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IDENTIFICATION AND ROBUST CONTROL: Bridging the Gap

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Abstract

The topics of Identification and Robust Control have a rich history and have reached a level of considerable maturity. A difficulty, however, is that the two fields have evolved along different lines and now have many incompatibilities. The aim of this paper is to raise awareness to this problem and to suggest alternative formulations that may lead to a better match between the fields. One possible line of attack on the problem is proposed together with illustrations showing the merits of approaching this problem from alternative perspectives.

1 Introduction

The subject of robust control has been central to research endeavours in the area of systems and control over the past 4 or 5 decades, see e.g. (Zhou *et al.*, 1995; Dahleh and Diaz-Bobillo, 1995; Bhattacharyya *et al.*, 1995). Modern robust control design procedures include H_∞ methods, (Doyle *et al.*, 1989; Grimble, 1994; Stoorvogel, 1992; Balas *et al.*, 1993), ℓ_1 methods (Dahleh and Diaz-Bobillo, 1995) and so on.

A typical starting point for contemporary robust control algorithms is that one has available hard bounds on the model uncertainty. This has led to substantial interest in the problem of obtaining error bounds from experimental data, see, e.g. (Belforte *et al.*, 1990; Helmicki *et al.*, 1991; Makila and Partington, 1991; Milanese and Belforte, 1982; Norton, 1987; Lau *et al.*, 1992), etc. However, these bounds have been predicated on the existing formulations of robust control and have thus focused on hard bounds in the frequency domain. This formulation of the problem disregards the fact that virtually all of the existing literature on the subject of system identification has been based on a probabilistic noise assumption which leads to statistical error bounds, see e.g. (Söderström and Stoica, 1989; Ljung, 1987). This dichotomy is philosophically objectionable, even if in practice, users feel they have adequate answers to the dual problems of identification and robust control.

In the light of the above arguments, the aim of this paper is to put the case for a better match between robust control and system identification. This will actually lead us to reexamine the existing wisdom regarding both fields. We hasten to add that our objective is not necessarily to lay down a final solution to the question. Instead, we hope that by bringing the two fields

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together we can inspire continued R&D effort which takes account of the totality of the problem rather than allowing one side to "call the shots".

With respect to system identification, our aim is, not only to obtain an estimated model, but also to give a quantification of the expected errors associated with that model. We recognize two sources of errors. Firstly, disturbances and measurement noise (present in all real data) induce, so-called, *variance errors*. Secondly, the impossibility of obtaining a complete description of the system within the hypothesised model structure induces, so-called, *undermodelling* or *bias errors*. Our aim is to give a consistent paradigm for quantifying the effect of *both* these errors on the estimated model. The study of the first of these error sources is the basis of traditional statistical theory. However, the undermodelling error component lies outside the scope of traditional approaches. For this reason, there has been ongoing interest in this topic, especially in the context of identification for robust control design.

In recent literature, undermodelling errors have been studied from several alternative perspectives (see e.g. (Ninness and Goodwin, 1995)). The existing approaches can be broadly classified into two streams, depending upon the type of error bounds computed. On the one hand, by characterising the undermodelling error sources as deterministic one can obtain "hard" L_∞-type error bounds. On the other hand, by characterising the undermodelling error sources as stochastic one can obtain "soft" variance-type error bounds. We remark that this classification, however, does not exhaust the possibilities to assess the model quality; see e.g. (Ljung and Guo, 1997).

The stochastic characterisation of modelling errors due to noise has been widely accepted as having a high degree of validity in practice (Goodwin and Payne, 1977; Ljung, 1987; Söderström and Stoica, 1989). It is therefore of interest to examine the possibility of extending this approach to undermodelling. A first step towards a stochastic characterisation of undermodelling errors was proposed in (Goodwin and Salgado, 1989), in which the "true" system model was embedded in an underlying stochastic process in the frequency domain. A further step in developing this paradigm was suggested in (Goodwin *et al.*, 1992), where a maximum likelihood procedure was proposed for estimating the parameters in the distribution of the embedded process, thus providing a self-contained mechanism for going from observed plant data to estimated model and associated error bounds. A practical case study of this methodology is described in (Tøffner-Clausen, 1996). More recently, it has been proposed in (Goodwin *et al.*, 1999a) that a more appropriate characterisation of the undermodelling would be as a *multiplicative* error which forms a *non-stationary (random walk)* stochastic process *in the frequency domain*. An additional feature of this approach is that it significantly simplifies the estimation of the parameters of the embedded process and, in particular, yields a closed-form expression for the total model error quantification due to noise and undermodelling. We will give brief details of this formulation below.

Turning to robust control; traditionally this has assumed the presence of hard error bounds. This is clearly inconsistent with soft bounds of the type described above. We will thus suggest an alternative approach to robust control which makes use of the type of model error quantification provided by the identification procedures in common use. In particular, we will aim to use statistical confidence bounds. With this type of bound it is unreasonable to guarantee stability, instead the focus here is to minimize the variance from a nominal design, which, *inter alia* will ensure that stability is addressed in a probabilistic sense. The essential idea of the procedure we describe is to modify a nominal controller so as to minimize the expected variation (variance) of the actual system performance from an a priori given desired performance.

2 Model Description

We consider a single-input single-output linear dynamic system having an unknown transfer function $g(j\omega)$; $\omega \in (0, \infty)$. Our objective is to fit a model of the form $g_0(j\omega;\theta)$ to plant input-output data by estimating θ by $\hat{\theta}$.

We assume that there exists some (unknown) $\theta \in \mathbf{R}^p$ such that

$$g(j\omega) = g_0(j\omega;\theta) + \delta g(\omega), \tag{1}$$

where δg represents the undermodelling error. The core idea of the model error quantification procedure proposed here is to think of δg as a particular realization of a stochastic process; i.e. δg is given a probabilistic interpretation similar to that which we typically ascribe to noise and disturbances (Goodwin and Salgado, 1989; Goodwin *et al.*, 1992). More will be said below about the specifics of the frequency domain embedded process. Note that we do *not* assume the existence of a value $\theta_0 \in \mathbf{R}^p$ such that $g_0(j\omega;\theta_0) = g(j\omega)$. On the contrary, (1) implies that the model class $g_0(j\omega;\theta)$ is fundamentally inadequate. We suggest that this coincides with practical reality.

We will approach the problem of estimating the transfer function in two stages. We first show how $g(j\omega)$ can be estimated for ω belonging to a finite set of (not necessarily equally spaced) frequencies $\{\omega_1, \ldots, \omega_m\}$. Then, we show how the parameter vector θ can be estimated along with the embedded undermodelling parameters in the frequency domain.

3 Transfer Function Point Estimation

Suppose that we are given a set of sampled input-output data of length N with sampling period T. In going from time domain data to an estimate of $g(j\omega)$, we assume a periodic input comprising m sine waves of frequencies $\{\omega_1, \ldots \omega_m\}$[1]. (This is not crucial but simplifies the description of the algorithm.) Thus,

$$u(t) = \sum_{\ell=1}^{m} A_\ell \cos(\omega_\ell t),$$

where

$$\omega_\ell = \frac{k_\ell 2\pi}{NT}, \quad \text{with } k_\ell \in \{1, \ldots, N\}, \tag{2}$$

We aim to obtain a first "raw" estimate of the transfer function at these frequencies. Assuming that the data are collected under steady state conditions, the corresponding sampled output response is given by

$$y(kT) = \sum_{\ell=1}^{m} A_\ell g^R(\omega_\ell) \cos(\omega_\ell kT) - \sum_{\ell=1}^{m} A_\ell g^I(\omega_\ell) \sin(\omega_\ell kT) + v(kT),$$

where the true frequency response of the plant satisfies $g(j\omega_\ell) = g^R(\omega_\ell) + jg^I(\omega_\ell)$ and $\{v(kT)\}_{k=1}^N$ is a noise process.

[1] For simplicity we assume that no $\omega_\ell = 0$ and that the phases are all zero. The theory can be trivially extended otherwise.

The above formulation suggests that g^R and g^I can be estimated using correlation methods. Specifically, we propose

$$\hat{g}^R(\omega_\ell) = \frac{2}{A_\ell N}\sum_{k=1}^{N} y(kT)\cos(\omega_\ell kT), \tag{3}$$

$$\hat{g}^I(\omega_\ell) = -\frac{2}{A_\ell N}\sum_{k=1}^{N} y(kT)\sin(\omega_\ell kT). \tag{4}$$

Actually, it can be readily shown that these are the optimal least squares estimates. For future reference, we call the estimates given in (3), (4), "transfer function point estimates".

4 Stochastic Embedding of the Noise Process

Note that bias errors need not be considered at this stage since we are fitting the model on a frequency by frequency basis. Within this context, it is standard in system identification to assume that the noise process $\{v(kT)\}_{k=1}^{N}$ is a realization of a time-domain stochastic process. For simplicity we assume here that the noise is a realization of a stationary uncorrelated sequence of variance δ_v^2, that is

$$E\{VV'\} = \delta_v^2 I, \tag{5}$$

where $V = [v(T), v(2T), \ldots, v(NT)]'$. Clearly other (coloured noise) processes can be dealt with similarly.

5 Quantification of Errors in Transfer Function Point Estimates

We can use standard least squares theory to quantify the effect of V on the estimated frequency response. We start by writing equations (3), (4) in vector form as

$$\hat{G} = (\Phi'\Phi)^{-1}\Phi' Y, \tag{6}$$

where

$$\begin{aligned}
\hat{G} &= \left[\hat{g}^R(\omega_1), \hat{g}^I(\omega_1), \ldots, \hat{g}^R(\omega_m), \hat{g}^I(\omega_m)\right]', \\
Y &= [y(T), y(2T), \ldots, y(NT)]' \\
\Phi &= \begin{bmatrix} \cos(\omega_1 T) & \sin(\omega_1 T) & \ldots & \sin(\omega_m T) \\ \cos(\omega_1 2T) & \sin(\omega_1 2T) & \ldots & \sin(\omega_m 2T) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(\omega_1 NT) & \sin(\omega_1 NT) & \ldots & \sin(\omega_m NT) \end{bmatrix} \\
&\quad \times \ \mathrm{diag}\,[\mathrm{A}_1, -\mathrm{A}_1, \mathrm{A}_2, -\mathrm{A}_2, \ldots, \mathrm{A}_\mathrm{m}, -\mathrm{A}_\mathrm{m}].
\end{aligned}$$

Note that $\Phi'\Phi = \mathrm{diag}\,[A_1^2, A_1^2, \ldots, A_m^2, A_m^2]N/2$, due to nature of the input signal.

Using (5), the error in the estimated transfer function can be quantified as (Goodwin and Payne, 1977, Theorem 2.3.1)

$$E\left\{(\hat{G} - G)(\hat{G} - G)'\right\} = \left(\Phi'\Phi\right)^{-1}\sigma_v^2,$$

where $G = \left[g^R(\omega_1), g^I(\omega_1), \ldots, g^R(\omega_m), g^I(\omega_m)\right]'$. Since $\Phi'\Phi$ is diagonal, the stochastic process

$$\left[\begin{array}{c} \tilde{g}^R(\omega_\ell) \\ \tilde{g}^I(\omega_\ell) \end{array}\right] = \left[\begin{array}{c} \hat{g}^R(\omega_\ell) \\ \hat{g}^I(\omega_\ell) \end{array}\right] - \left[\begin{array}{c} g^R(\omega_\ell) \\ g^I(\omega_\ell) \end{array}\right] \tag{7}$$

is an uncorrelated process in the frequency domain having *non-stationary* variance

$$\begin{aligned} E\left\{\left[\begin{array}{c} \tilde{g}^R(\omega_\ell) \\ \tilde{g}^I(\omega_\ell) \end{array}\right]\left[\begin{array}{c} \tilde{g}^R(\omega_n) \\ \tilde{g}^I(\omega_n) \end{array}\right]'\right\} &= D_\ell \delta(\omega_\ell - \omega_n), \\ D_\ell &\triangleq \frac{2\sigma_v^2}{A_\ell^2 N}\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]. \end{aligned} \tag{8}$$

Clearly (8) depends on the unknown quantity σ_v^2. However, an unbiased estimate of σ_v^2 can be obtained by standard statistical methods (Goodwin and Payne, 1977, Theorem 2.4.1) as

$$\hat{\sigma}_v^2 = \frac{1}{N-2m}(Y - \Phi\hat{G})'(Y - \Phi\hat{G}). \tag{9}$$

Finally, the covariance (8) can be estimated by replacing σ_v^2 by its unbiased estimate (9).

6 Stochastic Embedding of the Transfer Function

It has been common practice in identification (indeed throughout all of statistics) to assume that the only source of errors is random noise. However, as remarked earlier, we do not wish to assume that $g(j\omega_\ell) = g_0(j\omega_\ell;\theta)$ for any value of θ. For instance it can be argued, (Goodwin *et al.*, 1992), that the "optimal" model order is obtained when the bias errors are comparable to variance errors. Thus presumably one is ignoring the reality of 50% of the errors if one neglects bias errors. The crucial question is then, how should we think of the undermodelling error. To get a "feel" for this problem consider the simple case of an unmodelled pole; i.e.,

$$g(s) = g_0(s)\left(\frac{1}{\tau s + 1}\right),$$

which yields

$$g(s) - g_0(s) = g_0(s)\left(\frac{\tau s}{\tau s + 1}\right). \tag{10}$$

This, admittedly trivial, expression reveals two important qualitative features:

- Multiplicative errors are scaled and thus do not have units.
- The multiplicative error, in this example, has a magnitude which *grows* with frequency (at least up until the frequency $1/\tau$, which we take to be beyond the frequencies of interest).

Of course, in practice, we cannot expect to have such a simple description for the undermodelling as in (10). However, we use this to make a leap of faith and hypothesise that the undermodelled process can be embedded in a *non-stationary* stochastic process whose variance grows with frequency. At first glance this may appear bizarre but, it is perhaps no more bizarre than assuming the noise process is a stochastic process. (After all, it was Albert Einstein who said "God doesn't play dice"). The simplest non-stationary process with increasing variance is

a random walk in the frequency domain. This process has the conceptual advantage of having high complexity in the *realization space* but a *relatively simple probabilistic description.*

To be precise, we will assume that there exists a θ (unknown), a $\bar{\theta}$ (known) and a σ_ε^2 (unknown) such that the real and imaginary parts of $g(j\omega)$ have the form

$$\begin{array}{rcl} g^R(\omega) & = & g_0^R(\omega;\theta) + g_0^R(\omega;\bar{\theta})\lambda^R(\omega) \ , \\ g^I(\omega) & = & g_0^I(\omega;\theta) + g_0^I(\omega;\bar{\theta})\lambda^I(\omega) \ , \end{array} \tag{11}$$

where $\{\lambda^R(\omega)\}$ and $\{\lambda^I(\omega)\}$ are two zero-mean independent processes having uncorrelated increments, that is

$$\lambda^R(\omega) = \int_0^\omega d\varepsilon(s) \ , \qquad \lambda^I(\omega) = \int_0^\omega d\varepsilon(s) \ , \tag{12}$$

with $E\{d\varepsilon(s)d\varepsilon(s)\} = \sigma_\varepsilon^2 ds$, and

$$\begin{array}{rcl} E\{\lambda^R(\omega_\ell)\lambda^R(\omega_n)\} & = & E\{\lambda^I(\omega_\ell)\lambda^I(\omega_n)\} \\ & = & \displaystyle\int_0^{\min(\omega_\ell,\omega_n)} \sigma_\varepsilon^2 ds \\ & = & \min(\omega_\ell,\omega_n)\sigma_\varepsilon^2 \ . \end{array} \tag{13}$$

The value $\bar{\theta}$ in (11) could be taken, for example, as any a priori estimate of the nominal model parameters. One could alternatively take the value $\bar{\theta} = \theta$ as unknown. This, however, would lead to a significant increase in the complexity of work required to estimate θ. Since the model is an abstraction of reality, it seems that taking $\bar{\theta}$ known is a reasonable simplification. (Actually in the sequel we will take $\bar{\theta}$ as the estimate provided by a second least squares step.)

Finally, for purposes of illustration, we will model the nominal transfer function $g_0(j\omega;\theta)$ in terms of a given set of basis functions $b_1(\omega), \ldots, b_p(\omega)$. For example, we could use Orthonormal Bases (Wahlberg, 1991; Van den Hof *et al.*, 1993; Ninness and Gustafsson, 1997).

We define the vectors $B(\omega) = [b_1(\omega), \ldots, b_p(\omega)]$, $B^R(\omega) = \left[b_1^R(\omega), \ldots, b_p^R(\omega)\right]$ and $B^I(\omega) = \left[b_1^I(\omega), \ldots, b_p^I(\omega)\right]$ where the latter two vectors contain the real and imaginary parts of the basis functions. We thus assume (11) to be parameterised as

$$\begin{array}{rcl} g^R(\omega) & = & B^R(\omega)\theta + B^R(\omega)\bar{\theta}\lambda^R(\omega) \ , \\ g^I(\omega) & = & B^I(\omega)\theta + B^I(\omega)\bar{\theta}\lambda^I(\omega) \ . \end{array} \tag{14}$$

7 Parameter Estimation in the Model

We next use the frequency domain data $\hat{G}$ given by (6) to estimate the parameter θ. From (7) and (8) we have that the observed quantities $\hat{g}^R(\omega_\ell), \hat{g}^I(\omega_\ell)$ satisfy

$$\left[\begin{array}{c} \hat{g}^R(\omega_\ell) \\ \hat{g}^I(\omega_\ell) \end{array}\right] = \left[\begin{array}{c} g^R(\omega_\ell) \\ g^I(\omega_\ell) \end{array}\right] + \left[\begin{array}{c} \tilde{g}^R(\omega_\ell) \\ \tilde{g}^I(\omega_\ell) \end{array}\right] \ , \tag{15}$$

where $\{\tilde{g}^R(\omega_\ell), \tilde{g}^I(\omega_\ell)\}_{\ell=1}^m$ is an uncorrelated vector process (in the frequency domain) having covariance D_ℓ as in (8). Combining (15), (14) and (12), we have

$$\begin{array}{rcl} \hat{g}^R(\omega_\ell) & = & B^R(\omega_\ell)\theta + B^R(\omega_\ell)\bar{\theta}\lambda^R(\omega_\ell) + \tilde{g}^R(\omega_\ell) \ , \\ \hat{g}^I(\omega_\ell) & = & B^I(\omega_\ell)\theta + B^I(\omega_\ell)\bar{\theta}\lambda^I(\omega_\ell) + \tilde{g}^I(\omega_\ell) \ , \end{array} \tag{16}$$

where λ^R and λ^I are given by (12). In this model, $\{\hat{g}^R, \hat{g}^I\}$ is the given set of m observed quantities, $\{\tilde{g}^R, \tilde{g}^I\}$ are white noise processes (in the frequency domain) having variance (8), where σ_v^2 can be reasonably estimated as in (9), and $\{\lambda^R, \lambda^I\}$ are uncorrelated processes (in the frequency domain) having unknown covariance structure (13). We will add the further assumption that

$$\{\tilde{g}^R, \tilde{g}^I\} \text{ and } \{\lambda^{\mathrm{R}}, \lambda^{\mathrm{I}}\} \text{ are uncorrelated .} \tag{17}$$

To write (16), (12) in vector form, we define

$$\begin{aligned} \Lambda &= \operatorname{diag}\left[\lambda^{\mathrm{R}}(\omega_1), \lambda^{\mathrm{I}}(\omega_1), \ldots, \lambda^{\mathrm{R}}(\omega_{\mathrm{m}}), \lambda^{\mathrm{I}}(\omega_{\mathrm{m}})\right] \\ \tilde{G} &= \begin{bmatrix} \tilde{g}^R(\omega_1) \\ \tilde{g}^I(\omega_1) \\ \vdots \\ \tilde{g}^R(\omega_m) \\ \tilde{g}^I(\omega_m) \end{bmatrix}, \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} B^R(\omega_1) \\ B^I(\omega_1) \\ \vdots \\ B^R(\omega_m) \\ B^I(\omega_m) \end{bmatrix}. \end{aligned}$$

Then, the model (16) becomes

$$\hat{G} = \mathcal{B}\theta + \Lambda\mathcal{B}\bar{\theta} + \tilde{G}. \tag{18}$$

We then have:

Lemma 1. *(Parameter Estimation) An unbiased estimator for θ is given by the least-squares estimate*

$$\hat{\theta} = (\mathcal{B}'\mathcal{B})^{-1}\mathcal{B}'\hat{G} \ , \tag{19}$$

with associated model:

$$\hat{G}_0(\omega) = B(\omega)\hat{\theta} \tag{20}$$

The error covariance for $\hat{\theta}$ satisfies:

$$E\{(\hat{\theta} - \theta)(\hat{\theta} - \theta)'\} = (\mathcal{B}'\mathcal{B})^{-1}\mathcal{B}'(\Omega\sigma_\varepsilon^2 + A\sigma_v^2)\mathcal{B}(\mathcal{B}'\mathcal{B})^{-1} \ , \tag{21}$$

where

$$\Omega = \operatorname{diag}[\mathcal{B}\bar{\theta}] \left(\begin{bmatrix} \omega_1 & \omega_1 & \ldots & \omega_1 \\ \omega_1 & \omega_2 & \ldots & \omega_2 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1 & \omega_2 & \ldots & \omega_m \end{bmatrix} \otimes \mathrm{I}^{2\times 2} \right) \operatorname{diag}[\mathcal{B}\bar{\theta}] \tag{22}$$

$$A = \frac{2}{N}\operatorname{diag}[\mathrm{A}_1, \mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_2, \ldots, \mathrm{A}_{\mathrm{m}}, \mathrm{A}_{\mathrm{m}}]^{-2} \ . \tag{23}$$

Proof. From (19) and (18) we have that $\hat{\theta} = (\mathcal{B}'\mathcal{B})^{-1}\mathcal{B}'(\mathcal{B}\theta + \Lambda\mathcal{B}\bar{\theta} + \tilde{G})$.

It follows that

$$\hat{\theta} - \theta = (\mathcal{B}'\mathcal{B})^{-1}\mathcal{B}'(\Lambda\mathcal{B}\bar{\theta} + \tilde{G}) \tag{24}$$

using (17), we obtain

$$E\{(\hat{\theta}-\theta)(\hat{\theta}-\theta)'\} = (\mathcal{B}'\mathcal{B})^{-1}\mathcal{B}'[E\{\Lambda\mathcal{B}\bar{\theta}\bar{\theta}'\mathcal{B}'\Lambda\} + E\{\tilde{G}\tilde{G}'\}]\mathcal{B}(\mathcal{B}'\mathcal{B})^{-1} \ ,$$

which yields (21) on noting that

$$E\{\Lambda\mathcal{B}\bar{\theta}\bar{\theta}'\mathcal{B}'\Lambda\} = \Omega\sigma_\varepsilon^2 \ , \tag{25}$$

which follows from (13), and

$$E\{\tilde{G}\tilde{G}'\} = A\sigma_v^2 \ , \tag{26}$$

which follows from (8).

□

An unbiased estimate of the covariance error given in (21) is obtained using the estimate of σ_v^2 given by (9), and the following estimate of the undermodelling parameter σ_ε^2.

Lemma 2. *(Estimation of σ_ε^2) An unbiased estimate of σ_ε^2 is*

$$\hat{\sigma}_\varepsilon^2 = \frac{(\hat{G}-\mathcal{B}\hat{\theta})'(\hat{G}-\mathcal{B}\hat{\theta})}{trace[(I-\mathcal{B}(\mathcal{B}'\mathcal{B})^{-1}\mathcal{B}')\Omega]} - \frac{trace[(I-\mathcal{B}(\mathcal{B}'\mathcal{B}^{-1}\mathcal{B}')A]}{trace[(I-\mathcal{B}(\mathcal{B}'\mathcal{B}^{-1}\mathcal{B}')\Omega]}\sigma_v^2 \ . \tag{27}$$

Proof. By introducing the shorthand notation $P = I-\mathcal{B}(\mathcal{B}'\mathcal{B})^{-1}\mathcal{B}'$ we write, from (18) and (19),

$$\begin{aligned} E\{(\hat{G}-\mathcal{B}\hat{\theta})'(\hat{G}-\mathcal{B}\hat{\theta})\} &= E\{(\Lambda\mathcal{B}\bar{\theta}+\tilde{G})'P(\Lambda\mathcal{B}\bar{\theta}+\tilde{G})\} \\ &= \text{trace}[PE\{\Lambda\mathcal{B}\bar{\theta}\bar{\theta}'\mathcal{B}'\Lambda+\tilde{G}\tilde{G}'\}] \qquad \text{(using (17))} \\ &= \text{trace}[P(\Omega\sigma_\varepsilon^2+A\sigma_v^2)] \qquad \text{(using (25) and (26))} \\ &= \text{trace}[P\Omega]\sigma_\varepsilon^2 + \text{trace}[PA]\sigma_v^2 \ , \end{aligned}$$

and hence $E\{\hat{\sigma}_\varepsilon^2\} = \sigma_\varepsilon^2$ as claimed. □

Here again, we can replace σ_v^2 in (27) by its unbiased estimate from (9).

8 Quantification of Modelling Errors

The final estimated model is $\hat{G}_0(\omega)$ as in (20). The associated total modelling error (including the effects of noise and undermodelling) at any frequency ω_n is

$$G_e(\omega_n) = \begin{bmatrix} g_e^R(\omega_n) \\ g_e^I(\omega_n) \end{bmatrix} \triangleq \hat{G}_0(\omega_n) - G(\omega_n) = B(\omega_n)\hat{\theta} - G(\omega_n) \ , \tag{28}$$

where

$$G(\omega_n) = \begin{bmatrix} g^R(\omega_n) \\ g^I(\omega_n) \end{bmatrix} \text{ and } B(\omega_n) = \begin{bmatrix} B^R(\omega_n) \\ B^I(\omega_n) \end{bmatrix} \ .$$

The following result gives expressions for $G_e(\omega_n)$ and its covariance under the proposed non-stationary stochastic embedding.

Theorem 1. *(Modelling Error Quantification) At any frequency ω_n, the modelling error $G_e(\omega_n)$ in (28) has the form*

$$G_e(\omega_n) = B(\omega_n)Q\tilde{G} + [B(\omega_n)Q\Lambda\mathcal{B} - L(\omega_n)B(\omega_n)]\bar{\theta} \ , \tag{29}$$

where $Q = (\mathcal{B}'\mathcal{B})^{-1}\mathcal{B}'$ and $L(\omega_n) = diag[\lambda^R(\omega_n), \lambda^I(\omega_n)]$.

Without loss of generality, we assume that ω_n is such that $\omega_{k-1} \leq \omega_n < \omega_k$, where ω_{k-1} and ω_k are two consecutive frequencies of the test set (2). Then the error covariance $\Gamma(\omega_n) \triangleq E\{G_e(j\omega_n)G_e'(j\omega_n)\}$ is given by the expression

$$\Gamma(\omega_n) = K_v(\omega_n)\sigma_v^2 + K_\varepsilon(\omega_n)\sigma_\varepsilon^2 \ , \tag{30}$$

where

$$\begin{aligned} K_v(\omega_n) &= B(\omega_n)QAQ'B'(\omega_n) \\ K_\varepsilon(\omega_n) &= B(\omega_n)Q\Omega Q'B'(\omega_n) + (diag[B(\omega_n)\bar{\theta}])^2\omega_n - \Psi(\omega_n) - \Psi'(\omega_n) \end{aligned} \tag{31}$$

with A, Ω given in (23), (22), respectively, and

$$\Psi(\omega_n) = B(\omega_n)Q \begin{bmatrix} diag[B(\omega_1)\bar{\theta}]\omega_1 \\ \vdots \\ diag[B(\omega_{k-1})\bar{\theta}]\omega_{k-1} \\ diag[B(\omega_k)\bar{\theta}]\omega_n \\ \vdots \\ diag[B(\omega_m)\bar{\theta}]\omega_n \end{bmatrix} diag[B(\omega_n)\bar{\theta}] \ .$$

Proof. The expression (29) is obtained by replacing $\hat{\theta} - \theta$ from (24) in

$$\begin{aligned} G_e &= B(\omega_n)\hat{\theta} - B(\omega_n)\theta - L(\omega_n)B(\omega_n)\bar{\theta} \ , \\ &= B(\omega_n)(\hat{\theta} - \theta) - L(\omega_n)B(\omega_n)\bar{\theta} \ . \end{aligned}$$

Using (17), (25) and (26), we have

$$\begin{aligned} \Gamma &= E\{G_e{G_e}'\} \\ &= E\{[B(\omega_n)Q\Lambda\mathcal{B}\bar{\theta} - L(\omega_n)B(\omega_n)\bar{\theta}][B(\omega_n)Q\Lambda\mathcal{B}\bar{\theta} - L(\omega_n)B(\omega_n)\bar{\theta}]'\} \\ &\quad + E\{[B(\omega_n)Q\tilde{G}][B(\omega_n)Q\tilde{G}]'\} \\ &= B(\omega_n)QAQ'B(\omega_n)'\sigma_v^2 + B(\omega_n)Q\Omega Q'B(\omega_n)'\sigma_\varepsilon^2 \\ &\quad + \underbrace{E\{[L(\omega_n)B(\omega_n)\bar{\theta}][L(\omega_n)B(\omega_n)\bar{\theta}]'\}}_{(*)} \\ &\quad - \underbrace{B(\omega_n)QE\{\Lambda\mathcal{B}\bar{\theta}[L(\omega_n)B(\omega_n)\bar{\theta}]'\}}_{(**)} - (**)' \ . \end{aligned}$$

From (25) and (22) evaluated at the single frequency ω_n we have $(*) = (\text{diag}[B(\omega_n)\bar{\theta}])^2\omega_n\sigma_\varepsilon^2$. In a similar way it can be shown that $(**) = \Psi(\omega_n)\sigma_\varepsilon^2$. Putting everything together yields (30)–(31). □

When using (30) in practice one may simply replace σ_v^2 and σ_ε^2 by the unbiased estimates of these quantities given in (9) and (27).

Theorem 1 gives a simple expression for model error quantification. The component of this quantification due to noise is relatively standard. However, the key novelty here is that we have treated noise and undermodelling in a consistent framework. In particular, the first term on the RHS of (30) describes the effect of *noise* whilst the second captures the effect of *undermodelling*. Of course, the utility of the result ultimately depends on its capacity to represent the kinds of real errors met in practice. Perhaps the embedded process we have used is excessively simple but, at least, we believe it is a step in the right direction. Anyway, to again quote Einstein 'God may be subtle but He isn't plain mean'. Indeed, simulation studies with different types of systems has shown that the methodology described above gives remarkably good (and quite intricate) quantifications of model error - see Figure 1 for a selection of estimated models and associated error quantifications.

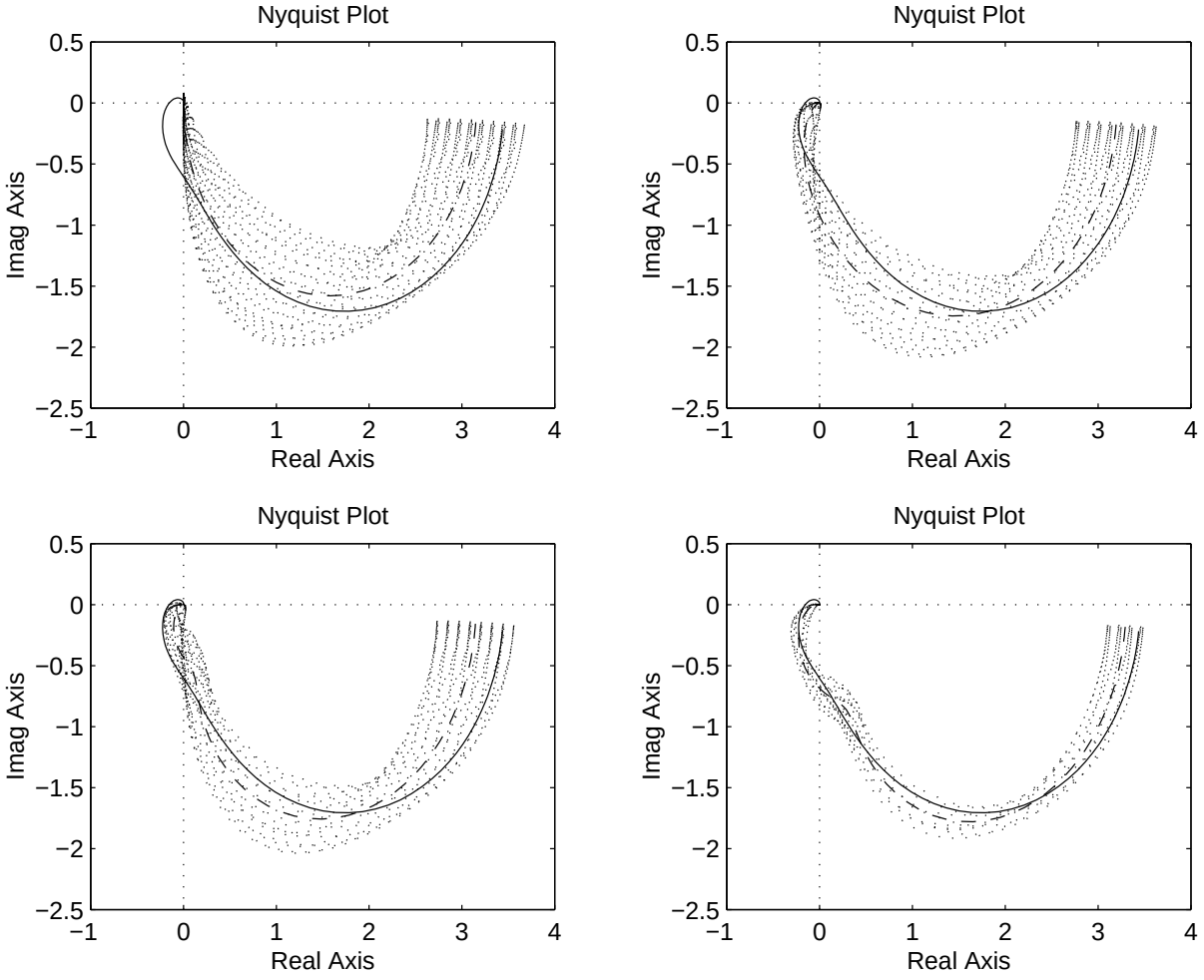

Figure 1: Example of Model Error Quantification. (Since this is a simulation example, we can show the true frequency response - the solid line. The estimated models are given by the dashed lines and the cloud of dots correspond to the expected total model errors at the 70% confidence level. The different plots correspond to increasing model complexity and it can be seen that the corresponding model error quantification, in this example, gives a good indication of the actual errors).

9 Robust Controller Design

We next turn to the question of robust control design. Since, we have argued above for a stochastic quantification of total model errors (due to noise and undermodelling) resulting from the identification phase, our aim here will be to utilize this error quantification as the basis for robust control design. This will again lead us to depart from traditional thinking on this problem.

As a starting point, let us assume that $\hat{G}_0(\omega)$ is used to obtain a preliminary design which leads to acceptable performance *when applied to the nominal model*. This design will typically account for the usual control design issues such as non-minimum phase behaviour, the available input range, unstable poles etc. (Goodwin *et al.*, 1999b). Let us say that this has been achieved with a nominal controller C_0 and that the corresponding nominal sensitivity function is S_0. Of course, the true plant is assumed to satisfy (28)-(30) and hence the value S_0 will thus not be achieved in practice. Thus there will be some variability of the achieved sensitivity S from S_0.

Our strategy will be to modify the nominal controller so as to minimize the expected variation of the actual system performance from the a priori desired performance as measured by S_0.

At this point it is convenient to change the notation for the nominal estimated model by introducing

$$\bar{G}_0(j\omega) = B^R(\omega)\hat{\theta} + jB^I(\omega)\hat{\theta}\ ;\ \bar{G}_0(s) = \frac{N_0(s)}{D_0(s)}$$

where $N_0(s)$ and $D_0(s)$ are polynomials with $D_0(s)$ monic. We also define the true plant transfer function as $\bar{G}(j\omega) = g^R(\omega) + jg^I(\omega)$ and the error between the estimated nominal model and true system as $\bar{G}_e(j\omega) = g_e^R(\omega) + jg_e^I(\omega)$.

We assume that we design a nominal controller $C_0(s) = \frac{P(s)}{L(s)}$ to stabilize the nominal model $\bar{G}_0(s) = \frac{N_0(s)}{D_0(s)}$. The nominal closed-loop poles are assumed to satisfy the following Diophantine equation

$$D_0(s)L(s) + N_0(s)P(s) = E(s) \tag{32}$$

where $E(s)$ is stable. The nominal sensitivity function $S_0(s)$ for this design is then given by

$$S_0(s) = \frac{D_0(s)L(s)}{E(s)} \tag{33}$$

and the complementary sensitivity function is

$$T_0(s) = \frac{N_0(s)P(s)}{E(s)} \tag{34}$$

the achieved sensitivity, S_1 using C_0 applied to the true plant will be given by

$$S_1 = \frac{1}{1 + C_0\bar{G}} \tag{35}$$

Our intention for robust design is to adjust the controller (i.e. replace C_0 by some other controller C, leading to a sensitivity S_2) so that the 'distance' between the sensitivity S_2 and S_0 is minimized. Such a 'mission' is difficult using the description of sensitivity in (35) because of

the nonlinear dependence on C_0 and $\bar{G}$. This difficulty is mitigated by using the Youla parameterisation of all stabilizing controllers, (Youla *et al.*, 1976). In terms of this parameterisation, the class of all controllers that stabilize the nominal model $\bar{G}_0(s)$ can be expressed as

$$C(s) = \frac{P(s) + D_0(s)\tilde{Q}(s)}{L(s) - N_0(s)\tilde{Q}(s)} \tag{36}$$

where $\tilde{Q}(s)$ is a stable and proper transfer function. We will adjust $C(s)$ by choosing $\tilde{Q}(s)$. For notational convenience, we also introduce $Q_0(s)$ and $Q(s)$ defined respectively by

$$\begin{aligned} Q_0(s) &= \frac{C_0(s)}{1 + C_0(s)\bar{G}_0(s)} \\ &= \frac{D_0(s)P(s)}{E(s)} \end{aligned} \tag{37}$$

$$\begin{aligned} Q(s) &= \frac{C(s)}{1 + C(s)\bar{G}_0(s)} \\ &= \frac{D_0(s)P(s)}{E(s)} + \frac{D_0(s)^2}{E(s)}\tilde{Q}(s) \end{aligned} \tag{38}$$

Note that the sensitivity function S_2 obtained using $C(s)$ applied to the true plant is

$$\begin{aligned} S_2(s) &= \frac{1}{1 + C(s)\bar{G}(s)} \\ &= \frac{1 - Q(s)\bar{G}_0(s)}{1 + Q(s)\bar{G}_e(s)} \end{aligned} \tag{39}$$

Here S_2 is a function of the uncertainty $\bar{G}_e(s)$. Observe that S_2 and S_0 denote respectively the sensitivity achieved when the plant is $\bar{G}$ and the controller is parameterized by Q, and the sensitivity when the plant is $\bar{G}_0$ and the controller is parameterized by Q_0. Our design will be aimed at minimizing an appropriate measure of the following error with respect to $\tilde{Q}$

$$S_2 - S_0 = \frac{1 - Q(s)\bar{G}_0(s)}{1 + Q(s)\bar{G}_e(s)} - (1 - Q_0(s)\bar{G}_0(s)) \tag{40}$$

Unfortunately, $(S_2 - S_0)$ is a nonlinear function of both Q and $\bar{G}_e$ which makes direct minimization of the error function a difficult task leading to non-convex solutions (Goodwin and Miller, 1998). As an alternative, we apply a standard procedure in optimization by choosing a suitable weighting function on the error to achieve linearisation. A suitable candidate for the weighting function is $W_1(s) = 1 + Q(s)\bar{G}_e(s)$ leading to

$$W_1(s)(S_2(s) - S_0(s)) = (1 - Q(s)\bar{G}_0(s)) - (1 - Q_0(s)\bar{G}_0(s))(1 + Q(s)\bar{G}_e(s)) \tag{41}$$

Actually, this turns out to be simply a particular approximation to $S_2 - S_0$.

Selecting the L_2 norm as the measure of the error, we define the following loss function to be minimized with respect to $\tilde{Q}$

$$\begin{aligned} J_2 &= ||E\left\{|W_1(S_2 - S_0)|^2\right\}|| = \int_{-\infty}^{\infty} E\left\{|W_1(j\omega)(S_2(j\omega) - S_0(j\omega)|^2\right\} d\omega \\ &= \int_{-\infty}^{\infty} E\left\{|(1 - Q(j\omega)\bar{G}_0(j\omega)) - (1 - Q_0(j\omega)\bar{G}_0(j\omega))(1 + Q(j\omega)\bar{G}_e(j\omega))|^2\right\} d\omega \end{aligned} \tag{42}$$

The weighted error function has the following properties.

Lemma 3. *Let $Q(j\omega)$ be defined by Equation (38). Then if $\tilde{Q}$ is stable, the weighted error is stable and has the following form*

$$W_1(S_2 - S_0) = -\frac{D_0 N_0}{E}\tilde{Q} - \left(\frac{LP}{E^2} + \frac{S_0}{E}\tilde{Q}\right)\bar{G}_e D_0^2 \tag{43}$$

Proof. The result follows on substituting Equations (40) and (38) into Equation (41) and observing that the denominator of the error consists of the polynomial E and the denominator of $\tilde{Q}$. □

Lemma 4. *For the modelling errors described in sect. 8, the loss function $E\left\{||W_1(S_2 - S_0)||_2^2\right\}$ can be written as*

$$J_2 = \int_{-\infty}^{\infty} |\frac{D_0 N_0}{E}|^2 |\tilde{Q}|^2 d\omega + \int_{-\infty}^{\infty} |\frac{LP}{E^2} + \frac{S_0}{E}\tilde{Q}|^2 |D_0^2| E\left\{|\bar{G}_e|^2\right\} d\omega \tag{44}$$

Proof. The result follows from Proposition 1 by observing that the model errors are unbiased, i.e. $E\{\bar{G}_e\} = 0$. □

Remark 1. *This loss function has intuitive appeal. The first term on the right hand side represents the 'bias' error. It can be seen that this term is zero if $\tilde{Q} = 0$. On the other hand, the second term in (44) represents the 'variance' error. For $E(|\bar{G}_e|^2) = 0$ (no model uncertainty), clearly the minimum of the loss function is reached if $\tilde{Q} = 0$, i.e. we leave the controller unaltered. As the magnitude of the uncertainty increases, the minimization will place more emphasis on the reduction of the 'variance' error.*

The methodology to be presented below applies equally well to stable and unstable open loop systems (see (Wang and Goodwin, 1998)). However, for the sake of clarity, we will consider the stable open loop case. In this special case, we can make the choices $E(s) = D_0(s)^2,\ P(s) = 0,\ L(s) = D_0(s)$. Then (38) becomes $Q(s) = Q_0(s) + \tilde{Q}(s)$ and (44) becomes

$$J_2 = \int_{-\infty}^{\infty} \{|\bar{G}_0(j\omega)|^2 |\tilde{Q}(j\omega)|^2 + |S_0(j\omega)Q_0(j\omega) + S_0(j\omega)\tilde{Q}(j\omega)|^2 \text{trace}[\Gamma(\omega)]\} d\omega \tag{45}$$

where $\Gamma(\omega)$ is the covariance of the total modelling error as in (30).

Let $\alpha = G_0$, $\beta = S_0 Q_0$, $\xi = \text{trace}[\Gamma]$ and $\gamma = S_0$. Equation (44) can be written in a more compact form as

$$||E\left\{|W_1(S_2 - S_0)|^2\right\}|| = \int_{-\infty}^{\infty} |\alpha|^2 |\tilde{Q}|^2 d\omega + \int_{-\infty}^{\infty} |\beta + \gamma\tilde{Q}|^2 \xi d\omega \tag{46}$$

We then have the following result.

Theorem 2. *Assume*

- *$\bar{G}_0$ is strictly proper with no zeros on the imaginary axis;*
- *ξ has as analytical spectral density which can be factorised as $\xi = \sigma(j\omega)\sigma(-j\omega)$.*

Then $|\alpha|^2 + |\gamma|^2\xi$ has a spectral factor which we label H and the loss function J_2 is minimized by choosing

$$\tilde{Q} = -\frac{1}{H} \times \textit{Stable part of } \frac{\xi|\delta|^2 Q_0}{H^*}$$

Proof. The result follows on using standard procedures for completing the square (Kucera, 1979). □

10 Incorporating Integral Action

The methodology given above can be readily extended to include integral action. Assuming that the nominal controller C_0 contains an integrator, from Equation (36), the final controller will also do so if $\tilde{Q}$ has the form

$$\tilde{Q}(s) = s\tilde{Q}'(s) \tag{47}$$

with $\tilde{Q}'$ stable and strictly proper. There are a number of ways to enforce this constraint. A particularly simple way is to change the loss function to

$$J' = \int_{-\infty}^{\infty} E\left\{|W_2|^2|S_2 - S_0|^2\right\} dw. \tag{48}$$

where

$$|W_2|^2 = \frac{|W_1|^2}{\omega^2} = \frac{|1 + Q\bar{G}_e|^2}{\omega^2}$$

Remark 2. *The term $\frac{1}{\omega^2}$ acts as an additional weighting function on the weighted error (41). This weighting function tends to infinity at zero frequency which puts heavy weight on the accuracy of the optimal solution at zero frequency. The finiteness of J' at $\omega = 0$ is ensured by the existence of integrators in both the nominal and robust controllers, as a consequence, both S_0 and S_2 contain $j\omega$ as a factor. In principle, we can select additional weighting functions in Equation (41) to reflect other performance requirements for the robust controller.*

Since C_0 contains an integrator, we can write $L(j\omega) = j\omega L'(j\omega)$. Then we obtain the following results which describe the robust controller in the case of integral action.

Theorem 3. *Under the same conditions as proposition 3, the value for $\tilde{Q}'$ is*

$$\tilde{Q}' = -\frac{1}{H} * \textit{stable part of } \left\{\frac{\xi|\delta|^2 Q_0}{sH^*}\right\}$$

11 Embellishments

As stated earlier, the robust control algorithm can be extended to the open loop unstable case. Another embellishment is to replace the true error quantification ξ by a suitable approximation. The latter may be useful for cases when ξ is presented in the form of experimental data or when the plant has a transcendental transfer function.

We write J_2 as

$$J_2 = \int_{-\infty}^{\infty} |H|^2|\tilde{Q} + \frac{\xi|S_0|^2 Q_0}{|H|^2}|^2 d\omega + \int_{-\infty}^{\infty} \xi|S_0|^2|Q_0|^2(1 - \frac{\xi|S_0|^2}{|H|^2})d\omega. \tag{49}$$

Let

$$F(j\omega) = \frac{\xi(\omega)|S_0(j\omega)|^2}{|H(j\omega)|^2}$$

Note that $F(j\omega)$ is an even function of ω which then can be approximated by

$$F(j\omega) \approx \tilde{F}(j\omega) + \tilde{F}(-j\omega) \tag{50}$$

where $\tilde{F}(s)$ is a stable, rational transfer function of some predetermined degree. Since the second term of righthand side of the cost function is independent of $\tilde{Q}$, the minimum of the cost function is achieved by setting

$$\tilde{Q}(s) \approx -(\tilde{F}(s)Q_0(s) + R(s)) \tag{51}$$

where $R(s)$ is the stable part of the partial fraction of $\tilde{F}(-s)Q_0(s)$.

The approximation of the even function $F(j\omega)$ in the form of Equation (50) is in the same spirit as ideas used in the context of Wiener filtering (see (Lee, 1960)). Specifically, we assume $\tilde{F}(s)$ has the following structure

$$\tilde{F}(s) = \sum_{i=1}^{N} c_i L_i(s) \tag{52}$$

where $L_i(s)$, $i = 1, 2, \ldots, N$, are suitably chosen orthonormal functions. $L_i(j\omega)$ satisfies the orthonormal properties

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |L_i(j\omega)|^2 d\omega = 1 \tag{53}$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} L_i(j\omega)L_k(j\omega)d\omega = 0 \tag{54}$$

where $k \neq i$. The coefficients of the approximation in Equation (52) are then given by, for $i = 1, 2, \ldots, N$,

$$c_i = \frac{1}{4\pi}\int_{-\infty}^{\infty} F(\omega)(L_i(j\omega) + L_i(-j\omega))d\omega \tag{55}$$

When the uncertainty $\xi(\omega)$ is given as discrete frequency response data, ie $\xi(\omega_k)$ ($k = 0, 1, 2, \ldots M$) or the nominal model $G_0(s)$ contains pure delays, $\bar{G}_0(j\omega)$ and $S_0(j\omega)$ are discretized into $\bar{G}_0(j\omega_k)$ and $S_0(j\omega_k)$, $k = 0, 1, \ldots M$. Then

$$F(j\omega_k) = \frac{\xi(\omega_k)|S_0(j\omega_k)|^2}{|H(j\omega_k)|^2} \tag{56}$$

where

$$|H(j\omega_k)|^2 = G_0(j\omega_k)G_0(-j\omega_k) + \xi(\omega_k)S_0(j\omega_k)S_0(-j\omega_k)$$

Equation (55) can then be approximated by using numerical integration to yield

$$c_i \approx \frac{1}{\pi}\sum_{k=0}^{M} F(j\omega_k)Real(L_i(j\omega_k))(\omega_k - \omega_{k-1}) \tag{57}$$

A particular choice for the functions, $L_i(s)$ are the Laguerre functions where for some $p > 0$

$$L_i(s) = \sqrt{2p}\frac{(s-p)^{i-1}}{(s+p)^i} \tag{58}$$

Alternatively one could use Kautz networks (Kautz, 1992) which allow the description of arbitrary stable poles including complex poles. For $\zeta_i > 0$ and $\omega_i > 0$, $i = 1, 2, 3...\frac{N}{2}$, the general form of Kautz filters are given by

$$L_{2i-1}(s) = \sqrt{2\zeta_i\omega_i}\frac{[s^2 - 2\zeta_1\omega_1 s + \omega_1^2]\dots[s^2 - 2\zeta_{i-1}\omega_{i-1}s + \omega_{i-1}^2][s + \omega_i]}{[s^2 + 2\zeta_1\omega_1 s + \omega_1^2]\dots[s^2 + 2\zeta_{i-1}\omega_{i-1}s + \omega_{i-1}^2][s^2 + 2\zeta_i\omega_i s + \omega_i^2]} \tag{59}$$

and

$$L_{2i}(s) = \sqrt{2\zeta_i\omega_i}\frac{[s^2 - 2\zeta_1\omega_1 s + \omega_1^2]\dots[s^2 - 2\zeta_{i-1}\omega_{i-1}s + \omega_{i-1}^2][s - \omega_i]}{[s^2 + 2\zeta_1\omega_1 s + \omega_1^2]\dots[s^2 + 2\zeta_{i-1}\omega_{i-1}s + \omega_{i-1}^2][s^2 + 2\zeta_i\omega_i s + \omega_i^2]} \tag{60}$$

12 Illustrative Example

As an illustration of the above procedure, we will apply it to a real laboratory heat exchange. This heat exchange is used for undergraduate control experiments and, inter-alia, has a pure time delay. The true system is, of course, unknown in this case but it is approximately of the form:

$$\bar{G}(s) \simeq \frac{Ke^{-sT}}{(\tau s + 1)}$$

where, depending on the operating point, K,T,τ lie in the ranges [1.5, 2.2]; [0.1, 0.2] and [0.38, 0.42] respectively.

Data was collected from the system by injecting sinusoidal signals as described in Section 3 - see Figure 2. We elect to model the system by a second order Laguerre Model. Figure 3 shows the Nyquist plot of the empirical transfer function estimates, $\hat{G}(\omega)$ (*the stars*), the estimated model $\bar{G}_0(j\omega)$ (*the solid line*) and total model error quantification via (30) (*the cloud of dots*). The nominal estimated model and nominal controller were given by

$$\bar{G}_\theta(s) = \frac{-3.4s + 33.7}{s^2 + 9.2s + 21.3}$$

$$Q_0(s) = \frac{s^2 + 9.2s + 21.3}{(s + 10)^2} * \frac{100}{33.7}$$

(As a matter of interest, the poles of the Laguerre basis functions were obtained by minimizing the total error as described via Γ in (30).)

The procedure described in section 11 was used to obtain an approximation based on the uncertainty description - see Figure 4. The controller was redesigned using the procedure outlined in sections 10 and 11 leading to the "robust" controller

$$Q(s) = \frac{2.04(s^4 + 22.1s^3 + 177.8s^2 + 623.3s + 809.1)}{(s + 9.95)^2(s + 5.1)^2}$$

Figure 5 shows the step responses of the physical system using the nominal and robust controllers at different operating points. The enhanced performance of the robust controller is evident.

This example is admittedly simple but it serves to illustrate that the two phases of identification and robust control design are indeed made compatible by the suggested formulation.

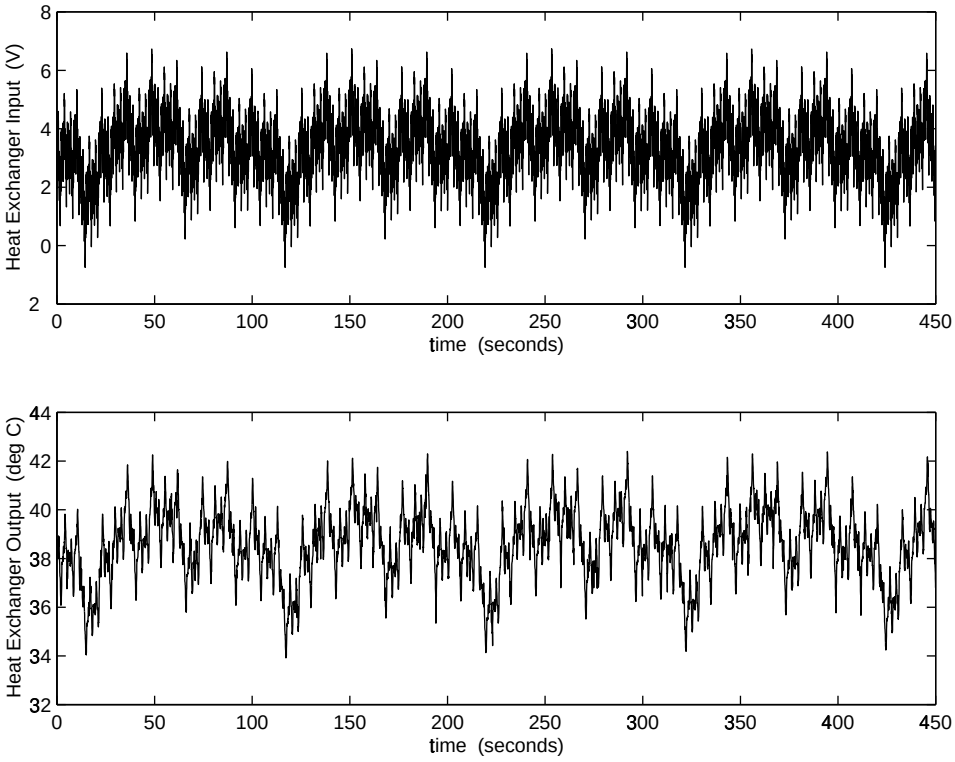

Figure 2: Plant Input-Output Data.

13 Conclusions

This paper has proposed a solution to the robust control problem which consistently combines system identification and robust control design. We have also shown how undermodelling and noise can be treated in a philosophically consistent fashion within the system identification phase. Apart from the technical issue posed, the paper also serves to survey many ideas that underlie aspects of modelling, identification, and control system design.

It is hoped that the paper might inspire others to take a fresh (and unprejudiced) look at the problems of identification and robust control from alternative perspectives from those which are currently favoured.

There are so many open questions for those who are looking for interesting research topics. These include:

- Extension of the undermodelling procedures to cover the common forms of undermodelling found in practice (including nonlinear elements such as deadzones, hysteresis, stiction, etc)
- Adding a stability constraint to the robust design procedure
- Deliniating those cases where the controller performance is limited by plant characteristics (non-minimum phase behaviour etc) from those where model uncertainty is the key limiting factor
- Implications for the design of experiments during the identification stage.
- Robust model predictive control accounting for control and state constraints.

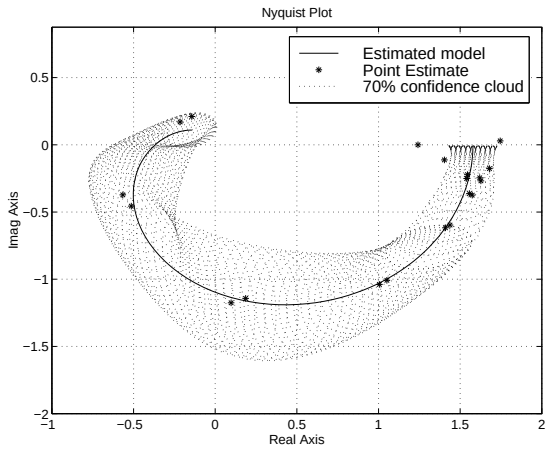

Figure 3: Estimated Frequency Response.

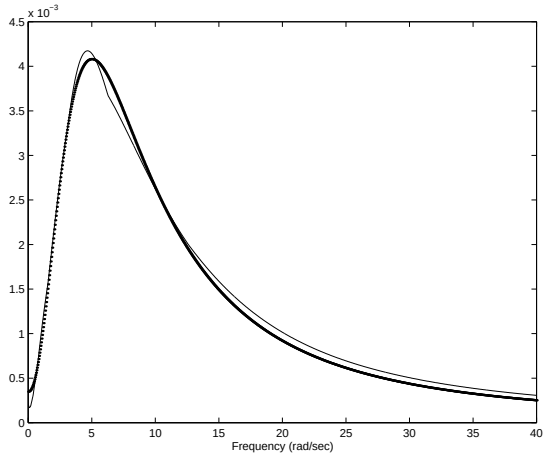

Figure 4: Approximating the Uncertainty.

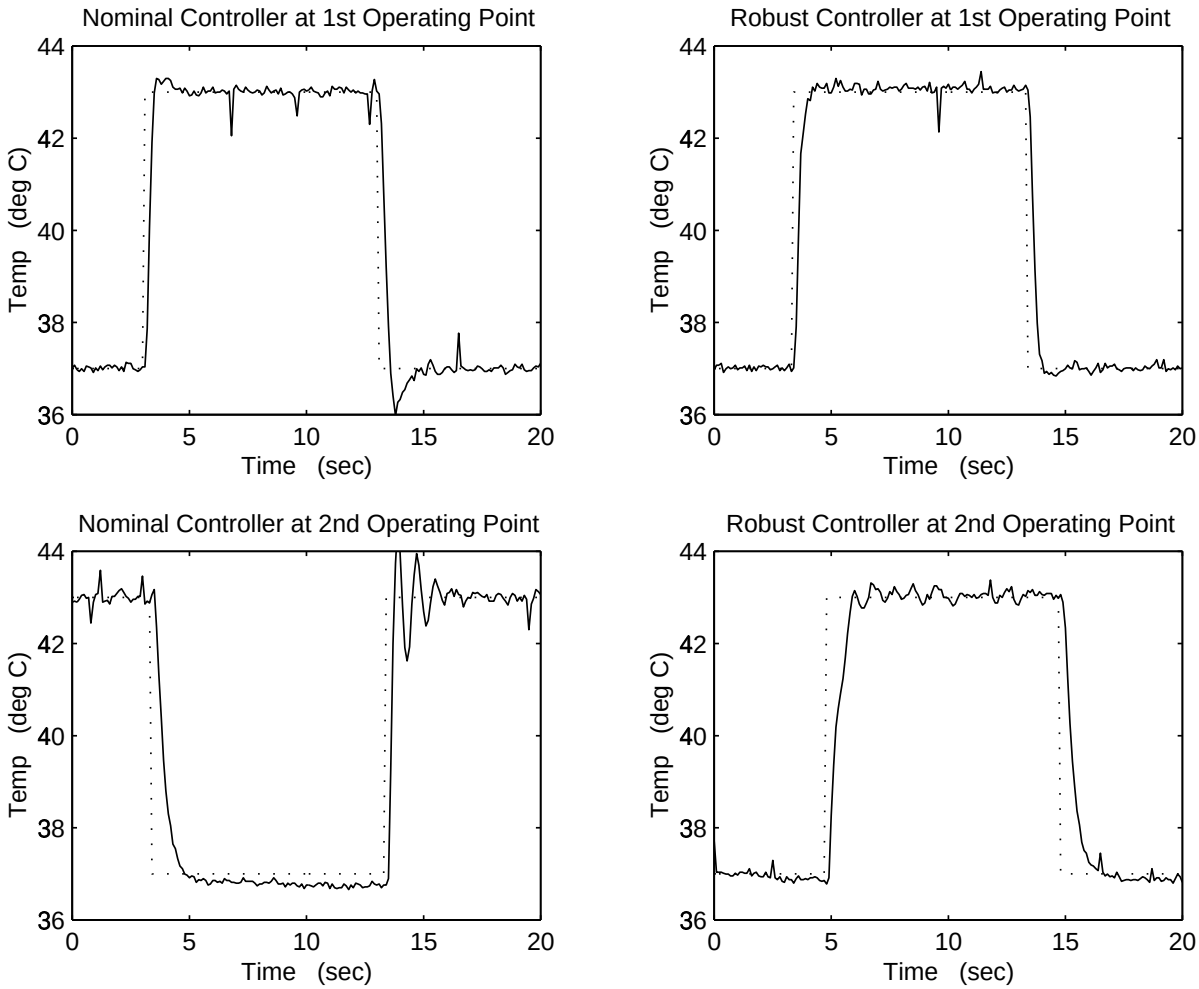

Figure 5: Step Responses with Nominal and Robust Controllers

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