Partial Lipschitz Nonlinear Sliding Mode Observers

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Abstract

The stability of a nonlinear observer for systems with uncertainties usually requires some sufficient conditions. The Lipschitz condition is a restrictive condition which many classes of systems may not satisfy. In this paper we consider a class of systems with two uncertain parts; one which satisfies the Lipschitz condition, whilst the other does not satisfy the Lipschitz condition but is a bounded uncertainty. Sliding mode theory is applied to yield feedforward compensation control to stabilize the error estimation system with non-Lipschitz uncertainty. New sufficient conditions for stability of the Thau observer are proposed. These conditions ensure the stability of the nonlinear observer by selecting a suitable observer gain matrix.

1 Introduction

A state observer for nonlinear systems was presented by Thau (1973) and extended by Kou *et al.* (1975). These methods do not include a systematic technique for the construction of the observer. However these observers satisfy a sufficient condition for the asymptotic stability of error system. In fact, there is not a straightforward method for selecting the observer gain to satisfy the sufficient condition. Rajamani (1998) and Rajamani *et al.* (1998) studied the Thau observer by considering the distance of unobservability (Eising, 1984) and the matrix condition number of the eignvector matrix of the error system.

Walcott and Żak (1987, 1990) discussed the state observation of nonlinear dynamic systems with bounded nonlinearities/uncertainties. They presented an observer design method using Lyapunov and min-max methods. Their approach requires the matching condition and is linked to the strictly positive real condition. Yaz and Azemi (1993) presented a method for designing an observer for nonlinear deterministic and stochastic systems, and used the continuous (boundary layer) gain given by Walcott and Żak (1987). Edwards and Spurgeon (1994) modified the Utkin observer (1992) and extended the discontinuous observer to nonlinear systems. They developed a robust discontinuous observer.

Dorling and Zinober (1983) compared the full and reduced order Luenberger observers with the Utkin observer. They reported some difficulty in the selection of an appropriate constant switched gain to ensure that the sliding mode occurs, and discussed the elimination of chattering. However, the unmatched uncertainty was shown to affect the ideal dynamics prescribed by the chosen sliding surface.

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Koshkouei and Zinober (1995) presented methods for designing an asymptotically stable observer, and studied the existence of the sliding mode and stability of state reconstruction systems of MIMO linear systems including a disturbance input.

In this paper we consider nonlinear systems with two uncertain parts; one part satisfying the Lipschitz condition, whilst the other does not, but is bounded. We generalize the condition given by Rajamani (1998) and Rajamani *et al.* (1998), and also propose a new sufficient condition for selecting the observer gain matrix without needing the solution of the Riccati equation. The sliding mode observer is presented and the stability of the error estimation systems is ensured if conditions such as matched uncertainty and the Lipschitz condition hold. Without these conditions, the method guarantees only that the trajectories enter a ball, with centre an equilibrium point, in finite time, and remain inside thereafter (Koshkouei and Zinober, 1998).

A sufficient condition for satisfying the Thau condition is given in Section 2. The nonlinear sliding mode observer is discussed in Section 3. An example illustrating the results is presented in Section 4.

In this paper $\sigma_M(\cdot)$, $\sigma_m(\cdot)$ $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ refer to the largest singular value, the smallest singular value, the largest and the smallest eigenvalue of $(\cdot)$, respectively. We also use p.d., p.d.s. and u.p.d.s. for positive definite, positive definite symmetric and unique positive definite symmetric. In addition, $P_1 > P_2$ indicates that $P_1 - P_2$ is a p.d. matrix and the H_∞ norm is defined as $\|G\|_\infty = \sup\limits_{\omega} \sigma_{\max}(G(iw))$.

2 Nonlinear Lipschitz Observer Design

Consider the nonlinear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + f(t,x,u) + \Gamma\xi(t) && (1)\\ y(t) &= Cx(t) && (2) \end{aligned}$$

where $x \in \mathbb{R}^n$ is the state variable, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $u \in \mathbb{R}^m$ is the control, $C \in \mathbb{R}^{m\times n}$, $y \in \mathbb{R}^m$ is the output, $\Gamma \in \mathbb{R}^{n\times m}$ the perturbation input map and $\xi \in \mathbb{R}^m$ the bounded disturbance input, i.e. there exists a positive real number M such that $\|\xi\| \le M$. $f(t,x,u)$ is an uncertain nonlinear function which satisfies the Lipschitz condition, with respect to x, with Lipschitz constant L, i.e.

$$\|f(t,x_1,u) - f(t,x_2,u))\| \le L\|x_1 - x_2\| \tag{3}$$

We assume that (A, C) is an observable pair. Suppose first that $\Gamma = 0$. Then a robust observer for the system (1)-(2) may be selected as

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + H(y - \hat{y}) + f(t,\hat{x},u) && (4)\\ \hat{y} &= C\hat{x} && (5) \end{aligned}$$

where $H \in \mathbb{R}^{n\times m}$ is the observer gain matrix. The state estimation error is defined as $e = x - \hat{x}$. Subtracting (1) from (4) gives the dynamical reconstruction error system

$$\begin{aligned} \dot{e} &= (A - HC)e + f(t,x,u) - f(t,\hat{x},u) && (6)\\ e_y &= Ce && (7) \end{aligned}$$

where $e_y = y - \hat{y}$ is the output reconstruction error. Let P be the u.p.d.s. solution of the Lyapunov equation (LE)

$$(A - HC)^T P + P(A - HC) = -Q \tag{8}$$

with Q an arbitrary p.d.s. matrix. The well-known Thau condition (1973) for the stability of the system (6) is

$$L < \frac{\lambda_{\min(Q)}}{2\lambda_{\max(P)}} \tag{9}$$

The maximum value of the ratio $\lambda_{\min}(Q)/\lambda_{\max}(P)$ is obtained for $Q = I$ (Patel and Toda, 1980) and then

$$L < \frac{1}{2\lambda_{\max}(P)} \tag{10}$$

The straightforward method for selecting the observer gain H to satisfy (9) or (10) is not known. Here we present a method to clarify the relationship between the eigenvalues $(A - HC)$ and L. For simplicity assume $A_c = A - HC$.

Theorem 2.1 *Assume* $\lambda_{\max}(A_c + A_c^T) < 0$. *Then condition (9) is satisfied if*

$$L < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(Q)}|\lambda_{\max}(A_c + A_c^T)| \tag{11}$$

Proof: Since $A_c < 0$, $\lambda_{\max}(A_c + A_c^T) < 0$ and $Q > 0$, we have (Lancaster, 1970)

$$\lambda_{\max}(P) \leq \frac{\lambda_{\max}(Q)}{|\lambda_{\max}(A_c + A_c^T)|}$$

So

$$\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \geq \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(Q)}|\lambda_{\max}(A_c + A_c^T)| \tag{12}$$

yielding (9).

The condition $\lambda_{\max}(A_c + A_c^T) < 0$ is a limitation condition. This condition is satisfied in rare cases such as stable diagonal matrices and stable symmetric matrices.

Consider the Lyapunov function

$$V(e) = e^T P e \tag{13}$$

Then

$$\dot{V} \quad = \quad e^T \left(A_c^T P + P A_c\right) + e^T P \left(f(t,x,u) - f(t,\hat{x},u)\right) + \left(f(t,x,u) - f(t,\hat{x},u) P e\right) \tag{14}$$

Since for any matrices X, Y and any positive number $\epsilon > 0$,

$$\epsilon \left(\frac{1}{\epsilon}X - Y\right)^T \left(\frac{1}{\epsilon}X - Y\right) \geq 0$$

then

$$X^T Y + Y^T X \leq \frac{1}{\epsilon} X^T X + \epsilon Y^T Y, \qquad \forall \epsilon > 0$$

For any $\epsilon > 0$

$$\begin{aligned} e^T P\left(f(t,x,u)-f(t,\hat{x},u)\right)+\left(f(t,x,u)-f(t,\hat{x},u)\right) Pe &\leq \epsilon e^T PPe + \frac{1}{\epsilon}\left\|f(t,x,u) - f(t,\hat{x},u)\right\|^2 \\ &= \epsilon e^T PPe + \frac{1}{\epsilon} L^2 \left\|x - \hat{x}\right\|^2 \\ &= \epsilon e^T P^2 e + \frac{1}{\epsilon} L^2 \|e\|^2 \end{aligned} \tag{15}$$

Therefore (14) and (15) imply

$$\dot{V} \leq e^T \left(A_c^T P + PA_c + \epsilon P^2 + \frac{1}{\epsilon} L^2 I \right) e \tag{16}$$

Hence, if

$$A_c^T P + PA_c + \epsilon P^2 + \frac{1}{\epsilon} L^2 I < 0 \tag{17}$$

then $\dot{V} < 0$. When $\epsilon = L^2$ the result is the same as given by Rajamani (1998) and Rajamani *et al.* (1998). Let $\alpha > 0$. If P is a solution of the algebraic Riccati equation (ARE)

$$A_c^T P + PA_c + \epsilon P^2 + \left(\frac{1}{\epsilon} L^2 + \alpha \right) I = 0 \tag{18}$$

then P satisfies the inequality (17). The algebraic Riccati inequality (ARI) condition (17) can be transformed into the linear matrix inequality problem (LMI)

$$\begin{bmatrix} A_c^T P + PA_c + \epsilon P^2 + \alpha I & L \\ L & -\epsilon \end{bmatrix} < 0 \tag{19}$$

Now we present necessary and sufficient conditions for the existence of a p.d.s. solution of the ARE (18).

Theorem 2.2 *Assume $G = \tilde{C}(sI - A_c)^{-1}$ with $\tilde{C} = \sqrt{\alpha + L^2/\epsilon} I$. Then $P = P^T$ satisfying the ARE (18) if and only if $\|G\|_\infty < 1/\sqrt{\epsilon}$. Moreover, P is a p.d. matrix.*

Proof: The proof results straightforwardly from Theorem 3.7.1 in the book by Green and Limebeer (1995). Since A_c is a stable matrix and P is the observability gramian of

$$\left(A_c, \begin{bmatrix} \tilde{C} \\ \sqrt{\epsilon} P \end{bmatrix} \right),$$

P is a p.d. matrix.

If $\epsilon = L^2$, then a necessary and sufficient condition for the existence of a p.d.s. solution P of the ARE (18) is that $\|G\|_\infty < 1/L$. The following lemma links the spectral condition on the appropriate Hamiltonian matrix with a p.d. solution of the ARE (18).

Lemma 2.1 *Assume*

$$H = \begin{bmatrix} A_c & \epsilon I \\ -\left(\frac{1}{\epsilon} L^2 + \alpha \right) I & -A_c^T \end{bmatrix}$$

The Hamiltonian matrix H has no eigenvalue on the imaginary axis if and only if the ARE (18) has a p.d.s. solution P.

Proof: The proof is directly obtained from Lemma 3.7.3 in the book by Green and Limebeer (1995).

We now find a relationship between the Lipschitz constant L and the eigenvalues of A_c. The following lemma is needed.

Lemma 2.2 *(Shapiro, 1974) Assume A is a stable matrix. Let P be the solution of the LE*

$$A^T P + PA = -Q \tag{20}$$

where Q is a p.d.s. matrix. Then the following inequalities hold

$$\lambda_{\max}(P) \geq \frac{\lambda_{\max}(Q)}{2\sigma_{\max}(A)}, \qquad \lambda_{\min}(P) \geq \frac{\lambda_{\min}(Q)}{2\sigma_{\min}(A)} \tag{21}$$

Theorem 2.3 *Let $\alpha > 0$, $\epsilon > 0$ and $A_c = A - HC$ with A and C as in (6). Assume that $\sigma_{\min}(A_c) \geq \sqrt{L^2 + \epsilon\alpha}$ and P is a p.d.s. solution of (18). Then*

$$\frac{\sigma_{\max}(A_c) - \sqrt{\sigma_{\max}^2(A_c) - (L^2 + \epsilon\alpha)}}{\epsilon} \leq \lambda_{\max}(P) \leq \frac{\sigma_{\max}(A_c) + \sqrt{\sigma_{\max}^2(A_c) - (L^2 + \epsilon\alpha)}}{\epsilon} \tag{22}$$

$$\frac{\sigma_{\min}(A_c) - \sqrt{\sigma_{\min}^2(A_c) - (L^2 + \epsilon\alpha)}}{\epsilon} \leq \lambda_{\min}(P) \leq \frac{\sigma_{\min}(A_c) + \sqrt{\sigma_{\min}^2(A_c) - (L^2 + \epsilon\alpha)}}{\epsilon} \tag{23}$$

Proof: Consider (18). A_c is a stable matrix. Using Lemma 2.2 with $Q = \epsilon P^2 + \left(\frac{1}{\epsilon}L^2 + \alpha\right) I$ yields

$$\begin{aligned} \lambda_{\max}(P) &\geq \frac{\frac{1}{\epsilon}L^2 + \alpha + \epsilon\lambda_{\max}(P^2)}{2\sigma_{\max}(A_c)} \\ &= \frac{\frac{1}{\epsilon}L^2 + \alpha + \epsilon\lambda_{\max}^2(P)}{2\sigma_{\max}(A_c)} \end{aligned} \tag{24}$$

which equivalent to

$$\epsilon\lambda_{\max}^2(P) - 2\sigma_{\max}(A_c)\lambda_{\max}(P) + \frac{1}{\epsilon}L^2 + \alpha \leq 0 \tag{25}$$

The inequality (25) holds if and only if (22) is satisfied. From (21)

$$\epsilon\lambda_{\min}^2(P) - 2\sigma_{\min}(A_c)\lambda_{\min}(P) + \frac{1}{\epsilon}L^2 + \alpha \leq 0 \tag{26}$$

which gives (23).

The condition $\sigma_{\min}(A_c) \geq \sqrt{L^2 + \epsilon\alpha}$ is a necessary condition for satisfying (22) and (23). If

$$L < \frac{\epsilon}{2\left(\sigma_{\max}(A_c) + \sqrt{\sigma_{\max}^2(A_c) - (L^2 + \epsilon\alpha)}\right)}$$

then

$$L < \frac{1}{2\lambda_{\max}(P)} \tag{27}$$

where P is a p.d.s. solution of (8) or (18) with $Q = \epsilon P^2 + \left(\frac{1}{\epsilon}L^2 + \alpha\right) I$. However, if (27) holds, then the error system may not be stable, because P in (10) yields the maximum value $\lambda_{\min}(Q)/\lambda_{\max}(P)$ for $Q = I$ (and not $Q = \epsilon P^2 + \left(\frac{1}{\epsilon}L^2 + \alpha\right) I$).

Lemma 2.3 *Assume P is a p.d.s. solution of the LE*

$$A_c^T P + PA_c + \frac{1}{\epsilon}L^2 I + Q = 0 \tag{28}$$

where $Q > 0$ is an arbitrary p.d.s. matrix and $\epsilon > 0$. Then P is also the solution of ARI (17) if

$$\epsilon < \frac{\lambda_{\min}(Q)}{\lambda^2_{\max}(P)} = L_p \tag{29}$$

Proof: Assume P is a p.d.s. solution of (27). Then for any vector e

$$\begin{aligned} e^T \left(A_c^T P + PA_c + \epsilon P^2 + \frac{1}{\epsilon}L^2 I\right) e &\leq e^T \left(-Q + \epsilon P^2\right) e \\ &\leq e^T \left(-\lambda_{min}(Q) + \epsilon \lambda^2_{\max}(P)\right) e \\ &< 0 \end{aligned}$$

The following theorem is a direct result of Lemma 2.3.

Theorem 2.4 *Assume $\epsilon > 0$ and P is a p.d.s. solution of the LE (28). The error system (6) is asymptotically stable if*

$$\epsilon < \frac{\lambda_{\min}(Q)}{\lambda^2_{\max}(P)} \tag{30}$$

The gain observer matrix H can be found straightforwardly by using the LQ method. The following theorem gives a method for selecting the observer gain matrix H so that the stability of the error system (35) is guaranteed.

Theorem 2.5 *Let P be a p.d.s. solution of the ARE*

$$AP + PA^T - 2PC^T R^{-1} CP + \frac{1}{\epsilon}L^2 I + Q = 0 \tag{31}$$

where R and Q are p.d.s. matrices. The error system (35) is asymptotically stable with $H = PC^T R^{-1}$ if

$$\epsilon < \frac{\lambda_{\min}(Q)}{\lambda^2_{\max}(P)} \tag{32}$$

Proof: Let $H = PC^T R^{-1}$. Then (31) can be rewritten as (28). Since the condition (17) holds, Theorem 2.4 yields the desired result.

Remark 2.1 The proof of Lemma 2.3 shows that condition (29) can be changed to the weaker condition $Q - \epsilon P^2 > 0$. Then the results of Lemma 2.3, Theorems 2.4 and 2.5 hold for $Q - \epsilon P^2 > 0$ rather than (29), (30) and (32).

3 Nonlinear Sliding Mode Observer Design

Now consider $f(t) \neq 0$. In this case, the observer (4) may not guarantees an asymptotically stable error system (6). However, this can be accomplished by utilizing sliding mode observer techniques.

Sliding observers potentially offer advantages similar to those of sliding controllers, in particular, inherent robustness to parametric uncertainty and straightforward application to important classes of systems. Here, suitable state estimation of the system (1) is considered so that the estimate of the state is close to the actual state. This yields a reconstruction error system which is asymptotically stable or ultimately bounded. An observer for the system (1) is assumed to be in the form

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + H(y - \hat{y}) + f(t, \hat{x}, u) && (33) \\ \hat{y} &= C\hat{x} && (34) \end{aligned}$$

where $v \in \mathbb{R}^m$ is an external discontinuous feedforward compensation signal and $\Lambda \in \mathbb{R}^{n\times m}$ is the feedforward injection map such that $C\Lambda$ is a nonsingular matrix. Subtracting (1) from (34) gives the dynamical reconstruction error system

$$\begin{aligned} \dot{e} &= (A - HC)e + f(t, x, u) - f(t, \hat{x}, u) + \Gamma\xi - \Lambda v && (35) \\ e_y &= Ce && (36) \end{aligned}$$

The ideal sliding mode for the system (34) satisfies $e_y = 0,\ \dot{e}_y = 0$ (Utkin, 1992). The virtual equivalent feedforward input is given by

$$v_{eq} = (C\Lambda)^{-1}C\left(Ae + f(t, x, u) - f(t, \hat{x}, u) + \Gamma\xi\right) \tag{37}$$

From (35) and (37) the reduced order system

$$\dot{e} = (I - \Lambda(C\Lambda)^{-1}C)Ae + (I - \Lambda(C\Lambda)^{-1}C)\left(\Gamma\xi + f(t, x, u) - f(t, \hat{x}, u)\right) \tag{38}$$

is obtained, with m of the eigenvalues of $(I - \Lambda(C\Lambda)^{-1}C)A$ zero and the $n - m$ remaining eigenvalues assignable (Utkin, 1992).

Assumption: (Matching Condition)
Assume that there exists an $m \times m$ matrix D such that

$$\Gamma = \Lambda D \tag{39}$$

then the disturbance and the uncertain nonlinear term do not affect the reduced order systems because of (38).

Suppose that the observer gain matrix H is selected satisfying (18). Let

$$v = W\frac{Ce}{\|Ce\|} \tag{40}$$

where W is an $m \times m$ diagonal p.d. matrix with

$$\lambda_{\min}(W) \geq M\|D\|\frac{\lambda_{\max}(CP^{-1}C^T)}{\lambda_{\min}(CP^{-1}C^T)} \tag{41}$$

and P is a p.d.s. solution of the ARE (18). Let

$$\Lambda = P^{-1}C^TW^{-1} \tag{42}$$

Noting that $C\Lambda$ is a nonsingular matrix and W is a p.d. matrix, $C\Lambda W$ is nonsingular and

$$\lambda_{\min}(C\Lambda W) = \lambda_{\min}(CP^{-1}C^T) \neq 0$$

The quadratic stability of the reconstruction error system is guaranteed by (41) and (42). If $Ce \neq 0$, then the time derivative $\dot{V}$ satisfies

$$\begin{aligned}
\dot{V} &= e^T\left(A_c^TP + PA_c\right) + e^TP\left(f(t,x,u) - f(t,\hat{x},u)\right) + \left(f(t,x,u) - f(t,\hat{x},u)\right)Pe \\
&\quad + 2e^TC^TW^{-1}D\xi - 2e^TC^T\frac{Ce}{\|Ce\|} \\
&\leq e^T\left(A_c^TP + PA_c + \epsilon P^2 + \frac{1}{\epsilon}L^2I\right)e + 2\left\|e^TC^T\right\|\left(\|W^{-1}D\|M - 1\right) \\
&\leq e^T\left(A_c^TP + PA_c + \epsilon P^2 + \frac{1}{\epsilon}L^2I\right)e + 2\left\|e^TC^T\right\|\left(\frac{1}{\lambda_{\min}(W)}\|D\|M - 1\right) \\
&< 0
\end{aligned} \tag{43}$$

since

$$\lambda_{\min}(W) \geq M\|D\|\frac{\lambda_{\max}(CP^{-1}C^T)}{\lambda_{\min}(CP^{-1}C^T)} \geq M\|D\|$$

and P is satisfied (18). If $Ce = 0$, $v = v_{eq}$ and

$$\begin{aligned}
\dot{V} &= e^T\left(A_c^TP + PA_c\right) + e^TP\left(f(t,x,u) - f(t,\hat{x},u)\right) + \left(f(t,x,u) - f(t,\hat{x},u)Pe\right) \\
&\quad +2e^TPP^{-1}C^TW^{-1}D\xi - 2e^TPP^{-1}C^TW^{-1}v_{eq} \\
&\leq -\alpha e^Te + 2e^TC^TW^{-1}D\xi - 2e^TC^TW^{-1}v_{eq} \\
&= -\alpha e^Te \\
&< 0
\end{aligned} \tag{44}$$

4 Example

Consider an observer with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

(Rajamani, 1998). Assume that the uncertainty term is a Lipschitz function with Lipschitz constant L. Let $Q = 5I$, $R = 1$ and $\epsilon = 0.05$. Then

$$H = PC^TR^{-1} = \begin{bmatrix} 2.3420 \\ 1.8948 \end{bmatrix}$$

where

$$P = \begin{bmatrix} 9.3227 & 2.3420 \\ 2.3420 & 1.8948 \end{bmatrix}$$

is the u.p.d.s. solution of the ARE (31) with the eigenvalues 9.9995 and 1.2180. The eigenvalues of $A_c = A - HC$ are -0.5797 and -2.3151. Since

$$0.05 = \epsilon < \frac{\lambda_{\min}(Q)}{\lambda^2_{\max}(P)} = L_p = 0.0501$$

the condition (29) is satisfied. Therefore P is also the solution of ARI (17). In fact, the eigenvalues of

$$AP + PA^T - 2PC^T R^{-1} CP + \frac{1}{\epsilon} L^2 I + \epsilon P^2$$

are -0.0005 and -4.9258. So the error system is asymptotically stable for $L \leq 0.2536$.

Now let $\epsilon = 0.005$. The eigenvalues of the p.d.s. solution P of the ARE (31) are 31.6089 and 3.0624. So $L_p = 0.05004$ and the condition (29) is satisfied if $L \leq 0.3250$ which guarantees that P is also the solution of the ARI (17). The eigenvalues of

$$AP + PA^T - 2PC^T R^{-1} CP + \frac{1}{\epsilon} L^2 I + \epsilon P^2$$

are -0.0044 and -4.9531. In this case, the observer gain vector is

$$H = \begin{bmatrix} 4.1486 \\ 3.6787 \end{bmatrix}$$

The eigenvalues of $A - HC$ are -0.8149 and -3.8637. If we set $\epsilon = 0.001$, then $L_p = 0.0010$. The system is asymptotically stable if $L \leq 0.2410$. The observer gain is then

$$H = \begin{bmatrix} 6.1383 \\ 5.6586 \end{bmatrix}$$

and the eigenvalues $A - HC$ are -0.8909 and -5.7678.

This method also allows us to select the observer gain vector H with prespecified eigenvalues of A_c and then test the stability condition. Suppose that it is desired to find the observer gain H so that the eigenvalues of $A_c = A - HC$ are $-6.2840 \pm 5.3911i$, then

$$H = \begin{bmatrix} 69.5526 \\ 11.5680 \end{bmatrix}$$

Let $Q = 5I$ and $\epsilon = 0.001$. The eigenvalues of the p.d.s. matrix P, the p.d.s. solution of (28), are 70.3252 and 0.9925. P is also the solution of the Lyapunov inequality (17) because the eigenvalues

$$A_c P + PA_c^T + \frac{1}{\epsilon} L^2 I + \epsilon P^2$$

are -0.0544 and -4.9990 with $L = 0.14$. The condition (30) is satisfied for values of $L \leq 0.14$. By selecting $\epsilon = 0.01$, the condition (30) is satisfied for $L \leq 0.1670$.

5 Conclusions

A nonlinear observer design approach for a class of nonlinear systems has been studied in this paper. The uncertainties and nonlinearities of this class of systems include two parts; one satisfying the Lipschitz condition and the second not satisfying the Lipschitz condition but is bounded. We have presented some criteria to test the stability of the error system when the system equation includes uncertainty satisfying the Lipschitz condition. If uncertainties in the system contain a bounded uncertainty, a discontinuous feedforward compensation input is needed to compensate the presence of the time-dependent uncertainty which does not satisfy the Lipschitz condition. The stability of the error system is guaranteed with a discontinuous feedforward input and some conditions on the uncertainty.

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