Stabilization of Singularly Perturbed Linearly Systems with Delay and Saturating Control

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Abstract

The paper deals with the feedback control law design methodology that applies to singularly perturbed linearly systems with time-delay control under saturation constraints. The results obtained by a scalar inequalities allows us to investigate a variety of control problems.

1. Introduction

There has been much interest in the last few years concerning stabilization of linear systems with time-delay in control and saturating actuators. It is well known that stabilizability of such systems but with perturbed parameter is equivalent to the common definition of stabilizability plus the added conditions.

The problem of stabilizing of linear time-delay systems with saturating controls has been studied by Chen, Wang and Lu [1],[2] and Shen and Kung [3]. A comprehensive treatment on singular perturbation theory has been developed by Kokotovic, Khalil et Reilly [6]. The singular perturbed systems with small delays, associated with fast subsystem has been analysed by E. Fridmann [7].

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In this paper we propose two feedback strategies for a singular perturbed systems with time delay and saturating controls. Delay parameter is associated to the slow variables. Several sufficient conditions expressed in term of norm or matrix measures are derived to guarantee the stability of such systems.

The remainder of this paper is organized as follow. In section 2 we precise formulate our problem and the main results will be proven in sections 3 and 4. An example is given in section 5 to illustrate our results. A brief concluding remarks is made in section 6.

2. Problem formulation

Consider the linear system:

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t)$$
$$\varepsilon\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) \qquad t \geq 0$$

(2.1) where $x_i \in \Re^{n_i}, i = 1,2, u \in \Re^m$ and $A_{ij}, B_i \; i, j = 1, 2$ are real matrices of appropriate dimensions; $\varepsilon > 0$ is a small parameter.

The presence of the small parameter ε, in system (2.1), shows that the evolution of the variables x_2 is "faster" than the one of the variables x_1.

Our aim is to stabilize the system (2.1) by using controls of the form:

$$u(t) = F_1x_1(t-\tau) + F_2x_2(t)$$

(2.2) If in (2.1) we take ε=0 and assume that A_{22} is an invertible matrix we obtain the so called reduced subsystem $\dot{x}_1(t) = \tilde{A}x_1(t) + \tilde{B}u(t)$

(2.3)

where $\tilde{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}$ and $\tilde{B} = B_1 - A_{12}A_{22}^{-1}B_2$.

Also, to system (2.1.) we associate the so called "boundary layer subsystem" or "fast subsystem".

$$x_2'(\sigma) = A_{22}x_2(\sigma) + B_2u(\sigma)$$

(2.4)

where $\sigma = \dfrac{t}{\varepsilon}$.

It is know [9] that to design a stabilizing feedback law $u(t) = F_1x_1(t) + F_2x_2(t)$ for system (2.1.) we can design separately the feedback gain $\tilde{F}$ and F_2 such that $\tilde{A} + \tilde{B}\tilde{F}$ and $A_{22} + B_2F_2$ be stable matrices. A stabilizing feedback gain for the full system (2.1) is obtained by taking $F_1 = \left(I_m + F_2A_{22}^{-1}B_2\right)\tilde{F} + F_2A_{22}^{-1}A_{21}$.

In this paper we proposed a similar method to design a stabilizing feedback law of type (2.2) based on designing of a stabilizing feedback gain for the boundary layer subsystem (2.4) and of a time delayed stabilizing state feedback for the reduced subsystem (2.3).

In section 4 the problem of designing of a stabilizing control law (2.2) is investigated in the presence of saturating actuators.

3. A time delayed stabilizing composite control

The main result of this section is:

Theorem 3.1 . Assume that the feedback gains F_2 and $\tilde{F}$ are such that:

$$\left|e^{(A_{22}+B_2F_2)t}\right| \le \beta_2 e^{-\alpha_2 t},\ t \ge 0 \text{ and}$$

$$\left|\tilde{\phi}(t)\right| \le \beta_1 e^{-\alpha_1 t},\ \ \beta_i \ge 1, \alpha_i > 0, i = 1,2$$

$\tilde{\phi}(.)$ being the fundamental matrix solution of the linear differential equation with lower dimension.

$$x_1'(t) = \left[\tilde{A} - \tilde{B}F_2(A_{22} + B_2F_2)^{-1}A_{21}\right]x_1(t) + \tilde{B}\tilde{F}x_1(t-\tau) \tag{3.1}$$

Set $F_1 = (I_m + F_2A_{22}^{-1}B_2)\tilde{F}$ and consider law

$$u(t) = F_1x_1(t-\tau) + F_2x_2(t) \tag{3.2}$$

Then for arbitrary $\varepsilon \in (0, \varepsilon_0)$, where ε_0 can be estimated, the control (3.8) stabilizes the system (2.1).

Remark Design of control law (3.2) has two steps :

-first we choose the gain matrix F_2 such that to stabilize the faster modes to meet design specifications,

- in second step we choose $\tilde{F}$ such that the control law :

$$u(t) = -F_2(A_{22} + B_2F_2)^{-1}A_{21}x_1(t) + \tilde{F}x_1(t-\tau)$$

must stabilize the slow modes so as to meet constraints imposed by some requirements.

Proof. The system obtained coupling the control (3.2) to the system (2.1) is :

$$\dot{x}_1(t) = A_{11}x_1(t) + B_1F_1x_1(t-\tau) + (A_{12} + B_1F_2)x_2(t)$$
$$\varepsilon\dot{x}_2(t) = A_{21}x_1(t) + B_2F_1x_1(t-\tau) + (A_{22} + B_2F_2)x_2(t)$$

(3.3)

Let $x_1(t,\varepsilon), x_2(t,\varepsilon)$ be a solution of the system (3.3) which verifies $x_1(0,\varepsilon) = x_1^0,\ x_2(0,\varepsilon) = x_2^0\ x_1(t,\varepsilon) = \varphi_1(t), -\tau \le t < 0$.

By constants variation formula, we deduce from the second equation of (3.3), that

$$x_2(t,\varepsilon) = e^{(A_{22}+B_2F_2)\frac{t}{\varepsilon}}x_2^0 + \frac{1}{\varepsilon}\int_0^t e^{(A_{22}+B_2F_2)\frac{t-s}{\varepsilon}}(A_{21}x_1(s,\varepsilon) + B_2F_1x_1(s-\tau,\varepsilon))ds$$

(3.4)

Further, the first equation of the system (3.3) can be rewritten

$$\dot{x}_1(t) = \left[A_{11} - (A_{12} + B_1F_2)(A_{22} + B_2F_2)^{-1}A_{21}\right]x_1(t) +$$
$$+\left[B_1 - (A_{12} + B_1F_2)(A_{22} + B_2F_2)^{-1}B_2\right]F_1x_1(t-\tau) +$$
$$+(A_{12} + B_1F_2)x_2(t) + (A_{12} + B_1F_2)(A_{22} + B_2F_2)^{-1}(A_{21}x_1(t)$$
$$+B_2F_1x_1(t-\tau))$$

(3.5)

We can write:

$$A_{11} - (A_{12} + B_1F_2)(A_{22} + B_2F_2)^{-1}A_{21} = \tilde{A} - \tilde{B}F_2(A_{22} + B_2F_2)^{-1}A_{21}$$
$$B_1 - (A_{12} + B_1F_2)(A_{22} + B_2F_2)^{-1}B_2 = \tilde{B}\left[I_m + F_2A_{22}^{-1}B_2\right]^{-1}$$

The fact that $I_m - F_2(A_{22} + B_2F_2)^{-1}B_2 = (I_m + F_2A_{22}^{-1}B_2)^{-1}$ can be verify by direct calculations.

Thus, the equation (3.5) becomes:

$$\dot{x}_1(t,\varepsilon) = \left[\tilde{A} - \tilde{B}F_2(A_{22} + B_2F_2)^{-1}A_{21}\right]x_1(t,\varepsilon) + \tilde{B}\tilde{F}x_1(t-\tau,\varepsilon) +$$
$$+(A_{12} + B_1F_2)x_2(t,\varepsilon) + (A_{12} + B_1F_2)(A_{22} + B_2F_2)^{-1}[A_{21}x_1(t,\varepsilon) + B_2F_1x_1(t-\tau,\varepsilon)]$$

Hence we have the representation formula [7], [8] :

$$x_1(t,\varepsilon) = \tilde{\phi}(t)x_1^0 + \int_0^t \tilde{\phi}(t-s)\tilde{B}\tilde{F}\hat{\varphi}(s)ds + \int_0^t \tilde{\phi}(t-s)(A_{12} + B_1F_2)x_2(s,\varepsilon)ds +$$
$$+\int_0^t \tilde{\phi}(t-s)(A_{12} + B_1F_2)(A_{22} + B_2F_2)^{-1}[A_{21}x_1(s,\varepsilon) + B_2F_1x_1(s-\tau,\varepsilon)]ds$$

where $\hat{\varphi}(s) = \begin{cases} \varphi(s-\tau) & \text{for} \quad 0 \le s \le \tau \\ 0 \quad \text{for} & s > \tau \end{cases}$

Taking into account (3.4) we write further:

$$x_1(t,\varepsilon)=\tilde{\phi}(t)x_1^0+\int_0^t\tilde{\phi}(t-s)\tilde{B}\tilde{F}\hat{\phi}(s)ds+\int_0^t\tilde{\phi}(t-s)(A_{12}+B_1F_2)e^{(A_{22}+B_2F_2)\frac{s}{\varepsilon}}x_2^0ds+$$
$$+\frac{1}{\varepsilon}\int_0^t\tilde{\phi}(t-s)(A_{12}+B_1F_2)\int_0^s e^{(A_{22}+B_2F_2)\frac{s-\sigma}{\varepsilon}}(A_{21}x_1(\sigma,\varepsilon)+B_2F_1x_1(\sigma-\tau,\varepsilon))d\sigma ds+$$
$$+\int_0^t\tilde{\phi}(t-s)(A_{12}+B_1F_2)(A_{22}+B_2F_2)^{-1}[A_{21}x_1(s,\varepsilon)+B_2F_1x_1(s-\tau,\varepsilon)]ds$$

After a change of the order of integration, the last but one term may be written:

$$\frac{1}{\varepsilon}\int_0^t\tilde{\phi}(t-s)(A_{12}+B_1F_2)\int_0^s e^{(A_{22}+B_2F_2)\frac{s-\sigma}{\varepsilon}}\left[A_{21}x_1(\sigma,\varepsilon)+B_2F_1x_1(\sigma-\tau,\varepsilon)\right]d\sigma ds=$$
$$=\frac{1}{\varepsilon}\int_0^t\int_s^t\tilde{\phi}(t-\sigma)(A_{12}+B_1F_2)e^{(A_{22}+B_2F_2)\frac{\sigma-s}{\varepsilon}}d\sigma[A_{21}x_1(s,\varepsilon)+B_2F_1x_1(\sigma-\tau,\varepsilon)]ds=$$
$$=\int_0^t\int_s^t\tilde{\phi}(t-\sigma)(A_{12}+B_1F_2)(A_{22}+B_2F_2)^{-1}\left(\frac{d}{d\sigma}e^{(A_{22}+B_2F_2)\frac{\sigma-s}{\varepsilon}}\right)d\sigma[A_{21}x_1(s,\varepsilon)+$$
$$+B_2F_1x_1(s-\tau,\varepsilon)]ds=$$
$$=-\int_0^t\tilde{\phi}(t-s)(A_{12}+B_1F_2)(A_{22}+B_2F_2)^{-1}[A_{21}x_1(s,\varepsilon)+B_2F_1x_1(s-\tau,\varepsilon)]ds+$$
$$+\int_0^t M(t,s,\varepsilon)[A_{21}x_1(s,\varepsilon)+B_2F_1x_1(s-\tau,\varepsilon)]ds$$

Where:

$$M(t,s,\varepsilon)=(A_{12}+B_1F_2)(A_{22}+B_2F_2)^{-1}e^{(A_{22}+B_2F_2)\frac{t-s}{\varepsilon}}+$$
$$+\int_s^t\left\{\tilde{\phi}(t-\sigma)\left[\tilde{A}-\tilde{B}F_2(A_{22}+B_2F_2)^{-1}A_{21}\right]+\tilde{\phi}(t-\sigma-\tau)\tilde{B}\tilde{F}\right\}*$$
$$*(A_{12}+B_1F_2)(A_{22}+B_2F_2)^{-1}e^{(A_{22}+B_2F_2)\frac{\sigma-s}{\varepsilon}}d\sigma$$
(3.6)

Now (3.5) becomes :

$$x_1(t,\varepsilon)=\tilde{\phi}(t)x_1^0+\int_0^t\tilde{\phi}(t-s)\tilde{B}\tilde{F}\hat{\phi}(s)ds+\int_0^t\tilde{\phi}(t-s)(A_{12}+B_1F_2)e^{(A_{22}+B_2F_2)\frac{s}{\varepsilon}}x_2^0ds$$
$$+\int_0^t M(t,s,\varepsilon)[A_{21}x_1(s,\varepsilon)+B_2F_1x_1(s-\tau,\varepsilon)]ds$$

By direct estimates (3.6) we deduce

$$\left|M(t,s,\varepsilon)\right|\le\gamma_1\varepsilon e^{-\alpha_1(t-s)}+\gamma_2 e^{-\alpha_2\left(\frac{t-s}{\varepsilon}\right)}$$

where γ_1 , γ_2 $>$ 0

Thus, from (3.6) we deduce

$$\left|x_1(t,\varepsilon)\right| \le \gamma_3 e^{-\alpha_1 t}\left[\left|x_1^0\right| + \varepsilon\left|x_0^2\right| + \left\|\varphi\right\|\right] + \int_0^t \left[\gamma_1 \varepsilon e^{-\alpha_1(t-s)} + \gamma_2 e^{-\alpha_2\left(\frac{t-s}{\varepsilon}\right)}\right] *$$

$$*\left(\left|A_{21}\right|\left|x_1(s,\varepsilon)\right| + \left|B_2F_1\right|\left|x_1(s-\tau,\varepsilon)\right|\right)ds$$

(3.7)

Let T>0, $\theta \in (0,1)$ be fixed. Denote $\rho(T,\varepsilon) = \sup\limits_{t\in[0,T]} e^{\theta\alpha_1 t}\left|x_1(t,\varepsilon)\right|$ then, from (3.7) we deduce

$$e^{\theta\alpha_1 t}\left|x_1(t,\varepsilon)\right| \le \gamma_3\left[\left|x_1^0\right| + \varepsilon\left|x_2^0\right| + \left\|\varphi\right\|\right] + \rho(T,\varepsilon)\int_0^t \left[\gamma_1 \varepsilon e^{\alpha_1(1-\theta)\frac{t-s}{\varepsilon}} + \gamma_2 e^{-(\alpha_2-\varepsilon\theta\alpha_1)\frac{t-s}{\varepsilon}}\right]$$

$$\times\left(\left|A_{21}\right| + e^{\alpha_1\tau}\left|B_2F_1\right|\right)ds$$

Hence

$$e^{\theta\alpha_1 t}\left|x_1(t,\varepsilon)\right| \le \gamma_3\left[\left|x_1^0\right| + \varepsilon\left|x_2^0\right| + \left\|\varphi\right\|\right] + \varepsilon\left[\frac{\gamma_1}{1-\theta\alpha_1} + \frac{\gamma_2}{\alpha_2 - \varepsilon\theta\alpha_1}\right]$$

$$\times\left(\left|A_{21}\right| + e^{\alpha_1\tau}\left|B_2F_1\right|\right)\rho(T,\varepsilon), \quad (\forall)t \in [0,T] \Rightarrow$$

$$\rho(T,\varepsilon) \le \gamma_3\left[\left|x_1^0\right| + \varepsilon\left|x_2^0\right| + \left\|\varphi\right\|\right] + \varepsilon\left[\frac{\gamma_1}{1-\theta\alpha_1} + \frac{\gamma_2}{\alpha_2 - \varepsilon\theta\alpha_1}\right]$$

$$\times\left(\left|A_{21}\right| + e^{\alpha_1\tau}\left|B_2F_1\right|\right)\rho(T,\varepsilon)$$

Choose :

$$\varepsilon_0 = \min\left[\frac{\alpha_2}{2\alpha_1}, \gamma_0\right] \quad \text{where}$$

$$\frac{1}{\gamma_0} = 2\frac{\beta_2}{\alpha_2}\left|(A_{12} + B_1F_2)(A_{22} + B_2F_2)^{-1}\right|\left\{1 + \frac{\beta_1}{(1-\theta)\alpha_1}\right.$$

$$\left.\times\left[\left|\tilde{A} - \tilde{B}F_2(A_2 + B_2F_2)^{-1}A_{21}\right| + e^{\alpha_1\tau}\left|\tilde{B}\tilde{F}\right|\right]\right\}\left(\left|A_{21}\right| + e^{\alpha_1\tau}\left|B_2F_1\right|\right), \quad \theta \in (0,1)$$

It is clear that if $\varepsilon \in (0,\varepsilon_0)$ we have

$$\rho(T,\varepsilon) \le \gamma_4\gamma_3\left[\left|x_1^0\right| + \varepsilon\left|x_2^0\right| + \left\|\varphi_1\right\|\right]$$

Where $\gamma_3, \gamma_4 > 0$ and thus

$$\left|x_1(t,\varepsilon)\right| \le \gamma_3\gamma_4 e^{-\alpha_1\theta t}\left[\left|x_1^0\right| + \varepsilon\left|x_2^0\right| + \left\|\varphi\right\|\right],\ t\in[0,T]$$
(3.8)

Since γ_3,γ_4 are not depending upon T, we conclude that (3.8) holds for $t\in[0,\infty]$.

On the other hand from (3.3) we deduce

$$\left|x_2(t,\varepsilon)\right| \le \beta_2 e^{-\alpha_2\left(\frac{t}{\varepsilon}\right)}\left|x_2^0\right| + \frac{1}{\varepsilon}\beta_2\int_0^t e^{-\alpha_2\left(\frac{t-s}{\varepsilon}\right)}\left[\left|A_{21}\right|\left\|x_1(0,\varepsilon)\right| + \left|B_2F_1\right|\left\|x_1(s-\tau,\varepsilon)\right|\right]ds$$

and taking into account (3.16) we obtain finally

$\left|x_1(t,\varepsilon)\right| \le \beta_2 e^{-\alpha_2\frac{t}{\varepsilon}}\left|x_2^0\right| + \gamma_5 e^{-\alpha_1\theta t}\left[\left|x_1^0\right| + \varepsilon\left|x_2^0\right| + \left\|\varphi\right\|\right]$ and the proof is complete.

4. Stabilization of a singular perturbed linear system with delay and saturation control

Now we consider the following system containing saturation control :

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u'$$
$$\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u' \qquad t\ge 0$$
(4.1)

where $u \to u' = \mathrm{Sat}\ u(t)$ is defined [3] by :

$$\mathrm{Sat}\ u(t) = [\mathrm{sat}\,u_1, \mathrm{sat}\,u_2, \mathrm{sat}\,u_3, ..., \mathrm{sat}\,u_m]^T$$

$$\mathrm{sat}\,u_i = \begin{cases} u_i & \text{if } u_i\in[u_1,u_2] \\ u_1 & \text{if } u_i < u_1 \\ u_2 & \text{if } u_i > u_2 \end{cases}$$

Our aim is to design a state feedback control

$$u(t) = F_1x_1(t-\tau) + F_2x_2(t)$$
(4.2)

which to stabilize the system (4.1).

By the same procedure is that used in section 3, we will separately design the feedback gain matrices $\tilde{F}$ and F_2 for low-order system (2.3) and for boundary layer system (2.4) which are free ε small parameter.

The results of this section are a extension of [2], [3] works applied to singular perturbed system.

Throughout this section we take vector norm $|x| = \sum_{i=1}^{n} |x_i|$ (such as L_1 norm in $\Re^n$) and the adequate norm for matrix A which is defined by the relation :

$$|A| = \max_{i \in \{1,\ldots n\}} \sum_{j=1}^{n} |A_{ij}|$$

Assume matrix $F_2 \in \Re^{mxn2}$ is chosen such that the control law $u(t) = \frac{1}{2} F_2 x(t)$ stabilizes the boundary layer system (2.4) and

$$\left| e^{(A_{22} + \frac{1}{2} B_2 F_2)t} \right| \le \beta_2 e^{-\alpha_2 t}$$

$$t \ge 0, \beta_2 \ge 1, \alpha_2 > 0$$

(4.3)

are satisfied.

Also we chose feedback matrix $\tilde{F} \in \Re^{mxn_1}$ that the control law

$$u(t) = -\frac{1}{2} F_2 (A_{22} + \frac{1}{2} B_2 F_2)^{-1} A_{21} x_1(t) + \frac{1}{2} \tilde{F} x_1(t - \tau)$$

stabilize the low order system (2.3) and such that the fundamental matrix of solution for the closed loop system

$$|\tilde{\phi}(t)| \le \beta_1 e^{-\alpha_1 t}$$

$$t \ge 0, \beta_1 \ge 1, \alpha_1 > 0$$

(4.4)

is satisfied.

Theorem 4.1 Assume that :

$$\alpha_1 > \frac{3}{2} \beta_1 \left| (I + \frac{1}{2} F_2 A_{22}^{-1} B_2) \tilde{F} \right| \left| \tilde{B} (I + \frac{1}{2} F_2 A_{22}^{-1} B_2)^{-1} \right|$$

and

$$\alpha_2 > \frac{1}{2} \beta_2 |B_2| |F_2|$$

are satisfied.

Then the control law

$$u(t) = \left(I + \frac{1}{2} F_2 A_{22}^{-1} B_2 \right) \tilde{F} x_1(t-\tau) + F_2 x_2(t)$$

(4.5)

stabilize the system (4.1) for arbitrary $\varepsilon > 0$ sufficiently small.

Besides the solutions of closed loop system satisfy

$$|x_1(t,\varepsilon)| \le \hat{\beta}_1 e^{-\theta_1 \alpha_1 \frac{t}{\varepsilon}} [|x_1^0| + \varepsilon |x_2^0| + \|\varphi\|]$$

and

$$|x_2(t,\varepsilon)| \le \hat{\beta}_2 e^{-\theta_2 \alpha_2 \frac{t}{\varepsilon}} |x_2^0| + \hat{\beta}_3 e^{-\theta_1 \alpha_1 t} \left[|x_1^0| + \varepsilon |x_2^0| + \|\varphi\| \right], \ \forall t \ge 0$$

where $\theta_i \in (0,1)$ $i = 1,2$ such that

$$(1-\theta_1)\alpha_1 > \frac{3}{2} \beta_1 \left| \left(I + \frac{1}{2} F_2 A_{22}^{-1} B_2 \right) \tilde{F} \right| \left| \tilde{B} \left(I + \frac{1}{2} F_2 A_{22}^{-1} B_2 \right)^{-1} \right|$$

$$(1-\theta_2)\alpha_2 > \frac{1}{2} \beta_2 |B_2| |F_2|$$

$$\tau < \frac{1}{\theta \alpha_1}$$

Proof : After taking

$$F_1 = \left(I + \frac{1}{2} F_2 A_{22}^{-1} B_2 \right) \tilde{F}$$

and substituting (4.5) into (4.1) the closed loop is obtained as:

$$x_1'(t,\varepsilon) = A_{11} x_1(t,\varepsilon) + \frac{1}{2} B_1 F_1 x_1(t-\tau,\varepsilon) + (A_{12} + \frac{1}{2} B_2 F_2) x_2(t,\varepsilon) + \\ + B_2 \left(\text{Sat } u(t) - \frac{1}{2} u(t) \right)$$

$$\varepsilon x_2'(t,\varepsilon) = A_{21} x_1(t,\varepsilon) + \frac{1}{2} B_2 F_1 x_1(t-\tau,\varepsilon) + (A_{22} + \frac{1}{2} B_2 F_2) x_2(t,\varepsilon) + \\ + B_2 \left(\text{Sat } u(t) - \frac{1}{2} u(t) \right) \tag{4.6}$$

The second equation of (4.6) leads to the relation:

$$x_2(t,\varepsilon) = e^{(A_{22}+\frac{1}{2}B_2F_2)\frac{t}{\varepsilon}x_2^0} + \frac{1}{2}\int_0^t e^{(A_{22}+\frac{1}{2}B_2F_2)\frac{t-s}{\varepsilon}}[A_{21}x_1(s,\varepsilon) + \frac{1}{2}B_2F_1x_1(s-\tau,\varepsilon)]ds$$

$$+\frac{1}{\varepsilon}\int_0^t e^{(A_{22}+\frac{1}{2}B_2F_2)\frac{t-s}{\varepsilon}}B_2[\mathrm{Sat}\ u(s) - \frac{1}{2}u(s)]ds$$

(4.7)

The first equation of (4.6) could be rewritten as:

$$x_1^{'}(t,\varepsilon) = \left[A_{11} - \left(A_{12} + \frac{1}{2}B_1F_2\right)\left(A_{22} + \frac{1}{2}B_2F_2\right)^{-1}A_{21}\right]x_1(t,\varepsilon)$$

$$+\frac{1}{2}\left[B_1 - \left(A_{12} + \frac{1}{2}B_1F_2\right)\left(A_{22} + \frac{1}{2}B_2F_2\right)^{-1}B_2\right]F_1x_1(t-\tau,\varepsilon)$$

$$+\left(A_{12} + \frac{1}{2}B_1F_2\right)x_2(t,\varepsilon) + \left(A_{12} + \frac{1}{2}B_1F_2\right)\left(A_{22} + \frac{1}{2}B_2F_2\right)^{-1}$$

$$*\left[A_{21}x_1(t,\varepsilon) + \frac{1}{2}B_2F_1x_1(t-\tau,\varepsilon)\right] + B_1(\mathrm{Sat}\ u(t) - \frac{1}{2}u(t))$$

As before in the proof of theorem (3.1) the above equation could be written in a more explicit form :

$$x_1^{'}(t,\varepsilon) = \left[\tilde{A} - \frac{1}{2}\tilde{B}F_2\left(A_{22} + \frac{1}{2}B_2F_2\right)^{-1}A_{21}\right]x_1(t,\varepsilon) + \frac{1}{2}\tilde{B}\tilde{F}x_1(t-\tau,\varepsilon) + \left(A_{12} + \frac{1}{2}B_1F_2\right)x_2(t,\varepsilon)$$

$$+\left(A_{12} + \frac{1}{2}B_1F_2\right)\left(A_{22} + \frac{1}{2}B_2F_2\right)^{-1}[A_{21}x_1(t,\varepsilon) + \frac{1}{2}B_2F_1x_1(t-\tau,\varepsilon)] + B_1[\mathrm{Sat}\ u(t) - \frac{1}{2}u(t)]$$

By using of constants variation formula we obtain the following relation:

$$x_1(t,\varepsilon) = \tilde{\phi}(t)x_1^0 + \frac{1}{2}\int_0^t \tilde{\phi}(t-s)\tilde{B}\tilde{F}\$(s)ds + \int_0^t \tilde{\phi}(t-s)(A_{12} + \frac{1}{2}B_1F_2)x_2(s,\varepsilon)ds +$$

$$+\int_0^t \tilde{\phi}(t-s)\left(A_{12} + \frac{1}{2}B_1F_2\right)\left(A_{22} + \frac{1}{2}B_2F_2\right)^{-1}\left(A_{21}x_1(s,\varepsilon) + \frac{1}{2}B_2F_1x_1(t-s,\varepsilon)ds\right) +$$

$$+\int_0^t \tilde{\phi}(t-s)B_1[\mathrm{Sat}\ u(s) - \frac{1}{2}u(s)]ds$$

(4.8)

By eliminating of $x_2(t,\varepsilon)$ from (4.7) and (4.8) and using integration by parts (see proof of theorem 3.1):

$$x_1(t,\varepsilon)=\tilde{\phi}(t)x_1^0+\frac{1}{2}\int_0^t\tilde{\phi}(t-s)\tilde{B}\tilde{F}\hat{\phi}(s)ds+\int_0^t\tilde{\phi}(t-s)(A_{12}+\frac{1}{2}B_1F_2)e^{(A_{22}+\frac{1}{2}B_2F_2)\frac{s}{\varepsilon}}ds\ \ x_2^0+$$

$$+\int_0^t\hat{M}(t,s,\varepsilon)\left(A_{21}x_1(s,\varepsilon)+\frac{1}{2}B_2F_1x_1(s-\tau,\varepsilon)\right)ds+\int_0^t\hat{M}_1(t,s,\varepsilon)[\mathrm{Sat}\ u(s)-\frac{1}{2}u(s)]ds$$

where

$$\hat{M}(t,s,\varepsilon)=\left(A_{12}+\frac{1}{2}B_1F_2\right)\left(A_{22}+\frac{1}{2}B_2F_2\right)^{-1}e^{(A_{22}+\frac{1}{2}B_2F_2)\frac{t-s}{\varepsilon}}+$$

$$+\int_0^t\left\{\tilde{\phi}(t-\sigma)\left[\tilde{A}-\frac{1}{2}\tilde{B}F_2\left(A_{22}+\frac{1}{2}B_2F_2\right)^{-1}A_{21}\right]+\frac{1}{2}\tilde{\phi}(t,-\sigma-\tau)\tilde{B}\tilde{F}\right\}*$$

$$*\left(A_{12}+\frac{1}{2}B_1F_2\right)\left(A_{22}+\frac{1}{2}B_2F_2\right)^{-1}e^{(A_{22}+\frac{1}{2}B_2F_2)\frac{\sigma-s}{\varepsilon}}d\sigma$$

$$M_1(t,s,\varepsilon)=\tilde{\phi}(t-s)\tilde{B}\left(I+\frac{1}{2}F_2A_{22}^{-1}B_2\right)^{-1}+\hat{M}(t,s,\varepsilon)B_2$$

Direct evaluation gives:

$$\left|\hat{M}(t,s,\varepsilon)\right|\le\hat{\gamma}_1\varepsilon e^{-\alpha_1(t-s)}+\hat{\gamma}_2e^{-\alpha_2\frac{t-s}{\varepsilon}}$$

$$\left|\hat{M}_1(t,s,\varepsilon)\right|\le\left[\beta_1\left|\tilde{B}\left(I+\frac{1}{2}B_2F_2A_{22}^{-1}\right)^{-1}\right|+\hat{\gamma}_1\left|B_2\right|\varepsilon\right]e^{-\alpha_1(t-s)}+\gamma_2\left|B_2\right|e^{-\alpha_2\frac{t-s}{\varepsilon}}$$

where $\hat{\gamma}_1,\ \hat{\gamma}_2\ >\ 0$

Taking into account above evaluations and using [3] :

$$\left|\mathrm{sat}\ u-\frac{1}{2}u\right|\le\frac{1}{2}|u|$$

we obtain:

$$\left|\mathbf{x_1(t,\varepsilon)}\right| \le \hat{\gamma}_3 \mathbf{e}^{-\alpha_1 t}\left[\left|\mathbf{x}_1^0\right| + \varepsilon\left|\mathbf{x}_2^0\right| + \|\varphi\|\right] + \int_0^t \left[\hat{\gamma}_1 \varepsilon \mathbf{e}^{-\alpha_1(t-s)} + \hat{\gamma}_2 \mathbf{e}^{-\alpha_2\frac{t-s}{\varepsilon}}\right]$$

$$\left[\left|\mathbf{A}_{21}\right|\left|\mathbf{x_1(s,\varepsilon)}\right| + \frac{1}{2}\left|\mathbf{B}_2\mathbf{F}_1\right|\left|\mathbf{x_1}(\mathbf{s}-\tau,\varepsilon)\right|\right]\mathbf{ds}$$

$$+\frac{1}{2}\left|\mathbf{F}_1\right|\int_0^t \left\{\left[\tilde{\mathbf{B}}\left|\left(\mathbf{I} + \frac{1}{2}\mathbf{F}_2\mathbf{A}_{22}^{-1}\mathbf{B}_2\right)^{-1}\right| + \hat{\gamma}_1\varepsilon\left|\mathbf{B}_2\right|\right]\mathbf{e}^{-\alpha_1(t-s)} + \hat{\gamma}_2\left|\mathbf{B}_2\right|\mathbf{e}^{-\alpha_2\frac{t-s}{\varepsilon}}\right\}$$

$$\times\left|\mathbf{x_1}(\mathbf{s}-\tau,\varepsilon)\right|\mathbf{ds} + \frac{1}{2}\left|\mathbf{F}_2\right| *$$

$$\int_0^t \left\{\left[\tilde{\mathbf{B}}\left|\left(\mathbf{I} + \frac{1}{2}\mathbf{F}_2\mathbf{A}_{22}^{-1}\mathbf{B}_2\right)^{-1}\right| + \hat{\gamma}_1\varepsilon\left|\mathbf{B}_2\right|\right]\mathbf{e}^{-\alpha_1(t-s)} + \hat{\gamma}_2\left|\mathbf{B}_2\right|\mathbf{e}^{-\alpha_2\frac{t-s}{\varepsilon}}\right\}\left|\mathbf{x_2(s,\varepsilon)}\right|\mathbf{ds}$$

We choose $\theta_1, \theta_2 \in (0,1)$ where the conditions in the statement fulfilled.

Under the assmption that T>0, $\varepsilon > 0$, denote:

$$\rho_1(T,\varepsilon) = \sup_{t\in[0,T]} e^{\theta_1\alpha_1 t}\left|x_1(t,\varepsilon)\right|$$

$$\rho_2(T,\varepsilon) = \sup_{t\in[0,T]} e^{\theta_2\alpha_2\frac{t}{\varepsilon}}\left|x_2(t,\varepsilon)\right|$$

From (4.10) it follows that:

$$e^{\theta_1\alpha_1 t}\left|x_1(t,\varepsilon)\right| \le \hat{\gamma}_3\left[\left|x_1^0\right| + \varepsilon\left|x_2^0\right| + \|\varphi\|\right] + \rho_1(T,\varepsilon)\int_0^t\left[\hat{\gamma}_1\varepsilon e^{-\alpha_1(1-\theta_1)(t-s)} + \hat{\gamma}_2 e^{-(\alpha_2-\varepsilon\theta_1\alpha_1)\frac{t-s}{\varepsilon}}\right]$$

$$\times\left[\left|A_{21}\right| + \frac{1}{2}\left|B_2F_1\right|e^{\theta_1\alpha_1\tau}\right]ds + \frac{1}{2}\left|F_1\right|\rho_1(T,\varepsilon)\int_0^t\left\{\tilde{B}\left|\left(I + \frac{1}{2}F_2A_{22}^{-1}B_2\right)^{-1}\right| + \varepsilon\left|B_2\right|\hat{\gamma}_1 e^{-\alpha_1(1-\theta_1)(t-s)}\right.$$

$$\left.+\hat{\gamma}_2\left|B_2\right|e^{-(\alpha_2-\varepsilon\theta_1\alpha_1)\frac{t-s}{\varepsilon}}e^{\theta_1\alpha_1\tau}\right\}ds + \frac{1}{2}\left|F_2\right|\rho_2(T,\varepsilon)\int_0^t\left\{\tilde{B}\left|\left(I + \frac{1}{2}F_2A_{22}^{-1}B_2\right)^{-1}\right| + \varepsilon\left|B_2\right|\hat{\gamma}_1 e^{-(\theta_2\alpha_2-\varepsilon\theta_1\alpha_1)\frac{s}{\varepsilon}}\right.$$

$$\left.+\hat{\gamma}_2\left|B_2\right|e^{-\alpha_2(1-\theta_2)\frac{t-s}{\varepsilon}}\right\}ds$$

It is revealed that:

$$e^{\theta_1\alpha_1 t}\left|x_1(t,\varepsilon)\right| \le \$_3\left[\left|x_1^0\right| + \varepsilon\left|x_2^0\right| + \|\varphi\|\right] + \left(\frac{\varepsilon\gamma_1}{\alpha_1(1-\theta_1)} + \frac{\varepsilon\$_2}{\alpha_2 - \varepsilon\theta_1\alpha_1}\right)$$

$$\times\left[\left|A_{21}\right| + \frac{1}{2}\left|B_2F_1\right|e^{\alpha_1\tau}\right]\rho_1(T,\varepsilon) + \frac{1}{2}\left|F_1\right|\left[\left|\tilde{B}\left(I + \frac{1}{2}F_2A_{22}^{-1}B_2\right)^{-1}\right| + \varepsilon\left|B_2\right|\$_1 + \frac{\$_2\left|B_2\right|\varepsilon}{\alpha_2 - \varepsilon\theta_1\alpha_1}\right]$$

$$\times e^{\theta_1\alpha_1\tau}\rho_1(T,\varepsilon) + \frac{1}{2}\left|F_2\right|\varepsilon\left(\left|\tilde{B}\left(I + \frac{1}{2}F_2A_{22}^{-1}B_2\right)^{-1}\right| + \varepsilon\left|B_2\right|\$_2 + \frac{\$_2\left|B_2\right|\varepsilon}{\alpha_2(1-\theta_2)}\right)\rho_2(T,\varepsilon)$$

$$(\forall)t \in [0,T]$$

Consequently:

$$\rho_1(T,\varepsilon) \le \$_3\left(\left|x_1^0\right| + \varepsilon\left|x_2^0\right| + \|\varphi\|\right) + \left[\frac{1}{2}\beta_1 e^{\theta_1\alpha_1\tau}\left|F_1\right|\frac{\left|\tilde{B}\left(I + \frac{1}{2}F_2A_{22}^{-1}B_2\right)^{-1}\right|}{\alpha_1(1-\theta_1)} + \varepsilon\$_v\right]\rho_1(T,\varepsilon)$$

$$+\varepsilon\$_s\rho_2(T,\varepsilon)$$

(4.11)

where we defined $\$_4$, $\$_5$ > 0

Other way from (4.7) we have a similar expression

$$\rho_2(T,\varepsilon) \le \beta_2\left|x_2^0\right| + \$_6\rho_1(T,\varepsilon) + \$_7\rho_2(T,\varepsilon)$$

(4.12)

where $\$_6$, $\$_7$ > 0

Now this proof can be handled similarly that in theorem 3.1 and finally we find a result for saturating case.

5. Example

Let us consider a simple singular perturbed system [6] with input delay and saturated control :

$$\dot{x}_1(t) = x_2(t)$$

$$\varepsilon\dot{x}_2(t) = -x_1(t) - x_2(t) + \text{sat}\ u(t-\tau)$$

(5.1)

If we neglect time delay and saturating contribution a composite control result :

$$u(t) = -5x_1(t) - x_2(t)$$

with $F_2 = -1$ and $F_0 = -2$ for a desired eigenvalue spectrum $(-3, -2/\varepsilon)$.

When we assume $\tau = 0.2$ and the system is saturated at level ± 1 then (4.5) became :

$$u(t) = -\frac{15}{4} x_1(t-\tau) - x_2(t)$$

We choose : $\alpha_1 = 2,\ \ \alpha_2 = 2,\ \ \beta_1 = \frac{1}{\sqrt{6}},\ \ \beta_2 = 1$

This control stabilizes (4.1) when the inequalities of the theorem 4.1 are true :

$$2 > \frac{3}{2}\frac{1}{\sqrt{6}}\left|(1+\frac{1}{2}(-1)(-1)(1))(-\frac{5}{2})\right|\left|(1)(1+\frac{1}{2}(-1)(-1)(1))\right| \cong 1.531$$

$$2 > \frac{1}{2}(1)|1||-1| = \frac{1}{2}$$

Thus, one can conclude that the stability of system (4.10 is controlled by (4.5).

6. <u>Conclusion</u>

Time - delay and actuator saturation could lead to instability of closed - loop systems. In this paper two linear feedback laws both state and output feedback are constructed to that achieve stabilization of singular perturbed systems with time delay control and position and rate limited actuators. The sufficient conditions for stability are derived and these inequalities allow us to evaluate the transient behaviour of the stabilized systems with a simple algorithm.

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