Proceedings of the 7th Mediterranean Conference on Control and Automation (MED99) Haifa, Israel - June 28-30, 1999

An Estimate to the Energy Function of a Rigid Robot with a Stabilizing PD Controller

Amit Ailon[•*] and Michael I. Gil'[#]
[•]Department of Electrical and Computer Engineering,
Ben Gurion University of the Negev, Beer Sheva 84105, Israel
[#]Department of Mathematics, Ben Gurion University
of the Negev, Beer Sheva 84105, Israel

Abstract

This study presents an explicit upper bound to the energy function of an *n-degree of freedom rigid robot* while it is under the action of a PD controller. The resulting upper bound is an exponential function that reflects the effect of the controller gains on the form of the system response. A tuning-rule for setting the controller gains and adjusting the system rate of convergence towards the desired operating point in any given ball, centered at the system equilibrium point, has been demonstrated. As shown, the effect of the controller structure on the proposed upper bound is similar to the one resulted in the case of a second-order linear system.

1 Introduction

The passivity property of a robot was stressed first in (Ortega and Spong, 1989). Later on, several papers (Takegaki and Arimoto, 1981; Tomei, 1991; Ailon and Ortega 1993; Berghuis and Nijmeijer 1995; Ortega *et al.*, 1995) have studied and demonstrated simple state and output controllers which are based on the particular properties of the robot, for the set-point tracking control tasks. All these control schemes guarantee *global asymptotic stability*, but do not give any quantitative evaluation concerning the behavior of the trajectory and the nature of the convergence of the robot towards its final target.

As indicated in Khalil (1996, Ch. 3), provided the origin of the linearization of a given nonlinear system is exponentially stable and the norm of the solution is bounded in a given region by some class of a KL function, the solution is bounded by an exponential function whose parameters may be dependent on the initial condition. But as far as the robot manipulator is concerned, still there is a need to have some method for carrying out an explicit and as tight estimate as possible, to the norm of the system trajectory.

To be more specific, the following questions are essential in robotic applications: how fast the system moves in the direction of the final target, and how the feedback gains can be tuned in order to change the system's rate of convergence to comply with the user needs.

In this paper we consider some issues associated with the energy function of a rigid robots under the action of a PD controller. Actually, the starting point of this study is (Wen and Bayard, 1988). In this reference an exponential stability was demonstrated using a *Lyapunov function* which includes a *cross term* of the position and the generalized momentum vectors. In

[*]E-mail: amit@bguee.bgu.ac.il

the present study we also insert a cross term in the Lyapunov candidate function, but unlike in (Wen and Bayard, 1988), the *scalar product* here is of the position and the velocity (rather than the generalized momentum) vectors. This allows us later on to present an explicit formula which reflects the effects of controller gains on the form of the system response. Further, this formulation provides us with a simple procedure for tuning the controller gains according to possible design specifications.

In other words, if the design objective is to ensure that the robot will move fast enough towards the target in a point-to-point task, then by employing the proposed procedure, which in a way resembles the pole-placement method in a second order linear system, the design goal can easily be accomplished.

2 Preliminaries

Using the Lagrangian formulation the model of an *n-degree of freedom rigid robot* is (Spong and Vidyasagar, 1989)

$$D(q(t))\ \ddot{q} + C(q(t), \dot{q}\ (t))\ \dot{q}\ (t) + g(q(t)) = u(t); t \geq 0, q\ (0) = q_0, \dot{q}\ (0) = \dot{q}_0,$$

where $q \in \Re^n$ is the vector of generalized coordinates, $D(q)$ is the inertia matrix, $C(q, \dot{q})\ \dot{q}$ represents the Coriolis and the centrifugal forces, $g(q)$ is the gravitational vector, and $u \in \Re^n$ is the applied torques vector. With $x = \left[x_1^T, x_2^T\right]^T = \left[q^T, \dot{q}^T\right]^T$ we have:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= D^{-1}(x_1)\left(-C(x_1, x_2)x_2 - g(x_1) + u\right), \end{aligned} \tag{1}$$

Along this paper we consider two controllers. In the first case the feedback is given by

$$u = -k\left(x_1 - x_{1d}\right) - bx_2 + g(x_1), \tag{2}$$

with the controller gains $k, b > 0$. The vector x_{1d} is the robot set-point. Hence, applying (2) in (1) we obtain the closed-lop system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= D^{-1}(x_1)(-C(x_1, x_2)x_2 - k\left(x_1 - x_{1d}\right) - bx_2). \end{aligned} \tag{3}$$

The more practical case, is associated with a feedback that was proposed in (Tomei, 1991)

$$u = -K\left(x_1 - x_{1d}\right) - Bx_2 + g(x_{1d}), \tag{4}$$

with $K = K^T, B = B^T > 0$. Using (4), (1) becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= D^{-1}(x_1)\left(-C(x_1, x_2)x_2 - g(x_1) - K\left(x_1 - x_{1d}\right) - Bx_2 + g(x_{1d})\right). \end{aligned} \tag{5}$$

We shall use the notation $\|x\|$ for the Euclidean norm of x and respectively $\|G\|_i$ denotes the induced norm of a matrix G. It is always possible to determine finite positive constants M_1, M_2, m_1, m_2, and k_c such that for all $x_1, x_2 \in \Re^n$ the following hold:

$$M_1 \leq \|D(x_1)\|_i \leq M_2; m_1 \leq \left\|D^{-1}(x_1)\right\|_i \leq m_2; \|C(x_1, x_2)\|_i \leq k_c \|x_2\|. \tag{6}$$

Following (Tomei, 1991), the system (5) is globally asymptotically stable if

$$\lambda_{\min}(K) > \beta \doteq x_1 \in \Re^n \sup \left\|\frac{\partial g\left(x_1\right)}{\partial x_1}\right\|_i. \tag{7}$$

It is clear therefore that (3) is globally asymptotically stable for all $k, b > 0$.

We assume without loss of generality that the robot set-point is $x_{1d} = 0$.

3 Solution Estimates

Recall that we take $x_{1d} = 0$, consider the scalar function

$$V_1(x) = \frac{1}{2}\left(k\left\|x_1\right\|^2 + x_2^T D(x_1) x_2\right) + \gamma x_2^T x_1 = \frac{1}{2} x^T \left[\begin{array}{cc} k & \gamma \\ \gamma & D(x_1) \end{array}\right] x. \tag{8}$$

The derivative of $V_1(\cdot)$ along the trajectory of (3) is given by

$$\begin{aligned} \dot{V}_1(x) &= -(b-\gamma)\left\|x_2\right\|^2 - \gamma k x_1^T D^{-1}(x_1) x_1 \\ &\quad + \gamma x_1^T D^{-1}(x_1)(-C(x_1, x_2) - bI_n) x_2, \end{aligned} \tag{9}$$

where I_n is the $n \times n$ identity matrix.

Select arbitrarily $r > 0$. We shall show that a positive number $\gamma > 0$ can be selected in (8) such that $V_1(\cdot) > 0$ and $\dot{V}_1(\cdot) < 0$ for all $x \in B_r$, $x \neq 0$, where B_r is a ball with radius r centered at $x = 0$.

To this end we firstly select a number γ such that the following conditions are satisfied:

$$\Phi \doteq \left[\begin{array}{cc} \gamma k m_1 & -\frac{1}{2}\gamma m_2 (k_c r + b) \\ -\frac{1}{2}\gamma m_2 (k_c r + b) & b - \gamma \end{array}\right] > 0; \Theta_1 \doteq \left[\begin{array}{cc} k & -\gamma \\ -\gamma & M_1 \end{array}\right] > 0, \tag{10}$$

where m_1, m_2, and k_c are given in (6). Recalling that $k, b > 0$ we have that

$$k M_1 > \gamma^2 \Rightarrow \Theta_1 > 0; k m_1 (b - \gamma) > \gamma m_2^2 (k_c r + b)^2 / 4 \Rightarrow \Phi > 0. \tag{11}$$

Hence, (here and below $\sqrt{\cdot}$ is a non-negative number)

$$0 < \gamma < \min\left\{\sqrt{k M_1}, \frac{k b m_1}{k m_1 + m_2^2 (k_c r + b)^2 / 4}\right\} \Rightarrow \Phi, \Theta_1 > 0. \tag{12}$$

But then, using (6) and (8)-(10) we conclude that

$$\begin{aligned} V_1(x) &\geq \frac{1}{2}[\left\|x_1\right\|, \left\|x_2\right\|]\Theta_1[\left\|x_1\right\|, \left\|x_2\right\|]^T \geq \frac{1}{2}\lambda_{\min}(\Theta_1)\left\|x\right\|^2; \forall x \in B_r \\ \dot{V}_1(x) &\leq -[\left\|x_1\right\|, \left\|x_2\right\|]\Phi[\left\|x_1\right\|, \left\|x_2\right\|]^T \leq -\lambda_{\min}(\Phi)\left\|x\right\|^2; \forall x \in B_r, \end{aligned} \tag{13}$$

and thus $V_1(x)$ is a *Lyapunov function* in the domain B_r.

Next, in view of (6) and (8), and using (12)

$$V_1(x) \leq \frac{1}{2} M_2 \left\|x_2\right\|^2 + \frac{1}{2} k \left\|x_1\right\|^2 + \gamma \left\|x_2\right\| \left\|x_1\right\| \leq \frac{1}{2}\lambda_{\max}(\Theta_2)\left\|x\right\|^2, \tag{14}$$

where

$$\Theta_2 \doteq \left[\begin{array}{cc} k & \gamma \\ \gamma & M_2 \end{array}\right] > 0. \tag{15}$$

Lemma 3.1. Fix arbitrarily $r > 0$ and select γ such that (12) is satisfied. Then, for each $\left\|x(0)\right\| \in B_\eta$ with

$$\eta \doteq (\lambda_{\min}(\Theta_1) / \lambda_{\max}(\Theta_2))^{1/2} r \leq r \tag{16}$$

the norm of the solution of (3) satisfies

$$\left\|x(t)\right\| \leq a \left\|x(0)\right\| \exp(-\lambda t); \forall t \geq 0, \tag{17}$$

with

$$a \doteq \left(\lambda_{\max}\left(\Theta_2\right)/\lambda_{\min}\left(\Theta_1\right)\right)^{1/2};\lambda \doteq \lambda_{\min}\left(\Phi\right)/\lambda_{\max}\left(\Theta_2\right), \tag{18}$$

where Θ_i and Φ are given in (10) and (15).

Proof. Observing (13)-(14), the lemma follows from (Vidyasagar, 1993, Ch. 5). ◇◇

Remark. It should be emphasized that the parameters η, a, and λ depend on r.

Next we shall extend the previous results to the second case, namely, the closed-loop system (5). In order to establish the exponential upper bound on the norm of the solution, we impose on the selected stiffness coefficient matrix K the following condition

$$\lambda_{\min}\left(K\right) > \beta m_2/m_1, \tag{19}$$

which is of course stronger then (7).

Consider the scalar function

$$V_2(x) = \frac{1}{2}\left(x_2^T D(x_1)x_2 + x_1^T K x_1\right) + \gamma x_2^T K x_1 + P\left(x_1\right) - x_1^T g\left(0\right) - U\left(0\right), \tag{20}$$

with $\partial P\left(x_1\right)/\partial x_1 = g^T\left(x_1\right)$. The derivative of $V_2\left(\cdot\right)$ along the trajectory of (5) is

$$\begin{aligned} \dot{V}_2\left(x\right) &= -x_2^T\left(B-\gamma K\right)x_2 - \gamma x_1^T K D^{-1}(x_1) K x_1 \\ &\quad -\gamma x_1^T K D^{-1}(x_1)(g\left(x_1\right) - g\left(0\right)) \\ &\quad +\gamma x_1^T K D^{-1}(x_1)(-C(x_1,x_2) - BI_n)x_2. \end{aligned} \tag{21}$$

In the present case we are looking for a number γ that satisfies

$$\begin{aligned} \Phi &\doteq \begin{bmatrix} \gamma\lambda_{\min}(K)(m_1\lambda_{\min}\left(K\right) - m_2\beta) & -\frac{1}{2}\gamma m_2\lambda_{\max}(K)(k_c r + \lambda_{\max}(B)) \\ -\frac{1}{2}\gamma m_2\lambda_{\max}(K)(k_c r + \lambda_{\max}(B)) & \lambda_{\min}(B) - \gamma\lambda_{\max}(K) \end{bmatrix} > 0; \\ \Theta_1 &\doteq \begin{bmatrix} \lambda_{\min}(K) - \beta & -\gamma\lambda_{\max}(K) \\ -\gamma\lambda_{\max}(K) & M_1 \end{bmatrix} > 0, \end{aligned} \tag{22}$$

where we recall that by the mean value theorem β in (7) satisfies $\left\|g\left(x_1\right) - g\left(0\right)\right\| \le \beta\left\|x_1\right\|$ for all $x_1 \in \Re^n$ (Tomei, 1991). Hence, we have the following implications

$$\begin{aligned} \left(\lambda_{\min}\left(K\right) - \beta\right)M_1 &> \gamma^2\lambda_{\max}^2\left(K\right) \Rightarrow \Theta_1 > 0; \\ \gamma\lambda_{\min}\left(K\right)\left(m_1\lambda_{\min}\left(K\right) - m_2\beta\right)\left(\lambda_{\min}\left(B\right) - \gamma\lambda_{\max}\left(K\right)\right) &> \\ \left(\gamma m_2\lambda_{\max}\left(K\right)\left(k_c r + \lambda_{\max}\left(B\right)\right)\right)^2/4 &\Rightarrow \Phi > 0. \end{aligned} \tag{23}$$

Therefore recalling the condition (19), if γ is selected such that

$$\begin{aligned} 0 &< \gamma\lambda_{\max}\left(K\right) < \min\left\{\sqrt{\left(\lambda_{\min}\left(K\right) - \beta\right)M_1}, \right. \\ &\left. \frac{\xi\lambda_{\min}\left(B\right)}{\xi + m_2^2\lambda_{\max}\left(K\right)\left(k_c r + \lambda_{\max}\left(B\right)\right)^2/4}\right\}, \end{aligned} \tag{24}$$

where $\xi \doteq \lambda_{\min}\left(K\right)\left(m_1\lambda_{\min}\left(K\right) - m_2\beta\right)$, then (22) is satisfied and $V_2(x)$ is a Lyapunov function in the domain B_r.

Next, similar to the previous case we have now

$$V_2(x) \le \frac{1}{2}\lambda_{\max}\left(\Theta_2\right)\left\|x\right\|^2, \tag{25}$$

where

$$\Theta_2 \doteq \begin{bmatrix} \lambda_{\max}(K)+\beta & \gamma\lambda_{\min}(K) \\ \gamma\lambda_{\min}(K) & M_2 \end{bmatrix} > 0. \tag{26}$$

Thus, by replacing the matrices Φ, Θ_1, and Θ_2 in (10) and (15) with those presented in (22) and (26), while γ is selected such that (24) holds, the statement of lemma 3.1 is applicable to the system (5).

Remark. If the stiffness coefficient matrix in (4) is selected as $K = kI_n$ the cross term in the energy function (20) can be reduced to $\gamma k x_2^T x_1$ and the relevant positive definite matrix in (21) becomes $\gamma k D^{-1}(x_1)$ and the structures of the matrices Φ and Θ_i and equation (24) are respectively simplified.

4 Some Related Results and a Simple Gain Tuning-Rule

We shall illustrate below how the previous results can be applied for adjusting the system rate of convergence. In particular we shall draw an interesting analogy between some features associated with the dynamics of the robot and a second order linear system.

To make the presentation more transparent we consider below the closed-loop system (5). However, as previously the extension of the results to the system (4) can be accomplished in a straightforward manner.

Consider a second-order linear system

$$\begin{bmatrix} \dot{\zeta}_1(t) \\ \dot{\zeta}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/M_2 & -b/M_2 \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}; \zeta(0) = \begin{bmatrix} \zeta_1(0) \\ \zeta_2(0) \end{bmatrix}, \tag{27}$$

where M_2 is given in (6), and as previously $k, b > 0$. Equation (27) can be regarded as the state-space model of a 'one-link robot' $M_2\ddot{\zeta}_1 = -k\zeta_1 - b\dot{\zeta}_1$ with a moment of inertia M_2. This linear system enables us to draw some interesting results regarding the effect of the controller gains k and b on the response of the system (3).

Clearly, an upper bound of the form (17) can be presented for the norm of the solution of the linear system (27). Then, observing (18) it can be shown that $\lambda < \lambda_l$ and $a > a_l$, where the subscript l stands for the linear system (27). In other words, the upper bound of the response of the multi-link manipulator can not be tighter than the one associated with (27). Moreover, we can state the following.

Lemma 4.1. A constant ϖ can be determined such that for any pair of positive gains $\{k, b\}$, for any $r > 0$ and γ that satisfies (12), the exponent in (17)-(18) satisfies $\lambda < \varpi$.

Proof. Since $\lambda < \lambda_l$, it is sufficient to show that there exists ϖ such that for any pair $\{k, b\}$, $\lambda_l < \varpi$. Indeed, solving the *Lyapunov matrix equation* $A^T P + PA = -I_2$, for $P = [p_{i,j}]$ where A is the system matrix in (27), we obtain

$$p_{11} = \frac{k^2/M_2 + k + b^2/M_2}{2kb/M_2}; \, p_{12} = p_{21} = \frac{1}{2k/M_2}; p_{22} = \frac{k/M_2 + 1}{2kb/M_2^2}.$$

Clearly $p_{11} > (k^2 + b^2)/2kb$. But since $k^2 + b^2 \geq 2kb$ we have $p_{11} > 1$, and thus $\lambda_{\max}(P) > 1$. The rest of the proof follows from (Vidyasagar, 1993, Ch.5). ◇◇

However, by employing a pole-placement technique it appears that for a proper selection of the gains k and b, we can assign arbitrarily the dominant eigenvalue of the system matrix in (27), and hence the time constant of the slow mode of the linear system becomes as short as needed. In this regard we can adopt the following procedure.

The solution of the characteristic equation $s^2 + sb/M_2 + k/M_2 = 0$ of (27) is

$$s_{1,2} = \left(-b/M_2 \pm \sqrt{(b/M_2)^2 - 4k/M_2}\right) \Big/ 2, \tag{28}$$

and thus $s_1 = \left(-b/M_2 + \sqrt{(b/M_2)^2 - 4k/M_2}\right) \Big/ 2$ is the dominant pole. Select arbitrarily an $\alpha > 1$ and replace the pair of gains $\{k, b\}$ in (27) by the pair $\{\bar{k}, \bar{b}\} = \{\alpha k, \sqrt{\alpha} b\}$. Then, the dominant pole of the linear system with the new gains is given by

$$\sigma_1 = \sqrt{\alpha}\left(-b/M_2 + \sqrt{(b/M_2)^2 - 4k/M_2}\right) \Big/ 2 = \sqrt{\alpha} s_1, \tag{29}$$

and thus the time-constant of the slower mode becomes shorter.

Motivated by this idea we adopt in the sequel a finer approach that allows us to show that a proper change in the controller gains k and b induces a similar effect on the response of the robotic system.

To this end we further simplify the model of the one-link robot in (27) as follows:

$$M\ddot{\zeta}_1 = -b\dot{\zeta}_1, \tag{30}$$

where the positive constant M is to be determined later on. The system (30) has an *equilibrium subspace*. However, the *kinetic energy* $U_1(\cdot)$ of the mechanical system represented by this equation decays exponentially as

$$U_1(\dot{\zeta}_1) = M\dot{\zeta}_1^2/2 = M\exp\left(-\left(2b/M\right)t\right)\dot{\zeta}_1^2\left(0\right)/2. \tag{31}$$

The rate of decay of $U_1(\cdot)$ is described by the equation

$$\dot{U}_1\left(\dot{\zeta}_1\left(t\right)\right) = -b\left\|\dot{\zeta}_1\left(t\right)\right\|^2 = -b\exp\left(-\left(2b/M\right)t\right)\dot{\zeta}_1^2\left(0\right). \tag{32}$$

Let a scalar $\dot{\zeta}_1(0)$ be selected such that

$$V_1\left(x\left(0\right)\right) \le U_1(\dot{\zeta}_1\left(0\right)) = M\dot{\zeta}_1^2\left(0\right)/2, \tag{33}$$

where $V_1(\cdot)$ is given in (8).

Lemma 4.2. Consider the systems (3) with the associated function $V_1(\cdot)$ in (8). Fix arbitrarily $r > 0$, and select $\gamma > 0$ such that (12) holds. Take M in (30) such that

$$\Phi > b\Theta_2/M, \tag{34}$$

where Φ and Θ_2 are given respectively in (10) and (15). Suppose that $\|x(0)\| \in B_\eta$ where η is given in (16) and let $\left[\zeta_1(0), \dot{\zeta}_1(0)\right]^T = \left[\zeta_1(0), \zeta_2(0)\right]^T$, the initial state vector of (30), be taken such that (33) holds. Then, the following is satisfied

$$V_1\left(x\left(t\right)\right) \le U_1\left(\dot{\zeta}_1\left(t\right)\right) = M\exp\left(-\left(2b/M\right)t\right)\dot{\zeta}_1^2\left(0\right)/2; \forall t \ge 0. \tag{35}$$

Proof. Observing (33) if the lemma claim is false, there must exist $t_1 \ge 0$ such that $V_1(\tau) \le U_1(\tau)$ for all $\tau \le t_1$, $V_1(t_1) = U_1(t_1)$ and $\dot{V}_1(t_1) > \dot{U}_1(t_1)$. Since $U_1(\dot{\zeta}_1(t))$ is a monotonically decreasing function, $\|x(t_1)\| \le r$. Hence, from (13)

$$\dot{V}_1\left(x\right) \le -\left[\|x_1\|, \|x_2\|\right]\Phi\left[\|x_1\|, \|x_2\|\right]^T. \tag{36}$$

Since $V_1(t_1) = U_1(t_1)$ we have using (6), (8), (15), and (31)

$$\begin{aligned}\dot{\zeta}_1^2(t_1) &\leq \frac{k}{M}\|x_1(t_1)\|^2 + \frac{M_2}{M}\|x_2(t_1)\|^2 + \frac{2\gamma}{M}\|x_1(t_1)\|\,\|x_2(t_1)\| \\ &= \frac{1}{M}\left[\|x_1(t_1)\|, \|x_2(t_1)\|\right]\Theta_2\left[\|x_1(t_1)\|, \|x_2(t_1)\|\right]^T. \qquad (37)\end{aligned}$$

Hence, in view of the second equation in (13), (33)-(34), and (36)-(37)

$$\begin{aligned}\dot{U}_1(t_1) &= -b\left\|\dot{\zeta}_1(t_1)\right\|^2 \\ &\geq -\frac{1}{M}b[\|x_1(t_1)\|, \|x_2(t_1)\|]\Theta_2[\|x_1(t_1)\|, \|x_2(t_1)\|]^T > \dot{V}_1(t_1). \qquad (38)\end{aligned}$$

But (38) contradicts the assumption $\dot{V}_1(t_1) > \dot{U}_1(t_1)$, and we complete the proof. $\diamondsuit\diamondsuit$

For further applications consider the following procedure. Let, for a given pair of gains $\{k, b\}$ and for a fixed $r > 0$, a constant $\gamma > 0$ be selected, and respectively a constant M be determined such that (34) and hence (35), are satisfied. Define arbitrary $\alpha > 1$ and let the gains in the systems (3) and (30) be replaced as follow: $k \to \bar{k} = \alpha k$, and $b \to \bar{b} = \sqrt{\alpha}b$. Take $\bar{r} = \sqrt{\alpha}r$ and $\bar{\gamma} = \sqrt{\alpha}\gamma$, and by observing (8), (10), (15), and (31) define $\bar{V}_1(\cdot)$, $\bar{\Phi}$, $\bar{\Theta}_1$, $\bar{\Theta}_2$, and $\bar{U}_1(\cdot)$ by using the new constants $\bar{k} = \alpha k$, $\bar{b} = \sqrt{\alpha}b$, $\bar{r} = \sqrt{\alpha}r$, and $\bar{\gamma} = \sqrt{\alpha}\gamma$. Then, we have the following result.

Lemma 4.3. Consider the system (3) with $\{k, b\} \to \{\bar{k}, \bar{b}\}$. Suppose that $\|x(0)\| \in B_{\bar{\eta}}$ where $\bar{\eta}$ is given in (16) with $r \to \bar{r}$, $\Theta_1 \to \bar{\Theta}_1$, and $\Theta_2 \to \bar{\Theta}_2$. Let $\dot{\zeta}_1(0)$, the initial velocity of (30) with $b \to \bar{b}$, be taken such that $\bar{V}_1(x(0)) \leq \bar{U}_1(\dot{\zeta}_1(0))$. Then,

$$\bar{V}_1(x(t)) \leq \bar{U}_1(\zeta(t)) = M\exp\left(-\left(2\sqrt{\alpha}b/M\right)t\right)\dot{\zeta}_1^2(0)/2; \forall t \geq 0. \qquad (39)$$

Proof. Since M was selected such that (34) holds, following the proof of lemma 4.2, to establish the present lemma it is sufficient to show that

$$\Phi > b\Theta_2/M \Rightarrow \bar{\Phi} > \bar{b}\bar{\Theta}_2/M. \qquad (40)$$

Recalling (10) and (15), we have

$$\Phi > b\Theta_2/M \Rightarrow \begin{bmatrix} \gamma k m_1 - \frac{bk}{M} & -\frac{1}{2}\gamma m_2(k_c r + b) - \frac{b\gamma}{M} \\ -\frac{1}{2}\gamma m_2(k_c r + b) - \frac{b\gamma}{M} & b - \gamma - \frac{bM_2}{M} \end{bmatrix} > 0. \qquad (41)$$

Moreover, the left hand-side of (41) implies

$$\gamma k m_1 - \frac{bk}{M} > 0; \left(\gamma k m_1 - \frac{bk}{M}\right)\left(b - \gamma - \frac{bM_2}{M}\right) > \left(\frac{1}{2}\gamma m_2(k_c r + b) + \frac{b\gamma}{M}\right)^2. \qquad (42)$$

But recalling that $\bar{k} = \alpha k$, $\bar{b} = \sqrt{\alpha}b$, $\bar{r} = \sqrt{\alpha}r$, and $\bar{\gamma} = \sqrt{\alpha}\gamma$, (42) yields

$$\bar{\gamma}\bar{k}m_1 - \frac{\bar{b}\bar{k}}{M} > 0; \left(\bar{\gamma}\bar{k}m_1 - \frac{\bar{b}\bar{k}}{M}\right)\left(\bar{b} - \bar{\gamma} - \frac{\bar{b}M_2}{M}\right) > \left(\frac{1}{2}\bar{\gamma}m_2(k_c\bar{r} + \bar{b}) + \frac{\bar{b}\bar{\gamma}}{M}\right)^2, \qquad (43)$$

which ensures that the right hand-side of (40) is satisfied. $\diamondsuit\diamondsuit$

In view of lemma 4.3 and equations (29) and (28) we conclude the following. As far as the upper bound in (39) is concerned, the adjustment of the controller gains of the robotic system according to the rule $k \to \bar{k} = \alpha k$, $b \to \bar{b} = \sqrt{\alpha}b$, yields a result which resembles the effect of a similar change in the gains in the linear system (27).

Remarks. The *fictitious one-link robot* (30) allows us to select the controller gains and to derive the upper bound (35). A similar model of a one-link robot is used in (Ailon, 1995). However in this study the upper-bound is obtained by *increasing* in a proper way the moment of inertia M in (30), while in the last reference the estimate to the robot energy (in the presence of *non-conservative forces*) is obtained by *reducing* the moment of inertia of the one-link robot whose energy function provides the upper-bound.

5 Conclusions

This study presents an estimate to the energy function of a rigid robot with a PD controller in a point-to-point task. The resulting exponential upper bound exhibits an explicit relation between the controller structure and the system rate of convergence. A simple procedure for regulating the controller gains has been established.

In view of the present approach, in the case of model uncertainty, as long as the upper and lower bounds in (6) are available the procedure which allows us to select the desired controller parameters and to evaluate the form of the system response, is still applicable.

More efficient procedures that may propose 'tighter' estimates to the energy function for more complicated models like a flexible-joint robot, are subject to future research.

References

Ailon, A. and R. Ortega (1993). "An observer-based set-point controller for robot manipulators with flexible joints," *Systems and Control Letters*, **21**, pp. 329-335.

Ailon, A. (1995). "Estimate of flexible-joint robot model response times and some control applications," *Systems and Control Letters*, **26**, pp. 79-85.

Berghuis, H. and H. Nijmeijer (1993). "Global regulation of robots using only position measurements," *Systems and Control Letters*, **21**, pp. 289-293.

Kelly, R., R. Ortega, A. Ailon and A. Loria (1994). "Global regulation of flexible joint robots using approximate differentiation," *IEEE Trans. Atomat. Contr.*, **39**, pp. 1222-1224.

Khalil, H. K. (1996) *Nonlinear Systems*, New Jersey, Prentice Hall.

Ortega, R. and M. W. Spong (1989). "Adaptive motion control of rigid robots: a tutorial," *Automatica*, **25**, pp. 877-888.

Ortega, R., A. Loria, R. Kelly and L. Praly (1995). "On passivity-based output feedback global stabilization of Euler-Lagarnge systems," *Int. J. Robust Nonlinear Contr.*, **5**, pp. 313-323.

Spong, M. and M. Vidyasagar (1989). *Robot Dynamics and Control*, New York, Wiley.

Takegaki, M. and S. Arimoto (1981). "A new feedback method for dynamic control manipulators," *ASME J. Dyn. Systems Meas. Contr.*, **102**, pp. 119-125.

Tomei, P., (1991). "A simple PD controller for robots with elastic joints," *IEEE Trans. Atomat. Cont.*, **36**, pp. 1208-1213.

Vidyasagar, M. (1993). *Nonlinear Systems Analysis*, New Jersey, Prentice Hall.

Wen, J. T. and D. S. Bayard (1988). "New class of control laws for robotic manipulators Part 1. Non-adaptive case," *Int. J. Contr.*, **47**, pp. 1361-1385.