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Practical Stability of Synchronized Chaotic Attractors and its Control

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Abstract

We discussed the application of the concept of practical stability to chaotic synchronized attractors located at the invariant subspaces. Additionally, we present the controlling method which allows enlarging the practical stability regions.

1 Introduction

One of the fundamental problems in practical applications of chaotic dynamics is the problem of stability of chaotic attractor A. e.g., chaos synchronization (Fujisaka and Yamada, 1983; Afraimovich *et al.*, 1986; Pecora and Carroll, 1990; Anishchenko *et al.*, 1991; Lai and Grebogi, 1993; Kapitaniak, 1994; Carroll, 1995; Chen, 1996; Carroll *et al.*, 1996; Kapitaniaak, 1996). The basin of attraction $\beta(A)$ is the set of points whose ω -limit set is contained in A. In Milnor's definition (Milnor, 1985) of an attractor the basin of attraction need not include the whole neighborhood of the attractor, i.e., we say that A is a Milnor attractor if $\beta(A)$ has positive Lebesgue measure. For example, riddled basins (Alexander *et al.*, 1992; Ott *et al.*, 1994; Heagy *et al.*, 1996; Kapitaniak, 1995; Kapitaniak and Chua, 1996) which have recently been found in practical physical systems (they usually occur in the systems with a simple class of symmetry like most of the chaos synchronization procedures (Heagy *et al.*, 1996; Kapitaniak, 1995; Kapitaniak and Chua, 1996), have positive Lebesgue measure but do not contain any neighborhood of the attractor. If the basin of attraction contains the neighborhood of A, then attractor is asymptotically stable.

Most of the chaotic attractors which can be met in the practical engineering systems are quasiattractors, i.e., the limiting sets enclosing the periodic orbits of different topological types, structurally unstable homoclinic trajectories, etc. Practical systems are mainly quasihyperbolic, i.e., many different types of attractors co-exist in the phase space.

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Existing definitions of the stability of chaotic attractors can not always give sufficient practical information about the behaviour of the real engineering system which is under the influence of both permanently acting and short time impulse-like perturbations. Even the property of the asymptotic stability of the synchronized chaotic state can be practically insufficient as the basin of attraction of the asymptotically stable chaotic attractor can be so small that perturbations can take trajectory out of it to the basin of another attractor.

In this paper we introduce the concepts of the practical stability (Bogusz, 1966; Kapitaniak and Brindley, 1998) of the chaotic synchronized attractors located at the invariant manifold.

The paper is organized as follows. In Sec. 2 we recall the definition of practical stability of chaotic attractors and describe its possible importance in the study of the stability of the synchronized chaotic systems. The examples of practically stable and unstable attractors are shown in Sec. 3. Section 3 also introduces the controlling procedure which allows to extend the regions of practical stability in the parameter space. Finally, we summarize our results in Sec. 4.

2 Practical stability of chaotic attractors

Consider a dynamical system given by

$$\frac{dx}{dt} = f(x,t) \tag{1}$$

which for initial conditions $x(t_0) = x_0 \in \omega$, where ω is an open set, has asymptotically stable attractor $A \in \mathcal{R}^n$.

Definition

(1) Let the system (1) be under the influence of permanently acting perturbations $p(x,t)$ so the perturbed system is in the form

$$\frac{dx}{dt} = f(x,t) + p(x,t). \tag{2}$$

(2) Let perturbation function $p(x,t)$ fulfill condition

$$||p(x,t)|| \le \delta$$

where $\delta > 0$.

(3) Let Ω be a closed, bounded set such that $A \in \Omega$ and $\omega \in \Omega$.

If for all initial conditions $x(t_0) = x_0 \in \omega$, all functions $p(x,t)$ and all $t \ge t_0$, $x(t) \in \Omega$ then attractor A is practically stable (in relation to sets ω, Ω and perturbations $p(x,t)$). $\diamond$

In this definition function $p(x,t)$ describes all continuously acting perturbations. Set ω defines limits of both uncertainties in initial conditions and short time perturbations. Perturbed trajectories $x(t)$ of the system (2) evolve in the region of the phase space given by the set Ω which is usually larger than attractor A of unperturbed system (1).

If attractor A is a fixed point (Bogusz, 1966) and in the absence of permanently acting perturbations ($p(x,t) = 0$) the above definition is equivalent to the definition of the stability in the sense of Lagrange in relation to the set Ω. In every case the practical stability is independent of the stability in the sense of Lyapunov.

As many engineering systems operate in a finite time we introduced in (Kapitaniak and Brindley, 1998) the weaker definition of practical stability in finite time but this problem will not be discussed here.

The application of the above definition will be illustrated on the example of unidirectionally coupled systems

$$\frac{dx}{dt} = f(x) \tag{3a}$$

$$\frac{dy}{dt} = f(y) + d(x - y) \tag{3b}$$

where $x, y, \in \mathcal{R}^n$, $d \in \mathcal{R}^n$ is constant. For $d = 0$ both x and y subsystems evolve on the asymptotically stable chaotic attractor A. It is well-known that there exists value of d for which x and y subsystems can synchronize, i.e., $x = y$ (Kapitaniak, 1996). In the synchronized state chaotic attractor A located on the invariant manifold $x = y$ has to be asymptotically stable in the $2n$-dimensional phase space of the coupled system (3).

Introducing new variable $e = x - y$ we can rewrite eq.(3) as follows

$$\frac{dx}{dt} = f(x) \tag{4a}$$

$$\frac{de}{dt} = f(x) - f(y) - de \tag{4b}$$

where eq.(4a) describes evolution on the chaotic attractor A and eq.(4b) evolution transverse to it. In the synchronized state when $x = y$ $e = 0$ is a fixed point of eq.(4b).

Suppose there exists a function $V(e,t) \in \mathcal{C}^1$ given for all e, $x, y \in A$ and $t \geq 0$ such that

(1)

$$V(e,t) > 0$$

for $e \neq 0$.

(2)

$$\frac{dV(e,t)}{dt} \leq 0$$

for $e \in \mathcal{R}^n - \omega$.

(3)

$$V(e_1,t_1) < V(e_2,t_2)$$

for $e_1 \in \omega$, $e_2 \in \mathcal{R}^n - \Omega$ and $t_1 < t_2$.

We have the following result:

Theorem

If there exists a function $V(e,t)$ which fulfills conditions (1-3) for all $x \in A$, then attractor A located on the invariant manifold $x = y$ is practically stable (in relation to sets ω, Ω and perturbations $p(x,t)$.

Proof Consider the solution $e(t)$ for initial conditions $x(t_0) \in \omega$. As set ω is open and $\omega \subset \Omega$ there exists $t_1 > t_0$, such that for all $x, y \in A$ and $e(t_1) \in \omega$. If for $t > t_1$ solution $e(t)$ for all $x \in A$ stays in Ω then the theorem is true. Suppose that $x(t)$ leaves set Ω and for $t_2 > t_1$, $e(t_2) \in \mathcal{R}^n - \Omega$. From the condition (3) one can write

$$V[e(t_1), t_1] < V[e(t_2), t_2].$$

This means that the function $V(e,t)$ grows along $e(t)$, but this is impossible as based on conditions (1-2) $V(e,t)$ is a nongrowing function. This proves theorem.

3 Controlling practical stability

In this section we present the controlling procedure which allows to enlarge the regions of practical stability in the parameters space of the considered system. Although our controlling method is general we introduce it using the particular but representative for chaos synchronization problems example.

3.1 Example of practical stability and instability

We consider dynamics of two coupled one-dimensional maps (a three-parameter family of two-dimensional piecewise linear endomorphism) (Maistrenko and Kapitaniak, 1996; Kapitaniak and Maistrenko, 1998)

$$F_{l,p} = \begin{cases} f_{l,p}(x_n) + d(y_n - x_n) & : \quad x_{n+1} = px_n + \frac{l}{2}\left(1 - \frac{p}{l}\right)\left(|x_n + \frac{1}{l}| - |x_n - \frac{1}{l}|\right) + d(y_n - x_n) \\ f_{l,p}(y_n) + d(x_n - y_n) & : \quad y_{n+1} = py_n + \frac{l}{2}\left(1 - \frac{p}{l}\right)\left(|y_n + \frac{1}{l}| - |y_n - \frac{1}{l}|\right) + d(x_n - y_n) \end{cases} \tag{5}$$

where $l, p, d \in \mathcal{R}$ which consists of two identical linearly coupled one-dimensional maps.

If parameter point (l,p) belongs to the region of $\prod_0$ given by $\{l > 1, -l/(l-1) < p \leq (1+(1+4l^2)^{1/2}/2l\}$ one-dimensional map $f_{l,p}$ has two symmetrical chaotic attractors $\Gamma^{(+)} = [1+p(l-1)/l, 1]$ and $\Gamma^{(-)} = [-1, -(1+p(l-1)/l)]$. Consider the stability of the corresponding main diagonal attractors $A^{(+)} = \{x = y \in \Gamma^{(+)}\}$ and $A^{(-)} = \{x = y \in \Gamma^{(-)}\}$. Due to the symmetry we can restrict ourselves to the consideration of one of them, let us say $A^{(+)}$, (results for the second one would be the same).

In (Maistrenko and Kapitaniak, 1996) it was shown that the chaotic attractors $A^{(+)}$ and $A^{(-)}$ at the main diagonal are asymptotically stable if

$$|p - d| < 1$$

and

$$|l - d|^k |p - d| < 1$$

where

$$(l,p) \in \left\{ l > 1, -\frac{l+1}{l} < p \leq -\frac{1+\sqrt{1+4l^2}}{2l} \right\} \qquad if \qquad k = 2$$

and

$$(l,p) \in \left\{ l > 1, -\frac{l^k - 1}{(l-1)l^{k-1}} < p \leq -\frac{l^{k-1} - 1}{(l-1)l^{k-2}} \right\} \qquad if \qquad k = 3, \ldots$$

The boundaries of asymptotic stability region in the d-parameter space are given by the relation

$$|l - d||p - d|^{\frac{\mu}{1-\mu}} = 1$$

where the measure $\mu = \mu_{l,p}(\{|x| > 1/l\})$. It should be mentioned here that at some parameter values invariant measure $\mu_{l,p}$ of the map $f_{l,p}$ can be easily constructed (Maistrenko and Kapitaniak, 1996).

An example of asymptotically stable attractor $A^{(+)}$ at the main diagonal can be found in (Maistrenko and Kapitaniak, 1996; Kapitaniak and Maistrenko, 1998). The basin of attractor $A^{(+)}$ is shown in white while the basin of the other attractor $A^{(-)}$ (not shown in this figure) is indicated in black. If we define set ω and Ω in the way $\omega, \Omega \subset \beta(A^{(+)})$ and allow perturbations $p(x,t)$ to evolve only in Ω attractor $A^{(+)}$ is practically stable in relation to sets ω, Ω and perturbations $p(x,t)$. If the sets ω and Ω are too small from practical point of view and we have to consider $\omega, \Omega \not\subset \beta(A^{(+)})$ attractor $A^{(+)}$)is no more practically unstable.

3.2 Controlling procedure

We can enlarge the d-parameter space region of practical stability if we adopt the controlling procedure described in (Lai, 1996). The set Ω contains a fraction f^+ (larger one) of initial conditions which yield trajectories that asymptote to attractor $A^{(+)}$, and the remaining initial conditions (smaller fraction f^-), asymptote to attractor $A^{(-)}$. Our goal is to enlarge the fraction f^+ to 1 so the typical trajectory originating in Ω will asymptote at $A^{(+)}$. To achieve this goal we assume that one of the system parameters, let us say d can be adjusted finely around a nominal value d_0, i.e., $d \in [d_0 + \Delta p, p_0 - \Delta p]$, where $\Delta p / p_0 << 1$.

To allow the practical stability control we consider the region $\sum$ ($\Omega \in \sum$) of the phase space and we build the "bush-like" structure of paths to the set Ω. First consider the randomly chosen initial point in $\sum$ such that it generates a trajectory leading to Ω. We call this trajectory the root path 1 and denote it by $X_0, X_1, ..., X_\Omega$, where $X_\Omega \in \Omega$. Next we choose the second trajectory to Ω from an arbitrary initial condition $Y_0 \in \sum$. For this second path, we examine if it approaches to Ω directly (not coming close to the path 1), in which case we call it root path 2. If Y_n falls into a suitably small neighborhood of some points along root path 1 before it comes close to Ω in this case the trajectory $Y_0, Y_1, ..., Y_n$ is the secondary path of the root path 1. This procedure can be repeated for initial conditions chosen on a uniform grid of size δ in $\sum$. If the trajectory goes to the undesirable attractor $A^{(-)}$, we simply disregard this trajectory in bush building procedure. Finally a hierarchy of paths to Ω in $\sum$ can contain, say, N_r paths. Each root path i can have some secondary paths, and on each secondary path there can be third-order paths, etc. The only difference between the described bush structure and one given in (Lai, 1996), is that in our case paths terminate in the set Ω not on the desired attractor $A^{(+)}$ so they are shorter and require less computer memory to storage them.

To control a trajectory to direct it to the set Ω after it comes close to a path on the bush, we use a simple feedback procedure. Suppose that the trajectory of the N-dimensional map $x_{n+1} = M(x_n, p)$ (a map (5) is an example of such system) originated from leaving the set Ω falls into a ϵ-neighborhood of a point y_n on the bush at some later time n, i.e., $|x_n - y_n| \leq \epsilon$. Let $y_n, y_{n+1}, ..., y_\Omega$ be a path on the bush. In the neighborhood of this path, we can consider the following linearized dynamics

$$\Delta x_{x+1} = DM(x_n, d)\Delta x_n + \frac{\partial M}{\partial d}\Delta d_n,$$

where $\Delta x_n = x_n - y_n$, $\Delta d_n = d_n - d_0$, and the Jacobian $DM(x_n, d)$ and the vector $\partial M / \partial d$ are calculated at $x_n = y_n$ and $d_n = d_0$. If we choose a unit vector u in the phase space and let $u\Delta x_{n+1} = 0$, we obtain the d- parameter perturbation necessary to achieve control

$$\Delta d_n = -\frac{uDM(x_n, d)\Delta x_n}{u\frac{\partial M}{\partial d}}. \tag{6}$$

The unit vector can be chosen arbitrarily provided that (i) it is not orthogonal to x_{n+1}, and (ii) the denominator in (6) is not zero.

We have controlled the system (1) and consider the set $\Omega = \{(x, y); x \in (0.5, 0.9), x - 0.3 < y < x + 0.3\}$ (without control about 18 % of the initial condition originated in Ω leave it). We assumed that an accessible parameter can be slightly perturbed around its nominal value $-d = 0.95$ as we took the maximal allowed parameter perturbation Δd_{max} to be 10^{-2}. We built a bush of paths to Ω by using a grid of 600×600 initial conditions in the region $\mathcal{L} = \{(x, y); x \in (0.45, 0.65), x - 0.5 < y < x + 0.5\}$ and in the controlling procedure took $\epsilon = 10^{-3}$. With such a control we managed to ensure that only 2 % of trajectories is not asymptotized in Ω.

Practically we observed that in the case when the set Ω has the property that all the points which belong to it and to the basin of undisered attractor A^{-} have in its neighborhood points which belong to the basin of desired attractor $A^{(+)}$ our controlling method is nearly 100 % effective. Unfortunately, in most systems such Ω set will be no more larger than the basin of $A^{(+)}$.

4 Conclusions

We hope that the concept of practical stability can be very useful in the study of chaos synchronization problems. Having the practical device we usually can estimate the bounds of possible continuously acting and short time perturbations and define sets ω and Ω. If the synchronized chaotic state is practically stable in relation to the considered perturbation we can be sure that the evolution of the system will not leave the attractor further than allowed by the boundaries of the set Ω.

Property of the practical stability of the synchronized chaotic state seems to be essential for practical applications of chaos synchronization like for example secure communication.

The described controlling procedure of practical stability allows the trajectories leaving the set Ω to be diverted into it. The application of this method to the practically unstable systems makes the loss of stability only temporal. Unfortunately, this method is only effective when Ω is not much larger than the basins of attraction of the considered attractor.

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