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Input-Output Models for a Class of Nonlinear Systems: Questions and Answers

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Abstract

In this paper we investigate the possibility of having an input-output model, having a specific structure, for observable multi-input multi-output systems with vector relative degree. The interest in this input-output form arises from the fact that the model has been extensively used in control design, including sliding mode control. Since the subclass of systems having this specific structure is extremely restrictive, we suggest an alternative approach.

1 Introduction

Atassi and Khalil (1997) pose but do not answer the following question: what subclass of nonlinear state-space systems will admit the input-output (i/o) model of the form

$$y_i^{(n_i)} = p_i(\cdot) + \sum_{j=1}^{p} q_{ij}(\cdot) u_j^{m_j}, \qquad 1 \leq i \leq p \tag{1}$$

where $p_i(\cdot)$ and $q_{ij}(\cdot)$ are functions of $y_k, \ldots, y_k^{(n_k-1)}, u_k, \ldots, u_k^{(m_k-1)}$, $k = 1, \ldots, p$, and the matrix $Q(\cdot) = [q_{ij}(\cdot)]$ is nonsingular over the domain of interest. Note that $y^{(k)}$ (resp. $u^{(k)}$), $k \in \mathbb{N}$, denotes the k-th time-derivative of y (resp. u). The interest in the form (1) comes from the fact that this model has been used extensively in control design such as in adaptive control (Khalil, 1996), in robust servomechanism design (Mahmoud and Kahlil, 1997) and in variable structure control (Oh and Khalil, 1995). Their motivation was to demonstrate that, as in the single-input single-output case, any observable system with vector relative degree has an input-output model that fits the form (1). The assumption of vector relative degree, although quite restrictive, plays a crucial role in making possible a straightforward extension of many results developed for single-input single-output systems. Unfortunately, this turned out not to be true with respect to the problem considered, as shown via counterexamples in (Atassi and Khalil, 1997). They then suggested, again through examples, that by extra differentiation of part of the input-output equations one can make the model fit the form (1). By differentiating

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the system-defining input-output equations one gets a model, which is transfer equivalent to the original system, but not in the irreducible (or minimal) form. The state equations that correspond to this input-output model are not accessible (Lin, *et al.*) which is probably not desirable. Note that the difficulties linked with the extra differentiation in output tracking were mentioned by Atassi and Khalil (1997) as well.

In this paper we will first indicate a subclass of state equations that fits the form (1). Unfortunately, this subclass of state-space systems is extremely restrictive. Next, we suggest an alternative approach to extra differentiation, motivated by the observation that the form (1) in (Khalil, 1996; Mahmoud and Khalil, 1997; Oh and Khalil, 1995) is desirable for the reason that it is easily invertible (i.e. it can be easily solved for the highest derivatives of the control). Our approach is based upon van der Schaft's algorithm converting the set of i/o differential equations into an equivalent but row-reduced form (with respect to input variables), which yields the inversion of an (almost) arbitrary set of i/o differential invertible systems and not just the restrictive subclass of invertible systems, having the vector relative degree.

2 The class of nonlinear state-space systems

Consider a nonlinear system of the form

$$\begin{array}{rcl} \dot{x} & = & f(x) + \sum_{j=1}^{p} g_j(x)u_j \\ y_i & = & h_i(x), i = 1, \ldots, p \end{array} \tag{2}$$

where $x = (x_1, \ldots, x_n)$ is the state, $u = (u_1, \ldots, u_p) \in I\!R^p$ is the control, $y = (y_1, \ldots, y_p) \in I\!R^p$ is the output, and $f, g_1 \ldots, g_p$ are smooth vector fields, while $h_1, \ldots, h_p$ are smooth functions. We make the following assumptions concerning (2):

Ass 1 The system (2) is locally observable.

Ass 2 The system (2) has a well-defined vector relative degree $(r_1, \ldots, r_p)$ on an open set $\mathcal{X}$ of the state space; see (Isidori, 1995) for the definition.

By observability we mean that the state can be derived from y, u and their time derivatives. Then, there exist p integers $n_1, \ldots, n_p$, called observability indices such that (Krener and Respondek, 1989)

- $n_1 \geq n_2 \geq \cdots \geq n_p$
- $\sum n_i = n$
- $dim[\text{span}\{y_i, \dot{y}_i, \ldots, y_i^{(n_i-1)}, i = 1, \ldots, p\}] = n$

The assumption of well-defined vector relative degree incorporates the assumption that the decoupling matrix is nonsingular. It is known (Nijmeijer and van der Schaft, 1990) that system (2) satisfying assumption Ass 2 can, after a change of state variables, be expressed as follows:

$$\begin{array}{rcl} \dot{x}_{i1} & = & x_{i2} \\ & \vdots & \\ \dot{x}_{ir_{i-1}} & = & x_{ir_i} \\ \dot{x}_{ir_i} & = & a_i(x, z) + \sum_{j=1}^{p} b_{ij}(x, z)u_j \\ \dot{z} & = & c(x, z) + d(x, z)u \\ y_i & = & x_{i1}, i = 1, \ldots, p \end{array} \tag{3}$$

Moreover, if the distribution $G = \text{span}\{g_1, \ldots, g_p\}$ is involutive, then $d(x, z) = 0$.

In (3) part of the state components are directly related to the outputs:

$$\begin{aligned} x_{i1} &= y_i \\ x_{i2} &= \dot{y}_i \\ &\vdots \\ x_{ir_i} &= y_i^{(r_i-1)}, i = 1, \dots, p \end{aligned} \tag{4}$$

The other state coordinates can be obtained from

$$\begin{aligned} y_i^{(r_i)} &= a_i(x,z) + \textstyle\sum_{j=1}^p b_{ij}(x,z)u_j \\ y_i^{(r_i+1)} &= a_i^1(x,z,u) + \textstyle\sum_{j=1}^p b_{ij}(x,z)\dot{u}_j \\ y_i^{(r_i+2)} &= a_i^2(x,z,u,\dot{u}) + \textstyle\sum_{j=1}^p b_{ij}(x,z)\ddot{u}_j \\ &\vdots \\ y_i^{(n_i-1)} &= a_i^{n_i-r_i-1}(x,z,u,\dot{u},\dots,u^{(n_i-r_i-2)}) + \textstyle\sum_{j=1}^p b_{ij}(x,z)u_j^{(n_i-r_i-1)} \end{aligned} \tag{5}$$

which from the observability assumption (Ass 1) yields

$$z = \xi(y_i, \dots, y_i^{(n_i-1)}, i = 1, \dots, p, u, \dots, u^{(\mu-1)}) \tag{6}$$

where $\mu = \max(n_1 - r_1, \dots, n_p - r_p)$.

Finally, we obtain

$$y_i^{(n_i)} = a_i^{(n_i-r_i)}(x,z,u,\dot{u},\dots,u^{(n_i-r_i-1)}) + \sum_{j=1}^{p} b_{ij}(x,z)u_j^{(n_i-r_i)} \tag{7}$$

which, after substituting x and z from (4) and (6), gives an i/o model, corresponding to the state-space equation (2) under the assumptions Ass 1 and 2:

$$y_i^{(n_i)} = p_i(\cdot) + \sum_{j=1}^{p} q_{ij}(\cdot)u_j^{n_i-r_i}, \qquad 1 \le i \le p. \tag{8}$$

Here $p_i(\cdot)$ and $q_{ij}(\cdot)$ are functions of $y_k, \dots, y_k^{(n_k-1)}$, $k = 1, \dots, p$, $u, \dots, u^{(\mu-1)}$.

Note that the paper (Atassi and Khalil, 1997) suggests the use of van der Schaft's algorithm to eliminate the states in order to get the i/o representation of (2). Though it might be useful for (2) in its most general form, we do not believe that this is necessary for systems having the vector relative degree and being in the normal form.

2.1 A subclass of the state equations that fits the form (1)

It is clear from (8) (and this confirms the observation of (Atassi and Khalil, 1997) obtained via counterexamples) that the i/o model of system (2) satisfying Ass 1 and 2 does not necessarily fit the form (1). The next theorem provides a special case when the state-space system (2) admits the i/o model that fits the form (1).

Theorem 1 *An observable state-space system of the form (2) with a well-defined vector relative degree $(r_1, \dots, r_p)$ admits an input-output model of the form (1), if the following condition is satisfied:*

$$n_1 - r_1 = n_2 - r_2 = \dots = n_p - r_p$$

where n_i, $i = 1, \ldots, p$ *are the observability indices of the system (2). Moreover, the matrix* $Q(\cdot)$ *in the form (1) is actually obtained from the decoupling matrix of (2) provided the state variables* x *and* z *are expressed, via (4) and (6), in terms of inputs, outputs and their time derivatives.*

Of course, the converse is not necessarily true. There exist even single-input single output systems having the form (1), not having a state-space realization.

Proof. Denote $n_1 - r_1 (= n_2 - r_2 = \cdots = n_p - r_p)$ by m. Under the observability assumption we now obtain from (6)

$$z = \gamma(y_i, \ldots, y_i^{(n_i-1)}, u_i \ldots, u_i^{(m-1)}, i = 1, \ldots, p) \tag{9}$$

This yields

$$y_i^{(n_i)} = p_i(\cdot) + \sum_{j=1}^{p} q_{ij}(\cdot) u_j^m, \qquad 1 \le i \le p \tag{10}$$

where $p_i(\cdot)$ and $q_{ij}(\cdot)$ are functions of $y_k, \ldots, y_k^{(n_k-1)}, u_k, \ldots, u_k^{(m-1)}$, for $k = 1, \ldots, p$, and the matrix $Q(\cdot) = [q_{ij}(\cdot)]$, the decoupling matrix of the system, is nonsingular over the domain of interest because of Ass 2. ∎

Note that the state space system with a diagonal decoupling matrix does not necessarily admit an i/o model of the form (1).

3 Alternative approach to extra differentiation

The i/o model (8), obtained from the normal form (3) by eliminating the state variables x and z, can be implicit in the highest derivatives of the inputs, depending on the value $\mu = max(n_1 - r_1, \ldots, n_p - r_p)$. As demonstrated by Example 2 in (Atassi and Khalil, 1997), one can bring the i/o equation by extra differentiations into the form (1).

As mentioned by Atassi and Khalil (1997), converting the i/o equation to look like (1), shows that (8) is invertible. If it is invertible (and obviously it is because of Ass 2), the inverse can be found by differentiating certain equations (perhaps repeatedly) until one reaches the stage where the highest derivatives of the inputs are multiplied by a nonsingular matrix.

It seems to us that the form (1) is necessary basically for the reason that it is easily invertible. So, we believe that this form is not actually necessary if we could solve (8) with respect to the highest derivatives of input. In this section we recall the result of van der Schaft (1989a) about inverting the arbitrary set of input/output differential equations

$$\varphi_i(y, \dot{y}, \ldots, y^{(n)}, u, \dot{u}, \ldots, u^{(n)}) = 0, \qquad i = 1, \ldots, p \tag{11}$$

which of course includes (8) but is much more general. Applying the result of van der Schaft, there is no need for any assumption about the vector relative degree (Ass 2).

Let us first recall some terminology. The order τ_i of $\varphi_i(\cdot)$ with respect to the input variables u is defined as the largest integer such that

$$\frac{\partial \varphi_i}{\partial u_j^{(\tau_i)}} \ne 0 \tag{12}$$

for some $j \in \{1, \ldots, p\}$, i.e. τ_i is the highest time-derivative of the input component appearing non-trivially in φ_i.

Now define the $p \times p$ matrix $A(y, \dot{y}, \ldots, y^{(n)}, u, \dot{u}, \ldots, u^{(n)})$ as the matrix with (i, j)-th element given by (12).

Definition 1. *The set of the higher-order i/o differential equations (11) is said to be* locally row-reduced *with respect to control if for all* $(y, \dot{y}, \ldots, y^{(n)}, u, \dot{u}, \ldots, u^{(n)})$ *about the solution point* $(\bar{y}, \bar{u})$

$$\mathrm{rank}\, A(\cdot) = p.$$

In the linear case, the row-reduced form means that the polynomial matrix, defining the set of equations (11), is row proper.

Theorem 2. *(Van der Schaft, 1989b) A set of higher order differential equations having a solution point* $(\bar{y}, \bar{u})$*, is, under certain constant rank assumptions, locally equivalent about* $(\bar{y}, \bar{u})$ *to a set of higher-order differential equations of the form*

$$\tilde{\varphi}_i(y, \dot{y}, \ldots, y^{(n)}, u, \dot{u}, \ldots, u^{(n)}) = 0, \qquad i = 1, \ldots, \bar{p} \tag{13}$$

$$\tilde{\varphi}_i(y, \ldots, y^{(n)}) = 0, \qquad i = \bar{p} + 1, \ldots, p \tag{14}$$

satisfying

$$\tau_1 \leq \tau_2 \cdots \leq \tau_{\bar{p}}$$

where τ_i*'s are the orders of* $\tilde{\varphi}_i$*'s and for* $i = 1, \ldots \bar{p}$

$$\frac{\partial \tilde{\varphi}_i(\cdot)}{\partial u_i^{(\tau_i)}} \neq 0, \tag{15}$$

and

$$\frac{\partial \tilde{\varphi}_i(\cdot)}{\partial u_j^{(\tau)}} = 0, \tag{16}$$

for $\tau_j \leq \tau \leq \tau_i,\ j < i$.

As an immediate consequence of the above theorem one has the following definition:

Definition 2. *(Van der Schaft, 1989b) Consider an i/o differential system (11) which can be transformed into the form (13) and (14). The system is called invertible if* $\bar{p} = p$.

Let d be the largest integer such that $\tau_d = 0$. Then by the implicit function theorem we can solve locally for $u_1, \ldots, u_d$ from the first d equations of (13):

$$u_i = k_i(u_{d+1}, \ldots, u_p, y_1, \ldots, y_p), \qquad i = 1, \ldots, d \tag{17}$$

Furthermore, assuming invertibility, we obtain from the other equations

$$u_i^{(\tau_i)} = k_i(y, \ldots, y^{(k-1)}, u_j, \ldots, u_j^{(\tau_j - 1)}, j = d+1, \ldots, p), \qquad i = d+1, \ldots, p \tag{18}$$

4 Examples

Below we will consider two input-output models, studied in (Atassi and Khalil, 1997). Both models are global, with the first one having $n_1 = 3, n_2 = 1, m_1 = m_2 = 1$, and the second one having $n_1 = n_2 = 2, m_1 = m_2 = 1$. In both cases, however, the matrix Q is singular. We will show, how one can invert those models on the basis of the results by van der Schaft (1989b).

Example 1. Consider the i/o model

$$\begin{aligned} &y_1^{(3)} - 2y_2^{(1)}y_2 + y_1^{(2)} - y_2^2 - u_1 - au_2 - u_1^{(1)} - au_2^{(1)} = 0 \\ &y_2^{(1)} - y_1^{(2)} + y_2^2 + (1-b)u_1 + (a-1)u_2 = 0 \end{aligned}$$

In order to invert the model, we have to transform it via equivalence transformations into an equivalent form (13) satisfying (15)-(16). First, we permute the equations (to obtain $0 = \tau_1 \leq \tau_2 = 1$) and then replace the equation $\varphi_2(\cdot) = 0$ by an equation $\varphi_1^{(1)}(\cdot) + (1-b)\varphi_2(\cdot)$ which does not depend anymore on $u_1^{(1)}$. That way we have obtained the desired equations satisfying (15)-(16)

$$\begin{gathered} y_2^{(1)} - y_1^{(2)} + y_2^2 + (1-b)u_1 + (a-1)u_2 = 0 \\ -by_1^{(3)} + 2by_2y_2^{(1)} + y_2^{(2)} + (1-b)y_1^{(2)} - (1-b)y_2^2 - (1-b)u_1 - a(1-b)u_2 - (1-b)u_2^{(1)} = 0. \end{gathered}$$

Finally, the system of equations can be solved for u_1 and $u_2^{(1)}$:

$$u_1 = \frac{1}{1-b}[y_1^{(2)} - y_2^{(1)} - y_2^2 + (1-a)u_2]$$

$$u_2^{(1)} = \frac{1}{1-b}[-by_1^{(3)} + 2by_2y_2^{(1)} + y_2^{(2)}] - u_1 + y_1^{(2)} - y_2^2 - au_2$$

Example 2. Consider the other example

$$\begin{gathered} y_1^{(2)} - y_2^2 - y_2^{(1)} + (b-1)u_1 + (1-a)u_2 = 0 \\ y_2^{(2)} + y_2^{(1)} - bu_1 - u_2 - bu_1^{(1)} - u_2^{(1)} = 0 \end{gathered}$$

We replace the equation $\varphi_2(\cdot) = 0$ by

$$\varphi_2(\cdot) + \frac{b}{b-1}\varphi_1^{(1)}(\cdot)$$

to obtain

$$y_1^{(2)} - y_2^2 - y_2^{(1)} + (b-1)u_1 + (1-a)u_2 = 0$$

$$y_2^{(2)} + y_2^{(1)} - bu_1 - u_2 - u_2^{(1)} + \frac{b}{b-1}[y_1^{(3)} - 2y_2y_2^{(1)} - y_2^{(2)}] + \frac{b(1-a)}{b-1}u_2^{(1)} = 0$$

Finally the system of equations can be solved for u_1 and $u_2^{(1)}$:

$$u_1 = \frac{1}{1-b}[y_1^{(2)} - y_2^2 - y_2^{(1)} - (1-a)u_2]$$

$$u_2^{(1)} = \frac{1-b}{1-ab}[y_2^{(2)} + y_2^{(1)} - bu_1 - u_2] - \frac{b}{1-ab}[y_1^{(3)} - 2y_2y_2^{(1)} - y_2^{(2)}]$$

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