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Near Optimal PLL Design for Decision Feedback Carrier and Timing Recovery

Oded Yaniv*
Faculty of Engineering,
Dept. of Ele. Eng. Systems,
Tel Aviv University,
Tel Aviv 69 978, Israel

Dan Raphaeli†
Faculty of Engineering,
Dept. of Ele. Eng. Systems,
Tel Aviv University,
Tel Aviv 69 978, Israel

Abstract

A new design method is presented for the design of PLL loop filters for carrier recovery, bit timing or other synchronization loops given phase noise spectrum and noise level. Unlike the conventional designs, our design incorporates a possible large decision delay and S-curve slope uncertainty. Large decision delays frequently exists in modern receivers due to, for example, a convolutional decoder or an equalizer. The new design also applies to coherent optical communications where delay in the loop limits the laser line width. We provide an easy to use complete design procedure for second order loops. We also introduce a design procedure for higher order loops for near-optimal performance. We show that using the traditional second order loop is suboptimal when there is a delay in the loop, and also show large improvements, either in the amount of allowed delay, or the phase error variance in the presence of delay.

1 Introduction

The phase locked loop (PLL) principle is being successfully used for decades for tracking the carrier phase and the bit timing. First or second order loops are sufficient in most cases. Optimal design of PLL without delay in the presence of oscillator phase noise is well known (Holmes, 1982; Lindsey, 1972). Most modern communication receivers incorporate coding and/or equalization and/or partial response detectors, and it is advantageous or sometimes necessary to use the output of the decoder or equalizer for data detection before phase or timing error information is produced for the synchronization loop (Chang and Srinivasagopalan, 1980; Macdonald and Anderson, 1991; Premjiand Taylor, 1987; Sari Hikmet *et al.* 1987). The decoder and/or equalizer creates delay into the operation of the PLL used for the synchronization, and for the case when such delay becomes problematic, several authors proposed combined detection and phase tracking, for example (Macdonald and Anderson, 1991; Simmonsand McLane, 1995), or use less reliable tentative decisions (Ungerboeck, 1982). Between the two loops, the major problem is in the carrier tracking loop since it needs to be wide enough to track the oscillator phase noise. The timing loop works at the symbol rate rather than carrier frequency, therefore its phase noise is normally lower and the loop is allowed to be narrow. However, sufficient delay which

*Email: *yaniv@eng.tau.ac.il.*
†Email: *danr@eng.tau.ac.il*

can be caused by the decoder (for example turbo decoder) can be problematic even for timing loops. The problem of loop design becomes complicated when there is a large uncertainty in the phase detector S-curve slope, which translates to uncertainty in the loop gain. Causes for such uncertainty are numerous, such as residual errors after AGC (or no AGC) in mobile receivers or in burst mode receivers, error rate change in decision feedback loops, timing errors, and ISI.

When significant delay is incorporated into the PLL, the second order loop which is traditionally used is far from being optimal and a new loop filter design is desired. The design presented in this paper is very close to optimal with respect to the mean square error of the phase in the presence of a known delay, phase noise spectrum, requirements for specific gain and phase margins and given loop gain uncertainty. These margins should be kept for any gain (within the range of uncertainty) of the PLL open loop. These combined design constrains are known in the feedback control community as mixed H_2/H_∞ synthesis with output feedback and plant uncertainty.

We use the notation upper gain margin for the maximum amount of loop gain which the PLL can loose without loosing stability, and lower gain margin for maximum increase in loop gain without loosing stability. Both gain and phase margins ensure fast settling step response and eliminate closed loop resonances. For third and higher order PLLs, lower as well as upper gain margins are mandatory in order to guarantee the stability of the PLL.

A general treatment of optimal controller design for loop having only rational transfer functions in the loop is given in (Shaked, 1976). Design of optimal PLL with pure delay can be executed to an arbitrary accuracy using a Padé approximation of high enough order (Friedland, 1996). The outcome of course will be a complex loop filter, but second order approximation leads to satisfactory results. Unfortunately, for a large delay the optimal design will not satisfy the margins constrains. The approach taken here to solve the optimization is a design process composed of two steps. The first step is the solution of the optimal controller for PLL with delay when a Padé approximation replaces the delay. The second step is based on the feedback synthesis theory known as QFT (Borghesani *et al.* 1994; Horowitz, 1992; Horowitz, 1991). The QFT technique is applied to modify the loop filter designed in the first step to satisfy the margins constrains. Finally it was shown how to design an optimal PI loop filter, and it was shown that the optimal loop filter of the PI form is not satisfactory in case of significant delay and/or reasonable gain uncertainty.

The proposed design methodology suits also other fields such as optical communication using coherent detection and RF synthesizers. Although the theory developed here takes place in the continuous time, the same approach can be used for a discrete time PLL.

Most of the work on PLL with delay was done in the framework of optical communications. In (Bary, 1992) the loop filter complexity was bypassed, for the usual laser phase noise spectrum, $\approx 1/f^2$, assuming the loop filter is of the PI form, $2\xi\omega_n + \omega_n^2$, and a design technique to calculate the optimal ω_n was presented. In (Norimatsu and Katsushi, 1991), first and second Padé approximations were used to estimate the degradation of the phase noise variance compared to zero delay, the loop filter again is of the PI form. Treatment of the effect of time delay on the over-all phase error variance was also discussed in (Grany 1987). Here again the same simplified PI loop filter was used and the optimal criterion was the parameter ω_n which was calculated numerically. The significance of the loop delay on the stability of discrete time PLL was discussed quantitatively in (Bergmans, 1995). Finally we would like to mention that loop delay also degrades the PLL loop pull-in rage. For a quantitative discussion based on a simple loop filter see (Morideand Sari, 1987). The structure of this paper is as follows. After the problem statement, an algorithm for high order loop will be developed. Next, an independent design procedure is given for loops having the PI form (second order loops).

2 Statement of the Problem

There are various forms for PLLs, however, without loss of generality we can treat the basic PLL form used for tracking a sinusoid of frequency ω_0. The PLL model used here is depicted schematically in Fig. 1a. It consists of a phase detector, loop filter $F(s)$, VCO and an optional pure delay which represents the undesired effect, for example, of a decision delay in a decision feedback loop. The inputs to the phase detector are two signals: The sum of the carrier with phase modulation or phase noise $\theta(t)$ and noise $n(t)$

$$y(t) = \sqrt{2}A\sin(\omega_0 t + \theta(t)) + n(t),$$

and the VCO output

$$v(t) = \sqrt{2}\cos(\omega_0 t + \hat{\theta}(t)).$$

The output of the phase detector, assuming it includes an appropriate low-pass filter, is

$$e(t) = A\sin(\theta - \hat{\theta}(t)) + n(t).$$

In other forms of PLL the function $\sin(x)$ may be replaced with other appropriate functions which are frequently called S-curve. When tracking, the PLL can be approximated for small phase errors by the linear model as depicted schematically in Fig. 1b, and its open loop transfer function is

$$L(s) = A\frac{1}{s}F(s)e^{-sT}. \tag{1}$$

Our problem is to design a loop filter, $F(s)$, which minimizes the phase error variance σ_e^2, subject to the following data and constrains:

- The power spectral density of the noise, n, is $\Phi_n(\omega)$.
- The power spectral density of the phase modulation or phase noise, θ, is $\Phi_\theta(\omega)$. We assume that θ and n are uncorrelated.
- The open loop delay is T.
- The phase detector gain, A, is fixed but only known to belong to an interval $A \in [A_1, A_2]$ where A_1 and A_2 are known (it reflects for example AGC inaccuracies). Note that if A changes slowly within its allowed interval, the closed loop response in the time range, where A is about A_0, will be approximately as if $A = A_0$.
- The open loop response should have some gain and phase margins in order to guarantee a well damped closed loop response. These margins are defined here by a constant γ or alternatively by a constant δ such that

$$\left|\frac{L(j\omega)}{1 + L(j\omega)}\right| \leq \gamma \text{ or } \left|\frac{1}{1 + L(j\omega)}\right| \leq \delta \tag{2}$$

  for all real ω and $A \in [A_1, A_2]$.

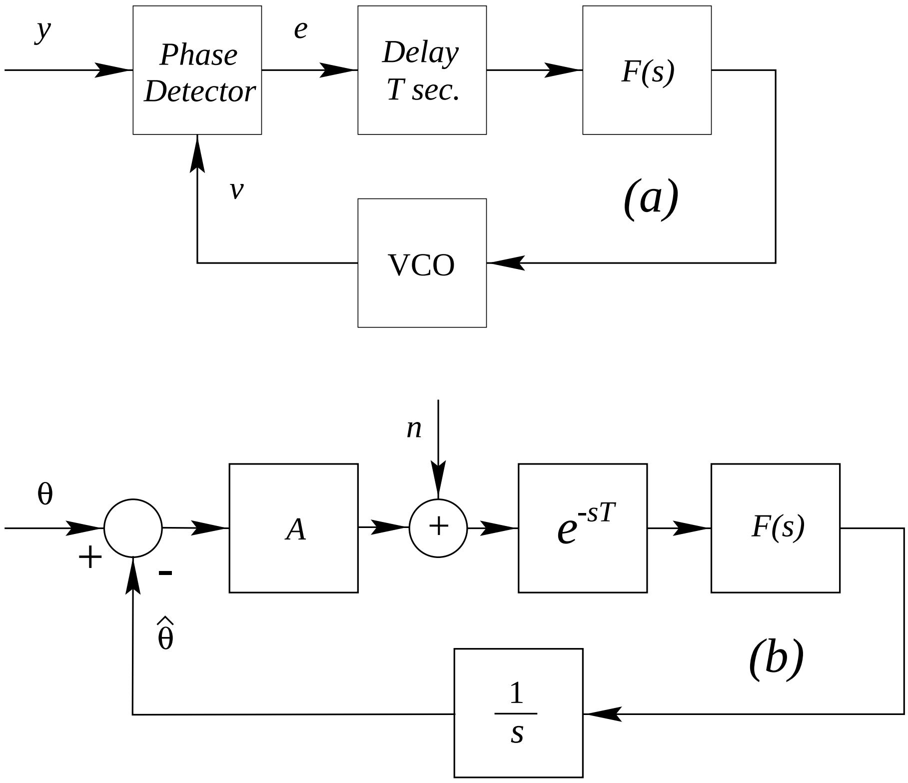

Figure 1: (a) PLL schematic model, (b) its linear approximation

The parameter γ determines the gain and phase margins by

$$20\log\frac{\gamma+1}{\gamma}[dB];\ \ 2\sin^{-1}\frac{1}{2\gamma}[deg]$$

and the parameter δ determines the gain and phase margins by

$$20\log\frac{\delta}{\delta-1}[dB];\ \ 2\sin^{-1}\frac{1}{2\delta}[deg] \tag{3}$$

For example if $\delta = 1.4$ and there is no gain uncertainty, the guaranteed phase and gain margin are 45° and $10dB$ and the guaranteed damping factor (assuming 2nd order model) is 0.4; if one adds $8dB$ gain uncertainty, (that is $20\log\frac{A_2}{A_1} = 8$) the guaranteed phase margin will not change but the gain margin will be $18dB$ for the low A_1 and $10dB$ for A_2. For the correlation between the margins, damping ratio and closed-loop time response such as step, overshoot, etc.; see (D'Azzo and Houpis,1988). Due to the phase detector gain uncertainty, σ_e^2 is not fixed but depends on the phase detector gain. The filter $F(s)$ we search for, minimizes the maximum of σ_e^2 for some $A_0 \in [A_1, A_2]$ subjected to the constrains listed above. Note that σ_e as a function of A_0 is expected to achieve its minimum value over all $A_0 \in [A_1, A_2]$ for $A_0 = A_1$.

3 The Proposed Algorithm

The Laplace transform of $e(t)$ is given by

$$e(s) = \frac{L(s)}{1+L(s)}n(s) + \frac{1}{1+L(s)}\theta(s),$$

and its variance (assuming zero mean) is

$$\sigma_e^2 = \frac{1}{2\pi}\int_{\infty}^{\infty}\left|\frac{L(j\omega)}{1+L(j\omega)}\right|^2 \Phi_n(\omega)d\omega + \frac{1}{2\pi}\int_{\infty}^{\infty}\left|\frac{1}{1+L(j\omega)}\right|^2 \Phi_\theta(\omega)d\omega. \tag{4}$$

From now on the factor $1/2/\pi$ will be ignored in this section. The solution for $F(s)$ which minimizes (4) where the margin conditions are ignored and the pure delay is approximated by a rational transfer function, is a standard stationary filtering problem. For a review and extensions see (Shaked, 1976). The algorithm which is based on coprime factorization and controller parameterization is now described (Doyle *et al.* 1992), then modified to cope with our purpose. But first a notation: since σ_e^2 depends only on $L(s)$ we shall incorporate, for simplicity of the representation, the free integrator into $F(s)$. The open loop will then be

$$L(s) = A\tilde{F}(s)e^{-sT}, \text{ where } \tilde{F}(s) = \frac{F(s)}{s}.$$

Let us assume for simplicity that $P(s)$ is a rational transfer function approximation of e^{-sT} and $P(s) = N(s)/M(s)$ is a coprime factorization over the family of all stable, rational and proper transfer functions (for example if a first order Padé approximation is used, that is $P(s) = \frac{1-sT/2}{1+sT/2}$ then $N = \frac{1-sT/2}{1+sT/2}$ and $M = 1$ are appropriate). Then there exists two transfer functions, $X(s)$ and $Y(s)$ belonging to the same family satisfying

$$N(s)X(s) + M(s)Y(s) = 1,$$

(for the example above, $X(s) = 1$, $Y(s) = \frac{sT}{1+sT/2}$) and $\tilde{F}(s)$ stabilizes the PLL if and only if

$$\tilde{F}(s) = \frac{X + MQ}{Y - NQ} \tag{5}$$

where Q is any stable proper and rational function (Doyle *et al.* 1992).

Let us denote the spectral factorization of Φ_n and Φ_θ by

$$\Phi_n = \phi_n(s)\phi_n(-s), \quad \Phi_\theta = \phi_\theta(s)\phi_\theta(-s) \tag{6}$$

where $\phi_n(s)$ and $\phi_\theta(s)$ are proper minimum-phase stable transfer functions. Substituting equations (5,6) in equation (4) gives

$$\sigma_e^2 = \int_{\infty}^{\infty} |M(Y - NQ)\phi_n|^2 d\omega + \int_{\infty}^{\infty} |N(X + MQ)\phi_\theta|^2 d\omega. \tag{7}$$

Using the notation $NM = U_{ap}U_{mp}$ where U_{ap} is all-pass and U_{mp} a stable minimum-phase, the integrand of equation (7) at $s = j\omega$ reduces to

$$\begin{aligned}
&|MY\phi_n - U_{ap}U_{mp}\phi_n Q|^2 + |NX\phi_\theta + U_{ap}U_{mp}\phi_\theta Q|^2 = \\
&\left|U_{ap}^{-1}MY\phi_n - U_{mp}\phi_n Q\right|^2 + \left|U_{ap}^{-1}NX\phi_\theta + U_{mp}\phi_\theta Q\right|^2 = \\
&\left|\left(U_{ap}^{-1}MY\phi_n\right)_{un} + \left(U_{ap}^{-1}MY\phi_n\right)_{st} - U_{mp}\phi_n Q\right|^2 + \\
&\qquad\qquad \left|\left(U_{ap}^{-1}NX\phi_\theta\right)_{un} + \left(U_{ap}^{-1}NX\phi_\theta\right)_{st} + U_{mp}\phi_\theta Q\right|^2,
\end{aligned} \tag{8}$$

where the subscript un stamps for the unstable part of the transfer function and st for its stable part. Since our design parameter Q must be stable, the Parseval theorem gives

$$\begin{aligned}\sigma_e^2 &= \int_{\infty}^{\infty} \left(\left| \left(U_{ap}^{-1} M Y \phi_n \right)_{un} \right|^2 + \left| \left(U_{ap}^{-1} M Y \phi_n \right)_{st} - U_{mp} \phi_n Q \right|^2 \right) \\ &+ \int_{\infty}^{\infty} \left(\left| \left(U_{ap}^{-1} N X \phi_\theta \right)_{un} \right|^2 + \left| \left(U_{ap}^{-1} N X \phi_\theta \right)_{st} + U_{mp} \phi_\theta Q \right|^2 \right) d\omega. \end{aligned} \tag{9}$$

From equation (9) it is clear that Q minimizes σ_e^2 if and only if it minimizes

$$\begin{aligned}\sigma_{e1}^2 &= \int_{\infty}^{\infty} \left(\left| \left(U_{ap}^{-1} M Y \phi_n \right)_{st} - U_{mp} \phi_n Q \right|^2 + \left| \left(U_{ap}^{-1} N X \phi_\theta \right)_{st} + U_{mp} \phi_\theta Q \right|^2 \right) d\omega \\ &\stackrel{\text{def}}{=} \int_{\infty}^{\infty} \left(|a(s) + b(s)Q|^2 + |c(s) + d(s)Q|^2 \right)_{s=j\omega} d\omega, \end{aligned} \tag{10}$$

where the last equality is used to define the transfer functions $a(s)$, $b(s)$ $c(s)$ and $d(s)$, respectively. By simple complex arithmetic and using the notation $*$ to denote the operation $W^*(s) = W(-s)$, it can be shown that

$$\sigma_{e1}^2 = \int_{\infty}^{\infty} \left(|\alpha + \beta Q|^2 + aa^* + bb^* - \alpha\alpha^* \right)_{s=j\omega} d\omega \tag{11}$$

where $\beta(s)$ and $\alpha(s)$ satisfy the following equations

$$\beta\beta^* = bb^* + dd^*, \tag{12}$$

$$\alpha^*\beta = ba^* + dc^*. \tag{13}$$

Equation (12) has power spectral density form, hence β is minimum phase and stable. Therefore the stable Q which minimizes σ_e^2 and σ_{e1}^2 denoted here by Q_{opt} is the one that minimizes

$$\sigma_{e2}^2 = \int_{\infty}^{\infty} \left(|\alpha_{un}|^2 + |\alpha_{st} + \beta Q|^2 \right) d\omega, \tag{14}$$

therefore

$$Q_{opt} = -\frac{\alpha_{st}(s)}{\beta(s)}, \tag{15}$$

and the optimal filter, $\tilde{F}_{opt}(s)$, is given by equation (5)

$$\tilde{F}_{opt}(s) = \frac{X + MQ_{opt}}{Y - NQ_{opt}}. \tag{16}$$

Clearly $\tilde{F}_{opt}(s)$ is the solution we seek for only if the closed loop satisfies the gain and phase margin specification for all possible loop gains. However if the open loop gain interval is large and/or the desired margins are large and/or the delay is too large compared to the PLL open loop bandwidth, $\tilde{F}_{opt}(s)$ will not be a satisfactory solution. It might even destabilize the system for some of the possible open loop gains (most likely for high gains). Our next step is devoted to show how to synthesize an appropriate $F(s)$ by modifying Q_{opt}.

Let us assume that the gain and phase margin specifications are of the following form (the other margin form, equation (2), is treated similarly)

$$\left| \frac{L}{1+L} \right| = \left| \frac{APF}{1+APF} \right|_{s=j\omega} \leq \gamma, \ \ \forall\, A \in [A_1, A_2]. \tag{17}$$

Using the notation

$$Q = Q_{opt} - \frac{\lambda(s)}{\beta(s)}, \tag{18}$$

and assuming that L can be approximation by PF

$$\begin{aligned} \left|\frac{L}{1+L}\right| &= \left|\frac{AN(X+MQ)}{M(Y-NQ)+AN(X+MQ)}\right| \\ &= \left|\frac{AN(X+MQ_{opt}) - \frac{ANM}{\beta}\lambda(s)}{M(Y-NQ_{opt})+AN(X+MQ_{opt}) + \frac{MN-ANM}{\beta}\lambda(s)}\right|. \end{aligned} \tag{19}$$

Hence inequality (17) reduces to the following binary inequality on the transfer function $\lambda(s)$

$$\left|\frac{AN(X+MQ_{opt}) - \frac{ANM}{\beta}\lambda(s)}{M(Y-NQ_{opt})+AN(X+MQ_{opt}) + \frac{MN-ANM}{\beta}\lambda(s)}\right|_{s=j\omega} \leq \gamma, \;\; \forall\, A \in [A_1, A_2]. \tag{20}$$

Moreover, since

$$\sigma_e^2\left(Q_{opt} - \frac{\lambda(j\omega)}{\beta(j\omega)}\right) - \sigma_e^2(Q_{opt}) = |\lambda(j\omega)|^2\,, \tag{21}$$

the *2-norm* of $\sigma_e^2(Q_{opt})$ is less than the *2-norm* of $\sigma_e^2(Q)$ by the *2-norm* of $\lambda(s)$. Inequality (20) and equation (21) translate our problem into the following problem: Find a stable transfer function, $\lambda(s)$, whose *2-norm* is as small as possible such that inequality (20) is true for all ω. The solution we seek will be $\tilde{F}$ of equation (5), where Q is defined in equation (18). This problem, with some modification, can be solved within the framework of the feedback synthesis theory known as QFT (Borghesani *et al.* 1994; Horowitz, 1992; Horowitz, 1991). The QFT technique modified to our problem, as stated above, is now described with the help of an example.

3.1 Example 1

The example parameters are (units are radians and seconds): $\Phi_\theta = 50^2/\omega^4$, $\Phi_n = 0.01$, open loop delay $T = 0.01$ which is approximated by a second order Padé approximation (Friedland, 1996)

$$e^{-sT} = \frac{(1-sT/4)^2}{(1+sT/4)^2},$$

and AGC gain, A, which can be any value in the interval $A \in [1, 2]$. The margins constraint is of the form $\left|\frac{L}{1+L}\right| < 3dB$, which guarantee $45°$ phase margin and $5dB$ gain margin for $A = 2$ and $11dB$ for $A = 1$. These margins are about the lowest one can choose for proper PLL operation (Martin, 1997).

For the AGC gain $A = 1$, the optimal filter, $F(s)$, calculated by the algorithm described above is

$$F(s) = \frac{150(s+33.3)(s+400)^2}{s(s^2+750s+600^2)}.$$

Using Bode plot of the open loop, $L(s)$, of equation (1), is shown in Fig. 2. It also includes the open loop where the Padé approximation replaces the pure delay. Clearly the phase margin

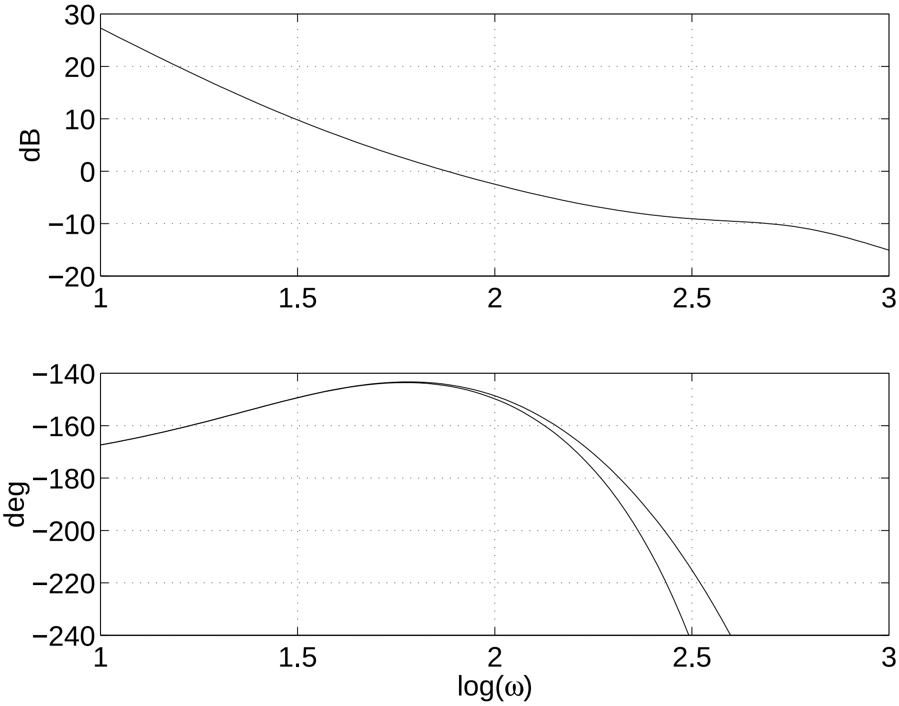

Figure 2: Bode plot of the optimal open loop without AGC uncertainty and margin constraints, the phase plot is for the delayed open loop (upper) and Padé approximation (lower)

constraint is not satisfied, one can show that the gain and phase margins are approximately $35°$ and $7.5dB$, respectively, which is $10°$ and $3.5dB$ less than required by the phase margin specified $\gamma = 3dB$. The other transfer function involved in calculating $F(s)$ where:

$$\begin{aligned} X &= 1, \;\; Y = \frac{sT}{s^2T^2/16 + sT/2 + 1}, \\ M &= 1, \;\; N = \frac{s^2T^2/16 - sT/2 + 1}{s^2T^2/16 + sT/2 + 1}, \\ Q &= \frac{s(50 - s)}{s^2 + 100s + 5000}, \;\; \beta = \frac{0.01s^2}{s^2 + 100s + 5000}. \end{aligned}$$

The expression for $L(s)$, $Q(s)$ and $\beta(s)$ are reduced order models of the original transfer functions. The *2-norm* of the cost function, σ_e^2, see equation (4) is 0.16^2. The next step is to design $\lambda(s)$, which is a two step procedure. First we calculate inequality (20). This inequality on $\lambda(j\omega)$, for each frequency ω and fixed A is a circle in the complex plane using real-imaginary coordinates (Chait and Yaniv, 1993). The intersection of all these circles over all A's in the specified interval $[A_1, A_2]$ is the region in the complex plane in which $\lambda(j\omega)$ is allowed to take values. These regions are shown in Fig. 3 using amplitude and phase coordinates instead of real imaginary coordinates. For example at $\omega = 70$, $\lambda(j70)$ should be inside the closed curve marked 70; at $\omega = 100$, $\lambda(j100)$ should be inside the closed curve marked 100; and at $\omega = 30$ $\lambda(j30)$ should be below the curve marked 30, which is in-fact a closed curve in the real imaginary plane.

The next step is to design $\lambda(s)$ such that $\lambda(j\omega)$ is within the allowed region and its *2-norm* is as small as possible. This process is a trial and error process known as loop shaping. One

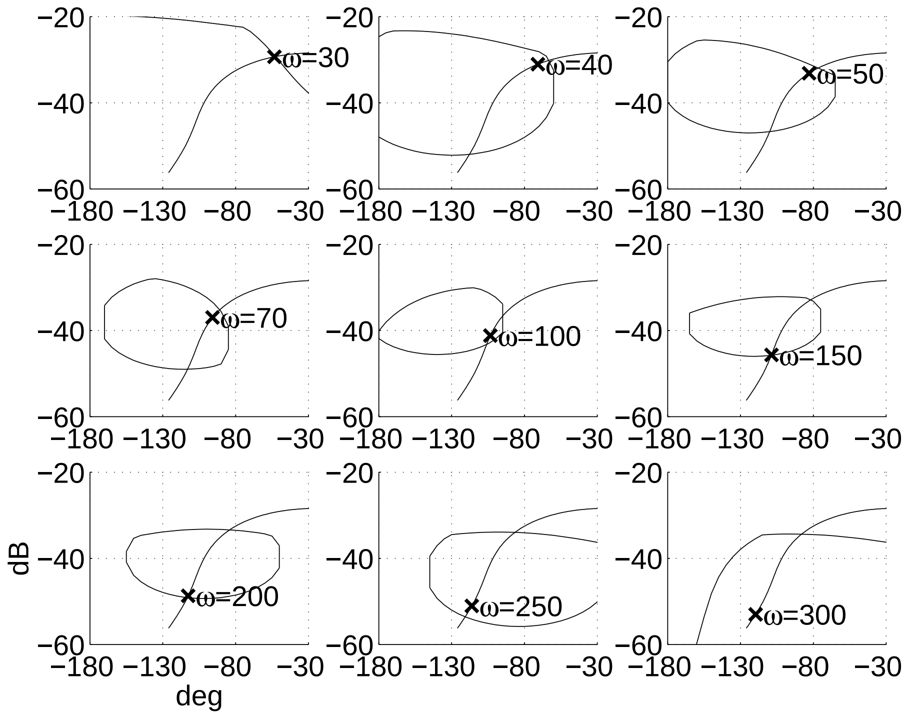

Figure 3: Complex plane regions for $\lambda(j\omega)$ at some frequencies. The plot of $\lambda(j\omega)$ vs. ω is shown and the appropriate $\lambda(\omega)$ is marked by $\times$

can start with a second order transfer function and iterate on its parameters, then add more elements such as lead, lags etc. until a satisfactory result is obtained. For our example the shaped $\lambda(s)$ is

$$\lambda(s) = \frac{440(s+7)}{(s+600)(s^2+53s+37^2)}.$$

This designed $\lambda(j\omega)$ appears in each sub-plot in Fig. 3 and the relative frequencies are marked by $\times$, clearly at each frequency it is within its allowed region. The reduced order loop filter is

$$F_r(s) = 140\frac{(s+205)(s+20)}{s(s+700)}.$$

The PLL was simulated for a phase step for $A=1$ and $A=2$, the simulation is shown in Fig. 4.

Fig. 5 compares between the loop filter $F(s)$ which was calculated for a fixed A and $F_r(s)$ which also satisfies the margin constrains. The comparison uses the Nichols chart of the open loop $L(s)$, equation (1), instead of the Bode plot because the phase and gain margins are easily compared. Also depicted in Fig. 5 is a closed region. This region means that $L(j\omega)$ must be, in all frequencies, outside it in order to satisfy the margin conditions $\left|\frac{L}{1+L}\right| < 3dB$. Clearly the solution F_r satisfies the margin constrains while F does not. The jitter result using F_r for $A=1$ is $\sigma_e = 0.224$ which is $2.8dB$ more than the result using F ($\sigma_e = 0.161$), the solution which ignores the margin specs and uncertainty. We like to compare our result to the performance of a PI controller (second order loop), which was numerically optimized under the margins and uncertainty limits. The PI loop filter is $(31 + 450/s)$, with $\sigma_e = 0.334$ which is $3.5dB$ more than 0.224.

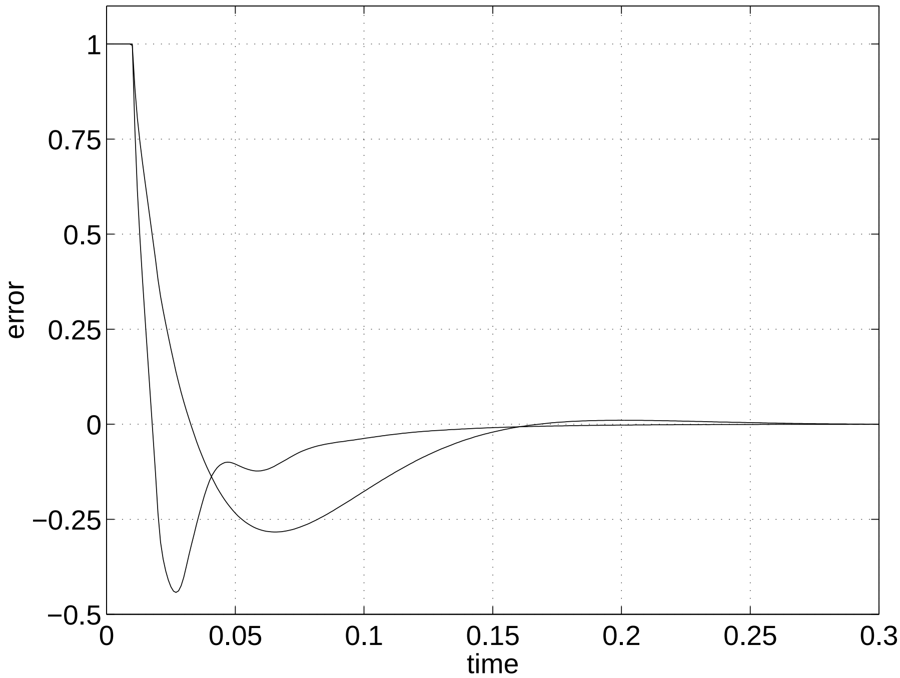

Figure 4: Time domain simulation for a phase step, faster response for $A = 2$, slower for $A = 1$

3.2 Example 2

This is a practical design example for coded-modulation system employed by the company HeliOss Communication Inc., Waltham MA, USA, who build a very high speed, 155Mbps, microwave link at around 30GHz for transmission of SDH/SONET. They use convolutionally coded QAM modulation $40GHz$. The relevant parameters are as follows: The required minimum E_b/N_0 of the coded bits is 11dB, and the decoder delay is 77 bits. Assuming correct symbols fed back, The normalized noise spectral density is $\phi_n = -93dB/Hz$. The measured phase noise spectrum is shown in Fig. 6. In the relevant frequency range in can be approximated by the function transfer function $756/w^4$ also shown in Fig. 6. The noise assumed white, and the system delay is $0.5 \cdot 10^{-6}$sec. It is required to design the PLL filter, $F(s)$, such that phase margin of $40°$ will be guaranteed when the AGC uncertainty can be any value in the interval $[1, 2]$ (6dB range). We have computed the optimal loop (with no margin constrains). The optimal solution has gain margin $9dB$ and phase margin $38°$, which does not satisfy the closed loop requirements. The phase error result using this loop is $\sigma = 2.3$. The result of the PI loop design using Fig. 11 for $40°$ and $6dB$ is $a = -32.3dB$ and $b = 7$ which gives

$$0.0243\frac{7Ts+1}{T^2s^2},$$

and the phase error result is $\sigma = 9.5$. The result of the optimal design subject to the margins and uncertainty constrains using the technique presented here is (s in krad/sec.),

$$F(s) = 1550\frac{(s+7100)(s+4350)(s+1600)(s+420)}{s^2(s+1100)(s^2+23000s+123000)},$$

and the phase error result is $\sigma = 4.9$, which is $5.75dB$ improvment. A comparison between the open loop of the PI solution and optimal one is shown in Fig. 7

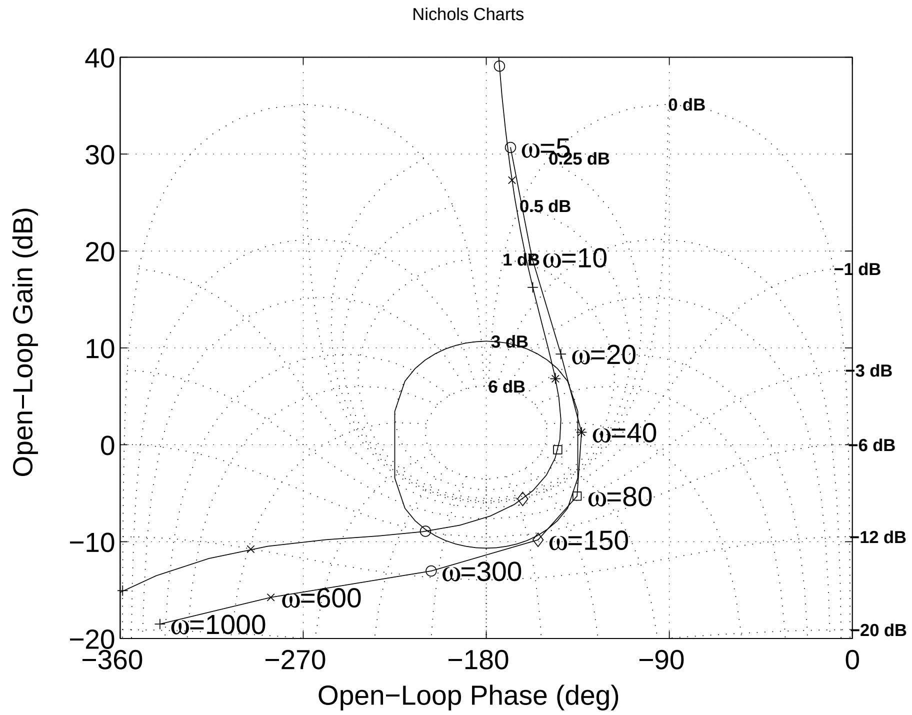

Figure 5: Comparison between the design $F_r(s)$ which satisfies the margin constrains and the optimal one, $F(s)$, which does not satisfy these constrains.

3.3 The Solution for $\Phi_\theta(\omega) \propto \omega^{-4}$ and white noise

Since the near-optimal design method described above is quite complex, we have chosen a very common case of parameters and solved it fully. The result is a cook-book for PLL design with delay, which can be used if the phase noise spectrum can be approximated as $\Phi_\theta(\omega) \propto \omega^{-4}$ and if the margins assumed here are appropriate. Let

$$\Phi_\theta(\omega) = \frac{B_0^2}{\omega^4} \;\text{ and }\; \Phi_n(\omega) = N_0 \tag{22}$$

where B_0 is constant and N_0 is the usual white noise density. Using the partition

$$L(s) = e^{-sT} L_0(s),$$

equation (4)

$$\begin{aligned} \sigma_e^2 &= \frac{1}{2\pi}\int_{\infty}^{\infty} \left| \frac{L_0(j\omega)}{1+e^{-j\omega T}L_0(j\omega)} \right|^2 N_0 d\omega + \frac{1}{2\pi}\int_{\infty}^{\infty} \left| \frac{1}{1+e^{-j\omega T}L_0(j\omega)} \right|^2 \frac{B_0^2}{\omega^4} d\omega \\ &= \frac{N_0}{2\pi T}\left(\int_{\infty}^{\infty} \left| \frac{L_0(j\Omega/T)}{1+e^{-j\Omega}L_0(j\Omega/T)} \right|^2 + \int_{\infty}^{\infty} \left| \frac{1}{1+e^{-j\Omega}L_0(j\Omega/T)} \right|^2 \frac{B_n^2}{\Omega^4} \right) d\Omega \end{aligned} \tag{23}$$

where

$$B_n^2 = \frac{B_0^2 T^4}{N_0}.$$

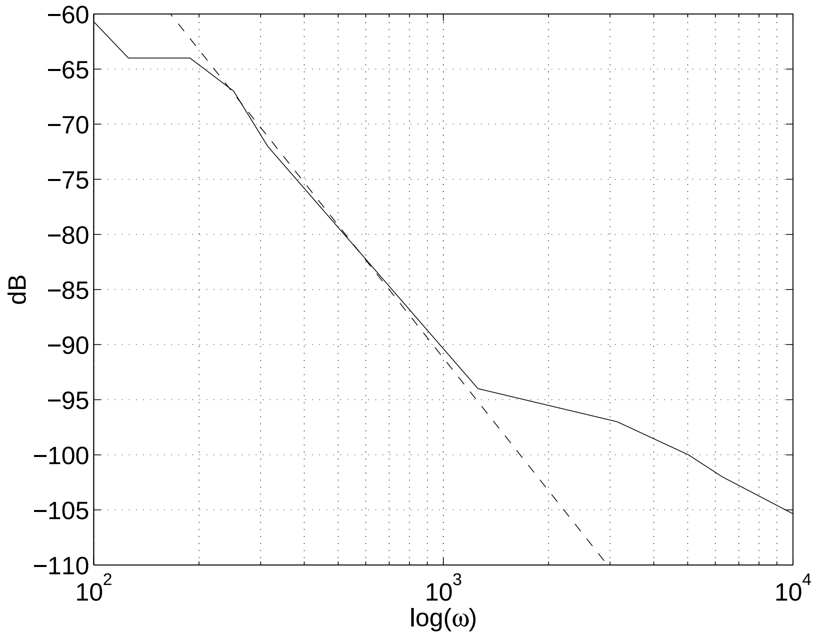

Figure 6: Phase noise spectral density (rigid) and $\propto 1/\omega^4$ (dashed)

Clearly $L_0(j\Omega/T)$ which minimizes σ_e^2 of equation (23), depends only on B_n, in the sense that if $L_0(s)$ minimizes σ_e^2 for $T = 1$ then $L_0(sT_0)$ minimizes σ_e^2 for $T = T_0$, moreover for given B_n^2

$$\sigma_e^2(T, N_0) = \frac{N_0}{T}\sigma_e^2(T = 1, N_0 = 1). \tag{24}$$

We therefore use the design technique developed here to present a PLL designer $L_0(s)$'s which suits different B_n^2's. We limit ourselves to $L_0(s)$'s which have only two free integrators in the origin, phase margin $40°$ and uncertainty $6dB$ (it is equivalent to phase margin $40°$ and gain margin $16dB$ which is in the reasonable PLL operation range).

By trial and error it was found that for $B_n \leq 0.02$ the optimal solution satisfies the margin specs, therefore the case where $B_n \leq 0.02$ is not an interesting case here. From the other hand, as B_n increases, $L_0(s)$ converges to a single solution. We found that $L_0(s)$ approximately stays constant for $B_n \geq 0.15$. Our designs are summarized below:

$$\begin{aligned}
L_0(B_n = 0.02) &= \frac{0.22(s+4)^2(s+0.091)}{s^2(s^2+8.0s+19.4)},\\
L_0(B_n = 0.03) &= \frac{0.29(s+1.0)(s+0.46)(s+0.1094)}{(s+1.8)(s+0.36)s^2},\\
L_0(B_n = 0.04) &= \frac{0.34(s+1.36)(s+0.43)(s+0.11)}{(s+2.5)(s+0.36)s^2},\\
L_0(B_n = 0.05) &= \frac{1.09(s+3.48)(s+2.6)(s+0.43)(s+0.106)}{(s+13.7)(s+3.7)(s+0.30)s^2},\\
L_0(B_n = 0.06) &= \frac{0.78(s+5.08)(s+0.46)(s+0.087)(s^2+5.8s+8.7)}{s^2(s+7.6)(s+0.28)(s^2+7.8s+22.0)}\\
L_0(B_n = 0.08) &= \frac{0.68(s+0.078)(s^2+1.57s+0.84)}{(s+4.6)(s+0.34)s^2}
\end{aligned}$$

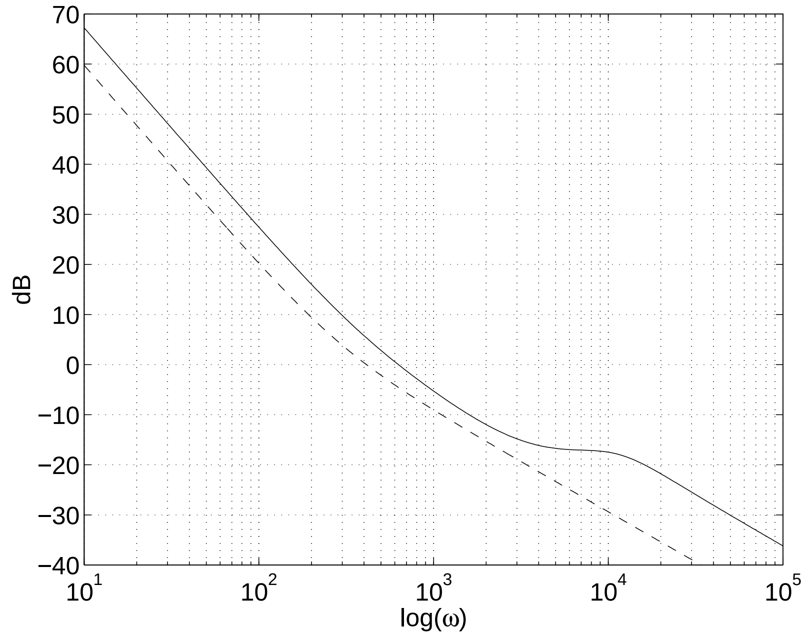

Figure 7: Bode plot comparison between the PI structured $F(s)$ and "optimal" $F(s)$ (dashed)

$$L_0(B_n = 0.1) \;=\; \frac{0.79(s+0.17)(s+0.035)(s^2+1.77s+1.63)}{(s+4.81)(s+1.0)(s+0.055)s^2},$$
$$L_0(B_n = 0.15) \;=\; \frac{0.79(s+0.17)(s+0.035)(s^2+1.77s+1.63)}{(s+4.8)(s+1.0)(s+0.055)s^2}.$$

The σ_e^2 values for $T = 1$ as a function of B_n are given in Fig. 8 below. Note that σ_n^2 converges as $B_n \to \infty$, because $L_0(s)$ converges as $B_n \to \infty$. Based on the above results, a cook book for delayed PLL design is

1. Calculate B_n.
2. If $B_n \leq 0.02$ use PI or any optimal existing technique.
3. If $B_n \geq 0.15$ your open loop is $L_0(sT)$ using $L_0(B_n = 0.15)$.
4. If $0.02 < B_n < 0.15$ pick the closest $L_0(s)$ from the table above, and your open loop is $L_0(sT)$.
5. Calculate σ_e^2 via Fig. 8 and equation (24).

4 Loop Filters Having a PI Form

A *reduced order* loop filter is a loop filter which has less poles and zeros than the optimal loop filter. There are three reasons for using a reduced order loop filter, these are: (*i*) reduction of computation effort in real time; (*ii*) the design of a reduced order loop filter may be simpler and faster; and (*iii*) the reduced order loop filter can be close enough to the optimal loop filter. The drawback of using a reduced order loop filter is when (*iii*) is not satisfied, that is, produce too much error compared to a non-reduced order design.

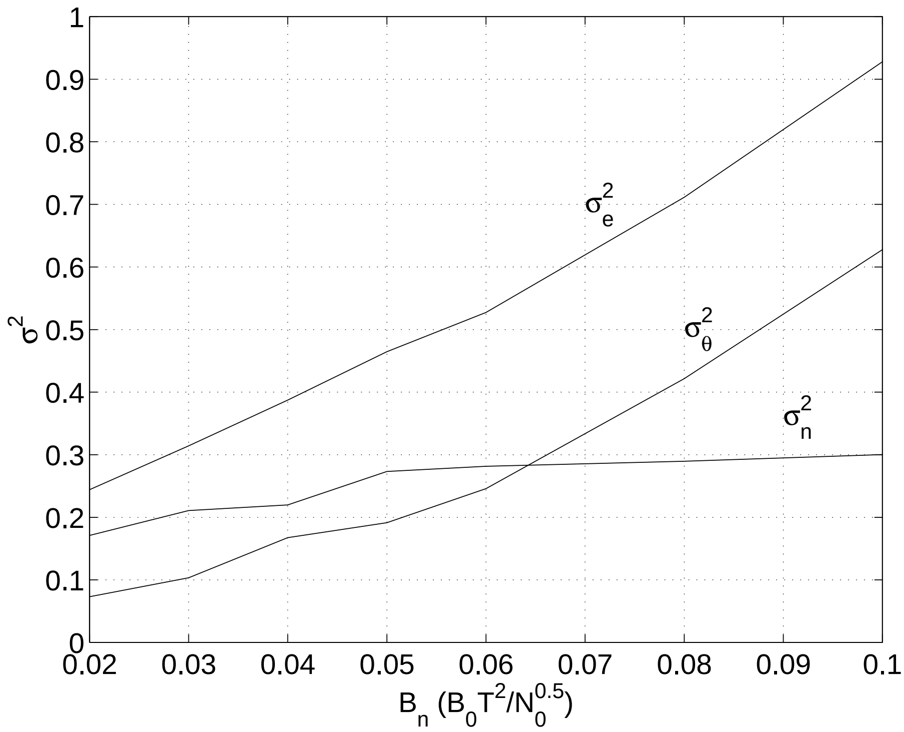

Figure 8: σ_e^2, its noise contribution σ_n^2 and its phase noise σ_θ^2 vs. B_n

The PLL open loop when the loop filter is PI can be written as follows

$$L(s) = \frac{e^{-sT}}{sT}\frac{a(1+bsT)}{sT} \tag{25}$$

and the two parameters to design are a and b. Since $L(j\omega)$ can be written as a function of ωT, the range of $L(j\omega)$ for all real ω does not depend on T. Therefore if margin specification of the form

$$\left|\frac{1}{1+L(j\omega)}\right| \leq \delta, \quad \forall\omega \tag{26}$$

is satisfied for some T it is satisfied for any T. This normalize the problem for the margin specification for all T, and $T = 1$ will be picked in the following. Let us now denote by ω_0 a frequency for which (26) is satisfied with equality. Explicitly there exists ω_0 such that

$$\left|1 + \frac{e^{-sT}}{sT}\frac{a(1+bsT)}{sT}\right|_{s=j\omega_0} = \frac{1}{\delta},$$

and ω_0 is an extremum point of, $|1 + L(j\omega)|$. Hence

$$a^2\frac{1+b^2\omega_0^2}{\omega_0^4} - 2a\frac{\cos(\omega_0) + b\omega_0\sin(\omega_0)}{\omega_0^2} + 1 - \frac{1}{\delta^2} = 0, \tag{27}$$

$$\frac{\partial}{\partial\omega}\left(a^2\frac{1+b^2\omega^2}{\omega^4} - 2a\frac{\cos(\omega) + b\omega\sin(\omega)}{\omega^2} + 1\right)_{\omega=\omega_0} = 0. \tag{28}$$

From equation (28)

$$a = \frac{\sin(\omega_0) - b\sin(\omega_0) - b\omega_0\cos(\omega_0) + 2(\cos(\omega_0) + b\omega_0\sin(\omega_0))/\omega_0}{2/\omega_0^3 + b^2/\omega_0}. \tag{29}$$

The solution of equation (27) and (29), for given δ, as a function of ω_0 is a curve $(a(\omega_0), b(\omega_0))$ in R^2. These curves are a function of the parameter δ which dictates phase margin ϕ according to equation (3), we shall therefore call them the δ-curves or ϕ-curves. These curves are depicted in Fig. 9. Clearly, the curves cannot intersect. Moreover, if we denote by D_ϕ the region inside the curve of phase margin ϕ then $D_{\phi_1} \subset D_{\phi_2}$ if $\phi_1 < \phi_2$ (equivalently if $\delta(\phi_1) > \delta(\phi_2)$). Therefore any (a, b) curve splits R^2 into two regions, D_ϕ in which inequality (26) is satisfied and its complement in which inequality (26) is not satisfied. For example, if a phase margin of 40° is required (which is equivalent to $\delta = 3.3dB$ and $10dB$ gain margin), then for $b = 15$ the allowed values for a are $-50.5 \le a \le -31.8$, and for $b = 10$, $-42.5 \le a \le -28.5$.

The extension to gain uncertainty is now strait forward: If it is known that the gain can increase by rdB then the allowed region, D_ϕ^r, is the intersection of D_ϕ and the region D_ϕ shifted down by rdB (to protect against possible gain increase of rdB). For example, if phase margin of 40° is required, $b = 15$, and $r = 14dB$, then $-56.5dB \le a \le -45.8dB$, and if $b = 10$ then $-42.5dB \le a \le -42.5dB$, that is, no tolerance in a. Therefore if $r > 14dB$, $b \le 10$ cannot be used. The maximum gain range a PI loop filter can tolerate as a function of the phase margin for different values of b can easily be retrieved from Fig. 9. For example at 40° and $b = 15$, the gain uncertainty range can be $19dB$, that is, in order to handle $19dB$ uncertainty with $b = 15$ the chosen gain must be $a = (-28.5 - 19)dB$ and the gain margin of $L(s)$ is between $10dB$ for the maximum gain and $29dB$ for the minimum gain. If for example the phase margin is 42° and the gain range is $10dB$, then $b \ge 9$ must be picked in order to satisfy inequality (26) by all possible $L(s)$ which suffers from $10dB$ gain uncertainty.

Now let us suppose that σ_e^2 where a PI loop filter is used, has a unique minimum, which does not satisfy given margin constrains, ϕ. Then the a, b pair which minimize σ_e^2 subjected to the margin constraint, ϕ, must lie on the surface of D_ϕ, that is, on the ϕ-curve. In that case the design process reduces into an extremum problem with a single parameter and single minimum as follows:

1. Pick the curve (a, b) from Fig. 9 for the chosen phase margin specification, and modify it to the appropriate gain uncertainty as described above.

2. Find the extremum of $\sigma_e^2(a, b)$

$$\begin{aligned}
\sigma_e^2(a,b) &= \frac{1}{2\pi}\int_\infty^\infty \left|\frac{e^{-sT}a(1+bsT)}{(sT)^2 + e^{-sT}a(1+bsT)}\right|^2_{s=j\omega} \Phi_n(\omega)d\omega \\
&+ \frac{1}{2\pi}\int_\infty^\infty \left|\frac{(sT)^2}{(sT)^2 + e^{-sT}a(1+bsT)}\right|^2_{s=j\omega} \Phi_\theta(\omega)d\omega \\
&= \frac{1}{2\pi T}\int_\infty^\infty \left|\frac{a(1+bs)}{s^2 + e^{-s}a(1+bs)}\right|^2_{s=j\omega} \Phi_n(\omega/T)d\omega \\
&+ \frac{1}{2\pi T}\int_\infty^\infty \left|\frac{s^2}{s^2 + e^{-s}a(1+bs)}\right|^2_{s=j\omega} \Phi_\theta(\omega/T)d\omega,
\end{aligned} \tag{30}$$

along the (a, b) curve picked in 1.

3. The PI optimal loop filter will then be

$$F(s) = \frac{a(1+bTs)}{A_1 T^2 s}$$

where $AP = Ae^{-sT}/s$ and $A \in [A_1, A_2]$.

4.1 The PI Solution for $\Phi_\theta(\omega) \propto \omega^{-4}$ and white noise

We treat here the case

$$\Phi_\theta(\omega) = \frac{B_0^2}{\omega^4} \ \text{ and } \ \Phi_n(\omega) = N_0 \tag{31}$$

where B_0 is constant and N_0 is the usual white noise density. Substituting in equation (30) gives

$$\sigma_e^2 \ = \ \frac{1}{2\pi T}\int_\infty^\infty \left|\frac{a(1+bs)\sqrt{N_0}}{s^2+e^{-s}a(1+bs)}\right|^2_{s=j\omega} d\omega + \frac{1}{2\pi T}\int_\infty^\infty \left|\frac{s^2 B_0 T^2/\omega^2}{s^2+e^{-s}a(1+bs)}\right|^2_{s=j\omega} d\omega. \tag{32}$$

Clearly

$$\sigma_e^2(T, N_0, B_0) = B_0^2 T^3 \sigma_e^2(1, n, 1), \quad n = \frac{N_0}{B_0^2 T^4}, \tag{33}$$

thus the a, b pairs which minimize σ_e^2 depend only on the single parameter n. Note that $n = B_n^{-2}$ defined in equation (24) but we use n for clarity. Let $\{a_0(n), b_0(n)\}$ be the point that minimizes σ_e^2 as a function of n. The curve $\{a_0(n), b_0(n)\}$ is plotted on top of the ϕ-curves in Fig. 9. Let us further denote the intersection point of the curve $\{a_0(n), b_0(n)\}$ with a ϕ-curve by $a_0(\phi)$, $b_0(\phi)$ and $n_0(\phi)$. For example, if phase margin of 40° is required assuming no gain uncertainty, then $\sqrt{n_0} = 13.8$, $a_0 = -26.3dB$ and $b_0 = 7.8$. $a_0(n), b_0(n)$ were calculated as follows: first σ_e^2 in equation (32) is written as

$$\sigma_e^2 = nB_L + \sigma_\theta^2.$$

Hence, $a_0(n), b_0(n)$ minimizes σ_e^2 for some n if

$$n = -\frac{\partial\sigma_\theta^2/\partial a}{\partial B_L/\partial a} \ \text{ and } \ n = -\frac{\partial\sigma_\theta^2/\partial b}{\partial B_L/\partial b}. \tag{34}$$

The two partial derivative ratios in expressions (34) where calculated along each of the ϕ-curves in Fig. 9 and it was found that they have a unique intersection, whose n value is written on its ϕ-curve in Fig. 9. This proves, numerically, that $\sigma_e^2(n)$ has a unique minimum. Moreover, we observe that $n(\phi)$ is a monotonicaly increasing function of ϕ. The same results are depicted in Fig. 10 which includes a graph of n_0 as a function of b_0 and a graph of $b_0(\phi)$.

Fig. 10II also shows b which minimizes $\sigma_e^2(1, n = 0, 1)$ on the ϕ-curve. Since the two curves in Fig. 10II almost coincide, and the solution for constrained minimization of σ_e^2 for $n = 0$ must lie on the ϕ-curve, the $a_0(\phi), b_0(\phi)$ pair, for a very good approximation, minimize $\sigma_e(1, n_1, 1)$ for any $n_1 \le n_0(\phi)$. But this will not be the case if uncertainty is introduced. Fig. 11 depicts modified ϕ-curves for phase margin of 40° and uncertainties between $0dB$ and $18dB$ every $2dB$. $\sqrt{n_0}$ at the intersection of $a_0(n), b_0(n)$ with the modified ϕ-curve is marked on each curve. For $n_1 < n_0$ the a, b pairs which minimize $\sigma_e(1, n_1, 1)$ on the modified ϕ-curve move along that curve toward the point marked $\circ$ which is the minimum point for $n_1 = 0$. Finally, Fig. 12 depicts $\sqrt{n}\sigma_n$, σ_θ, σ_e and $\sqrt{n}$ on the point $a_0(\phi), b_0(\phi)$ as a function of ϕ.

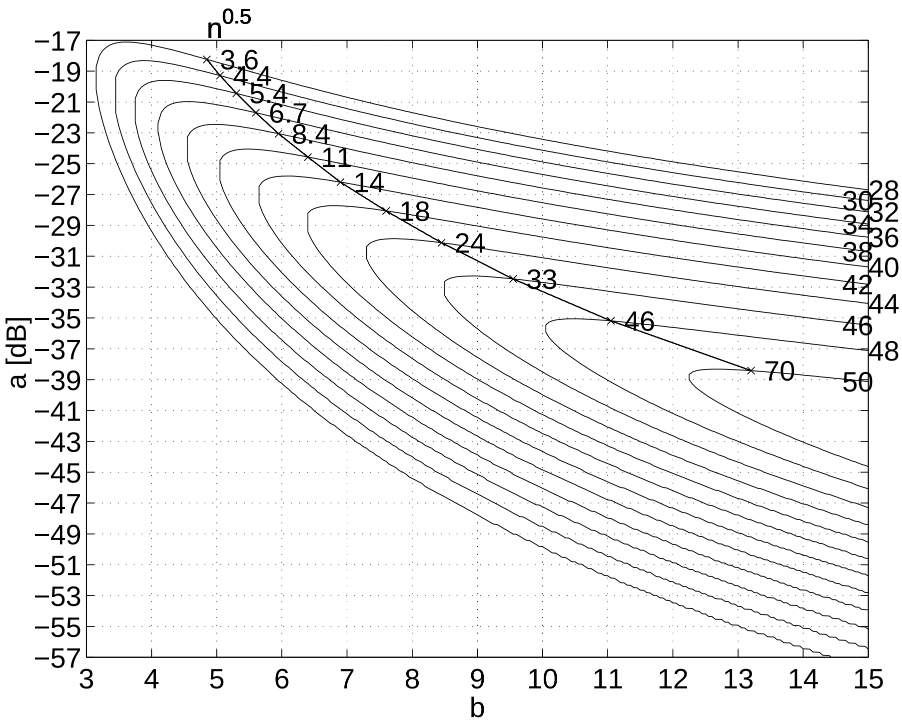

Figure 9: (a, b) curves for different phase margins, marked on its right side. Also Location of a, b's which minimize σ_e^2 on the ϕ-curves and their $\sqrt{n}$ values

4.1.1 Tradeoff amongst reduced-order, delay time and phase noise

The first tradeoff is based on equation (32) which states that when N_0 is small enough then the thermal noise contribution in equation (33) is neglected and therefore $\sigma_e^2 \propto B_0^2 T^3$.

The next tradeoff we are interested in is by how much σ_e^2 can be reduced by an loop filter designed by the method of section 3 compared to a PI loop filter. The answer provided here is based on an example whose parameters are: $\Phi_\theta = 50^2/\omega^4$, Φ_n can be neglected, open loop delay $T = 0.01$ and gain uncertainty, A, in the interval $A \in [1, 2.5]$, that is, $8dB$ uncertainty. The margins specification is of the form $|1 + L|^{-1} < 3.3dB$, which guarantee $40°$ phase margin and $10dB$ gain margin for $A = 2.5$ and $18dB$ for $A = 1$.

Using Fig. 11 for $8dB$ uncertainty, $a = -34.3dB$ and $b = 7$. For that PI loop filter $\sigma_e^2 = 0.57$. Using the suboptimal methodology described herein the loop filter is

$$F_r(s) = 32.7\frac{(s + 120)(s + 570)(s^2 + 11.4s + 38)}{s^2(s + 300)(s + 28)}$$

for which $\sigma_e^2 = 0.28$. This figure is half of that figure when an optimal PI loop filter is used. By equation (33), it is equivalent to $3dB$ reduction of the phase noise spectral density or 25% in the delay time.

5 Conclusions

We have presented a design method for near optimal PLL taking into consideration the phase noise, the thermal noise, the undesired but unavoidable loop delay caused by delayed decisions

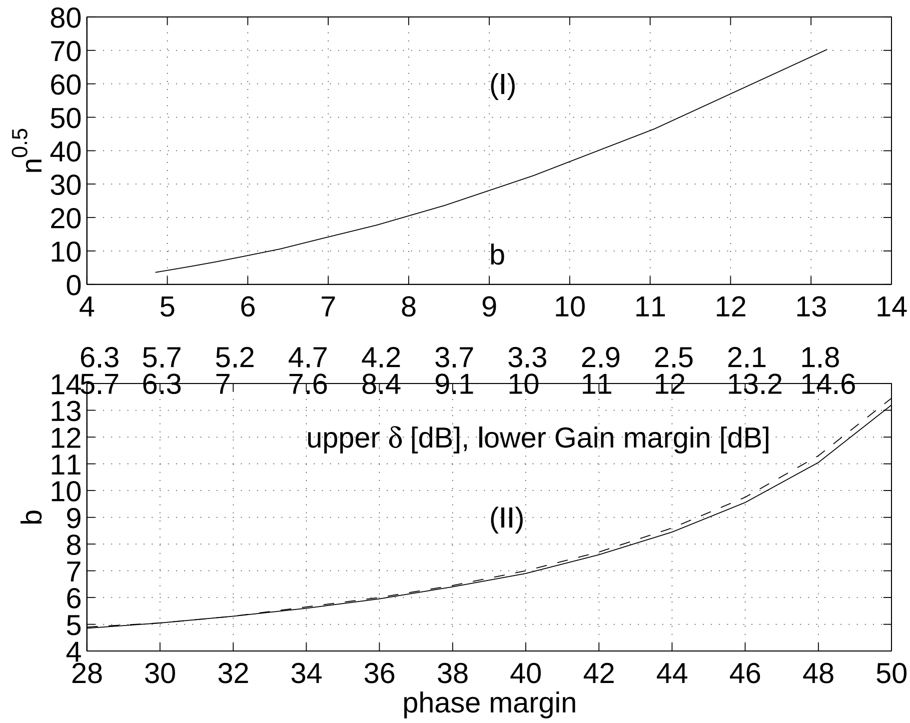

Figure 10: (I) $\sqrt{n_0}$ as a function of b_0, (II) $b_0(\phi)$, rigid line (a_0 can be picked from Fig. 9) and b which minimizes $\sigma_e^2(1, n = 0, 1)$ on the same ϕ-curve, dashed line

and margins for protection from gain uncertainty and insuring good step response. The method is general and can be used with any PLL. We find its main application in carrier tracking since a wide loop bandwidth is required to track the phase noise. We do not limit the loop order to be second order, and we demonstrate a large performance gain with respect to a well designed second order loop.

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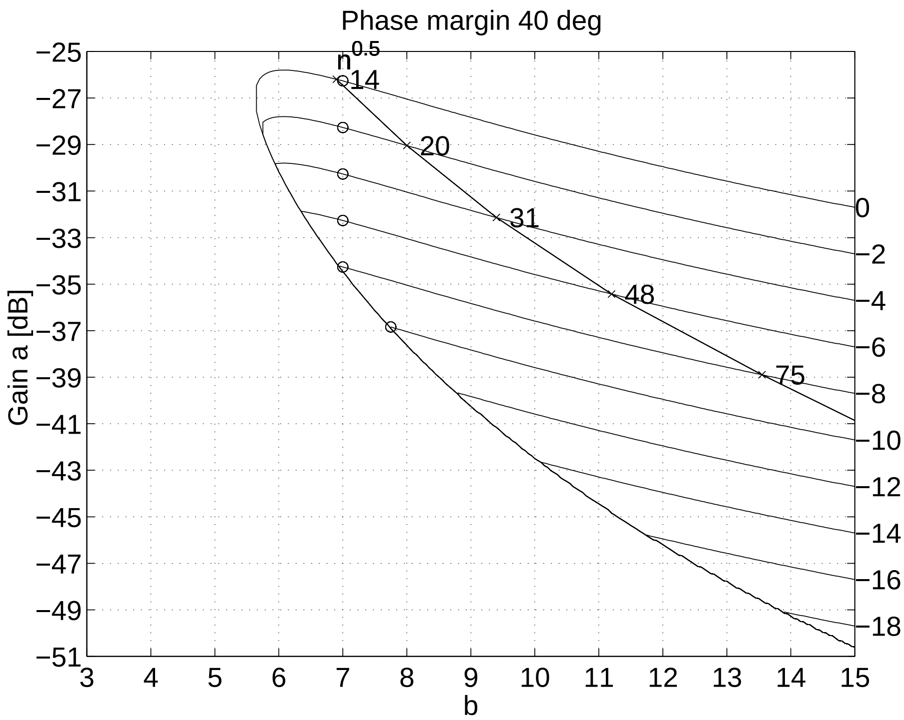

Figure 11: Modified ϕ-curves for phase margin 40° and gain uncertainties from $0dB$ to $18dB$ every $2dB$, also shown the curve $a_0(n), b_0(n)$ and the points n_0 at the intersections, marked $\times$. The points marked $\circ$ are the points that minimize $\sigma_e^2(1,0,1)$ on the modified ϕ curve

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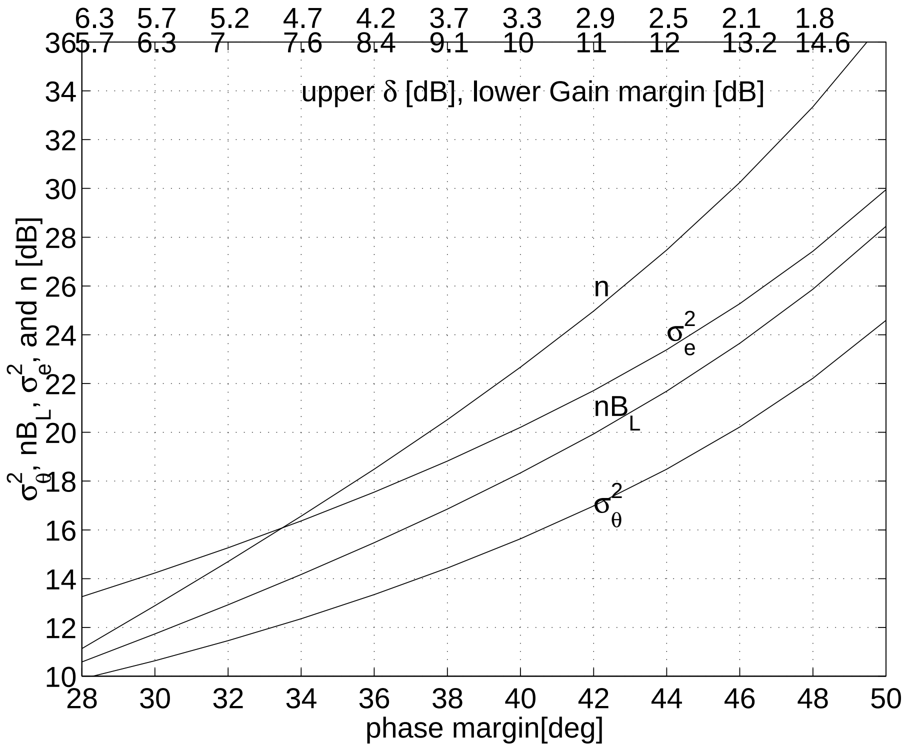

Figure 12: The phase error σ_e^2, the phase noise contribution σ_θ^2, the thermal noise contribution $n_0 B_L$ and n_0 vs. phase margin

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