Proceedings of the 7th Mediterranean Conference on Control and Automation (MED99) Haifa, Israel - June 28-30, 1999

Input-Output Stability of Systems Governed by Nonlinear Second Order Evolution Equations in Hilbert Spaces

M. I. Gil'
Department of Mathematics
Ben Gurion University of the Negev
P.0. Box 653, Beer-Sheva 84105, Israel

Abstract

We consider systems governed by nonlinear second order evolution equations in a Hilbert space and establish explicit conditions for the input-output stability.

1 Introduction

The present paper is devoted to the input-output stability of nonlinear second order evolution equations in a Hilbert space. It is well-known that wide classes of distributed systems are governed by such equations. Many works deal with the Lyapunov stability of the first and second order nonlinear evolution equations (see for instance (Krasnosel'skii *et al.*, 1983; Kunimatsu and Sano, 1994; Lakshmikantham *et al.*, 1989; Likhtarnilov and Yakubovich, 1983) and references given therein). At the same time, the input-output stability of nonlinear systems governed by the second order evolution equations to the best of our knowledge was not investigated.

Let E be a real Hilbert space with a scalar product $(.,.)_E$, and X, Y be arbitrary Banach spaces, $\|.\|_X$ means the norm in X. The concepts of functional analysis used by us can be found in the books (Ahiezer and Glazman, 1969) and (Tanabe, 1997). Denote by $L(X) \equiv L^2(R_+, X)$ the space of X-valued functions defined on $R_+ \equiv [0, \infty)$ and equipped with the norm

$$\|v\|_{L(X)} = [\int_0^\infty \|v(t)\|_X^2 dt]^{1/2} \ (v \in L(X)).$$

In addition, $C([0,l], X)$ is the space of all continuous X-valued functions defined on a segment $[0,l]$ $(l < \infty)$ with the sup-norm, $C^k([0,l], X)$ $(k = 1, 2, ...)$ is the space of all X-valued functions, whose derivatives of the order k are in $C([0,l], X)$.

Furthermore, let A and B be closed linear operators in E. Consider in E the system

$$\ddot{w}(t) + A\dot{w}(t) + B^2 w(t) = F(w(t), \zeta(t), t),$$
$$y(t) = N_S w(t) + N_I \zeta(t) \ (t \geq 0), \tag{1.1}$$

where $w(t) \in E$ is the state, $\zeta(t) \in X$ is the input, $y(t) \in Y$ is the output, and F maps $E \times X \times R_+$ into E. In addition, $N_S : E \to Y$, $N_I : X \to Y$ are bounded linear operators.

Take the zero initial conditions

$$w(0) = 0, \ \dot{w}(0) = 0. \tag{1.2}$$

The definition of a solution to problem (1.1), (1.2) and conditions for solution existence and uniqueness are given below. In the following definition, the solution existence and uniqueness are assumed.

Definition 1.1 *System (1.1) is said to be input-output L^2-stable, if under conditions (1.2), for any $\delta > 0$, there is a constant $\epsilon > 0$, independent of an input $\zeta \in L^2(R_+, X)$, such that the condition $\|\zeta\|_{L(X)} \leq \delta$, implies the inequality $\|y\|_{L(Y)} \leq \epsilon$ for the corresponding output y.*

System (1.1) is said to be input-output L^2-stable with finite gain and zero bias (L^2-stable zb), if there exists a positive constant γ_2, independent of $\zeta \in L(X)$, such that

$$\|y\|_{L(Y)} \leq \gamma_2 \|\zeta\|_{L(X)}. \tag{1.3}$$

These definitions generalize the corresponding definitions for systems with lumped parameters cf. the paper (Vidyasagar, 1993). About some other approaches to the input-output stability see (Georgiou and Smith, 1997) and references therein.

In the present paper, explicit conditions for the L^2-input-output stability of system (1.1) are derived. Moreover, in Section 6 below, we separate a class of distributed systems satisfying the generalized Aizerman conjecture in the input-output version. About the history of the Aizerman conjecture for the finite dimensional systems see for instance (Aizerman, 1949; Reissig *et al.*, 1974); Gil', 1983; 1994; Voronov, 1979), etc. The input-output version of Aizerman's conjecture for finite dimensional systems was considered in (Vidyasagar, 1993) and (Gil' and Ailon, 1998).

A few words about the contents. In Sections 2 and 3 we discuss the solvability of considered equations. The main result of the paper-Theorem 4.1 is presented in Section 4. In Section 5 we specialize the main result for equations whose operators are pencils of selfadjoint operators. Besides, the coefficients of the pencils are matrices. Systems with positive impulse functions and the generalized Aizerman conjecture in the input-output version are considered in Section 6. In Section 7, the relevant examples are collected.

2 Preliminaries

Denote by $D(A)$ the domain of a linear operator A and by $\sigma(A)$ the spectrum of A.

It is assumed that operators A and B in (1.1) have in E dense domains and the relation

$$(Ax, x)_E \geq a(x, x)_E \ (x \in D(A)) \tag{2.1}$$

holds with a real constant a. Let $Re\ B$, $Im\ B$ be the Hermitian components of B. That is, $Re\ B$, $Im\ B$ are selfadjoint operators, such that $B = Re\ B + iIm\ B$. We will suppose that

$$B \text{ is invertible and } Im\ B \text{ is bounded.} \tag{2.2}$$

In other words, $D(B^*) = D(B)$ and the extension of the operator $B - B^*$ from $D(B)$ to E is bounded. The star means the adjointness. In addition, it is assumed that

$$D(A) \supseteq D(B). \tag{2.3}$$

Consider in E the system

$$\dot{z}_1 = -Az_1 - Bz_2, \dot{z}_2 = Bz_1. \tag{2.4}$$

This system can be written in the space $H = E \oplus E$ as

$$\dot{z} = Tz \ \ (z = (z_1, z_2)) \tag{2.5}$$

with the operator

$$T = \begin{pmatrix} -A & -B \\ B & 0 \end{pmatrix},$$

and

$$D(T) = \{x = (x_1, x_2) \in H : x_1 \in D(A), x_2 \in D(B)\}.$$

Since $2ab \le a^2 + b^2$ for all real a, b, we have by (2.1) the relations

$$(Tz, z)_H = -(Az_1, z_1)_E - ((B - B^*)z_2, z_1)_E/2 \le$$

$$-a\|z_1\|_E^2 + \|Im\ B\|_E\|z_1\|_E\|z_2\|_E/2 \le \gamma(z, z)_H\ (z = (z_1, z_2) \in D(T),\ \gamma = const). \tag{2.6}$$

We need the notion of the m-dissipative operator (Tanabe, 1997, pp. 310 and 311). Denote by I_H, I_E the unit operators in H and E respectively.

Lemma 2.1 *Let conditions (2.1-2.3) hold. Then there is a constant γ, such that the operator $T - \gamma I_H$ is m-dissipative.*

Proof: Consider the system

$$-Ay_1 - By_2 = f_1,\ By_1 = f_2 \tag{2.7}$$

with given $f_1, f_2 \in E$. Clearly, $y_1 = B^{-1}f_2$. Since $D(A) \supseteq D(B)$, we have $AB^{-1}f_2 \in E$. Thus (2.7) gives the relation $y_2 = B^{-1}(f_1 + Ay_1) = B^{-1}(f_1 + AB^{-1}y_2)$. Therefore, for any $f_1, f_2 \in E$ system (2.7) has a solution. But (2.7) is equivalent to the equation $Ty = f$ with $f = (f_1, f_2),\ y = (y_1, y_2)$. Hence, it follows that T is invertible. So according to (2.6) $T - \gamma I_H$ is m-dissipative in H. $\Box$

Since T is m-dissipative, due to the well-known Theorem 3.1.5 (Tanabe, 1979, p. 62), T generates a strongly continuous semigroup. Thus, the Cauchy problem for equation (2.5) is well-posed. This means that for any $z_0 \in D(T)$, equation (2.5) has a solution $z(t) : R_+ \to D(T)$ with a strong derivative $\dot{z}(t) \in H$, and $z(t) \to z_0$ as $t \to 0+$ in the strong topology. Besides, $z(t) = (z_1(t), z_2(t))$ with $z_1(t) \in D(A), z_2(t) \in D(B)\ (t \ge 0)$.

Put in (2.4) $u(t) = B^{-1}z_2(t)$. Then $u(t) \in D(B^2)$. By (2.4) we have $\dot{u}(t) = z_1(t) \in D(A)$, and $\ddot{u}(t) = \dot{z}_1(t) \in E\ (t \ge 0)$. System (2.4) takes the form

$$\ddot{u} + A\dot{u} + B^2u = 0\ (t \ge 0). \tag{2.8}$$

We thus have proved the following

Lemma 2.2 *Let conditions (2.1)-(2.3) hold. Then for any finite $l > 0$ and all*

$$u_0 \in D(B^2) \text{ and } u_1 \in D(A), \tag{2.9}$$

there is a unique function $u \in C^2([0, l], E)$ satisfying (2.8), such that

$$u(t) \in D(B^2),\ \dot{u}(t) \in D(A)\ (0 \le t \le l). \tag{2.10}$$

In addition,

$$\lim_{t\to 0+} u(t) = u_0,\ \lim_{t\to 0+} \dot{u}(t) = u_1. \tag{2.11}$$

We need the following result

Lemma 2.3 *Let S be a positive definite selfadjoint operator in E and R be a linear bounded one. Then $(S + R^*)^{1/2} - (S + R)^{1/2}$ is a bounded operator provided $Re\sigma(S + R) > 0$.*

Proof: Due to Corollary 1.4.5 from (Henry, 1981), $T = S + R$ is a sectorial operator. Due to Theorem 1.4.2 (Henry, 1981) we have $T^{1/2} = TJ$, where

$$J = \pi^{-1} \int_0^\infty s^{-1/2}(T + I_E s)^{-1} ds.$$

Clearly,

$$J - J^* = \pi^{-1} \int_0^\infty s^{-1/2}[(T+s)^{-1} - (T^*+s)^{-1}]ds = \pi^{-1} \int_0^\infty s^{-1/2}(T+s)^{-1}(R-R^*)(T^*+s)^{-1}ds.$$

Hence,

$$T(J - J^*) = \pi^{-1} \int_0^\infty s^{-1/2} T(T + s)^{-1}(R - R^*)(T^* + s)^{-1} ds.$$

Since $\sigma(T)$ has no points on the negative halfline, $\|T(T+s)^{-1}\|$ is uniformly bounded on $[0, \infty)$. So

$$\|T(J - J^*)\| = \pi^{-1} \int_0^\infty s^{-1/2}\|T(T + s)^{-1}(R + R^*)\|\|(T^* + s)^{-1}\|ds < \infty.$$

Furthermore,

$$T^{1/2} - (T^*)^{1/2} = T(J - J^*) + (T - T^*)J^*.$$

Since $T - T^*$ and J^* are bounded, we have the required result. $\square$

3 Existence of mild solutions

We will say that $K(t)$ *is the impulse (Green) function* of equation (2.8) if it satisfies the equation

$$\ddot{K}(t) + A\dot{K}(t) + B^2 K(t) = 0$$

with the initial condition

$$K(0) = 0, \dot{K}(0) = I.$$

This means that for any $h \in D(B^2)$, the function $u(t) = K(t)h$ is in $C^2([0, l], E)$ for any finite l, and satisfies equation (2.8) and the relations $u(0) = 0, \dot{u}(0) = h$. The existence of $K(t)$ under conditions (2.1)-(2.3) is due to Lemma 2.2.

Lemma 3.1 *Under conditions (2.1)-(2.3), let a twice continuously differentiable function $v : R_+ \to E$ satisfy the problem*

$$\ddot{v} + A\dot{v} + B^2 v = f(t) \ (t \geq 0) \tag{3.1}$$

$$v(0) = \dot{v}(0) = 0 \tag{3.2}$$

with a given bounded continuous function $f : R_+ \to E$. Then the formula

$$v(t) = \int_0^t K(t - s)f(s)ds,$$

is valid, where K is the Green function to (2.8). Moreover,

$$K(t) = \frac{1}{2\pi i} \int_{-i\infty+c_0}^{i\infty+c_0} e^{\lambda t}(\lambda^2 I_E + A\lambda + B^2)^{-1} d\lambda \ (c_0 = const). \tag{3.3}$$

The integral is understood in the sense of the inverse Laplace transformation.

Proof: As it was above proven, the Cauchy problem for equation (2.5) is well-posed. Thanks to the variation constant formula (Krein, 1971, Section 1.6, formula (6.2)), any solution of equation (3.1) has the exponential growth. Consequently, we can apply the Laplace transformation to equation (3.1). It gives

$$(\lambda^2 I_E + A\lambda + B^2)\tilde{v}(\lambda) = \tilde{f}(\lambda),$$

where $\tilde{v}(\lambda), \tilde{f}(\lambda)$ are the Laplace transforms to u and f, respectively. Further, since $T - \gamma I_H$ is dissipative, $\lambda^2 I_E + A\lambda + B^2$ is invertible in E for any λ from the half-plane $Re\lambda > \gamma$. Hence,

$$\tilde{v}(\lambda) = (\lambda^2 I_E + A\lambda + B^2)^{-1}\tilde{f}(\lambda) \ (Re\lambda > \gamma). \tag{3.4}$$

By virtue of the inverse Laplace transformation and the property of the convolution we arrive at the required result. □

Now consider the problem

$$\ddot{u} + A\dot{u} + B^2 u = F_0(u(t), t); \ u(0) = 0, \ \dot{u}(0) = 0 \ (t \geq 0), \tag{3.5}$$

where F_0 maps $E \times R_+$ into E. Due to Lemma 3.1, any function $u \in C^2([0, l], E)$ $(l < \infty)$ satisfying the problem (3.5) (if it exists), also satisfies the equation

$$u(t) = \int_0^t K(t-s)F_0(u(s), s)ds. \tag{3.6}$$

Using this relation, *we will call a continuous function* $u(t) : R_+ \to E$ *satisfying the integral equation (3.6) for all* $t \geq 0$, *a mild solution* to problem (3.5).

Lemma 3.2 *Under conditions (2.1)-(2.3), let* $F_0(x, .) \in L^2([0, l], E)$ *for any* $x \in E$ *and a positive* $l < \infty$. *In addition, let* F_0 *have the Lipschitz property*

$$\|F_0(x,t) - F_0(y,t)\|_E \leq L\|x - y\|_E \ (L = const, \ x, y \in E, \ t \geq 0). \tag{3.7}$$

Then problem (3.5) has a unique mild solution.

Proof: Take a fixed $l < \infty$. Define in $C([0, l], E)$ a mapping Ψ by

$$(\Psi(z))(t) = \int_0^t K(t-s)F(z(s), s)ds.$$

Then

$$\|\Psi(z) - \Psi(y)\|_E \leq \int_0^t \|K(t-s)\|_E \|F(z(s), s) - F(y(s), s)\|_E ds \leq$$

$$\int_0^t \|K(t-s)\|_E L\|z(s) - y(s)\|_E ds \ (z, y \in C([0, l], E)).$$

Now the result is due to the contraction mapping theorem (cf. the trivial Lemma 15.3.1 from (Gil', 1998)). □

4 The main result

In the sequel *it is assumed that for any* $\zeta \in L^2(R_+, X)$ *the function* $F_0(x,t) \equiv F(x, \zeta(t), t)$ *satisfies the Lipschitz condition (3.7) and* $F_0(x, .) \in L^2([0,l], E)$ for all $x \in E$ and positive $l < \infty$. Clearly, constant L in (3.7) depends on ζ, in general.

In addition, C_+ means the closed right half-plane.

Theorem 4.1 *Let the conditions (2.1)-(2.3) and*

$$\|F(s,z,t)\|_E \leq q_S\|s\|_E + q_I\|z\|_X \ (q_S,\ q_I = const,\ s \in E, z \in X, t \geq 0) \tag{4.1}$$

be fulfilled. In addition, let $\tilde{K}(\lambda) \equiv (\lambda^2 + A\lambda + B^2)^{-1}$ *be regular on* C_+ *and*

$$M(K) \equiv \max_{-\infty \leq \omega \leq \infty} \|\tilde{K}(i\omega)\|_E < q_S^{-1}. \tag{4.2}$$

Then system (1.1) is input-output L^2*-stable zb. Moreover, the constant* γ_2 *in (1.3) can be taken as*

$$\gamma_2 = (1 - M(K)q_S)^{-1} M(K) q_I \|N_S\|_{E \to Y} + \|N_I\|_{X \to Y}.$$

Firstly, let us prove the following

Lemma 4.2 *Under conditions (2.1)-(2.3), let* $\tilde{K}$ *be regular on* C_+*. If, in addition,* $M(K) < \infty$*, then function* v*, defined by the formula*

$$v(t) = \int_0^t K(t-s) f(s) ds$$

with $f \in L(E)$*, satisfies the inequality* $\|v\|_{L(E)} \leq M(K)\|f\|_{L(E)}$.

This result is due to the Parseval equality and equality (3.4).

Proof of Theorem 4.1: The existence and uniqueness of mild solutions are due to Lemma 3.2. Since $\tilde{K}$ is regular in C_+, we can take in (3.3), $c_0 = 0$. Hence, it easily follows that for a sufficiently small $\epsilon > 0$,

$$\|K(t)\|_E \leq const\ e^{-\epsilon t}\ (t \geq 0). \tag{4.3}$$

Furthermore, introduce the scalar-valued function $z_l(t)$ defined by the relations $z_l(t) = 1$ for $0 \leq t \leq l < \infty$ and $z_l(t) = 0$ for $t > l$. Consider the equation

$$u_l(t) = \int_0^t K(t-s) z_l(s) F(u_l(s), \zeta(s), s) ds = \int_0^l K(t-s) z_l(s) F(u_l(s), \zeta(s), s) ds \ (t \geq l) \tag{4.4}$$

Thanks to (4.3) the solution u_l of (4.4) is in $L(E)$. Due to (3.6) and the previous lemma, condition (4.1) implies the relations

$$\|u_l\|_{L(E)} \leq M(K)\|F(u_l, \zeta, .)\|_{L(E)} \leq M(K)(q_S\|u_l\|_{L(E)} + q_I\|\zeta\|_{L(X)}).$$

Therefore, (4.2) implies

$$\|u_l\|_{L(E)} \leq (1 - M(K)q_S)^{-1} q_I M(K) \|\zeta\|_{L(X)}.$$

But, clearly, $\|u_l\|_{L(E)} \to \|w\|_{L(E)}$ as $l \to \infty$. This proves the required result. □

5 Equations with selfadjoint operators

In this section we are going to specialize Theorem 4.1 in the case when operators A, B are represented by pencils of selfadjoint operators. Namely, let $E = H_0^n$ be an orthogonal sum of the same Hilbert spaces $H_1 = ... = H_n = H_0$. We will assume that

$$A = \sum_{k=0}^{m} a_k S^k, B^2 = \sum_{k=0}^{l} b_k S^k \ (m \leq l/2), \tag{5.1}$$

where a_k, b_j $(k = 0, ..., m;\ j = 0, ..., l)$ are real constant $n \times n$-matrices, S *is a selfadjoint positive definite operator in* E *commuting with matrices* a_k, b_j $(k = 0, ..., m;\ j = 0, ..., l)$. So

$$\beta(S) \equiv \inf \sigma(S) > 0.$$

System (1.1) takes the form

$$\ddot{w}(t) + \sum_{k=0}^{m} a_k S^k \dot{w}(t) + \sum_{k=0}^{l} b_k S^k w(t) = F(w(t), \zeta(t), t),$$

$$y(t) = N_S w(t) + N_I \zeta(t). \tag{5.2}$$

In addition, assume that

$$a_m + a_m^* > 0;\ b_l = b_l^* > 0;\ a_k + a_k^* \geq 0;\ b_j = b_j^* \geq 0\ (k = 1, ..., m-1;\ j = 1, ..., l-1) \tag{5.3}$$

Besides, a_0, b_0 are arbitrary provided that $Re\sigma(B^2) > 0$. It is simple to check that (5.3) implies (2.1). In addition, due to the Lemma 2.3 under the conditions (5.3), operator B satisfy relation (2.2). Furthermore, clearly that $D(A) = D(S^m), D(B) = D(S^{l/2})$ and, thus, condition (2.3) holds.

Introduce the matrix pencil

$$Q(s, \lambda) = \lambda^2 + \lambda \sum_{k=0}^{m} a_k s^k + \sum_{k=0}^{l} b_k s^k \ (s \in \sigma(S);\ \lambda \in \mathbf{C}).$$

In this section it is supposed that $Re\sigma(B^2) > 0$ *and for any fixed* $s \in \sigma(S)$*, all the roots of* $\det Q(s, \lambda)$ *are in the open left half-plane.* To formulate the result, put

$$M_Q = \max_{\omega \in R^1,\, s \in \sigma(S)} \|Q^{-1}(s, i\omega)\|_{C^n}.$$

Lemma 5.1 *Under conditions (5.3) and (4.1), let*

$$q_S M_Q < 1. \tag{5.4}$$

Then system (5.2) is input-output L^2*-stable zb. Moreover, the constant* γ_2 *in (1.3) can be taken as*

$$\gamma_2 = (1 - M_Q q_S)^{-1} M_Q q_I \|N_S\|_{E \to Y} + \|N_I\|_{X \to Y}.$$

Proof: Omitting simple calculations, we have

$$\tilde{K}(\lambda) = [\lambda^2 + \lambda \sum_{k=0}^{m} a_k S^k + \sum_{k=0}^{l} b_k S^k]^{-1}.$$

Hence, due to Lemma 6.3.1 from (Gil', 1995, p. 179)

$$M(K) \equiv \max_{\omega \in \mathbf{R}} \|\tilde{K}(i\omega)\|_{C^n} = M_Q. \tag{5.5}$$

Now Theorem 4.1 yields the result. □

Let W be a linear operator in a complex Euclidean space $\mathbf{C}^n$ with the eigenvalues $\lambda_1(W), \ldots,$ $\lambda_n(W)$. We apply the following estimate for the resolvent from (Gil', 1995, p. 9, Corollary 1.2.4) (see also (Gil', 1998, p. 353)):

$$\|(W - \lambda I_{C^n})^{-1}\|_{C^n} \leq \sum_{k=0}^{n-1} \frac{g^k(W)}{\sqrt{k!}\rho^{k+1}(W, \lambda)} \text{ for all regular } \lambda, \tag{5.6}$$

where $\rho(W, \lambda)$ is the distance between the spectrum $\sigma(W)$ of W and a complex point λ, and

$$g(W) = (N^2(W) - \sum_{k=1}^{n} |\lambda_k(W)|^2)^{1/2}.$$

Here $N(W)$ is the Frobenius (Hilbert-Schmidt) norm of W, i.e. $N^2(W) = Trace(WW^*)$. If W is a normal matrix: $WW^* = W^*W$, then $g(W) = 0$. The following relations:

$$g(W) \leq \sqrt{1/2}N(W^* - W) \text{ and } g(We^{i\tau} + zI_{C^n}) = g(W) \text{ for every } \tau \in \mathbf{R},\ z \in \mathbf{C} \tag{5.7}$$

are true (Gil', 1995, Section 1.1). Here I_{C^n} is the unit matrix.

So we have by (5.6)

$$\|Q^{-1}(s, i\omega)\|_{C^n} \leq \sum_{k=0}^{n-1} \frac{g^k(Q(s, i\omega))}{\sqrt{k!}\rho_0^{k+1}(Q(s, i\omega))}$$

where $\rho_0(Q(s, i\omega))$ is the smallest modulus of the eigenvalues of matrix $Q(s, i\omega)$ with fixed s, ω. Now Lemma 5.1 yields

Theorem 5.2 *Under conditions (5.3) and (4.1), let*

$$\Gamma(Q) \equiv \sup_{\omega \in \mathbf{R}, s \in \sigma(S)} \sum_{k=0}^{n-1} \frac{g^k(Q(s, i\omega))}{\sqrt{k!}\rho_0^{k+1}(Q(s, i\omega))} < q_S^{-1}. \tag{5.8}$$

Then system (5.2) is input-output L^2-stable zb. Moreover, the constant γ_2 in (1.3) can be taken as

$$\gamma_2 = (1 - \Gamma(Q)q_S)^{-1}\Gamma(Q)q_I\|N_S\|_{E \to Y} + \|N_I\|_{X \to Y}.$$

For all $s \in \sigma(S), \lambda \in \mathbf{C}$, let $Q(s, \lambda)$ be a normal matrix, then $g(Q(\lambda, s)) = 0$ and

$$\Gamma(Q) = \sup_{\omega \in \mathbf{R}, s \in \sigma(S)} \rho_0^{-1}(Q(s, i\omega)). \tag{5.9}$$

In particular, if $n = 1$, then $Q(s, i\omega)$ is a polynomial in ω, and

$$\Gamma(Q) = \sup_{\omega \in \mathbf{R}, s \in \sigma(S)} |Q(s, i\omega)|^{-1}. \tag{5.10}$$

6 Systems with positive impulse functions

In this section we consider in E system (5.2) with $n = 1$ and the constants

$$a_m, b_l > 0,\ a_k, b_j \geq 0\ (k = 1, ..., m-1;\ j = 1, ...l-1,\ m \leq l/2). \tag{6.1}$$

Constants a_0, b_0 are arbitrary real, such that $Re\sigma(B^2) > 0$.

So $Q(s, \lambda)$ is a scalar polynomial whose roots for all $s \in \sigma(S)$ lie in the open left half-plane, and S is a positive definite selfadjoint operator in E, again.

The following problem can be considered as a generalization of the Aizerman conjecture in the input-output version:

Problem 1: *To separate a class of systems (5.2) such that the asymptotic stability of the linear system*

$$\ddot{w} + \sum_{k=0}^{m} a_k S^k \dot{w} + \sum_{k=0}^{l} b_k S^k w = q_1 w \tag{6.2}$$

with some $q_1 \in [0, q_S]$ provides the input-output L^2-stability of system (5.2) under condition (4.1).

Let us write

$$K(t, s) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{ti\omega} Q^{-1}(s, i\omega) d\omega.$$

Assume that

$$max_{s \in \sigma(S)} |K(t, s)| = K(t, s_0) \geq 0, \tag{6.3}$$

where s_0 does not depend on t.

Theorem 6.1 *Let condition (6.3) be satisfied. Then $Q(s_0, 0) > 0$. If, in addition,*

$$q_S < Q(s_0, 0), \tag{6.4}$$

then system (5.2) under (6.1) is input-output L^2-stable zb provided condition (4.1) holds.

Below we will check that this theorem separate a class of systems satisfying Problem 1.

The proof of this theorem is divided into a series of lemmata

Lemma 6.2 *Let $w_0(t)$ be a scalar-valued function defined by the convolution*

$$w_0(t) = \int_0^t W(t-s) y(s) ds \tag{6.5}$$

with a scalar integrable kernel $W(t)$ and a scalar-valued function $y \in L^2(R_+)$. In addition, let the Laplace transform $\tilde{W}(\lambda)$ to $W(t)$ exist and regular in C_+. Then the relation

$$\|w\|_{L^2(R_+)} \leq \max_{\omega} |\tilde{W}(i\omega)| \|y\|_{L^2(R_+)}$$

is valid.

Proof: Applying the Laplace transformation to equation (6.5), we have $\tilde{w}(\lambda) = \tilde{W}(\lambda)\tilde{y}(\lambda)$, where $\tilde{w}(\lambda), \tilde{y}(\lambda)$ are the Laplace transforms to $w_0(t), y(t)$. Hence, the result is due to the Parseval equality. □

Lemma 6.3 *Let $\tilde{W}(\lambda)$ be the Laplace transform to a positive function $W \in L^1(R_+)$. Then*

$$\max_{s\in\mathbf{R}} |\tilde{W}(is)| = \tilde{W}(0) = \int_0^\infty W(t)dt > 0.$$

Proof: We have

$$|\tilde{W}(is)|^2 = |\int_0^\infty e^{-its}W(t)dt|^2 = (\int_0^\infty cos(ts)W(t)dt)^2 + (\int_0^\infty sin(ts)W(t)dt)^2.$$

Hence,

$$|\tilde{W}(is)|^2 = \int_0^\infty \int_0^\infty W(t_1)W(t)(cos(t_1 s)cos(ts) + sin(ts)sin(t_1 s))dt_1\, dt =$$

$$\int_0^\infty \int_0^\infty W(t_1)W(t)\, cos[(t-t_1)s]\, dt_1\, dt \le \int_0^\infty W(t_1)dt_1 \int_0^\infty W(t)\, dt = \tilde{W}^2(0).$$

As claimed. □

Corollary 6.4 *Let $w_0(t)$ be a scalar-valued function defined by (6.5) with a positive kernel $W(t) \in L^1(R_+)$ and a function $y \in L^2(R_+)$. Then the Laplace transform $\tilde{W}(\lambda)$ to $W(t)$ has the property $\tilde{W}(0) > 0$ and the relation $\|w\|_{L^2(R_+)} \le \tilde{W}(0)\|y\|_{L^2(R_+)}$ is valid.*

Proof of Theorem 6.1: Rewrite equation (5.2) in the form

$$w(t) = \int_0^t K(t-\tau)F(w(\tau), \zeta(\tau), \tau)d\tau, \tag{6.6}$$

where $K(t)$ again is the Green function. Let E_s be the resolution of the identity for S. Due to the definition of the function of a selfadjoint operator cf. (Ahiezer and Glazman, 1969), we easily get

$$K(t) = \int_{-\infty}^\infty K(t,s)dE_s.$$

Therefore, by virtue of condition (6.3),

$$\|K(t)\|_E = max_{s\in\sigma(S)}|K(t,s)| = K(t,s_0) \ge 0.$$

It follows from (6.6) that

$$\|w(t)\|_E \le \int_0^t K(t-\tau, s_0)\|F(w(\tau), \zeta(\tau), \tau)\|_E d\tau.$$

But (4.1) implies

$$\|w(t)\|_E \le \int_0^t K(t-\tau, s_0)(q_S\|w(\tau)\|_E + q_I\|\zeta(\tau)\|_X)d\tau. \tag{6.7}$$

Now Corollary 6.4 yields

$$\|w\|_{L(E)} \le Q^{-1}(s_0, 0)(q_S\|w\|_{L(E)} + q_I\|\zeta\|_{L(X)}).$$

Thanks to (6.4),

$$\|w\|_{L(E)} \le (1 - Q^{-1}(s_0,0)q_S)^{-1}q_I Q^{-1}(s_0,0)\|\zeta\|_{L(X)}.$$

This inequality proves the required result. □

Corollary 6.5 *Let conditions (4.1), (6.1) and (6.3) be satisfied. If, in addition, equation (6.2) with $q_1 = q_S$ is asymptotically stable in E, then system (5.2) is input-output L^2-stable zb.*

Thus, Theorem 6.1 specifies a class of equations that satisfy Problem 1.

Consider now in E the system

$$w_{tt} + 2Sw_t + cS^2 w = F(w, \zeta(t), t) \ (0 < c = const < 1)$$

$$y(t) = N_S w(t) + N_I \zeta(t) \tag{6.8}$$

with a positive definite selfadjoint operator S.

Corollary 6.6 *Let the conditions (4.1) and*

$$c\beta^2(S) > q_S$$

hold. Then system (6.8) is input-output L^2-stable zb.

In fact, under consideration, the roots of $P(\lambda, s) = \lambda^2 + 2s\lambda + cs^2$, are

$$r_{1,2}(s) = -s(1 \pm b) \text{ with } b = \sqrt{1-c},$$

and

$$K(t, s) = (2bs)^{-1}[e^{-st(1-b)} - e^{-st(1+b)}] \geq 0.$$

So the derivative in s is

$$K_s(t, s) = -(2bs^2)^{-1}[((1-b)ts + 1)e^{-st(1-b)} - ((1+b)ts + 1)e^{-st(1+b)}].$$

The function $(1 + z)e^{-z}$ $(z \geq 0)$ decreases. Therefore,

$$((1-b)ts + 1)e^{-st(1-b)} - ((1+b)ts + 1)e^{-st(1+b)} \geq 0 \ (t, s > 0).$$

Thus

$$K_s(t, s) \leq 0 \ (s > 0) \text{ and } \max_{s \geq \beta(S)} K(t, s) = K(t, \beta(S)) \geq 0.$$

So due to Theorem 6.1 if $Q(\beta(S), 0)) = c\beta^2(S) > q_S$, then system (6.8) is input-output L^2-stable zb, as claimed.

7 Examples

In the following examples the existence and uniqueness of mild solutions are assumed.

Example 7.1 *Take $X = Y = C[0, 1]$ and consider in the real space $E = L^2[0, 1]$ the problem*

$$w_{tt} + 2a_0 w_t + b_0 w - b_1 w_{xx} = F(w, \zeta(x, t), x, t) \ (0 < x < 1)$$

$$w(0, t) = w(1, t) = 0 \ (t > 0), \tag{7.1}$$

$$y(x, t) = Mw(x, t) + \zeta(x, t), \tag{7.2}$$

where M, a_0, b_0, b_1 are real constants and $a_0, b_1 > 0$. In addition, the continuous function $F : \mathbf{R}^2 \times [0, 1] \times R_+ \to \mathbf{R}$ satisfies the condition

$$|F(v, z, x, t)| \leq q_S|v| + q_I|z| \ (v, z \in \mathbf{R};\ t \geq 0;\ 0 \leq x \leq 1). \tag{7.3}$$

So condition (4.1) is fulfilled. On the set

$$D(S) = \{w \in L^2[0,1] : w'' \in L^2[0,1] : w(0) = w(1) = 0\} \tag{7.4}$$

define the operator S by the equation $(Sw)(x) = -w''(x)$. Then (7.1) takes the form

$$w_{tt} + 2a_0 w_t + b_0 w + b_1 Sw = F(w, \zeta(t), x, t).$$

Clearly,

$$\sigma(S) = \{\pi^2 k^2;\ k = 1, 2, ...\}. \tag{7.5}$$

So $\beta(S) = \pi^2$. Assume that

$$b_0 + b_1 \pi^2 > 0. \tag{7.6}$$

Under condition (7.6), for any $s \geq \beta(S)$ all the roots of the polynomial

$$Q(s, \lambda) = \lambda^2 + 2a_0\lambda + b_0 + b_1 s. \tag{7.7}$$

are in the open left-plane, and

$$\inf_{s \geq 0, \omega \in \mathbf{R}} |Q(s, i\omega)| \geq b_0 + b_1\pi^2.$$

So due to Lemma 5.1 the inequality $q_S < b_0 + b_1\pi^2$ provides the input-output L^2-stability of system (7.1), (7.2).

Example 7.2 *Consider system (7.1), (7.2) in space* $E = L^2([0,1], \mathbf{R}^n)$ *of*

vector-valued functions, assuming now that a_0, b_0, b_1 are now $n \times n$-matrices and M is an $m \times n$ matrix. In addition, $a_0 = a_0^* > 0,\ b_1 = b_1^* > 0$ and the continuous function

$$F : \mathbf{R}^n \times \mathbf{R}^m \times [0,1] \times R_+ \to \mathbf{R}^n$$

satisfies the condition

$$\|F(v, z, x, t)\|_{R^n} \leq q_I \|z\|_{R^m} + q_S \|v\|_{R^n}\ (z \in \mathbf{R}^m;\ v \in \mathbf{R}^n;\ t \geq 0;\ 0 \leq x \leq 1)$$

Take $X = Y = C([0,1], \mathbf{R}^m)$. Then condition (4.1) holds. For the sake of simplicity assume that either

$$a_0 \text{ commutes with } b_0 \text{ and } b_1, \tag{7.8}$$

or

$$b_1 \text{ commutes with } a_0 \text{ and } b_0. \tag{7.9}$$

Further, on the set

$$D(S) = \{w \in L^2([0,1], \mathbf{R}^n) : w'' \in L^2([0,1], \mathbf{C}^n) : w(0) = w(1) = 0\}$$

define the operator S by the equality $(Sw)(x) = -w''(x)$. Then $\sigma(S)$ is given by (7.5).

Define $Q(s, \lambda)$ by (7.7) and assume that for each $s \geq 0$, *det* $Q(s, \lambda)$ is a Hurwitz polynomial. In the case (7.8), due to Corollary 1.3.9 from (Gil', 1995) according to (5.7) we have

$$g(Q(s, i\omega)) = g(b_0 + b_1 s) \leq \sqrt{1/2} N(b_0 - b_0^*)\ (\omega,\ s \in \mathbf{R}).$$

Similarly, in the case (7.9),

$$g(Q(s,i\omega)) = g(a_0 i\omega + b_0) \le \sqrt{1/2}N(b_0 + b_0^*) \ (\omega,\ s \in \mathbf{R}).$$

Thus, $g(Q(s,i\omega)) \le g_0$, where $g_0 = \sqrt{1/2}N(b_0 + b_0^*)$ in the case (7.9), and $g_0 = \sqrt{1/2}N(b_0 - b_0^*)$ in the case (7.8).

With fixed s, ω, let $\rho_0(Q(s,i\omega))$ be the smallest modulus of the eigenvalues of matrix $Q(s,i\omega)$. Put

$$\tilde{\rho} = \inf_{\omega\in\mathbf{R},\ k=1,2,\dots} \rho_0(Q(k,i\omega)).$$

Then according to (5.8),

$$\Gamma(Q) \le \sum_{k=0}^{n-1} \frac{g_0^k}{\sqrt{k!}\tilde{\rho}^{k+1}}.$$

So due to Theorem 5.2, the inequality

$$q_S \sum_{k=0}^{n-1} \frac{g_0^k}{\sqrt{k!}\tilde{\rho}^{k+1}} < 1$$

provides the input-output L^2 stability zb of system (7.1), (7.2) in the considered case.

Example 7.3 *Take $X = Y = C[0,1]$ and consider in the real space $E = L^2[0,1]$ the problem*

$$w_{tt} - 2w_{txx} + cw_{xxxx} = F(w, \zeta(x,t), x, t) \ (t > 0;\ 0 < x < 1;\ c \in (0,1))$$
$$w^{(k)}(0,t) = w^{(k)}(1,t) = 0 \ (k = 0,2;\ t \ge 0), \tag{7.10}$$

together with equation (7.2), where M is a constant. Here the real continuous function $F : \mathbf{R}^2 \times [0,1] \times R_+ \to \mathbf{R}$ satisfies condition (7.3). On set (7.4) define the operator S by the equation $(Sw)(x) = -w''(x)$. Then (7.10) takes the form (6.7). Further, clearly, condition (7.3) implies relation (4.1). Thanks to Corollary 6.6 and (7.5), under condition $c\pi^4 > q_S$, system (7.10) is input-output L^2-stable zb.

8 Concluding remarks

A lot of papers and books are devoted to solvability of the linear second order equation of the type (2.8) (see the books (Fattorini, 1985) and (Skhlyar, 1997), and references therein). For example, in the paper (Sandefur, 1983), it is supposed that $A = A_1 + A_2, B^2 = A_1A_2$ with linear operators A_1, A_2. The paper (Engler, 1985) is devoted to the case $A = aA_1 + bI_E, B^2 = cA_1 + dI_E$ where a, b, c, d are numbers. In the book (Skhlyar, 1997), A and B are normal commuting operators. In the paper (Engel, 1994) the following restrictions are imposed: B is selfadjoint and positive definite, $Re(Bx, A^{-1}x) \ge 0$ for $x \ge 0$. In the paper (Neubrander, 1986), it is assumed that $D(A) \subseteq D(B^2)$. The paper (Chen and Triggiani, 1989) is devoted to equation (2.8) with positive definite A and B. In addition, $c_1A^\alpha \le B^2 \le c_2A^\alpha$ with $1/2 \le \alpha \le 1$. At the same time Lemma 2.2 gives the solvability conditions (2.1)-(2.3).

Theorems 4.1, 5.2 and 6.1 and give explicit input-output stability conditions.

Acknowledgment

I am very grateful to late Professors M. A. Aizerman and A. A. Voronov for their interest in and approval of my investigations. I also very indebted to the Associate Editor of MCSS for his very helpful remarks.

References

Ahiezer, N. I. and I. M. Glazman (1969). *Theory of Linear Operators in a Hilbert Space.* Pitman Advanced Publishing Program, Boston.

Aizerman, M. A. (1969). "On a conjecture from absolute stability theory," *Uspehi Mat. Nauk*, **4**, no. 4. In Russian.

Chen, S. and R. Triggiani (1989). "Proof of extensions of two conjectures on structural damping for elastic systems," *Pacific J. Math.*, **136**, no. 1, pp. 15–55.

Daleckii, Yu. L. and M. G. Krein (1974). *Stability of Solutions of Differential Equations in Banach Space*, Amer. Math. Soc., Providence, R. I.

Engel, K.-J. (1994). "On dissipative wave equations in Hilbert space," *J. of Math. Anal. Appl.*, **184**, pp. 302–316.

Engler, H., F. Neubrander, and J. Sandefur (1985). "Strongly damped second order equations," *Lecture Notes in Mathematics*, no. 1248, Springer-Verlag, pp. 52–62.

Fattorini, H. (1985). *Second Order Linear Differential Equations in Banach Spaces*, North Holland, Amsterdam.

Georgiou, T. and M. Smith (1997). "Robustness analysis of nonlinear feedback systems. An input-output approach," *IEEE Trans. Aut. Contr.*, **42**, pp. 1200–1221.

Gil', M. I. (1983). "On a class of one-contour systems which are absolutely stable in the Hurwitz angle," *Automation and Remote Control*, no. 10, pp. 70–75.

Gil', M. I. (1994). "Class of absolutely stable multivariable systems," *Int. J. Systems Sci.*, **25**, no. 3, pp. 613–617.

Gil', M. I. (1995). *Norm Estimations for Operator-valued Functions and Applications.* Marcel Dekker, Inc. New York.

Gil', M. I. (1998). *Stability of Finite and Infinite Dimensional Systems*, Kluwer Academic Publishers, Boston-Dordrecht-London.

Gil', M. I. and A. Ailon (1998). "The input-output version of Aizerman's conjecture," *Int. J. Robust and Nonlinear Control*, **8**, pp. 1212–1226.

Henry, D. (1981). *Geometric Theory of Semilinear Parabolic Equations*, Lectures Notes in Mathematics, no. 840, Springer-Verlag, Berlin.

Krein, S. G. (1971). *Linear Differential Equations in Banach space*, AMS, Providence, R. I.

Krasnosel'skii, M. A., A. Pokrovskii, and A. G. Zarubin (1983). "On a principle bounded regimes absence in absolute stability of distributed parameters systems," *Automation and Remote Control*, no. 3, pp. 13-19.

Kunimatsu, N. and H. Sano (1994). "Compensator design of semilinear parabolic systems," *Int. J. Control*, **60**, no. 2, pp. 243–263.

Lakshmikantham, V., S. Leela, and A. A. Martynyuk (1989). *Stability Analysis of Nonlinear Systems*, Marsel Dekker, Inc., New York.

Likhtarnikov, A. L. and V. A. Yakubovich (1983). "Absolute stability of nonlinear systems," in V. Rezvan, *Absolute stability of Control Systems with Delay* (Supplement), Nauka, Moscow, In Russian.

Neubrander, F. (1986). "Well-posedness of higher order abstract Cauchy problems," *Transactions of the AMS*, **295**, no. 1, pp.257–290.

Reissig, R., G. Sansone, and R. Conti (1974). *Nonlinear Differential Equations of High Order*, Leiden, Noordhoff International Publishing.

Sandefur, J. T. (1983). "Existence and uniqueness of solutions of second order nonlinear differential equations," *SIAM J. Math. Anal.*, **14**, no. 3, pp. 477–487.

Skhlyar, A. Y. (1997). *Complete Second Order Linear Differential Equations in Hilbert Spaces*, Operator Theory, Advances and Applications, Vol. 92, Birkhäuser, Basel.

Tanabe, H. (1979). *Equations of Evolution*, Pitman, London-San-Francisco-Melburn.

Tanabe, H. (1997). *Functional Analytic Methods for Partial Differential Equations*, Marcel Dekker Inc., New York.

Vidyasagar, M. (1993). *Nonlinear Systems Analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, 2nd edition.

Voronov, A. A. (1979). *Stability, Controllability, Observability*, Nauka, Moscow, In Russian.