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Nonlinear observers for a class of differential delay systems

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Abstract

This paper focuses on the design of observers for a class of nonlinear systems with time-varying delay. Sufficient convergence conditions are established from the Lyapunov-Krasovskii theory. These conditions are linked to the existence of a positive definite matrix satisfying a certain Riccati equation. Using an $\mathcal{H}_\infty$ theory result, we propose sufficient conditions to guarantee such an existence.

1 Introduction

Numerous control processes encountered for example in biology, mechanic or chemistry (Kolmanovskii and Nosov, 1986; Malek-Zavarei and Jamshidi, 1987) involve delays. Their presence may affect the performance of control laws or even be a source of instability. Often, the control of such systems, includes the design of an observer which must asymptotically estimate the state variables of the system from the output and the input measurement (Bhat and Koivo, 1976). During the last decades, reconstruction of the state variables of systems with time delays has been the subject of many papers (Bhat and Koivo, 1976; Fairman and Kumar, 1986; Gressang and Lamont, 1975; Hamidi-Hashemi and Leondes, 1979; Kamen, 1982; Lee and Olbrot, 1981; Pearson and Fiagbedzi, 1989; Salomon, 1980; Pourboghrat and Chyung, 1984; Watanabe and Ouchi, 1985; Watanabe, 1986).

These results, all given in the case of linear systems, have not been extended for nonlinear systems with time-varying delays. This paper is a contribution in this area. In the case of systems without delays, recent developements have been realized in the design of observers for nonlinear systems, with an objectif to find less restrictive conditions to ensure the asymptotic convergence of the observer (Raghavan and Hedrick, 1994; Rajamani, 1998). This study generalize the approach given in those papers. The proposed observer is an extension of Luenberger observers to a class of nonlinear systems with time-varying delay. The analysis of its convergence is done by using Lyapunov-Krasovskii theory. The obtained sufficient convergence conditions involve the existence of a positive definite matrix satisfying a certain Riccati equation and therefore 'algebrize' convergence results. The existence proofs are constructive, and hence lead to the prescription of a hole class of observers gains.

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The paper is organized as follows: in Section 2 we describe the class of systems considered and recall some basic notions. In Section 3, sufficient conditions to guarantee the convergence of the observer are established. Finally, Section 4 gives conclusions.

2 System description and Preliminaries

The system under investigation is described by the following equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \bar{A}x(t-h(t)) + f\big(t, x(t), u\big) + g\big(t, x(t-h(t)), u\big) \\ y &= Cx(t) \\ x(t) &= \phi(t), \quad t \in [-H, 0] \end{aligned} \tag{1}$$

where A and $\bar{A}$ are matrices of $\mathbb{R}^{n\times n}(\mathbb{R})$, $C \in \mathbb{R}^{q\times n}(\mathbb{R})$. $y \in \mathbb{R}^q$ represents the measurements of the system. $u \in \mathbb{R}^m$ is the input. $h(.)$ which represents the delay, is a scalar differential function. It is supposed to be known and to satisfy $0 \leq h(t) \leq H$ for all $t > 0$. f and g are given nonlinear continuous functions, respectively k_f and k_g-lipschitzian with respect of their second argument, i.e.

$$|f(t, \phi, u) - f(t, \psi, u)| \leq k_f \|\phi - \psi\|$$

and

$$|g(t, \phi, u) - g(t, \psi, u)| \leq k_g \|\phi - \psi\|,$$

$\forall t \in \mathbb{R}$, $\forall \phi$, $\psi \in \mathcal{C}([-H, 0], \mathbb{R}^n)$. $\mathcal{C}([-H, 0], \mathbb{R}^n)$ is the banach space of continuous function mapping $[-H, 0]$ into $\mathbb{R}^n$, with the norm $\|\phi\| = \sup\limits_{t\in[-H,0]} |\phi(t)|$. The euclidean norm of $\phi(t) \in \mathbb{R}^n$ is denoted by $|\phi(t)|$. We also suppose that

$$f(t, 0, 0)=g(t, 0, 0)=0, \ \ \forall t \in \mathbb{R}.$$

Before proceeding further, we will give some preliminary results. Let us consider the nonlinear delay systems of the general form:

$$\dot{x}(t) = f(t, x_t) \tag{2}$$

where $f : \mathbb{R} \times \mathcal{C}([-H, 0], \mathbb{R}^n) \mapsto \mathbb{R}^n$ is continuous in the first argument, lipschitzian in the second and satisfy $f(t, 0) = 0$ for all $t \in \mathbb{R}$.
For $t \geq \sigma - H$, we denote by $x(\sigma, \phi)(t)$, its solution at time t with initial data ϕ, specified at time σ, i.e., $x(\sigma, \phi)(\sigma + \theta) = \phi(\theta)$, $\forall \theta \in [-H, 0]$. For $\theta \in [-H, 0]$,

$$x_t(\theta) = x(t + \theta)$$

and represents the state of the delay system.
For all δ positif, let us denote by $\mathcal{B}(0, \delta)$, the ball

$$\mathcal{B}(0, \delta)=\{\phi \in \mathcal{C}([-H, 0], \mathbb{R}^n)/\|\phi\| < \delta\}.$$

$\mathcal{A}$ will designate in the following, the class of scalar nondecreasing functions $\alpha \in \mathcal{C}([0, \infty), \mathbb{R})$, satisfying $\alpha(s) > 0$ for $s > 0$ and $\alpha(0) = 0$.

Definition

The equilibrium solution, $x \equiv 0$ of the delay differential equation (2) is said to be :

1. stable, if for any $\sigma \in \mathbb{R}$, $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, \sigma)$ such that $\phi \in \mathcal{B}(0, \delta)$ implies $x_t(\sigma, \phi) \in \mathcal{B}(0, \varepsilon)$ for $t \geq \sigma$.
2. asymptotically stable, if it is stable and there exists $b_0 = b_0(\sigma) > 0$ such that $\phi \in \mathcal{B}(0, b_0)$ implies $x_t(\sigma, \phi) \to 0$ as $t \to \infty$.
3. $\mathcal{N}$-robustly asymptotically stable if the equilibrium solution is asymptotically stable for all delay functions $h(.)$ of the set $\mathcal{N} = \{h(t) \in [-H, 0] : \dot{h}(t) < 1\}$.

Definition

Let $V : \mathbb{R} \times \mathcal{B}(0, \delta) \to \mathbb{R}$ be a continuous functional such that $V(t, 0) = 0$. The functional $(t, \phi) \to V(t, \phi)$ is said :

1. to be positive definite, if there is a function α in $\mathcal{A}$ such that $V(t, \phi) \geq \alpha(|\phi(0)|)$, for all $t \in \mathbb{R}$, $\phi \in \mathcal{B}(0, \delta)$.
2. to have infinitesimal upper bound, if there is a function $\alpha \in \mathcal{A}$ such that

   $V(t, \phi) \leq \alpha(||\phi||)$, for all $t \in \mathbb{R}$, $\forall \phi \in \mathcal{B}(0, \delta)$.

Theorem 2.1. *(Kolmanovskii and Myshkis, 1992)*
Assume that for some positive constant H, there exists a positive definite continuous functional $((t, \phi) \to V(t, \phi)) : \mathbb{R} \times \mathcal{B}(0, \delta) \to \mathbb{R}$ which has infinitesimal upper bound and whose derivative $\dot{V}$ is a negative definite functional on $\mathbb{R} \times \mathcal{B}(0, \delta)$. Then the trivial solution of (2) is asymptotically stable.

The following notations will be used throughout the paper. For $v \in \mathbb{R}^n$, v^T denote the transpose of v. If M is a positive definite matrix, then $M^{\frac{1}{2}}$ denote a square root of M. For any matrix M, M^T designate the transpose of this matrix. The $\mathbb{R}^n$-valued identity matrix will be denoted by I_n.

3 Main results

Consider the system given by (1). We define our observer as follows:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \bar{A}\hat{x}(t - h(t)) + f(t, \hat{x}(t), u) + g(t, \hat{x}(t - h(t)), u) + L(y(t) - C\hat{x}(t)). \tag{3}$$

where the observed state is denoted by $\hat{x}$, and $L \in \mathbb{R}^{n \times q}$ is the observer gain matrix.
Then the error in the state estimate, $e = x - \hat{x}$, has the following dynamics :

$$\dot{e}(t) = (A - LC)\, e(t) + \bar{A}e(t - h(t)) + F(t, e(t), u) + G\,(t, e(t - h(t)), u) \tag{4}$$

with

$$F\big(t, e(t), u\big) = f\big(t, x(t), u\big) - f\big(t, x(t) - e(t), u\big)$$

and

$$G\big(t, e(t-h(t)), u\big) = g\big(t, x(t-h(t)), u\big) - g\big(t, x(t-h(t)) - e(t-h(t)), u\big).$$

We can remark that F and G are respectively k_f and k_g lipschitzian with respect of their second componant. Moreover, we can note that

$$F(t,0,u) = G(t,0,u) = 0, \ \ \forall t \in \mathbb{R}, u \in \mathbb{R}^m.$$

The problem now consists of choosing the observer gain L so that (4) is made asymptotically stable. The following result present a sufficient condition to achieve this goal.

Theorem 3.1.

Consider the system (1) with, $\forall t$, $0 \le h(t) \le H$, $\dot h(t) < 1$ and its observer (3). If there exists a pair of symmetric positive definite matrices P and Q such that :

$$AP + PA^T + P\Big(\gamma_g I_n - \frac{1}{\varepsilon}C^T C\Big)P + \frac{1}{\beta}\bar A \bar A^T + \gamma_f I_n + Q = 0 \tag{5}$$

with $\beta < 1 - \dot h(t)$, $\forall t \in \mathbb{R}$, $\gamma_g = 2 + k_g^2$ and $\gamma_f = \dfrac{1}{\beta} + k_f^2$ then, the observer (3), with the matrix gain

$$L = \frac{1}{2\varepsilon}PC^T, \tag{6}$$

is asymptotically convergent.

Proof of Theorem 3.1

Suppose in a first time, that there exists an $\varepsilon > 0$ such that the Riccati equation (5) is satisfied and substitute the matrix gain L by its expression (6). This implies that the equation (5) becomes

$$(A - LC)P + P(A-LC)^T + \gamma_g P^2 + \frac{1}{\beta}\bar A \bar A^T + \gamma_f I_n + Q = 0.$$

By multiplying this equation on the left and right by P^{-1}, we get

$$P^{-1}(A-LC) + (A-LC)^T P^{-1} + \gamma_g I_n + P^{-1}\big(\frac{1}{\beta}\bar A \bar A^T + \gamma_f I_n\big)P^{-1} + P^{-1}QP^{-1} = 0. \tag{7}$$

Let us define the Lyapunov-Krasovskii $V : C(t, [-H, 0], \mathbb{R}^n) \mapsto \mathbb{R}$, by

$$V(t,\psi) = \psi(0)^T P^{-1}\psi(0) + (1 + k_g^2)\int_{-h(t)}^{0} \psi(\theta)^T \psi(\theta) d\theta. \tag{8}$$

We first note that the functional (8) is positive definite. Indeed

$$\begin{aligned} V(t,\psi) &= \psi(0)^T P^{-1}\psi(0) + \big(1 + k_g^2\big)\int_{-h(t)}^{0} \psi(\theta)^T \psi(\theta) d\theta \\ &\ge \lambda_{min}(P^{-1})|\psi(0)|^2. \end{aligned}$$

Moreover, V can be upper bounded as $V(t,\psi) \leq \zeta \|\psi\|^2$, where ζ is a positive constant. Indeed for all $\psi \in \mathcal{C}([-H,0],\mathbb{R}^n)$

$$V(t,\psi) \leq |\psi(0)|^2 \lambda_{\max}(P^{-1}) + \max_{s\in[-H,0]} |\psi(s)|^2 H\big(1+k_g^2\big).$$

By choosing $\zeta \geq \lambda_{\max}(P^{-1}) + H\big(1+k_g^2\big)$ we get what we announced.

If we derivate V along the trajectories of the system (4), we get

$$\begin{aligned}
\dot{V}(t,e_t) &= e(t)^T\Big((A-LC)^T P^{-1} + P^{-1}(A-LC) + (1+k_g^2)I_n\Big)e(t) \\
&+ 2e(t)^T P^{-1}\bar{A}e(t-h(t)) + 2e(t)^T P^{-1}F(t,e(t),u) + 2e(t)^T P^{-1}G(t,e(t-h(t)),u) \\
&- \big(1-\dot{h}(t)\big)\big(1+k_g^2 I_n\big)e(t-h(t))^T e(t-h(t)).
\end{aligned}$$

Using the assumption on F and the following Young's inequality

$$2u^T v \leq \varepsilon u^T u + \frac{1}{\varepsilon} v^T v \quad \forall u,v \in \mathbb{R}^n, \forall \varepsilon > 0,$$

gives

$$\begin{aligned}
2F\big(t,e(t),u\big)^T P^{-1}e(t) &\leq \frac{1}{k_f^2}|F\big(t,e(t),u\big)|^2 + k_f^2 e(t)^T P^{-1}P^{-1}e(t) \\
&\leq |e(t)|^2 + k_f^2 e(t)^T P^{-1}P^{-1}e(t).
\end{aligned} \tag{9}$$

Proceeding in the same manner, we obtain

$$\begin{aligned}
2G\big(t,e(t-h(t)),u\big)^T P^{-1}e(t) &\leq \frac{1}{1-\dot{h}(t)}|P^{-1}e(t)|^2 + (1-\dot{h}(t))|G\big(t,e(t-h(t)),u\big)|^2 \\
&\leq \frac{1}{1-\dot{h}(t)} e(t)^T P^{-1}P^{-1}e(t) + (1-\dot{h}(t))k_g^2|e(t-h(t))|^2.
\end{aligned} \tag{10}$$

By a completion of the squares

$$\begin{aligned}
&2e(t)^T P^{-1}\bar{A}e(t-h(t)) - (1-\dot{h}(t))|e(t-h(t))|^2 = \frac{1}{\big(1-\dot{h}(t)\big)} e(t)^T P^{-1}\bar{A}\bar{A}^T P^{-1}e(t) \\
&-(1-\dot{h}(t))\Big(e(t-h(t)) - \frac{1}{\big(1-\dot{h}(t)\big)}\bar{A}^T P^{-1}e(t)\Big)^T\Big(e(t-h(t)) - \frac{1}{\big(1-\dot{h}(t)\big)}\bar{A}^T P^{-1}e(t)\Big).
\end{aligned}$$

Since $1-\dot{h}(t) \geq \beta > 0$,

$$2e(t)^T P^{-1}\bar{A}e(t-h(t)) - (1-\dot{h}(t))|e(t-h(t))|^2 \leq \frac{1}{\beta} e(t)^T P^{-1}\bar{A}\bar{A}^T P^{-1}e(t). \tag{11}$$

From (9)(10)(11),

$$\begin{aligned}\dot{V}(t,e_t) &\leq e(t)^T\Big(\big(A-LC\big)^T P^{-1} + P^{-1}\big(A-LC\big) + (2+k_g^2)I_n \\ &+ P^{-1}\Big(\tfrac{1}{\beta}\bar{A}\bar{A}^T + (k_f^2+\tfrac{1}{\beta})I_n\Big)P^{-1}\Big)\, e(t).\end{aligned}$$

Then, by (7), with $\gamma_g = 2+k_g^2$ and $\gamma_f = \dfrac{1}{\beta}+k_f^2$,

$$\dot{V}(t,e_t) \leq -e(t)'\left(P^{-1}QP^{-1}\right)e(t).$$

This concludes the proof of Theorem 3.1.

Remark
We first note that the conditions obtained are independant of the delay. We also note that with the concept of robustness introduced in the second section, we can conclude that if the conditions of the Theorem 3.1 are satisfied, then the observer is $\mathcal{N}$-robustly asymptotically convergent.

In the following we state sufficient conditions to ensure the existence of a solution to the Riccati equation (5). To achieve this goal, we use a classical result of $\mathcal{H}_\infty$ theory. For sake of completeness, we recall it here.

Lemma 3.1 (Strict real bounded lemma, (Petersen *et al.*, 1991)).
Consider a continuous-time transfer function of realization $\mathcal{H}(s) = C(sI_n - A)^{-1}B$. The following statements are equivalent :

1. *A is stable and $\|\mathcal{H}(s)\|_\infty = \sup_{w\in\mathbb{R}} \|\mathcal{H}(jw)\|_2 < 1$.*
2. *The Riccati equation*
$$A^TP + PA + PBB^TP + C^TC = 0$$
has a stabilizing solution $P \geq 0$ (i.e. $A+BB^TP$ is stable).
3. *There exists a matrix $\bar{P} > 0$ such that*
$$A^T\bar{P} + \bar{P}A + \bar{P}BB^T\bar{P} + C^TC < 0.$$
Furthermore, if these statements hold then $P < \bar{P}$.

We then have the following result.

Theorem 3.2.
If A is stable and

$$\|\big(sI_n - A^T\big)^{-1}\big(\gamma_g I_n - \frac{1}{\varepsilon}C^TC\big)^{\frac{1}{2}}\|_\infty < \frac{1}{\|\big(\frac{1}{\beta}\bar{A}\bar{A}^T + \gamma_f I_n\big)^{\frac{1}{2}}\|_2} \tag{12}$$

then the equation (5) has a solution.

Proof of Theorem 3.2
Let us denote

$$G(s) = (sI_n - A^T)^{-1}(\gamma_g I_n - \frac{1}{\varepsilon}C^T C)^{\frac{1}{2}} \quad \text{and} \quad \Upsilon = \|G(s)\|_\infty.$$

Form (12), $\Upsilon < \dfrac{1}{\|(\frac{1}{\beta}\bar{A}\bar{A}^T + \gamma_f I_n)^{\frac{1}{2}}\|_2}$. We choose the matrix Q so that

$$\frac{1}{\beta}\bar{A}\bar{A}^T + \gamma_f I_n + Q < \frac{1}{\Upsilon^2} I_n.$$

Therefore, we have

$$\|(\frac{1}{\beta}\bar{A}\bar{A}^T + \gamma_f I_n + Q)^{\frac{1}{2}} G(s)\|_\infty \leq \|(\frac{1}{\beta}\bar{A}\bar{A}^T + \gamma_f I_n + Q)\|_2^{\frac{1}{2}} \; \|G(s)\|_\infty \tag{13}$$

By the choice of Q,

$$\|\frac{1}{\beta}\bar{A}\bar{A}^T + \gamma_f I_n + Q\|_2^{\frac{1}{2}} < \frac{1}{\Upsilon}. \tag{14}$$

From (13) and (14),

$$\|(\frac{1}{\beta}\bar{A}\bar{A}^T + \gamma_f I_n + Q)^{\frac{1}{2}}(sI_n - A^T)^{-1}(\gamma_g I_n - \frac{1}{\varepsilon}C^T C)^{\frac{1}{2}}\|_\infty < 1.$$

Thus, by lemma 3.1, the Riccati equation (5) has a solution.

This completes the proof of Theorem 3.2.

4 Conclusion

In this paper we have presented observer for a class of nonlinear systems with time-varying delay. The analysis of its convergence was obtained form Lyapunov-Krasovskii and $\mathcal{H}_\infty$ theory. Sufficient conditions, independant of the delay, expressed in terms of the existence of a certain Riccati equation and of a frequency domain criterion have been obtained.

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