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Tuning via Measurements of the Squared Error

Felipe M Pait*
pait@lac.usp.br
Universidade de São Paulo
Laboratório de Automação e Controle – PEE
São Paulo SP 05508–900 Brasil

Abstract

Given data x, we wish to adjust a parameter vector $p(t)$ so as to minimize $z(t) = x^\top p - y$ as best we can in some norm sense. If y (or equivalently z) were available, we might choose as our cost function the integral of z^2 and minimize it using standard least-squares algorithms. We consider the case when neither y nor z are available; rather, at each instant we are able to choose $p(t)$ and measure $z^2(t)$, that is to say, z's magnitude but not its sign.

Key Words: Adaptive Control; Parameter Estimation; Direct Control.

1 On gradient and least-squares tuning

Given data $x : [0, \bar{t}) \to \mathbb{R}^n$, we wish to adjust a parameter vector $p(t)$ so as to minimize

$$z(t) = x^\top p - y \tag{1}$$

as best we can in some norm sense. If $y : [0, \bar{t}) \to \mathbb{R}$ (or equivalently z) were available, we might choose as our cost function the integral of z^2 and minimize it using standard least-squares algorithms. We consider the case when neither y nor z are available; rather, at each instant we are able to choose $p(t)$ and measure $z^2(t)$, that is to say, z's magnitude but not its sign. The garden-variety least-squares algorithm depends on explicit knowledge of y; and its recursive version relies, as do other recursive gradient-type methods, on $z(t)$ to specify the direction of the adjustment. In this note we propose a recursive algorithm to tune estimates of $p(t)$. One application we have in mind is direct adaptive control (Pait, 1999).

Running the risk of boring the reader, we shall briefly review some standard parameter estimation concepts. With M symmetric, positive-semidefinite, define

$$J(t) = \frac{1}{2}\hat{p}^\top M \hat{p} + \frac{1}{2}\int_0^t d\tau\, (x^\top \hat{p} - y)^2.$$

$J(t)$ represents a cost to be minimized at a given instant t by choosing an appropriate $\hat{p}$, which must satisfy

$$\frac{\partial J}{\partial \hat{p}^\top}(t) = M\hat{p} + \int_0^t d\tau\, x(x^\top \hat{p} - y) = 0,$$

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so that

$$\hat{p} = \left(M + \int_0^t d\tau \, x x^\top\right)^{-1} \int_0^t d\tau \, x y \tag{2}$$

is the solution to the least-squares estimation problem. If $M > 0$ this solution is unique, and because the Hessian

$$\frac{\partial^2 J}{\partial \hat{p}^\top \partial \hat{p}}(t) = M + \int_0^t d\tau \, x x^\top > 0,$$

$J(t)$ is a convex function of $\hat{p}$ and the solution is a global minimum.

Equation (2) can be used to construct a recursive solution $\hat{p}$ to the least-squares problem. However, most important is that the solution to the least-squares problem exist and be unique. Once this condition is satisfied, a number of techniques can be used, assuming that y is a noisy measurement of the scalar product of data x with an unknown parameter vector p_*, namely, $y = x^\top p_* + w$, so that $\hat{p}$ can be interpreted as an estimate of p_*. If y were available, perhaps the simplest way of constructing a recursive estimate $\hat{p}$ would be to choose as tuning error $\hat{z} = x^\top \hat{p} - y$ and set

$$\dot{\hat{p}} = -x\hat{z}.$$

Then $x^\top(\hat{p} - p_*) = \hat{z} + w$, so the derivative of

$$V = \frac{1}{2}(\hat{p} - p_*)^\top (\hat{p} - p_*),$$

one-half of the squared parameter error, is

$$\begin{aligned} \dot{V} &= (\hat{p} - p_*)^\top \dot{\hat{p}} = -(\hat{p} - p_*)^\top x \hat{z} \\ &= -\hat{z}^2 - \hat{z} w. \end{aligned}$$

Using the inequality $|\hat{z}w| \leq \frac{1}{2}\hat{z}^2 + \frac{1}{2}w^2$ and integrating results in

$$V(t) \leq V(0) - \frac{1}{2}\int_0^t \hat{z}^2 + \frac{1}{2}\int_0^t w^2.$$

Because $V \geq 0$, this shows that the energy of the signal $\hat{z}$ is limited by that of w plus a finite constant; in particular if $w \in \mathcal{L}^2$ then $\hat{z} \in \mathcal{L}^2$ as well, and $\hat{p}$ is bounded. If for all t we choose the adjustable parameters p equal to p_*'s estimate $\hat{p}$, which of course we are free to do, then $\hat{z} = z$ and

$$z = x^\top p - y \in \mathcal{L}^2 \text{ and } p \in \mathcal{L}^\infty.$$

Remark: If p are parameters of a direct or indirect adaptive feedback controller, then $p = \hat{p}$ can be considered an expression of the certainty-equivalence principle. The distinction between the adjustable parameters p and the parameter estimate $\hat{p}$ may appear recherché, but it plays a crucial role in the solution of the more complex problem treated in §2 of this paper.

2 Gradient tuning

The method just described requires that we set the tuning error equal to z, that is to say, z must be known in sign as well as magnitude. If only the latter were known, we might choose as our tuning error

$$e_T(\hat{p}, t) = (\hat{p} - p)^\top x x^\top (\hat{p} - p) - z^2. \tag{3}$$

Intuition motivates the definition of e_T above: if the sign of z is unknown, then at each instant we might equally believe that $y = x^\top p + |z|$ or that $y = x^\top p - |z|$. Unable to minimize $x^\top \hat{p} - y$ as before, we content ourselves with trying to keep the product $(x^\top \hat{p} - x^\top p - |z|)(x^\top \hat{p} - x^\top p + |z|) = e_T$ small. Using $\hat{z} - z = x^\top(\hat{p} - p)$, a useful formula that relates e_T to the parameter estimation error can be derived:

$$\begin{aligned} e_T &= (x^\top \hat{p} - x^\top p - z)(x^\top \hat{p} - x^\top p + z) \\ &= (x^\top(\hat{p} - p) - z)\hat{z}. \end{aligned}$$

Notice that e_T is a function of $\hat{p}$ as well as of time. The control engineer's instinct is to tune the parameter estimate in the direction of the negative "gradient" of e_T^2:

$$\boxed{\dot{\hat{p}} = -xx^\top(\hat{p} - p)e_T(\hat{p}, t).} \tag{4}$$

Along the trajectories of (4), the derivative of the positive function V used in §1 is

$$\begin{aligned} \dot{V} &= -(\hat{p} - p_*)^\top x e_T x^\top(\hat{p} - p) \\ &= -(\hat{z} + w)\hat{z} \ \underbrace{(x^\top(\hat{p} - p) - z)x^\top(\hat{p} - p)}. \end{aligned}$$

For the analysis to carry through as in §1, it would be necessary to choose p in a manner that ensures that the underbraced term stays positive. This is unfortunately not feasible without further assumptions about p_*. The following proposition indicates that gradient tuning is nonetheless a reasonable route to follow.

Proposition 1 *Suppose that $w = 0$, that $x(t)$ is continuous, and that there exist instants t_1, t_2, t_3, and t_4, $t_i < t_1 < t_2 < t_3 < t_4 < t_f$, such that $\int_{t_1}^{t_2} xx^\top > 0$ and $\int_{t_3}^{t_4} xx^\top > 0$. Let p be chosen so that $p(t) = p_0 + p_1(t) + \alpha(t)x(t)/|x(t)|$, where p_0 is constant, $p_1(t)$ is orthogonal to $x(t)$, and $\alpha(t)$ is such that if $x(t_5)/|x(t_5)| = x(t_6)/|x(t_6)|$ for some t_5 and t_6 then $\alpha(t_5) \neq \alpha(t_6)$. Then*

$$J_{[t_i,t_f]}(\hat{p}) = \int_{t_i}^{t_f} e_T(\hat{p}, t) = 0$$

has a unique minimum at $\hat{p} = p_*$.

Proof: If $w = 0$, then $\hat{z} = x^\top(\hat{p} - p_*)$ and $e_T = x^\top(\hat{p} - 2p + p_*)x^\top(\hat{p} - p_*)$, so

$$J_{[t_i,t_f]}(\hat{p}) = (\hat{p} - p_*)^\top \left(\int_{t_i}^{t_f} xx^\top \left(x^\top(\hat{p} - 2p + p_*)\right)^2 \right) (\hat{p} - p_*).$$

Clearly $J(\cdot) \geq 0$ and $J(p_*) = 0$. For there to be another minimizing solution, it is necessary that, for some constant x_i,

$$\int_{t_i}^{t_f} (x_i^\top x)^2 \left(x^\top(\hat{p} - 2p + p_*)\right)^2 = 0,$$

that is to say, $\xi(t) = x^\top(\hat{p} - 2p + p_*) = 0$ when $x_i^\top x \neq 0$. But by assumption there must exist τ_1 and τ_2 such that $x_i^\top x(\tau_1)$ and $x_i^\top x(\tau_2)$ are both nonzero. With p as chosen

$$\xi(t) = x^\top(\hat{p} + p_* - 2p_0 - 2p_1(t) - 2\alpha(t)x(t)/|x(t)|)$$

so $\xi(t) = 0$ implies

$$\alpha(t)|x| = \frac{1}{2}x^\top(\hat{p} + p_* - 2p_0),$$

which by construction of α can only happen either for $t = \tau_1$ or for $t = \tau_2$. But the support of $x_i^\top x$ must have open interior because $\int_{t_i}^{t_f} xx^\top > 0$ and $x(t)$ is continuous, so $x_i^\top \xi(t)$ must be nonzero on an open set, hence no other minimizing solution can exist. $\square$

The requirements Proposition 1 makes on $x(\cdot)$ are not excessively demanding, being much less restrictive than a persistency of excitation condition. However the description of $p(t)$ does not furnish an explicit algorithm for choosing it in a manner that ensures uniqueness of the minimum. Work is in progress to develop a complete algorithm and study its properties in the presence of noise. An alternative approach is presented in the sequel.

3 Convexity

In this section we approach the problem by searching for conditions for global convexity of the cost function

$$J(t) = \frac{1}{2}\hat{p}^\top M\hat{p} + \frac{1}{4}\int_0^t d\tau\, e_T^2(\hat{p},\tau).$$

The $\hat{p}$ that minimizes J must satisfy

$$\frac{\partial J}{\partial \hat{p}}(t) = \hat{p}^\top M + \int_0^t d\tau\, e_T(\hat{p},\tau)(\hat{p}-p(\tau))^\top xx^\top = 0.$$

If the Hessian

$$\frac{\partial^2 J}{\partial \hat{p}^\top \partial \hat{p}}(t) = M + \int_0^t d\tau\, \left(2(x^\top(\tau)(\hat{p}-p(\tau))^2 + e_T\right) xx^\top(\tau) > 0,$$

then $J(t)$ is a convex function of $\hat{p}$ and has a global minimum. Given an arbitrary vector v_i, let

$$h_i = m_i + \int_0^t d\tau\, \xi_i(\tau)\left(2(x^\top(\tau)(\hat{p}-p(\tau))^2 + e_T\right)$$

where $m_i = v_i^\top M v_i$ and $\xi_i = (v_i^\top x)^2$. The minimum of h_i over all $\hat{p}$ occurs when

$$\frac{\partial h_i}{\partial \hat{p}^\top}(t) = 6\int_0^t d\tau\, \xi_i(\tau) xx^\top(\hat{p}-p) = 0,$$

namely, at $\hat{p} = \bar{p}$ where

$$\bar{p} = \left(\int_0^t d\tau\, \xi_i xx^\top\right)^{-1}\int_0^t d\tau\, \xi_i xx^\top.$$

So

$$\begin{aligned} h_{i_{\min}} &= 3\int \xi_i (x^\top p)^2 - \int \xi_i z^2 - 3\bar{p}^\top \left(\int_0^t d\tau\, \xi_i xx^\top\right)\bar{p} \\ &= \int \xi_i \left(3\left(x^\top(p-\bar{p})\right)^2 - z^2\right). \end{aligned}$$

We are, of course, assuming that $\int \xi_i xx^\top$ is invertible. This assumption is more restrictive than the excitation condition in Proposition 1, but greatly simplifies the resulting equations.

The patient reader will not complain overmuch if we continue the orgy of derivatives and compute

$$\dot{h}_{i_{\min}} = 3\xi_i \left(x^\top(p-\bar{p})\right)^2 - \xi_i z^2.$$

We are concerned with the magnitude of the output $\bar{z}$ of the system

$$\frac{d}{dt}(p - \bar{p}) = -\left(\int_0^t d\tau\, \xi_i x x^\top\right)^{-1} \xi_i x x^\top (p - \bar{p}) + \dot{p}$$
$$\bar{z} = x^\top (p - \bar{p})$$

As long as $|\bar{z}|^2 > \frac{1}{3}|z|^2$, $\dot{h}_{i_{\min}}$ remains positive, the minimum value attained by the Hessian is increased, and J stays convex! This needs to happen for all directions v_i, of course.

We are finally in a position to justify the fastidious distinction between p and p_*'s estimate $\hat{p}$. For if $\hat{p}(t)$ were the output of some successful tuner, $\dot{\hat{p}}$ would become small. Thus if we were to set $p(t) = \hat{p}(t)$, the tuner's success would render $\bar{z}$ small and thus possibly undermine convexity — the basis on which its success rests.

4 Concluding Remarks

We have presented conditions under which tuning when the estimation error is known in magnitude only is feasible. Besides the intrinsic interest of an estimation problem under partial output information (reminiscent of recent work on stabilization of linear systems for which only the magnitudes of outputs are measured (Nesic and Sontag, 1998)), our motivation lies in the possibility of using such tuners to develop novel direct adaptive control algorithms that can be applied more broadly than the usual model-reference adaptive controllers. Direct adaptive control using concepts from linear-quadratic regulator theory is the subject of a companion paper (Pait, 1999). Whether tuners based on the ideas discussed here will indeed bear fruit for the latter application will depend crucially on their properties, which are currently under investigation.

References

Nesic, D. and E. D. Sontag (1998). "*Input-to-State Stabilization of Linear Systems with Positive Outputs*," Nonlinear Control Abstracts NCA 2-3-980316, (`http://www.math.rutgers.edu/~sontag/papers.html`).

Pait, F. M. (1999). "*On the Design of Direct Adaptive Controllers*," To appear in the 7th IEEE Mediterranean Conference on Control and Automation.