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State Space and Internal Models in Discrete-Time LQ Regulator Design *

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Abstract

Discrete-time state space models are proposed for direct implementation of the discrete-time linear-quadratic regulator (DLQR) with not all the state variables but only the output of the plant measured. The case of non zero both the set point and disturbance is considered, by using an appropriate internal model corrector. Including this corrector to the augmented plant the considered case is transformed to usual DLQR stabilisation problem with zero set point. Using the proposed state determination and internal model corrector the modified DLQR design technique, giving a partially prescribed pole placement, is described. Finally, the method is illustrated in an example.

1 Introduction

The linear quadratic regulator (LQR), both in the continuous- and discrete -time versions is now a classical problem being the subject of many papers and books e.g. remind here only two early papers of Kalman (1960) and of Letov (1960) and two contemporary books: of Anderson and Moore (1990) and of Dorato *et al* (1995). The latter book contains a compact recapitulation of the state of art concerning this problem.

It should be realised that usual LQR stabilisation problem in both the versions takes only into account the transients resulting from initial states with no external excitation. This means that the zero set point is then considered. The so called LQ tracking problems admit external excitations (non-zero set point and/or disturbance) but have unrealistic assumption that the excitations are known in advance in the whole control horizon.

Another possibility for accounting a non zero set point and/or disturbance is the internal model approach of Francis and Wonham (1976).

In the present paper the original state space models are proposed and used for derivation of the discrete-time transfer function (TF) of the regulator with using discrete-time, linear quadratic regulator (DLQR) technique. The modified DLQR problem is formulated in which the internal model corrector is used. Especially the integrator and/or oscillator are internal model correctors corresponding to constant and/or sinusoidal excitations. Further on, the modified DLQR design

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is proposed making it possible to obtain a partially prescribed pole placement of the closed-loop (CL) system. Finally, the method is illustrated in an example.

The contribution of the paper is in the state space model proposal which together with applied internal model corrector and an appropriate performance index creates the modified DLQR design, giving a partially prescribed pole placement of the CL system.

2 State space Model $\mathcal{I}$

Consider the discrete-time plant described by the transfer function (TF)

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_0 z^l + b_1 z^{l-1} + ... + b_l}{z^n + a_1 z^{n-1} + ... + a_n} \tag{1}$$

where $l < n,\ Y(z) = \mathcal{Z}[y(t)]$, $U(z) = \mathcal{Z}[u(t)]$, $\mathcal{Z}$ is the symbol of the Z transform; $y(t)$ and $u(t)$ are the output and input signals, $t = 0, 1, 2...$ is the discrete time and $a_n \neq 0$. Assume that the numerator and denominator of (1) are relatively prime polynomials. Determine the state $\mathcal{I}$ in the form

$$\begin{aligned}
&\check{x}_1(t) = y(t+n-m-1) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\check{x}_{n-m}(t) = y(t) \\
&\check{x}_{n-m+1}(t) = y(t-1) + \delta_1 u(t-1) \\
&\check{x}_{n-m+2}(t) = y(t-2) + \delta_1 u(t-2) + \delta_2 u(t-1) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\check{x}_n(t) = y(t-m) + \delta_1 u(t-m) + ... + \delta_m u(t-1)
\end{aligned} \tag{2}$$

where m is an integer such that $l \leq m \leq n-1$ and $\delta_i, i = 1, 2, ..., m$ are constant coefficients which will be determined further on.

Replacing t in (2) by $t+1$, using notation (2) as well as resulting from (1) equation

$$\begin{aligned}
y(t+n-m) + ... + a_{n-m} y(t) + a_{n-m+1} y(t-1) + ... + a_n y(t-m) = \\
= b_0 u(t+l-m) + b_1 u(t+l-m-1) + ... + b_l u(t-m)
\end{aligned} \tag{3}$$

we obtain the state space model $\mathcal{I}$, n-dimensional in the form

$$\check{x}(t+1) = \check{A}\check{x}(t) + \check{B}u(t), \qquad y(t) = \check{C}\check{x}(t) \tag{4}$$

where $\check{x}(t) = [\check{x}_1(t), \check{x}_2(t), ..., \check{x}_n(t)]^T$, $u(t)$, $y(t)$ are scalars,

$$\check{A} = \begin{bmatrix} -a_1 & , & -a_2 & , ..., & -a_{n-1} & , & -a_n \\ 1 & , & 0 & , ..., & 0 & , & 0 \\ 0 & , & 1 & , ..., & 0 & , & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & , & 0 & , ..., & 1 & , & 0 \end{bmatrix} \tag{5}$$

and $\check{B}$ and $\check{C}$ are column and row vectors, respectively, with components $\check{B}_i$ and $\check{C}_i$, $i = 1, 2, ..., n$ determined by

$\check{B}_1 = b_0$,
$\check{B}_i = 0, \quad$ for $\ 2 \leq i \leq n-m$,

$$\tag{6}$$

$\check{B}_i = \delta_{i-n+m}, \quad$ for $\; n-m+1 \leq i \leq n$
$\check{C}_{n-m} = 1$ and $\; \check{C}_i = 0$ for $\; i \neq n-m$

The constants δ_i are chosen so that the terms $u(t+i)$ for $i \neq 0$ are eliminated in (4). After derivations we obtain

$$\delta_i = -\frac{1}{a_n}(b_{l-i+1} + a_{n-i+1}\delta_1 + a_{n-i+2}\delta_2 + ... + a_{n-1}\delta_{i-1}), \tag{7}$$

$$i = 2, 3, ..., m, \;\; \delta_1 = -\frac{b_l}{a_n}$$

3 Regulator-Observer Transfer Function

From solving DLQR problem described by (4) - (7) and the performance index

$$\check{I} = \sum_{t=0}^{N} [\check{x}^T(t+1)\check{Q}\check{x}(t+1) + ru^2(t)] \tag{8}$$

(where $\check{Q}$ is a symmetric semipositive weighting matrix of the state, r is a small positive number and $N \to \infty$) the following feedback law is obtained

$$u = -\check{k}\check{x} = -\check{k}_1\check{x}_1 - \check{k}_2\check{x}_2 - ... - \check{k}_n\check{x}_n \tag{9}$$

Here $\check{k}$ is n-dimensional vector with constant components $\check{k}_i, \;\; i = 1, 2, ..., n$, which may be calculated using e.g. *dlqr* MATLAB function.

The regulator-observer equation and corresponding TF $\check{R}(z)$ may be derived by substituting (2) to (9) and rearranging the terms containing $u(t+i)$.

$$\check{R}(z) = -\frac{U(z)}{Y(z)} = -\frac{\check{k}_1 z^{n-1} + \check{k}_2 z^{n-2} + ... + \check{k}_n}{z^m + p_1 z^{m-1} + ... + p_m} \tag{10}$$

where p_i, $i = 1, 2, ...m$ are dependent upon δ_i and $\check{k}_i$. For $m = n-1$ the TF $\check{R}(z)$ is proper with $(n-1)$-th order polynomials both in numerator and denominator.

The closed-loop (CL) system composed of the plant (4) and feedback (9) is optimal in steady state and has n-th order characteristic equation. However, then all the state variables must be measured which usually is not possible. Note, that the CL system composed of the plant (1) and regulator (10) is of order $(n+m)$. Therefore the latter, second system may have worse performances than the first one. The first CL system is optimal in steady state ($N \to \infty$). Then, there arises the question: whether and in which sense the second system is optimal ? The answer to this question will be given in Section 5.

4 State Space Model $\mathcal{II}$

Consider the plant described by the TF (1). The assumption $a_n \neq 0$ is now not needed. Determine the state $\mathcal{II}$ in the form

$$\begin{array}{lll} \hat{x}_1(t) = y(t+n-m-1), & \hat{x}_2(t) = y(t+n-m-2), ..., & \hat{x}_{n-m}(t) = y(t), \\ \hat{x}_{n-m+1}(t) = y(t-1), & \hat{x}_{n-m+2}(t) = y(t-2), ..., & \hat{x}_n(t) = y(t-m), \\ \hat{x}_{n+1}(t) = u(t-1), & \hat{x}_{n+2}(t) = u(t-2), ..., & \hat{x}_{n+m}(t) = u(t-m) \end{array} \tag{11}$$

where $l \leq m \leq n-1$ and the notation used is the same as previously.

Replacing t in (11) by $t+1$, using notation (11) and the equation (3), we now obtain the state space model $\mathcal{II}$ in the form

$$\hat{x}(t+1) = \hat{A}\hat{x}(t) + \hat{B}u(t), \qquad y(t) = \hat{C}\hat{x}(t) \tag{12}$$

where $\hat{x}(t) = [\hat{x}_1(t), \hat{x}_2(t), ..., \hat{x}_{n+m}(t)]^T$ is $(n+m)$-dimensional and the formulas determining the matrix $\hat{A}$ and vectors $\hat{B}, \hat{C}$ are dependent upon relation between m and l.

It may be shown that the elements $\hat{A}_{i,j}$ of the $(n+m) \times (n+m)$ matrix $\hat{A}$ are determined as follows:

$$\begin{array}{ll}
\hat{A}_{1,j} = -a_j, & \text{for } 1 \leq j \leq n; \\
\hat{A}_{1,j} = 0, & \text{for } n+1 \leq j \leq n+m-l-1, \quad n+1 \leq n+m-l-1; \\
\hat{A}_{i,j} = b_{j-n-m+l}, & \text{for } n+m-l \leq j \leq n+m, \quad m > l; \\
\hat{A}_{i,j} = 0, & \text{for } 2 \leq i \leq n, \quad j \neq i-1; \\
\hat{A}_{i,i-1} = 1, & \text{for } 2 \leq i \leq n; \\
\hat{A}_{n+1,j} = 0, & \text{for } 1 \leq j \leq n+m; \\
\hat{A}_{i,j} = 0, & \text{for } n+2 \leq i \leq n+m, \quad j \neq i-1; \\
\hat{A}_{i,i-1} = 1, & \text{for } n+2 \leq i \leq n+m.
\end{array} \tag{13}$$

The column and row vectors $\hat{B}$ and $\hat{C}$ with components $\hat{B}_i$ and $\hat{C}_i$, $i = 1, 2, ..., n+m$, respectively, are determined by

$$\begin{array}{l}
\hat{B}_1 = b_0, \text{ for } m = l; \quad \hat{B}_1 = 0, \text{ for } m > l; \\
\hat{B}_i = 0, \text{ for } i \geq 2, \quad i \neq n+1, \quad \hat{B}_{n+1} = 1; \\
\hat{C}_{n-m} = 1; \quad \hat{C}_i = 0, \text{ for } i \neq n-m.
\end{array} \tag{14}$$

Note that both the state space models: $\mathcal{I}$ (n-dimensional described in Section 2) and $\mathcal{II}$ ($(n+m)$-dimensional described in this Section) have the same TF (1). The first model is controllable and observable, while the second is controllable but not observable.

From solving DLQR problem described by (12)-(14) and the performance index

$$\hat{I} = \sum_{t=0}^{N} [\hat{x}^T(t+1)\hat{Q}\hat{x}(t+1) + ru^2(t)] \tag{15}$$

(where $\hat{Q}$ is a symmetric, semipositive matrix, r is a small positive number and $N \to \infty$) the following feedback law is obtained

$$u = -\hat{k}\hat{x} = -\hat{k}_1\hat{x}_1 - \hat{k}_2\hat{x}_2 - ... - \hat{k}_{n+m}\hat{x}_{n+m} \tag{16}$$

The regulator-observer TF results from substitution of (11) into (16) and has the form

$$\hat{R}(z) = -\frac{\hat{k}_1 z^{n-1} + \hat{k}_2 z^{n-2} + ... + \hat{k}_n}{z^m + \hat{k}_{n+1} z^{m-1} + ... + \hat{k}_{n+m}} \tag{17}$$

Thus, for $m = n-1$ the TF $\hat{R}(z)$ is proper, while for $m = n-2$ it has the order of numerator higher by one of that of denominator and is not implementable.

5 Discussion of the Solutions

Note that the CL system composed of the plant described by the state space model $\mathcal{II}$ (12)-(14) and the feedback (16) realises the optimal steady state DLQR solution. Note that in the feedback (16) all the $(n+m)$ state variables are known at time t which results from determination (11) of the state $\hat{x}(t)$. The regulator (17) results directly from (16) and (11), and the state space model $\mathcal{II}$ described by (12)-(14) describes the plant (1). Therefore, it results:

Corollary 1. The CL system composed of the plant (1) and regulator (17) realises the optimal steady state DLQR solution of the problem (12) - (15).

Another situation is in the case of the CL system composed of the plant (1) and the regulator (10). Though, the latter regulator results from substitution (2) to (9) the resulting CL system is of $(n+m)$-th order, while the CL system (4)-(7), (9) is of n-th order. This means that the CL system (1), (10) does not realise the optimal steady state DLQR solution of the problem (4), (8). But, both the state space models, $\mathcal{I}$ (4)-(7) and $\mathcal{II}$ (12)-(14) describe the same plant (1). For $m = n-1$ the regulators (10) and (17) have the same structure i.e. the numerator and denominator polynomials of (10) and (17) have the same orders. If additionally $\check{x}^T\check{Q}\check{x} \equiv \hat{x}^T\hat{Q}\hat{x}$ then both the problems have the same plant, performance index and structure of resulting regulators. Therefore we have:

Corollary 2. If $\check{x}^T\check{Q}\check{x} \equiv \hat{x}^T\hat{Q}\hat{x}$ and $m = n-1$, then both the regulators (10) and (17) are the same and realise the optimal steady state DLQR solution of the problem (12) - (15).

From comparison of (2) and (11) it results that the components of the state $\check{x}(t)$ are determined by some linear combinations of components of the state $\hat{x}(t)$, i.e. $\check{x}(t) = H\hat{x}(t)$, where H is the $n \times (n+m)$ matrix with elements dependent upon $\delta_i, \quad i = 1, 2, ..., m$. Thus the equality $\check{x}^T\check{Q}\check{x} \equiv \hat{x}\hat{Q}\hat{x}$ is fulfilled if $\hat{Q} = H^T\check{Q}H$.

5.1 Pole placement Choice

By means of the appropriate choice of the matrix $\hat{Q}$ e.g. for the state space model $\mathcal{II}$ it is possible to establish an appropriate locus of $(n-1)$ roots of CL system. Really, if we choose the matrix $\hat{Q}$ so that

$$\hat{Q} = f^T f, \qquad f = [f_1, f_2, ..., f_n, 0, ..., 0] \tag{18}$$

(where $dim f = n+m$ and f_i, $i = 1, 2, ..., n$ are appropriately chosen) then the performance index (15) takes the form

$$I = \sum_{t=0}^{N} [\varepsilon^2(t+1) + ru^2(t)], \qquad N \to \infty \tag{19}$$

where

$$\varepsilon(t) = f\hat{x}(t), \qquad \varepsilon^2(t) = \hat{x}^T(t)Q\hat{x}(t) \tag{20}$$

Substituting (11) and (18) into (20) and then replacing t by $t+m$ we obtain

$$f_1 y(t+n-1) + f_2 y(t+n-2) + ... + f_n y(t) = \varepsilon(t+m) \tag{21}$$

Minimisation of (19) especially for $r \to 0$ means that $\varepsilon(t)$ and also $\varepsilon(t+m)$ tends to zero when $t \to \infty$. This means that when r tends to zero, then $(n-1)$ roots of the characteristic equation of the CL system determined by the steady state DLQR solution of the problem (12)-(14), (19) tend to the roots of the equation

$$f_1 z^{n-1} + f_2 z^{n-2} + ... + f_n = 0 \tag{22}$$

Thus, by means of an appropriate choice of the coefficients f_i, $i = 1, 2, ..., n$, we may place $(n-1)$ roots of the CL system, freely.

Corollary 3. The solution of the steady state DLQR problem (12)-(14), (19) for $r \to 0$ determines the CL system for which $(n-1)$ roots of the characteristic equation tend to the roots of the equation (22).

In applications one may use an appropriate lower order polynomial (22) e.g. of the second order establishing only two roots of the characteristic equation.

The stability of the CL system resulting from solving the steady-state DLQR problem (12)-(14), (19) is a necessary demand. Below, one from the known sufficient stability conditions is reminded (compare (Dorato *et al*, 1995, p. 176)).

Stability condition. The CL system resulting from solving the steady-state LQR problem (12)-(14), (19) is asymptotically stable if the pair $(\hat{A}, \hat{B})$ is controllable and the pair $(\hat{A}, f)$ is observable.

6 Modified DLQR Problem

Till now, the CL system with zero set point was considered. This results from the fact that the usual steady-state DLQR problem takes only into account the transients resulting from initial states with zero external excitation.

Consider the CL system shown in Fig. 1. Assume that the signals $w(t)$ and $d(t)$ are different, but both fulfil the following difference equation

$$\tilde{u}(t+p) + d_1\tilde{u}(t+p-1) + ... + d_p\tilde{u}(t) = 0 \tag{23}$$

This means that the equation (23) is fulfilled if we substitute $\tilde{u}(t) = w(t)$ or $\tilde{u}(t) = d(t)$. The fact that the signals w and d are different results from different initial conditions of the equation (23) used for generating both the signals.

If for instance $w_1(t) = \bar{w} = const$ and $z_1(t) = Asin(\omega t + \varphi)$ then the equation (23) takes the

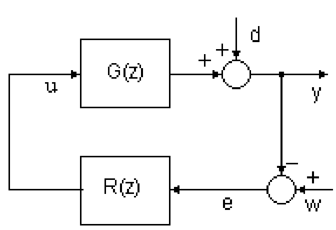

Figure 1: Closed-loop system.

form

$$\tilde{u}(t+3) - (1 + 2cos\omega)\tilde{u}(t+2) + (1 + 2cos\omega)\tilde{u}(t+1) - \tilde{u}(t) = 0 \tag{24}$$

The signals $w_1(t)$ and $z_1(t)$ with any $\bar{w}$, A and φ are the solutions of the equation (24) with appropriate initial conditions.

Consider the corrector described by the TF

$$\frac{U(z)}{V(z)} = C(z) = \frac{1}{z^p + d_1 z^{p-1} + \ldots + d_p} \tag{25}$$

which in denominator has the polynomial corresponding to the equation (23); $V(z) = \mathcal{Z}[v(t)]$ is the input of the corrector.

The augmented plant described by the TF

$$\frac{E(z)}{V(z)} = \bar{G}(z) = C(z)G(z) \tag{26}$$

is of $(n+p)$ order and has all the modes of the plant (1) and of the corrector (25). Thus, the error $e = y_1 + d - w$ appearing in the CL system shown in Fig.1 can be obtained directly as the output of the augmented plant (26) for some appropriate initial conditions. This fact justifies the used notation $E(z) = \mathcal{Z}[e(t)]$ for the augmented plant output.

Let

$$x^*(t+1) = A^* x^*(t) + B^* v(t), \qquad e(t) = C^* x^*(t) \tag{27}$$

be the state equation of the augmented plant (26) derived by using the formulas (11)-(14). It means that the state components $x_i^*(t)$ are determined by the formulas (11) in which the variables $\hat{x}_i(t)$, $y(t)$, $u(t)$ and n are replaced by $x_i^*(t), e(t), v(t)$ and $n+p$, respectively, and the elements of $(n+p+m) \times (n+p+m)$ matrix A^* and of $(n+p+m)$ vector B^* are determined by the appropriately modified formulas (13), (14).

Using the performance index

$$\bar{I} = \sum_{t=0}^{N} [x^{*T}(t+1) Q^* x^*(t+1) + r v^2(t)], \qquad N \to \infty \tag{28}$$

where $Q^* = f^T f$, by the appropriate choice of $(n+p)$ components of the $(n+p+m)$ vector f, we may establish maximum $(n+p-1)$ roots of the CL system.

It is easy to note that the formulas (27), (28) determine the usual steady state DLQR stabilisation problem with zero set point.

Thus, the considered problem with non zero w and d may be solved using a modified MATLAB function (called *dlqr*1, which uses fast conergence of the solution of dynamic Riccati equation) for the case of zero excitations. It is worthwhile to note that the *dlqr* MATLAB function, used to the DLQR problem with the state space model $\mathcal{II}$ usually gives no result, therefore this function needs some modification. On the other hand the solution may be usually obtained by means of the function *dlqr* when the state space model $\mathcal{I}$ is used.

Applying an appropriate MATLAB function to the matrices A^*, B^*, Q^* and r we obtain similarly as (16)

$$v(t) = -kx^*(t) = -k_1 x_1^*(t) - k_2 x_2^*(t) - \ldots - k_{n+p+m} x_{n+p+m}^*(t) \tag{29}$$

The regulator-observer TF results from substituting the state components $x_i^*(t)$ determined by modified formulas (11) into (29)

$$R_1(z) = \frac{V(z)}{E(z)} = -\frac{k_1 z^{n+p-1} + k_2 z^{n+p-2} + \ldots + k_{n+p}}{z^m + k_{n+p+1} z^{m-1} + \ldots + k_{n+p+m}} \tag{30}$$

Finally the regulator for the CL system which contains the observer and the internal model corrector takes the form

$$R(z) = \frac{U(z)}{E(z)} = C(z)R_1(z) \tag{31}$$

It is easy to note that for $m = n-1$ the TF $R(z)$ is proper with polynomials of $(n+p-1)$ order both in numerator and denominator.

The internal model equation should be of not to high order since it influences the order of the CL system. Since the DLQR solution for a small r determines usually a high gain feedback control, therefore the inclusion of the internal model corrector in some cases may be not needed.

7 Example

Let the discrete-time plant $G(z)$ shown in Fig. 1 results from discretization of the continuous-time plant $K(s)$ with zero order hold and sampling period $h = 0.1$. Let

$$K(s) = \frac{1}{s^2+2s+3} \qquad G(z) = \frac{b_0 z + b_1}{z^2 + a_1 z + a_2} \tag{32}$$

where $b_0 = 0.00467$, $b_1 = 0.00437$, $a_1 = -1.79161$, $a_2 = 0.818731$. We would like to design the regulator $R(z)$ for which the CL system works sufficiently accurately for any non zero constant set point $w = const \neq 0$ and for sinusoidal disturbance $\bar{d}(\bar{t}) = Asin(2\bar{t} + \varphi)$ with any amplitude A and phase φ ($\bar{t}$-denotes the continuous time).

Further on, three different regulators shall be designed.

7.1 DLQR Without Corrector

For the plant $G(z)$ determined by (32), the formulas (11) with $n = 2$, $m = 1$ may be used for determination of the state components.

Assuming $f = [1, 0, 0]$ and choosing $r = 0.001$, the following TF of the regulator has been obtained.

$$R_1(z) = \frac{U(z)}{Y(z)} = -\frac{k_1' z + k_2'}{z + k_3'} \tag{33}$$

where $k_1' = 65.4283$, $k_2' = -45.3770$, $k_3' = 0.2422$ were calculated using a modified *dlqr*1 MATLAB function. The roots of the characteristic equation of the CL system (32), (33) are: $z_1' = 0.6219 + j0.2684, z_2' = 0.6219 - j0.2684, z_3' = 0$.

The regulator (33), designed using usual steady state solution of DLQR problem with zero excitation, was applied in the system shown in Fig. 1 having assumed non zero signals w and $d(t)$. The results of simulations performed with SIMULINK are shown in Fig. 2.

7.2 DLQR with Internal Model Corrector

The internal model difference equation for considered signals $w = const \neq 0$, $\bar{d}(\bar{t}) = Asin(2\bar{t}+\varphi)$ takes the form (24) with $\omega = 0.2$. The latter value results from accounting that for sampling period $h = 0.1$ it is $\bar{t} = 0.1t$. The corresponding internal model corrector (25) takes the form

$$C(z) = \frac{1}{z^3 - 2.9601z^2 + 2.9601z - 1} \tag{34}$$

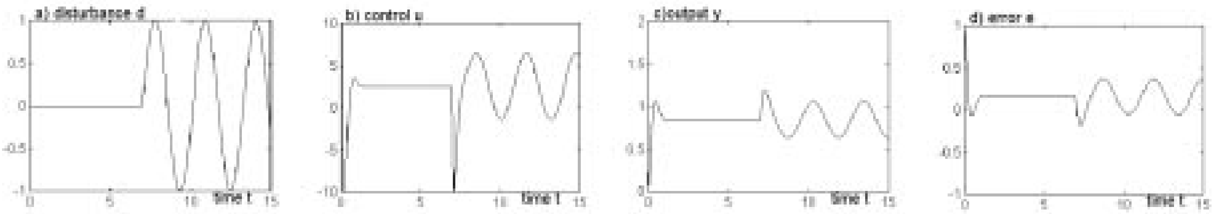

Figure 2: Results of simulations for CL system with regulator $R_1(z)$.

Including the corrector (34) to the plant $G(z)$ we obtain the augmented plant $\bar{G}(z) = C(z)G(z)$ of the fifth order. To derive the state space model $\mathcal{II}$ (27) we use the formulas (11) with $n+p = 5$, $m = 1$ and $e(t) = y_1(t) + d(t) - w(t)$ $(d(t) = \bar{d}(0.1t))$. Determining

$$x_1^*(t) = e(t+3), \quad x_2^*(t) = e(t+2), \quad x_3^*(t) = e(t+1), \quad x_4^*(t) = e(t),$$

$$x_5^*(t) = e(t-1), \quad x_6^*(t) = v(t-1) \tag{35}$$

and using (13), (14) (or deriving from (35) and the difference equation describing the augmented plant $\bar{G}(z)$ with components b_i and a_i in numerator and denominator, respectively) we obtain

$$x^*(t+1) = \begin{bmatrix} -a_1, & -a_2, & -a_3, & -a_4, & -a_5, & b_1 \\ 1, & 0, & 0, & 0, & 0, & 0, \\ 0, & 1, & 0, & 0, & 0, & 0, \\ 0, & 0, & 1, & 0, & 0, & 0, \\ 0, & 0, & 0, & 1, & 0, & 0, \\ 0, & 0, & 0, & 0, & 0, & 0, \end{bmatrix} x^*(t) + \begin{bmatrix} b_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v(t) \tag{36}$$

Assuming $f = [0, 0, 0, 1, 0, 0]$ and $r = 0.001$ we obtain finally

$$R_2(z) = -C(z)\frac{k_1 z^4 + k_2 z^3 + k_3 z^2 + k_4 z + k_5}{z + k_6} \tag{37}$$

where $k_1 = 336.364$, $k_2 = -970.798$, $k_3 = 1117.01$, $k_4 = -595.849$, $k_5 = 123.076$, $k_6 = 0.657$ were calculated as previously. The roots of the characteristic equation of the CL system (32), (34), (37) (and of the CL system (36), (29)) are: $z_1 = 0.5583 + j0.5445$, $z_2 = 0.5583 - j0.5445$, $z_3 = 0.4730 + j0.2383$, $z_4 = 0.4730 - j0.2383$, $z_5 = 0.4612$, $z_6 = 0$.

The results of simulations for the CL system (32), (34), (37) are shown in Fig.3.

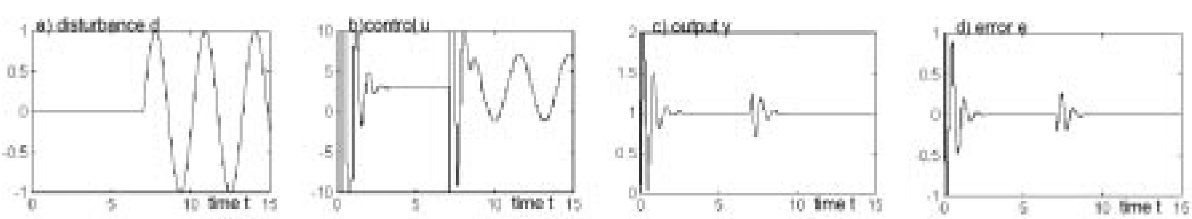

Figure 3: Results of simulations for CL system with regulator $R_2(z)$.

7.3 DLQR with Partially Established Roots

To decrease the oscillations in the transient period now the vector f will be chosen so that two roots of the CL system correspond to the roots of the CT system $s_1 = -0.5+j1.2$ $s_2 = -0.5-j1.2$ (this pair of roots, if it is dominant, give approximately 25% overshot of the step response). Assuming $f = [0, 0, 1, -1.8888, 0.9048, 0]$ (which establishes two roots $z_1 = exp(-0.05+j0.12) = 0.9444 + j0.1139$ and $z_2 = exp(-0.05 - j0.12) = 0.9444 - j0.1139$ of the CL system) and using the same models (34), (36) we obtain

$$R_3(z) = -C(z)\frac{\bar{k}_1 z^4 + \bar{k}_2 z^3 + \bar{k}_3 z^2 + \bar{k}_4 z + \bar{k}_5}{z + \bar{k}_6} \tag{38}$$

where $\bar{k}_1 = 152.9672$, $\bar{k}_2 = -514.6322$, $\bar{k}_3 = 654.4318$, $\bar{k}_4 = -373.6783$, $\bar{k}_5 = 81.1897$, $\bar{k}_6 = 0.4333$ were calculated as previously. The roots of the characteristic equation of the CL system (32), (34), (38) (and of the CL system (36) with corresponding state feedback) are: $\bar{z}_1 = 0.9444+j0.1139$, $\bar{z}_2 = 0.9444-j0.1139$, $\bar{z}_3 = 0.5968+j0.4059$, $\bar{z}_2 = 0.5968-j0.4059$, $\bar{z}_5 = 0.5215$, $\bar{z}_6 = 0$

The results of simulations for the CL system (32), (34), (38) are shown in Fig.4

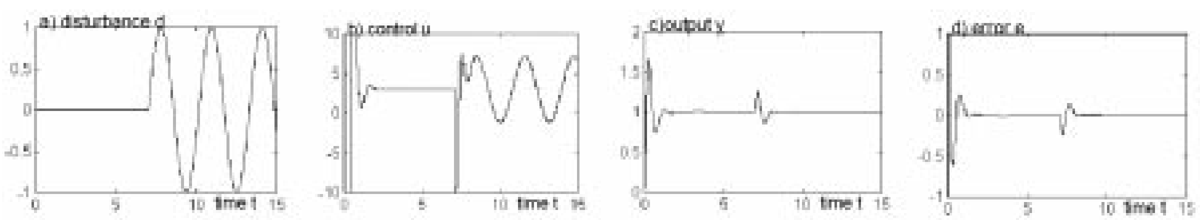

Figure 4: Results of simulations for CL system with regulator $R_3(z)$.

7.4 Results of Simulations

For all the three obtained regulators the responses of the CL system to the set point $w(t) = \mathbf{1}(t)$ ($\mathbf{1}(t) = 0$ for $t < 0$ and $\mathbf{1}(t) = 1$ for $t \geq 0$) and to the disturbance $\bar{d}(\bar{t}) = \mathbf{1}(\bar{t}-7)Asin(2(\bar{t}-7))$ were calculated. From Fig. 2 it is shown that the CL system with regulator $R_1(z)$ has an error in steady state both for the constant set point and the sinusoidal disturbance. The results of Fig. 3 show that the CL system with regulator $R_2(z)$ having internal model corrector has no steady state errors for both the signals w and $\bar{d}(\bar{t})$. From Fig. 4 it results that the CL system having regulator $R_3(z)$ designed with additional partial choice of pole placement has better response in the transient interval.

It is worthwhile to note that in all the three considered cases the same regulators were obtained by using the state space model $\mathcal{I}$. The condition was that in derivations the same performance index was used. In this case the usual *dlqr* MATLAB function gave good results.

Note, that the CL system working with each of the designed regulator has one zero root of its characteristic equation. Additionally, it becomes that for each designed regulator the remaining non zero roots are the same as for the CL system resulting from the model $\mathcal{I}$ in which all the state components are available. When only the output is available and the designed regulators are implemented then the CL systems have their order increased respectively by one and have the additional zero root. It looks that it is a regularity which may be the subject of further researches.

8 Conclusions

The proposed state space models make it possible to derive the regulator transfer function implementable in feedback control with not all the states but only the output variable measured.

The appropriate choice of the weighting matrix Q makes it possible to establish a partial placement of the poles of the CL system.

The problem with non-zero both the set point and the disturbance fulfilling the internal model equation, may be transformed to the usual steady state DLQR stabilisation problem with zero set point, by including the internal model corrector to the augmented plant and using one of the proposed state space model. In this manner the solution of the usual steady state DLQR stabilisation problem may be used for designing the regulator for the system with non-zero set point and disturbance.

The integrator and/or oscillator corrector is a special case of the internal model corrector. DLQR technique with using integrator-oscillator correctors makes it possible to design regulators working accurately in steady state for any constant and sinusoidal set point and/or disturbance with given frequency.

Similar modified LQR design method can be applied for continuous-time systems. The main difference is in the state space model in which in the place of output and control variables evaluated at times back shifted, some appropriate integrals of output and control should appear.

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