Proceedings of the 7th Mediterranean Conference on Control and Automation (MED99) Haifa, Israel - June 28-30, 1999

Exponential stabilization of vibrating systems by collocated feedback

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Abstract: We consider regular linear systems described by $\dot{x} = Ax + Bu$, $y = B^*_\Lambda x$, where A generates a strongly continuous semigroup on the Hilbert space X and A is essentially skew-adjoint and dissipative. This means that the domains of A^* and A are equal and $A^* + A = -Q$, where Q is a bounded nonnegative operator. The control operator B is possibly unbounded, but admissible and B^*_Λ is the Λ-extension of B^*. Such a description fits many wave and beam equations and it has been shown for many particular cases that the feedback $u = -\kappa y$, with $\kappa > 0$, stabilizes the system, strongly or even exponentially. We show, by means of a counterexample, that if B is sufficiently unbounded, then such a feedback may be unsuitable: the closed-loop semigroup may even grow exponentially. However, if κ is sufficiently small, and if the original system is exactly controllable and observable, then the closed-loop system is exponentially stable. The above assumptions may be relaxed in various directions, for example, regularity may be replaced by well-posedness, exact controllability may be replaced by optimizability etc.

Keywords: well-posed linear systems, regular linear systems, positive-real transfer functions, exact controllability and observability, collocated sensors and actuators.

1. The main results

In this paper, we consider the stabilization of a special class of well-posed linear systems, as described below. To specify our terminology and notation, we recall that for any well-posed linear system Σ with input space U, state space X and output space Y, all Hilbert spaces, the state trajectories $z \in C([0,\infty), X)$ are described by the differential equation

$$\dot{z}(t) = Az(t) + Bu(t), \tag{1.1}$$

where $u \in L^2_{loc}([0,\infty), U)$ is the input function. The operator $A : \mathcal{D}(A) \to X$ is the generator of a strongly continuous semigroup of operators $\mathbb{T}$ on X and the (possibly unbounded) operator B

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is an admissible control operator for $\mathbb{T}$. In general, the output function y is in $L^2_{loc}([0,\infty),Y)$. If $u=0$ and $z(0)\in\mathcal{D}(A)$, then y is given by

$$y(t) = Cz(t) \qquad \forall\, t\geq 0,$$

where $C:\mathcal{D}(A)\rightarrow Y$ is an admissible observation operator for $\mathbb{T}$. If $z(0)=0$, then the input and output functions u and y are related by the formula

$$\hat{y}(s) = \mathbf{G}(s)\hat{u}(s), \tag{1.2}$$

where a hat denotes the Laplace transform and $\mathbf{G}$ is the transfer function of Σ. The formula (1.2) holds for all $s\in\mathbb{C}$ with $\mathrm{Re}\, s$ sufficiently large. We refer to Section 3 for more details and references on well-posed linear systems. Now we specify the special class of systems studied in this paper.

Assumption ESAD. The operator A is *essentially skew-adjoint and dissipative*, which means that $\mathcal{D}(A)=\mathcal{D}(A^*)$ and there exists a $Q\in\mathcal{L}(X)$ with $Q\geq 0$ such that

$$Ax + A^*x = -Qx \qquad \forall\, x\in\mathcal{D}(A). \tag{1.3}$$

This implies that $\mathbb{T}$ is a contraction semigroup. Note that A is a bounded perturbation of the skew-adjoint operator $A+\frac{1}{2}Q$. Such a model is often used to describe the dynamics of oscillating systems, such as waves or flexible structures (in most cases, $Q=0$, so that $\mathbb{T}$ is unitary).

Assumption COL. $Y=U$ and $C=B^*$.

In the literature on stabilization of flexible structures, a very popular way of implementing actuators and sensors is through collocated pairs, i.e., an actuator and sensor pair act at the same physical position. This often leads to assumption COL being satisfied, often with a finite-dimensional U.

Our aim is to show that for certain numbers $\kappa>0$, the static output feedback law $u=-\kappa y+v$ exponentially stabilizes the system, where v is the new input function. The closed-loop system Σ^κ is shown as a block diagram in Figure 1 (see Section 3 for details on output feedback). Denoting the semigroup of the closed-loop system by $\mathbb{T}^\kappa$, so that $\mathbb{T}^\kappa_t\in\mathcal{L}(X)$, by *exponential stability* of Σ^κ we mean that there exist $M\geq 1$ and $\alpha>0$ such that $\|\mathbb{T}^\kappa_t\|\leq Me^{-\alpha t}$ for all $t\geq 0$.

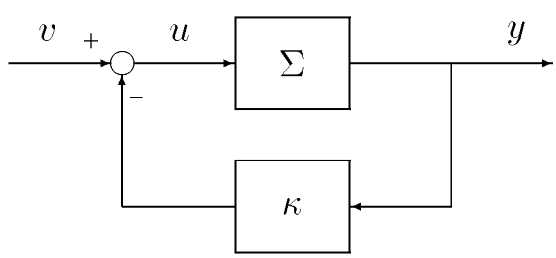

Figure 1: The open-loop system Σ with negative output feedback via κ. If the number $\kappa>0$ is sufficiently small, then this is a new well-posed linear system Σ^κ, called the closed-loop system, which is input-output stable. If Σ is exactly controllable and exactly observable, then Σ^κ is exponentially stable.

We are also interested in three stability-related concepts which concern the whole closed-loop system, not only its semigroup. We denote by B^κ the control operator of Σ^κ, by C^κ its observation operator and by $\mathbf{G}^\kappa$ its transfer function.

- *input stability* means that for any $v \in L^2([0,\infty),U)$, the state trajectory of Σ^κ corresponding to the initial state zero and the input function v is bounded. This property is also known as *infinite-time admissibility* of B^κ.

- *output stability* means that for any $x_0 \in X$, the output function of Σ^κ corresponding to the initial state x_0 and the input function zero is in $L^2([0,\infty),U)$. This property is also known as *infinite-time admissibility* of C^κ.

- *input-output stability* means that for any $v \in L^2([0,\infty),U)$, the output function of Σ^κ corresponding to the initial state zero and the input function v is in $L^2([0,\infty),U)$. Equivalently, $\mathbf{G}^\kappa \in H^\infty(\mathcal{L}(U))$, the space of bounded analytic $\mathcal{L}(U)$-valued functions on the open right half-plane.

Theorem 1.1. *Let Σ be a well-posed linear system satisfying assumptions* ESAD *and* COL. *Then there exists a $\kappa_0 > 0$ (possibly $\kappa_0 = \infty$) such that for all $\kappa \in (0,\kappa_0)$, the feedback law $u = -\kappa y + v$ (where u and y are the input and the output of Σ) leads to a closed-loop system Σ^κ with the following properties:*

(a) $\mathbb{T}^\kappa$ *is a semigroup of contractions.*

(b) Σ^κ *is input stable.*

(c) Σ^κ *is output stable.*

(d) Σ^κ *is input-output stable.*

Note that in this theorem there are no controllability or observability assumptions. The proof is given in Section 4, together with the proof of our main result:

Theorem 1.2. *With the assumptions and the notation of Theorem* 1.1, *if Σ is exactly controllable and exactly observable, then Σ^κ is exponentially stable.*

In fact, this result is only a corollary of a result proven in Section 4, in which the assumptions are weaker than exact controllability and observability.

In all the published examples that we are aware of, the feedback $u = -\kappa y + v$ is stabilizing for all $\kappa > 0$, at least in the input-output sense, and often strongly or exponentially. In Section 4 we give an example of a simple open-loop system Σ which fits into our framework, moreover it is regular with feedthrough operator zero (see Section 3 for definitions), but for which the feedback $u = -\kappa y + v$ is only exponentially stabilizing for sufficiently small $\kappa > 0$. For too large a κ, the closed-loop semigroup $\mathbb{T}^\kappa$ will have a positive exponential growth rate.

2 Comments on the literature and a self-contained presentation of the finite-dimensional case

Many models of controlled flexible structures satisfy assumptions ESAD and COL. The feedback $u = -\kappa y + v$ is very simple to implement and it is often used in the stabilization of these structures: we shall list relevant references below. Thus, our results may be regarded as an abstract unifying theory. However, it must be pointed out that not all the examples of this type in the literature satisfy our assumptions, since sometimes the open-loop system is not well-posed.

Much of the early work on the stabilization of flexible structures concerned finite-dimensional systems, see for example Benhabib et al [11] and Joshi [25] and the references therein. They used the feedback $u = -\kappa B^* z + v$ because of its simplicity and its nice robustness properties. Indeed, it works for all systems with a positive-real transfer function, as we shall explain below (see also Vidyasagar and Desoer [19]). We think that it will be instructive to give here a short self-contained presentation of the finite-dimensional version of our main result (Theorem 1.2), without any claims to novelty. The infinite-dimensional argument will go along the same lines, only with much more technicalities.

Recall that a square matrix-valued transfer function $\mathbf{G}$, analytic on the open right half-plane $\mathbb{C}_0$, is called *positive-real* if $\overline{\mathbf{G}(s)} = \mathbf{G}(\overline{s})$ and

$$\mathbf{G}(s)^* + \mathbf{G}(s) \geq 0 \qquad \forall\, s \in \mathbb{C}_0 . \tag{2.1}$$

Positive-real transfer functions were introduced in electrical network theory, but they have strong connections with systems theory formulated in state space, see Anderson and Vongpanitlerd [1]. In particular, if the real square matrix A is dissipative, then for any real matrix B of appropriate dimensions, $\mathbf{G}(s) = B^*(sI - A)^{-1}B$ is positive-real. Such transfer functions often occur as models of flexible structures with collocated actuators and sensors, see [11].

Now consider an arbitrary square matrix-valued transfer function $\mathbf{G}$. If the closed-loop system with transfer function $\mathbf{G}^\kappa$ is obtained from the open-loop system with transfer function $\mathbf{G}$ via the feedback $u = -\kappa y + v$, then $\mathbf{G}^\kappa = \mathbf{G}(I + \kappa\mathbf{G})^{-1}$. We need the following simple sufficient condition for $\mathbf{G}^\kappa$ to be bounded on $\mathbb{C}_0$, a fact which is written as $\mathbf{G}^\kappa \in H^\infty$:

Lemma 2.1. *Suppose that $cI + \mathbf{G}$ is positive-real for some $c \geq 0$, and denote $\kappa_0 = \frac{1}{c}$ (for $c = 0$, take $\kappa_0 = \infty$). Then for any $\kappa \in (0, \kappa_0)$, $\mathbf{G}^\kappa \in H^\infty$.*

Proof. Denoting $a = \frac{1}{\kappa}$ and $T(s) = aI + \frac{1}{2}(\mathbf{G}(s)^* + \mathbf{G}(s))$, we have $T(s) \geq (a - c)I > 0$. Hence, for any $u \in U$ with $\|u\| = 1$ and any $s \in \mathbb{C}_0$,

$$\|(aI + \mathbf{G}(s))u\| \;\geq\; \mathrm{Re}\,\langle (aI + \mathbf{G}(s))u, u\rangle \;=\; \langle T(s)u, u\rangle \;\geq\; a - c ,$$

whence

$$\left\|(aI + \mathbf{G}(s))^{-1}\right\| \;\leq\; \frac{1}{a - c} ,$$

so that $(aI + \mathbf{G})^{-1} \in H^\infty$. We rewrite $\mathbf{G}^\kappa$ in the form

$$\mathbf{G}^\kappa \;=\; a\left[I - a(aI + \mathbf{G})^{-1}\right] ,$$

which shows that $\mathbf{G}^\kappa \in H^\infty$. ■

Now consider a finite-dimensional system Σ described by the equations

$$\begin{cases} \dot{z}(t) &=& Az(t) + Bu(t), \\ y(t) &=& B^*z(t) + Du(t), \end{cases}$$

where A, B and D are real matrices of appropriate dimensions and $A^* + A \leq 0$, so that assumptions ESAD and COL are satisfied.

If the number κ is such that $I + \kappa D$ is invertible, then the feedback $u = -\kappa y + v$ leads to the closed-loop system Σ^κ described by the equations

$$\left\{ \begin{array}{rcl} \dot{z}(t) & = & (A - B\kappa(I+\kappa D)^{-1}B^*)\, z(t) + B(I+\kappa D)^{-1}v(t), \\ y(t) & = & (I+\kappa D)^{-1}B^* z(t) + D(I+\kappa D)^{-1}v(t). \end{array} \right. \tag{2.2}$$

We are interested in conditions that guarantee that the matrix

$$A^\kappa = A - B\kappa(I+\kappa D)^{-1}B^*$$

appearing in (2.2) is stable, i.e., its eigenvalues are in the left open half-plane. The finite-dimensional version of Theorem 1.2 reads as follows:

Proposition 2.2. *With the above notation, if (A, B) is controllable and (A, B^*) is observable, then there exists a $\kappa_0 > 0$ such that for all $\kappa \in (0, \kappa_0)$, A^κ is stable.*

Like Theorem 1.2, this proposition is only a corollary of a stronger result which is a little more complicated to state. The stronger version of Theorem 1.2 is stated and proved in Section 4. Its finite-dimensional counterpart is stated and proved below. To state it, we introduce the notation

$$\kappa_0 = \frac{1}{c}, \quad \text{where} \quad c = \|E^+\|, \quad E = -\frac{1}{2}(D^* + D). \tag{2.3}$$

Here, E^+ is the positive part of E, i.e., $E^+ = EP^+$ where P^+ is the spectral projector corresponding to all the positive eigenvalues of E (hence $E^+ \geq E$ but $\|E^+\| \leq \|E\|$). Note that if $E \leq 0$ then $c = 0$, and then we put $\kappa_0 = \infty$.

Proposition 2.3. *With the above notation, if (A, B) is stabilizable and (A, B^*) is detectable, then for all $\kappa \in (0, \kappa_0)$, A^κ is stable.*

Proof. The transfer function of Σ is $\mathbf{G}(s) = B^*(sI - A)^{-1}B + D$ and we have

$$\mathbf{G}(s)^* + \mathbf{G}(s) \geq D^* + D \qquad \forall\, s \in \mathbb{C}_0.$$

This implies that, denoting $E = -\frac{1}{2}(D^* + D)$, $\mathbf{G} + E$ is positive-real, and hence $\mathbf{G} + E^+$ is positive-real. Then for $c = \|E^+\|$, $cI + \mathbf{G}$ is positive-real. (A simpler but more restrictive choice would be $c = \|E\|$.) The transfer function of the closed-loop system Σ^κ from (2.2) is $\mathbf{G}^\kappa$. By Lemma 2.1, we have $\mathbf{G}^\kappa \in H^\infty$ for all $\kappa \in (0, \kappa_0)$. It is well known that if a finite-dimensional system is stabilizable, detectable and input-output stable, then it is stable. ∎

We now return to the discussion of the infinite-dimensional case. In this paper we replace the concept of positive-real transfer function with the more general concept of a positive transfer function. An analytic $\mathcal{L}(U)$-valued function on $\mathbb{C}_0$ is called a *positive transfer function* if (2.1) holds. Note that for simplicity we have dropped the condition concerning complex conjugates, since it is not needed in our arguments: in particular, it is not needed in Lemma 2.1. (Defining the complex conjugate of an operator is a bit awkward and not necessary.) In the finite-dimensional case, this slight generalization amounts to dropping the requirement that the system matrices should be real. Note that if the generator A is dissipative and $B \in \mathcal{L}(U, X)$ (i.e., B is bounded), then $\mathbf{G}(s) = B^*(sI - A)^{-1}B$ is a positive transfer function. This is not always true for unbounded B, as we shall show.

The first PDE (partial differential equation) examples fitting into our class assumed that A was dissipative, B was bounded and the open-loop system had the transfer function $\mathbf{G} = B^*(sI - A)^{-1}B$ (i.e., $D = 0$), see Bailey and Hubbard [5], Balakrishnan [6], [7], Russell [37], Slemrod [42], [43], [44]. In this case, $\mathbf{G}$ is positive and (by Lemma 2.1) the feedback $u = -\kappa y + v$ always stabilizes in an input-output sense. Of course, the most desirable type of stability is exponential stability, and if $A^* + A = 0$ then for this we need the system to be exactly controllable (equivalently, exactly observable). This is the setup studied in Haraux [23].

However, early on it was realized that exact controllability is not achievable with a bounded B if U is finite-dimensional and if A has infinitely many eigenvalues on the imaginary axis. This is illustrated in Russell [37] with a PDE model of an undamped string. He showed that the closed-loop eigenvalues λ_n are in the open left half-plane, but $\mathrm{Re}\,\lambda_n \to 0$ as $n \to \infty$. Similar results for a beam example can be found in Slemrod [44]. More generally, it is known that if U is finite-dimensional, B is bounded and A has infinitely many unstable eigenvalues, we can never achieve exponential stability, see Gibson [20] or Triggiani [49]. So for this class the best we can hope for is strong stability. Early results giving sufficient conditions under which $A - BB^*$ generates a weakly or strongly stable semigroup using LaSalle's principle can be found in Slemrod [42]. These were sharpened by Benchimol in [10] who used the canonical decomposition of contraction semigroups due to Szökefalvi-Nagy and Foias. He showed that if A generates a contraction semigroup and $B \in \mathcal{L}(U, X)$, a sufficient condition for $A - BB^*$ to generate a weakly stable semigroup is that

$$\|\mathbb{T}_t x\| = \|x\| = \|\mathbb{T}_t^* x\| \text{ and } B^* \mathbb{T}_t^* x = 0 \quad \forall\, t \geq 0 \text{ implies that } \quad x = 0. \tag{2.4}$$

In the above line, $\mathbb{T}_t^*$ may be interchanged with $\mathbb{T}_t$. If, in addition, A has compact resolvent, then (2.4) implies strong stability. The above result for weak stability was also obtained by Batty and Phong [8]. They improved the above sufficient condition for strong stability obtaining: if the spectrum of A has at most countable many points of intersection with the imaginary axis, then $A - BB^*$ generates a strongly stable semigroup if and only if (2.4) holds. It is worthwhile noting that, while the assumption that B be bounded is restrictive, it does not exclude PDE's with boundary control, see Slemrod [44], You [60] and Chapter 9 of Oostveen [31]. The last reference contains more results on the control of systems with dissipative A, bounded B and observation operator B^*, concerning Riccati equations, the Nehari problem and the robustness of strong stabilization in the gap metric and with respect to nonlinear perturbations (see also Oostveen and Curtain [32], [33], [34]).

Many examples of flexible beams, plates and hybrid structures have been shown to be exponentially stabilizable by static output feedback, see for example Chen [12], [13]. The approach is a classical Lyapunov one, with the key step being the appropriate PDE formulation so that the energy of the system can play the role of a Lyapunov functional. If one examines these examples carefully, one can recognize that they fit into our framework (the assumptions ESAD and COL are satisfied) and they use the feedback $u = -\kappa y + v$ for stabilization.

3 Some background on infinite-dimensional systems

In this section we gather, for easy reference, some basic facts about admissible control and observation operators, about well-posed and regular linear systems, their transfer functions, well-posed triples of operators and closed-loop systems. For proofs and for more details we refer to the literature.

We assume that X is a Hilbert space and $A : \mathcal{D}(A) \to X$ is the generator of a strongly continuous semigroup $\mathbb{T}$ on X. We define the Hilbert space X_1 as $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, where $\beta \in \rho(A)$ is fixed (this norm is equivalent to the graph norm). The Hilbert space X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1} z\|$. This space is isomorphic to $\mathcal{D}(A^*)^*$, and we have

$$X_1 \subset X \subset X_{-1}\,, \tag{3.1}$$

densely and with continuous embeddings. $\mathbb{T}$ extends to a semigroup on X_{-1}, denoted by the same symbol. The generator of this extended semigroup is an extension of A, whose domain is X, so that $A : X \to X_{-1}$.

We assume that U is a Hilbert space and $B \in \mathcal{L}(U, X_{-1})$ is an *admissible control operator* for $\mathbb{T}$, defined as in Weiss [51]. This means that if z is the solution of $\dot{z}(t) = Az(t) + Bu(t)$, as in (1.1), which is an equation in X_{-1}, with $z(0) = z_0 \in X$ and $u \in L^2([0,\infty), U)$, then $z(t) \in X$ for all $t \geq 0$. In this case, z is a continuous X-valued function of t. We have

$$z(t) = \mathbb{T}_t z_0 + \Phi_t u\,, \tag{3.2}$$

where $\Phi_t \in \mathcal{L}(L^2([0,\infty), U), X)$ is defined by

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma)\, d\sigma\,. \tag{3.3}$$

The above integration is done in X_{-1}, but the result is in X. The Laplace transform of z is

$$\hat{z}(s) = (sI - A)^{-1} \left[z_0 + B\hat{u}(s) \right].$$

B is called *bounded* if $B \in \mathcal{L}(U, X)$ (and unbounded otherwise). If B is an admissible control operator for $\mathbb{T}$, then $(sI - A)^{-1} B \in \mathcal{L}(U, X)$ for all s with $\operatorname{Re} s$ sufficiently large. Moreover, there exist positive constants δ, ω such that

$$\|(sI - A)^{-1} B\|_{\mathcal{L}(U,X)} \leq \frac{\delta}{\sqrt{\operatorname{Re} s}} \qquad \forall\, \operatorname{Re} s > \omega\,, \tag{3.4}$$

and if $\mathbb{T}$ is normal then (3.4) implies admissibility, see [56].

We assume that Y is another Hilbert space and $C \in \mathcal{L}(X_1, Y)$ is an *admissible observation operator* for $\mathbb{T}$, defined as in Weiss [52]. This means that for every $T > 0$ there exists a $K_T \geq 0$ such that

$$\int_0^T \|C\,\mathbb{T}_t z_0\|^2 dt \leq K_T^2 \|z_0\|^2 \qquad \forall\, z_0 \in \mathcal{D}(A)\,. \tag{3.5}$$

C is called *bounded* if it can be extended such that $C \in \mathcal{L}(X, Y)$.

We regard $L^2_{loc}([0,\infty), Y)$ as a Fréchet space with the seminorms being the L^2 norms on the intervals $[0, n]$, $n \in \mathbb{N}$. Then the admissibility of C means that there is a continuous operator $\Psi : X \to L^2_{loc}([0,\infty), Y)$ such that

$$(\Psi z_0)(t) = C\,\mathbb{T}_t z_0 \qquad \forall\, z_0 \in \mathcal{D}(A)\,. \tag{3.6}$$

The operator Ψ is completely determined by (3.6), because $\mathcal{D}(A)$ is dense in X. We introduce an extension of C, called the Λ-extension of C, defined by

$$C_\Lambda z_0 = \lim_{\lambda \to +\infty} C \lambda (\lambda I - A)^{-1} z_0\,, \tag{3.7}$$

whose domain $\mathcal{D}(C_\Lambda)$ consists of all $z_0 \in X$ for which the limit exists. Take $\lambda_0 \in \mathbb{R}$ such that $[\lambda_0, \infty) \subset \rho(A)$. We define on $\mathcal{D}(C_\Lambda)$ the norm

$$\|x_0\|_{\mathcal{D}(C_\Lambda)} = \|x_0\|_X + \sup_{\lambda \geq \lambda_0} \|C\lambda(\lambda I - A)^{-1} x_0\|_Y ,$$

with which it becomes a Banach space. We have

$$\mathcal{D}(A) \subset \mathcal{D}(C_\Lambda) \subset X , \tag{3.8}$$

with continuous embeddings. We shall also use the *weak Λ-extension* of C, $C_{\Lambda w}$. It is defined as in (3.7), but replacing the strong limit by the weak limit.

If we replace C by C_Λ, formula (3.6) becomes true for all $z_0 \in X$ and for almost every $t \geq 0$. If $y = \Psi z_0$, then its Laplace transform is

$$\hat{y}(s) = C(sI - A)^{-1} z_0 . \tag{3.9}$$

The following duality result holds: if $\mathbb{T}$ is a semigroup on X with generator A, then $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$, if and only if $B^* : \mathcal{D}(A^*) \to U$ is an admissible observation operator for the dual semigroup $\mathbb{T}^*$.

By a *well-posed linear system* we mean a linear time-invariant system such that on any finite time interval, the operator from the initial state and the input function to the final state and the output function is bounded. The input, state and output spaces are Hilbert spaces, and the input and output functions are of class L^2_{loc}. To express this more clearly, let us denote by U the input space, by X the state space and by Y the output space of a well-posed linear system Σ. The input and output functions u and y are locally L^2 functions with values in U and in Y. The state trajectory z is an X-valued function. The boundedness property mentioned earlier means that for every $\tau > 0$ there is a $c_\tau \geq 0$ such that

$$\|z(\tau)\|^2 + \int_0^\tau \|y(t)\|^2 \, dt \leq c_\tau^2 \left(\|z(0)\|^2 + \int_0^\tau \|u(t)\|^2 \, dt \right) \tag{3.10}$$

(with c_τ independent of $z(0)$ and of u). For the detailed definition, background and examples we refer to Salamon [41], [40], Staffans [45], [46], [48], Weiss [54], [55], Avalos and Weiss [3] and Weiss[2] [59].

We recall some necessary facts about well-posed linear systems. Let Σ be such a system, with input space U, state space X and output space Y. Then there are operators A, B, C satisfying the assumptions in the previous discussion, which are related to Σ in the following way: First of all, the state trajectories of Σ satisfy the equation (1.1), so that they are given by (3.2). $\mathbb{T}$ is called the *semigroup* of Σ, A is called its *semigroup generator* and B is called the *control operator* of Σ. If u is the input function of Σ, z_0 is its initial state and y is the corresponding output function, then

$$y = \Psi z_0 + \mathbb{F} u . \tag{3.11}$$

Here, Ψ is an operator as in (3.6), and C is called the *observation operator* of Σ.

The operator $\mathbb{F}$ appearing above is easiest to represent using Laplace transforms. An operator-valued analytic function is called *well-posed* if its domain contains a right half-plane in $\mathbb{C}$ such that the function is uniformly bounded on this half-plane. We do not distinguish between two well-posed functions if one is a restriction of the other (to a smaller domain in $\mathbb{C}$). There exists a unique $\mathcal{L}(U,Y)$-valued well-posed function $\mathbf{G}$, called the *transfer function* of Σ, which determines $\mathbb{F}$ as follows: if $u \in L^2([0,\infty),U)$ and $y = \mathbb{F}u$, then y has a Laplace transform $\hat{y}$ and, for $\mathrm{Re}\, s$ sufficiently large,

$$\hat{y}(s) = \mathbf{G}(s)\hat{u}(s)\,.$$

This determines $\mathbb{F}$, since L^2 is dense in L^2_{loc}. Because of the identification mentioned earlier, by a transfer function we mean in fact an equivalence class of analytic functions. We have

$$\mathbf{G}(s) - \mathbf{G}(\beta) = C\left[(sI-A)^{-1} - (\beta I - A)^{-1}\right]B\,, \tag{3.12}$$

for any s,β in the open right half-plane determined by the growth bound of $\mathbb{T}$. This shows that $\mathbf{G}$ is determined by A, B and C up to an additive constant operator.

We call (A,B,C) the *generating triple* of Σ. In the time domain, the output function y corresponding to the input function u and state trajectory z is given by

$$y(t) = C_\Lambda\left[z(t) - (\beta I - A)^{-1}Bu(t)\right] + \mathbf{G}(\beta)u(t)\,, \tag{3.13}$$

valid for almost every $t \geq 0$, if $\mathrm{Re}\,\beta$ is larger than the growth bound of the semigroup. Thus, the system Σ is completely determined (via (1.1) and (3.13)) by its generating triple (A,B,C) and by the value of its transfer function at one point.

Definition 3.1. Let U, X and Y be complex Hilbert spaces. A triple of operators (A,B,C) is called *well-posed* on U, X and Y, if there exists a well-posed linear system Σ with input space U, state space X and output space Y, such that (A,B,C) is the generating triple of Σ.

This definition is taken from Curtain and Weiss [16]. It is useful to have a list of explicit conditions that A, B and C have to satisfy in order to constitute a well–posed triple. The following result was proven in [16].

Proposition 3.2. *A triple of operators (A,B,C) is well-posed on U, X and Y if and only if the following conditions are satisfied:*

(1) *A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X,*
(2) *$B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$,*
(3) *$C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$,*
(4) *some (hence every) transfer function $\mathbf{G}$ associated with (A,B,C) (i.e., satisfying (3.12)) is a bounded $\mathcal{L}(U,Y)$-valued function on some right half-plane.*

The well-posed linear system Σ is called *regular* if the limit

$$\lim_{s\to+\infty} \mathbf{G}(s)\mathrm{v} = D\mathrm{v}$$

exists for every $\mathrm{v} \in U$, where s is real (see [54]). In this case, the operator $D \in \mathcal{L}(U,Y)$ is called the *feedthrough operator* of Σ. Regularity is equivalent to the fact that the product $C_\Lambda(sI-A)^{-1}B$ makes sense, for some (hence, for every) $s \in \rho(A)$. In this case,

$$\mathbf{G}(s) = C_\Lambda(sI-A)^{-1}B + D\,, \tag{3.14}$$

as in finite dimensions. Moreover, the function y from (3.11) satisfies, for almost every $t \geq 0$,

$$y(t) = C_\Lambda z(t) + Du(t)\,, \tag{3.15}$$

where z is the state trajectory of the system (compare this with (3.13)). The operators A, B, C, D are called the *generating operators* of Σ, because they determine Σ via (1.1) and (3.15). (A, B, C) is called a *regular triple* if $A, B, C, 0$ are the generating operators of a regular linear system. Equivalently, A generates a semigroup, B and C are admissible, the product $C_\Lambda(sI - A)^{-1}B$ exists and it is bounded on some right half-plane (see Section 2 of [55]). In particular, if A is a generator, one of B and C is admissible and the other is bounded, then (A, B, C) is regular triple.

Let be Σ be a well-posed linear system with generating triple (A, B, C) and transfer function $\mathbf{G}$. An operator $K \in \mathcal{L}(Y, U)$ is called an *admissible feedback operator* for Σ (or for $\mathbf{G}$) if $I - \mathbf{G}K$ has a well-posed inverse (equivalently, if $I - K\mathbf{G}$ has a well-posed inverse). If this is the case, then the system with output feedback $u = Ky + v$ (see Figure 1, but with $-K$ in place of κ), is well-posed (its input is v, its state and output are the same as for Σ). This new system is called the *closed-loop system* corresponding to Σ and K, and it is denoted by Σ^K. Its transfer function is $\mathbf{G}^K = \mathbf{G}(I - K\mathbf{G})^{-1} = (I - \mathbf{G}K)^{-1}\mathbf{G}$. We have that $-K$ is an admissible feedback operator for Σ^K and the corresponding closed-loop system is Σ. Let us denote by (A^K, B^K, C^K) the generating triple of Σ^K. Then for every $x_0 \in \mathcal{D}(A^K)$ and for every $z_0 \in \mathcal{D}(A)$,

$$A^K x_0 = \left(A + BKC^K\right) x_0, \qquad A z_0 = \left(A^K - B^K KC\right) z_0 .$$

Any interconnection of finitely many well-posed linear systems can be thought of as a closed-loop system in the above sense.

To obtain explicit formulas for the generating operators of Σ^K, we need to assume regularity. We shall need the following result from [55].

Theorem 3.3. *Let Σ be a regular linear system with generating operators A, B, C, D and suppose that K is an admissible feedback operator for Σ and $I - DK$ is boundedly invertible. Then the resulting closed-loop system Σ^K is also regular and its generating operators A^K, B^K, C^K, D^K are given by*

$$\begin{aligned}
A^K x &= (A + BK(I - DK)^{-1}C_\Lambda)x \qquad \forall\, x \in \mathcal{D}(A^K),\\
\mathcal{D}(A^K) &= \{x \in \mathcal{D}(C_\Lambda) \mid (A + BK(I - DK)^{-1}C_\Lambda)x \in X\},\\
B^K &= B(I - KD)^{-1},\ \ D^K = D(I - KD)^{-1},\\
C^K x &= (I - DK)^{-1}C_\Lambda x \qquad \forall\, x \in \mathcal{D}(A^K).
\end{aligned}$$

The Λ-extension of C^K with respect to A^K is related to the Λ-extension of C with respect to A through

$$C^K_\Lambda x = C_\Lambda(I - DK)^{-1}x \qquad \forall\, x \in \mathcal{D}(C_\Lambda) = \mathcal{D}(C^K_\Lambda).$$

Moreover, W^K_1, the closure of $\mathcal{D}(A^K)$ in $\mathcal{D}(C^K_\Lambda) = \mathcal{D}(C_\Lambda)$, equals W_1, the closure of $\mathcal{D}(A)$ in $\mathcal{D}(C_\Lambda)$.

The proof of the following result is the same as that of Lemma 2.1 in Rebarber and Townley [36].

Lemma 3.4. *Let Σ be a regular linear system with generating operators A, B, C, D and transfer function $\mathbf{G}$. Suppose that K is an admissible feedback operator for Σ such that $(I - DK)$ is boundedly invertible. If λ is in the spectrum of the closed-loop generator A^K, then either $\lambda \in \sigma(A)$ or $1 \in \sigma(\mathbf{G}(\lambda)K)$.*

We recall the definitions of exact controllability and observability.

Definition 3.5. Let A be the generator of a strongly continuous semigroup $\mathbb{T}$ on X and let $B \in \mathcal{L}(U, X_{-1})$ be an admissible control operator for $\mathbb{T}$. The pair (A, B) is *exactly controllable* in time $T > 0$, if for every $x_0 \in X$ there exists a $u \in L^2([0, T], U)$ such that

$$\Phi_T u = \int_0^T \mathbb{T}_{T-\sigma} B u(\sigma) d\sigma = x_0 .$$

(A, B) is *exactly controllable* if it is exactly controllable in time T, for some $T > 0$.

Suppose that A is the generator of the strongly continuous semigroup $\mathbb{T}$ on X and $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$. Of course, this is equivalent to C^* being an admissible control operator for the dual semigroup $\mathbb{T}^*$. We say that (A, C) is *exactly observable* (in time T) if (A^*, C^*) is exactly controllable (in time T).

For more details on exact observability in an operator-theoretic setting we refer to Hansen and Weiss [22], Russell and Weiss [39], Avdonin and Ivanov [4] and the references therein. In the PDE setting, the relevant literature is overwhelming, and we mention the books of Lions [29], Lagnese and Lions [28] and Komornik [27] and the paper of Bardos, Lebeau and Rauch [9].

The following invariance result is taken from Section 6 of [55].

Proposition 3.6. *Let Σ be a well-posed linear system, let K be an admissible feedback operator for Σ and let Σ^K be the corresponding closed-loop system. We denote by (A, B, C) the generating triple of Σ and by (A^K, B^K, C^K) the generating triple of Σ^K. Then the following holds:*

(a) *(A, B) is exactly controllable in time T, if and only if (A^K, B^K) has the same property.*

(b) *(A, C) is exactly observable in time T, if and only if (A^K, C^K) has the same property.*

We recall some concepts which are often used in the optimal control literature, following the formulation in Weiss and Rebarber [58].

Definition 3.7. Let A be the generator of a strongly continuous semigroup $\mathbb{T}$ on X and suppose that $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$. Then (A, B) is *optimizable* if for every $z_0 \in X$ there exists a $u \in L^2([0, \infty), U)$ such that the state trajectory z defined in (3.2) is in $L^2([0, \infty), Z)$.

Let $C \in \mathcal{L}(X_1, Y)$ be an admissible observation operator for $\mathbb{T}$. Then (A, C) is *estimatable* if (A^*, C^*) is optimizable.

It was shown in [58] that optimizability and estimatability are invariant under output feedback:

Proposition 3.8. *With the notation from Proposition* 3.6, *(A, B) is optimizable if and only if (A^K, B^K) is optimizable and (A, C) is estimatable if and only if (A^K, C^K) is estimatable.*

We quote from [58] the following characterization of exponential stability:

Theorem 3.9. *A well-posed linear system is exponentially stable if and only if it is optimizable, estimatable and input-output stable.*

It follows from this theorem that a well-posed linear system is exponentially stable if it is exactly controllable, exactly observable and input-output stable.

4 Exponential stabilization

We formulate our results for well-posed linear systems, which is the more general context, but we also explain the consequences for the more restrictive but simpler case of regular linear systems.

Theorem 4.1. *Suppose that the well-posed linear system* Σ *satisfies* ESAD *and* COL. *We denote by* $\mathbf{G}$ *the transfer function of* Σ. *Then there exist operators* $E = E^* \in \mathcal{L}(U)$ *such that* $\mathbf{G} + E$ *is a positive transfer function, i.e.,*

$$\mathbf{G}(s)^* + \mathbf{G}(s) + 2E \geq 0 \qquad \forall\, s \in \mathbb{C}_0 . \tag{4.1}$$

In particular, if Σ *is weakly regular and its feedthrough operator is* D, *then one such operator* E *is determined by*

$$\langle Ev, v\rangle = -\frac{1}{2}\langle (D^* + D)v, v\rangle + \lim_{\lambda\to+\infty} \lambda \|(\lambda I - A)^{-1}Bv\|^2 \qquad \forall\, v \in U . \tag{4.2}$$

Proof. We shall use the following identity:

$$(\overline{s}I - A^*)^{-1} + (sI - A)^{-1} = (\overline{s}I - A^*)^{-1}[2(\operatorname{Re} s)I + Q](sI - A)^{-1} . \tag{4.3}$$

The transfer function $\mathbf{G}$ of Σ satisfies (see (3.12))

$$\mathbf{G}(s) - \mathbf{G}(\beta) = B^*[(sI - A)^{-1} - (\beta I - A)^{-1}]B. \tag{4.4}$$

Using this we calculate

$$\begin{aligned}
&[\mathbf{G}^*(s) + \mathbf{G}(s)] - [\mathbf{G}^*(\beta) + \mathbf{G}(\beta)] \\
&\quad = B^*[(\overline{s}I - A^*)^{-1} + (sI - A)^{-1}]B - B^*[(\overline{\beta}I - A^*)^{-1} + (\beta I - A)^{-1}]B \\
&\quad = B^*(\overline{s}I - A^*)^{-1}[2(\operatorname{Re} s)I + Q](sI - A)^{-1}B \\
&\qquad\quad - B^*(\overline{\beta}I - A^*)^{-1}[2(\operatorname{Re} \beta)I + Q](\beta I - A)^{-1}B,
\end{aligned}$$

where we have used (4.3). Rearranging the above formula we obtain

$$\begin{aligned}
&[\mathbf{G}^*(s) + \mathbf{G}(s)] - B^*(\overline{s}I - A^*)^{-1}[2(\operatorname{Re} s)I + Q](sI - A)^{-1}B = \\
&[\mathbf{G}^*(\beta) + \mathbf{G}(\beta)] - B^*(\overline{\beta}I - A^*)^{-1}[2(\operatorname{Re} \beta)I + Q](\beta I - A)^{-1}B.
\end{aligned} \tag{4.5}$$

Thus both sides are equal to a bounded, self-adjoint operator on X, which depends neither on s nor on β. We denote this operator by $-2E$. Now since

$$B^*(\overline{s}I - A^*)^{-1}[2(\operatorname{Re} s)I + Q](sI - A)^{-1}B \geq 0 \qquad \forall\, s \in \mathbb{C}_0 ,$$

we deduce that

$$2E + \mathbf{G}^*(s) + \mathbf{G}(s) \geq 0 \qquad \forall\, s \in \mathbb{C}_0 .$$

Suppose now that Σ is weakly regular with feedthrough operator D. Then taking $s = \lambda$ in (4.5) to be real and positive, we obtain

$$\begin{aligned}
-2\langle Ev, v\rangle &= \lim_{\lambda\to+\infty} \langle (\mathbf{G}(\lambda)^* + \mathbf{G}(\lambda))v, v\rangle - \lim_{\lambda\to+\infty} 2\lambda \|(\lambda I - A)^{-1}Bv\|^2 \\
&\quad + \lim_{\lambda\to+\infty} \langle Q(\lambda I - A)^{-1}Bv, (\lambda I - A)^{-1}Bv\rangle \\
&= -\langle (D + D^*)v, v\rangle - 2 \lim_{\lambda\to+\infty} \lambda \|(\lambda I - A)^{-1}Bv\|^2,
\end{aligned}$$

since the inequality (3.4) shows that the limit containing Q is zero. ■

We remark that it follows from this theorem that $cI + \mathbf{G}$ is a positive transfer function for $c = \|E^+\|$, where $\|E^+\|$ is the positive part of E, as in Section 2.

We know that the admissibility of B implies

$$\|(\lambda I - A)^{-1}B\|_{\mathcal{L}(U,X)} \leq \frac{\delta}{\lambda^{0.5}} \tag{4.6}$$

for some $\delta > 0$ and for all sufficiently large λ, see (3.4). If B is bounded, then we may replace the exponent 0.5 appearing above with 1. Thus, it is natural to consider operators B for which 0.5 may be replaced by a larger number. It turns out that this has interesting consequences.

Proposition 4.2. *Let Σ be a well-posed linear system satisfying* ESAD *and* COL. *If B is not maximally unbounded, i.e., if there exist numbers $\delta > 0$ and $\nu > 0.5$ such that for all sufficiently large $\lambda > 0$*

$$\|(\lambda I - A)^{-1}B\|_{\mathcal{L}(U,X)} \leq \frac{\delta}{\lambda^{\nu}}, \tag{4.7}$$

then Σ is regular and $B^*_\Lambda(sI - A)^{-1}B$ is a positive transfer function.

Proof. If (4.7) holds then from $\mathbf{G}'(s) = B^*(sI - A)^{-2}B$ (valid for all $s \in \mathbb{C}_0$) and from $\|B^*(\lambda I - A)^{-1}\|_{\mathcal{L}(X,U)} \leq \frac{\delta'}{\lambda^{0.5}}$ (the dual version of (4.6), a consequence of the admissibility of B^* as an observation operator for $\mathbb{T}$) we see that

$$\|\mathbf{G}'(\lambda)\|_{\mathcal{L}(U)} \leq \frac{\delta\delta'}{\lambda^{1+\varepsilon}}, \quad \text{where} \quad \varepsilon = \nu - \frac{1}{2} > 0\,,$$

for all sufficiently large $\lambda > 0$. By integration we obtain

$$\|\mathbf{G}(\lambda_1) - \mathbf{G}(\lambda_2)\|_{\mathcal{L}(U)} \leq \frac{\delta\delta'}{\varepsilon}\left|\frac{1}{\lambda_1^\varepsilon} - \frac{1}{\lambda_2^\varepsilon}\right|,$$

so that Σ is regular. Let us denote by D its feedthrough operator. It follows from (4.2) and (4.7) that we may take $E = -\frac{1}{2}(D^* + D)$ and (by Theorem 4.1) $\mathbf{G} + E$ is a positive transfer function. This implies that $\mathbf{G} - D$ is also a positive tranfer function. Using (3.14) (with $C = B^*$) gives the desired conclusion. ■

The following example shows the existence of a regular linear system with feedthrough operator $D = 0$ for which the operator E is nonzero.

Example 4.3. Consider the usual realization of a delay line of length h, $h > 0$, as given, e.g., on p. 831 of [54]. The state space of this system Σ_0 is $X = L^2[-h, 0]$, the semigroup is the left shift operator with zero entering from the right, with the generator

$$A_0 = \frac{d}{d\xi}, \quad \mathcal{D}(A_0) = \{x \in H^1(-h, 0) \mid x(0) = 0\}.$$

The control operator is $B = \delta_0$ and the observation operator is $C = \delta^*_{-h}$, which means that $Cx = x(-h)$ for $x \in D(A_0)$. The feedthrough operator is zero and the transfer function of this system is $\mathbf{G}_0(s) = e^{-hs}$. We call the input w and the output y. We close a positive unity feedback loop around this delay line, meaning that $w = y + u$, where u is the new input function. This leads to a new well-posed linear system Σ with the transfer function

$$\mathbf{G}(s) = \frac{e^{-hs}}{1 - e^{-hs}}.$$

The semigroup $\mathbb{T}$ of this new system is the periodic left shift semigroup on X, which is unitary. The generating operators of Σ can be computed directly, or using the feedback theory in [55]. The generator of $\mathbb{T}$ is again $A = \frac{d}{d\xi}$, but now the domain is

$$\mathcal{D}(A) = \{x \in H^1(-h, 0) \mid x(-h) = x(0)\}.$$

It is easy to see that this A is skew-adjoint. The operators B and C remain basically the same, but of course the new C is defined on the new $\mathcal{D}(A)$ and this results in $C = B^*$. Thus, the system Σ fits into the framework of this paper (it satisfies ESAD and COL). Moreover, Σ is regular and its feedthrough operator is zero.

It is not difficult to verify that we have

$$\mathbf{G}(i\omega)^* + \mathbf{G}(i\omega) = -1 \qquad \forall\, \omega \in \mathbb{R},$$

so that $\mathbf{G}$ is not positive-real, but $\frac{1}{2} + \mathbf{G}$ is. If we close a negative feedback loop around Σ by putting $u = -\kappa y + v$, where v is the new input, then we get the closed-loop transfer function

$$\mathbf{G}^\kappa(s) = \frac{e^{-hs}}{1 - (1-\kappa)e^{-hs}}.$$

This transfer function is bounded on some right half-plane, since

$$|\mathbf{G}^\kappa(s)| \leq \frac{e^{-hx}}{|1 - (1-\kappa)e^{-hx}|}, \qquad x = \operatorname{Re} s,$$

and so the closed-loop system is well-posed. $\mathbf{G}^\kappa$ is stable for $0 < \kappa < 2$, but for $\kappa \geq 2$ the transfer function has unstable poles. This shows that the closed-loop semigroup becomes unstable. Moreover, the larger κ becomes, the more unstable the closed-loop system becomes (its poles move to the right).

Theorem 4.4. *Suppose that Σ is a well-posed linear system with input and output space U. Assume that its transfer function $\mathbf{G}$ is such that for some $c \geq 0$, $cI + \mathbf{G}$ is a positive transfer function. Denote $\kappa_0 = \frac{1}{c}$ (for $c = 0$, take $\kappa_0 = \infty$). Then for any $\kappa \in (0, \kappa_0)$ the operator $K = -\kappa I$ is an admissible feedback operator for Σ and the corresponding closed-loop system Σ^κ is input-output stable, i.e., $\mathbf{G}^\kappa = \mathbf{G}(I + \kappa\mathbf{G})^{-1} \in H^\infty$. Moreover, if Σ is optimizable and estimatable, then the closed-loop system is exponentially stable.*

Proof. The proof of the fact that $\mathbf{G}^\kappa \in H^\infty$ for all $\kappa \in (0, \kappa_0)$ is exactly like the proof of Lemma 2.1. Now applying Theorem 3.9, we can conclude exponential stability if Σ^K is optimizable and estimatable. The latter follows from the optimizability and estimatability of Σ using Proposition 3.8. ∎

As we remarked previously, $cI + \mathbf{G}$ is a positive transfer function for $c = \|E^+\|$ and Theorem 1.2 is a corollary of the above theorem using this c.

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