Optimal Guidance with Time Delay for Continuous Time Systems

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Abstract

The guidance problem with continuous time-delayed control is considered. The interception conflict we are dealing with involves a pursuer continuously measuring his relative position and velocity , and who is able to control his own acceleration subject to a given pure time delay , and an evader who can apply constant maneuvers. The cost function for this optimal control problem includes the control effort and a quadratically weighted version of the miss distance and final relative velocity.

Within this setup , several guidance problems are formulated and analytically solved : proportional navigation (PN), augmented PN (APN), and augmented optimal rendezvous (AOR). The solution is obtained by applying results previously published by the authors to the corresponding discrete-time guidance problems and by using an alternative derivation based on the theory of continuous-time linear quadratic optimal control with an input delay. The resulting new guidance laws are compared to the numerically classical guidance laws. The examples demonstrate the advantage of the optimal guidance laws which take the delay into account over the classical ones.

1. Introduction

The problem of minimum effort guidance has been formulated and solved in (Bryson and Ho, 1975). For the case of ideal pursuer and stationary evader (Zarchan 1990) the resulting guidance law is the well known *Proportional Navigation law (PN)* whereas for maneuvering targets it turns out to be the *Augmented Proportional Navigation law(APN).*

In (Bryson and Ho, 1975) it was shown that when the pursuer is non-ideal and is characterized by a single time constant , a guidance law emerges which has the structure of *APN* but has a time varying navigation constant.

The case where the pursuer has very fast dynamics so that its time constant can be neglected , but its acceleration commands are subject to a pure time delay has not been considered , in an optimal control framework to the best of our knowledge.

In the present work , the ideal pursuer and evader , with a pure time delay of the pursuer's acceleration command is considered. Two solution methods are compared. The first is a limiting case of the sampled data guidance law (Gitizadeh , et. al. 1999) , and the second is a direct method. The resulting guidance law performance is evaluated using simulations. The results are encouraging and ask for realistic applications.

In (Gitizadeh , et. al. 1999) the *Proportional Navigation law(P.N.) for discrete time systems with time delays* was considered. *T*he continuous-time case is a limiting situation of the solution when $h \rightarrow 0$ (h being the sampling time). To check our results , we calculate the *P.N.*

guidance law for non-maneuvering targets , using the corresponding theory of linear-quadratic one sided optimal control problem.

The guidance laws of *Augmented Proportional Navigation (APN) and the Augmented Optimal Rendezvous laws(AOR) for continuous time systems with time delays* are also considered.

2. Continuous Time System with Time Delay

In (Gitizadeh , et. al. 1999) we find that if τ , the time delay , is longer than the sampling time (h) , then we define an integer d and τ' as :

$$\tau = (d-1)h + \tau' \qquad 0 \le \tau' < h \tag{2.1}$$

The optimal control with time delay for $k \in \{0,1,2,...,N-d\}$ is :

$$u_k = -(g_1 x_{1k} + g_2 x_{2k} + g_3 u(t_{k-d}) + \sum_{j=1}^{d-1} g_4 u(t_{k-j}) + g_5 w_k) \tag{2.1a}$$

where x_{1k} and x_{2k} , the components of X_k , are the relative displacement and velocity , respectively ; u_k , w_k are the corresponding normal accelerations of the pursuer and evader.

defining $t_{go}=t_f - t$, $t_{go\tau} = t_{go} - \tau$, we get :

$$g_1 = 6b(h - ct_{go\tau}^2 + (hc-2)t_{go\tau} + c\tau'(\tau'-h)) / Den \tag{2.1b}$$

$$\begin{aligned} g_2 = {} & (-4bct_{go\tau}^3 + ((h-2\tau)c-4)3bt_{go\tau}^2 + ((h^2+6h\tau-6h\tau'+6\tau'^2)c - 12\tau + 6h)bt_{go\tau} + \\ & (-6h\tau + h^2 - 3\tau' h + 6\tau\tau' + 2\tau'^2)c\tau' + 6h\tau)b - 12c) / Den \end{aligned} \tag{2.1c}$$

$$\begin{aligned} g_3 = {} & \tau'[-4bct_{go\tau}^3 + ((3\tau' - 6\tau + 3h)c - 12)bt_{go\tau}^2 + ((h^2 + 6h\tau - 9h\tau' + 6\tau'^2)c + 6\tau' - \\ & 12\tau + 6h)bt_{go\tau} + (-6h\tau + h^2 - 3\tau' h + 6\tau\tau' + 2\tau'^2)c\tau' + 6h\tau - 3h\tau')b - 12c] / Den \end{aligned} \tag{2.1d}$$

$$\begin{aligned} g_4 = {} & [-4hbct_{go\tau}^3 + (-6bjh^2 + (6\tau h^2 - 12h)b)t_{go\tau}^2 + ((6h^3c - 12h^2)bj + \\ & h(-6h\tau' + 6\tau'^2 - 2h^2)c + 12h^2)b)t_{go\tau} + ((\tau'^2 + h\tau')c + h)6h^2bj + \\ & ((4h^2 - 6h\tau + 2h\tau'^2)h\tau' c - 3h^3)b - 12ch] / Den \end{aligned} \tag{2.1e}$$

$$\begin{aligned} g_5 = {} & [-bct_{go\tau}^4 + (-4c\tau - 6)bt_{go\tau}^3 + ((-3\tau^2 + h^2 + 3\tau h - 3h\tau' + 3\tau'^2)c - 12\tau + 3h)bt_{go\tau}^2 + \\ & (((2\tau'^3 + (6\tau - 3h)\tau'^2 + (-6h\tau + h^2)\tau' + \tau(h^2 + 3h\tau))c + \tau(3h + 6\tau) + 3h\tau)b - 12c) \\ & t_{go\tau} + ((2\tau\tau'^2 + 3\tau\tau'(\tau - h) + \tau(h^2 - 3h\tau)\tau')c + 3\tau^3 h)b - 12c\tau] / Den \end{aligned} \tag{2.1f}$$

$$Den = -bct_{go\tau}^4 - 4bt_{go\tau}^3 + (h^2 + 3\tau'^2 - 3h\tau')cbt_{go\tau}^2 + (((2\tau'^2 + h^2 - 3\tau'h)\tau'c + h^2)b$$
$$-12c)t_{go\tau} + (2\tau'^2 + h^2 - 3\tau'^2 h)\tau'b - 12 \tag{2.1h}$$

For $k \in \{ N\text{-}d+1 , N\text{-}d+2 , ... , N \}$ we have $u_k = 0$.

If the time delay is very large with respect to the sampling time , then the number of states is accordingly large. Thus we now consider the continuous-time case as a limiting case of the solution in (Gitizadeh , et. al. 1999 - equation 5.28). Since the delay τ is given by $\tau = (d-1)h+\tau'$ where $0 \le \tau' < h$, then taking $h \to 0$ implies $\tau' \to 0$.

We should , however , be careful about taking $h \to 0$. By definition , the relation $\frac{t_f}{h} = N$ must be maintained , thus be letting $h \to 0$ we make $N \to \infty$. We denote jh by τ_1 in the sequel. In the limit g_3 tend to zero and :

$$u(t) = -(g_1(t)x_1(t) + g_2(t)x_2(t) + \int_0^\tau g_4(t,\tau_1)u(t-\tau_1)d\tau_1 + g_5(t)w(t)) \tag{2.2}$$

where:

$$g_1(t_{go}) = \frac{6bt_{go\tau}(2 + ct_{go\tau})}{bct_{go\tau}^4 + 4bt_{go\tau}^3 + 12ct_{go\tau} + 12} \tag{2.3}$$

$$g_2(t_{go}) = \frac{2(6c + 3bc\tau t_{go\tau}^2 + 2bct_{go\tau}^3 + 6b\tau t_{go\tau} + 6bt_{go\tau}^2)}{bct_{go\tau}^4 + 4bt_{go\tau}^3 + 12ct_{go\tau} + 12} \tag{2.4}$$

$$g_4(t_{go},\tau_1) = \frac{2(2bt_{go\tau}^2 + 3bct_{go\tau}^3 + 6c + 6b\tau_1 t_{go\tau}(1 + ct_{go\tau}))}{bct_{go\tau}^4 + 4bt_{go\tau}^3 + 12ct_{go\tau} + 12} \tag{2.5}$$

$$g_5(t_{go}) = \frac{t_{go}(12c + 6bt_{go\tau}^2 + 6b\tau t_{go\tau} + bct_{go\tau}^3 + 3bc\tau t_{go\tau}^2)}{bct_{go\tau}^4 + 4bt_{go\tau}^3 + 12ct_{go\tau} + 12} \tag{2.6}$$

3. Non-maneuvering target

3.1 Time Delayed Continuous Proportional Navigation (TCPN)

In this case $w = 0$ and $c = 0$, and from (2.3 - 2.6) we find that:

$$g_1 = \frac{3b(t_{go} - \tau)}{3 + b(t_{go} - \tau)^3} \tag{3.1}$$

$$g_2 = \frac{3bt_{go}(t_{go} - \tau)}{3 + b(t_{go} - \tau)^3} = t_{go} g_1 \tag{3.2}$$

$$g_4 = \frac{3b(t_{go} - \tau)(t_{go} - \tau + \tau_1)}{3 + b(t_{go} - \tau)^3} = (t_{go} - \tau + \tau_1) g_1 \tag{3.3}$$

We define $\mu = t\text{-}\tau_1$, thus :

$$u(t) = -(g_1(t)x_1(t) + g_2(t)x_2(t) + \int_{t-\tau}^{t} g_4(t,\mu)u(\mu)d\mu) \tag{3.4}$$

where:

$$g_1 = \frac{3b(t_{go} - \tau)}{3 + b(t_{go} - \tau)^3} \tag{3.5}$$

$$g_2 = \frac{3bt_{go}(t_{go} - \tau)}{3 + b(t_{go} - \tau)^3} = t_{go} g_1 \tag{3.6}$$

$$g_4 = \frac{3b(t_{go} - \tau)(t_{go} - \tau + t - \mu)}{3 + b(t_{go} - \tau)^3} = (t_f - \tau - \mu) g_1 \tag{3.7}$$

We describe the optimal control law as function of the line-of-sight-rate (Gitizadeh , et. al. 1998). We get :

$$u(t) = -N' (V_C \dot{\lambda} + \frac{1}{t_{go}^2} \int_{t-\tau}^{t} (t_f - \tau - \mu) u(\mu) d\mu) \tag{3.8}$$

Where :

$$N' = \frac{3t_{go}^2 (t_{go} - \tau)}{\frac{3}{b} + (t_{go} - \tau)^3} \tag{3.9}$$

is the navigation constant for a continuous time system with time delay. This results shows that if we take $b \to \infty$ (for perfect hit) , we find that the $N' \to \infty$ as $t_{go} \to \tau$. Therefore we should take finite b to ensure finite guidance gains.

3.2 Time Delayed Continuous Optimal Rendezvous (TDCOR)

The perfect rendezvous for non-maneuvering target is obtained from (2.2 - 2.5) by letting

$$b \to \infty \quad c \to \infty \tag{3.10}$$

We obtain the following feedback gains ($w = 0$) :

$$g_1 = \frac{6}{(t_{go} - \tau)^2} \tag{3.11}$$

$$g_2 = \frac{2(2t_{go} + \tau)}{(t_{go} - \tau)^2} \tag{3.12}$$

$$g_4(t_{go}, \tau_1) = \frac{2(2t_{go} - 2\tau + 3\tau_1)}{(t_{go} - \tau)^2} \tag{3.13}$$

We define $\mu = t\text{-}\tau_1$, thus :

$$u(t) = -(g_1(t)x_1(t) + g_2(t)x_2(t) + \int_{t-\tau}^{t} g_4(t,\mu)u(\mu)d\mu) \tag{3.14}$$

when:

$$g_1 = \frac{6}{(t_{go} - \tau)^2} \tag{3.15}$$

$$g_2 = \frac{2(2t_{go} + \tau)}{(t_{go} - \tau)^2} \tag{3.16}$$

$$g_4(t_{go}, s) = \frac{2(2t_{go} - 2\tau + 3t - 3\mu)}{(t_{go} - \tau)^2} \tag{3.17}$$

We describe the optimal control law as function of the line-of-sight angle and its rate. We get :

$$u_i = -V_c(\overline{g1}\lambda_i + \overline{g2}\,\dot{\lambda}_i) - \int_{t-\tau}^{t} g_4(t_{go},\mu)u(\mu)d\mu \tag{3.18}$$

where:

$$\overline{g_1} = \frac{2}{t_{go} - \tau} \tag{3.19}$$

$$\overline{g_2} = \frac{2t_{go}(2t_{go} + \tau)}{(t_{go} - \tau)^2} \tag{3.20}$$

$$g_4(t_{go}, s) = \frac{2(2t_{go} - 2\tau + 3t - 3\mu)}{(t_{go} - \tau)^2} \tag{3.21}$$

This result shows that if we take $b, c \to \infty$, we find that the guidance gains tend to infinity as $t_{go} \to \tau$. Therefore finite *b* and *c* should be taken as to obtain bounded gains.

4. Control with input delay (stationary target)

For checking of our results for TDCPN and TDCOR we describe the optimal control for this case by another method. Consider the following system :

$$\dot{X} = AX + B_1 w + B_2 u(t-\tau)$$
$$J = X^t(t_f)H(t_f)X(t_f) + \int_{t0}^{tf} (X^t(t)Q(t)X(t) + u^t u - \gamma^2 w^t w)dt \qquad (4.1)$$

The general case of finite γ where *w* is not available for the controller is quite complex. The coresponding controller turns out to be infinite dimensional in an intricate way (Pila , et. al. 1996) . In our case where $\gamma \to \infty$ (one sided optimal control problem) , and where *w* is we known (in fact we assume for simplicity that $w \equiv 0$), a rather simple solution can be obtained.

Defining:

$$\tilde{u}(t) = u(t-\tau) \qquad (4.2)$$

We readily obtain from the well known standard LQ problem (Bryson and Ho, 1975) that:

$$\tilde{u}(t) = -B_2^t P(t)X(t) \qquad (4.3)$$

where:

$$-\dot{P} = A^t P + PA - PB_2 B_{21}^t P + Q \qquad P(t_f) = H(t_f) \qquad (4.4)$$

Remark :

We assume[1] that in $t \in [-\tau, 0]$:

$$u(t) = u_{trim} \qquad (4.5)$$

Therefore from (4.2) and (4.3) we get:

$$u(t) = \tilde{u}(t+\tau) = -B_2^t P(t+\tau)X(t+\tau) \;\; for \;\; t_0 \le t \le t_f - \tau \qquad (4.6)$$

Defining $\tilde{P}(t) = P(t+\tau)$ leads to :

[1]Eq. (4.5) follows from the fact that in $t \in [-\tau, 0]$ the missile is assumed to be trimmed and the control commands are not to be optimized. It should be noted that due to the time delay of the input (control signal) , the states X(t) are influenced by u_{trim} during $t \in [-\tau, 0]$. We also note the fact that the control signal is zero for the τ seconds prior to target approach because then the states are not dependent on u(t).

$$\tilde{P}(t-\tau) = P(t) \tag{4.7}$$

thus:

$$u(t) = -B_2^t \tilde{P}(t) X(t+\tau) \tag{4.8}$$

Since τ is constant , $\tilde{P}(t)$ satisfies :

$$-\dot{\tilde{P}} = A^t \tilde{P} + \tilde{P}A - \tilde{P}B_2 B_{21}^t \tilde{P} + Q \qquad \tilde{P}(t_f - \tau) = H(t_f) \tag{4.9}$$

However , since $w(t) = 0$ (Non-maneuvering target) , we can estimate $X(t+\tau)$ by :

$$X(t+\tau) = \Phi(t+\tau, t)X(t) + \int_t^{t+\tau} \Phi(t+\tau, s) B_2 u(s-\tau) ds \tag{4.10}$$

where $\Phi(t+\tau,s)=e^{A(t+\tau-s)}$ and $\Phi(t+\tau,t)=e^{A\tau}$. Thus we get:

$$u(t) = -B_2^t \tilde{P}(t)(e^{A\tau} X(t) + \int_t^{t+\tau} e^{A(t+\tau-s)} B_2 u(s-\tau) ds) \tag{4.11}$$

If we define μ = s -τ , we obtain:

$$u(t) = -B_2^t \tilde{P}(t)(e^{A\tau} X(t) + \int_{t-\tau}^{t} e^{A(t-\mu)} B_2 u(\mu) d\mu) \tag{4.12}$$

Note that in our case :

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad , \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ e^{A\tau} &= I + A\tau = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \quad \text{since} \quad A^2 = 0 \end{aligned} \tag{4.13}$$

From (4.12) we get:

$$u(t) = -\begin{bmatrix} 0 & 1 \end{bmatrix} \tilde{P}(t)(\begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \int_{t-\tau}^{t} \begin{bmatrix} 1 & t-\mu \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\mu) d\mu) \tag{4.14}$$

where:

$$-\dot{\tilde{P}} = A^t \tilde{P} + \tilde{P}A - \tilde{P}B_2 B_{21}^t \tilde{P} + Q \tag{4.15a}$$

$$\tilde{P}(t_f - \tau) = H(t_f) = \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} \tag{4.15b}$$

Since in our case $Q = 0$, we can define $S(t) = \tilde{P}^{-1}(t)$. Multiplying (4.15a) by S from bothsides , we get:

$$\dot{S} = SA^t + AS - B_2B_2^t \qquad S(t_f - \tau) = \begin{bmatrix} 1/b & 0 \\ 0 & 1/c \end{bmatrix} \tag{4.16}$$

thus :

$$\begin{bmatrix} \dot{s}_{11} & \dot{s}_{12} \\ \dot{s}_{12} & \dot{s}_{22} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2s_{12} & s_{22} \\ s_{22} & -1 \end{bmatrix} \tag{4.17}$$

We define $t_{go} = t_f - t$ and solve the above equation. We get :

$$\begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} = \begin{bmatrix} \frac{(t_{go}-\tau)^3}{3} + \frac{(t_{go}-\tau)^2}{2} + \frac{1}{b} & -\frac{(t_{go}-\tau)^2}{2} - \frac{t_{go}-\tau}{c} \\ -\frac{(t_{go}-\tau)^2}{2} - \frac{t_{go}-\tau}{c} & t_{go} - \tau + \frac{1}{c} \end{bmatrix} \tag{4.18}$$

Then(define : $t_{go\tau} = t_{go} - \tau$)

$$\tilde{P} = S^{-1} = \frac{1}{bct_{go\tau}^4 + 4bt_{go\tau}^3 + 12ct_{go\tau} + 12}\begin{bmatrix} 12b(ct_{go\tau}+1) & 6bt_{go\tau}(ct_{go\tau}+2) \\ 6bt_{go\tau}(ct_{go\tau}+2) & 4(bct_{go\tau}^3 + 3bt_{go\tau}^2 + 3c \end{bmatrix} \tag{4.19}$$

The control is :

$$u(t) = -\begin{bmatrix} 0 & 1 \end{bmatrix}\frac{1}{bct_{go\tau}^4 + 4bt_{go\tau}^3 + 12ct_{go\tau} + 12}\begin{bmatrix} 12b(ct_{go\tau}+1) & 6bt_{go\tau}(ct_{go\tau}+2) \\ 6bt_{go\tau}(ct_{go\tau}+2) & 4(bct_{go\tau}^3 + 3bt_{go\tau}^2 + 3c \end{bmatrix}$$
$$(\begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \int_{t-\tau}^{t}\begin{bmatrix} 1 & t-\mu \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}u(\mu)d\mu) \tag{4.20}$$

or:

$$u(t) = -\frac{\begin{bmatrix} 6bt_{go\tau}(ct_{go\tau}+2) & 4(bct_{go\tau}^3 + 3bt_{go\tau}^2 + 3c) \end{bmatrix}}{bct_{go\tau}^4 + 4bt_{go\tau}^3 + 12ct_{go\tau} + 12}\begin{bmatrix} x_1(t) + \tau x_2(t) + \int_{t-\tau}^{t}(t-\mu)u(\mu)d\mu \\ x_2(t) + \int_{t-\tau}^{t}u(\mu)d\mu \end{bmatrix} \tag{4.21}$$

4.1 Time Delayed Continuous Proportional Navigation (TCPN)

In this case $c = 0$, and from (4.21) we find that:

$$u(t) = -\frac{\begin{bmatrix} 3bt_{go\tau} & 3bt_{go\tau}^2 \end{bmatrix}}{bt_{go\tau}^3 + 3} \begin{bmatrix} x_1(t) + \tau x_2(t) + \int_{t-\tau}^{t} (t-\mu)u(\mu)d\mu \\ x_2(t) + \int_{t-\tau}^{t} u(\mu)d\mu \end{bmatrix} \quad (4.22)$$

or:

$$u(t) = -(g_1(t)x1(t) + g_2(t)x2(t) + \int_{t-\tau}^{t} \overline{g_3(t,\mu)}u(\mu)d\mu) \quad (4.23)$$

where:

$$g_1 = \frac{3b(t_{go} - \tau)}{3 + b(t_{go} - \tau)^3} \quad (4.24)$$

$$g_2 = \frac{3bt_{go}(t_{go} - \tau)}{3 + b(t_{go} - \tau)^3} = t_{go}g_1 \quad (4.25)$$

$$\overline{g_3} = \frac{3b(t_{go} - \tau)(t_{go} - \tau + t - \mu)}{3 + b(t_{go} - \tau)^3} = (t_f - \tau - \mu)g_1 \quad (4.26)$$

Which is the same optimal control law of (3.4 -3.7)

4.2 Time Delayed Continuous Optimal Rendezvous (TDCOR)

The perfect rendezvous is obtained from (4.21) by letting:

$$b \to \infty \quad c \to \infty \quad (4.27)$$

We obtain :

$$u(t) = -\frac{\begin{bmatrix} 6 & 4t_{go\tau} \end{bmatrix}}{t_{go\tau}^2} \begin{bmatrix} x_1(t) + \tau x_2(t) + \int_{t-\tau}^{t} (t-\mu)u(\mu)d\mu \\ x_2(t) + \int_{t-\tau}^{t} u(\mu)d\mu \end{bmatrix} \quad (4.28)$$

or:

$$u(t) = -(g_1(t)x1(t) + g_2(t)x2(t) + \int_{t-\tau}^{t} \overline{g_3(t,\mu)}u(\mu)d\mu) \quad (4.29)$$

where:

$$g_1 = \frac{6}{(t_{go} - \tau)^2} \quad (4.30)$$

$$g_2 = \frac{2(2t_{go} + \tau)}{(t_{go} - \tau)^2} \tag{4.31}$$

$$\overline{g_3}(t_{go}, \mu) = \frac{2(2t_{go} - 2\tau + 3t - 3\mu)}{(t_{go} - \tau)^2} \tag{4.32}$$

Which is the same optimal control law of (3.14 -3.17)

5. Maneuvering target

5.1 The case without velocity weighting - Time Delayed Continuous Augmented Proportional Navigation (TCAPN)

In this case $c = 0$, and from (2.2 - 2.6) we find that

$$g_1 = \frac{3b(t_{go} - \tau)}{3 + b(t_{go} - \tau)^3} \tag{5.1}$$

$$g_2 = \frac{3bt_{go}(t_{go} - \tau)}{3 + b(t_{go} - \tau)^3} = t_{go} g_1 \tag{5.2}$$

$$g_4 = \frac{3b(t_{go} - \tau)(t_{go} - \tau + \tau_1)}{3 + b(t_{go} - \tau)^3} = (t_{go} - \tau + \tau_1) g_1 \tag{5.3}$$

$$g_5 = \frac{3bt_{go}^2(t_{go} - \tau)}{2(3 + b(t_{go} - \tau)^3)} = \frac{t_{go}^2}{2} g_1 \tag{5.4}$$

We describe the optimal control law as function of the line-of-sight-rate. We get :

$$\begin{aligned} u(t) &= -N'\,(V_C\,\dot{\lambda} + \frac{1}{t_{go}^2}\int_0^{\tau}(t_{go} - \tau + \tau_1)u(t - \tau_1)d\tau_1 + \frac{1}{2t_{go}^2}w(t)) = \\ &= -\,N'\,(V_C\,\dot{\lambda} + \frac{1}{t_{go}^2}\int_{t-\tau}^{t}(t_{go} - \tau + t - s)u(s)ds + \frac{1}{2}w(t)) \end{aligned} \tag{5.5}$$

Where :

$$N' = \frac{3t_{go}^2(t_{go} - \tau)}{\frac{3}{b} + (t_{go} - \tau)^3} \tag{5.6}$$

We should be take finite *b* to ensure finite guidance gains.

5.2 Perfect intercept with terminal velocity weighting - Time Delayed Continuous Augmented Optimal Rendezvous (TDCAOR)

The perfect rendezvous is obtained from (2.2 - 2.6) by letting

$$b \to \infty \quad c \to \infty \tag{5.7}$$

We obtain the following feedback gains :

$$g_1 = \frac{6}{(t_{go} - \tau)^2} \tag{5.8}$$

$$g_2 = \frac{2(2t_{go} + \tau)}{(t_{go} - \tau)^2} \tag{5.9}$$

$$g_4(t_{go}, \tau_1) = \frac{2(2t_{go} - 2\tau + 3\tau_1)}{(t_{go} - \tau)^2} \tag{5.10}$$

$$g_5 = \frac{t_{go}(t_{go} + 2\tau)}{(t_{go} - \tau)^2} \tag{5.11}$$

We describe the optimal control low as function of the line-of-sight and its rate . We get :

$$\begin{aligned} u_i &= -v_c(\overline{g1}\lambda_i + \overline{g2}\dot{\lambda}_i) - \int_0^{\tau} g_4(t_{go}, \tau_1)u(t - \tau_1)d\tau_1 - g_5(t)w(t) \\ &= -v_c(\overline{g1}\lambda_i + \overline{g2}\dot{\lambda}_i) - \int_{t-\tau}^{t} g_4(t_{go}, s)u(s)ds - g_5(t)w(t) \end{aligned} \tag{5.12}$$

where $\overline{g_1}$, $\overline{g_2}$, $g_4(t_{go}, s)$ is given in (3.19 -3.21) and g_6 is given in (5.11).

This result shows that if we take $b, c \to \infty$, we find that the guidance gains tend to infinity as $t_{go} \to \tau$. Therefore finite *b* and *c* should be taken as to obtain bounded gains.

6. Numerical Simulations

6.1 Perfect Intercepts - Numerical Simulations for Continuous Time System with Time Delay :

In this section the effects of two guidance laws for the perfect intercepts case will be analyzed. The following Simulink diagram describes the Perfect Intercepts case. In this example the closing velocity is *300 m/s* and $t_f = 0.5$ *sec.*

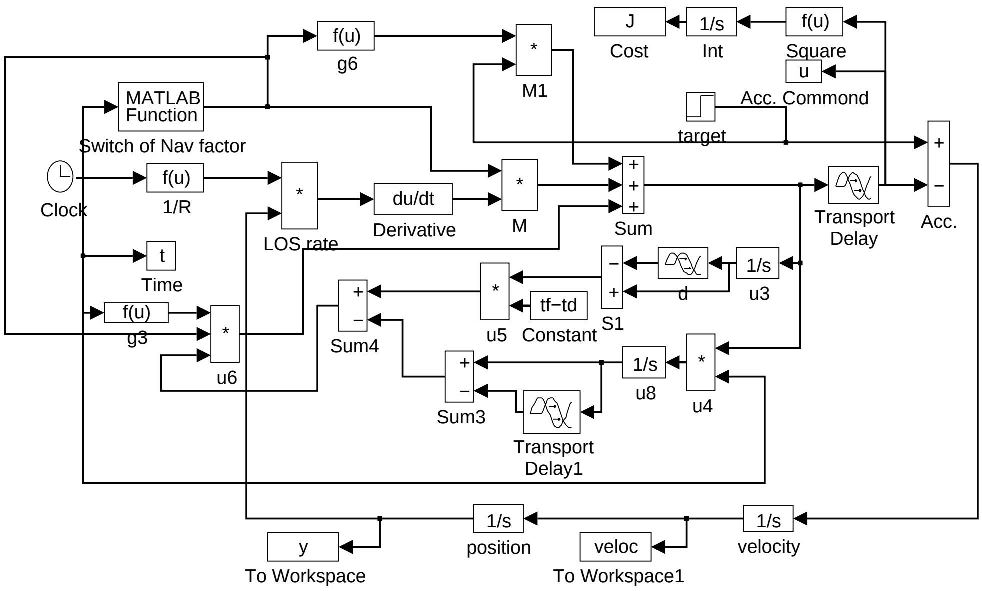

Fig. 6.1 The Simulink diagram describe the Perfect Rendezvous case for Continuous Time System with Time Delay

The following guidance laws are compared:

***i.* C*APN* (Continuous Augmented Proportional Navigation) :**
We use a sampled version of the proportional navigation guidance law with $N'=3$.

***ii. TCAPN* (Time Delayed Continuous Augmented Proportional navigation):**
The optimal control law in Eq. 5.5-5.6.

Initial conditions are : $\tau = 0.2$ *[sec]* , $y_0 = 20$ *[m]* , $v_0 = 26.2$ *[m/s]* $b = 1e6$.

Target is maneuvering with: $w(t) = -10g \qquad t \geq 0$

The results for this case are depicted in Fig. 6.2a-d. Fig. 6.2a-d illustrate the performance of the guidance laws in the presence of this evasive maneuver. In Fig. 6.2a the relative separation is in meters , while Fig. 6.2b shows the missile acceleration and Fig. 6.2c the relative velocity in m/sec. The control effort $J = \frac{1}{2}\int_0^{tf} u^2(t)dt$ is shown in Fig. 6.2d where $J(t_N)$ is the overall control effort. In all subfigures the solid line shows the *TAPCN* , and dotted line shows the *CAPN*.

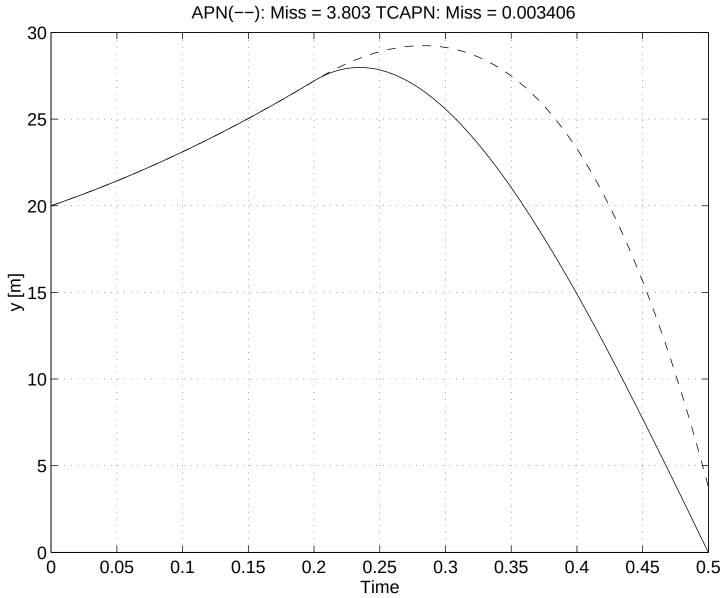

Fig. 6.2a The relative separation is in meters for Perfect Intercepts in Continuous Time System with Time Delay

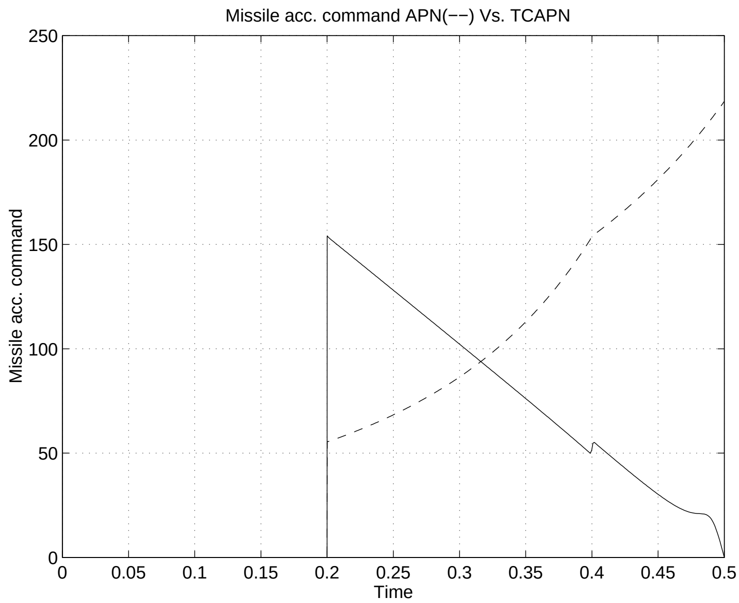

Fig. 6.2b The missile acceleration for Perfect Intercepts in Continuous Time System with Time Delay

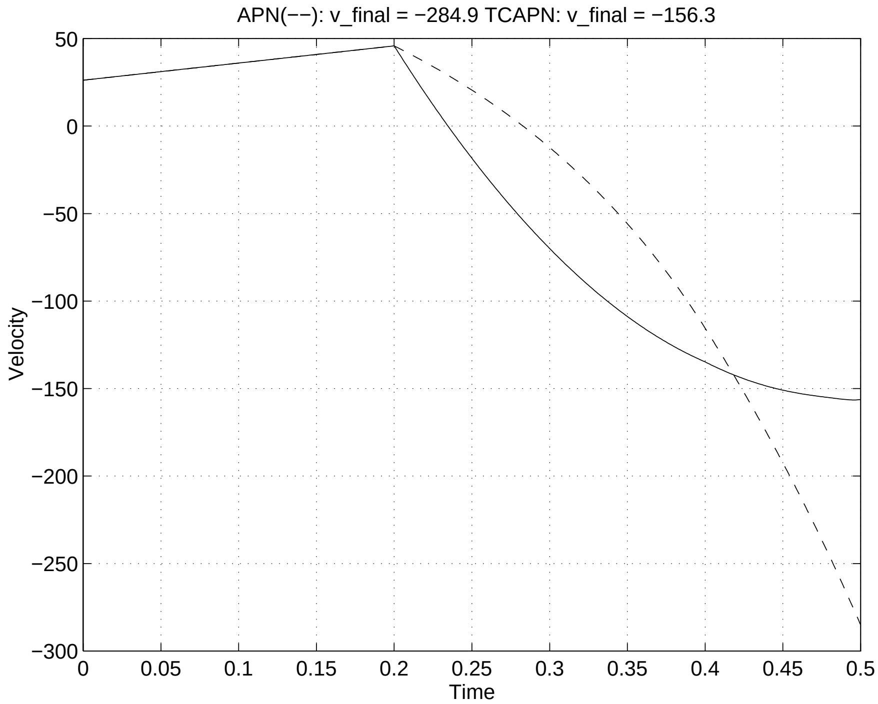

Fig. 6.2c The relative velocity in m/sec for Perfect Intercepts in Continuous Time System with Time Delay

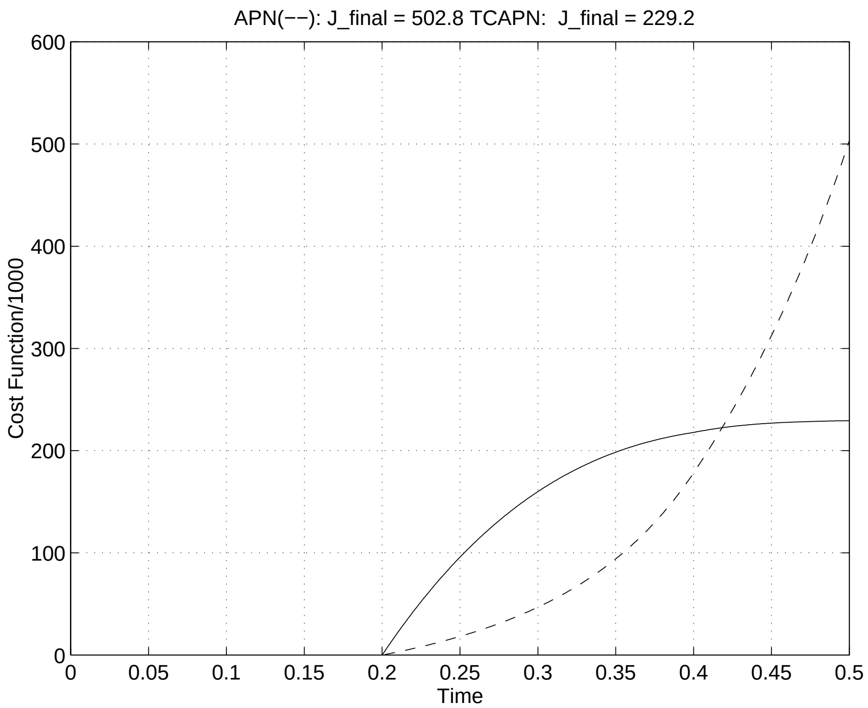

Fig. 6.2d The overall control effort for Perfect Intercepts in Continuous Time System with Time Delay

The miss-distance is 0.003m for *TCAPN* , and 3.8m for *APN*. Moreover *APN* lead to larger cost than the *TCAPN* .

If the target is evasively maneuvering with: $w(t) = \begin{cases} 10g & t < t_f - 0.3 \\ -10g & t \geq t_f - 0.3 \end{cases}$

The results of numerical simulations for this case are depicted in Fig. 6.3a-d. In all subfigures the solid line shows the *TAPCN* , and dotted line shows the *CAPN*.

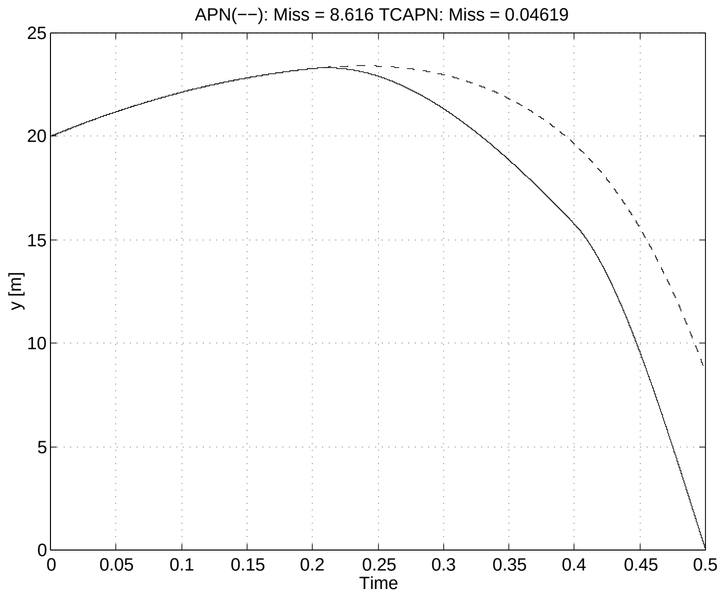

Fig. 6.3a The relative separation is in meters for Perfect Intercepts in Continuous Time System with Time Delay

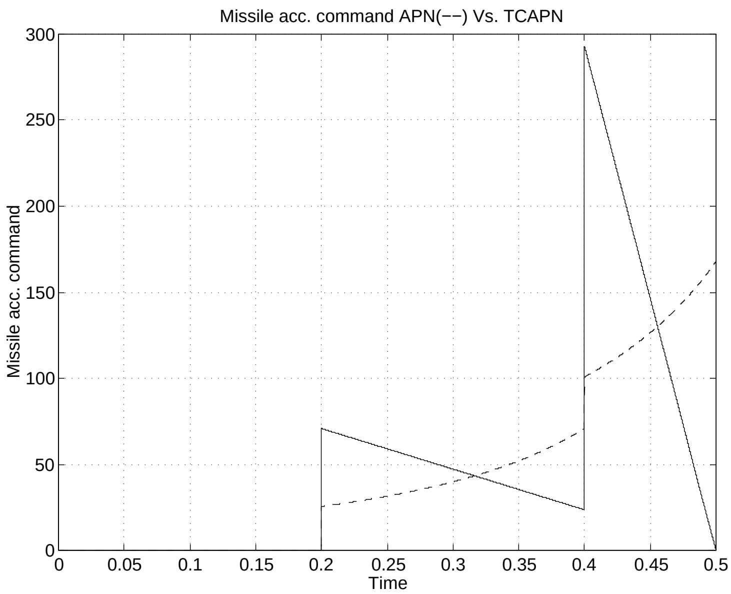

Fig. 6.3b The missile acceleration for Perfect Intercepts in Continuous Time System with Time Delay

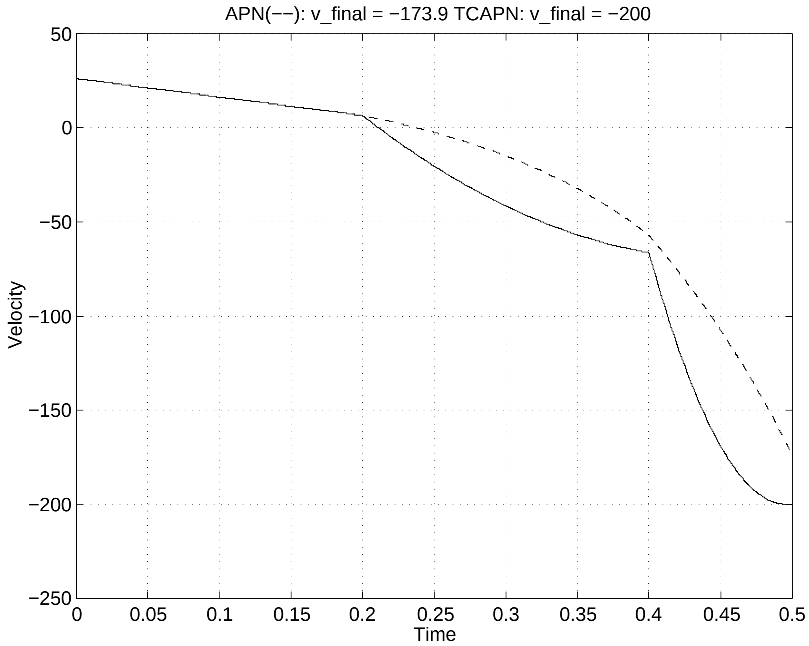

Fig. 6.3c The relative velocity in m/sec for Perfect Intercepts in Continuous Time System with Time Delay

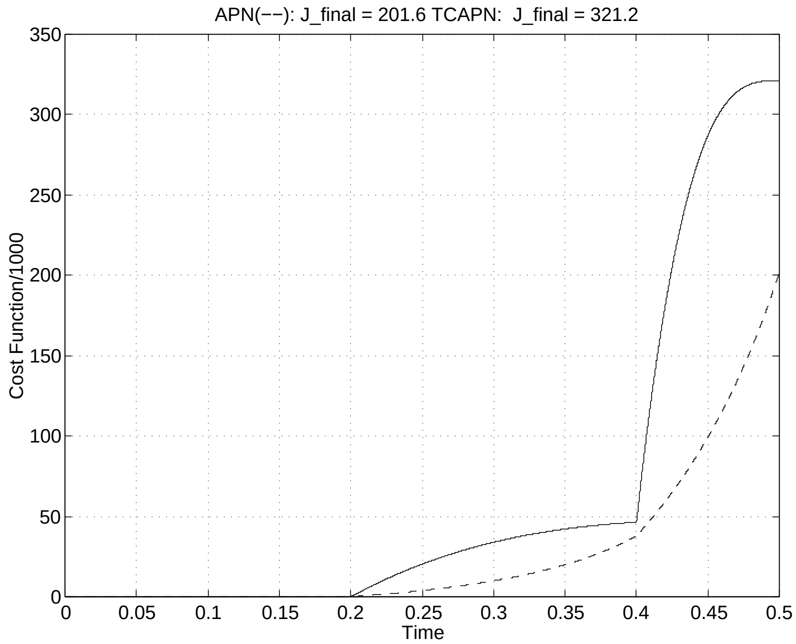

Fig. 6.3d The overall control effort for Perfect Intercepts in Continuous Time System with Time Delay

The miss-distance is 0.04m for *TCAPN* , and 8.6m for *APN*. Moreover *APN* lead to larger cost than the *TCAPN* .

The Fig. 6.4a-b describes the relative separation in meters and the overall control effort for perfect intercepts in continuous time system with time delay as function of t_{go} for b = 10 , 100 , 1000 , 1e6 .

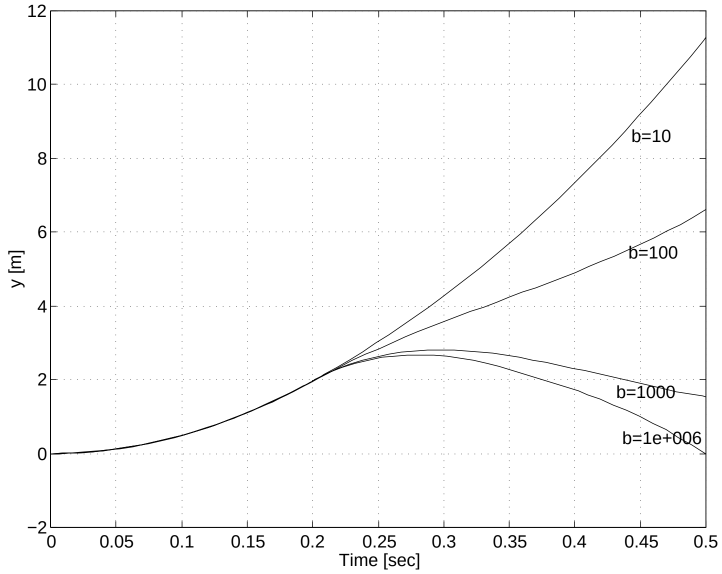

Fig. 6.4a The relative separation is in meters for Perfect Intercepts in Continuous Time System with Time Delay for b = 10 , 100 , 1000 , 1e6 .

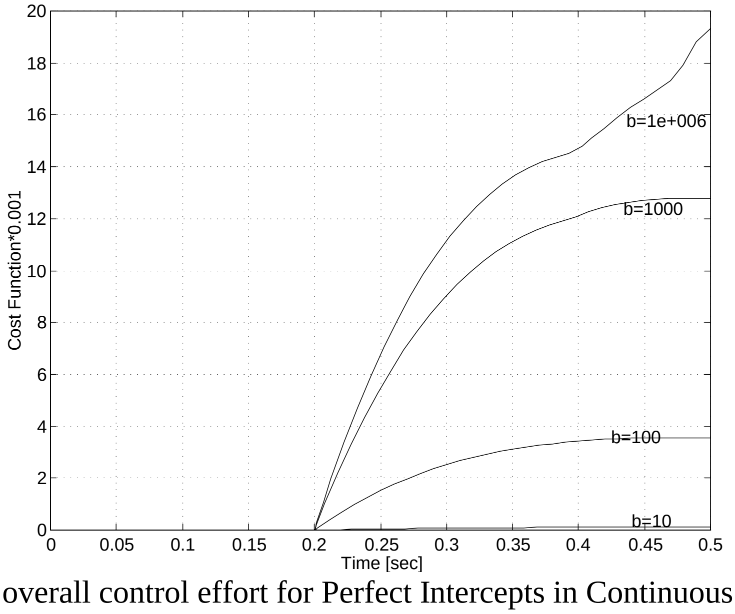

Fig. 6.4b The overall control effort for Perfect Intercepts in Continuous Time System with Time Delay for b = 10 , 100 , 1000 , 1e6 .

As could be expected , the overall control effort lead to larger when *b* is large and then the relative separation is small.

6.2 Perfect Rendezvous - Numerical Simulations for Continuous Time System with Time Delay :

In this section the version of tow guidance laws for the perfect rendezvous case will be analyzed .The following Simulink diagram describes the perfect rendezvous case. In this example the closing velocity is 300 m/s and t_f is 0.5 sec.

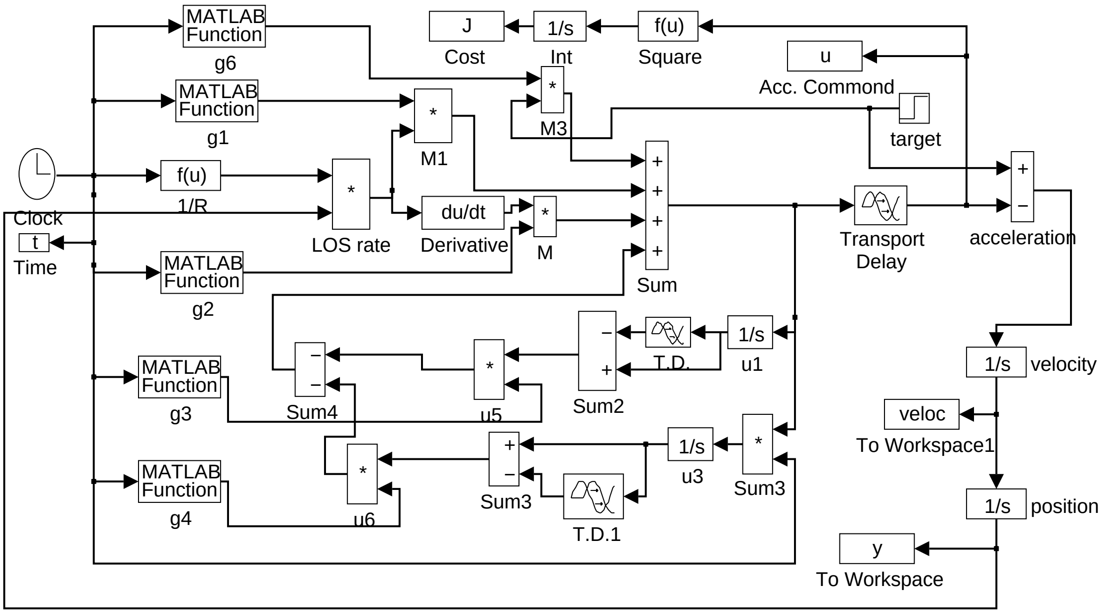

Fig. 6.5 The Simulink diagram describe the Perfect Rendezvous case in Continuous Time System

The following guidance laws are compared:

***i)* AOR(Augmented Optimal Rendezvous)**

We use a sampled version of the Augmented Optimal Rendezvous guidance law.

***ii)* TCAOR (Time Delayed Discrete Augmented Optimal Rendezvous)**

The optimal control law in Eq. 5.12.

Initial conditions are : $\tau = 0.13$ [sec] , $y_0 = 20$ [m] , $v_0 = 26.2$ [m/s] , b=1e4 , c=1e5

Target is maneuvering with:

$$w(t) = \begin{cases} 5g & t < t_f - 0.1 \\ -5g & t \geq t_f - 0.1 \end{cases}$$

The results of numerical simulations for this case are depicted in Fig. 6.6a-d. In all subfigures the solid line shows the *TCAOR* , and dotted line shows the *CAOR*.

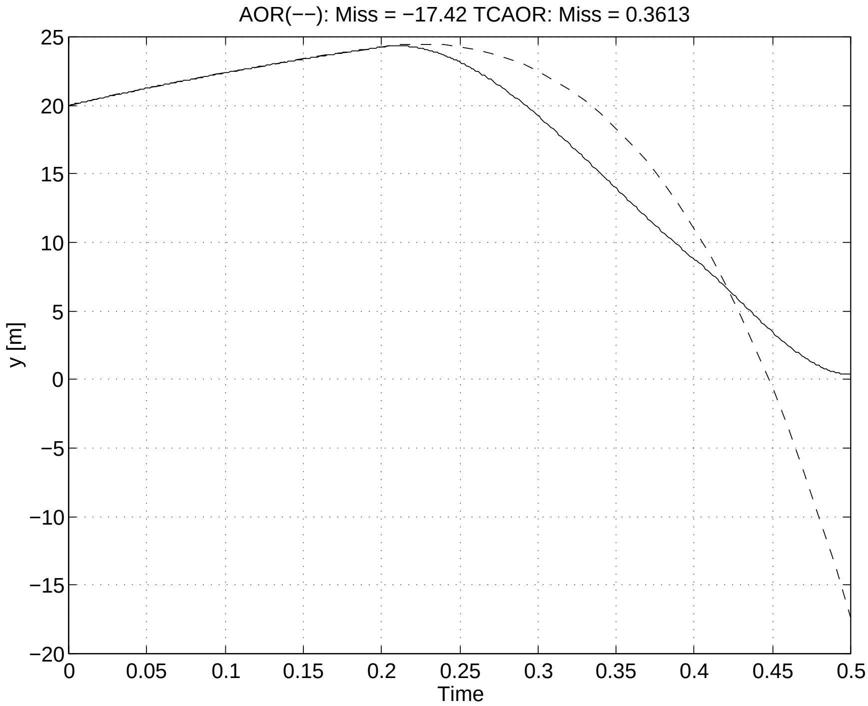

Fig. 6.6a The relative separation is in meters for Optimal Rendezvous in Continuous Time System with Time Delay

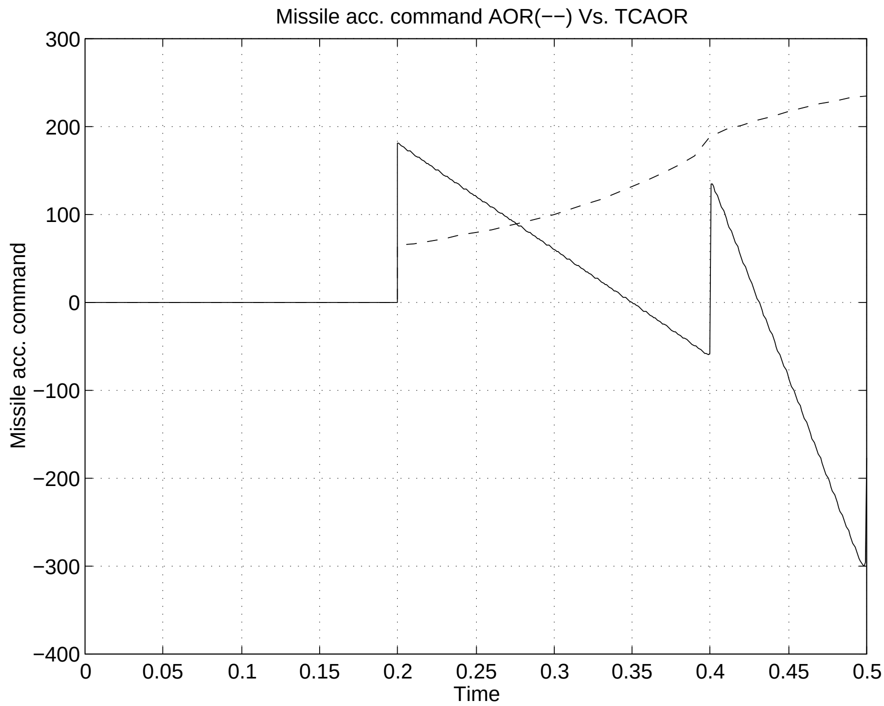

Fig. 6.6b The missile acceleration for Optimal Rendezvous in Continuous Time System with Time Delay

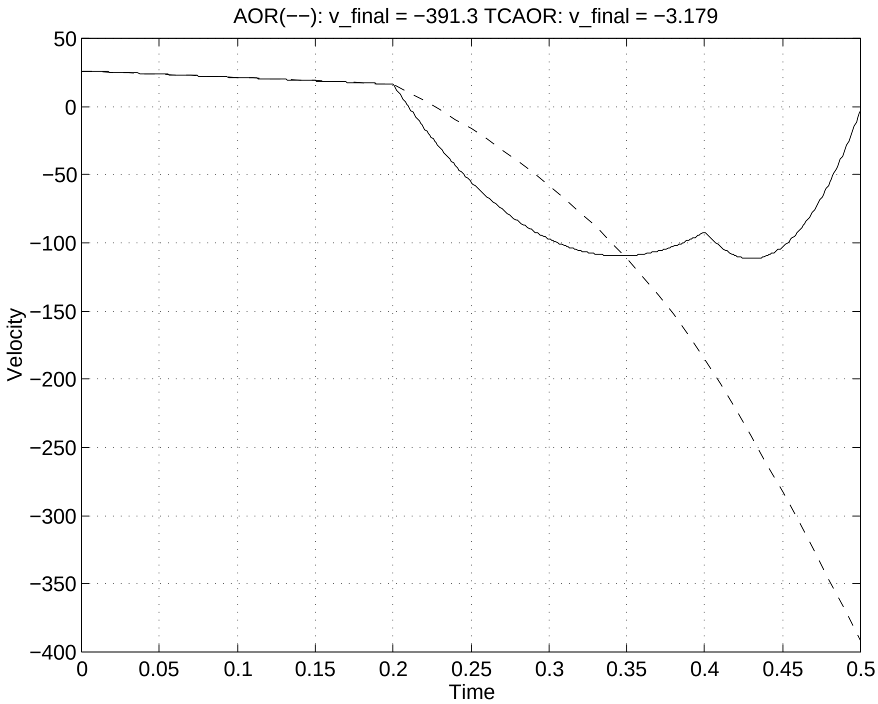

Fig. 6.6c The relative velocity in m/sec for Optimal Rendezvous in Continuous Time System with Time Delay

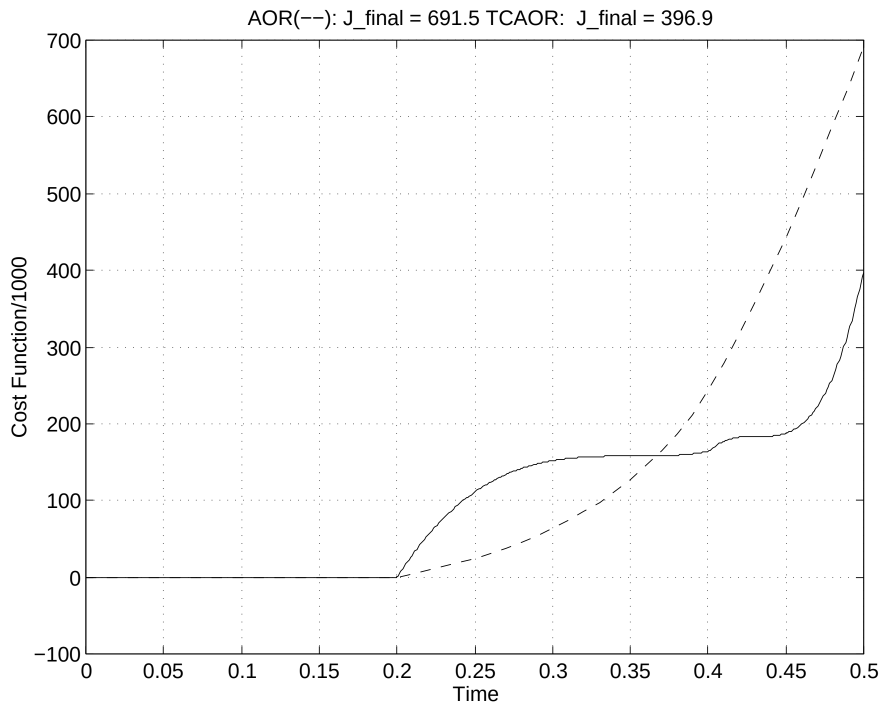

Fig. 6.6d The overall control effort for Optimal Rendezvous in Continuous Time System with Time Delay

The miss-distance is 0.04m for *TCAOR* , and 18m for C*AOR* , the relative velocity is 3 m/sec for *TCAOR* , and 390 m/sec for C*AOR.* Moreover C*AOR* lead to larger cost than the *TCAOR* .

The Fig. 6.7a-c describes the relative separation in meters and the relative velocity in m/sec and the overall control effort for perfect intercepts in continuous time system with time delay as function of t_{go} for b = 10 , 1000 , 100000 , c=1e6 .

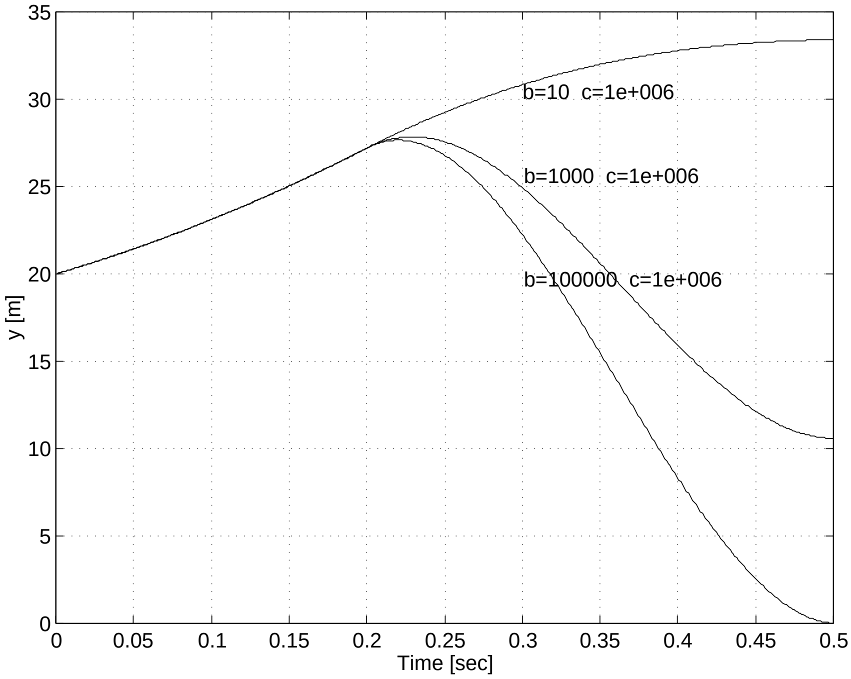

Fig. 6.7a The relative separation in meters for Optimal Rendezvous in Continuous Time System with Time Delay for b = 10 , 1000 , 100000 , c=1e6 .

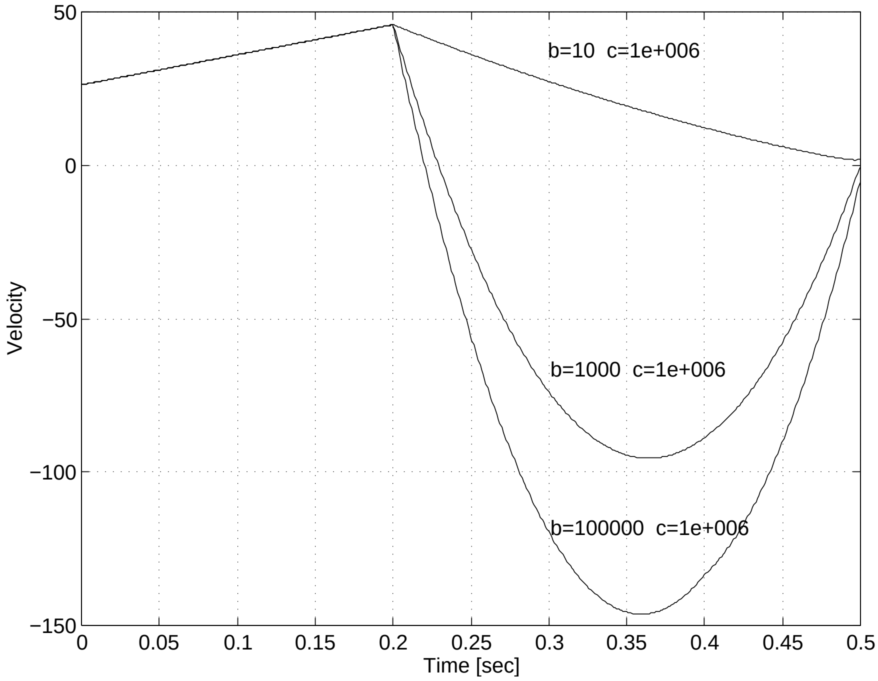

Fig. 6.7b The the relative velocity in m/sec for Optimal Rendezvous in Continuous Time System with Time Delay for b = 10 , 1000 , 100000 , c=1e6 .

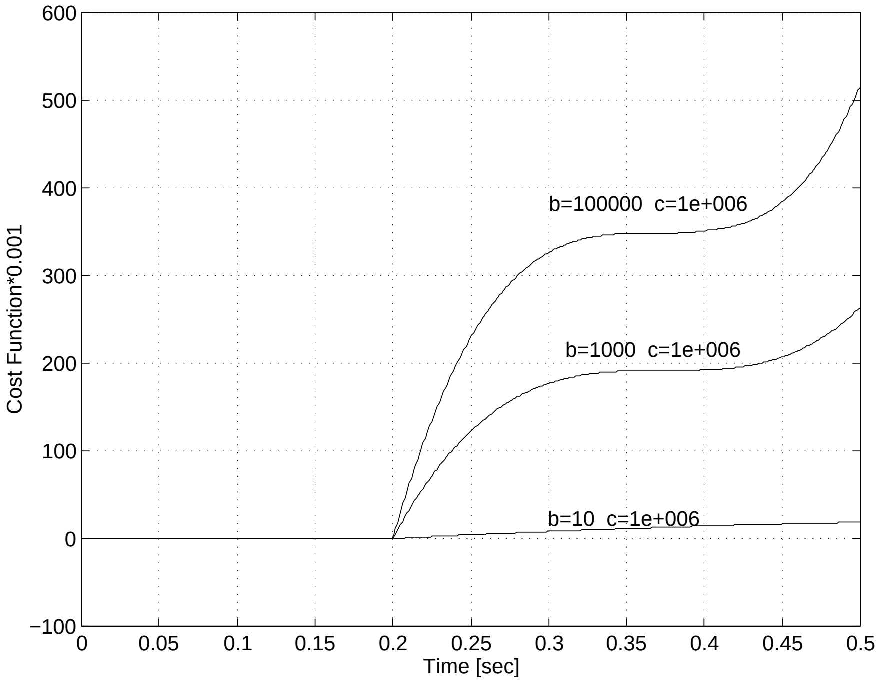

Fig. 6.7c The overall control effort for Optimal Rendezvous in Continuous Time System with Time Delay for b = 10 , 1000 , 100000 , c=1e6 .

As could be expected , the overall control effort lead to larger when *b,c* is large and then the relative separation and the relative velocity are small.

7. **Summary and Conclusions**

In the present paper we described the *Augmented Proportional Navigation (APN) and the Augmented Optimal Rendezvous laws(AOR) for continuous time systems with time delays.* The continuous-time case is a limiting situation of the discrete-time delay results as $h \rightarrow 0$. As a check , we also calculated the *P.N.* gains using the corresponding linear-quadratic optimal control thory and obtained an identical solution.

Our results show that if we take $b \rightarrow \infty$ for *APN* and $b, c \rightarrow \infty$ for *AOR* , we find that the guidance gains tend to infinity as $t_{go} \rightarrow \tau$. Therefore finite b and c should be taken to obtain bounded gains.

The results of the simulations are promising and show a clear advantage of our guidance laws , over guidance laws which neglect the delay.

REFERENCES:

Bryson A. E., and Ho Y.C., "*Applied Optimal Control*", Hemisphere Pub. Cor., N.Y. 1975.

Gitizadeh, R., Yaesh I., and J.Z. Ben-Asher, *Discrete-Time Optimal Guidance*" , Proceedings of the 38th Israel Annual Israel Conference on Aerospace Sciences, 1998 pp. 137-144.

Gitizadeh, R., Yaesh I., and J.Z. Ben-Asher, "*Discrete-Time Optimal Guidance with Time Delay*" , Proceedings of the 39th Israel Annual Israel Conference on Aerospace Sciences, 1999 pp. 186-197.

Pila, A.W., Shaked, U., and deSouza, C.E., Robust H_∞ Control Of Continuous Time-Varying Linear Systems With Time Delay, *Proceedings of the 35th IEEE Conference On Decision And Control*, Kobe Japan, Dec. 1996., pp.1368-1369

Zarchan, P., "*Tactical and Strategic Missile Guidance*", AIAA, Inc, Wasington DC, 1990.