Proceedings of the 7th Mediterranean Conference on Control and Automation (MED99) Haifa, Israel - June 28-30, 1999

Nonminimum Phase Output Tracking via Sliding Mode Control: Stable System Center Technique

Ilya A. Shkolnikov*
Dept. of Electrical and Computer Eng.
The Univ. of Alabama in Huntsville
Huntsville, AL 35899

Yuri B. Shtessel†
Dept. of Electrical and Computer Eng.
The Univ. of Alabama in Huntsville
Huntsville, AL 35899

Abstract

Nonlinear output tracking in multi-input/multi-output (MIMO) nonminimum phase systems with matched nonlinearities as well as matched and unmatched disturbances is considered in sliding modes. The output tracking problem has been transformed to an equivalent state control problem. The nonminimum phase output tracking problem is solved using an extension of the method of system center for nonminimum phase systems and the dynamic sliding manifold technique. The asymptotic motion of the output tracking error with given eigenvalue placement for noncausal output tracking is provided in absence of unmatched disturbance. Linear bounded error dynamics with desired eigenvalue placement forced by unmatched disturbance and an arbitrary reference output profile are provided for causal output tracking in sliding mode. The theoretical results are illustrated on two numerical examples.

1 Introduction

Many real life control systems have a nonminimum phase nature. It is known that a nonlinear control system is of nonminimum phase if its internal or *zero dynamics* (Isidori , 1995) is unstable. A nonminimum phase nature of a plant restricts application of the powerful nonlinear control techniques such as *feedback linearization control* (Isidori , 1995) and *sliding mode control* (Utkin , 1992), (DeCarlo *et al.* , 1988). In this paper, we approach the approximate and exact output tracking of an arbitrary reference profile (given in real time) with special consern for robustness, the profile with the finite known number of nonzero derivatives under presence of unmatched disturbances of the same property will be asymptotically followed.

Quite a few works that address the problem of nonlinear nonminimum phase output tracking were recently introduced in the literature. The main feature of the techniques proposed in (Isidori and Byrnes , 1990),(Devasia *et al.* , 1996),(Hunt *et al.* , 1996) is that they allow for non-causal inputs (reference profiles known *a priori* or given by a stable exosystem). Another approach investigates the problem for the differentially flat systems (Fliess and Sira-Ramírez , 1998). A number of indirect regulation schemes of the nonminimum phase output with different approximation techniques have been proposed in (Hauser *et al.* , 1992), (Benvenuti *et al.* , 1993), (Azam and Singh , 1994), including a SMC algorithm with output redefinition (Gopalswamy and

*Email: *elia@ece.uah.edu*
†Email: *shtessel@ece.uah.edu*

Hedrick , 1993). This output redefinition leads to a system with stable zero dynamics (minimum phase).

This work employs the system basis transformation technique (Shtessel , 1997), which identifies such "prescribed" profiles for new state variables, so that the state tracking yields the output tracking, and the new system has a regular form (Utkin , 1992), convenient for a SMC design. The reference state profiles are solutions of a system of differential-agebraic equations (DAE) known as the *system center* (Shtessel , 1997). It is proved that if the original system is of a nonminimum phase then the corresponding equations of the system center are unstable. This prevents successful tracking reference state profiles in the sliding mode. The bounded solutions of unstable system of DAE can be obtained using the method of undetermined coefficients (the one that has been employed in (Hunt *et al.* , 1996)) for the case of matched nonlinearities and non-causal reference inputs. As for the arbitrary reference profiles, here it's proposed to employ a linear dynamic extension of a system center to make it stable and provide linear bounded error dynamics with desired eigenvalue placement in the sliding mode. As for unmatched disturbances, they can be compensated for using a SMC with dynamic sliding manifold (DSM) (Sira-Ramírez , 1993),(Shtessel , 1997),(Shtessel , 1998).

This work is an extension of the works (Shtessel , 1997),(Shtessel , 1998) to the MIMO nonminimum phase systems. Concerning the approach developed, there is no output redefinition or solution of unstable differential equation required. Addressing causal nonlinear nonminimum phase output tracking, the dynamic sliding mode controller joints features of a conventional sliding mode controller (insensitivity to matched disturbances and nonlinearities) and a conventional dynamic compensator (accomodation to unmatched disturbances).

2 Problem Formulation

The following plant is considered

$$\dot{x} = Ax + Bu + F(x,t), \;\; y = Gx, \tag{1}$$

where $x \in \Re^n$ (n -dimensional Euclidian space) and $y \in \Re^m$ are state and controlled output vectors respectively, $u \in \Re^m$ is a control. $A, D \in \Re^{n\times n}, \; B \in \Re^{n\times m}, \; G \in \Re^{m\times n}$ are known constant matrices, $rank(B) = m$. $\{A, B\}$ is a controllable pair,$\{A, G\}$ is an observable one, $\forall x, t$: $F(x,t) \in \Re^n$ is a nonlinear time-dependent vector function, so that $F(x,t) = DF_1(x) + F_2(t)$, where $|F_1(x)| < N_1$ is a nonlinear bounded known matched function ($Im(D) \subseteq Im(B)$, (Utkin , 1992)), and $|F_2(t)| < N_2$ is an uncertain but bounded and smooth enough unmatched disturbance; N_1, N_2 are positive constants. The plant is supposed to be a nonminimum phase, meaning that input-output linearization model has unstable internal dynamics (Isidori , 1995).

Given first a known and then an arbitrary output reference profile $\forall t$: $y^*(t) \in \Re^m$, we wish to design a sliding mode controller to achieve a given motion of the output tracking error ($e(t) = y^*(t) - y(t)$) in sliding mode for the system (1). The error dynamics will be in a linear time invariant differential equations format. Given eigenvalue placement must be provided to the error dynamics in sliding mode. Asymptotic non-causal output tracking should be provided in absence of unmatched disturbance. Effect of first k derivatives of an output reference profile $y^*(t)$ given in real time as well as of p derivatives of unmatched disturbance to the output tracking error dynamics should be cancelled out.

3 Transformation of the system basis

The output tracking can be transformed to an equivalent state control problem as the following. Utilising nonsingular transformation (Shtessel , 1997)

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} - \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} x, \tag{2}$$

the nonminimum phase system 1 can be transformed to the form

$$\left\{ \begin{array}{rcl} \dot{z}_1 & = & A_{11}z_1 + A_{12}z_2 - f_1(t) + \dot{x}_1^* - A_{11}x_1^* - A_{12}x_2^*, \\ \dot{z}_2 & = & A_{21}z_1 + A_{22}z_2 - B_2u + \dot{x}_2^* - A_{21}x_1^* - A_{22}x_2^* - D_2F_1(z, x^*) - f_2(t) \\ e & = & G_1z_1 + G_2z_2 + y^* - G_1x_1^* - G_2x_2^*, \; z_1 \in \Re^{n-m}, z_2, e \in \Re^m \end{array} \right. \tag{3}$$

where nonsingular linear transformation (Utkin , 1992)

$$M = \left[\begin{array}{c} M_1 \\ M_2 \end{array} \right] \tag{4}$$

is specified as follows

$$MB = \begin{bmatrix} M_1B \\ M_2B \end{bmatrix} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, detB_2 \neq 0,$$
$$MAM^{-1} = \begin{bmatrix} A_{11} A_{12} \\ A_{21} A_{22} \end{bmatrix}, MD = \begin{bmatrix} 0 \\ D_2 \end{bmatrix}$$
$$GM^{-1} = [G_1 \vdots G_2].$$

If reference state variables profiles x_1^*, x_2^* satisfy the linear system of DAE (Shtessel , 1997), (Brenan *et al.* , 1995)

$$\left\{ \begin{array}{rcl} \dot{x}_1^* & = & A_{11}x_1^* + A_{12}x_2^*, \\ 0 & = & y^* - G_1x_1^* - G_2x_2^*, \end{array} \right. \tag{5}$$

which has been named *the system center* (Shtessel , 1997), then the system 3 will have the form

$$\left\{ \begin{array}{rcl} \dot{z}_1 & = & A_{11}z_1 + A_{12}z_2 - f_1(t), \\ \dot{z}_2 & = & A_{21}z_1 + A_{22}z_2 - B_2u + \dot{x}_2^* - A_{21}x_1^* - A_{22}x_2^* - D_2F_1(z, x^*) - f_2(t) \\ e & = & G_1z_1 + G_2z_2, \end{array} \right. \tag{6}$$

Thus, maintaining asymptotic stability of the system 6 at the origin, we provide asymptotic output tracking the system (1), while stability of the system (6) will give us the bounded output tracking error dynamics. The problem of nonminimum phase output tracking via SMC was solved for a known reference profile and SISO case in (Shtessel , 1998). It was proved that an asymptotic output tracking in the system (1) via conventional sliding mode controller is not possible even in absence of unmatched disturbance because of nonminimum phase nature of a plant. In Section 4, we'll design a conventional SMC for the system (6) and show how the nonminimum phase condition influences the SMC to be designed.

4 Design of the SMC

The following proposition has been proved in the work (Shtessel , 1997)

Proposition 1 *The sliding mode exists in a sliding manifold of the form*

$$\sigma = z_2 + Cz_1 = 0, \;\; \sigma \in \Re^m, \;\; C \in \Re^{m\times(n-m)}, \tag{7}$$

in the system (6), under control law of the form

$$u = \hat{u}_{eq} + B_2^{-1} R \cdot SIGN(\sigma), \tag{8}$$

$$\hat{u}_{eq} = B_2^{-1}((A_{21} + CA_{11})z_1 + (A_{22} + CA_{12})z_2 + \dot{x}_2^* - A_{21}x_1^* - A_{22}x_2^*), \tag{9}$$

where $R = diag\{sign(\rho_i)\}$, $SIGN(\sigma) = [sign(\sigma_1), sign(\sigma_2), \ldots, sign(\sigma_m)]^T$, $i = \overline{1,m}$, $\{x_1^*, x_2^*\}$ *are given by (5) and*

$$\rho_i > \max\left|\sum_{j=1}^{m} b_{ij}\Delta F_j\right|, \;\; i = \overline{1,m}, \Delta F_j = D_2F_1(z, x^*) + f_2(t) - Cf_1(t),$$

where b_{ij} *are elements of matrix* B_2

Corollary 1 *If the conditions of Proposition 1 are met for the system (6) and* $f_1(t) \equiv 0$ *then the output tracking error* $e(t) = y^*(t) - y(t)$ *is described in the sliding mode by the linear time-invariant homogeneous system of differential equations*

$$\left\{\begin{array}{rcl} \dot{z}_1 & = & (A_{11} - A_{12}C)z_1, \\ e & = & (G_1 - G_2C)z_1. \end{array}\right. \tag{10}$$

Here, the matrix C can be chosen to provide given eigenvalue placement to the matrix $A_{11} - A_{12}C$. However, as it will be proved further, if the original system (1) is of nonmimnimum phase then the corresponding system center of the form (5) is unstable. This fact prevents successful tracking in sliding mode, since the expression (9) becomes unbounded. In the next section we investigate properties of a system center of the form (5) of a nonminimum phase system of the form (6).

5 The system center equations, zero dynamics and nonminimum phase condition

Correlation between properties of the system center and the system's internal dynamics will be illustrated on the following example. Given

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & u, \\ y & = & x_1 - \alpha x_2, \end{array}\right.$$

the goal is to provide $\lim\limits_{t\to\infty}(y^*(t) - y(t)) = 0$.

If $\alpha = 0$, then relative degree of initial system equals to its order, i.e. $r = n = 2$. In this case we have no zero(internal) dynamics, and in normal form (Isidori , 1995) this system can be rewritten as

$$\left\{\begin{array}{rcl} \dot{\xi}_1 & = & \dot{\xi}_2, \\ \dot{\xi}_2 & = & u, \\ y & = & \xi_1. \end{array}\right.$$

If $\alpha \neq 0$, then relative degree is $r = 1$, and the system has the following normal form

$$\begin{cases} \dot{\xi}_1 &= -\frac{1}{\alpha}\xi_1 + \frac{1}{\alpha}\eta - \alpha u, \\ \dot{\eta} &= -\frac{1}{\alpha}\xi_1 + \frac{1}{\alpha}\eta, \\ y &= \xi_1. \end{cases}$$

The second equation in this system represents *internal dynamics* of initial system, and in case $\xi_1 \to 0$ we obtain the following zero dynamics

$$\dot{\eta} = \frac{1}{\alpha}\eta.$$

If $\alpha < 0$, then we have stable zero dynamics . This is a minimum phase case.

If $\alpha > 0$, then we have unstable zero dynamics and nonminimum phase case.

At this point, let's consider the system center equations for initial system and apply the same analysis. Applying nonlinear transformation described in Section 3, we obtain the following system of DAE for reference state variable profiles

$$\begin{cases} \dot{x}_1^* &= x_2^*, \\ y^* &= x_1^* - \alpha x_2^*. \end{cases}$$

If $\alpha = 0$, then $x_1^* = y^*,\ x_2^* = \dot{y}^*$, and we have only the set of algebraic expressions to obtain $\{x_1^*, x_2^*\}$.

If $\alpha \neq 0$, then the system of DAE can be solved as follows

$$\begin{cases} \dot{x}_1^* &= \frac{1}{\alpha}x_1^* - \frac{1}{\alpha}y^*, \\ x_2^* &= \frac{1}{\alpha}x_1^* - \frac{1}{\alpha}y^*. \end{cases}$$

If $\alpha < 0$, then we have stable system center, and we can obtain bounded profiles for $\{x_1^*, x_2^*\}$.

If $\alpha > 0$, then we have unstable system center, and for causal reference output profile $y^*(t)$ we'll have unbounded solutions for $\{x_1^*, x_2^*\}$. It leads to unbounded control (8),(9), which in this case is obtained as $u = c(x_2^* - x_2) + \dot{x}_2^* + \rho \cdot sign(\sigma),\ \sigma = (x_2^* - x_2) + c(x_1^* - x_1),\ \rho, c \in \Re^1$.

Further, we'll have stated and proved the following generalizations, which we'll use later on solving the system center equations.

Theorem 1 *If the system (1) is observable, then a solution of the system (5) of DAE exists.*

Proof: See the Appendix. □

Theorem 2 *If the system (1) has total relative degree r, $r \leq n$, then a system center of the form (5) can be expressed by the system of $(n - r)$ differential and r algebraic equations.*

Proof: See the Appendix. □

Futher in the text, we will imply under the term "stability" of a system its bounded-input-bounded-output behavior

Theorem 3 *The original system (1) is of nonminimum phase, if and only if the system center (5) is unstable.*

Proof: See the Appendix. □

So, one can conclude that for any controllable and observable nonminimum phase system of the form (1), being transfomed to the form (6), a solvable but unstable system of DAE is obtained in the form (5) for the state reference profiles to be tracked. The next task is to identify the bounded state reference profiles such that it could provide a successful output tracking. Two approaches to obtaining stable solutions of the system center are presented below. The first approach is to find bounded soluiton of the system center given *a priori* output reference profile. The second is based on dynamic extention of the system center in order to achieve its stability.

6 Bounded solutions of the system center via method of undetermined coefficients

Looking for a bounded solution of unstable equations of the system center (5), assume $|G_2| \neq 0$, then we a solution of the system of DAE (5) is identified as follows

$$\left\{ \begin{array}{lcl} \dot{x}_1^* & = & (A_{11} - A_{12}G_2^{-1}G_1)x_1^* + A_{12}G_2^{-1}y^*, \\ x_2^* & = & G_2^{-1}(y^* - G_1x_1^*). \end{array} \right. \tag{11}$$

Remark In a singular case when $|G_2| = 0$, we can transform either the system (1) or the system center (5) to an appropriate form use one of the following approaches

a) choose another matrix M in (4), and obtain the nonsingular matrix G_2; if possible;

b) apply the transformation technique by (Brenan *et al.* , 1995) to "reduce" the system (5) to pure differential form;

c) transform the initial system (1) to the *normal form* (Isidori , 1995) $\{x\} \xrightarrow{\Phi} \{\xi,\ \eta\}$, $x \in \Re^n,\ \{\xi,\eta\} \in \Re^n$, $\xi = (\xi^1,\ldots,\xi^m) \in \Re^r,\ \xi^i = (\xi_1^i,\ldots,\xi_{r_i}^i),\ \ r_i + r_2 + \ldots + r_n = r$, $r < n,\ i = \overline{1,m}$, $\eta \in \Re^{n-r}$, where r is known to be the *total relative degree* (Isidori , 1995) of the system (1).

Then in new basis $\{\xi,\ \eta\}$ the system center can be rewritten as

$$\left\{ \begin{array}{lcl} y_i^* & = & \xi_1^{*i}, \\ \dot{y}_i^* & = & \xi_2^{*i}, \\ & \vdots & \\ y^{*(r_i-1)} & = & \xi_{r_i}^{*i},\ i = \overline{1,m}, \\ \dot{\eta}^* & = & Q_2\eta^* + Q_1\xi^*. \end{array} \right. \tag{12}$$

For initial nonminimum phase system (1) the matrix $(A_{11} - A_{12}G_2^{-1}G_1)$ in (11) or Q_2 in (12) are non-Hurwitz as it's followed from Theorem 2. However, as stated above, we're to get bounded state reference profiles $\{x_1^*, x_2^*\}$ or $\{\xi^*, \eta^*\}$. The key idea in designing a bounded solution is to find a bounded particular solution of the unstable system (11) or (12), when a bounded output reference profile $y^*(t)$ given on time interval $[0,T]$ can be periodically extended and approximated with given accuracy by a finite Fourier series. Let we have known $y^*(t)$ for $[0,T]$ as

$$y_i^*(t) = \sum_{k=1}^{N} \alpha_{ik} \sin(\omega_k t) + \sum_{k=1}^{N} \beta_{ik} \cos(\omega_k t), \; i = \overline{1,m}. \tag{13}$$

For $y^*(t)$ in the form (13) we can use the method of undetermined coefficients to derive the bounded particular solution of linear nonhomogeneous differential part of the system (11) or (12), which we can present by general form

$$\dot{x}_1^* = A^* x_1^* + D^* y^*. \tag{14}$$

In case, when the roots(eigenvalues) of $|A^* - \lambda I| = 0$ don't contain $\pm j\omega_k$ (so called *non-resonance case*), we can apply the following algorithm.

Step 1 Construct a bounded particular solution of system (14) in the form

$$x_{i1}^* = \sum_{k=1}^{r} \alpha_{ik}^1 \sin(\omega_k t) + \sum_{k=1}^{r} \beta_{ik}^1 \cos(\omega_k t), \; i = \overline{1, n-m}. \tag{15}$$

Step 2 Substitute form (15) for x_1^* into (14) and solve for the constraints in x_1^* by equating terms. Denote $(n-m) \times r$ matrices to be determined as $[\alpha^1]$ and $[\beta^1]$; $diag[\omega]$ is a diagonal matrix of $(\omega_1, \ldots \omega_r)$, and $[\alpha]$, $[\beta]$ are known $m \times r$ matrices from (13). After equating the terms we'll have the following system of linear algebraic equations

$$\left\{ \begin{array}{rcl} [\alpha^1] diag[\omega] & = & A^*[\beta^1] + D^*[\beta] \\ -[\beta^1] diag[\omega] & = & A^*[\alpha^1] + D^*[\alpha] \end{array} \right. \tag{16}$$

According to Theorem 1, solution of the system (16) exists. After calculating x_1^* in the form (15), we need to substitute it into the second equation in (11) to obtain the bounded soluiton for x_2^*. In case when the system center has been determined by (12), we'll have the analogous procedure. To avoid a resonace case, when the roots of characteristic polynomial for the system (14) contain $\pm j\omega_k$, we need to choose another transformation basis (4), as known (Utkin , 1992), it's not unique. An example of application of the algorithm designed will be presented in Section 9.

7 Stable system center design via dynamic extension

The approach presented in Section 6 is developed for noncausal nonminimum phase output tracking. In this section we introduce a modification of the system center technique for causal nonminimum phase output tracking in sliding modes.The key idea is to use a dynamic extension of the system center that achieves a stable system center as well as provides the output tracking error dynamics with given eigenvalue placement in the sliding manifold (7). The concept will be demonstrated on the following example. Given

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_1 + x_2, \\ \dot{x}_2 & = & -x_1 + x_2 + \sin(0.3 x_2) + u + f_2(t), \\ y & = & x_2, \end{array} \right. \tag{17}$$

we wish to provide causal nonminimum phase output tracking the reference profile y^* of ramp type ($\ddot{y}^*(t) = 0$, almost everywhere) given in real time in presence of matched bounded disturbance $f_2(t)$. Applying the transformation (2) with identity matrix M to the system (17), one can obtain equations of the system center (5) as the following

$$\left\{ \begin{array}{rcl} \dot{x}_1^* & = & x_1^* + x_2^*, \\ y^* & = & x_2^*, \end{array} \right.$$

which are unstable. Dynamically extending them to the format

$$\begin{cases} \dot{x}_1^* = x_1^* + x_2^*, \\ y^* = x_2^*, \\ -2.4\dot{g}^* - g^* = 3.4\ddot{x}_1^*, \end{cases} \tag{18}$$

which is rewritten as follows

$$\begin{cases} \ddot{x}_1^* + 1.4\dot{x}_1^* + x_1^* &= -(2.4\dot{y}^* + y^*), \\ x_2^* &= y^*, \end{cases} \tag{19}$$

we obtain obviously stable profiles $\{x_1^*, x_2^*\}$. Given equations of the system center (18), the system (17) is transformed to the following format

$$\begin{cases} \dot{z}_1 &= z_1 + z_2 - g^*(t), \\ \dot{z}_2 &= -z_1 + z_2 - u - sin(0.3(x_2^* - z_2)) - f_2(t) + \dot{x}_2^* + x_1^* - x_2^*, \\ e &= z_2. \end{cases} \tag{20}$$

Equivalent control that provides the system (20) motion in the sliding manifold

$$\sigma = z_2 + 3z_1 = 0, \tag{21}$$

is derived as $u_{eq} = 2z_1 + 4z_2 - \sin(0.3(x_2^* - z_2)) + \dot{x}_2^* + x_1^* - x_2^* - f_2(t)$ and is obviously bounded given bounded solution of the system center (19) and bounded $f_2(t)$. So, the control law of the form (7), (8), that provides existence of the sliding mode to the system (20) is realizable. The following output tracking error dynamics of the system (20) in the sliding manifold (21) are obtained on the basis of equations (18), (20)

$$\begin{cases} \dot{e} = -2e + 3g^*, \\ \ddot{g}_1^* + 1.4\dot{g}_1^* + g_1^* = 0, g^*(0) = y^*(0),\ \dot{g}^*(0) = \dot{y}^*(0). \end{cases}$$

Thus, for the system (17) and $y^*(t)$ of a ramp type we've got asymptotic error convergence to zero with given eigenvalue placement. (Here, we select them for g^* decay to be $\lambda = \{-0.7 \pm 0.712j\}$), and for error e to be $\lambda = -2$.) The results observed in the concept-demonstration example are generalized for the system (1) with matched disturbance into the following theorem.

Theorem 4 *For the nonminimum phase system (6) with $|G_2| \neq 0$ and $f_1 \equiv 0$, $\exists$ matrices $T_{k-1}, T_{k-2}, \ldots, T_0 \in \Re^{(n-m)\times(n-m)}$, such that*

1) the state tracking reference profiles $\{x_1^*, x_2^*\}$ are described by the following stable equations of the system center

$$\begin{cases} \dot{x}_1^* = (A_{11} - A_{12}G_2^{-1}G_1)x_1^* + A_{12}G_2^{-1}y^* - g^*, \\ T_{k-1}g^{*(k-1)} + T_{k-2}g^{*(k-2)} + \ldots + T_1\dot{g}^* + T_0g^* = x_1^{*(k)}, \\ x_2^* = G_2^{-1}(y^* - G_1x_1^*); \end{cases} \tag{22}$$

2) any output reference profile $y^*(t)$ given in real time with zero high derivatives ($y^{*(i)} \equiv 0,\ i \geq k$) will be asymptotically followed in sliding mode by the SMC (7),(8), in accordance with the following linear system of differential equations

$$\begin{cases} \dot{z}_1 &= (A_{11} - A_{12}C)z_1 - g^*, \\ e &= (G_1 - G_2C)z_1. \end{cases} \tag{23}$$

where the matrix C is chosen to provide given eigenvalue placement and $\lim_{t\to\infty} g^*(t) = 0$, ($g^*$ asymptotically converges to zero with given eigenvalue placement as well).

Proof: See the Appendix. □

Remark In case, when total relative degree of the system (1) is greater than m and $|G_2| \equiv 0$, according to Theorem 2, the conventional system center can be expressed by $(n-r)$ differential and r algebraic equations. In this case, one can build an analogous modification of the system center (12) to that of given by Theorem 4.

8 Compensation for unmatched disturbance via DSM

Theorem 4 gives tools for SMC design providing asymptotic nonminimum phase output tracking in sliding mode in absence of unmathced disturbance $f_1(t)$. In order to compensate for unmatched disturbance in sliding mode, a dynamic sliding manifold is designed in this section and tailored to the dynamic extention of system center.

Given equations (22) the system (1) is presented in a new basis (2) as follows:

$$\left\{ \begin{array}{rcl} \dot{z}_1 & = & A_{11}z_1 + A_{12}z_2 - g^* - f_1(t), \\ \dot{z}_2 & = & A_{21}z_1 + A_{22}z_2 - B_2 u + \dot{x}_2^* - A_{21}x_1^* - A_{22}x_2^* - D_2 F_1(z, x^*) - f_2(t) \\ e & = & G_1 z_1 + G_2 z_2, \end{array} \right. \tag{24}$$

and, according to Theorem 4, the control (7),(8) can provide asymptotic convergence e to zero only in absence of unmatched disturbance $f_1(t)$. In the work (Shtessel , 1997), it has been shown that a dynamic siliding manifold (DSM) (Sira-Ramírez , 1993), (Shtessel , 1997), (Shtessel , 1998) can be used instead of conventional sliding manifold (7) to compensate for unmatched disturbance $f_1(t)$ in a system of the form (6) in sliding mode.

Similar to that of work (Shtessel , 1997) we introduce the DSM for the system (24) as follows

$$\left\{ \begin{array}{l} \sigma = z_2 - C\sigma_1 = 0,\ z_2 \in \Re^m,\ z_1, \sigma_1 \in \Re^{n-m},\ C \in \Re^{m\times(n-m)} \\ \sigma_1^{(k)} + P_{k-1}\sigma_1^{(k-1)} + \ldots + P_1\dot{\sigma}_1 + P_0\sigma_1 = Q_k z_1^{(k)} + Q_{k-1}z_1^{(k-1)} + \ldots + Q_1\dot{z}_1 + Q_0 z_1, \\ P_{k-1}, \ldots, P_0 \in \Re^{(n-m)\times(n-m)},\ Q_k, \ldots, Q_0 \in \Re^{(n-m)\times(n-m)}. \end{array} \right. \tag{25}$$

So, instead of reducing the order of the system (24) motion in sliding, we expand it to compensate for unmatched disturbance $f_1(t)$ with zero high derivatives $f_1^{(i)} \equiv 0,\ i \geq k$ and provide the desirable tracking error dynamics. The following theorem will give us a poof to existence of sliding mode and compensation for unmatched disturbance.

Theorem 5 *For the system (24) with $|G_2| \neq 0,\ |A_{11}| \neq 0$ and G_1 of full rank, exists matrices $P_{k-1}, \ldots, P_0 \in \Re^{(n-m)\times(n-m)},\ Q_k, \ldots, Q_0 \in \Re^{(n-m)\times(n-m)}$, such that*

1) the sliding mode exists for the system (24) in a DSM of the form (25) under bounded control law

$$\begin{array}{rcl} u & = & \hat{u}_{eq} + B_2^{-1}R \cdot SIGN(\sigma), \\ \hat{u}_{eq} & = & B_2^{-1}(-C\dot{\sigma}_1 + A_{21}z_1 + A_{22}z_2 + \dot{x}_2^* - A_{21}x_1^* - A_{22}x_2^*), \end{array} \tag{26}$$

$$\rho_i > \max\left|\sum_{j=1}^{m} b_{ij}\Delta F_j\right|,\ i = \overline{1,m}, \Delta F_j = D_2 F_1(z, x^*) + f_2(t),$$

where $R = diag\{sign(\rho_i)\},\ SIGN(\sigma) = [sign(\sigma_1), sign(\sigma_2), \ldots, sign(\sigma_m)]^T,\ i = \overline{1,m},\ b_{ij}$ are elements of matrix B_2.

2) the corresponding output error dynamics are not effected by unmatched bounded disturbance $f_1(t)$ $(f_1^{(i)}(t) \equiv 0,\ i \geq k)$ and is asymptotically stable with given eigenvalue placement.

Proof: See the Appendix. $\square$

Remark It's possible to extend the results of *Theorem 5* for the cases $|A_{11}| = 0,\ \mathrm{rank}(G_1) < n-m$ or $|G_2| = 0$, and prove existence of the sliding mode in a DSM of the form (25) able to reject the error from any unmatched disturbance $f_1(t)$ with zero high derivatives $f_1^{(i)}(t) \equiv 0,\ i \geq k$.

The algorithm of causal nonminimum phase output tracking for a system of the form (1) using transformation (2), (22) and control law (27) is demonstrated on the second example in *Section 9*.

9 Examples

Example 1: Let's consider again the example of *Section 5* with matched disturbance term

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= u + f_2(t),\ \forall t : |f_2(t)| \leq 1, \\ y &= x_1 - x_2. \end{cases}$$

The goal is to track output reference profile given ahead of time $y(t) \to y^*(t) = \sin(t)$. Applying method of undetermined coefficients, we find a stable particular solution of the system center

$$\begin{cases} x_1^* &= 0.5(sin(t) + cos(t)), \\ x_2^* &= 0.5(-sin(t) + cos(t)). \end{cases}$$

and a control $u(x,t) = u_{eq} + \rho sign(\sigma)$, $\rho = 2$, $u_{eq} = 4(x_2^* - x_2) + \dot{x}_2^*$, $\sigma = (x_2^* - x_2) + 4(x_1^* - x_1)$.

The results of the simulation with the SMC designed are shown in Figs. 1-4.

Example 2 Let's consider the following MIMO tutorial example of the nonminimum phase system with arbitrary reference output profiles

$$\begin{cases} \dot{x}_1 &= x_1 + x_2 + x_3 + f_1(t), \\ \dot{x}_2 &= -x_1 + x_2 + \sin(3x_2) + u_1 + f_2(t), \\ \dot{x}_3 &= x_2 - 2x_3 + \cos(3x_3) + u_2 + f_2(t). \end{cases} \quad \begin{cases} y_1 &= -x_1 + x_2, \\ y_2 &= x_2, \end{cases}$$

This is $n = 3$ dimentional system with $m = 2$ inputs and outputs, with total relative degree $r = 2$ and unstable zero dynamics $\dot{x}_1 = 2x_1$. We wish to achieve asymptotic output tracking reference profiles (y_1^*, y_2^*) with zero high derivatives beginning with $k = 3$ in presence of bounded matched disturbance $f_2(t)$ and constant unmatched disturbance $f_1(t) = N_1$. Applying the transformation (2) with identity matrix M in (4) to this system, select the system center equations as follows

$$\begin{cases} \dot{x}_1^* &= 2x_1^* + y_1^* + y_2^* - g^*, \\ x_2^* &= y_1^* + x_1^*, \\ x_3^* &= y_2^*, \end{cases}$$

with dynamic extension $-6.5\ddot{g}^* - 7\dot{g}^* - 3g^* = 7.5x_1^{*(3)}$. Transforming equations of the system center into the form

$$\begin{cases} x_1^{*(3)} + 6\ddot{x}_1^* + 11\dot{x}_1^* + 6x_1^* = -\left(6.5(\ddot{y}_1^* + \ddot{y}_2^*) + 7(\dot{y}_1^* + \dot{y}_2^*) + 3(y_1^* + y_2^*)\right), \\ x_2^* = y_1^* + x_1^*, \\ x_3^* = y_2^*, \end{cases}$$

for $y^{*(k)} = \begin{bmatrix} y_1^{*(k)} \\ y_2^{*(k)} \end{bmatrix} \equiv 0, \forall k \geq 3$, we obtain obviously bounded solutions $\{x_1^*, x_2^*, x_3^*\}$.
Selecting a DSM in the form

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{cases} \dot{\sigma}_{11} - 0.5\sigma_{11} & = & 0.95\dot{z}_1 + 0.5z_1, \\ \dot{\sigma}_{12} - 0.5\sigma_{12} & = & 1.95\dot{z}_1, \end{cases}$$

we obtain equations of error dynamics in sliding mode

$$\begin{cases} \ddot{e}_1 + 1.4\dot{e}_1 + e_1 & = & 1.95(\dot{g}^* + \dot{f}_1(t)), \quad g^{*(3)} + 6\ddot{g}^* + 11\dot{g}^* + 6g^* = 0, \\ \ddot{e}_2 + 1.4\dot{e}_2 + e_2 & = & 1.95(\dot{g}^* + \dot{f}_1(t)), \quad g^{*(i)}(0) = y_1^{*(i)}(0) + y_2^{*(i)}(0), i = 0, 1, 2; \end{cases}$$

where $e_1 = -z_1 + z_2,\ e_2 = z_3$ asymptotically converge to zero with eigenvalues $\lambda = \{-0.7 \pm 0.712j\}$ for both errors and $\lambda^* = \{-1, -2, -3\}$ for g^* , as long as $f_1(t)$ is unknown constant. The results of simulations are given in Figs. 5-9 with the control law $u = 5 \cdot SIGN(\sigma)$.

10 Concslusions

Nonlinear nonminimum phase MIMO output tracking problem is addressed via sliding mode control. A sliding mode controller has been designed to provide robust tracking to the nonminimum phase system with matched uncertain nonlinear terms as well as matched and unmatched external disturbances using the method of system center and the dynamic sliding manifold technique. Such a controller is shown to be insensitive to matched disturbances and uncertain nonlinearities, and allows to cancel out effect to the output tracking error from an arbitrary reference input and unmatched disturbance with finite number of nonzero derivatives. Future research will extend the developed approach and the dynamic sliding manifold technique to the nonlinear nonminimum phase output tracking case with unmatched nonlinearities.

APPENDIX

A: *Proof of Theorem 1*

For the system to be solved in the form

$$\begin{bmatrix} I_{(n-m)\times(n-m)} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1^* \\ \dot{x}_2^* \end{bmatrix} = \begin{bmatrix} A_{11} A_{12} \\ G_1\ G_2 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} + \begin{bmatrix} 0 \\ -y^*(t) \end{bmatrix}.$$

one can apply the following nonsingular coordinate transformations

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \end{bmatrix},$$

$$\begin{bmatrix} v_1^1 \\ v_2^1 \end{bmatrix} = A \begin{bmatrix} v_1^2 \\ v_2^2 \end{bmatrix},$$

$$\vdots$$

$$\begin{bmatrix} v_1^{n-1} \\ v_2^{n-1} \end{bmatrix} = A^{n-1} \begin{bmatrix} v_1^n \\ v_2^n \end{bmatrix}.$$

After first transformation we'll have

$$\begin{bmatrix} M_1 \\ 0 \end{bmatrix} \begin{bmatrix} \dot{v}_1^1 \\ \dot{v}_2^1 \end{bmatrix} - \begin{bmatrix} M_1 A \\ G \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \end{bmatrix} = \begin{bmatrix} 0 \\ -y^*(t) \end{bmatrix},$$

and after first differentiation

$$\begin{bmatrix} M_1 \\ G \end{bmatrix} \begin{bmatrix} \dot{v}_1^1 \\ \dot{v}_2^1 \end{bmatrix} - \begin{bmatrix} M_1 A \\ 0 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\dot{y}^*(t) \end{bmatrix}.$$

Applying to the last system of differential equations all other transformations consequently, we'll have got the system of n^2 differential equations of general type

$$S\dot{v} - Pv = f,$$

where S will have the form

$$\begin{bmatrix} M_1 \\ G \\ M_1 A \\ GA \\ \vdots \\ M_1 A^{n-1} \\ GA^{n-1} \end{bmatrix}$$

and $rank(S) = n$, since (A, G) is an observable pair. Therefore, there exists pseudoinverse transformation $S^+ = [S^T S]^{-1} S^T$, such that

$$\dot{v} = S^+ Pv + S^+ f.$$

Thus, we "reduce" initial system of DAE to solvable system of n^2 differential equations. $\square$

B: *Proof of Theorem 2*

First of all, let's prove that if $r = n$, than the DAE can be transformed to pure algebraic form. For the initial system in the form

$$\begin{cases} \dot{x} &= Ax + \sum_{j=1}^{m} b_j u_j \\ y_1 &= g_1 x \\ &\vdots \\ y_m &= g_m x \end{cases}$$

where b_j is j-column of $B \in \Re^{n\times m}$, and g_i is i-row of $G \in \Re^{m\times n}$, we apply a coordinate transformation (Isidori , 1995)

$$\xi = \Phi x,\ \Phi = \begin{vmatrix} g_1 \\ g_1 A \\ \vdots \\ g_1 A^{r_1 - 1} \\ \vdots \\ g_m \\ \vdots \\ g_m A^{r_m - 1} \end{vmatrix}. \tag{27}$$

In new basis $\xi = (\xi^1, \ldots, \xi^m),\ \xi^i = (\xi_1^i, \ldots, \xi_{r_i}^i)$,
$r_i + r_2 + \ldots + r_n = n,\ i = \overline{1,m}$ the system will has the form

$$\left\{ \begin{array}{rcl} \dot{\xi}_1^i & = & \xi_2^i \\ & \vdots & \\ \dot{\xi}_{r_i-1}^i & = & \xi_{r_i}^i \\ \dot{\xi}_{r_i}^i & = & \bar{b}_i\xi + \sum_{j-1}^m \bar{a}_{ij}u_j \end{array} \right. ,i = \overline{1,m},$$

$$\bar{A} = \left| \begin{array}{ccc} g_1 A^{r_1-1}b_1 & \ldots & g_1 A^{r_1-1}b_m \\ \vdots & & \vdots \\ g_m A^{r_m-1}b_1 & \ldots & g_m A^{r_m-1}b_m \end{array} \right|$$

$$\bar{b}_i = g_i \Phi^{-1} A^{r_i},\ i = \overline{1,m}$$

and after feedback transformation $u = \bar{A}^{-1}[-\bar{B}\xi + v]$ we'll have our system in Brunowsky canonical form

$$\begin{array}{rcl} \dot{\xi} & = & A'\xi + B'v \\ y_i & = & \xi_1^i,\ i = \overline{1,m} \end{array} ,$$

where $A' = diag\{A_1, \ldots, A_m\}$,
$B' = diag\{b_1, \ldots, b_m\},\ A_i \in \Re^{r_i \times r_i},\ b_i \in \Re^{r_i \times 1}$

$$A_i = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & 1\ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \end{bmatrix},$$

$$b_i = col(0, \ldots, 0, 1).$$

Finally, we need to permute the rows of the last system, to obtain the form presented in *Section 3*. At last, applying the transformation of *Section 3*, we'll have the system center in the form of n algebraic expressions

$$\left\{ \begin{array}{rcl} y_i^* & = & \xi_1^{*i} \\ y^{*(k)} & = & \xi_{k+1}^{*i},\ k = \overline{1, r_i - 1} \end{array} \right. .$$

In case when $r_1 + r_2 + \ldots + r_m = r < n$, after the above transformations, we'll have only r algebraic expressions for r reference state variable profiles, while another set of $(n-r)$ state variables (and corresponding reference profiles) can be obtained in the form

$$\dot{\eta} = Q_1\xi + Q_2\eta,$$

where matrix $Q = [Q_1 Q_2],\ Q \in \Re^{(n-r)\times n}$ can be chosen such that a transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \Phi' x,\ \Phi' = \left| \begin{array}{c} \Phi_1 \\ \Phi_2 \end{array} \right|,\ Q = \Phi_2 A \Phi'^{-1} \tag{28}$$

is nonsingular, and $q_i b_j = 0,\ \forall i = \overline{1,(n-r)},\ j = \overline{1,m}$, q_i is i-row of matrix Q (Isidori , 1995). Here, matrix Φ_1 has the form (27). Thus, we'll have the following $(n-r)$ differential equations for leftover part of reference state variable profiles

$$\dot{\eta}^* = Q_2\eta^* + Q_1\xi^*,\ \eta^* \in \Re^{(n-r)}, \xi^* \in \Re^r. \square$$

C: *Proof of Theorem 3*

As we can easily see from the proof of *Theorem 2* that, after the transformation applied, the zero dynamics of initial system ($\xi \to 0$) has the form $\dot{\eta} = Q_2\eta$, and the differential part of the system center has the form $\dot{\eta}^* = Q_2\eta^* + Q_1\xi^*$. If this is the nonminimum phase case, we have unstable zero dynamics, and, hence, unstable system center and vice versa. $\square$

D: *Proof of Theorem 4*

First. Let's prove the fact that the system center (22) can be chosen to be stable and g^* will be asymptotically tend to zero with given eigenvalue placement under the condition that $y^{*(i)} \equiv 0, i \geq k$. Let's rewrite the first equation in (22) as

$$g^* = -\dot{x}_1^* + S_1 x_1^* + S_2 y^*, \tag{29}$$

where we denote $S_1 = A_{11} - A_{12}G_2^{-1}G_1$, and $S_2 = a_{12}G_2^{-1}$. Now, let's differentiate (29) $(k-1)$ times, then we obtain

$$\left\{ \begin{array}{rcl} \dot{g}^* & = & -\ddot{x}_1^* + S_1\dot{x}_1^* + S_2\dot{y}^*, \\ & \vdots & \\ g^{*(k-1)} & = & -x_1^{*(k)} + S_1 x_1^{*(k-1)} + S_2 y^{*(k-1)}, \end{array} \right. \tag{30}$$

and substituting (29),(30) into the second equation in (22) we'll have

$$(T_{k-1}+I)x_1^{*(k)} + (T_{k-2}-T_{k-1}S_1)x_1^{*(k-1)} + \ldots + (T_0 - T_1S_1)\dot{x}_1^* - T_0S_1x_1^* = -T_{k-1}S_2y^{*(k-1)} - \ldots - T_0S_2y^*. \tag{31}$$

It's obvious that selecting a set of matrices $T_0, T_1, \ldots, T_{k-1}$, the solution of homogeneous part in (31) can be made asymptotically stable. Consequently a forced response will be bounded bounded input $y^{*(i)}(t), i = \overline{0, k-1}$. According to the third equation in (22) we obtain also bounded x_2^* given bounded (x_1^*, y^*).

Second. Differentiating (29) k times and the second equation in (22) once and combining results we'll have

$$(T_{k-1}+I)g^{*(k)} + (T_{k-2} - T_{k-1}S_1)g^{*(k-1)} + \ldots + (T_0 - T_1S_1)\dot{g}^* - T_0S_1g^* = S_2y^{*(k)} \equiv 0$$

hence, selecting $T_0, T_1, \ldots, T_{k-1}$ we provide asymptotic convergence g^* to zero with given eigenvalue placement.

Third. We've got the stable system center (22) and the bounded control $\hat{u}_{eq}$ of the form (9). Now, we're to prove existence of the sliding mode in the sliding manifold (7). As we can observe, the system (1) after the new transformation (2),(22) has the form (24). The forms (3),(5) and (22),(24) is similar with respect to control synthesis and conditions described in *Proposition 1*. So, the sliding mode will exist to the system (24) in the sliding manifold (7) under control law (8). System (24) motion in sliding manifold will have the form

$$\left\{ \begin{array}{rcl} \dot{z}_1 & = & (A_{11} - A_{12}C)z_1 - g^*, \\ e & = & (G_1 - G_2C)z_1. \end{array} \right. \tag{32}$$

Selecting matrix C, we can provide any eigenvalue placement for the error dynamics (32). Since $g^*(t) \to 0,\ t \to \infty$, we'll have asymptotic error convergence to zero. Thus, any bounded output reference profile $y^*(t)$ with bounded $(k-1)$ derivatives and zero high derivatives will be asymptotically followed in sliding mode under bounded control law. $\square$

E: *Proof of Theorem 5*

First. Let's prove bounded behaviour of the system (24) in sliding mode. According to (24),(25), the system (24) motion in sliding mode will be described as follows

$$\begin{cases} \dot{z}_1 = A_{11}z_1 + A_{12}C\sigma_1 - f_1 - g^*, \\ \sigma_1^{(k)} + P_{k-1}\sigma_1^{(k-1)} + \ldots + P_1\dot{\sigma}_1 + P_0\sigma_1 = Q_k z_1^{(k)} + Q_{k-1}z_1^{(k-1)} + \ldots + Q_1\dot{z}_1 + Q_0 z_1, \\ e = G_1 z_1 + G_2 C\sigma_1. \end{cases} \quad (33)$$

Applying superposition principle to the linear system (33), without loss of generality we can consider the system (33) responce to the input f_1 only, excluding $g6*$, moreover g^* approaches zero with given eigenvalue placement. We introduce the following sets of matrices

$$P_k^1 = Q_k,\ P_{k-1}^1 = P_k^1 A_{11} + Q_{k-1}, \ldots, P_1^1 = P_2^1 A_{11} + Q_1,\ P_{k-1}^{"} = P_{k-1} - P_k^1 A_{12}C, \ldots, P_0^{"} = P_0 P_1^1 A_{12}C, \quad (34)$$

to meet the following equality

$$P_1^1 A_{11} + Q_0 = 0. \quad (35)$$

Substituting equations (34) and (35) into the second equation in (33)one can obtain

$$\sigma_1^{(k)} + P_{k-1}^{"}\sigma_1^{(k-1)} + \ldots + P_1^{"}\dot{\sigma}_1 + P_0^{"}\sigma_1 = -P_k^1 f_1^{(k-1)} - P_{k-1}^1 f_1^{(k-2)} - \ldots - P_1^1\dot{f}_1 + P_1^1 f_1. \quad (36)$$

Matrices $P_{k-1}, \ldots, P_0$ can be selected to make the homogeneous part of (36) to be asymptotically stable. Matrices $Q_k, \ldots, Q_1$ can be selected to make asymptotically stable equation

$$Q_k z_1^{(k)} + Q_{k-1}z_1^{(k-1)} + \ldots + Q_1\dot{z}_1 + Q_0 z_1 = 0.$$

Then given bounded $f_1^{(i)}(t),\ i = \overline{0, k-1}$, the system (33) is obviously stable.

Second. Let's prove existence of the sliding mode on on the surface (25) under bounded control law. For this prupose we choose a Lyapunov's function candidate as $V = \frac{1}{2}\sigma^T\sigma > 0$, then its derivative $\dot{V} = \sigma^T\dot{\sigma}$ is identified on the basis of equations (24-27) as follows

$$\dot{V} = \sigma^T(-R \cdot SIGN(\sigma) + \Delta F),\ \text{or}\dot{V} = -\sum_{k=1}^{m}\rho_k|\sigma_k| + \sigma^T\Delta F.$$

The control law makes $\dot{V} < 0$, if $\rho_i > \max\left|\sum_{j=1}^{m} b_{ij}\Delta F_j\right|$, $\Delta F = D_2F_1(z, x^*) + f_2(t)$. This control law will be realizible (bounded) if $\hat{u}_{eq}$ is realizable, i.e. $\dot{\sigma}_1$ is to be bounded. This is true, because the system (33) has been set to be stable, so a sliding mode exists on the surface (25) under control law (27).

Third. We'll prove that effect of first k derivatives of unmatched disturbance to the output error dynamics will be cancelled out. Since $\forall t : f_1(t) \in \Re^n - m$, let's introduce $e_1 \in \Re^{n-m},\ e_1 = [G_2C]^+e$ (where existence of the pseudoinverse matrix $[G_2C]^+$ will be proved later). From (33) we can obtain

$$e_1 = S_0 z_1 + \sigma_1, S_0 = [G_2C]^+, \quad (37)$$

$$\dot{z}_1 = S_1 z_1 + S_2 e_1, S_1 = A_{11} - A_{12}C[G_2C]^+G_1, S_2 = A_{12}C. \quad (38)$$

As far as $f_1^{(k)} \equiv 0$, from (33),(38) we can derive

$$z_1^{(k+1)} = S_1 z_1^{(k)} + S_2 e_1^{(k)},\ z_1^{(k+1)} = A_{11}z_1^{(k)} + S_2\sigma_1^{(k)}. \quad (39)$$

We should provide for $z_1^{(k+1)} - S_1 z_1^{(k)}$ to be

$$z_1^{(k+1)} - S_1 z_1^{(k)} = T_k e_1^{(k)} + T_{k-1} e_1^{(k-1)} + \ldots + T_1 \dot{e}_1 + T_0 e_1, \tag{40}$$

to get desired error dynamics with given eigenvalue placement

$$(T_k - S_2) e_1^{(k)} + T_{k-1} e_1^{(k-1)} + \ldots + T_1 \dot{e}_1 + T_0 e_1 = 0. \tag{41}$$

Selecting $T_i = P_i,\ i = \overline{1, k-1}$, from (33),(35),(37-40) we derive the following set of matrices

$$\left\{ \begin{array}{rcl} T_k & = & 2S_2 + I,\ I_{(n-m)\times(n-m)} - \text{identity matrix}, \\ Q_k & = & A_{11} - S_1 - (S_2 + I) S_0, \\ Q_{k-1} & = & -P_{k-1} S_0, \\ & \vdots & \\ Q_0 & = & -P_0 S_0, C = -G_2^{-1} G_1 [P_1^1 A_{11}]^{-1} P_0, \end{array} \right. \tag{42}$$

which provide us a DSM in the form (25) and the e_1 dynamics (41), which asymptotically converge to zero. Since G_1 has full rank, then according to (42) $G_2 C$ will be of full rank as well, and $[G_2 C]^+$ will do exist. Since $e = [G_2 C] e_1$, we'll have $\lim_{t \to \infty} e(t) = 0$ as well. □

References

Isidori, A. (1995). *Nonlinear Control Systems*, 3rd ed, Springer-Verlag, London.

Utkin, V.I. (1992). *Sliding Modes in Control Optimization.*,Springer-Verlag, Berlin.

DeCarlo, R.A., S.H.Zak, and G.P. Matthews (1988). "Variable stucture control of nonlinear mulivariable systems: A tutorial," *Proc. IEEE,* **76**, pp. 212-232, .

Isidori, A. and C.I. Byrnes (1990). "Output regulation of nonlinear systems," *IEEE Trans. Automat. Contr.,* **35**, no. 2, pp. 131-140.

Devasia, S. , D. Chen, and B. Paden (1996). "Nonlinear inversion-based output tracking," *IEEE Trans. Automat. Contr.,* **41**, no. 7, pp. 930-942.

Hunt, L.R., G. Meyer, and R.Su (1996). "Noncausal inverses for linear systems," *IEEE Trans. Automat. Contr.,* **41**, no. 4, pp. 608-611.

Fliess, M., H. Sira-Ramírez, and R. Márquez (1998). "Requlation of nonminimum-phase outputs: A flatness based approach" in *Perspectives in Control* (D. Normand-Cyrot, ed.), Springer-Verlag.

Hauser, J., S. Sastry, and G. Meyer (1992). "Nonlinear control design for slightly nonminimum phase systems: Application to V/STOL aircraft," *Automatica*, **28**, no. 4, pp. 665-679.

Benvenuti, L., M. D. Di Benedetto, and J. W. Grizzle (1993). "Approximate output tracking for nonlinear nonminimum phase systems with an application to flight control," it Journal of Nonlinear and Robust Control, **4**, pp. 397-414.

Azam, M., and S. N. Singh (1994). "Invertibility and trajectory control for nonlinear maneuvers of aircraft," *Jouranl of Guidance, Control, and Dynamics*, **17**, no. 1, pp. 192-200.

Gopalswamy, S. and J.K. Hedrick (1993). "Tracking nonlinear non-minimum phase systems using sliding control," *Int. J. Control,* **57**, no. 5, pp. 1141-1158.

Sira-Ramírez, H. (1993). "A dynamical variable structure control strategy in asymptotic output tracking problems," *IEEE Trans. Automat. Contr.,* **38**, no. 4, pp. 615-620.

Shtessel, Y.B. (1998). "Nonlinear nonminimum phase output tracking via dynamic sliding manifolds," *J. Franklin Inst.,* **335B**, no. 5, pp. 841-850.

Shtessel, Y.B. (1997). "Nonlinear output tracking in conventional and dynamic sliding manifolds," *IEEE Trans. Automat. Contr.,* **42**, no. 9, pp. 1282-1286.

Brenan, K.E., S.L. Campbell, and L.R. Petzold (1989). *Numerical Solution of Initial-Value Problems in DAE*, Elsevier, New York.

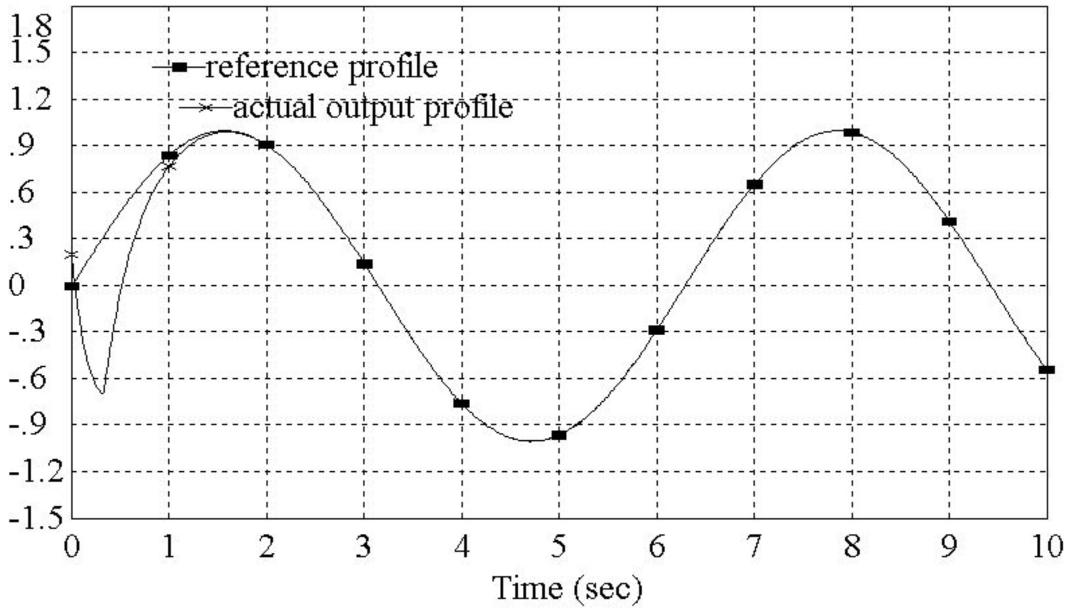

Figure 1: Output tracking

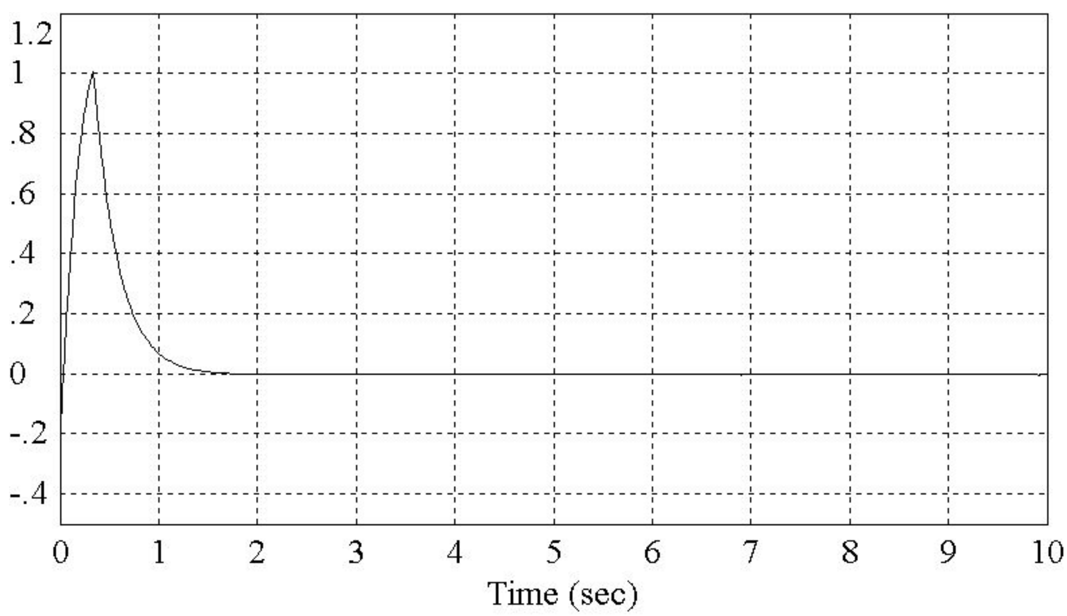

Figure 2: Output tracking error $e = y^* - y$

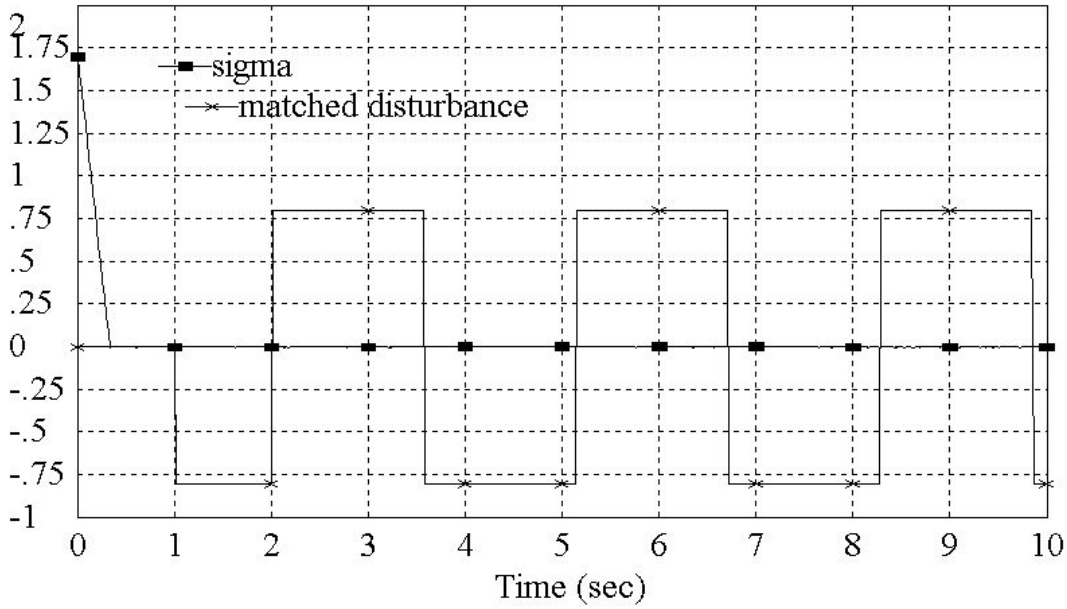

Figure 3: Slilding surface σ and matched disturbance f_2

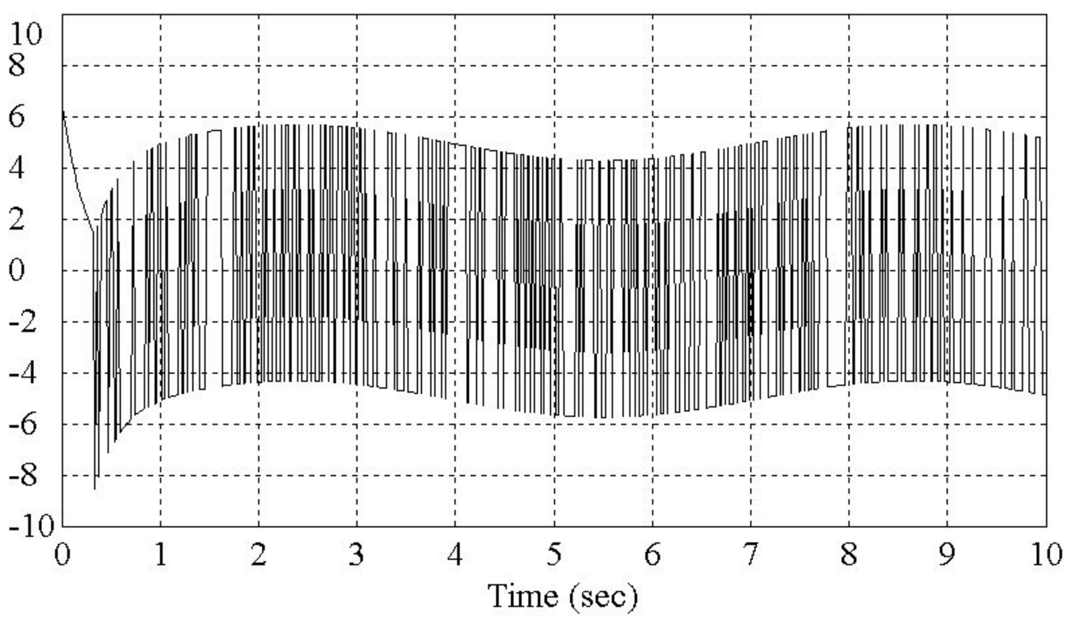

Figure 4: Control law $u(x,t)$

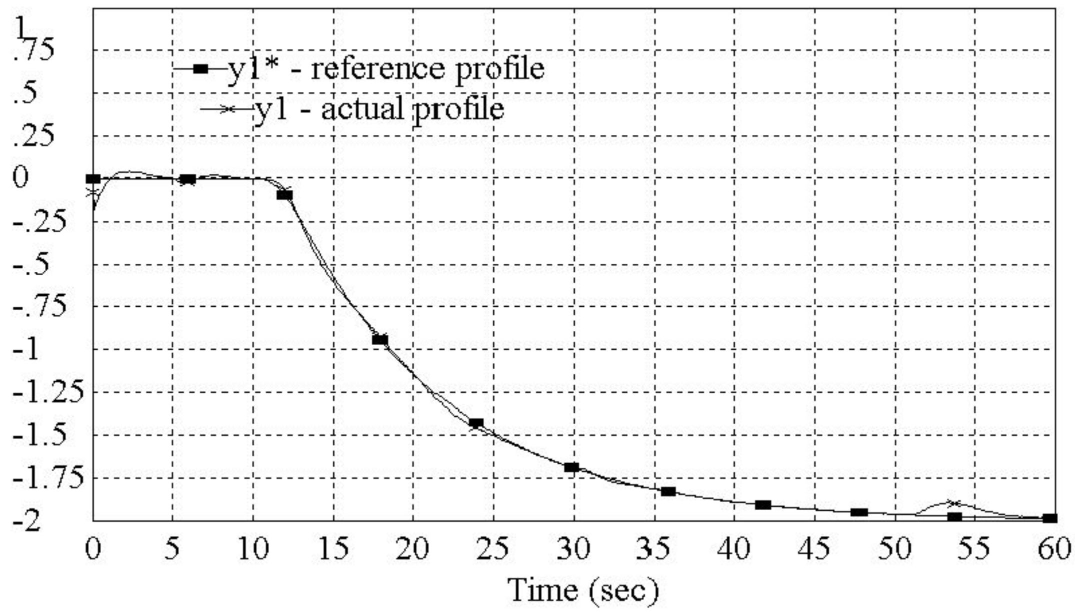

Figure 5: Output tracking the reference profile y_1^*

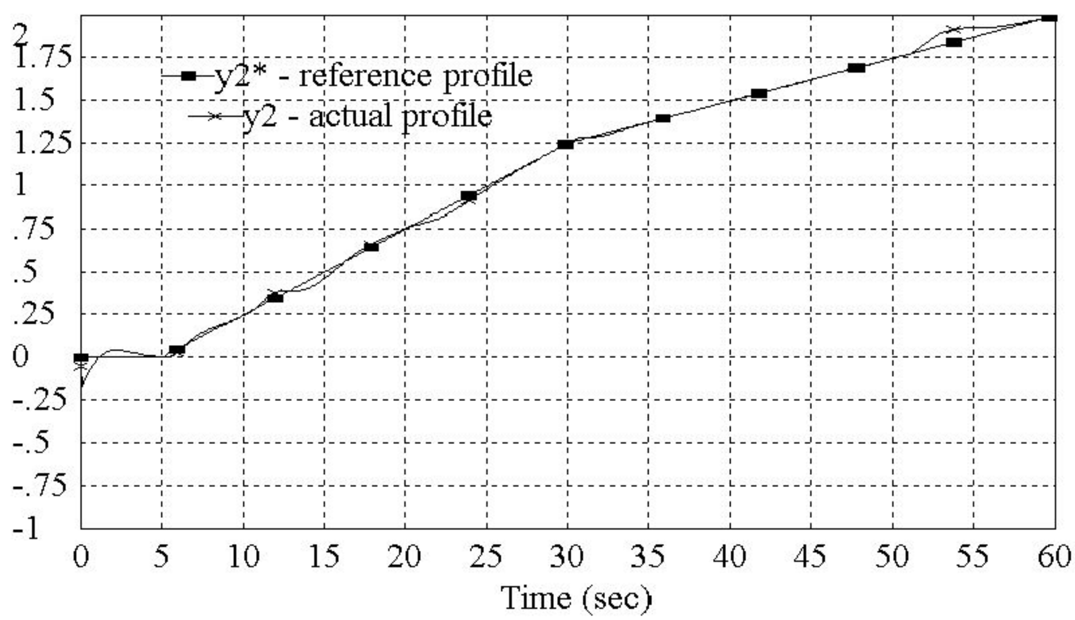

Figure 6: Output tracking the reference profile y_2^*

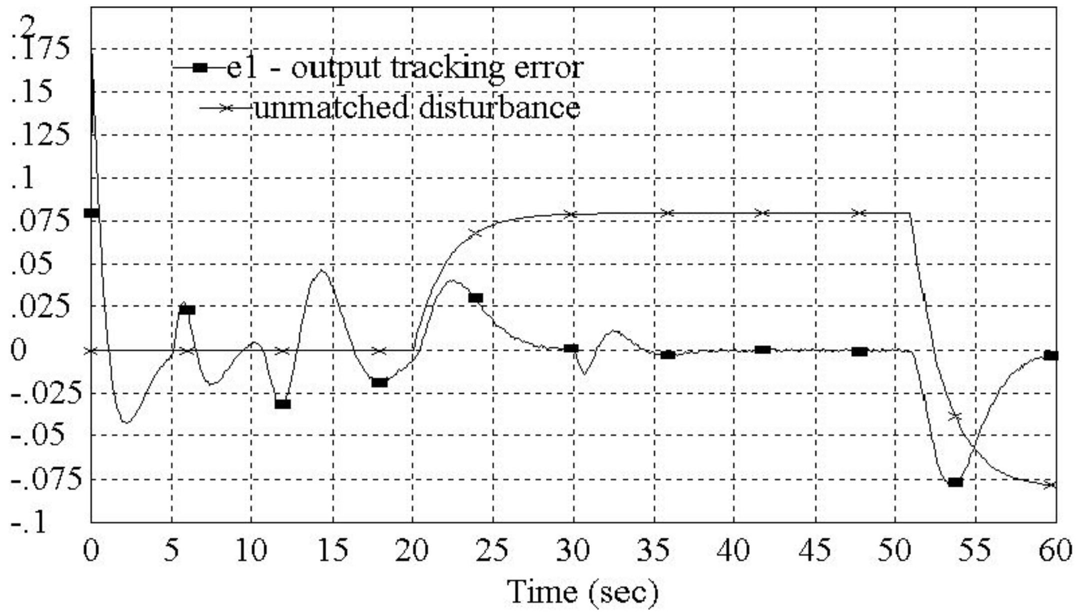

Figure 7: Output tracking error $e_1 = y_1^* - y_1$ and unmatched disturbance f_1.

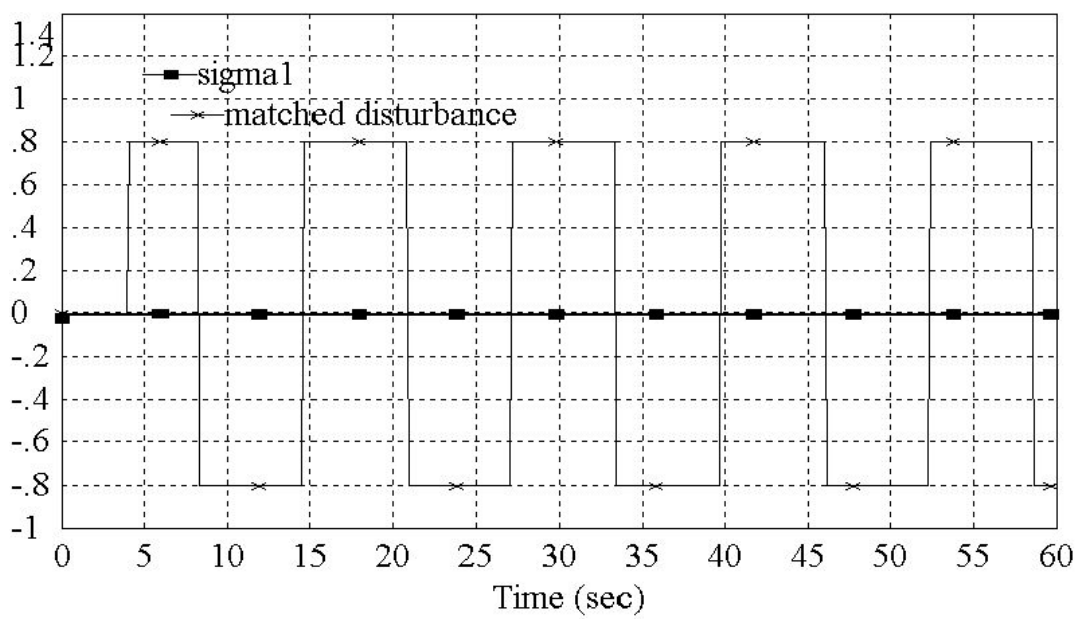

Figure 8: Sliding surface σ_1 and matched disturbance f_2.

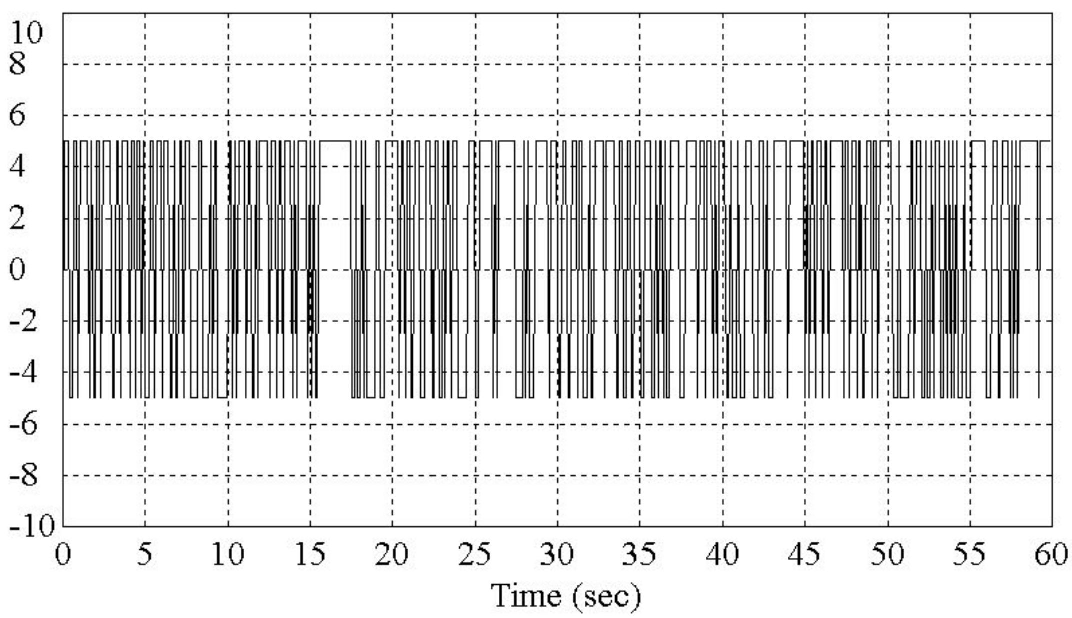

Figure 9: Control function u_1.