cost function is finite or minimum. This controller class is parameterized in terms of an arbitrary stable, real rational, and strictly proper matrix $Z(s)$. Meanwhile, the solutions of the standard problem corresponding to the three cases of the input signal are considered.

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