DEVELOPMENT OF THE MODAL REGULATOR DESIGN METHOD FOR A PLANT WITH INTERVAL PARAMETERS

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Abstract

We consider a problem of a modal P-regulator synthesis for a linear multivariable dynamical system with uncertain (interval) parameters. The designed regulator has to place all coefficients of the system characteristic polynomial within assigned intervals. We developed the approach proposed in (Dug *et al.*, 1990) and proved the direct correlation between system controllability and existence of a modal P- regulator.

1 INTRODUCTION

The problem of closed-loop stability is an important topic of feedback controller design when perturbancies and/or uncertainties can affect a control system. This paper is devoted to a solution of robust modal P-regulator synthesis problem for a plant with uncertain parameters described by intervals with the given bounds. On practice such a plant can be considered as a linear dynamical system with interval parameters. Two alternative methods of modal regulator synthesis for a control system with interval parameters in a state-space have been considered in (Dug *et al.*, 1990; Smag, 1997). In this paper we develop the approach (Dug *et al.*, 1990) and present a new proof of solvability conditions of the problem based on the latest research in the area of interval analysis.

2 PROBLEM STATEMENT

We consider a linear time invariant dynamical multivariable system in state-space

$$dx/dt = [A]x + [B]u \tag{1}$$

where $x = x(t)$ is an n state vector and $u = u(t)$ is an r input vector. The elements $[a_{ij}]$, $[b_{ik}]$ $(i, j=1,...,n, k=1,..,r)$ of $n \times n$ matrix $[A]$ and $n \times r$ matrix $[B]$ are interval (interval numbers) with the known upper and lower bounds (Moor, 1966). These matrices describe sets of matrices $A \in [A]$, $B \in [B]$ with real elements $a_{ij} \in [a_{ij}]$, $b_{ik} \in [b_{ik}]$.

It is necessary to find an $r \times n$ matrix K with real elements (gain coefficients k_{ij}, $i=1,...,r$, $j=1,..,n$) for a robust feedback state control (in other words, a modal P--regulator)

$$u = Kx \tag{2}$$

which ensures the inclusions

$$\det (sI - A - BK) \subseteq [D(s)] \tag{3}$$

for every real *A* Î [*A*], *B* Î [*B*]. In (3) [*D(s)*] is an assigned asymptotic stable *n* degree polynomial with interval coefficients $[d_i]$, *i=0,1,...,n-1*

$$[D(s)] = s^n + [d_{n-1}]s^{n-1} + ...+ [d_1]s + [d_0]. \qquad (4)$$

The mentioned interval polynomial may be described as a set of asymptotic stable polynomials $D(s) = s^n + d_{n-1}s^{n-1} + ...+ d_1 s + d_0$ with real coefficients d_i Î $[d_i]$. Different methods can be used in order to find interval coefficients of [*D(s)*] (Khar, 1979).

If a characteristic polynomial of closed-loop system *dx/dt* = (*A* +*BK*)*x* can be presented as

$$\det(sI - A - BK) = s^n + b_{n-1}s^{n-1} + ...+ b_1 s + b_0 \qquad (5)$$

then inclusion (3) takes the following form

$$b_i \text{ Î } [d_i], \ i = 0, ..., n-1. \qquad (6)$$

3 MAIN RESULT

For the given problem we consider two cases: *r* =1, *r* ³ 2.

<u>**Case *r =1***</u>. Suppose that [*B*] = [*b*] is a column vector, *K* = *k* is a row vector.

Definition 1 (Dug *et al.*, 1990). A pair ([*A*], [*b*]) is controllable for any *A* Î [*A*], *b* Î [*b*] if a square interval controllability matrix

$$[Y] = ([b], [A]*[b], ... , [A]^{n-1}*[b]) \qquad (7)$$

satisfies the condition

$$0 \notin \text{Det}\,[Y]\,. \qquad (8)$$

In (8) Det **<•>** denotes the interval extension (Moor, 1966) of the function det **<•>**. In (7) and further the sign '*' is a multiplication of two intervals . It is one of interval- arithmetical operations introduced in (Moor, 1966).

Consider an interval *n* ´ *n* matrix

$$[P] = [Y] * \begin{bmatrix} -[a_1] & -[a_2] & \cdots & -[a_{n-1}] & -1 \\ -[a_2] & -[a_3] & \cdots & -1 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ -1 & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad (9)$$

and an interval *n* row vector

$$[f] = ([d_o] \,\Theta\, [a_o], [d_1] \,\Theta\, [a_1], ... , [d_{n-1}] \,\Theta\, [a_{n-1}]) \qquad (10)$$

where $[a_{n-1}]$,..., $[a_o]$ are interval coefficients of the characteristic polynomial of the matrix [*A*]: $[f(s)] = s^n + [a_{n-1}]s^{n-1} + ...+ [a_1]s + [a_0]$ which may be considered as an interval extension of the rational function $f(s) = s^n + a_{n-1}s^{n-1} + ...+ a_1 s + a_0$, *A* Î [*A*]; Θ - is a nonstandart interval subtraction (Mark, 1977) for the intervals $[a] = [a_l, a_s]$ and $[b] = [b_l, b_s]$ defined as $[a]\,\Theta\,[b] = \{\min(a_l - b_l,\ a_s - b_s),\ \max(a_l - b_l,\ a_s - b_s)\}$.

Theorem. If a pair $([A],[b])$ is controllable and the following inclusion takes place for an assigned asymptotic stable interval polynomial (4)

$$\mathrm{M}[f]\,(\mathrm{M}[P])^{-1} * [P] \subset [f], \tag{11}$$

then a modal state regulator $u = kx$ exists and it can be calculated as follows:

$$k = \mathrm{M}[f]\,(\mathrm{M}[P])^{-1} \tag{12}$$

where $\mathrm{M}[\bullet]$ denotes a real matrix(vector) the elements of which are the midpoints (Moor, 1966) of the interval elements of matrix(vector). For an interval number [a] we have: $\mathrm{M}[a] = \mathrm{M}[a_l, a_s] = (a_l + a_s)/2$.

Proof. A pair $([A],[b])$ is referred to as a controllable if and only if all pairs (A,b), $A \in [A]$, $b \in [b]$ are controllable. It is known (Owens, 1978) that if a pair (A,b), $A \in [A]$, $b \in [b]$ is controllable, then a row vector k can be calculated as

$$k(b, Ab, \ldots (A)^{n-1}) \begin{bmatrix} -\boldsymbol{a}_1 & -\boldsymbol{a}_2 & \cdots & -\boldsymbol{a}_{n-1} & -1 \\ -\boldsymbol{a}_2 & -\boldsymbol{a}_3 & \cdots & -1 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ -1 & 0 & \cdots & 0 & 0 \end{bmatrix} +$$

$$+\ (\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}) = (d_0,\ d_1\ \ d_{n-1}) \tag{13}$$

where $\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$ are the coefficients of the characteristic polynomial of A; $d_0, d_1, \ldots, d_{n-1}$ are the coefficients of a chosen asymptotically stable polynomial $D(s) = s^n + d_{n-1}s^{n-1} + \ldots + d_1 s + d_0$ with $d_i \in [d_i]$, $i = 0, \ldots, n-1$.

The elements of the row vectors in the left hand side of (13) are rational functions of the element of k, A, b (Note that $\boldsymbol{a}_i$, $i = 0, \ldots, n-1$ are multilinear functions of the elements of A). We denote the left hand side of (13) by $f(k, A, b)$ and find an interval extension $F(k, [A], [b])$ of the function $f(k, A, b)$ for $A \in [A]$, $b \in [b]$. As a result we have the expression

$$F(k, [A], [b]) = k * [P] + ([\boldsymbol{a}_0], [\boldsymbol{a}_1], \ldots, [\boldsymbol{a}_{n-1}]) \tag{14}$$

where the $n \times n$ interval matrix $[P]$ is defined from (9); $[\boldsymbol{a}_i]$, $i = 0, \ldots, n-1$ are an interval extension of the rational function $\boldsymbol{a}_i(A)$, $A \in [A]$.

We need to find a real row vector k satisfying the following inclusion

$$k * [P] + ([\boldsymbol{a}_0], [\boldsymbol{a}_1], \ldots, [\boldsymbol{a}_{n-1}]) \in ([d_0], [d_1], \ldots, [d_{n-1}]) \tag{15}$$

Then for every $A \in [A]$ and $b \in [b]$ equality (13) with appropriate d_i, $i = 0, \ldots, n-1$ will be satisfied and all real d_i, $i = 0, \ldots, n-1$ will belong to the assign intervals $d_i \in [d_i]$, $i = 0, \ldots, n-1$.

The left hand side of (15) is a sum of an unknown interval row vector $k*[P]$ and the known interval row vector ($[a_0]$, $[a_1]$,...,$[a_{n-1}]$). Using the nonstandart subtraction Θ of interval arithmetic we can represent inclusion (15) as

$$k* [P] \ \hat{I} \ [f] \ . \tag{16}$$

The solution of (16) for a real n row vector k is known as a particular case of the total linear interval tolerance problem (Dobr et al., 1990) . In common case the solution can be an interval vector $[k]$. In our case we can calculate real k from (12) if M[P] is nonsingular n ´ n matrix and condition (11) takes place (Dobr et al., 1990). The Theorem is proved.

Corollary. As it follows from the Theorem the controllability of the pair ([A],[b]) is a necessary condition for a modal regulator $u = kx$ existence .

Remark 1. The elements of the interval vector [f] are so called proper intervals for which the inequalities $w[d_i] > w[a_i]$, i=0,...,n-1 hold. Here $w[\bullet]$ denotes a width of an interval number [•](Moor, 1966). For an interval number [a] we have: $w[a] = w[a_l, a_s] = (a_s - a_l)$. If some of the inequalities are not satisfied then we can increase the width of the appropriate interval coefficients $[d_i]$, i=0,...,n-1.

Remark 2. Condition (11) can be always guaranteed by increasing the widths of the assigned interval polynomial coefficients $[d_i]$. Therefore, a stabilizing (non-modal) state feedback control $u = kx$ exists if and only if the pair ([A],[b]) is controllable.

<u>Case r³ 2.</u>

Definition 2. System (1) is called a controllable if and only if for any A Î [A] and B Î [B] the following equality takes place

$$\text{rank} \ [\, B, AB, \ldots , A^{n-1} B] \ = n. \tag{17}$$

Suppose that for any A Î [A] and B Î [B] the pair ([A] , [B]) is controllable. Moreover, assume that for any A Î [A] and B Î [B] the pair (A,B) is a cyclic pair (Owens, 1978). Then we can (almost always) find a real r vector q which guarantees the controllability of the pair ([A] , [B]$*q$), i.e. the n ´ n interval controllability matrix $[Y_1] = ([B]*q, [A]*[B]*q, \ldots , [A]^{n-1}*[B]*q)$ satisfies the condition (8). Considering $[b] = [B]*q$ and using the Theorem we can calculate a real row vector k from formula (12). Then the r ´ n real matrix K results from the formula

$$K = qk. \tag{18}$$

If the cyclic condition is violated for some A^* Î [A] and B^* Î [B] , then we can use the property (Wonh, 1974) that 'almost always' a feedback matrix K exists for a controllable pair (A^*, B^*) such that the pair ($A^*+ B^*K$, B^*) is cyclic.

Thus, the controllability of the interval pair ([A] , [B]) is a basic solvability condition for existence of a stabilizing P-regulator.

In conclusion an algorithm may be used for modal P- regulator calculation.

Step 1. Analyse the controllability of a pair ([A] , [B]) for all A Î [A] and B Î [B]. If this pair is not controllable then the problem has no solution.

Step 2. If $r = 1$ then go to step 3 otherwise chose some real numbers as the elements of r vector q. Calculate $[b] = [B] * q$. If the pair $([A], [B]*q)$ is controllable then go to step 3 else chose another vector q.

Step 3. Analyse inclusions (11). If the inclusion are not hold for some $[d_i]$ then increase the width of the interval $[d_i]$.

Step 4. Calculate n vector k from formula (12).

Step 5. If $r = 1$ then $K = k$. If $r \geq 2$ then $K = qk$.

4 EXAMPLE

Consider a stabilization control problem for a helicopter longitudinal motion speed; the helicopter longitudinal motion can be described by linear dynamical state-space model (1) with $n = 3$, $r = 2$ and the matrices

$$[A] = \begin{bmatrix} [a_{11}] & [a_{12}] & [a_{13}] \\ [a_{21}] & [a_{22}] & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad [B] = \begin{bmatrix} [b_{11}] & 0 \\ 0 & [b_{22}] \\ 0 & 0 \end{bmatrix} \tag{19}$$

where the elements of $[A]$ and $[B]$ are (Khleb, 1980):

$$[a_{11}] = [-0.031, -0.0128], [a_{12}] = [-3.4, -0.1], [a_{21}] = [-0.00077, -0.0007],$$
$$[a_{22}] = [-0.32, -0.31], [a_{13}] = [-9.8, -9,8], [b_{11}] = [-18, -15],$$
$$[b_{22}] = [-3,3, -3]. \tag{20}$$

In the vector $x = (x_1, x_2, x_3)^t$ x_1 is a deviation of the longitudinal motion projection, x_2 is an angular speed deviation, x_3 is a pitch angular deviation.

It is necessary to find such a real 2×3 matrix K that for every real $A \in [A]$ and $B \in [B]$ the characteristic polynomial coefficients of the closed-loop matrix $A + BK$ are located within the interval coefficients of the given interval stable polynomial

$$[D(s)] = s^3 + [3, 4]s^2 + [2, 8]s + [0.5, 5.5]. \tag{21}$$

The controllability analysis of $([A], [B])$ shows that this pair is controllable for all $A \in [A]$ and $B \in [B]$.

We chose $q = (0.8, 1.2)^t$ and compute the vector

$$[b] = [B] * q = \begin{bmatrix} [-14.4, -12] \\ [-3.96, -3.6] \\ 0 \end{bmatrix}$$

The pair $([A], [b])$ is controllable because the condition from (8) is satisfied:
$0 \notin \mathrm{Det}\,([b], [A]*[b], [A]^2*[b])$. Then from formula (9) we can calculate

$$[P] = ([b],\ [A]*[b],\ [A]^2*[b])*\begin{bmatrix} -[a_1] & -[a_2] & -1 \\ -[a_2] & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} [-43.383, -30.527] & [-10.036, 4.5404] & [12, 14.4] \\ [-0.092, 0.095] & [-0.116, 0.2556] & [3.6, 3.96] \\ [-0.116, 0.256] & [3.6, 3.96] & 0 \end{bmatrix}$$

where $[a_1]$, $[a_2]$ are the coefficients of the characteristic polynomial of $[A]$: $[f(s)] = \det(sI-[A]) = s^3 + [a_2]\, s^2 + [a_1]s + [a_0]$ with $[a_2] = [0.3228, 0.351]$, $[a_1] = [0.00135, 0.00985]$, $[a_0] = [-0.007546, -0.00686]$.

We find row vector

$$[f] = ([d_o]\ \Theta\ [a_o], [d_1]\ \Theta\ [a_1], [d_2]\ \Theta\ [a_2]) = ([0.5075, 5.50686], [1.99865, 7.99015], [2.6772, 3.649])$$

and verify inclusions (11). The inclusions are guaranteed, therefore, we can calculate

$$k = \mathrm{M}[f]\,(\mathrm{M}[P])^{-1} = (-0.0793,\ 1.116,\ 1.2477)$$

Then the gain matrix of modal P-regulator can be computed as

$$K = qk = \begin{bmatrix} -0.0634 & 0.8228 & 0.9977 \\ -0.09576 & 1.3392 & 1.49625 \end{bmatrix}$$

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