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Following a path of varying curvature as an output regulation problem

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Abstract

Given a path of nonconstant curvature, local asymptotic stability can be proven for the general n-trailer whenever the curvature can be considered as the output of an exogenous dynamical system. It turns out that the controllers that provide convergence to zero of the tracking error chosen for the path following problem are composed of a prefeedback that input-output linearizes the system plus a linear part that can be chosen in an optimal way.

1 Introduction

In the path following problem for nonholonomic wheeled vehicles (see (Canudas de Wit, 1998) for a survey), the longitudinal dynamics, expressing how fast the path is covered, is normally of secondary importance with respect to the lateral dynamics expressing a notion of distance (i.e. a tracking error) of the vehicle form the path by means of a tracking criterion. This is equivalent to say that the longitudinal speed input can be an *a priori* given function, for example a nonnull constant. If the tracking error used is a scalar, the system to analyze is basically a SISO system with drift from the steering input to the tracking error. When the curvature of a path to follow can be modeled as the output of a neutrally stable dynamical system, then the path following problem can be formulated as an output regulation problem in the nonlinear setting proposed by (Isidori and Byrnes, 1990). The curvature can, in fact, be considered as a known exogenous disturbance and the output of the system, corresponding to the tracking error of the path following criterion, can be rendered independent from it by input-output linearizing the system with a static change of input. With the error independent from the curvature, if the relative degree of the system is well defined, the output zeroing manifold is the only invariant manifold that solves the regulation problem. This is equivalent to say that local asymptotic stability to the nonconstant steady state is achieved by and only by the controllers composed of a prefeedback that input-output linearizes the system plus a linear part that can be chosen in an optimal (linear) fashion. If we choose as tracking criterion the one proposed in (Altafini and Gutman, 1998) based on the so-called off-tracking distance, whose peculiarity is that it keeps the whole vehicle (and not a single guidepoint on the vehicle) at a reduced distance from the path, then the relative degree between the steering angle and the corresponding tracking error is equal to 2 whereas for the criteria normally used it is higher: for example taking as guidepoint the midpoint of the last axle would give a relative degree equal to $n+1$ in the

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n-trailer system. We choose input-output linearization because it is easier to obtain than input-state linearization, see (Sampei *et al.*, 1991), which requires more involved calculations for the n-trailer configuration that we are going to use. Moreover, input-state linearization is not achievable in case some of the trailers present kingpin hitching. We show that instead, under some regularity assumptions, the off-tracking criterion implies a well-defined relative degree also for this more general configuration.

It must be noticed that the whole analysis is *local* and that, due to the singularities of the Frenet frame representation used here, there is no way to formally prove a well-behaved transient even for admissible (wrong) initial conditions that are too close to the limits of the region of attraction. This can be intended as a direct consequence of the fact that the rigid motion of the "chained" mechanism occurs along a path of varying curvature which implies a variable width of the region of definition.

2 Kinematic model for the general n-trailer and Frenet frames

Suppose we have a general n-trailer system with m ($m \leq n$) of the trailers hooked at a distance M_i form the preceding axle, see Fig. 1. Assume that each body is composed of one single axle.

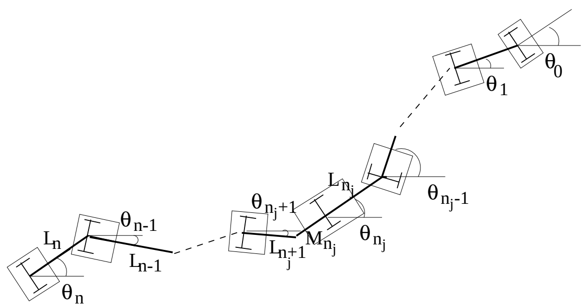


Figure 1: The general n-trailer system.

The nonholonomic constraints on the points P_i (below called *nonholonomic points*) originate from the assumption of rolling without slipping of the wheels.

If we call $n_1, \ldots, n_m$, $n_j < n_{j+1}$, $n_m < n$ the indices of the axles having nonnull off-hitching ($M_{n_j} \neq 0$) we can group together the axles between two consecutive steering wheels: $\{0, 1, \ldots, n_1\}, \ldots, \{n_{j-1}+1, n_{j-1}+2, \ldots, n_j - 1, n_j\}, \ldots, \{n_m+1, n_m+2, \ldots, n-1, n\}$. We do not consider the case of two consecutive axles having off-hitching. A kinematic model for this system in cartesian coordinates was obtained in (Altafini, 1998):

$$\dot{\theta}_{n_j+1} = \frac{v_{n_j+1}\tan(\theta_{n_j} - \theta_{n_j+1})}{L_{n_j+1}} - \frac{M_{n_j}\dot{\theta}_{n_j}}{L_{n_j+1}\cos(\theta_{n_j} - \theta_{n_j+1})} \tag{1}$$

$$v_{n_j+1} = v_{n_j}\cos(\theta_{n_j} - \theta_{n_j+1}) + M_{n_j}\sin(\theta_{n_j} - \theta_{n_j+1})\dot{\theta}_{n_j} \tag{2}$$

$$j \in \{1, \ldots, m\}$$

$$\dot{\theta}_{n_j+i} = \frac{v_{n_j+i}\tan(\theta_{n_j+i-1} - \theta_{n_j+i})}{L_{n_j+i}} \tag{3}$$

$$v_{n_j+i} = v_{n_j+i-1}\cos(\theta_{n_j+i-1} - \theta_{n_j+i}) \tag{4}$$

$$j \in \{0, 1, \ldots, m\}, \quad i \in \{2, 3, \ldots, n_{j+1} - n_j\}, \quad n_0 = 0 \text{ and } n_{m+1} = n.$$

where θ_i is the orientation angle of the $i-th$ axle, v_i its translational velocity, L_i is the distance between the i-th axle P_i and the hitching point of the same trailer $\beta_1 \triangleq \theta_0 - \theta_1$ is the steering angle. The n-trailer system has two inputs, corresponding to translational and steering actions of the car pulling the trailers. At the kinematic level, we can consider these two inputs to be the steering speed $\omega \triangleq \dot{\beta}_1$ and the translational speed v_n of the last trailer. The configuration state can be completed by considering the two coordinates of one of the points P_i.

Under the assumption that the path is sufficiently smooth and that the curvature has an upper bound, a particularly useful local frame to describe the lateral dynamics of the path following problem decoupled from the longitudinal one is the so-called Frenet frame i.e. a frame moving on the path having origin on the orthogonal projection of the point of interest. In (Altafini and Gutman, 1998), the tracking criterion introduced consists in considering $n+1$ frames simultaneously, one for each nonholonomic point. Each of the curvilinear frames (see

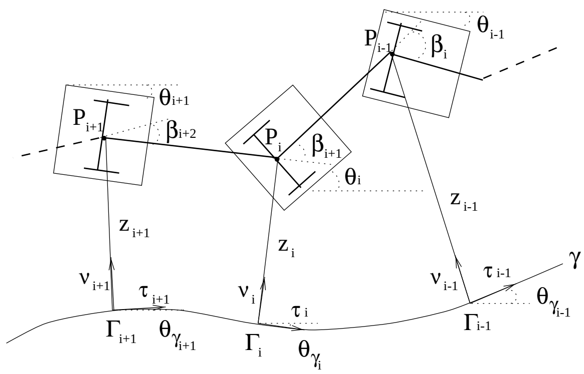


Figure 2: Frenet frames associated with the nonholonomic points P_i.

Fig. 2) is represented by two coordinates $(s_{\gamma_i}, \theta_{\gamma_i})$ where s_{γ_i} is the line integral along the path to follow, up to the actual projection of the point P_i on the path itself and θ_{γ_i} is the orientation of the frame with respect to the inertial frame. In the Frenet frame, the point P_i is represented by the signed distance z_i between the point itself and its orthogonal projection and by the relative orientation angle $\tilde{\theta}_i$.

The decoupling property of the Frenet frame has already been used by several authors for the path following problem (see (Sampei *et al.*, 1991; Micaelli and Samson, 1993)). We also use it but substituting the tracking criterion normally used $z_n \to 0$ (Samson, 1995) (or an equivalent one based on another of the distances z_i) with the *sum* of the signed distances:

$$\sum_{i=0}^{n} z_i \to 0 \tag{5}$$

It can be noticed that for a nonzero curvature neither the θ_i nor the θ_{γ_i} tend to a steady state in the path following problem, but their difference $\tilde{\theta}_i \triangleq \theta_i - \theta_{\gamma_i}$, $i \in \{0, 1, \ldots, n\}$ can have an equilibrium value if $\kappa_\gamma = const$. The same observation is valid also for the the angles $\beta_i \triangleq \theta_{i-1} - \theta_i$, $i \in \{1, \ldots, n\}$. Therefore it is convenient to transform the dynamic equations of θ_i and θ_{γ_i} into the corresponding equations for $\tilde{\theta}_i$ and β_i. We can group together all the 4 equations relative to each point P_i. When there is off-axle hitching the equations for the node

P_{n_j+1} (first node of each steering train, except for the driving cart) are:

$$\begin{bmatrix} \dot{s}_{\gamma_{n_j+1}} \\ \dot{z}_{n_j+1} \\ \dot{\tilde{\theta}}_{n_j+1} \\ \dot{\beta}_{n_j+1} \end{bmatrix} = v_{n_j+1} \begin{bmatrix} \frac{\cos\tilde{\theta}_{n_j+1}}{1-\kappa_\gamma(s_{\gamma_{n_j+1}})z_{n_j+1}} \\ \sin\tilde{\theta}_{n_j+1} \\ \frac{\tan\beta_{n_j+1}-\frac{M_{n_j}}{L_{n_j}}\tan\beta_{n_j}}{L_{n_j+1}\left(1+\frac{M_{n_j}}{L_{n_j}}\tan\beta_{n_j}\tan\beta_{n_j+1}\right)} - \frac{\cos\tilde{\theta}_{n_j+1}\,\kappa_\gamma(s_{\gamma_{n_j+1}})}{1-\kappa_\gamma(s_{\gamma_{n_j+1}})z_{n_j+1}} \\ \frac{\tan\beta_{n_j}}{L_{n_j}\cos\beta_{n_j+1}} - \frac{\tan\beta_{n_j+1}}{L_{n_j+1}} + \frac{M_{n_j}\tan\beta_{n_j}}{L_{n_j+1}L_{n_j}} \end{bmatrix} \tag{6}$$

$j \in \{0, 1, \ldots, m\}$, $n_0 = 0$. For the other nonholonomic points the corresponding M_{n_j+i} are 0 so the formulae simplify to:

$$\begin{bmatrix} \dot{s}_{\gamma_{n_j+i}} \\ \dot{z}_{n_j+i} \\ \dot{\tilde{\theta}}_{n_j+i} \\ \dot{\beta}_{n_j+i} \end{bmatrix} = v_{n_j+i} \begin{bmatrix} \frac{\cos\tilde{\theta}_{n_j+i}}{1-\kappa_\gamma(s_{\gamma_{n_j+i}})z_{n_j+i}} \\ \sin\tilde{\theta}_{n_j+i} \\ \frac{\tan\beta_{n_j+i}}{L_{n_j+i}} - \frac{\cos\tilde{\theta}_{n_j+i}\,\kappa_\gamma(s_{\gamma_{n_j+i}})}{1-\kappa_\gamma(s_{\gamma_{n_j+i}})z_{n_j+i}} \\ \frac{\tan\beta_{n_j+i-1}}{L_{n_j+i-1}\cos\beta_{n_j+i}} - \frac{\tan\beta_{n_j+i}}{L_{n_j+i}} \end{bmatrix} \tag{7}$$

$j \in \{0, 1, \ldots, m\}$, $i \in \{2, 3 \ldots, n_{j+1} - n_j\}$, $n_{m+1} = n$, where

$$v_{n_j+i} = \frac{v_n}{\prod_{k=j+1}^{m}\left(1+\frac{M_{n_k}}{L_{n_k}}\tan\beta_{n_k}\tan\beta_{n_k+1}\right)\prod_{k=n_j+i+1}^{n}(\cos\beta_k)} \tag{8}$$

$j \in \{0, 1, \ldots, m\}$, $i \in \{1, 2 \ldots, n_{j+1} - n_j\}$.

Since we are only interested in the problem of following a given path, the velocity v_n can be neglected as input and assumed to be a given (nonull) open-loop function, for example a constant. Considering only one of the actuators as free input implies then that the system we obtain has a drift component. Calling:

$$\boldsymbol{p}_i = \left[s_{\gamma_i}\ z_i\ \tilde{\theta}_i,\ \beta_i\right]^T$$

The subsystem (6)-(7) can be expressed more compactly as:

$$\dot{\boldsymbol{p}}_i = \mathcal{F}_i\left(\boldsymbol{p}_i,\ \underline{\beta}_{i+1},\ \kappa_\gamma(s_{\gamma_i}),\ v_n\right) \tag{9}$$

$i \in \{2, \ldots, n\}$, where we have defined:

$$\underline{\beta}_i \triangleq [\beta_i \ldots \beta_n]$$

For the nonholonomic points P_1 and P_0, the equations are function also of the steering input ω. Assuming $n_1 > 1$, we get

$$\dot{\boldsymbol{p}}_1 = \begin{bmatrix} \dot{s}_{\gamma_1} \\ \dot{z}_1 \\ \dot{\tilde{\theta}}_1 \\ \dot{\beta}_1 \end{bmatrix} = v_1 \begin{bmatrix} \frac{\cos\tilde{\theta}_1}{1-\kappa_\gamma(s_{\gamma_1})z_1} \\ \sin\tilde{\theta}_1 \\ \frac{\tan\beta_1}{L_1} - \frac{\cos\tilde{\theta}_1\,\kappa_\gamma(s_{\gamma_1})}{1-\kappa_\gamma(s_{\gamma_1})z_1} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\omega = \mathcal{F}_1\left(\boldsymbol{p}_1,\ \underline{\beta}_1,\ \kappa_\gamma(s_{\gamma_1}),\ v_n\right) + \mathcal{G}_1\,\omega$$

In P_0 there is no equation for β:

$$\dot{\boldsymbol{p}}_0 = \begin{bmatrix} \dot{s}_{\gamma 0} \\ \dot{z}_0 \\ \dot{\tilde{\theta}}_0 \end{bmatrix} = v_0 \begin{bmatrix} \frac{\cos\tilde{\theta}_0}{1-\kappa_\gamma(s_{\gamma 0})z_0} \\ \sin\tilde{\theta}_0 \\ \frac{\sin\beta_1}{L_1} - \frac{\cos\tilde{\theta}_0\,\kappa_\gamma(s_{\gamma 0})}{1-\kappa_\gamma(s_{\gamma 0})z_0} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega = \mathcal{F}_0\left(\boldsymbol{p}_0,\ \underline{\beta}_1,\ \kappa_\gamma(s_{\gamma 0}),\ v_n\right) + \mathcal{G}_0\,\omega$$

Again, v_1 and v_0 are calculated using (8).

The whole configuration state is represented by

$$\boldsymbol{p} = \left[\boldsymbol{p}_n,\ \boldsymbol{p}_{n-1},\ \dots,\boldsymbol{p}_0\right]^T,\quad \boldsymbol{p} \in \mathcal{D}$$

where the domain of definition $\mathcal{D}$ and the singularity locus of the general n-trailer are discussed in (Altafini, 1998) and the dynamic equations of the system are:

$$\dot{\boldsymbol{p}} = \begin{bmatrix} \mathcal{F}_n \\ \mathcal{F}_{n-1} \\ \vdots \\ \mathcal{F}_1 \\ \mathcal{F}_0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \mathcal{G}_1 \\ \mathcal{G}_0 \end{bmatrix} \omega = \mathcal{F}(\boldsymbol{p}) + \mathcal{G}\omega \tag{10}$$

In order to consider simultaneously the error distances of all the nonholonomic points P_i from the path, we take as output the sum of the $n+1$ signed distances z_i:

$$y = [0\,1\,0\,0\ \ 0\,1\,0\,0 \dots 0\,1\,0\,0\ \ 0\,1\,0]\,\boldsymbol{p} \overset{\triangle}{=} \mathcal{H}\boldsymbol{p} \tag{11}$$

Considering multiple frames on the same rigid body leads to redundant description of the system. To recover the original dynamic equations, a number of constraints must be added. They express the rigid body assumption with respect to the curvilinear abscissae representation and they depend on line integrals that cannot be resolved in closed form. However, for our stabilization purposes they can simply be neglected and we can work with the overparameterized system (10)-(11), see again (Altafini and Gutman, 1998) for a complete formulation.

3 Input-output feedback linearization

We need to introduce the notion of relative degree of a nonlinear system. For this and the other concepts used in the remaining of the paper, like Lie derivative, input-state and input-output feedback linearization, zero dynamics etc. we remand the reader to any standard text on nonlinear control systems, like (Isidori, 1995).

Definition 1 *The SISO nonlinear system*

$$\begin{aligned} \dot{x} &= F(x) + G(x)u \\ y &= H(x) \end{aligned}$$

where $F:\ D \to \mathbb{R}^{\mathrm{n}}$*,* $G:\ D \to \mathbb{R}^{\mathrm{n}}$ *and* $H:\ D \to \mathbb{R}$ *are sufficiently smooth functions in* $D \in \mathbb{R}^{\mathrm{n}}$ *, is said to have* relative degree $\mathrm{r}, 1 \le \mathrm{r} \le \mathrm{n}$ *if:*

1. $L_G L_F^{\mathrm{k}} H(x) = 0 \quad 0 \le \mathrm{k} < \mathrm{r} - 1$
2. $L_G L_F^{\mathrm{r}-1} H(x) \neq 0$

$\forall x$ in a region $D_0 \in D$.

In practice, the relative degree of a system corresponds to the number of times the output function $H(x)$ has to be derived in order to have the input appearing explicitly on it.

In what follow we will assume that $M_0 = 0$ i.e. that there is no off-axle connection on the driving unit. In fact, if $M_0 \neq 0$ the general n-trailer does not have a well-defined relative degree.

The following proposition can be proven by direct calculation

Proposition 1 *The n-trailer system (10)-(11) with the tracking criterion (5) has relative degree 2.*

For these "chains" the relative degree reflects the complexity of the simplest possible controller needed to stabilize the system.

The general n-trailer cannot be input-state feedback linearized i.e. cannot be transformed into the classical Brunowsky-like canonical form nor can be trivialized to a system without zero dynamics because of the off-axle hitches.

However, what really matters in the path following problem is the tracking error, in our case the output of the system, not the whole state.

The low relative degree suggests that input-output feedback linearization is easily attained for our system: in fact it is enough to derive the output (11) twice and cancel the corresponding dynamics by means of a change of input. From

$$y = \sum_{i=0}^{n} z_i$$

we get:

$$\begin{aligned}
\dot{y} &= L_{\mathcal{F}}\mathcal{H}\boldsymbol{p} = \sum_{i=0}^{n} v_i \sin\tilde{\theta}_i = \\
&= \sum_{j=0}^{m} \sum_{i=i}^{n_{j+1}-n_j} \frac{v_n \sin\tilde{\theta}_{n_j+i}}{\prod_{k=j+1}^{m}\left(1+\frac{M_{n_k}}{L_{n_k}}\tan\beta_{n_k}\tan\beta_{n_k+1}\right)\prod_{k=n_j+i+1}^{n}(\cos\beta_k)} + v_0\sin\tilde{\theta}_0
\end{aligned}$$

In order to have a well-defined relative degree, we have already assumed that the driving unit has no off-axle connection, i.e. $M_0 = 0$. For sake of simplicity, we require here also that $M_1 = 0$. The case with $M_1 \neq 0$ does not differ except for the more involved formulation of the domain of definition and will be treated in the example of Section 4.1. In fact, when differentiating the above expression a second time we need to isolate the terms in $\tilde{\theta}_0$ and β_1 whose derivatives introduce the input ω. With $M_0 = M_1 = 0$, it can be noticed that they appear only in the term $v_0 \sin\tilde{\theta}_0$. Calling $\mathfrak{p}$ the state obtained from $\boldsymbol{p}$ excluding β_1 and $\tilde{\theta}_0$, we have:

$$\begin{aligned}
\ddot{y} &= L^2_{\mathcal{F}}\mathcal{H}\boldsymbol{p} + L_{\mathcal{G}}L_{\mathcal{F}}\mathcal{H}\boldsymbol{p}\omega = \frac{\partial\dot{y}}{\partial\mathfrak{p}}\dot{\mathfrak{p}} + \frac{\partial\dot{y}}{\partial\beta_1}\dot{\beta}_1 + \frac{\partial\dot{y}}{\partial\tilde{\theta}_0}\dot{\tilde{\theta}}_0 = \\
&= \frac{\partial\dot{y}}{\partial\mathfrak{p}}\dot{\mathfrak{p}} + v_0^2\cos\tilde{\theta}_0\left(\frac{\sin\beta_1}{L_1} - \frac{\cos\tilde{\theta}_0\kappa_\gamma(s_{\gamma_0})}{1-\kappa_\gamma(s_{\gamma_0})z_0}\right) + v_0\left(\sin\tilde{\theta}_0\tan\beta_1 + \cos\tilde{\theta}_0\right)\omega
\end{aligned}$$

The term $\left(\sin\tilde{\theta}_0\tan\beta_1 + \cos\tilde{\theta}_0\right)$ is singular $\Longleftrightarrow \tan(\tilde{\theta}_0 - \beta_1) = \pm\infty \Longleftrightarrow \tilde{\theta}_0 - \beta_1 = \frac{\pi}{2}\, mod\, \pi$. Since $\tilde{\theta}_0 - \beta_1 = \theta_1 - \theta_{\gamma_0}$ the singularities are function of how much the path is "bending" between the projections on the path of P_0 and P_1. Therefore in $\mathcal{D} \cap \left\{(\tilde{\theta}_0, \beta_1) \; s.t. \; \tilde{\theta}_0 - \beta_1 \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right\}$ the input transformation:

$$\omega = \frac{-L_{\mathcal{F}}^2 \mathcal{H}\boldsymbol{p} + u}{L_{\mathcal{G}} L_{\mathcal{F}} \mathcal{H}\boldsymbol{p}} = \frac{-\left(\frac{\partial \dot{y}}{\partial \mathfrak{p}} \dot{\mathfrak{p}} + v_0^2 \cos\tilde{\theta}_0 \left(\frac{\sin\beta_1}{L_1} - \frac{\cos\tilde{\theta}_0 \kappa_\gamma(s_{\gamma_0})}{1-\kappa_\gamma(s_{\gamma_0}) z_0}\right)\right) + u}{v_0 \left(\sin\tilde{\theta}_0 \tan\beta_1 + \cos\tilde{\theta}_0\right)} \tag{12}$$

is a diffeomorphism that reduces the input-output dynamics to the chain of integrators

$$\ddot{y} = u \tag{13}$$

that can be stabilized using linear control theory provided that the system is minimum phase. The zero dynamics is obtained confining the dynamics of the system to the so-called *Output-Zeroing Manifold*

$$Z^* = \{\boldsymbol{p} \in \mathcal{D} \;\; s.t. \;\; y = \dot{y} = \ddot{y} = 0\}$$

In practice, it is obtained adding to the original system (10) the conditions $y = 0$, $\dot{y} = 0$ and the input

$$\omega = \frac{-L_{\mathcal{F}}^2 \mathcal{H}\boldsymbol{p}}{L_{\mathcal{G}} L_{\mathcal{F}} \mathcal{H}\boldsymbol{p}}$$

and it represents the part of the system equation which is not anymore connected to the output after the change of input. The motion of the system restricted to Z^* is obtained, for example, eliminating the variables z_0 and $\tilde{\theta}_0$ by means of

$$z_0 = -\sum_{1}^{n} z_i \tag{14}$$

$$\tilde{\theta}_0 = \arcsin\left(-\sum_{j=0}^{m} \sum_{i=i}^{n_{j+1}-n_j} \frac{v_n \sin\tilde{\theta}_{n_j+i}}{v_0 \prod_{k=j+1}^{m}\left(1 + \frac{M_{n_k}}{L_{n_k}} \tan\beta_{n_k} \tan\beta_{n_k+1}\right) \prod_{k=n_j+i+1}^{n} (\cos\beta_k)}\right)$$

The local asymptotic stability of the zero dynamics can be easily proven for a path of constant curvature using Lyapunov linearization. The chain of integrators (13) can now be stabilized in an "optimal" fashion for example using Linear Quadratic theory. Any output feedback of the form

$$u = k_1 y + k_2 \dot{y} \tag{15}$$

with $k_1 < 0$, $k_2 < 0$ is a locally asymptotically stabilizer for the whole system.

We will see in next Section that this condition is sufficient to prove asymptotic stability also for paths of varying curvature. To understand this, it can be noticed that input-output linearization renders the tracking error dynamics independent from the curvature κ_γ of the path. The zero dynamics instead is still function of the value of the curvature between s_{γ_0} and s_{γ_n} so, when κ_γ is varying, the dynamic system on Z^* is nonautonomous and does not have a constant equilibrium point to which to stabilize the system.

4 Following a path of varying curvature as an output regulation problem

In what follows we will try to asymptotically stabilize the system to paths whose curvature is varying in a given class of functions. We will treat the problem as an *Output Regulation Problem* in which the error $y(\cdot)$ has to asymptotically reject the variation of curvature $k_\gamma(s_\gamma)$ intended as a persistent input generated by a dynamical system. In the classical context of linear time-invariant, finite-dimensional systems, this geometric control problem was first solved by Davison (Davison, 1976) and Francis and Wonham (Francis and Wonham, 1976) based on the assumption that the external command can be modeled as the output of an autonomous system called the *exosystem*. The solution was then extended to the nonlinear case by Isidori and Byrnes (Isidori and Byrnes, 1990).

The presence of a known "disturbance" acting as a persistent input, called exogenous input, implies that the steady state of the system is varying depending only on the exogenous input and not on the initial conditions of the system (that have to be in an appropriate neighborhood of the origin). In our case the exogenous input of the system is the curvature function $k_\gamma(\cdot)$ of the path. To be consistent with our control problem, the curvature has to be upper bounded: in fact it is intuitively clear that too hight a curvature implies that a steady state for the tracking criterion (5) does not exist.

We need to introduce a few definitions and results whose proof can be looked up in (Isidori, 1995).

The properties of persistence in time and of boundedness of the exogenous input are compactly described by the notion of *neutral stability* of the exogenous system. A system is said neutrally stable if it is both Lyapunov stable and Poisson stable. Lyapunov stability is required for the boundedness of the states of the exogenous system while Poisson stability implies persistence of the trajectories. An example of neutrally stable system is any periodic system. A necessary condition for a system to be neutrally stable is that its first order approximation has all the eigenvalues on the imaginary axis. The assumption of Poisson stability can obviously be relaxed: this implies that the exogenous system is stable and has a subsystem that tends to an invariant manifold corresponding to the eigenvalues of the first order approximation that lie on the open left half of the complex plane. In our case, the exogenous system has to represent how the curvature is evolving along the path γ. Looking at system (10), we can see that at every time instant the curvature has $n+1$ "entries" in the equations, corresponding to the values of curvature in different positions along the path. For the nonholonomic point P_i, if the curvature function is given in terms of the curvilinear abscissa

$$\kappa_{\gamma_i} = \kappa_\gamma(s_{\gamma_i})$$

then we can think of it as generated by a dynamical system

$$\kappa'_{\gamma_i} = \frac{d\kappa_{\gamma_i}}{ds_{\gamma_i}} = \gamma\left(\kappa_{\gamma_i}\right) \qquad i = 0,\, 1, \ldots n \tag{16}$$

where the independent variable is the curvilinear abscissa s_{γ_i} and the output equation is the identity. In order to couple this exogenous system with the remaining part of the equations, we have to rescale it as a function of time, expressing s_{γ_i} as $s_{\gamma_i}(t)$ i.e. substituting the space derivatives of eq. (16) with the corresponding time derivatives:

$$\dot{\kappa}_{\gamma_i} = \frac{d\kappa_{\gamma_i}}{dt} = \frac{d\kappa_{\gamma_i}}{ds_{\gamma_i}}\frac{ds_{\gamma_i}}{dt} = v_{\gamma_i}\gamma\left(\kappa_{\gamma_i}\right) = \dot{s}_{\gamma_i}\gamma\left(\kappa_{\gamma_i}\right) \tag{17}$$

The exogenous equation is the same for all the nonholonomic points (since the reference path is the same) but the initial values of κ_{γ_i} are different since they express the value of the curvature at the initial curvilinear abscissa $s_{\gamma_i}(0)$. The presence of the term $\dot{s}_{\gamma_i}$ does not spoil the "exogenousness" of the system (17): it is in fact possible to rescale the whole system (10) as a function of the curvilinear abscissa yielding a completely time-independent system in which the terms $\dot{s}_{\gamma_i}$ obviously disappear. Provided we can prove well-posedness and asymptotic stability of the problem in the time-dependent scale, then in the formulation (17) the $\dot{s}_{\gamma_i}$ locally represent terms which are monotone, bounded and continuous (if the path has continuous curvature) since they represent the projections on the path of the translational velocities v_i of the nonholonomic points P_i. For example for forward motion ($v_i > 0$):

$$0 < \dot{s}_{\gamma_i} \leq v_i$$

Therefore the neutral stability of (16) implies the neutral stability of (17) and viceversa. In fact, the eigenvalues of the first order approximation are on the imaginary axis in both cases. For all times t we have $s_{\gamma_n}(t) < s_{\gamma_{n-1}}(t) < \ldots < s_{\gamma_0}(t)$ but the delay between $s_{\gamma_i}(t)$ and $s_{\gamma_{i-1}}(t)$ is variable according to the curvature of the path in the interval $s_{\gamma_{i-1}}(t) - s_{\gamma_i}(t)$ and to the position and orientation of the vehicle with respect to the path.

Calling $k_\gamma = [k_{\gamma_n} \ldots k_{\gamma_1}\, k_{\gamma_0}]^T$ and $s_\gamma = [s_{\gamma_n} \ldots s_{\gamma_1}\, s_{\gamma_0}]^T$, the complete system is then:

$$\begin{aligned} \dot{\boldsymbol{p}} &= \mathcal{F}(\boldsymbol{p},\ \kappa_\gamma) + \mathcal{G}(\boldsymbol{p})\omega && (18) \\ \dot{\kappa}_\gamma &= \dot{s}_\gamma^T\, \Gamma(\kappa_\gamma) && (19) \\ y &= \mathcal{H}\boldsymbol{p} && (20) \end{aligned}$$

where $\Gamma(\kappa_\gamma)$ has the diagonal structure:

$$\begin{bmatrix} \gamma(\kappa_{\gamma_n}) & & 0 \\ & \ddots & \\ 0 & & \gamma(\kappa_{\gamma_0}) \end{bmatrix}$$

In our case, the exogenous system has a single eigenvalue of multiplicity $n+1$, therefore relaxing the assumption of Poisson stability implies that we get asymptotic stability of the exosystem i.e. the curvature tends to a constant value as the curvilinear abscissa tends to infinity.

The right formulation for our case is named in (Isidori, 1995) the *Full Information Output Regulation Problem*, meaning with this expression that the whole state is measurable, together with the output of the exogenous system.

Given the nonlinear system (18) and the neutrally stable exogenous system (19), the output regulation problem is solvable if there exists a map $\alpha(\boldsymbol{p},\ \kappa_\gamma)$ such that:

P1. the equilibrium $\boldsymbol{p} = 0$ of

$$\dot{\boldsymbol{p}} = \mathcal{F}(\boldsymbol{p},\ 0) + \mathcal{G}(\boldsymbol{p})\alpha(\boldsymbol{p},\ 0)$$

is asymptotically stable in the first order approximation;

P2. there exist a neighborhood $V \subset \Pi \times K_\Gamma^0$ of $(0,\ 0)$ such that for each initial condition $(\boldsymbol{p}(0),\ \kappa_\gamma(0)) \in V$, the solution of (18) satisfies:

$$\lim_{t \to 0} \mathcal{H}\boldsymbol{p}(t) = 0$$

The statement P1 is a consequence of the Center Manifold Theory. In fact, given the system (18)-(19), we know that the eigenvalues of the exogenous system are on the imaginary axis and cannot be moved. So the problem is solvable if all the other eigenvalues of the system can be moved to the open left half of the complex plane by means of a state feedback on the endogenous input ω. If such a feedback can be found for $\kappa_\gamma = 0$, then the center manifold theory assures the existence of an invariant manifold in a neighborhood of the origin whose graph is the solution of an associated partial differential equation (see below). This is formulated in the following theorem:

Theorem 1 (Isidori, 1995) Given the neutrally stable system (19) and assuming the existence of an endogenous feedback law $\omega = \alpha(\boldsymbol{p}, 0)$, $\alpha(0, 0) = 0$ such that the equilibrium $\boldsymbol{p} = 0$ of

$$\mathcal{F}(\boldsymbol{p}, 0)) + \mathcal{G}(\boldsymbol{p})\, \alpha(\boldsymbol{p}, 0)$$

is asymptotically stable in the first order approximation, then there exist mappings $\boldsymbol{p} = \pi(\kappa_\gamma)$ and $\omega = \alpha(\pi(\kappa_\gamma), \kappa_\gamma)$ defined in a neighborhood $K_\Gamma^\circ \subset K_\Gamma$ of the origin with $\pi(0) = 0$ and $\alpha(0, 0) = 0$, which satisfy

$$\frac{\partial \pi}{\partial \kappa_\gamma} \Gamma(\kappa_\gamma) = \mathcal{F}(\pi(\kappa_\gamma), \kappa_\gamma)) + \mathcal{G}(\pi(\kappa_\gamma))\, \alpha(\pi(\kappa_\gamma), \kappa_\gamma)$$

$\forall \kappa_\gamma \in K_\Gamma^\circ$.

The theorem assures also the existence of a well-defined steady state response $\forall$ exogenous inputs in K_Γ°.

In practice, what P1 says is that the fulfillment of the partial differential equation is reduced to the the analysis of a linear system. Consider the Jacobian of $\mathcal{F}$ at the origin

$$\mathcal{F}_{e_0} = \left. \frac{\partial \mathcal{F}(\boldsymbol{p}, \kappa_\gamma)}{\partial \boldsymbol{p}} \right|_{(0,\, 0)}$$

From linear control theory it is deducted that the stabilizability of the pair $(\mathcal{F}_{e_0}, \mathcal{G})$ is also a necessary condition for the solution of P1.

The previous condition can be used to adapt the necessary and sufficient condition for the solution of the Full Information Output Regulation Problem provided in (Isidori, 1995) to our case.

Theorem 2 *The full information output regulation problem is solvable if and only if* $(\mathcal{F}_{e_0}, \mathcal{G})$ *is stabilizable and there exist mappings* $\boldsymbol{p} = \pi(\kappa_\gamma)$ *and* $\omega = \alpha(\pi(\kappa_\gamma), \kappa_\gamma)$ *with* $\pi(0) = 0$ *and* $\alpha(0, 0) = 0$*, both defined in a neighborhood* $K_\Gamma^\circ \subset K_\Gamma$ *satisfying the conditions:*

$$\frac{\partial \pi}{\partial \kappa_\gamma} \Gamma(\kappa_\gamma) = \mathcal{F}(\pi(\kappa_\gamma), \kappa_\gamma)) + \mathcal{G}(\pi(\kappa_\gamma))\, \alpha(\pi(\kappa_\gamma), \kappa_\gamma) \tag{21}$$

$$0 = \mathcal{H}\pi(\kappa_\gamma) \tag{22}$$

for all $\kappa_\gamma \in K_\Gamma^\circ$.

The conditions (21) and (22) express the fact that the mapping $\boldsymbol{p} = \pi(\kappa_\gamma)$ which is rendered locally invariant by the feedback law $\omega = \alpha(\pi(\kappa_\gamma), \kappa_\gamma)$ has to be an output zeroing manifold of the composite system. Due to the independence of the output equation (20) from the curvature κ_γ, the output zeroing property is not related to the exogenous system but only to the exact input-output feedback linearization. This is formalized in the following theorem.

Theorem 3 *Given the system (18)-(20), the Full Information Output Regulation Problem is solvable by and only by the controllers composed of a prefeedback that input-output exactly linearizes the system and of a stabilizing feedback for the resulting chain of integrators. All such controllers have the following structure:*

$$\omega = \frac{-L_{\mathcal{F}}^2 \mathcal{H}\boldsymbol{p} + k_1 y + k_2 \dot{y}}{L_{\mathcal{G}} L_{\mathcal{F}} \mathcal{H}\boldsymbol{p}} \tag{23}$$

$\forall\, k_1 < 0,\ k_2 < 0.$

Proof. From Section 3, we know that the input-output feedback linearizing controller transform the system into a chain of integrators. For this linear system, local asymptotic stability to paths of constant curvature (and so also along the straight line path $\kappa_\gamma = 0$) can be assured by means of the linear controller (15). Therefore, from Theorem 1 we deduce the existence of an invariant manifold characterized by the map $\boldsymbol{p} = \pi(\kappa_\gamma)$ such that on its graph the conditions (21)-(22) are satisfied. The necessity derives from the fact that in our case the output $y(\cdot)$ is independent of κ_γ. This implies that the condition (22) is satisfied only on the zero output manifold. Therefore the map $\pi(\cdot)$ whose graph solves the partial differential equation (21) is unique and corresponds to the static change of input that input-output linearizes the system.

$\diamondsuit$

Such a property is characteristic not only of our system (18)-(20) but of any control-affine SISO system with relative degree for which only disturbance rejection is required i.e. in which the exogenous system consists only of disturbances acting on the state space and not of signals to be tracked by the output. Although it strictly reflects the underlying method used in the proofs in (Isidori and Byrnes, 1990), this particular case is not explicitly mentioned in the literature. We state it as corollary to Theorem 3.

Corollary 1 *Given the neutrally stable exosystem*

$$\dot{\kappa} = \Gamma(\kappa),$$

assume the SISO control-affine nonlinear system

$$\begin{aligned} \dot{x} &= F(x,\, \kappa) + G(x\,\kappa)u \\ y &= H(x) \end{aligned}$$

where $F : D \to \mathbb{R}^{\mathrm{n}}$, $G : D \to \mathbb{R}^{\mathrm{n}}$ and $H : D \to \mathbb{R}$ are sufficiently smooth functions in $D \in \mathbb{R}^{\mathrm{n}}$ has relative degree r in a region $D_0 \subset D$, then the Full Information Output Regulation Problem is solvable if and only if the pair $\left(\frac{\partial F}{\partial x}\Big|_{(0\,0)},\, G(0\,0)\right)$ is stabilizable.

Proof. The argument is the same as Theorem 3. The assumption of relative degree assures the nonsingularity of $L_G L_F^{\mathrm{r}-1} H(x)$ in D_0 also under the exogenous input κ.

$\diamondsuit$

What this means is that in the case of well-defined relative degree there is no need to solve a partial differential equation to find the invariant manifold $\pi(\cdot)$, since the prefeedback (12) provides the unique solution.

Also the problem of following a path of constant curvature can be reformulated as an output regulation problem in which the exogenous input is a constant set point like the one generated by the system $\dot{\kappa}_\gamma = 0$.

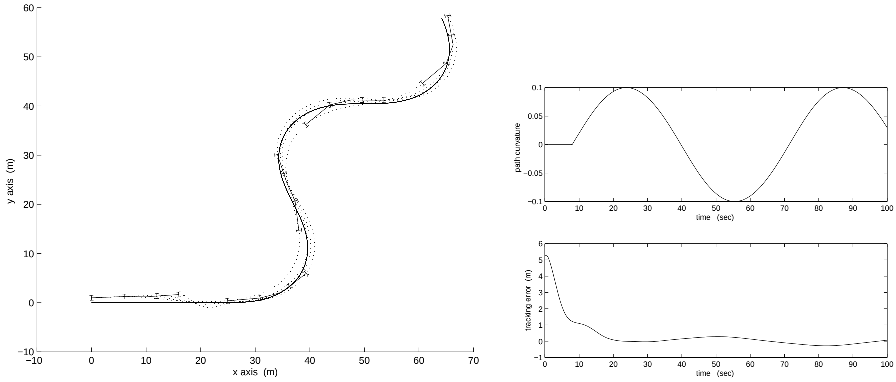
4.1 Example

Consider a car pulling two trailers the first of which has off-axle hooking. This configuration resembles in a more realistic way than the standard 3-trailer system for example the lorries that normally run on our highways. Here we have that $M_1 \neq 0$, therefore the term $L_{\mathcal{G}} L_{\mathcal{F}} \mathcal{H} \boldsymbol{p}$ instead of having the expression in the denominator of eq. (12) has the more complex one:

$$L_{\mathcal{G}} L_{\mathcal{F}} \mathcal{H} \boldsymbol{p} =$$
$$= v_3 \frac{\cos\beta_1 \cos\beta_2 \cos\beta_3 \left[\sin\tilde{\theta}_0 \left(\tan\beta_1 - \frac{M_1}{L_1}\tan\beta_2\right) + \cos\tilde{\theta}_0 \left(1 + \frac{M_1}{L_1}\tan\beta_1 \tan\beta_2\right)\right] + \sin\tilde{\theta}_1 \left(\frac{M_1}{L_1}\tan\beta_2\right)}{\left(1 + \frac{M_1}{L_1}\tan\beta_1\tan\beta_2\right)^2 \cos^2\beta_1 \cos^2\beta_2 \cos^2\beta_3}$$

In a neighborhood of $\boldsymbol{p} = 0$, $\cos\tilde{\theta}_0$ is the dominant term, therefore as in (12), we can conclude that there exist a subdomain of $\mathcal{D}$ in which the denominator $L_{\mathcal{G}} L_{\mathcal{F}} \mathcal{H} \boldsymbol{p}$ is nonsingular.

The different behaviors of the linear controller used in (Altafini and Gutman, 1998) and of the input-output linearizing controller (23) are compared for a sinusoidal path in Fig. 3 and Fig. 4. The linear controller cannot achieve any steady state even though the tracking error



(a)

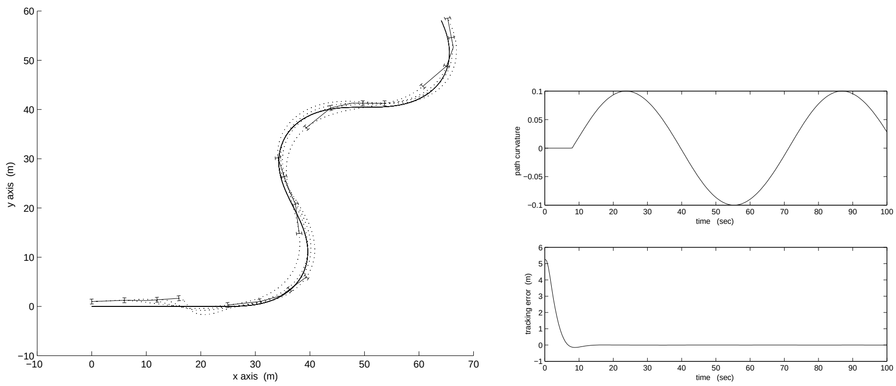
(b) path curvature (top) and tracking error (bottom)

Figure 3: Following a path of sinusoidal curvature with the linear controller proposed in (Altafini and Gutman, 1998).

remains bounded. For the second controller instead, the tracking error asymptotically converges to zero.

5 Conclusion

This paper proposes a controller that locally asymptotically stabilizes a kinematic vehicle to a path of smoothly varying curvature treating the problem as an output regulation problem in which the curvature is seen as a known disturbance affecting the state but not the output (i.e. the tracking error) of the system.



(a)

(b) path curvature (top) and tracking error (bottom)

Figure 4: Following the same sinusoidal path of Fig. 3 with the controller (23).

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