Proceedings of the 7th Mediterranean Conference on Control and Automation (MED99) Haifa, Israel - June 28-30, 1999

STRUCTURE OF OPTIMAL SOLUTIONS OF INFINITE DIMENSIONAL CONTROL PROBLEMS

Alexander J. Zaslavski

Department of Mathematics
Technion-Israel Institute of Technology
32000 Haifa, Israel

Abstract

In this paper we present several results concerning the structure of optimal solutions for infinite-dimensional optimal control problems. The primary area of applications of these problems concerns models of regional economic growth, cattle ranching models and systems with distributed parameters and boundary controls arising in certain engineering applications. We are concerned with the existence of an overtaking optimal trajectory over an infinite horizon. The existence result that we obtain extends the result of Carlson, Haurie and Jabrane (1987) to a situation where the trajectories are not necessary bounded. We show that an optimal trajectory defined on an interval $[0, T]$ is contained in a small neighborhood of the optimal steady-state in the weak topology for all $t \in [0, T] \setminus E$ where $E \subset [0, T]$ is a measurable set such that the Lebesgue measure of E does not exceed a constant which depends only on the neighborhood of the optimal steady-state and does not depend on T. Moreover, we show that the set E is a finite union of intervals and their number does not exceed a constant which depends only on the neighbborhood.

In this paper we present several results concerning the structure of optimal solutions for infinite-dimensional optimal control problems. The primary area of applications of these problems concerns models of regional economic growth discussed in (Isard and Liossatos, 1979) cattle ranching models proposed in (Derzko and Sethi, 1980) and systems with distributed parameters and boundary controls arising in certain engineering applications (Barbu, 1980; Fattorini, 1968). We are concerned with the existence of an overtaking optimal trajectory over an infinite horizon. The existence result that we obtain extends the result of Carlson, Haurie and Jabrane (1987) to a situation where the trajectories are not necessary bounded. We show that an optimal trajectory defined on an interval $[0, T]$ is contained in a small neighborhood of the optimal steady-state in the weak topology for all $t \in [0, T] \setminus E$ where $E \subset [0, T]$ is a measurable set such that the Lebesgue measure of E does not exceed a constant which depends only on the neighborhood of the optimal steady-state and does not depend on T. Moreover, we show that the set E is a finite union of intervals and their number does not exceed a constant which depends only on the neighborhood.

We consider a system described by the following input-output relationship:

$$x(t) = S(t)x_0 + \int_0^t S(t-s)Bu(s)ds, \ t \in I \tag{1}$$

where I is either $[0, \infty)$ or $[0, T]$ $(0 \le T < \infty)$, E and F are separable Hilbert spaces, $x_0 \in E$, $\{S(t) : \ t \ge 0\}$ is a strongly continuous semigroup on E with generator A, $u(\cdot) \in L^2_{loc}(I; F)$, the

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX space of all strongly measurable functions $u(\cdot) : I \to F$, which are square-integrable on every finite interval $\Delta \subset I$, and $B : F \to E$ is a bounded linear operator.

Thus $x(\cdot)$ is the mild solution of the state equation

$$x'(t) = Ax(t) + Bu(t),\ t \in I,$$

$$x(0) = x_0$$

where A is a possibly unbounded, closed and densely defined operator in E.

In addition we know (see (Balakrishnan, 1976)) that although a mild solution need not be absolutely continuous it does satisfy the following mild differential equation for any $y \in D(A^*)$:

$$(d/dt) < x(t), y >=< x(t), A^* y > + < Bu(t), y > \text{ a.e. } t \in I,$$

$$\lim_{t \to 0_+} < x(t), y >=< x_0, y >$$

where A^* is the adjoint operator associated with A, with domain $D(A^*)$.

We impose the following additional constraints on state and control:

$$x(t) \in X,\ t \in I \text{ where } X \text{ is a convex and closed subset of } E$$

and

$$u(t) \in U(x(t)) \subset F,\ t \in I \text{ where } U(\cdot) :\ X \to 2^F$$

is a point to set mapping which is convex valued and such that

$$\alpha U(x_1) + (1-\alpha)U(x_2) \subset U(\alpha x_1 + (1-\alpha)x_2),\ x_1, x_2 \in X,\ \alpha \in [0,1]$$

and if $u_n \in U(x_n),\ n = 1, 2, \ldots$ and $u_n \to u,\ x_n \to x$ as $n \to \infty$ in the weak topology, then $u \in U(x)$.

The perfomance of the system is evaluated by the cost functional

$$J(T_1, T_2, x, u) = \int_{T_1}^{T_2} f(x(t), u(t))dt$$

where $f :\ E \times F \to R^1$ is a convex functional which is lower semicontinuous on $E \times F$ and satisfies the following growth condition:

there exist $K_1 > 0$ and $K > 0$ such that:

(2) $\quad f(x,u) \geq K(||x||^2 + ||u||^2)$ for each $x \in E,\ u \in F$ satisfying $||x||^2 + ||u||^2 > K_1$.

A function $x :\ I \to E$ where I is either $[0, \infty)$ or $[0, T]$ $(T > 0)$ will be called a trajectory if there exists $u(\cdot) \in L^2_{loc}(I; F)$ (referred to as a control) such that the pair (x, u) satisfies (1) and

$$x(t) \in X,\ u(t) \in U(x(t)),\ t \in I.$$

A trajectory-control pair $\hat{x} :\ [0, \infty) \to E,\ \hat{u} :\ [0, \infty) \to F$ is overtaking (resp. weakly overtaking) optimal if for any other trajectory-control pair $x :\ [0, \infty) \to E,\ u \to [0, \infty) \to F$ satisfying $x(0) = \hat{x}(0)$

$$\limsup_{t \to \infty} [J(0, t, \hat{x}, \hat{u}) - J(0, t, x, u)] \leq 0,$$

$$(\text{resp. } \liminf_{t \to \infty} [J(0, t, \hat{x}, \hat{u}) - J(0, t, x, u)] \leq 0).$$

These are the optimality concepts used in (Brock and Haurie, 1976). For other contributions to the theory of optimal control on an infinite horizon making use of these solution concepts see (Carlson *et al.,* 1991; Feinstein and Luenberger, 1981; Leizarowitz, 1985; Zaslavski, 1995; Zaslavski, 1996a; Zaslavski, 1996b).

Assume the following

Assumption 1. The optimal steady state problem (OSSP) consisting of

$$\text{Min } f(x,u) \text{ over all } (x,u) \in E \times F \text{ satisfying}$$

$$0 =< x, A^* y > + < Bu, y > \text{ for any } y \in D(A^*),\ x \in X,\ u \in U(x)$$

has a solution $(\bar{x}, \bar{u})$ with $\bar{x}$ uniquely defined.

By the convexity assumptions already made on f, X, and U, the OSSP is a convex programming problem in a Hilbert space. Thus there exists $\bar{p} \in D(A^*)$ such that (see (Ekeland and Temam, 1976; Rockafellar, 1969))

$$f(\bar{x}, \bar{u}) \le f(x,u) - < x, A^* \bar{p} > - < Bu, \bar{p} >$$

for every $x \in X$ and $u \in U(x)$. Let $L: E \times F \to [0, \infty)$ be defined by

$$L(x,u) = f(x,u) - f(\bar{x}, \bar{u}) - < x, A^* \bar{p} > - < Bu, \bar{p} > \text{ if } x \in X \text{ and } u \in U(x),$$

$$L(x,u) = \infty \text{ otherwise.}$$

Then we have $L(\bar{x}, \bar{u}) = 0$. Furthermore, since L differs from f through an affine function of x and u, it still satisfies the growth property (2) with f replaced by L.

Let I be either $[0, \infty)$ or $[0, T]$ $(T > 0)$, $x: I \to E$, $u: I \to F$ be a trajectory-control pair and $T_1, T_2 \in I$, $T_1 < T_2$. We define

$$J_L(T_1, T_2, x, u) = \int_{T_1}^{T_2} L(x(t), u(t)) dt.$$

For a trajectory-control pair $x: [0, \infty) \to E$, $u: [0, \infty) \to F$ we define

$$J_L(0, \infty, x, u) = \int_0^{\infty} L(x(t), u(t)) dt.$$

For each $T > 0$ and each $z \in E$ we define

$$\sigma(z, T) = \inf\{J(0, T, x, u) : x: [0, T] \to E,\ u: [0, T] \to F$$

$$\text{is a trajectory-control pair, } x(0) = z\}.$$

Now we present the following five results established in (Zaslavski, 1996b).

Theorem 1. *Suppose that Assumption 1 holds and $x: [0, \infty) \to E$, $u: [0, \infty) \to F$ is a trajectory-control pair. Then one of the following relations holds:*

(i) $\sup\{|J(0, T, x, u) - T f(\bar{x}, \bar{u})| : T \in (0, \infty)\} < \infty$.

(ii) $J(0, T, x, u) - T f(\bar{x}, \bar{u}) \to \infty$ *as* $T \to \infty$. *Moreover (i) holds if and only if* $J_L(0, T, x, u) < \infty$.

Theorem 2. *Suppose that Assumption 1 holds and r_1, r_2, r_3 are positive numbers. Then there exist $\Delta, r > 0$ such that*

$$||x(t)|| \leq \Delta,\ t \in [0, T],\ J_L(0, T, x, u) \leq r$$

for each $T > 0$ and each trajectory-control pair $x:\ [0, T] \to E,\ u:\ [0, T] \to F$ which has the following properties:

(a) $||x(0)|| \leq r_2,\ J(0, T, x, u) \leq \sigma(x(0), T) + r_3$;

(b) there is a trajectory-control pair $y:\ [0, \infty) \to E,\ v:\ [0, \infty) \to F$ satisfying $y(0) = x(0)$, $J_L(0, \infty, y, v) \leq r_1$.

The following result is an extension of Theorem 1 in (Carlson *et al.,* 1987) to optimal trajectories defined on finite intervals.

Theorem 3. *Suppose that Assumption 1 holds, r_1, r_2, r_3 are positive numbers and V is a neighborhood of $\bar{x}$ in the weak topology. Then there exists a number $l > 0$ such that*

$$(T_2 - T_1)^{-1} \int_{T_1}^{T_2} x(t)dt \in V$$

for each $T \geq l$, each trajectory-control pair $x:\ [0, T] \to E,\ u:\ [0, T] \to F$ which has properties (a), (b) from Theorem 2 and each $T_1, T_2 \in [0, T]$ satisfying $T_2 - T_1 \geq l$.

Denote by $\mathcal{F}$ the set of all trajectory-control pairs $x:\ [0, \infty) \to E,\ u:\ [0, \infty) \to F$ such that

$$L(x(t), u(t)) = 0 \text{ a.e. on } [0, \infty).$$

We say that $\mathcal{F}$ has property $\mathcal{G}$ if for any neighborhood V of $\bar{x}$ in the weak topology there exists a number $t_v > 0$ such that $x(t) \in V$ for each $t \geq t_v$ and each trajectory-control pair $(x, u) \in \mathcal{F}$.

This property corresponds to property (S) in (Leizarowitz, 1985).

The following result describes the structure of “approximate” optimal solutions.

Theorem 4. *Suppose that Assumption 1 holds and $\mathcal{F}$ has property $\mathcal{G}$. Let r_1, r_2, r_3 be positive numbers and V be a neighborhood of $\bar{x}$ in the weak topology. Then there exist an integer $Q \geq 1$ and a number $l > 0$ such that for each $T > 0$ and each trajectory-control pair $x:\ [0, T] \to E$, $u:\ [0, T] \to F$, which has properties (a), (b) from Theorem 2, there exists a sequence of intervals $[b_j, c_j]$, $j = 1, \ldots q$ such that*

$$1 \leq q \leq Q,\ 0 < c_j - b_j \leq l,\ j = 1, \ldots q \text{ and}$$

$$x(t) \in V \text{ for each } t \in [0, T] \setminus \cup_{j=1}^{q} [b_j, c_j].$$

The following result is a generalization of Theorem 4 in in (Carlson *et al.,* 1987) which establishes the existence of an overtaking optimal solution in the subclass of bounded trajectories.

Theorem 5. *Suppose that Assumption 1 holds and $\mathcal{F}$ has property $\mathcal{G}$. Let $\tilde{x} : [0, \infty) \to E$, $\tilde{u}:\ [0, \infty) \to F$ be a trajectory-control pair satisfying $J_L(0, T, \tilde{x}, \tilde{u}) < \infty$. Then there exists an overtaking optimal trajectory-control pair $x^*:\ [0, \infty) \to E,\ u^*:\ [0, \infty) \to F$ such that $x^*(0) = \tilde{x}(0)$.*

For the proofs of Theorems 1-5 see (Zaslavski, 1996b).

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