Finite Horizon H_∞ Control of Systems with State Delays

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Abstract

The finite horizon H_∞ control of time-invariant linear systems with a finite number of point and distributed time-delays is considered. For controllers coupled Riccati type partial differential equations are derived. The solutions to these equations are related to the solutions of the associated Hamiltonian systems. For small time delays the solutions and the resulting controllers are approximated by series expansions in powers of the largest delay. Unlike the infinite horizon case, these approximations possess both regular and boundary layer terms. It is shown that the controller obtained by high-order approximations improves the performance of the system. The performance of the closed-loop system under the memoryless zero-approximation controller is analyzed.

Keywords: time-delay systems, $H_\infty-$ state-feedback control, asymptotic approximation, continuous-time systems, small delays .

1 Problem Formulation

Throughout this paper we denote by $|\cdot|$ the Euclidean norm of a vector or the appropriate norm of a matrix. Given $t_f > 0$, let $L_2[0, t_f]$ be the space of the square integrable functions with the norm $||\cdot||_{L_2}$ and let $C[a, b]$ be the space of the continuous functions on $[a, b]$ with the norm $|\cdot|_c$. We denote $x_t = x(t+\theta),\ y^t = y(t-\theta),\ \theta \in [-h, 0]$. Prime denotes the transpose of a matrix and $col\{x, y\}$ denotes a column vector with components x and y.

Consider the system

$$\dot{x}(t) = L(x_t(\cdot)) + Bu(t) + Dw(t), \quad z(t) = col\{Cx(t), u(t)\}, \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector , $u(t) \in \mathbf{R}^l$ is the control signal, $w(t) \in \mathbf{R}^q$ is the exogenous disturbance, and $z(t) \in \mathbf{R}^p$ is the observation vector, B, C and D are constant matrices of appropriate dimensions. The R^n-valued function $L(\cdot)$ which carries R^n-valued functions on $[-h, 0]$ into R^n is defined as follows:

$$L(x_t(\cdot)) = \sum_{i=0}^{r} A_i x_t(-h_i) + \int_{-h}^{0} A_{01}(s) x_t(s) ds, \tag{2}$$

where $-h = -h_r < -h_{r-1} < ... < -h_1 < -h_0 = 0$, $A_0, A_1, ..., A_r$ are constant matrices and $A_{01}(s)$ is a smooth enough matrix function.

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Given $\gamma > 0$, and assuming that $w \in L_2[0, t_f]$, we consider the following performance index

$$J = ||z||^2_{L_2} - \gamma^2 ||w||^2_{L_2}. \tag{3}$$

The problem is to find a state-feedback controller which ensures that $J \leq 0$ for all $w \in L_2[0, t_f]$ and for the zero initial conditions $x(\tau) = 0,\ \tau \leq 0$. This means that the H_∞-norm of (1), which is defined by the supremum over $w \in L_2[0, t_f]$ of the ratio between $||z||_{L_2}$ and $||w||_{L_2}$, is not greater than γ. In the infinite horizon case such a controller has been obtained by Bensoussan *et al.* (1992), Van Keulen (1993), Lee *et al.* (1994), Ge *et al.* (1996), Fridman and Shaked (1998). In (Bensoussan *et al.*,1992) and (Van Keulen, 1993) the controller has been obtained by solving Riccati operator equations. In (Lee *et al.*, 1994) and (Ge *et al.*, 1996), a delay-independent controller has been designed. In (Fridman and Shaked, 1998) for the case of one discrete time-delay, the controller has been derived from Riccati type partial differential equations (RPDE's) or inequalities, and the solution of the RPDE's has been approximated by expansions in the powers of the delay. In (Fridman and Shaked, 1999) a Bounded real lemma has been obtained for the case of a finite number of discrete and distributed time-delays. The LQ optimal control problem for a system with small time delay has been studied by Sannuti and Reddy (1973), where asymptotic series solution to the Hamiltonian system has been constructed. Asymptotic approximation to the solution of the initial value problem for the system with small delay has been constructed in (Vasilieva, 1962) (see also O'Malley, 1974).

In the present paper, we generalize the results of Fridman and Shaked (1998, 1999) to the finite horizon case. We obtain the required controllers by solving coupled RPDE's. We derive an asymptotic approximation to the solution of these RPDE's by expanding it in the powers of the largest delay. The resulting approximation is obtained by solving uncoupled low-order partial differential equations. The performance of the system with the controller that has been obtained using the zero approximation (the one that corresponds to zero delay) is analyzed when the open-loop system possesses a non-zero delay.

2 H_∞-Controller Design

Consider the following RPDE's with respect to the $n \times n$-matrices $P(t), Q(t,\xi)$ and $R(t,\xi,s)$:

$$\begin{array}{c} \dot{P}(t) + A_0'P(t) + P(t)A_0 + \sum_{i=1}^r A_i'Q'(t,-h_i) + \sum_{i=1}^r Q(t,-h_i)A_i \\ +P(t)SP(t) + C'C + \int_{-h}^0 Q(t,\theta)A_{01}(\theta)d\theta + \int_{-h}^0 A_{01}'(\theta)Q'(t,\theta)d\theta = 0, \end{array} \tag{4}$$

$$\frac{\partial}{\partial t}Q(t,\xi) + \frac{\partial}{\partial \xi}Q(t,\xi) = -[A_0' + PS]Q(t,\xi) - \sum_{i=1}^r A_i'R(t,-h_i,\xi) - \int_{-h}^0 A_{01}'(s)R(t,s,\xi)ds, \tag{5}$$

$$\frac{\partial}{\partial t}R(t,\xi,s) + \frac{\partial}{\partial \xi}R(t,\xi,s) + \frac{\partial}{\partial s}R(t,\xi,s) = -Q'(t,\xi)SQ(t,s), \tag{6}$$

$$P(t) = Q(t,0), \quad Q(t,\xi) = R(t,0,\xi), \quad R(t,\xi,s) = R'(t,s,\xi), \quad \xi \in [0,h],\ s \in [0,h], \tag{7}$$

$$P(t_f) = 0, \quad Q(t_f,\xi) = 0, \quad R(t_f,\xi,s) = 0, \tag{8}$$

where $S = \gamma^{-2}DD' - BB'$.

A solution of (4)-(8) is a triple of $n \times n$-matrices $\{P(t), Q(t,\xi), R(t,\xi,s)\}$ $t \in [0,t_f],\ \xi \in [-h,0], s \in [-h,0]$, where $P(t),\ Q(t,\xi)$ and $R(t,\xi,s)$ are continuous and piecewise continuously differentiable functions of their arguments that satisfy (4)-(8) for almost every t, ξ and s.

We show that the controller that solves the H_∞ control problem has the form:

$$u^*(t) = -B'[P(t)x(t) + \int_{-h}^{0} Q(t,\xi)F(x_t)(\xi)d\xi], \tag{9}$$

where

$$F(x_t)(\xi) = \sum_{i=1}^{r} A_i x_t(-h_i - \xi)\chi_i(\xi) + \int_{-h}^{\xi} A_{01}(p)x_t(p-\xi)dp, \tag{10}$$

and where χ_i is the indicator function for the set $[-h_i, 0]$, i.e. $\chi_i(\xi) = 1$ if $\xi \in [-h_i, 0]$ and $\chi_i(\xi) = 0$ otherwise. We obtain

Theorem 1 *Let (4)-(8) have a solution on $[0, t_f]$ for given $\gamma > 0$. Then, the controller of (9) solves the H_∞-control problem.*

Proof. Let $x(t)$ be a solution of (1). Consider the following Lyapunov-Krasovskii functional (Delfour, 1986):

$$\begin{aligned} V(t,x_t) = x(t)'P(t)x(t) + 2x'(t)\textstyle\int_{-h}^{0} Q(t,\xi)F(x_t)(\xi)d\xi \\ + \textstyle\int_{-h}^{0}\int_{-h}^{0} F'(x_t)(s)R(t,s,\xi)F(x_t)(\xi)dsd\xi. \end{aligned} \tag{11}$$

Then differentiating $V(t, x_t)$ with respect to t and integrating by parts, we obtain, similarly to Fridman and Shaked (1998), that

$$\frac{d}{dt}V(t,x_t) = -x'(t)C'Cx(t) - \gamma^2|w(t) - w^*(t)|^2 + \gamma^2|w(t)|^2 + |u(t) - u^*(t)|^2 - |u(t)|^2, \tag{12}$$

where

$$w^*(t) = \gamma^{-2}D'[P(t)x(t) + \int_{-h}^{0} Q(t,\xi)F(x_t)(\xi)d\xi].$$

It follows from (12) that

$$V(t_f, x_{t_f}) - V(0, x_0) + \int_0^{t_f} [|z|^2 - \gamma^2|w|^2]dt = -\gamma^2||w - w^*||_{L_2} + ||u - u^*||_{L_2}.$$

The latter relation implies $J \leq 0$ for $u = u^*$ and $x_0 = 0$ and completes the proof.

Consider next the associated Hamiltonian system:

$$\dot{x}(t) = L(x_t(\cdot)) + Sy(t), \quad \dot{y}(t) = -C'Cx(t) - \tilde{L}(y^t(\cdot)), \tag{13}$$

where

$$\tilde{L}(y^t(\cdot)) = \sum_{i=0}^{r} A_i' y^t(h_i) + \int_{-h}^{0} A_{01}'(s)y^t(-s)ds.$$

Notice that (13) depends on the future values of the adjoint vector y (similarly to the case of the state delay LQ problem. Consider the following boundary conditions for (13):

$$x_s = \phi, \quad y(t_f - \xi) = Q_f'(\xi)x(t_f) + \int_{-h}^{0} R_f(\xi,\tau)F(x_{t_f})(\tau)d\tau, \quad \xi \in [-h, 0], \quad 0 \leq s \leq t_f, \tag{14}$$

where $\phi \in C[-h, 0]$. A solution of (13) on the segment $[s, t_f]$ ($t_f > s$) is a pair of continuous functions $x : [s-h, t_f] \to R^n$ and $y : [s, t+h] \to R^n$, that is absolutely continuous and satisfies (13) on $[s, t_f]$.

We look for a solution to (13) and (14) of the form

$$y(t-\xi) = Q'(t,\xi)x(t) + \int_{-h}^{0} R(t,\xi,\tau)F(x_t)(\tau)d\tau, \quad \xi \in [-h,0]. \tag{15}$$

Setting $\xi = 0$ in (15) we get due to (7)

$$y(t) = P(t)x(t) + \int_{-h}^{0} Q(t,\tau)F(x_t)(\tau)d\tau. \tag{16}$$

The solvability of the RPDE's (4)-(8) is related to the existence of the solution (15) to the boundary value problem of (13) and (14) by the following

Lemma 1 *The system of (4)-(8) has a solution iff for every $s \in [0,t_f]$ and $\phi \in C[-h,0]$ the two point boundary value problem of (13) and (14) has a solution of the form (15) such that (16) holds, where $P(t)$, $Q(t,\xi)$ and $R(t,\xi,s)$ are continuous and piecewise continuously differentiable functions of their arguments.*

The proof of the lemma is omitted since it is similar to proof of Theorem 2 in (Fridman and Shaked, 1999).

3 Asymptotic Approximation of the H_∞ Controller

3.1. Asymptotic solutions to the RPDE's. As we have seen, in both the continuous time and the sampled-data cases, the H_∞ controller has been found by solving a set of coupled PRDE's. Finding a solution to the latter is not an easy task and we are, therefore, looking for a solution to the RPDE's in a form of asymptotic expansion in the powers of the delay h:

$$\begin{aligned} &P(t) = P_0(t) + h[P_1(t) + \Pi_{1P}(\tau)] + h^2[P_2(t) + \Pi_{2P}(\tau)] + ..., \\ &Q(t,h\zeta) = Q_0(t,\zeta) + h[Q_1(t,\zeta) + \Pi_{1Q}(\tau,\zeta)] + h^2[Q_2(t,\zeta) + \Pi_{2Q}(\tau,\zeta)] + ..., \\ &R(t,h\zeta,h\theta) = R_0(t,\zeta,\theta) + h[R_1(t,\zeta,\theta) + \Pi_{1R}(\tau,\zeta,\theta)] + h^2[R_2(t,\zeta,\theta) + \Pi_{2R}(\tau,\zeta,\theta)] + ..., \\ &\tau = \tfrac{t_f - t}{h}, \ \zeta \in [-1,0], \ \theta \in [-1,0]. \end{aligned} \tag{17}$$

The 'outer expansion' terms $\{P_i, Q_i, R_i\}$, $i = 0,1...$ constitute the major part of the solution that satisfies (4)-(7) for $t \in [0,t_f]$, $\theta \in [-1,0]$, $\zeta \in [-1,0]$. The boundary-layer correction terms Π_{iP}, Π_{iQ} and Π_{iR} will be chosen such that (17) satisfies the terminal conditions of (8) and that

$$|\Pi_{iP}(\tau)| + \sup_{\zeta\in[-1,0]} |\Pi_{iQ}(\tau,\zeta)| + \sup_{\zeta,\theta\in[-1,,0]} |\Pi_{iR}(\tau,\zeta,\theta)| \to 0 \quad \text{as} \quad \tau \to \infty. \tag{18}$$

Since τ is a stretched-time variable around $t = t_f$, (18) asserts that Π_{iP}, Π_{iQ} and Π_{iR} are essential only around $t = t_f$ and they thus provide a correction to the outer expansion at the terminal point $t = t_f$.

We substitute (17) in (4) and (7) and equate, separately, outer expansion and boundary-layer correction terms with the same powers of h. We notice that for $t = t_f - h\tau$, $\xi = h\zeta$ and $s = h\theta$ we have $\partial/\partial t = -h^{-1}\partial/\partial\tau$, $\partial/\partial\xi = h^{-1}\partial/\partial\zeta$ and $\partial/\partial s = h^{-1}\partial/\partial\theta$. Thus, for the zero-order terms we obtain from (5), (6) and (7):

$$Q_0(t,\zeta) = P_0(t), \quad R_0(t,\zeta,\theta) = P_0(t). \tag{19}$$

Then, from (4), we have

$$\dot{P}_0(t) + \sum_{i=0}^{r} A_i' P_0(t) + \sum_{i=0}^{r} P_0(t)A_i + P_0(t)SP_0(t) + C'C = 0, \quad P_0(t_f) = 0, \tag{20}$$

The latter is the well-known differential Riccati equation (DRE), that corresponds to (1) for $h = 0$.

Our main assumption is:

A1. For a specified value of $\gamma > 0$, the DRE of (20) has a bounded solution on $[0, t_f]$.

Assumption A1 means that the H_∞ state-feedback control problem for (1) without delay has a solution. If this were not the case, even P_0, the zero-order term in (17), would not exist.

To determine the first-order terms we start with the equations for Q_1:

$$\frac{\partial}{\partial \zeta} Q_1(t,\zeta) = -\mathcal{M}'(t)P_0(t) - \dot{P}_0(t), \quad Q_1(t,0) = P_1(t), \quad \mathcal{M} = \sum_{i=0}^{r} A_i + SP_0. \tag{21}$$

Then,

$$Q_1(t,\zeta) = P_1(t) - [\mathcal{M}'(t)P_0(t) + \dot{P}_0(t)]\zeta.$$

Substituting this expression into the equation for P_1, we obtain

$$\begin{array}{l} \dot{P}_1 + \mathcal{M}'P_1 + P_1\mathcal{M} + \sum_{i=1}^{r} g_i A_i'(P_0\mathcal{M} + \dot{P}_0) + \sum_{i=1}^{r} g_i(\mathcal{M}'P_0 + \dot{P}_0)A_i = 0, \\ P_1(t_f) + \Pi_{1P}(0) = 0, \quad g_i = h_i/h. \end{array} \tag{22}$$

It follows from (4) that $\dot{\Pi}_{1P}(\tau) = 0$. Since Π_{1P} vanishes for $\tau \to \infty$, we have $\Pi_{1P}(\tau) \equiv 0,\ \tau \geq 0$. Hence, $P_1(t_f) = 0$, and P_1 is a solution to the linear differential equation (22) with the latter terminal condition.

For Π_{1Q}, R_1 and Π_{1R} we obtain from (5), (6) and (19)

$$\begin{array}{l} \frac{\partial}{\partial\tau}\Pi_{1Q}(\tau,\zeta) - \frac{\partial}{\partial\zeta}\Pi_{1Q}(\tau,\zeta) = 0, \quad Q_1(t_f,\zeta) + \Pi_{1Q}(0,\zeta) = 0(\zeta), \\ \Pi_{1Q}(\tau,0) = \Pi_{1P}(\tau) = 0; \\ \frac{\partial}{\partial\zeta}R_1(t,\zeta,\theta) + \frac{\partial}{\partial\theta}R_1(t,\zeta,\theta) = -P_0(t)SP_0(t) - \dot{P}_0(t), \quad R_1(\tau,0,\theta) = Q_1(\tau,\theta); \end{array} \tag{23}$$

and

$$\begin{array}{c} \frac{\partial}{\partial\tau}\Pi_{1R}(\tau,\zeta,\theta) - \frac{\partial}{\partial\zeta}\Pi_{1R}(\tau,\zeta,\theta) - \frac{\partial}{\partial\theta}\Pi_{1R}(\tau,\zeta,\theta) = 0, \\ R_1(t_f,\zeta,\theta) + \Pi_{1R}(0,\zeta,\theta) = 0(\zeta,\theta), \quad \Pi_{1R}(\tau,0,\theta) = \Pi_{1Q}(\tau,\theta). \end{array} \tag{24}$$

Then, for $\tau \geq 0$ and $t \in [0, t_f]$, we find successively

$$\Pi_{1Q}(\tau,\zeta) = \left\{ \begin{array}{ll} -Q_1(t_f,\zeta+\tau), & \text{if } \tau \leq -\zeta, \\ 0, & \text{if } \tau > -\zeta; \end{array} \right.$$

$$R_1(t,\zeta,\theta) = R_1'(t,\theta,\zeta) = -\zeta[P_0(t)SP_0(t) + \dot{P}_0(t)] + Q_1(t,\theta-\zeta), \quad \zeta \geq \theta;$$

$$\Pi_{1R}(0,\zeta,\theta) = \theta\dot{P}_0(t_f);$$

$$\Pi_{1R}(\tau,\zeta,\theta) = \Pi_{1R}'(\tau,\theta,\zeta) = \left\{ \begin{array}{ll} \Pi_{1R}(0,\zeta+\tau,\theta+\tau), & \text{if } \tau \leq -\zeta,\ \theta \leq \zeta \\ \Pi_{1Q}(\tau+\zeta,\theta-\zeta), & \text{if } \tau > -\zeta, \theta \leq \zeta. \end{array} \right.$$

Therefore,

$$\Pi_{1Q}(\tau,\zeta) = 0, \quad \tau + \zeta > 0; \quad \Pi_{1R}(\tau,\zeta,\theta) = \Pi_{1Q}(\tau+\zeta,\theta-\zeta) = 0, \quad \tau + \theta > 0,\ \theta \leq \zeta.$$

The higher order terms of the outer expansions can be similarly found. We obtain next the boundary-layer terms and show by induction that

$$\begin{array}{r} \Pi_{iP}(\tau) = 0, \quad \tau > i-1; \quad \Pi_{iQ}(\tau,\zeta) = 0, \quad \tau + \zeta > i-1; \\ \Pi_{iR}(\tau,\zeta,\theta) = 0, \quad \tau + \theta > i-1,\ \theta \leq \zeta. \end{array} \tag{25}$$

We assume that (25) is satisfied for all $i \leq m-1$. Then we derive the following equations for Π_{mP}, Π_{mQ} and Π_{mR}:

$$\dot{\Pi}_{mP}(\tau) = f_m(\tau), \quad \Pi_{mP}(m-1) = 0;$$

$$\begin{array}{l} \frac{\partial}{\partial\tau}\Pi_{mQ}(\tau,\zeta) - \frac{\partial}{\partial\zeta}\Pi_{mQ}(\tau,\zeta) = \phi_m(\tau,\zeta) \\ \Pi_{mQ}(\tau,0) = \Pi_{mP}(\tau), \quad Q_m(t_f,\zeta) + \Pi_{mQ}(0,\zeta) = 0; \end{array}$$

$$\begin{array}{l} \frac{\partial}{\partial\tau}\Pi_{mR}(\tau,\zeta,\theta) - \frac{\partial}{\partial\zeta}\Pi_{mR}(\tau,\zeta,\theta) - \frac{\partial}{\partial\theta}\Pi_{mR}(\tau,\zeta,\theta) = \psi_m(\tau,\zeta,\theta), \\ \Pi_{mR}(\tau,0,\theta) = \Pi_{mQ}(\tau,\theta), \quad R_m(t_f,\zeta,\theta) + \Pi_{mR}(0,\zeta,\theta) = 0, \end{array}$$

where f_m and ϕ_m are known functions that vanish for $\tau > m-1$, and ψ_m is a known function that vanishes for $\tau + \theta > m-2,\ \theta \leq \zeta$.

From these equations we find

$$\Pi_{mP}(\tau) = \int_{m-1}^{\tau} f_m(s)ds,$$

and thus (25) for Π_{mP} holds since $f_m(s) = 0$ for $\tau > m-1$. Further

$$\Pi_{mQ}(\tau,\zeta) = \begin{cases} \Pi_{mQ}(0,\zeta+\tau) + \int_0^\tau \phi_m(s,-s+\tau+\zeta)ds & \text{if } \tau \leq -\zeta, \\ \Pi_{mP}(\zeta+\tau) - \int_0^\zeta \phi_m(-s+\tau+\zeta,s)ds & \text{if } \tau > -\zeta; \end{cases}$$

and Π_{mQ} satisfies (25) since $\Pi_{mP}(\zeta+\tau) = 0$ for $\zeta+\tau > m-1$ and $\phi_m(\tau,\zeta) = 0$ for $\tau > m-1$. Finally,

$$\begin{array}{l} \Pi_{mR}(\tau,\zeta,\theta) = \Pi'_{mR}(\tau,\theta,\zeta) \\ = \begin{cases} \int_0^\tau \psi_m(s,-s+\zeta+\tau,-s+\theta+\tau)ds + \Pi_{mR}(0,\zeta+\tau,\theta+\tau), & \text{if } \tau \leq -\zeta,\ \theta \leq \zeta \\ -\int_0^\zeta \psi_m(-s+\zeta+\tau,s,s+\theta-\zeta)ds + \Pi_{mQ}(\tau+\zeta,\theta-\zeta), & \text{if } \tau > -\zeta, \theta \leq \zeta. \end{cases} \end{array}$$

Conditions (25) for Π_{mR} readily follow from the latter expressions and the properties of Π_{mQ} and ψ_m.

3.2. Near-optimal continuous-time H_∞-control.

Theorem 2 *Under A1 the following holds for all small enough time-delay h :*
(i) *The system of (1)-(8) has a solution. This solution is approximated, for any integer m, by:*

$$\begin{array}{c} P(t) = P_0(t) + \sum_{i=1}^m h^i[P_i(t) + \Pi_{iP}(\tau)] + O(h^{m+1}), \\ Q(t,h\zeta) = P_0(t) + \sum_{i=1}^m h^i[Q_i(t,\zeta) + \Pi_{iQ}(\tau,\zeta)] + O(h^{m+1}), \\ R(t,h\zeta,h\theta) = P_0(t) + \sum_{i=1}^m h^i[R_i(t,\zeta,\theta) + \Pi_{iR}(\tau,\zeta,\theta)] + O(h^{m+1}), \\ \tau = \frac{t_f - t}{h},\ \zeta \in [-1,0],\ \theta \in [-1,0], \end{array} \tag{26}$$

where the boundary-layer terms satisfy (25), and $|O(h^{m+1})| \leq ch^{m+1}$, where c is a positive scalar which is independent of h, t, ζ and θ.
(ii) *The controller of (9) is approximated by*

$$\begin{array}{l} u(x_t) = u_m(x_t) + O(h^{m+1}), \quad u_m(x_t) = -B'[P_0(t) + \sum_{i=1}^m h^i[P_i(t) + \Pi_{iP}(\tau)]]x(t) \\ \qquad\qquad -B'h\int_{-1}^0 \left\{P_0(t) + \sum_{i=1}^{m-1} h^i[Q_i(t,\zeta) + \Pi_{iQ}(\tau,\zeta)]\right\}x(t+h\zeta)d\zeta. \end{array} \tag{27}$$

The approximate controller u_m guarantees an attenuation level of $\gamma + O(h^{m+1})$.

The proof of theorem is given in the Appendix. It follows from Theorem 2 that a high-order approximate controller improves the performance polynomially in the size of the small time-delay h.

Note that in (Ndiaye and Sorine, 1999) the delay sensitivity of J has been investigated. The gradient of J with respect to h at $h = 0$ has been computed there in terms of P_0.

4 The zero-order controller performance

We study the performance of the system under the zero-order controller $u_0(t)$ which solves the H_∞-control problem for (1) without delay.

Theorem 3 *Under A1 the controller u_0 for all small enough h leads to a performance level of γ.*

Proof . We start with the continuous-time case. Applying u_0 to (1), we obtain a system

$$\begin{array}{l} \dot{x}(t) = \bar{A}(t)x(t) + \sum_{i=1}^{r} A_i x(t-h_i) + Dw(t), \quad \bar{A}(t) = A_0 - BB'P_0(t), \\ z = \tilde{C}x(t), \ \tilde{C}(t) = col\{C, -B'P_0(t)\}. \end{array} \tag{28}$$

Note that in (28) only the matrices $\bar{A}(t)$ and $\tilde{C}(t)$ are time-varying and thus the corresponding $F(x_t)$ is given by (10) and is time-invariant. Similarly to (1) it can be proved that this closed-loop system has an induced L_2-gain less or equal to γ if the corresponding RPDE's of (4)- (7), where $A_0 = \bar{A}(t), C = \tilde{C}(t)$, and $S = DD'/\gamma^2$, have a solution. The existence of a solution to the resulting RPDE's, approximated by (26) with $m = 0$, can be proved similarly to (i) of Theorem 2.

Given $\gamma > 0$ and h, one should verify that the corresponding RPDE's have a solution in order to make certain that u_0 leads to a performance level of γ. This is not an easy task. That is why one may resort to more conservative, but computationally simpler, conditions in terms of differential linear matrix inequalities (DLMI) or Riccati differential inequalities (RDI) that were formulated for the case of one delay in (Shaked *et al.*, 1998) and can be easily generalized to the case of r delays.

Example. Consider the following system:

$$\dot{x}(t) = x(t) - x(t-h) + u + w, \quad z = col\{x, u\}. \tag{29}$$

For $h = 0$ we obtain

$$\dot{P}_0(t) - (1-\gamma^{-2})P_0^2 + 1 = 0, \quad P_0(t_f) = 0.$$

Choosing $\gamma = 1$ and $t_f = 1$ we find

$$\begin{array}{l} P_0 = 1-t, \quad P_1 = 2P_0, \quad Q_1 = 2P_0 + \zeta, \quad \Pi_1 Q = -(\tau+\zeta)\chi_1(\tau+\zeta), \quad R_1(t,\zeta,\theta) = \theta, \\ \Pi_1 R(\tau,\zeta,\theta) = \theta + \tau, \ \tau+\zeta \le 0, \quad \Pi_1 R(\tau,\zeta,\theta) = -(\tau+\theta)\chi_1(\tau+\theta), \end{array}$$

where $\theta \le \zeta$. We obtain

$$u_0(t) = t-1, \quad u_1(t) = u_0(t) + h(t-1)\Big[2x(t) - \int_{-1}^{0} x(t+h\zeta)d\zeta\Big].$$

Consider now the performance of (29) under $u = u_0$. Applying to the closed-loop system the delay-dependent criterion of (Shaked *et al.*, 1998), we find that u_0 leads (29) to $\gamma = 1$ for all $0 < h \le 0.29$, since the corresponding RDI's have bounded solutions on [0,1]. For $h = 0.3$ the solution to this RDI has an escape point.

5 Conclusions

The paper presents a solution to the state-feedback H_∞ control of linear time-invariant systems with state time-delays in the finite horizon case. The theory that has been developed in this paper shows that for small delays, similarly to the case of singularly perturbed systems (Pan and Basar, 1993) and (Fridman, 1996), our controllers are affected by the boundary-layer phenomenon. This fact requires evaluation of both, outer expansion and boundary-layer corrections.

6 Appendix

Proof of Theorem 2. (i) To prove the validity of (26) we consider the equations for the remainders

$$\begin{aligned}&h^{m+1}P_{m+1} = P - \textstyle\sum_{i=0}^m h^i P_i,\\ &h^{m+1}Q_{m+1}(t,\xi) = Q(t,\xi) - \textstyle\sum_{i=0}^m h^i[Q_i(t,h^{-1}\xi) + \Pi_{iQ}(h^{-1}t,h^{-1}\xi)],\\ &h^{m+1}R_{m+1}(t,\xi,s) = R(t,\xi,s) - \textstyle\sum_{i=0}^m h^i[R_i(t,h^{-1}\xi,h^{-1}s) + \Pi_{iR}(h^{-1}t,h^{-1}\xi,h^{-1}s)],\end{aligned}$$

in these expansions:

$$\begin{aligned}&\dot P_{m+1} + P_{m+1}\mathcal{M} + \mathcal{M}'P_{m+1} + \textstyle\sum_{i=1}^r A_i'[Q_{m+1}'(t,-h_i) - Q_{m+1}'(t,0)]\\ &\quad + \textstyle\sum_{i=1}^r [Q_{m+1}(t,-h_i) - Q_{m+1}(t,0)]A_i + E_m(t,h,hP_{m+1}(t)) = 0,\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{\partial}{\partial t}Q_{m+1}(t,\xi) + \frac{\partial}{\partial \xi}Q_{m+1}(t,\xi) = -\mathcal{M}'Q_{m+1}(t,\xi) - \textstyle\sum_{i=1}^r A_i'[R_{m+1}(t,-h_i,\xi) - R_{m+1}(t,0,\xi)]\\ +G_m(t,h,hQ_{m+1}(t,\xi)),\end{aligned} \tag{31}$$

$$\frac{\partial}{\partial \xi}R_{m+1}(\xi,s) + \frac{\partial}{\partial s}R_{m+1}(\xi,s) + K_m(t,h,\xi,s,hQ_{m+1}(t,\xi),hQ_{m+1}(t,s)) = 0, \tag{32}$$

$$\begin{aligned}P_{m+1}(t) = Q_{m+1}(t,0), \quad Q_{m+1}(t,\xi) = R_{m+1}(t,0,\xi), \quad R_{m+1}(t,\xi,s) = R_{m+1}'(t,s,\xi),\\ P_{m+1}(t_f) = 0, \quad Q_{m+1}(t_f,\xi) = 0, \quad R(t_f,\xi,s) = 0.\end{aligned} \tag{33}$$

Note that P_{m+1}, Q_{m+1} and R_{m+1} depend on h. The known matrix functions E_m, G_m and K_m are continuous on t,h,ξ,s and contain linear and quadratic terms in hP_{m+1} and hQ_{m+1}.

Let $\Phi(t,s)$ be the transition matrix of the system $\dot x(t) = -\mathcal{M}'(t)x(t)$. Denote by

$$\begin{aligned}\bar E_m(t) = E_m(t,h,hP_{m+1}(t)), \quad \bar G_m(t,\xi) = G_m(t,h,hQ_{m+1}(t,\xi)),\\ \bar K_m(t,\xi,s) = K_m(h,\xi,s,hQ_{m+1}(t,\xi),hQ_{m+1}(t,s)).\end{aligned}$$

Then, the system of (30)-(33) implies the following integral system for the determination of P_{m+1}, R_{m+1} and Q_{m+1}:

$$\begin{aligned}P_{m+1}(t) = -\textstyle\int_{t_f}^t \Phi(t,s)\Big\{ \textstyle\sum_{i=1}^r A_i'[Q_{m+1}'(s,-h_i) - Q_{m+1}'(s,0)]\\ + \textstyle\sum_{i=1}^r [Q_{m+1}(s,-h_i) - Q_{m+1}(s,0)]A_i + \bar E_m(s,h,hP_{m+1}(s))\Big\}\Phi'(t,s)ds,\end{aligned}$$

$$Q_{m+1}(t,\xi) = \begin{cases} \int_{t_f}^t \Phi(t,p)\bar G_m(p,p+t_f-t+\xi)dp & \text{if } t-\xi \ge t_f,\\ \Phi(t,t_f)P_{m+1}(t_f-t+\xi) + \int_0^\xi \Phi(t,p)\bar G_m(p-t_f+t-\xi,p)dp & \text{if } t-\xi < t_f;\end{cases}$$

$$\begin{aligned}R_{m+1}(t,\xi,s) = R_{m+1}'(t,s,\xi) = \textstyle\int_{t_f}^t \bar K_m(p,p+\xi+t_f-t,p+s+t_f-t)dp\\ +\bar R_{m+1}(t_f,\xi+t_f-t,t_f-t+s), \quad \text{if } t_f-t \le -\xi,\ s \le \xi;\end{aligned}$$

$$\begin{aligned}R_{m+1}(t,\xi,s) = R_{m+1}'(t,s,\xi) = -\textstyle\int_0^\zeta \bar K_m(p-\xi-t_f+t,p,p+s-\xi)dp\\ +Q_{m+1}(-t_f+t-\xi,s-\xi), \quad \text{if } t_f-t > -\xi,\ s \le \xi.\end{aligned}$$

Applying the contraction principle argument on the latter system, one can show that for all small enough h this system has a unique solution P_{m+1}, Q_{m+1} and R_{m+1}, continuously depending on h, t, s and ξ. Hence, the approximation of (26) is uniform on h, t, ζ and θ.

(ii) Eq(27) follows from (26) and the rest of (ii) is similar to (Fridman and Shaked, 1998).

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