Boundary Control of the Korteweg–de Vries–Burgers Equation: Further Results on Stabilization and Numerical Demonstration *

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Abstract

We consider the Korteweg–de Vries–Burgers (KdVB) equation on the interval $[0, 1]$. Motivated by poor decay rates of a recently proposed control law by Liu and Krstic which keeps some of the boundary conditions as homogeneous, we propose a strengthened set of feedback boundary conditions. We establish stability properties of the closed–loop system and illustrate the performance improvement by a simulation example.

Keywords — Korteweg–de Vries–Burgers equation, nonlinear boundary feedback control, global stabilization.

1 Introduction

The Korteweg–de Vries–Burgers (KdVB) equation is one of the simplest nonlinear mathematical models displaying the features of both dispersion and dissipation. It serves as a model of long waves in shallow water and some other physical phenomena. The usual and simplest setting in which the controlled and uncontrolled KdVB equation or the simpler KdV equation is considered is either the case of periodic boundary conditions (Biler, 1984b; Bona *et al.*, 1996, 1992; Russel and Zhang, 1995) or the case where the spatial domain is the whole real line (Biler, 1984a; Bona and Smith, 1975). As a next step in the analysis of a system it is natural to consider the controllability (Rosier, 1997) and stabilization (Zhang, 1994) on a bounded domain. In a recent work Liu and Krstic (1998) consider a boundary feedback stabilization problem for a KdVB equation on a finite spatial interval. Our paper is motivated by relatively poor performance of the controller in (Liu and Krstic, 1998) which we have observed in numerical simulations. In this paper we propose a more aggressive control law that achieves better performance. Our control law can be implemented via any of the following three variables actuated at one boundary with u held at zero at the other boundary: (u_x, u_{xx}), (u, u_x), (u, u_{xx}). The uncontrolled versions of some of these problems are known not to be asymptotically stable. An example of a physical problem where our control law would be implementable is the water channel setup with boundary

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actuation discussed in Rosier (1997). In Section 2 we prove the stability and existence of solutions of the resulting boundary controlled KdVB equation. In Section 3 we provide a numerical example after a brief description of the finite difference numerical method we used.

2 Stabilization

Consider the Korteweg–de Vries–Burgers equation

$$w_t - \epsilon w_{xx} + \delta w_{xxx} + w w_x = 0, \quad x \in [0,1], \quad t > 0, \tag{2.1}$$

with ϵ, $\delta > 0$ and with some initial data

$$w(x,0) = w_0(x), \quad x \in [0,1]. \tag{2.2}$$

Liu and Krstic (1998) proposed the control law

$$w(0,t) = 0, \tag{2.3}$$

$$w_x(1,t) = 0, \tag{2.4}$$

$$w_{xx}(1,t) = \frac{1}{\delta}\left(c + \frac{1}{9c}w^2(1,t)\right) w(1,t), \tag{2.5}$$

where $c > 0$, and showed that it globally asymptotically stabilizes the zero solution. Unfortunately, as we shall see in Section 3, the choice $w_x(1,t) = 0$ results in slow convergence to zero. For this reason, in this paper we seek and find a more aggressive boundary condition that also uses $w_x(1,t)$ for feedback:

$$w(0,t) = 0, \tag{2.6}$$

$$w_x(1,t) = -\frac{1}{\epsilon}\left(c + \frac{1}{9c}w^2(1,t)\right) w(1,t), \tag{2.7}$$

$$w_{xx}(1,t) = \frac{1}{\epsilon^2}\left(c + \frac{1}{9c}w^2(1,t)\right)^2 w(1,t). \tag{2.8}$$

It is clear that, since (2.7) and (2.8) are invertible functions, this control law can be implemented via any of the following three variables at the 1-boundary: (u_x, u_{xx}), (u, u_x), (u, u_{xx}).

Definition 1. *Let $\mathcal{H}_0^1(0,1) = \{w \in H^1(0,1) : w(0) = 0\}$. A function $w \in C\left([0,T]; \mathcal{H}_0^1(0,1)\right)$ is said to be a weak solution of equation (2.1), (2.2), (2.6), (2.7), (2.8) if w satisfies*

$$\begin{aligned}
&\int_0^T \Big((w,\varphi_t) - \epsilon\,(w_x,\varphi_x) - \delta\,(w_x,\varphi_{xx}) - (w w_x,\varphi)\Big)\,dt \\
&\quad = \int_0^T \left(\frac{\delta}{\epsilon^2}\left(c + \frac{1}{9c}w^2(1,t)\right)^2 + \left(c + \frac{1}{9c}w^2(1,t)\right)\right) w(1,t)\varphi(1,t)\,dt \\
&\qquad + \frac{\delta}{\epsilon}\int_0^T \left(c + \frac{1}{9c}w^2(1,t)\right) w(1,t)\varphi_x(1,t) - (w_0,\varphi(0))\;dt
\end{aligned} \tag{2.9}$$

for any $\varphi \in C^\infty\left([0,T]; H^2(0,1)\right)$ with $\varphi(0,t) = \varphi_x(0,t) = \varphi(x,T) = 0$, $(x,t) \in [0,1] \times [0,T]$, where $(\cdot,\cdot)$ denotes the usual scalar product in $L^2(0,1)$.

Theorem 1. *For any initial data $w_0 \in \mathcal{H}_0^1$ and for any $T > 0$ system* (2.1), (2.2), (2.6), (2.7), (2.8) *has a unique weak solution $w(x,t)$ with*

1. *Global exponential stability in the L^2–sense:*

$$\|w(t)\| \leq \|w_0\| e^{-\epsilon t}, \qquad \forall t \geq 0, \tag{2.10}$$

2. *Global asymptotic and semi–global exponential stability in the H^1-sense: there exist $k > 0$ such that*

$$\|w(t)\|_{H^1} \leq k \|w_0\|_{H^1} e^{k\|w_0\|_{H^1}^2} e^{-\epsilon t/2}, \qquad \forall t \geq 0. \tag{2.11}$$

Essentially the same statements hold for system (2.1), (2.2), (2.3), (2.4), (2.5) with different constant k. Since (2.10) and (2.11) are conservative energy estimates, they do not provide basis for a comparison of the two controllers.

Proof of Theorem 1. The proof of well–posedness goes exactly as in (Liu and Krstic, 1998). It is based on linearization of the system and the application of the Banach fixed point theorem.[1]

In order to prove the stability results we use energy estimates. These estimates are also part of the well–posedness proof as a priori estimates. The use of derivatives of order higher than the regularity claimed in this theorem is justified in (Liu and Krstic, 1998).

First take the L^2–inner product of (2.1) with w to obtain

$$\int_0^1 w_t w\, dx - \epsilon \int_0^1 w_{xx} w\, dx + \delta \int_0^1 w_{xxx} w\, dx + \int_0^1 w_x w^2\, dx = 0. \tag{2.12}$$

Using mainly integration by parts, we can write the various terms as

$$\int_0^1 w_t w\, dx = \frac{1}{2}\frac{d}{dt}\|w(t)\|^2, \tag{2.13}$$

$$\begin{aligned} -\epsilon \int_0^1 w_{xx} w\, dx &= -\epsilon\, w_x w|_0^1 + \epsilon \|w_x(t)\|^2 \\ &= -\epsilon w_x(1,t) w(1,t) + \epsilon \|w_x(t)\|^2 \\ &= \left(c + \frac{1}{9c} w^2(1,t)\right) w^2(1,t) + \epsilon \|w_x\|^2, \end{aligned} \tag{2.14}$$

$$\begin{aligned} \delta \int_0^1 w_{xxx} w\, dx &= \delta\, w_{xx} w|_0^1 - \delta \int_0^1 w_{xx} w_x\, dx \\ &= \delta \frac{1}{\epsilon^2} \left(c + \frac{1}{9c} w^2(1,t)\right)^2 w^2(1,t) - \frac{\delta}{2}\, w_x^2|_0^1 \\ &= \frac{\delta}{2\epsilon^2} \left(c + \frac{1}{9c} w^2(1,t)\right)^2 w^2(1,t) + \frac{\delta}{2} w_x^2(0,t), \end{aligned} \tag{2.15}$$

[1] For solutions corresponding to more regular initial conditions that are compatible with the boundary conditions further regularity statements hold and are stated in (Liu and Krstic, 1998). The regularity statements hold at least locally in time in our case too. Density argument shows that relaxing the regularity of the initial data we are left with the present form of the theorem.

and

$$\int_0^1 w_x w^2\,dx = \frac{1}{3}w^3(1,t) \le \frac{1}{18c}w^4(1,t) + \frac{c}{2}w^2(1,t). \tag{2.16}$$

Substituting (2.13)–(2.16) into (2.12) and simplifying the resulting inequality we obtain

$$\frac{d}{dt}\|w(t)\|^2 + 2\epsilon\|w_x\|^2 + \left(\frac{\delta c^2}{\epsilon^2} + c\right) w^2(1,t) + \left(\frac{1}{9c} + \frac{2\delta}{9\epsilon^2}\right) w^4(1,t) + \frac{\delta}{81c^2\epsilon^2}w^6(1,t) \le 0. \tag{2.17}$$

As a first consequence of (2.17) we obtain, using Poincaré's inequality, the inequality

$$\frac{d}{dt}\|w(t)\|^2 \le -2\epsilon\|w_x(t)\|^2 \le -2\epsilon\|w(t)\|^2, \tag{2.18}$$

which implies (2.10), i.e. the global exponential stability in the L^2 sense:

$$\|w(t)\| \le \|w_0\|e^{-\epsilon t}. \tag{2.19}$$

Returning back to (2.17), multiplying it by $e^{\epsilon t}$ and using (2.19) we get

$$\frac{d}{dt}\left(e^{\epsilon t}\|w(t)\|^2\right) + 2\epsilon e^{\epsilon t}\|w_x(t)\|^2 + e^{\epsilon t}\left(\frac{\delta c^2}{\epsilon^2} + c\right) w^2(1,t) + e^{\epsilon t}\left(\frac{1}{9c} + \frac{2\delta}{9\epsilon^2}\right) w^4(1,t)$$
$$+ \frac{\delta e^{\epsilon t}}{81c^2\epsilon^2}w^6(1,t) \le \epsilon e^{\epsilon t}\|w(t)\|^2 \le \epsilon\|w_0\|^2 e^{-\epsilon t}. \tag{2.20}$$

Integrating (2.20) with respect to time we obtain

$$e^{\epsilon t}\|w(t)\|^2 + \int_0^t e^{\epsilon\tau}\left(\|w_x(\tau)\|^2 + w^2(1,\tau) + w^4(1,\tau) + w^6(1,\tau)\right)\,d\tau \le M\|w_0\|^2. \tag{2.21}$$

Next, we take the L^2–inner product of (2.1) with $-w_{xx}$ to obtain

$$-\int_0^1 w_t w_{xx}\,dx + \epsilon\|w_{xx}\|^2 - \delta\int_0^1 w_{xxx}w_{xx}\,dx - \int_0^1 w w_x w_{xx}\,dx = 0. \tag{2.22}$$

The various terms of (2.22) can be written in the following way.

$$\begin{aligned}
-\int_0^1 w_t w_{xx}\,dx &= -\left. w_t w_x\right|_0^1 + \frac{1}{2}\frac{d}{dt}\|w_x\|^2 \\
&= -w_t(1,t)w_x(1,t) + \frac{1}{2}\frac{d}{dt}\|w_x\|^2 \\
&= \frac{1}{\epsilon}w_t(1,t)\left(c + \frac{1}{9c}w^2(1,t)\right) w(1,t) + \frac{1}{2}\frac{d}{dt}\|w_x\|^2 \\
&= \frac{1}{2}\frac{d}{dt}\left(\|w_x\|^2 + \frac{c}{\epsilon}w^2(1,t) + \frac{1}{18\epsilon c}w^4(1,t)\right),
\end{aligned} \tag{2.23}$$

$$\delta\int_0^1 w_{xxx}w_{xx}\,dx = \frac{\delta}{2}\left. w_{xx}^2\right|_0^1 = \frac{\delta}{2\epsilon^4}\left(c + \frac{1}{9c}w^2(1,t)\right)^4 w^2(1,t) - \frac{\delta}{2}w_{xx}^2(0,t), \tag{2.24}$$

$$\begin{aligned}\int_0^1 w_x w w_{xx}\,dx &\le \|w(t)\|_{L^\infty}\int_0^1 |w_x w_{xx}|\,dx\\ &\le \|w(t)\|_{L^\infty}\|w_x(t)\|\|w_{xx}(t)\|\\ &\le \frac{1}{2\epsilon}\|w(t)\|^2_{L^\infty}\|w_x(t)\|^2+\frac{\epsilon}{2}\|w_{xx}(t)\|^2\\ &\le \frac{1}{2\epsilon}\|w_x(t)\|^4+\frac{\epsilon}{2}\|w_{xx}(t)\|^2. \end{aligned}\tag{2.25}$$

Here, in the last step, we used the simple inequality $\|w\|_{L^\infty(0,1)} \le \|w_x\|$, which holds for $w \in \mathcal{H}_0^1(0,1)$. Introducing the notation

$$A(t) \equiv \frac{c}{\epsilon}w^2(1,t)+\frac{1}{18\epsilon c}w^4(1,t)+\|w_x(t)\|^2, \tag{2.26}$$

$$b(t) \equiv e^{\epsilon t}\left(w^2(1,t)+w^4(1,t)+w^6(1,t)+\|w_x(t)\|^2\right), \tag{2.27}$$

and substituting (2.23)–(2.25) into (2.22) we obtain

$$\begin{aligned}\frac{d}{dt}A(t)+\epsilon\|w_{xx}(t)\|^2 &\le \frac{\delta}{\epsilon^4}\left(c^2w^2(1,t)+\frac{2}{9}w^4(1,t)+\frac{1}{81c^2}w^6(1,t)\right)\\ &\quad\times\left(c^2+\frac{2}{9}w^2(1,t)+\frac{1}{81c^2}w^4(1,t)\right)-\delta w_{xx}^2(0,t)+\frac{1}{\epsilon}\|w_x(t)\|^4\\ &\le M\left(w^2(1,t)+w^4(1,t)+w^6(1,t)+\|w_x(t)\|^2\right)\\ &\quad+M\left(w^2(1,t)+w^4(1,t)+w^6(1,t)+\|w_x(t)\|^2\right)A(t). \end{aligned}\tag{2.28}$$

Omitting the nonnegative second term on the left, using definitions (2.26) and (2.27) and multiplying (2.28) by $e^{\epsilon t}$ we get

$$\frac{d}{dt}\left(e^{\epsilon t}A(t)\right) \le Mb(t)+Me^{-\epsilon t}b(t)e^{\epsilon t}A(t). \tag{2.29}$$

After integration we obtain from here

$$\begin{aligned}e^{\epsilon t}A(t) &\le A(0)+\int_0^t Mb(\tau)\,d\tau+\int_0^t Me^{-\epsilon t}b(\tau)\left(e^{\epsilon\tau}A(\tau)\right)\,d\tau\\ &\le A(0)+\int_0^t Mb(\tau)\,d\tau+\int_0^t Mb(\tau)\left(e^{\epsilon\tau}A(\tau)\right)\,d\tau. \end{aligned}\tag{2.30}$$

It follows now from Gronwall's inequality, estimate (2.21) and the definition of $A(t)$ and $b(t)$ that

$$\begin{aligned}e^{\epsilon t}A(t) &\le \left(A(0)+\int_0^t Mb(\tau)\,d\tau\right)\left(1+\int_0^t Mb(\tau)exp\left(\int_\tau^t Mb(s)\,ds\right)\,d\tau\right)\\ &\le \left(A(0)+M\|w_0\|^2\right)+\left(A(0)+M\|w_0\|^2\right)M\|w_0\|^2e^{M\|w_0\|^2} \le M\|w_0\|^2_{H^1}e^{M\|w_0\|^2_{H^1}}. \end{aligned}\tag{2.31}$$

Multiplying (2.31) by $e^{-\epsilon t}$, taking the square root, and using the definition of $A(t)$ one more time we arrive at the inequality

$$\|w(t)\|_{H^1} \le k\|w_0\|_{H^1}e^{k\|w_0\|^2_{H^1}}e^{-\epsilon t/2}, \tag{2.32}$$

which proves (2.11), the semi–global exponential stability in the H^1-sense. Due to the general Sobolev embedding theorem $H^\ell(\Omega) \subset C^k\left(\overline{\Omega}\right)$, which holds for $k \le \ell - \frac{n}{2}$, $\Omega \subset \mathbb{R}^n$, the solution $w(t,x)$ is continuous and bounded for all $t \ge 0$ and all $x \in [0,1]$. $\square$

3 A Numerical Example

In this section we compare the closed–loop system (2.1), (2.2), (2.6), (2.7), (2.8) to the open–loop system as well as to the system (2.1)–(2.5) through a numerical example. By the open–loop system we mean the KdVB equation (2.1) with boundary conditions

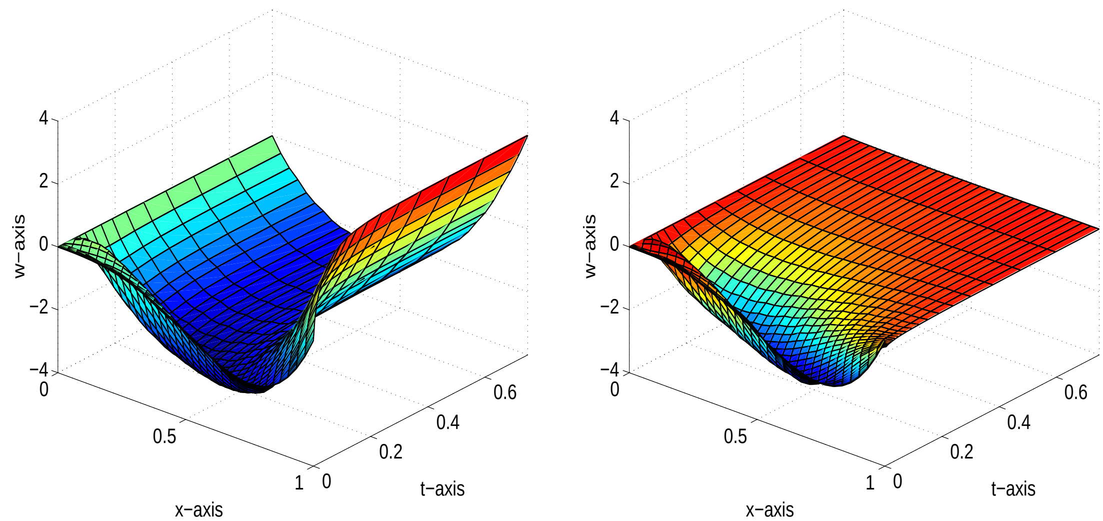
$$w(0,t) = 0, \tag{3.1}$$

$$w_x(1,t) = w_0'(1), \tag{3.2}$$

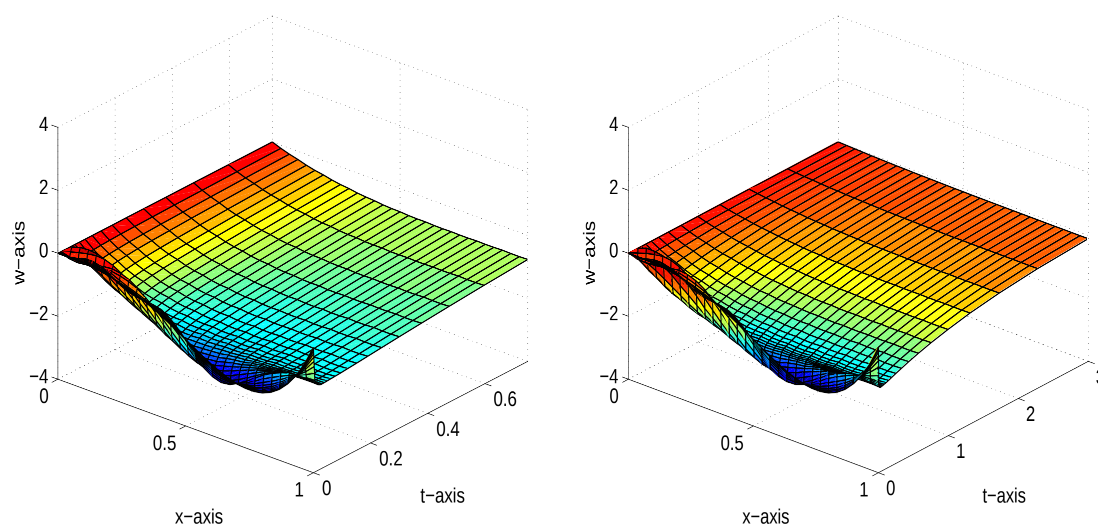
$$w_{xx}(1,t) = w_0''(1). \tag{3.3}$$

The existence of a solution of the uncontrolled system is obvious, at least on a finite time interval. It can be proven for example using Galerkin's method.

As a consequence of the third derivative in x and first derivative in t, it is necessary to use



(a) Uncontrolled System $T = .75$ (b) Two Derivatives Controlled, $T = .75$



(c) Controlled Second Derivative, $T = .75$ (d) Controlled Second Derivative, $T = 3$

Figure 1: Comparison of Solutions

very small time steps ($\approx 10^{-9}$) in order to balance the very small number in the denominator resulting from the cube of the spatial step. We are able to compensate in a certain extent the very small time steps by rescaling the equation, i.e. compressing the time domain. We consider, from the above reason, the scaled equation

$$u_t - \epsilon' u_{xx} + \delta' u_{xxx} + p u u_x = 0, \quad x \in [0,1], \quad t > 0, \tag{3.4}$$

with some initial data

$$u(x,0) = u_0(x), \quad u_0(0) = 0, \tag{3.5}$$

and in the controlled case with boundary condition

$$u(0,t) = 0, \tag{3.6}$$

$$u_x(1,t) = -\frac{p}{\epsilon'}\left(c + \frac{1}{9c}u^2(1,t)\right)u(1,t), \tag{3.7}$$

$$u_{xx}(1,t) = \frac{p^2}{\epsilon'^2}\left(c + \frac{1}{9c}u^2(1,t)\right)^2 u(1,t), \tag{3.8}$$

where ϵ', δ', c and p are positive constants. The transformation $u(x,t) \equiv w(x,pt)$ shows the equivalence of system (2.1), (2.2), (2.6), (2.7), (2.8) to (3.4)–(3.8) with $\epsilon \equiv \epsilon'/p$ and $\delta \equiv \delta'/p$.

Our numerical simulation is based on a fully discrete, implicit scheme of second order accuracy, using three time level quadratic approximation in time and central difference scheme in space, which is derived using the finite volume method (Ferziger and Peric, 1996). In accordance with the finite volume method we consider equation (3.4) integrated with respect to x over a small interval (which is called control volume, usually denoted by $[w,e]$ and whose center is a grid point). We obtain

$$\int_w^e u_t\, dx - \epsilon' \left. u_x\right|_w^e + \delta' \left. u_{xx}\right|_w^e + p\int_w^e u u_x\, dx = 0. \tag{3.9}$$

In an implicit scheme an important goal is to keep most of the expressions at the highest time level. Keeping this in mind, the last (quadratic) term of (3.9) is linearized in the following way:

$$\int_w^e u u_x\, dx = \left.\frac{1}{2}u^2\right|_w^e = \left.\frac{1}{2}u^{n+1}u^n\right|_w^e, \tag{3.10}$$

where the superscript $n+1$ denotes the highest time level. Using the notation $\vec{U}^n \equiv (u_1^n, u_2^n, \dots u_N^n)^T \equiv (u(t_n,x_1), u(t_n,x_2), \dots u(t_n,x_N))^T$ and the aforementioned discretization and linearization, (3.9) gives us the following difference equation on a uniform grid:

$$h\frac{3u_i^{n+1} - 4u_i^n + u_i^{n-1}}{k} - \epsilon' \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h} + \delta' \frac{u_{i+2}^{n+1} - 2u_{i+1}^{n+1} + 2u_{i-1}^{n+1} - u_{i-2}^{n+1}}{2h^2}$$
$$+ \frac{p}{8}\left(u_{i+1}^{n+1} + u_i^{n+1}\right)\left(u_{i+1}^n + u_i^n\right) - \frac{p}{8}\left(u_i^{n+1} + u_{i-1}^{n+1}\right)\left(u_i^n + u_{i-1}^n\right) = 0, \tag{3.11}$$

where $n = 2,3,\dots$, $i = 2,\dots,N-2$. We use the notation k for the increment in time and h for the space mesh size. Equation (3.11) provides us a linear system with a sparse, band state matrix. Two initial vectors are required to start our three time level scheme: $\vec{U}^1$ and $\vec{U}^2$. The first vector $\vec{U}^1$ is the initial function $u_0(x)$ itself, and $\vec{U}^2$ was obtained from this vector using the one step Euler method.

At the boundaries $x = 0$ and $x = 1$ we use half control volumes. The boundary conditions at $x = 1$ provide a straightforward modification in the linear system after the linearization

$$u_x^{n+1}\big|_{x=1} = g_x^n u_N^{n+1}, \tag{3.12}$$
$$u_{xx}^{n+1}\big|_{x=1} = g_{xx}^n u_N^{n+1}, \tag{3.13}$$

where

$$g_x^n \equiv -\frac{p}{\epsilon'}\left(c + \frac{1}{9c}\,(u_N^n)^2\right),$$
$$g_{xx}^n \equiv \frac{p^2}{\epsilon'^2}\left(c + \frac{1}{9c}\,(u_N^n)^2\right)^2. \tag{3.14}$$

At the boundary $x = 0$, in order to obtain the necessary value outside the interval $[0, 1]$, we use extrapolation based on the equation (3.4) reduced to the ODE

$$-\epsilon' u_{xx} + \delta' u_{xxx} = 0. \tag{3.15}$$

We omit the details of these simple calculations. The boundary conditions of the uncontrolled system and system (2.1)–(2.5) are handled similarly. The resulting sparse linear system is solved using the preconditioned BiConjugate Gradient Stabilized method implemented in a C++ templated library (Barrett *et al.*, 1994; Dongara *et al.*, 1994). The computation was performed on a 300MHz, 130Mb memory Sun SPARC Workstation and, due to the small time step, it required several hours to reach one time unit with the numerical solution.

As an example, we consider the (KdVB) equation (3.4) with parameters $\epsilon' = 1$, $\delta' = 10$, $p = 100$ and with initial function

$$u_0(x) = 20x^3(x - 1.001). \tag{3.16}$$

The time step we use is $k = 10^{-9}$ with final time $T = 10^{-2}$, and spatial step $h = 5 \times 10^{-3}$. In the case when only the second derivative is controlled a final time of $T = 3 \times 10^{-2}$ was

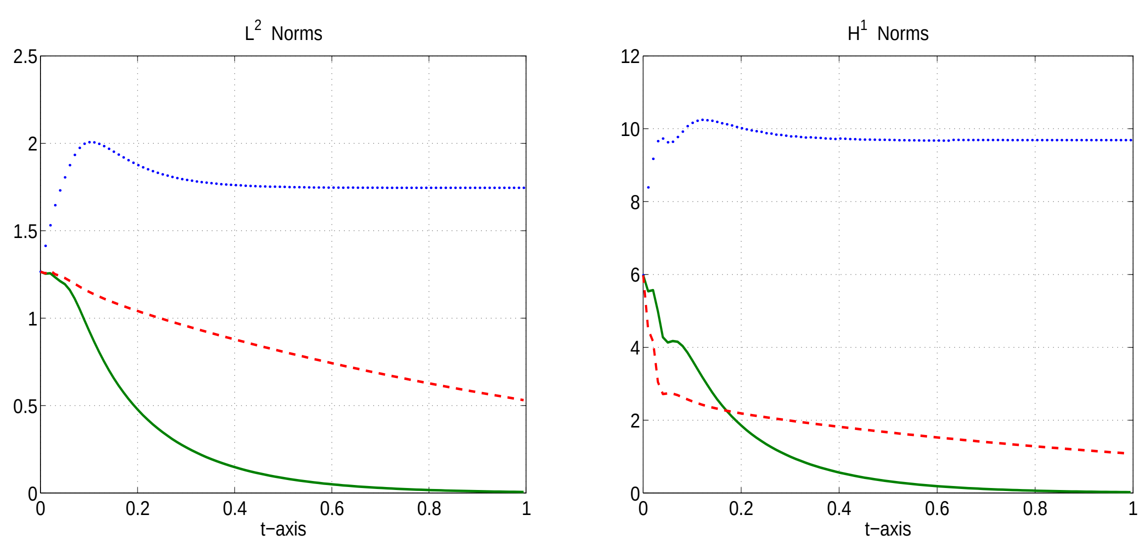


Figure 2: Comparison of Norms
$\cdots$: Uncontrolled, $--$: Controlled Second Derivative, —: Controlled First and Second Derivative

required in order to see significant convergence. The scaling $p = 100$ corresponds to an unscaled Korteweg–de Vries–Burgers system with parameters $\epsilon = 0.01$, $\delta = 0.1$ on a time interval $[0, 1]$. In the controlled case the control gain was $c = 0.1$. As we can see in Figure 1, the uncontrolled solution seems to converge to a nontrivial stationary solution. While both controlled systems converge to zero [parts (b) and (c) of Figure 1], the case when the first derivate is kept at zero at $x = 1$ and only the second derivative is controlled by feedback shows poor convergence relative to our controller (3.7)–(3.8). In fact, Figure 2 shows that the differences between the rates of convergence are significant both in the L^2 and in the H^1 sense.

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