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Reliable computation of the input-state-output relations in autoregressive representations of multivariable systems

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Abstract

Input-state-output analysis of systems with external variables on equal footing is pursued through a numerical algorithm for processing a set of linear differential equations in the form of an autoregressive representation. Instead of resorting to the computation of elementary polynomial operations, numerically robust routines from numerical linear algebra are used to compute an implicit state-space realization in the form of a minimal driving variable representation. The representation is used to detect candidate inputs among the external variables. The algorithm is based on polynomial matrix to state space conversions leading to application of well-proven methods of numerical linear algebra such as Gram-Schmidt orthonormalization, Householder transformations, and the singular value decomposition.

Keywords: Subspace methods, Numerical methods, Linear systems

1 Introduction

Analysis of dynamic systems relies on our ability to capture that signals describing a system at one time are interrelated, not only with other signals at that time, but in a special way with signals at other times. In continuous time, p dynamic relations among $p+m$ scalar signals are often formulated in a set of p scalar differential equations with $p+m$ scalar variables. The set of equations may be described in the form of a single equation

$$P\left(\frac{d}{dt}\right)w(t) \quad = \quad 0 \tag{1}$$

where $P \in \mathbb{R}^{p\times(p+m)}[s]$ is a polynomial matrix with full normal rank p. The vector w accommodates the $p+m$ scalar variables. A polynomial matrix equation of the form (1) is called the AR (autoregressive representation) and the entries of w are referred to as the external variables of the underlying system. The external variables w are said to be on equal footing; we draw no distinction between the input and output nature of the scalar entries in the vector w.

The main thrust of the paper is to present a reliable algorithm for minimal externally equivalent state-space realization of ARs, and to use the algorithm to determine all input-state-output relations defined on a given AR by considering w as a collection of input variables u and output variables y. External equivalence of ARs is studied in Blomberg and Ylinen (1983). For a complete list of externally equivalent operations on a general system of differential equations, see Schumacher (1988).

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Of course, not every entry in w may act as an input to a causal system. Candidate input-state-output relations are determined by $(p+m)$-dimensioned permutation matrices but not every such matrix imposes an input-state-output relation. The ith input-state-output relation may be specified in the form

$$\Sigma_i \;=\; \left\{ \begin{array}{rcl} \begin{bmatrix} y_i(t) \\ u_i(t) \end{bmatrix} & = & \Pi_i w(t) \\ \frac{d}{dt} x_i(t) & = & A_i x_i(t) + B_i u_i(t) \\ y_i(t) & = & C_i x(t) + D_i u_i(t) \end{array} \right. \tag{2}$$

where Π_i is the relevant permutation matrix. The situation is depicted in Figure 1.

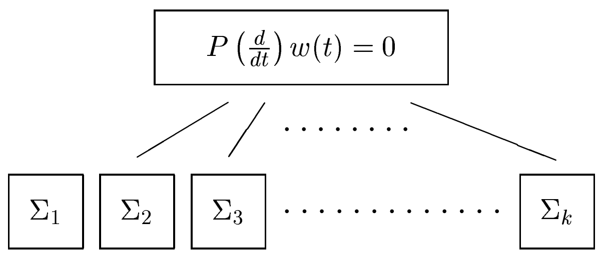

Figure 1: input-state-output relations

The main results are outlined in Section 2. Section 3 covers some standard material to be used in the sequel. Section 4 describes an algorithm for minimal externally equivalent state-space realization of (1). The algorithm is a prerequisite for the detection of the input-state-output relations whose specification is subject to the algorithm in Section 5. Last before conclusions, in Section 6 we discuss the observability properties of the minimal externally equivalent realization and realizations in the form (2).

2 Main results

The main results are *Algorithm 4.1* — a numerically reliable algorithm for state-space realization of ARs (autoregressive representations) associated with full normal row rank polynomial matrices — and the observation that, related to the AR in the form (1), all input-state-output relations Σ_i of the form (2) may be described by observable but not necessarily controllable state-space realizations with a common set of observability indices.

Algorithm 4.1 does not require the computation of elementary polynomial operations. The algorithm is based on invariant subspace methods with orthonormal bases. Computationally, the algorithm relies on Gram-Schmidt orthonormalization, Householder transformations, and the singular value decomposition.

3 Preliminaries

The invariant subspace methods in Algorithm 4.1 rely on the maximal controlled invariant subspace contained in a given subspace. A state-space realization

$$\begin{array}{rcl} \frac{d}{dt} x(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \tag{3}$$

is used as a conceptual tool to perform externally equivalent transformations on (1). The maximal controlled invariant subspace with respect to the pair (A, B) such that this subspace is contained in $\ker C$

is denoted by $\mathcal{V}^*(A,\ B,\ \ker C)$. The interest is in $\mathcal{V}^*(A,\ B,\ \ker C)$ and its orthogonal complement. Extensive details about invariant subspace methods are omitted and may be found in Wonham (1979), Hautus (1983), and the monograph by Basile and Marro (1992).

According to Rosenbrock (1970), the set of observability indices of a state-space realization (A, B, C, D) may be read off following a similarity transformation into a staircase form

$$\left[\begin{array}{cc} A & B \\ C & D \end{array}\right] \xrightarrow{T} \left[\begin{array}{cc|c} A_{11} & 0 & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & 0 & D \end{array}\right] \tag{4}$$

where (A_{11}, B_1, C_1, D) is an observable realization of the transfer function realized by (A, B, C, D). The similarity transformation T may rely on a product of orthogonal matrices in the form of Householder matrices whose successive right-hand side application transforms (A, B, C, D) into the form

$$\left[\begin{array}{ccccc|c} A_{11} & A_{12} & 0 & \cdots & 0 & B_1 \\ A_{21} & A_{22} & A_{23} & \cdots & 0 & B_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ A_{\mu-1,1} & A_{\mu-1,2} & A_{\mu-1,3} & \cdots & 0 & B_{\mu-1} \\ A_{\mu,1} & A_{\mu,2} & A_{\mu,3} & \cdots & 0 & B_{\mu} \\ A_{\mu+1,1} & A_{\mu+1,2} & A_{\mu+1,3} & \cdots & A_{\mu+1,\mu+1} & B_{\mu+1} \\ \hline C_1 & 0 & 0 & \cdots & 0 & D \end{array}\right] \tag{5}$$

with $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ for $i, j = 1, 2, \cdots, \mu$. The integers n_i are defined such that

$$\begin{array}{rcl} n_1 & = & \text{rank } C \\ n_2 & = & \text{rank } A_{i,i+1} \\ & \vdots & \\ n_\mu & = & \text{rank } A_{\mu-1,\mu} \end{array} \tag{6}$$

C_1 and $A_{i,i+1}$, $i = 1, 2, \cdots, \mu-1$, are in the lower echelon form. In particular, if (3) is observable, then $A_{\mu+1,\mu+1}$ is void.

4 State-space realization of ARs

A polynomial matrix $P \in \mathbb{R}^{p\times(p+m)}[s]$ may be defined in terms of a matrix polynomial

$$P(s) \quad = \quad P_0 + P_1 s + P_2 s^2 + \cdots + P_l s^l$$

and manipulated as an array of coefficient matrices $P_i \in \mathbb{R}^{p\times(p+m)}$, $i = 0,\ 1,\ \ldots,\ l$. The following algorithm is based on external equivalence of linear systems in the form (1). The result is a minimal externally equivalent state-space realization of (1). The concept of external equivalence implies that minimal realizations of ARs are in the form of observable but not necessarily controllable state-space realizations. If (A, B, C, D) and (A', B', C', D') are externally equivalent minimal realizations, then there exist nonsingular matrices S and R along with a matrix K such that

$$\left[\begin{array}{cc} A' & B' \\ C' & D' \end{array}\right] = \left[\begin{array}{cc} S^{-1}(A+BK)S & S^{-1}BR \\ (C+DK)S & DR \end{array}\right]. \tag{7}$$

Algorithm 4.1 (State-space realization of ARs) *Let $P \in \mathbb{R}^{p\times(p+m)}[s]$ be a full-row-rank polynomial matrix without zero columns.*

1. *Define A, B, C, H, $J \in \mathbb{R}^{\bullet\times\bullet}$ such that*

$$\begin{bmatrix} A & B \\ C & 0 \\ H & J \end{bmatrix} := \left[\begin{array}{cccccc|c} 0 & \cdots & \cdots & \cdots & \cdots & 0 & I \\ I & \ddots & & & & \vdots & 0 \\ 0 & \ddots & \ddots & & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ 0 & \cdots & 0 & I & \ddots & \vdots & 0 \\ P_{l-1} & \cdots & \cdots & P_1 & P_0 & 0 & P_l \\ \hline 0 & \cdots & \cdots & \cdots & 0 & I & 0 \\ \hline 0 & \cdots & \cdots & 0 & I & 0 & 0 \end{array}\right].$$

2. *Apply Algorithm 4.2 to A, B, C, H, J specified by Step 1. The result is described by A_{11}, B_{12}, H_1, $J_2 \in \mathbb{R}^{\bullet\times\bullet}$. Define new A, B, C, H, $J \in \mathbb{R}^{\bullet\times\bullet}$ such that*

$$\begin{bmatrix} A & B \\ C & 0 \\ H & J \end{bmatrix} := \begin{bmatrix} A_{11}^{\mathrm{T}} & H_1^{\mathrm{T}} \\ B_{12}^{\mathrm{T}} & 0 \\ 0 & -I \end{bmatrix}. \tag{8}$$

3. *Apply Algorithm 4.2 to A, B, C, H, J specified by (8). The result is described by A_{11}, B_{12}, H_1, $J_2 \in \mathbb{R}^{\bullet\times\bullet}$. Define A, B, C, D, $J \in \mathbb{R}^{\bullet\times\bullet}$ such that*

$$\begin{bmatrix} A & B \\ C & D \\ 0 & J \end{bmatrix} := \begin{bmatrix} A_{11}^{\mathrm{T}} & H_1^{\mathrm{T}} \\ B_{12}^{\mathrm{T}} & J_2^{\mathrm{T}} \\ 0 & I \end{bmatrix}. \tag{9}$$

4. *Apply the orthogonal matrix $T_v = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ —specified by (15) in Algorithm 4.2 — to obtain*

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ C & 0 & -I \\ 0 & T_1 & T_2 \end{array}\right] := \begin{bmatrix} A_{11}^{\mathrm{T}} & H_1^{\mathrm{T}} \\ B_{12}^{\mathrm{T}} & J_2^{\mathrm{T}} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & T_1 & T_2 \end{bmatrix}.$$

Then $P\left(\frac{d}{dt}\right) w(t) = 0$ admits an externally equivalent state-space realization

$$\begin{aligned} \tfrac{d}{dt}x(t) &= (A + B_2C)x(t) + B_1v(t) \\ w(t) &= T_2Cx(t) + T_1v(t) \end{aligned} \tag{10}$$

where w are the external variables, x are the state variables, and v are the driving variables. In addition, (10) has the minimal $\dim x$ and the minimal $\dim v$ amongst all externally equivalent realizations of $P\left(\frac{d}{dt}\right) w(t) = 0$.

Because of the necessity to introduce v, the state-space realization (10) is called the DVR (driving variable representation) of the system representation (1).

Throughout Algorithm 4.1 the dimensions of the externally equivalent realizations are successively deflated on the grounds that if a state-space realization (A, B, C, D) is a strongly observable[1], then

$$\text{rank} \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = \dim A + \text{rank} \begin{bmatrix} -B \\ D \end{bmatrix}.$$

The latter result is a part of a useful theorem in Hautus (1983). In the context of Algorithm 4.1, the full-column rank system matrix

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \tag{11}$$

guarantees existence of a certain unimodular transformation which validates the deflation. In particular, the deflation is subject to Algorithm 4.2, where (A, B, C, D) of (11) is defined by $(A_{22}, B_{21}, C_2, 0)$ of Step 3 and Step 4.

Algorithm 4.2 (Tool in Algorithm 4.1) *Let A, B, C, H, $J \in \mathbb{R}^{\bullet\times\bullet}$ specify*

$$\begin{aligned} \tfrac{d}{dt}x(t) &= Ax(t) + Bv(t) \\ 0 &= Cx(t) \\ w(t) &= Hx(t) + Jv(t) \end{aligned} \tag{12}$$

— a state-space realization with external variables w, state variables x, and driving variables v.

1. *Find an orthonormal basis $(\xi_1, \ldots, \xi_k)$ for $\mathcal{V}^*(A, B, \ker C)$ defined as the largest subspace $\mathcal{V}$ such that*

   (i) $\mathcal{V}$ *is controlled invariant for* (A, B),
   (ii) $\mathcal{V} \subset \ker C$.

2. *Wrap a feedback $v \longrightarrow v + Kx$ around (12) such that a state-space realization*

   $$\begin{aligned} \tfrac{d}{dt}x(t) &= (A + BK)x(t) + Bv(t) \\ 0 &= Cx(t) \\ w(t) &= (H + JK)x(t) + Jv(t) \end{aligned} \tag{13}$$

   is obtained such that

   $$\mathcal{V}^*(A + BK) = \mathcal{V}^*(A, B, \ker C)$$

   is a simple invariant subspace.

3. *Find $(\xi_{k+1}, \ldots, \xi_n)$ — the orthonormal complement to the basis of $\mathcal{V}^*$ — and apply the orthogonal matrix*

   $$T = \begin{bmatrix} \xi_1 \ldots \xi_k & \xi_{k+1} \ldots \xi_n \end{bmatrix}$$

[1] An observable state-space realization is *strongly observable* if the realization remains observable under arbitrary regular state feedback, see (7).

as a similarity transformation matrix in (13). In the new coordinates, (13) is described in the Kalman form

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} v(t) \\ 0 &= \begin{bmatrix} 0 & C_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ w(t) &= \begin{bmatrix} H_1 & H_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Jv(t) \end{aligned} \tag{14}$$

where $A_{11} \in \mathbb{R}^{k\times k}$.

4. *Orthogonally transform v in (14) such that — in the new coordinates — B_2 is in a column-compressed form. If T_v is the relevant orthogonal transformation matrix, then*

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{bmatrix} := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} T_v \tag{15}$$

   such that B_{21} has full column rank.

Then (12) is externally equivalent to the state-space realization

$$\begin{aligned} \tfrac{d}{dt}x(t) &= A_{11}x(t) + B_{12}v(t) \\ w(t) &= H_1x(t) + J_2v(t). \end{aligned} \tag{16}$$

In (12)–(16), x and v denote different quantities. For example, although the interpretation of x and v remains intact, the dimension of v in (16) is less than the dimension of the quantity denoted by v in (14). On the other hand, throughout the algorithm, w denotes one and the same quantity.

In (16), $\dim x$ and $\dim v$ need not be the minimal dimensions amongst all externally equivalent state-space realizations of (12); $\dim x$ is minimal following the second application of Algorithm 4.2 in Algorithm 4.1.

4.1 Example

This example illustrates Algorithm 4.1 on the problem of minimal externally equivalent realization of the system described in (1). The relevant polynomial matrix,

$$\begin{aligned} P(s) &= \begin{bmatrix} P_1(s) & P_2(s) \end{bmatrix} \\ P_1(s) &= \begin{bmatrix} -0.33s^2+7.35s+4.39 & 1.74s^2+1.89s+1.7 & -0.78s^2+3.82s-1 \\ 0.19s^2+0.674s+1.99 & 0.037s^2+0.524s+0.833 & 0.84s-0.78 \\ -0.19s^2-0.934s-5.29 & -0.037s^2+0.326s-1.29 & -0.45s+1.57 \end{bmatrix} \\ P_2(s) &= \begin{bmatrix} 2.61s^2-15.4s-8.4 & 2.02s^2+12.2s-10.4 \\ -0.19s^2-2.4s-3 & 0.44s^2+1.58s-4.5 \\ 0.19s^2+s-1.5 & -0.44s^2-0.79s+9.9 \end{bmatrix}, \end{aligned}$$

is depicted in Figure 2.

After the first application of Algorithm 4.2, the externally equivalent — yet non-minimal — state-space realization of (1) is in the form

$$\begin{aligned} \tfrac{d}{dt}x(t) &= A_{11}x(t) + B_{12}v(t) \\ w(t) &= H_1x(t) + J_2v(t) \end{aligned} \tag{17}$$

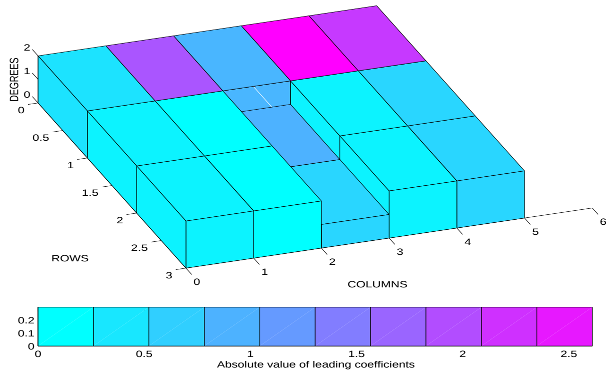

Figure 2: a "Lego" diagram of $P\left(\frac{d}{dt}\right)$

where

$$A_{11} = \begin{bmatrix} -3.9343 & -1.8439 & -1.2948 & 0.9519 & 8.4347 & 5.4079 & -6.0072 & 1.5706 & -1.9460 \\ 0.1048 & -0.0221 & 0.3310 & -0.2001 & -3.3537 & -0.6842 & 0.8980 & -0.6633 & 0.4515 \\ -0.0091 & 0.0961 & 0.0600 & -0.0516 & -0.3322 & -0.2577 & 0.0831 & -1.0669 & 0.0512 \\ -0.0062 & -0.0737 & -0.0902 & 0.0668 & 0.6952 & 0.2930 & -0.1815 & 0.1384 & 0.9012 \\ 0.0255 & 0.4602 & 0.5197 & -0.3902 & -3.9045 & -1.7354 & 0.0164 & -0.7782 & 0.5577 \\ 0.9799 & -0.5963 & -0.6297 & 0.4788 & 4.6254 & 2.1545 & -1.2011 & 0.9225 & -0.6640 \\ 0.1392 & -0.7998 & -0.2117 & 0.3647 & -0.6396 & 1.3914 & 0.6141 & -0.3160 & 0.1093 \\ 2.5244 & -0.0403 & -0.3952 & 0.3317 & 3.4177 & 0.8074 & 1.5872 & 0.7549 & -0.0210 \\ 0.2460 & 0.0227 & 0.1579 & -0.0629 & -1.6933 & -0.4160 & 0.7625 & -0.3845 & 0.2869 \end{bmatrix}$$

$$B_{12} = \begin{bmatrix} 0.0548 & -0.3051 \\ 0.1640 & -0.2455 \\ 0.0229 & -0.0342 \\ -0.0393 & 0.0588 \\ 0.2236 & -0.3348 \\ -0.2684 & 0.4018 \\ 0.1667 & 0.2266 \\ 0.2559 & -0.5904 \\ -0.8676 & -0.4104 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} 0.0391 & 0.8369 & -0.0227 & 0.0391 & -0.2225 & 0.2670 & 0.0692 & -0.0420 & 0.0193 \\ 0.0054 & -0.0227 & 0.9968 & 0.0054 & -0.0310 & 0.0372 & 0.0096 & -0.0059 & 0.0027 \\ -0.0093 & 0.0391 & 0.0054 & 0.9907 & 0.0533 & -0.0639 & -0.0166 & 0.0101 & -0.0046 \\ 0.0533 & -0.2225 & -0.0310 & 0.0533 & 0.6966 & 0.3640 & 0.0944 & -0.0573 & 0.0263 \\ -0.0639 & 0.2670 & 0.0372 & -0.0639 & 0.3640 & 0.5632 & -0.1133 & 0.0688 & -0.0315 \end{bmatrix}$$

and J_2 is a zero matrix.

After the second application of Algorithm 4.2, the externally equivalent — now minimal — state-space realization of (1) is in the form

$$\begin{aligned} \tfrac{d}{dt}x(t) &= Ax(t) + Bv(t) \\ w(t) &= Hx(t) + Jv(t) \end{aligned} \tag{18}$$

where A, B, H, and J are specified by

$$\left[\begin{array}{cc} A & B \\ H & J \end{array}\right] = \left[\begin{array}{rrrrr|rr} -3.0501 & 3.3934 & -2.4642 & 1.1425 & -0.2319 & -2.3112 & 6.1252 \\ 1.1593 & -1.1881 & 1.1629 & 1.0333 & -0.7791 & 1.3914 & -1.1079 \\ 0.7715 & 3.8256 & -0.0321 & 1.0318 & -0.4277 & -2.2707 & 4.5775 \\ -1.1503 & 5.0813 & -4.6348 & -0.2695 & 1.7085 & -4.6585 & 5.5119 \\ 1.6010 & -4.6996 & 0.9751 & -1.2808 & 0.6162 & 2.7191 & -6.9995 \\ \hline 0.1461 & 0.1084 & -0.2155 & -0.2791 & -0.5389 & 0.0678 & -0.7397 \\ -0.1432 & 0.3390 & 0.1717 & 0.3784 & 0.5432 & 0.1655 & -0.5442 \\ -0.2067 & -0.1553 & 0.0193 & 0.0599 & 0.0246 & -0.9358 & -0.1919 \\ 0.6455 & 0.3574 & 0.0843 & -0.0610 & 0.2635 & -0.2607 & -0.0535 \\ 0.0730 & 0.7425 & -0.1687 & 0.0227 & -0.2461 & -0.1558 & 0.3421 \end{array}\right].$$

In (17) and (18) we use the same abuse of notation as in (12)–(16).

5 Input-state-output relations in ARs

In this section, we present an algorithm for the computation of the state-space realizations related to the input-state-output relations within a given AR of the form (1).

Algorithm 5.1 (Input-state-output relations in a minimal DVR) *Consider a minimal DVR (F, G, H, J), and the following sequence of operations:*

1. *Permute the rows of J to obtain a nonsingular matrix J_2 such that*

$$\Pi w(t) = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} x(t) + \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} v(t).$$

2. *Apply a state-feedback transformation $F \longrightarrow F + GK$ where K is constructed such that (F, G, H, J) is transformed to*

$$\begin{aligned} \tfrac{d}{dt}x(t) &= (F+GK)x(t) + Gv(t) \\ \Pi w(t) &= \begin{bmatrix} H_1 + J_1 K \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} v(t). \end{aligned} \tag{19}$$

3. *Transform the present driving variables v in accordance with $u := J_2 v$ to bring (19) into*

$$\begin{aligned} \tfrac{d}{dt}x(t) &= (F+GK)x(t) + GJ_2^{-1}u(t) \\ \Pi w(t) &= \begin{bmatrix} H_1 + J_1 K \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} J_1 J_2^{-1} \\ I \end{bmatrix} u(t). \end{aligned}$$

Then $(F + GK, GJ_2^{-1}, H_1 + J_1K, J_1J_2^{-1})$ is a "minimal" state-space realization driven by external variables. This realization is related to the given minimal DVR (F, G, H, J) through

$$\begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \Pi w(t)$$

where Π is a non-unique but fixed permutation matrix obtained in Step 1.

As shown in Step 1, the total number k of input-state-output relations in a given minimal DVR, and thence in (1), depends on the number of combinations to choose a nonsingular J_2 from J, that is,

$$1 \le k \le \begin{pmatrix} p+m \\ m \end{pmatrix}. \tag{20}$$

5.1 Example

Let us illustrate Algorithm 5.1 on the problem of detection and specification of every input-state-output relation (2) in the system representation (1). The polynomial matrix P is displayed in Example 4.1.

We start off at the minimal DVR (18) and apply Algorithm 5.1 to all admissible permutation matrices Π_i. In agreement with (20), the true number of relations is $k = 9$. That is, one of the 10 relations is nonexistent since the relevant Π yields a singular J_2.

Table 1: DVR and input-state-output relations: the eigenvalues

DVR	-4.8912	-1.9901	-0.0190	$1.4883 + 2.0620i$	$1.4883 - 2.0620i$
345	-6.5255	$3.7205 + 4.6790i$	$3.7205 - 4.6790i$	-0.5080	1.0894
245	-5.4749	$-0.3803 + 0.8238i$	$-0.3803 - 0.8238i$	0.2919	4.5899
235	-5.2878	4.2668	-1.0133	-0.1743	0.8414
234	$1.6786 + 4.4807i$	$1.6786 - 4.4807i$	-0.2076	$0.5884 + 1.2782i$	$0.5884 - 1.2782i$
145	-6.8694	-3.4594	$1.1632 + 3.1445i$	$1.1632 - 3.1445i$	0.0041
135	-6.7409	$-0.0524 + 3.6321i$	$-0.0524 - 3.6321i$	$0.1841 + 0.2442i$	$0.1841 - 0.2442i$
134	-8.0647	$2.1589 + 6.9930i$	$2.1589 - 6.9930i$	-0.1704	2.7106
124	$-2.8104 + 5.4283i$	$-2.8104 - 5.4283i$	-1.1650	-0.1369	1.0167
123	$-2.2040 + 6.7448i$	$-2.2040 - 6.7448i$	1.5163	$-0.2379 + 0.0401i$	$-0.2379 - 0.0401i$

In Table 1, the first row specifies the eigenvalues of a minimal externally equivalent state-space realization of (1) while the remaining nine rows list the eigenvalues of the nine state-space realizations representing the nine input-state-output relations in (1). For example "345" denotes the relation where y, the vector of outputs, occupies the positions $\{3, 4, 5\}$ of the external variables vector w.

The application of Algorithm 5.1 revealed the nonexistence of an input-state-output relation between entries at the positions $\{1, 2, 5\}$ of w as outputs and $\{3, 4\}$ as inputs.

6 Observability properties

The minimal externally equivalent realization of (1) —in the form of a DVR — generalizes the realization of a left polynomial MFD (matrix fraction description). Recall that state-space realization of left MFDs may be organized to yield minimal (externally equivalent) state-space representations in the form of observable but not necessarily controllable realizations (Wolovich, 1971).

The DVR is not only observable but also strongly observable, that is, the DVR remains observable under arbitrary regular state feedback transformations. Similarly to minimal realizations of left MFDs, minimal DVRs need not be controllable. The set of observability indices related to a DVR (and the relevant input-state-output relations) may be studied as described in Section 3. Construction of the Hessenberg form (5) is useful to show that, given a fixed (1), the set of observability indices of either input-state-output relation equals the set of observability indices of the minimal DVR.

By Algorithm 5.1, the input-state-output relations are described in terms of observable but not necessarily controllable state-space realizations whose dimension equals the dimension of the minimal DVR. The dimension is the McMillan degree of the underlying system.

7 Conclusions

A reliable algorithm for minimal externally equivalent state-space realization of systems described by ARs (autoregressive representations) was presented. For uncontrollable systems the realizations are observable non-minimal state-space realizations because external equivalence of ARs covers both the controllable and uncontrollable dynamics. The algorithm can be useful in CAD of control systems (CADCS) based on polynomial matrices; it replaces the computation of elementary polynomial operations by reliable numerical methods such as Gram-Schmidt orthonormalization, Householder transformations, and the singular value decomposition.

The elementary polynomial operations are avoided by applying an innovative technique for state-space realization. The technique does not require polynomial matrices in a (row) reduced form. Minimal externally equivalent realizations are obtained in a successive conversion based on invariant subspace methods with orthonormal bases. A Matlab implementation of the algorithm is partly based on Grace *et al.* (1990) and the software appendix to Basile and Marro (1992). For comparison, the conventional algorithm (Wolovich, 1971) would require transformation to a row reduced form, modulo polynomial matrix division, and constant matrix inversion for specification of every single input-state-output relation, cf. Kraffer (1993).

Finally, observability properties of the minimal externally equivalent realizations were studied along with the observability properties of the underlying input-state-output relations. The resulting connections are useful in better understanding the applicability of polynomial matrices in control system design.

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