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Iterative Adaptive (Unfalsified) Control

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Abstract

Uncertainty model unfalsification is reviewed. Based on the concept of unfalsification, an iterative direct (unfalsified) adaptive control scheme is proposed which may alleviate some of the difficulties of iterative adaptation, *e.g.*, convergence.

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1 Introduction

Learning, apparently, is the result of making mistakes. The procedure is to make observations, construct a model, and validate the model against new data. Validation is perhaps a misnomer, as one can never prove that a model will be able to accurately predict the future. More precisely, the data can *falsify* a model, *i.e.*, the model may prove to be incapable of fully explaining the data. Hence, instead of validation, we use the more precise, but awkward term: *unfalsification.* In this paper we will show how unfalsification can be used for *direct iterative adaptive control*, *i.e.*, the control parameters are adjusted directly.

Uncertainty model unfalsification using finite time domain data, which is the underlying basis for the work presented in this paper, was first described by Poolla *et al.* (1992). Earlier work using frequency domain data was presented in (Smith and Doyle, 1989), and some precursers to unfalsification can be found in (Kosut *et al.*, 1992). Further extensions and applications to plant uncertainty model unfalsification can be found in (Kosut, 1995), (Kosut, 1996), (Kosut and Anderson, 1997), and (Livestone *et al*, 1995). A method employing a probabistic description of dynamic uncertainty is given in (Goodwin *et al.*, 1992). The origin of the ideas for direct controller unfalsification are presented by Safonov and Tsao (1997) and the references therein. The mathematical basis for unfalsification of linear-time-invariant systems can be found in (Grenander and Szebo, 1958) and (Foias and Frazho, 1990). Computations using convex programming is discussed in (Woodley *et al.*, 1998, 1999).

The roots of iterative adaptive control can be traced to the dual control concept (see, *e.g.*, (Åström and Wittenmark, 1995, Ch. 7)), which typically involves *indirect adaptation*, *i.e.*, identification followed by control parameter adjustment. A survey of iterative identification and control schemes is given in (Gevers, 1993). Of particular relevance to the work presented here – for purposes of comparison – is (Zang *et al.*, 1991), (Åström, 1993), and (Åström and Nilsson, 1994) which describe how data filters can be selected to make the identification and control criteria merge; the windsurfer approach to adaptation and learning, as described in (Lee *et al*, 1993, 1995), where the closed-loop bandwidth is gradually increased every iteration; and (Hjalmarsson *et al.*, 1994), which describes a direct iterative controller design method.

2 Iterative Adaptive Control

A generic iterative adaptive control system is depicted in figure 1. The adaptive part of the controller consists of a parameter estimator and a control design algorithm connected in series through a sample and hold. The latter is what makes the system "iterative." That is, the next controller design is based on data collected while the previous controller was in place.

The system consists of two feedback "loops" each operating at different sampling rates. The inner loop, operating at the fast rate, consists of the plant and controller, where u is the control input to the plant, y is the sensed output from the plant, and r is the reference command to the controller. The outer loop, operating at the slow rate, consists of the plant parameter estimator and control parameter design, The sequence of parameter estimates, $\bar{\theta}$, are produced at the end of every data collection interval of ℓ-samples, and hence, depend on the prior applied sequence of controller parameters, $\hat{\alpha}$, which are based on $\hat{\theta}$, the prior plant parameter sequence, and so on. Thus, $\hat{\theta}$ and $\hat{\alpha}$, are piece-wise constant vector sequences, *i.e.*, constant over every ℓ-samples. Specifically, during the i-th iteration (ℓ-data collection interval), that is, for $t = 1 + (i-1)\ell, \dots, i\ell$, let $\theta_\ell^i \in \mathbf{R}^p$ denote the plant parameters and let $\alpha_\ell^i \in \mathbf{R}^q$ denote the corresponding control parameters. The relation between θ_ℓ^i and α_ℓ^i is typically algebraic and

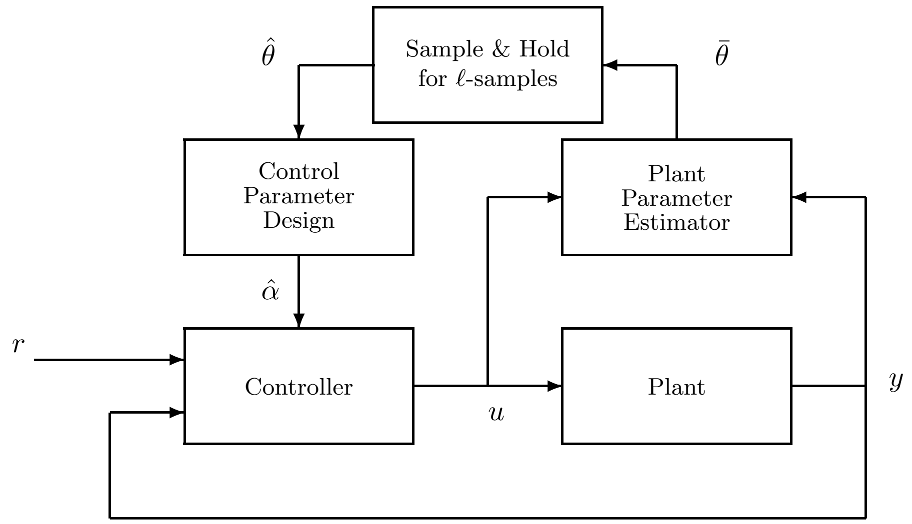

Figure 1: Iterative adaptive control system.

depends on the design procedure, *i.e.*,

$$\alpha_\ell^i = \alpha_\star(\theta_\ell^i) \tag{1}$$

Let $(y^i, u^i) \in \mathbf{S}^\ell \times \mathbf{S}^\ell$ denote[1] the plant output and input data recorded during the i-th data collection period, *i.e.*,

$$\begin{aligned} y^i &= \left\{ y_{1+(i-1)}, \ldots, y_{i\ell} \right\} \\ u^i &= \left\{ u_{1+(i-1)}, \ldots, u_{i\ell} \right\} \end{aligned} \tag{2}$$

After every data collection period of ℓ samples, a new parameter estimate, denoted by θ_ℓ^{i+1}, is determined by solving an optimization problem of the form[2],

$$\theta_\ell^{i+1} = \arg \min_{\theta \in \boldsymbol{\Theta}} \rho_\ell^i(\theta) \tag{3}$$

where $\boldsymbol{\Theta} \subset \mathbf{R}^p$ is the set of possible parameters, which in most cases is simply $\mathbf{R}^p$. The objective function, $\rho_\ell^i(\theta)$, depends on the data $(y^i, u^i) \in \mathbf{S}^\ell \times \mathbf{S}^\ell$, as denoted by the superscript i and the subscript ℓ, and on θ, where the precise dependence is determined by the estimation and/or control design criterion. For "indirect adaptive" schemes, the criterion is related to providing a good fit to the data based on an assumed uncertainty model for the plant system. Hence, $\alpha = \alpha_\star(\theta)$ simply denotes how the plant uncertainty model parameters relate to the corresponding control design parameters. For the "direct adaptive" schemes, as the name implies, the control parameter is itself adjusted, *i.e.*, $\alpha \equiv \theta$, and the criterion is directly related to closed-loop performance. Actually even in the indirect case, the criterion is constructed to be as close as possible to a closed-loop performance criterion or at least useful for control design.

[1] $\mathbf{S}^\ell$ denotes the set of scalar sequences of length ℓ, *i.e.*, $x \in \mathbf{S}^\ell \leftrightarrow x = \{ x_t \mid t \in [1, \ell] \}$ where t are the uniformy spaced integer samples.

[2] The optimization operation "arg min" as used here and throughout the paper is to be understood to mean the minimizing argument, or if there is no unique minimum, then "arg min" refers to the set of (local) minima, *i.e.*, $\arg\min f(x) = \{ x \mid f_x(x) = 0,\ f_{xx}(x) > 0 \}$.

2.1 Convergence

Convergence analysis is very difficult because the data $(y^i, u^i) \in \mathbf{S}^\ell \times \mathbf{S}^\ell$ depends on *all* past control parameter switchings $\{\ \alpha^1_\ell, \ldots, \alpha^i_\ell\ \}$, which in turn depend on all past plant parameter estimates $\{\ \theta^1_\ell, \ldots, \theta^i_\ell\ \}$. However, for long data collection periods, provided that all past controllers are stabilizing, the system memory of past controllers fades. Hence, in the limit, with infinite data collected during every iteration, the data collected during the i-th interval only depends on the last parameter values (α^i, θ^i), *i.e.*,

$$\lim_{\ell\to\infty} y^i = y(\alpha^i), \quad \lim_{\ell\to\infty} u^i = u(\alpha^i), \quad \alpha^i = \alpha_\star(\theta^i) \tag{4}$$

Similarly, the objective function becomes,

$$\lim_{\ell\to\infty} \rho^i_\ell(\theta) = \rho(\theta, \alpha^i), \quad \alpha^i = \alpha_\star(\theta^i) \tag{5}$$

So in the limiting case of infinite data, the parameter estimation step at iteration i is,

$$\theta^{i+1} = \arg \min_{\theta \in \boldsymbol{\Theta}} \rho(\theta, \alpha^i), \quad \alpha^i = \alpha_\star(\theta^i) \tag{6}$$

As observed in (Hjalmarsson *et al.*, 1995), for infinite data, convergent parameter values are equivalently fixed-points of the mapping $\Gamma : \mathbf{R}^p \mapsto \mathbf{R}^p$, where

$$\Gamma(\theta) = \arg \min_{\psi \in \boldsymbol{\Theta}} \rho(\psi, \alpha) \text{ subject to } \alpha = \alpha_\star(\theta) \tag{7}$$

Hence, if $\hat\theta$ is a fixed-point, then

$$\hat\theta = \arg \min_{\psi \in \boldsymbol{\Theta}} \rho(\psi, \hat\alpha), \quad \hat\alpha = \alpha_\star(\hat\theta) \tag{8}$$

and must satisfy the necessary condition for optimality, namely,

$$\left.\frac{\partial}{\partial\theta}\rho(\theta, \alpha_\star(\hat\theta))\right|_{\theta = \hat\theta} = 0 \tag{9}$$

However, the minimizer, θ_{opt}, of $\rho(\theta, \alpha_\star(\theta))$ satisfies,

$$\left.\frac{\partial}{\partial\theta}\rho(\theta, \alpha_\star(\theta))\right|_{\theta = \theta_{opt}} = 0 \tag{10}$$

As pointed out in (Hjalmarsson *et al.*, 1995), the fixed-point $\hat\theta$ is not likely to be the same as θ_{opt}. So it appears that iterative schemes have a built-in flaw. Even if the infinite-data estimation criterion, $\rho(\theta, \alpha_\star(\theta))$, is constructed to be a sensible control criterion, no iterative algorithm can be guaranteed to reach the minimum, or at least a local minimum. This has led some researchers to seek another path, *e.g.*, in (Hjalmarsson *et al.*, 1994) the authors show how to obtain an unbiased estimate of the gradient (and Hessian) of error signals with respect to control paramters by performing a series of specialized experiments. Incidentally, this "flawed" property of iterative schemes of adaptation is a recrudescence of the identical property of all slowly varying parameter adaptive algorithms, *e.g.*, see (Anderson *et al.*, 1986), (Phillips *et al.*, 1988), or the chapter on averaging analysis in (Åström and Wittenmark, 1995).

But all is not lost for iterative schemes – well, it may be for "convergence," but not for unfalsification. In fact, this leads to an interesting philosphical issue – a debate, perhaps – discussed briefly in section 8 on convergence vs. unfalsification.

3 Parametrization and Performance

3.1 Parametrization

We will assume throughout that the control is given by,

$$u = C(\alpha)(r - y) \tag{11}$$

where for fixed $\alpha \in \mathbf{R}^q$, $C(\alpha^i) \in \mathbf{LTI}$[3] The controller parameters, α, are to be adaptively adaptively adjusted. There are of course mant possible ways to parametrize the controller. For example, the control parameters, α, can consist of all the numerator and denominator transfer function coefficients up to a specified degree, thereby restricting the controller order. For example, all order-n controllers can be parametrized as follows:[4]

$$\begin{array}{rcl} C(\alpha) & = & N(\alpha)D(\alpha)^{-1} \\ N(\alpha) & = & b_0 + b_1 z^{-1} + \cdots + b_n z^{-n} \\ D(\alpha) & = & 1 + a_1 z^{-1} + \cdots + a_n z^{-n} \\ \alpha & = & [a_1 \cdots a_n \; b_0 \cdots b_n]^T \in \mathbf{R}^{2n+1} \end{array} \tag{12}$$

Another paramtrization is all the PI controllers, *i.e.*,

$$C(\alpha) = \alpha_P + \alpha_I \frac{z^{-1}}{1 - z^{-1}}, \qquad \alpha = [\alpha_P \; \alpha_I \;]^T \in \mathbf{R}^2 \tag{13}$$

3.2 Performance

We would like the closed-loop system to behave like the reference system,

$$y_{\text{ref}} = T_{\text{ref}} \; r \tag{14}$$

for a specified system $T_{\text{ref}} \in \mathbf{LTI}$. "Behave like" can have a variety of meanings. For example, it could mean that the output error, $y - T_{\text{ref}} \; r$, should be small relative to the size of the command r. An example of such a specification is that[5] ,

$$\|y - T_{\text{ref}} \; r\|_{rms} \leq \rho \|r\|_{rms}, \qquad \forall \|r\|_{rms} < \infty \tag{15}$$

In this case, we are not looking for a response to a *specific* r, such as a sinusoid at one frequency or a step, rather, for every possible r such that $\|r\|_{rms} < \infty$. For example, if the plant system is given by,

$$y = Pu \tag{16}$$

where $P \in \mathbf{LTI}$, and the controller is given by,

$$u = C(r - y) \tag{17}$$

with $C \in \mathbf{LTI}$, then (15) is equivalent to,

$$\|T(P, C) - T_{\text{ref}}\|_{\mathbf{H}_\infty} \leq \rho \tag{18}$$

[3] **LTI** is the set of linear-time-invariant systems with rational transfer functions.

[4] z^{-k} denotes the k-delay operator.

[5] The RMS-norm of a sequence (technically a semi-norm) is defined as $\|x\|_{rms} = \left(\lim_{\ell \to \infty} \frac{1}{\ell} \sum_{t=1}^{\ell} x_t^2 \right)^{1/2}$

with $T(P, C)$ given by,

$$T(P, C) = (1 + PC)^{-1} PC \tag{19}$$

This is clearly a measure of the error between the transfer functions of the closed-loop and reference systems. Hence, for a given controller parametrization, $C(\alpha) \in \mathbf{LTI}$, and a given plant parametric model, $P(\theta) \in \mathbf{LTI}$, the optimal control design (1) is,

$$\alpha_\star(\theta) = \arg\min_\alpha \|T(P(\theta), C(\alpha)) - T_{\text{ref}}\|_{\mathbf{H}_\infty} \tag{20}$$

Hence, the *designed* closed-loop system $T(P(\theta), C(\alpha_\star(\theta))$ is the closest $\mathbf{H}_\infty$ approximation to T_{ref}, the reference system. Since it is unrealistic to expect that the plant is in the model set, *i.e.*, $P \neq P(\theta)$, it follows that the actual, or *achieved* closed-loop system, $T(P, C(\alpha_\star(\theta))$, may be quite different. Of course the most desirable goal is,

$$\alpha_{opt} = \arg\min_\alpha \|T(P, C(\alpha)) - T_{\text{ref}}\|_{\mathbf{H}_\infty} \tag{21}$$

The performance measure can be modified to penalize control activity, *e.g.*,

$$\|y - T_{\text{ref}}\ r\|_{\text{r}ms}^2 + \lambda \|u\|_{rms}^2 \leq \rho^2 \|r\|_{\text{r}ms}^2, \quad \forall \|r\|_{\text{r}ms} < \infty \tag{22}$$

Again, if the plant and controller are LTI, then this measure is equivalent to,

$$\left\| \begin{bmatrix} T(P, C) - T_{\text{ref}} \\ \lambda Q(P, C) \end{bmatrix} \right\|_{\mathbf{H}_\infty} \leq \rho \tag{23}$$

where

$$Q(P, C) = (1 + PC)^{-1} C \tag{24}$$

4 Uncertainty Model Unfalsification

4.1 Unfalsification

The generic uncertainty model unfalsification problem is as follows:

> *Given scalar data sequences $e, v \in \mathbf{S}^\ell$, establish necessary and sufficient conditions for the existence of a disturbance sequence $w \in \mathbf{S}^\ell$ and a causal system Δ such that*
>
> $$w \in \mathbf{W}(\sigma), \quad \Delta \in \mathbf{\Delta}(\delta) \tag{25}$$
>
> *and which are consistent with the model*
>
> $$e_t = w_t + (\Delta v)_t, \quad t \in [1, \ell] \tag{26}$$

The sets $\mathbf{W}(\sigma)$ and $\mathbf{\Delta}(\delta)$ denote, respectively, a set of sequences with norm bounded by σ and a set of systems with gain bounded by δ.

4.2 Uncertainty Model Forms

The data sequence e is often obtained as the *prediction error* associated with an assumed model of the system, and v is a function of other sensed signals, the choice reflecting the type of dynamic uncertainty, or model error. For example, consider the standard prediction error form in (Ljung, 1987),

$$e = H^{-1}(y - Pu), \qquad P, H \in \mathbf{LTI} \tag{27}$$

If $v = u$, then Δ represents *additive* model error, *i.e.*, the uncertainty model set is given by,

$$\mathbf{J}(\sigma, \delta) = \{\ y, u \ \mid y = Pu + H\Delta u + Hw, \quad w \in \mathbf{W}(\sigma), \quad \Delta \in \mathbf{\Delta}(\delta)\ \}$$

If $v = Gu$, then Δ represents *multiplicative* model error, *i.e.*, the uncertainty model set becomes,

$$\mathbf{J}(\sigma, \delta) = \{\ y, u \ \mid y = Pu + H\Delta Gu + Hw, \quad w \in \mathbf{W}(\sigma), \quad \Delta \in \mathbf{\Delta}(\delta)\ \}$$

There are clearly many variations one could include, *e.g.*, combinations of additive and multiplicative model errors, co-prime factor uncertainty, and so on, ultimately leading to the uncertainty structures descibed by the more inclusive linear fractional representation familiar in robust control design, *e.g.*, (Newlin and Smith, 1998). In addition, the error could be obtained from a *parametric* prediction error model with parameters associated with transfer function coefficients which characterize the input/output and disturbance dynamics, *i.e.*,

$$e(\theta) = H(\theta)^{-1}\,(y - P(\theta)u) \tag{28}$$

with $\theta \in \mathbf{\Theta}$, the set of parameters for which the predictor is stable (Ljung, 1987).

4.3 Disturbance Uncertainty

There are many ways to characterize the disturbance set $\mathbf{W}(\sigma)$. For example, consider the following sets of finite sequences:

- **Rms-bounded noise**

$$\mathbf{W}_{\mathrm{rms}}(\sigma) = \left\{\ w \in \mathbf{S}^\ell \ \middle|\ \frac{1}{\ell}\|w\|^2 \le \sigma^2\ \right\} \tag{29}$$

- **Time-domain white noise** (Paganini, 1996)

$$\mathbf{W}_{\mathrm{wht\_time}}(\gamma, m) = \left\{\ w \in \mathbf{S}^\ell\ \mid |r_w(\tau)| \le \gamma r_w(0)\ \right\} \tag{30}$$

  where $r_w(\tau)$ is the auto-correlation of w,

$$r_w(\tau) = \frac{1}{\ell}\sum_{t=1}^{\ell-\tau} w_t w_{t+\tau}, \qquad \tau \in [0, m-1] \le \ell \tag{31}$$

  Observe that $r_w(0) = \|w\|^2/\ell$.

- **Frequency-domain white noise** (Massoumnia and Kosut, 1993)

$$\mathbf{W}_{\mathrm{wht\_freq}}(\sigma, \epsilon, m) = \left\{\ w \in \mathbf{S}^\ell \ \middle|\ |eig\,\{R_m(w)\}\,/\sigma^2 - 1| \le \epsilon\ \right\} \tag{32}$$

  where

$$R_m(w) = \begin{bmatrix} r_w(0) & \cdots & r_w(m-1) \\ \vdots & \ddots & \vdots \\ r_w(m-1) & \cdots & r_w(0) \end{bmatrix} \tag{33}$$

The disturbance set $\mathbf{W}_{\mathrm{rms}}(\sigma)$ is the simplest of choices for deterministically characterizing "noise." The main advantage is that it is a convex set and therefor easy to handle in optimization. However, there are no restrictions preventing correlation with inputs and so the "worst-case" can occur. As shown above, characterizations of deterministic sets which resemble white noise have been examined in (Massoumnia and Kosut, 1993) in the frequency domain with application to system identification and in (Paganini, 1996) for both time and frequency domains with application to robust control. The set $\mathbf{W}_{\mathrm{wht\_time}}(\gamma, m)$ is essentially one of the standard white noise test where γ is chosen from χ^2 distribution tables; m is the *lag window* used to smooth the correlation function. The set $\mathbf{W}_{\mathrm{wht\_freq}}(\sigma, \epsilon, m)$ is shown in (Massoumnia and Kosut, 1993) to also be useful for white noise testing; m again is the lag window, σ^2 is the rms-level of w and hence, the average level of the spectrum of w, and $\epsilon \in (0, 1)$ determines the "flatness" of the spectrum. Clearly these latter sets do preserve the character of white noise, but they are not convex. However, they are no worse than quadratic and so may be quite amenable to conjugate-gradient methods of optimization. The work reported in (Kruger and Poolla, 1998) shows a two-step procedure involving a Kalman filter for unfalsifiying stochastic disturbance signals.

4.4 Gain-Bounded Dynamic Uncertainty

Uncertain dynamics can also be characterized in a number of ways. Consider the following gain-bounded, time-invariant (**TI**) dynamic uncertainty sets:

- **Linear (LTI)**

$$\mathbf{\Delta}_{\mathbf{L}TI}(\delta) = \{\ \Delta \in \mathbf{LTI}\ \mid \|\Delta v\|_{rms} \leq \delta \|v\|_{rms},\ \ \forall \|v\|_{rms} < \infty\ \} \tag{34}$$

  Since $\Delta \in \mathbf{LTI}$, the gain bound condition is equivalent to the frequency domain bound:

$$|\Delta(e^{j\omega})| \leq \delta, \quad \omega \in [-\pi, \pi] \tag{35}$$

- **Incrementally nonlinear (INTI)**

$$\mathbf{\Delta}_{\mathrm{INTI}} = \{\ \Delta \in \mathbf{TI}\ \mid \|\Delta v_1 - \Delta v_2\|_{rms} \leq \delta \|v_1 - v_2\|_{rms},\ \ \forall \|v_1\|_{rms}, \|v_2\|_{rms} < \infty\ \} \tag{36}$$

- **Nonlinear (NTI)**

$$\mathbf{\Delta}_{\mathrm{NTI}} = \{\ \Delta \in \mathbf{TI}\ \mid \|\Delta v\|_{rms} \leq \delta \|v\|_{rms},\ \ \forall \|v\|_{\mathrm{rms}} < \infty\ \} \tag{37}$$

4.5 Unfalsification

Consequences of unfalsification are summarized in the following.

(i) **Finite-Data Test**

Given data sequences $e, v \in \mathbf{S}^\ell$, there exists a sequence $w \in \mathbf{S}^\ell$ and a causal system Δ such that,

$$e_t = w_t + (\Delta v)_t, \quad t \in [1, \ell] \tag{38}$$

with $w \in \mathbf{W}_{\mathrm{rms}}(\sigma)$ if and only if

$$\frac{1}{\ell}\|w\|^2 \leq \sigma^2 \tag{39}$$

and such that:

- $\Delta \in \mathbf{\Delta}_{\mathbf{LTI}}(\delta)$ *if and only if,*

$$\mathcal{T}\{e-w\}^T \mathcal{T}\{e-w\} - \delta^2 \mathcal{T}\{v\}^T \mathcal{T}\{v\} \leq 0 \tag{40}$$

with $(\mathcal{T}\{e\}, \mathcal{T}\{v\}, \mathcal{T}\{w\})$ *the* $\ell \times \ell$ *Toeplitz matrices formed from the sequences* (e, v, w), *respectively, e.g.,*

$$\mathcal{T}\{e\} = \begin{bmatrix} e_1 & 0 & \cdots & 0 \\ e_2 & e_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_\ell & e_{\ell-1} & \cdots & e_1 \end{bmatrix}$$

- $\Delta \in \mathbf{\Delta}_{\mathbf{INTI}}(\delta)$ *if and only if,* $\forall m - n = 0 : \ell$ *and* $\forall t \in [1, \ell]$,

$$\|(z^n - z^m)(e-w)\|_{\mathbf{L}_2[1,t]} \leq \delta \|(z^n - z^m)v\|_{\mathbf{L}_2[1,t]} \tag{41}$$

where z^k *is the k-forward shift operator, i.e., if* $x = \{\ x_1, x_2, \dots\ \}$ *then* $z^k x = \{\ 0, \dots, 0, x_1, x_2, \dots\ \}$ *with k-zeros.*

- $\Delta \in \mathbf{\Delta}_{\mathbf{NTI}}(\delta)$ *if and only if* $\forall t \in [1, \ell]$,

$$\|e - w\|_{\mathbf{L}_2[1,t]} \leq \delta \|v\|_{\mathbf{L}_2[1,t]} \tag{42}$$

(ii) **Uncertainty Tradeoff**

The (σ, δ) *boundary between falsified and unfalsified uncertainty models for a given finite data set with* $w \in \mathbf{W}_{\mathrm{rms}}$ *and* $\Delta \in \mathbf{\Delta}_\mu(\delta)$, $\mu = \mathbf{LTI},\ \mathbf{INTI},\ \mathbf{NTI}$ *is determined by solving:*

$$\sigma_\mu(\delta) = \min\left\{\ \ell^{-1}\|e - \Delta v\|_{\mathbf{L}_2[1,\ell]}\ \mid \Delta \in \mathbf{\Delta}_\mu(\delta)\ \right\}, \quad 0 \leq \delta \leq \delta_\mu \tag{43}$$

with

$$\delta_\mu = \min\{\ \delta\ \mid e_t = (\Delta v)_t,\ t \in [1, \ell],\ \Delta \in \mathbf{\Delta}_\mu(\delta)\ \} \tag{44}$$

(iii) **Nesting**

For all $\delta \geq 0$,

$$\sigma_{\mathbf{NTI}}(\delta) \leq \sigma_{\mathbf{INTI}}(\delta) \leq \sigma_{\mathbf{LTI}}(\delta) \tag{45}$$

and

$$\delta_{\mathbf{NTI}} < \delta_{\mathbf{INTI}} < \delta_{\mathbf{LTI}} \tag{46}$$

Comments

(1) The results in part (i) for $\Delta \in \mathbf{\Delta}_{\mathbf{LTI}}(\delta)$ and the necessity for $\Delta \in \mathbf{\Delta}_{\mathbf{NTI}}(\delta)$ (which is the same as the necessary and sufficient conditions for gain-bounded linear-time-varying (LTV) systems) is found in (Poolla *et al.*, 1992). Proof of the remaining results in (i) can be found in (Kosut and Anderson, 1997).

(2) The tradeoff and nesting results follow from convexity of the uncertainty sets. All the results can be extended when the error is formed from the ARX parametric prediction error model with efficient computations using LMIs, (Boyd *et al.*, 1994). Output error and other linear fractional parameter forms are not convex sets, and the nesting and tradeoff results are thus not guaranteed. Details are in (Kosut, 1995, 1996; Kosut and Anderson, 1997).

5 Controller Unfalsification

In a series of papers by Safonov *et al.*(see (Safonov and Tsao, 1997) and the references therein), it is shown how to *directly* falsify a candidate controller before it is implemented. The procedure for controller unfalsification is essentially the same as that for uncertainty model unfalsification, but applied to the closed-loop specification. Specifically, the closed-loop specification (15) can be viewed as the uncertainty model set,

$$\mathbf{J}(\rho) = \{ \ y, r \ \mid \|y - T_{\rm ref}\ r\|_{rms} \leq \rho \|r\|_{rms} \ , \quad \forall \|r\|_{rms} < \infty \ \} \tag{47}$$

The goal is to adjust the parameters $\alpha \in \mathbf{R}^q$ such that the controller,

$$u = C(\alpha)(r - y) \tag{48}$$

makes ρ as small as possible. From the previous discussion on unfalsification of uncertainty models, without any further assumptions about the plant system, the specification set is equivalently expressed as,

$$\mathbf{J}(\rho) = \{ \ y, r \ \mid y - T_{\rm ref}\ r = \Delta r, \quad \Delta \in \mathbf{\Delta_{NTI}}(\rho) \ \} \tag{49}$$

We could *impose* the additional assumption that the closed-loop system is LTI. Thus, setting $\Delta \in \mathbf{\Delta_{LTI}}(\rho)$, the specification becomes the LTI uncertainty set,

$$\mathbf{J}(\rho) = \left\{ \ y, r \ \middle| \ y - T_{\rm ref}\ r = \Delta r, \quad \|\Delta\|_{\mathbf{H}_\infty} \leq \rho \ \right\} \tag{50}$$

Although it may be of interest, and even important, to postulate both NTI and LTI uncertainty sets, for the remainder of this paper assume that $\Delta \in \mathbf{\Delta_{NTI}}(\rho)$, and hence, (49) is the closed-loop specification. Further discussion along these lines can be found in (Kosut and Anderson, 1997).

Let $(y^\ell, u^\ell, r^\ell) \in \mathbf{S}^\ell \times \mathbf{S}^\ell \times \mathbf{S}^\ell$ denote the measured data for $t \in [1, \ell]$, where ℓ is essentially the current time. Define the corresponding *measured* performance error by,

$$\rho^\ell_{\rm meas} = \min \left\{ \ \rho \ \middle| \ \varepsilon^\ell = \Delta r^\ell, \quad \Delta \in \mathbf{\Delta_{NTI}}(\rho) \ \right\} \tag{51}$$

where

$$\varepsilon^i = y^\ell - T_{\rm ref}\ r^\ell \tag{52}$$

¿From (42) it follows that the measured performance can be computes directly from,

$$\rho^\ell_{\rm meas} = \min \left\{ \ \rho \ \middle| \ \|\varepsilon^\ell\|_{\mathbf{L}_2[1,t]} \leq \rho \|r^\ell\|_{\mathbf{L}_2[1,t]} \ , \quad t \in [1, \ell] \ \right\} \tag{53}$$

This is a measure of the performance of the already implemented controller, which may have already switched (adapted) several times. The question arises, could the existing data record be informative about an untried *candidate controller*, say $C(\alpha)$, with respect to the measured performance? To answer the question, consider, the following "thought experiment:"

> *If a candidiate controller $C(\alpha)$ had produced the measured plant output/input data $(y^\ell, u^\ell) \in \mathbf{S}^\ell \times \mathbf{S}^\ell$, then the reference input would have been the sequence $r^\ell(\alpha)$ satisfying*
>
> $$u^\ell = C(\alpha)\left(r^\ell(\alpha) - y^\ell\right) \tag{54}$$
>
> *Assuming the indicated inverse exists,*
>
> $$r^\ell(\alpha) = y^\ell + C(\alpha)^{-1} u^\ell \tag{55}$$

Thus, the error, $\varepsilon^\ell(\alpha)$, would have been,

$$\begin{aligned} \varepsilon^\ell(\alpha) &= y^\ell - T_{\text{ref}}\; r^\ell(\alpha) \\ &= S_{\text{ref}}\; y^\ell - T_{\text{ref}} C(\alpha)^{-1} u^\ell \end{aligned} \tag{56}$$

with

$$S_{\text{ref}} = 1 - T_{\text{ref}}$$

Hence, the set of controller parameters that achieve a performance level, ρ, would have been,

$$\begin{aligned} \mathbf{A}^\ell(\rho) &= \left\{ \alpha \in \mathbf{R}^q \;\middle|\; \varepsilon^\ell(\alpha) = \Delta r^\ell(\alpha), \quad \Delta \in \mathbf{\Delta_{NTI}}(\rho) \right\} \\ &= \left\{ \alpha \in \mathbf{R}^q \;\middle|\; \|\varepsilon^\ell(\alpha)\|_{\mathbf{L}_2[1,t]} \le \rho \|r^\ell(\alpha)\|_{\mathbf{L}_2[1,t]}\;, \quad t \in [1,\ell] \right\} \end{aligned} \tag{57}$$

By this argument, $\mathbf{A}^\ell(\rho)$ is the the set of all controller parameters which are unfalsified by the available data, with respect to performance level ρ. This set can also be expressed as an intersection of sets, *i.e.*,

$$\mathbf{A}^\ell(\rho) = \bigcap_{t\in[1,\ell]} \left\{ \alpha \in \mathbf{R}^q \;\middle|\; \|\varepsilon^\ell(\alpha)\|_{\mathbf{L}_2[1,t]} \le \rho \|r^\ell(\alpha)\|_{\mathbf{L}_2[1,t]} \right\} \tag{58}$$

Hence, as we record more data, *i.e.*, as ℓ increases, the unfalsified parameter set, $\mathbf{A}^\ell(\rho)$, can only get smaller. The *falsified* parameter set, its complement, therefor, monotonically increases.

It remains to choose a controller parameter to implement in the next sample, $\ell + 1$. An aggressive choice is one which produces the smallest ρ, *i.e.*,

$$(\alpha^\ell_{\text{un}f}, \rho^\ell_{\text{un}f}) = \arg\min \left\{ \rho \;\middle|\; \alpha \in \mathbf{A}^\ell_{\text{un}f}(\rho) \right\} \tag{59}$$

A more cautious choice, reflecting the distribution of elements in $\mathbf{A}^\ell_{\text{un}f}$, is the average, or the geometric center of the set, *i.e.*,

$$(\alpha^\ell_{\text{un}f}, \rho^\ell_{\text{un}f}) = \arg avg \left\{ \rho \;\middle|\; \alpha \in \mathbf{A}^\ell_{\text{un}f}(\rho) \right\} \tag{60}$$

No matter the choice, the control parameters are updated whenever the unfalsified performance level, $\rho^\ell_{\text{un}f}$, is smaller than the measured performance, ρ^ℓ_{meas}. Thus, the controller parameter update rule is,

$$\alpha^{\ell+1} = \begin{cases} \alpha^\ell_{\text{un}f}, & \rho^\ell_{\text{un}f} < \rho^\ell_{\text{meas}} \\ \alpha^\ell, & \rho^\ell_{\text{un}f} \ge \rho^\ell_{\text{meas}} \end{cases} \tag{61}$$

Clearly if the exogenous inputs are not sufficiently rich from iteration to iteration, then it is likely that the control will not switch.

As discussed in (Safonov and Tsao, 1997), there are several advantages to this data-based control design approach:

1. The approach is nonconservative; i.e., it gives "if and only if" conditions on the candidate controller $C(\alpha)$ to be unfalsified.

2. The unfalsified set of candidate controllers is determined from past data only – *no candidate controller is implemented if it is falsified*, and the test is applied *without actually implementing the candidate controller* $C(\alpha)$.

   Equally important, if the test fails, those candidate controllers, $C(\alpha)$, which have been falsified, again without implementation, *can all be discarded from any future consideration.*

3. The test for controller unfalsification is "plant-model free." No plant model is needed to test its conditions. It depends only on the data, the controller, and the specification.

4. The data which falsifies a controller may be open loop data or data generated by some other control law which may or may not be in the parametric set.

5. Controller falsification implies falsification of any underlying uncertainty model for the plant model, based on the same data, which would have resulted in the same controller. The converse, however, is not true: a falsified uncertainty model of the plant does not imply falsification of a controller based on this falsified uncertainty model. As a result, using the same data set, direct controller unfalsification can produce less conservative control than plant unfalsification followed by robust control design.

Computional Issues

Solving for ρ_{meas} is clearly easy. However, the optimization problem for $(\rho_{\mathrm{un}f}, \alpha_{\mathrm{un}f})$ has two difficulties. First, it is not in general convex, hence, there is no guaranty of finding the optimum. For PID and/or lead-lag type controllers, which have a small number of parameters, a combinatorial search is very effective as has been shown in (Safonov and Tsao, 1997) and will be demonstrated in the example in section 7. Difficulties with the optimization are to be expected, because in essence, we are trying to solve the fixed-order control design problem, which is generically hard even when the plant is known. In the case here, the plant is not known, and the problem is compounded further by using data! But, as in the output error identification problem, there are some instances where there are no local minima for parameters restricted to a region where a certain transfer function is passive (Ljung, 1987, Ch.10,p.301). Even if this could be applied here, obviously more assumptions about the plant are required.

The second issue is that the problem size increases as time goes on because more data is recorded, effectively adding more constraints. We offer an approximate solution to this problem in the next section on iterative unfalsification. Essentially the data is only recorded over a fixed length window which slides along with current time.

Other methods for dealing with these computational difficulties are presented in (Woodley *et al.*, 1998, 1999). A reformulation of the performance specification allows the problem to be cast as a convex optimization. Dealing with the ever increasing problem size is addressed by developing recursive methods which provide outer and inner ellipsoid bounds on earlier data, hence compressing the earlier data into matrices on the order of the parameters. This is akin to least-squares estimation which compresses prior information into parameter sized covariance matrices.

6 Iterative Controller Unfalsification

In this section the unfalsification paradigm is used to develop an iterative direct adaptive controller.

During the i-th iteration (data collection period) the controller is held fixed at $C(\alpha^i) \in \mathbf{LTI}$, *i.e.*,

$$u = C(\alpha^i)(r - y) \tag{62}$$

Suppose each data collection period contains ℓ samples, where $(y^i, u^i, r^i) \in \mathbf{S}^\ell \times \mathbf{S}^\ell \times \mathbf{S}^\ell$ is the data measured. Define the corresponding *measured* performance error by,

$$\rho^i_{\mathrm{meas}} = \min \left\{ \rho \;\middle|\; \varepsilon^i = \Delta r^i, \quad \Delta \in \mathbf{\Delta}_{\mathbf{NTI}}(\rho) \right\} \tag{63}$$

where

$$\varepsilon^i = y^i - T_{\mathrm{ref}}\; r^i \tag{64}$$

> **Caveat emptor** – *This definition of measured performance for the i-th data collection period is reasonable only if the period data length, ℓ, is sufficiently large so as to make negligible any effects due to controller adjustments or exogenous disturbances in previous periods.* **Assume from now on that this is the case.**

¿From (42) it follows that the measured performance is given by,

$$\rho^i_{\mathrm{meas}} = \min \left\{ \rho \;\middle|\; \|\varepsilon^i\|_{\mathbf{L}_2[1,t]} \leq \rho \|r^i\|_{\mathbf{L}_2[1,t]}\;, \quad t \in [1, \ell] \right\} \tag{65}$$

Based soley on the data collected in the i-th period, the set of unfalsified controller parameters that achieve a performance level, ρ, is,

$$\mathbf{A}^i_\ell(\rho) = \left\{ \alpha \in \mathbf{R}^q \;\middle|\; \|\varepsilon^i(\alpha)\|_{\mathbf{L}_2[1,t]} \leq \rho \|r^i(\alpha)\|_{\mathbf{L}_2[1,t]}\;, \quad t \in [1, \ell] \right\} \tag{66}$$

It folows that the set of all controller parameters which are unfalsified, with respect to performance level ρ, up to and including the i-th interval, is the intersection of these sets, *i.e.*,

$$\mathbf{A}^i(\rho) = \bigcap_{j \in [1,i]} \mathbf{A}^i_\ell(\rho) \tag{67}$$

As before, it remains to choose a controller to implement in the next iteration. The aggressive choice produces the smallest ρ, *i.e.*,

$$(\alpha^i_{unf}, \rho^i_{unf}) = \arg\min \left\{ \rho \;\middle|\; \alpha \in \mathbf{A}^i(\rho) \right\} \tag{68}$$

whereas the cautious choice, reflecting the distribution of elements in $\mathbf{A}^i$, is the average, or the geometric center of the set, *i.e.*,

$$(\alpha^i_{unf}, \rho^i_{unf}) = \arg avg \left\{ \rho \;\middle|\; \alpha \in \mathbf{A}^i_{unf}(\rho) \right\} \tag{69}$$

We then propose to update the control parameters whenever the unfalsified performance level, ρ^i_{unf}, is smaller than the best measured performance,

$$\rho^k_{\mathrm{meas}} = \min_{j \in [1, i]} \rho^j_{\mathrm{meas}} \tag{70}$$

If not, then control is returned to $C(\alpha^k)$, the controller which produced the best measured performance. Thus, the controller parameter update rule is,

$$\alpha^{i+1} = \begin{cases} \alpha^i_{unf}, & \rho^i_{unf} < \rho^k_{\mathrm{meas}} \\ \\ \alpha^k, & \rho^i_{unf} \geq \rho^k_{\mathrm{meas}} \end{cases} \tag{71}$$

This is a slightly different procedure than in the previous "one-step-at-a-time" case. Here, because the control is held fixed at $C(\alpha^i)$ for a long time, we have (we assume) a good reading of the performance with this control. In the previous formualtion, the control can switch at every instant when new data is acquired.

7 Simulation Example: PI Control

The iterative procedure is simulated with the following nonlinear plant system:

$$
\begin{aligned}
y &= G\Big(v + N(u)\Big) \\
G &= \frac{.1z^{-1}}{1 - .4z^{-1}} \\
N(u) &= \begin{cases} 0, & |u| \leq d \\ u - d, & u > 0 \\ u + d, & u < 0 \end{cases} \\
\|v\|_{rms} &\leq \sigma
\end{aligned}
\tag{72}
$$

The plant system is thus a linear system, G, driven by an RMS-bounded disturbance, v, and controlled through a deadband nonlinearity, $N(\cdot)$, with deadband of size d. The control is given by the PI control,

$$u = C(\alpha)(r - y)$$

where

$$C(\alpha) = \alpha_P + \frac{\alpha_I z^{-1}}{1 - z^{-1}} \tag{73}$$

The reference system is

$$T_{\text{ref}} = \frac{(1-a)z^{-1}}{1 - az^{-1}}, \quad a = \exp(-2\pi f_{\text{ref}})$$

Figures 2-5 show the results of the simulations. Each figure has two rows and four columns. Each row corresponds to a different bandwidth (f_{ref}) of the reference system. The rows are as follows:

- row 1: the initial output response, before adaptation, compared to the reference system output.
- row 2: the final output response, after adaptation, compared to the reference system output.
- row 3: the per iteration values of ρ^i_{meas}, $\rho^i_{\text{un}f}$, and the $\mathbf{H}_\infty$-norm of the error between the linearized system and the reference system.
- row 4: the PI gains per iteration.

The simulations were performed under the following conditions:

- The control was initialized as the low gain integrator:

$$C = \frac{.01z^{-1}}{1 - z^{-1}}$$

- A single repeating cycle of the reference input is given by:

$$r = \begin{cases} 1 & t = 1 : 200 \\ -1 & t = 201 : 400 \\ 0 & t = 401 : 600 \end{cases}$$

- There are two cycles of 4 iterations each of this reference.
  - During cylce 1, the reference system bandwidth

    $$f_{\text{ref}} = .005 \text{ hz}$$

    The results are shown in column 1 of all the figures.
  - During cylce 2, the reference system bandwidth

    $$f_{\text{ref}} = .05 \text{ hz}$$

    The results are shown in column 2 of all the figures.
- The deadband width (d) and RMS-disturbance level (σ) were set as follows:

figure 2	$d = 0$	$\sigma = 0$
figure 3	$d = 0$	$\sigma = .1$
figure 4	$d = 1.5$	$\sigma = 0$
figure 5	$d = 1.5$	$\sigma = .1$

We see in all cases that the iterative unfalsified adaptation works very well despite some extreme variations and no prior knowledge about the plant system. Although not shown, the intermedite time responses are not very much different than the final responses (after 4 iterations).

8 Convergence vs. Unfalsification

There are several intriguing aspects of unfalsification as applied to direct adaptive control. First, existing data can be used to falsify an experiment you would like to perform, but cannot. Secondly, controllers can be proven to be unable to meet the closed-loop performance specification without being implemented. This reduces the set of unfalsified controllers, and this reduction is non-conservative. But what about convergence? The answer to this could be: why convergence? If adaptation is meant to be used in the face of highly uncertain systems, which may exhibit large variations over time and operating conditions, there is no convergence. We just keep throwing away bad controllers. A well respected American football player, when asked why he was such a good defender against the run, replied, "I just keep knock'n 'em down 'till I get to the one with the ball."

References

Anderson, B. D. O., R. R. Bitmead, C. R. Johnson, Jr., P. V. Kokotovic, R. L. Kosut, I. M. Y. Mareels, L. Praly, and B. D. Riedle, (1986). *Stability of Adaptive Systems: Passivity and Averaging Analysis*, MIT Press.

Åström, K. J. and B. Wittenmark (1995). *Adaptive Control*, Addison-Wesley.

Åström, K. J. and J. Nilsson (1994). "Analysis of a scheme for iterated identification and control," *Proc. IFAC Symp. on Identification*, pp. 171–176, Copenhagen, Denmark.

Åström, K. J. (1993). "Matching criteria for control and identification," *Proc. 2nd European Control Conference*, Groningen, Holland, June 1993, pp. 248–251.

Boyd, S., L. El Ghaoui, E. Feron, and V. Balakrishnan (1994). *Linear Matrix Inequalities in System and Control Theory*, SIAM Studies in Applied Mathematics.

Foias, C. and A. E. Frazho (1990). *The Commutant Lifting Approach to Interpolation Problems*, Birkhauser-Verlag.

Gevers, M. (1993). "Towards a joint design of identification and control," *Essays on Control: Perspectives in the Theory and its Applications*, editors: H.L. Trentelman and J.C. Willems, pp. 111-151, Birkhauser, Boston, MA.

Goodwin, G. C., M. Gevers, and B. Ninness (1992). "Quantifying the error in estimated transfer functions with applications to model order selection," *IEEE Trans. on Automatic Control*, **37**, no. 7, pp. 913–928.

Grenander, U. and G. Szebo (1958). *Toeplitz Forms and Their Applications*, Chelsea Publishing Co., New York.

Hjalmarsson, H., S. Gunnarsson, and M. Gevers (1994). "A convergent iterative restricted complexity control design scheme," *Proc. 1994 CDC*, Orlando, Florida, Dec. 1994, pp. 1735–1740.

Hjalmarsson, H., S. Gunnarsson, and M. Gevers (1995). "Optimality and sub-optimality of iterative identification and control design schemes," *Proc. 1995 ACC*, Seattle, Washington, June 1995, pp. 2559–2563.

Kosut, R. L., G. C. Goodwin, and M. P. Polis, *eds.* (1992) "Special Issue on System Identification for Robust Control Design," *IEEE Trans. on Automatic Control*, **37**, no. 7, pp. 899–1008.

Kosut, R. L. (1995). "Uncertainty model unfalsification: a system identification paradigm compatible with robust control design," *Proc. 1995 CDC*, New Orleans, LA, Dec. 1995.

Kosut, R. L. (1996). "Iterative adaptive robust control via uncertainty model unfalsification," *Proc. 1996 IFAC World Congress*, June, 1996, San Francisco, CA.

Kosut, R. L. and B. D. O. Anderson (1997). "Uncertainty model unfalsification," *Proc. 1997 CDC*, San Diego, CA, Dec, 1997.

Kruger, M. V. P. and K. Poolla (1998). "Validation of uncertainty models in the presence of stochastic disturbance signals," *Selected Topics in Modelling and Control*, **11**, pp. 1–8, Delft University Press, Dec. 1998.

Lee, W. S., B. D. O. Anderson, R. L. Kosut, and I. M. Y. Mareels (1993). "A new approach to adaptive robust control," *Int. J. of Adaptive Control and Signal Processing*, **7**, pp. 183–211.

Lee, W. S., B. D. O. Anderson, I. M. Y. Mareels, and R. L. Kosut (1995). "On some key issues in the windsurfer approach to robust adaptive control", *Automatica*, **31**, no. 11, pp. 1619–1636.

Livestone, M. M., M. A. Dahleh, and J. A. Farrell (1994). "A framework for robust control based model invalidation," *Proc. 1994 ACC*, Baltimore, MD, June 1994, pp. 3017–3020.

Ljung, L. (1987). *System Identification: Theory for the User*, Prentice-Hall.

Massoumnia, M. and R. L. Kosut (1993). "A family of norms for system identification problems," *Proc. 1993 ACC*, San Francisco, CA, June 1993, and *IEEE Trans. Aut. Control*, **39**, no. 5, pp. 1027–1031, May 1994.

Newlin, M. P. and R. S. Smith (1998). "A generalization of the structured singular value and its application to model validation," *IEEE Trans. Aut. Contr.*, **43**, no. 7, pp. 901–907.

Paganini, F. (1996). "A set-based approach for white noise modeling," *IEEE Trans. Aut. Contr.*, **41**, no. 10, pp. 1453–1465.

Phillips, S. M., R. L. Kosut, G. F. Franklin (1988). "An averaging analysis of discrete-time indirect adaptive control", *Proc. 1988 American Control Conference*, Atlanta, GA.

Poolla, K., P. Khargonnekar, A. Tikku, J. Krause, and K. Nagpal (1992). "A time-domain approach to model validation", *Proc. 1992 ACC*, Chicago, IL, June 1992, and *IEEE Trans. Aut. Contr.*, **39**, no. 5, pp. 951–959, May 1994.

Safonov, M. G. and T. C. Tsao (1997). "The unfalsified control concept and learning," *IEEE Trans. Aut. Contr.*, **42**, no. 6, pp. 843–847.

Smith, R. S. and J. C. Doyle (1989). "Model validation – a connection between robust control and identification," *Proc. 1989 ACC*, pp. 1435–1440, June 1989, and *IEEE Trans. Aut. Contr.*, **37**, pp. 942–952, July 1992.

Woodley, B. R., R. L. Kosut, and J. P. How (1998). "Uncertainty model unfalsification with simulation," *Proc. 1998 ACC*, Philadelphia, PA, June 24-26, 1998, pp. 2754–2755.

Woodley, B. R., J. P. How, and R. L. Kosut (1999). "Direct unfalsified controller design - solution via convex optimization," submitted 1999 ACC, San Diego, CA, June 2-4, 1999.

Zang, Z., R. R. Bitmead, and M. Gevers (1991). "$\mathbf{H}_2$ iterative model refinement and control robustness enhancement," *Proc. 30th IEEE CDC*, Brighton, UK, Dec. 1991, pp. 279–284.

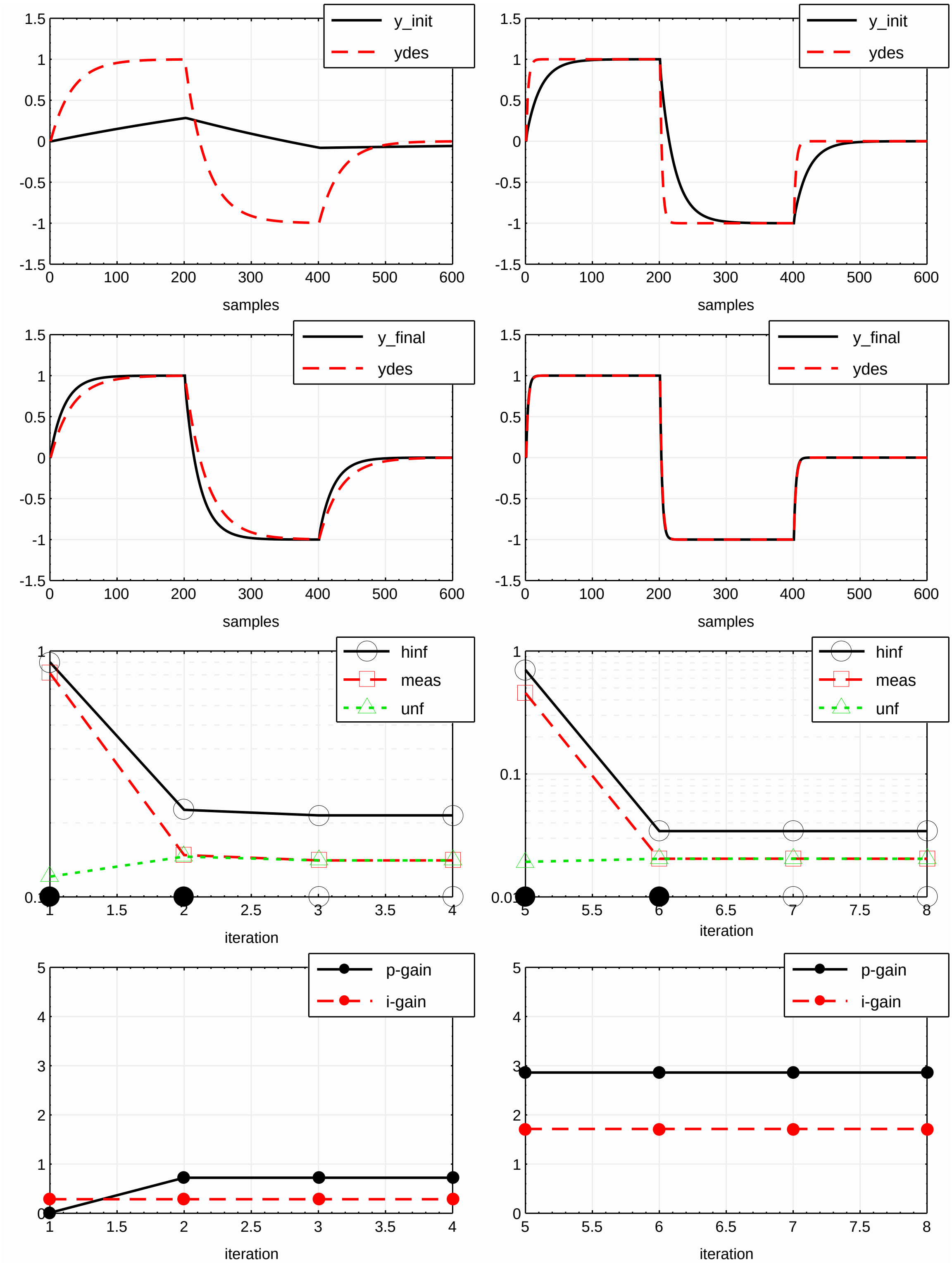

Figure 2: Deadband width $d = 0$; RMS-disturbance $\sigma = 0$; reference bandwidth $f_{\text{ref}} = .005$ hz (col 1), $f_{\text{ref}} = .05$ hz (col 2).

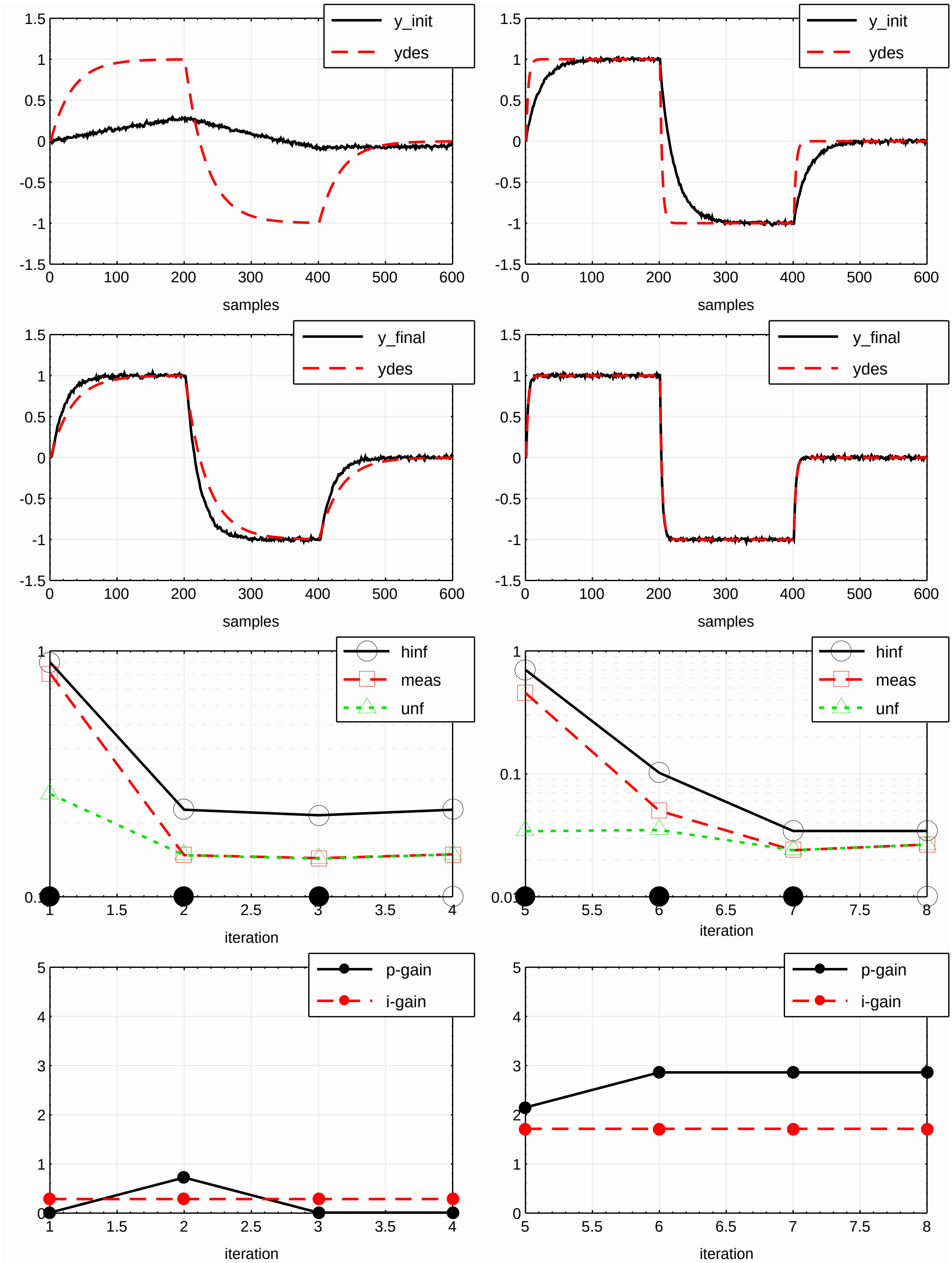

Figure 3: Deadband width $d = 0$; RMS-disturbance $\sigma = .1$; reference bandwidth $f_{\text{ref}} = .005$ hz (col 1), $f_{\text{ref}} = .05$ hz (col 2).

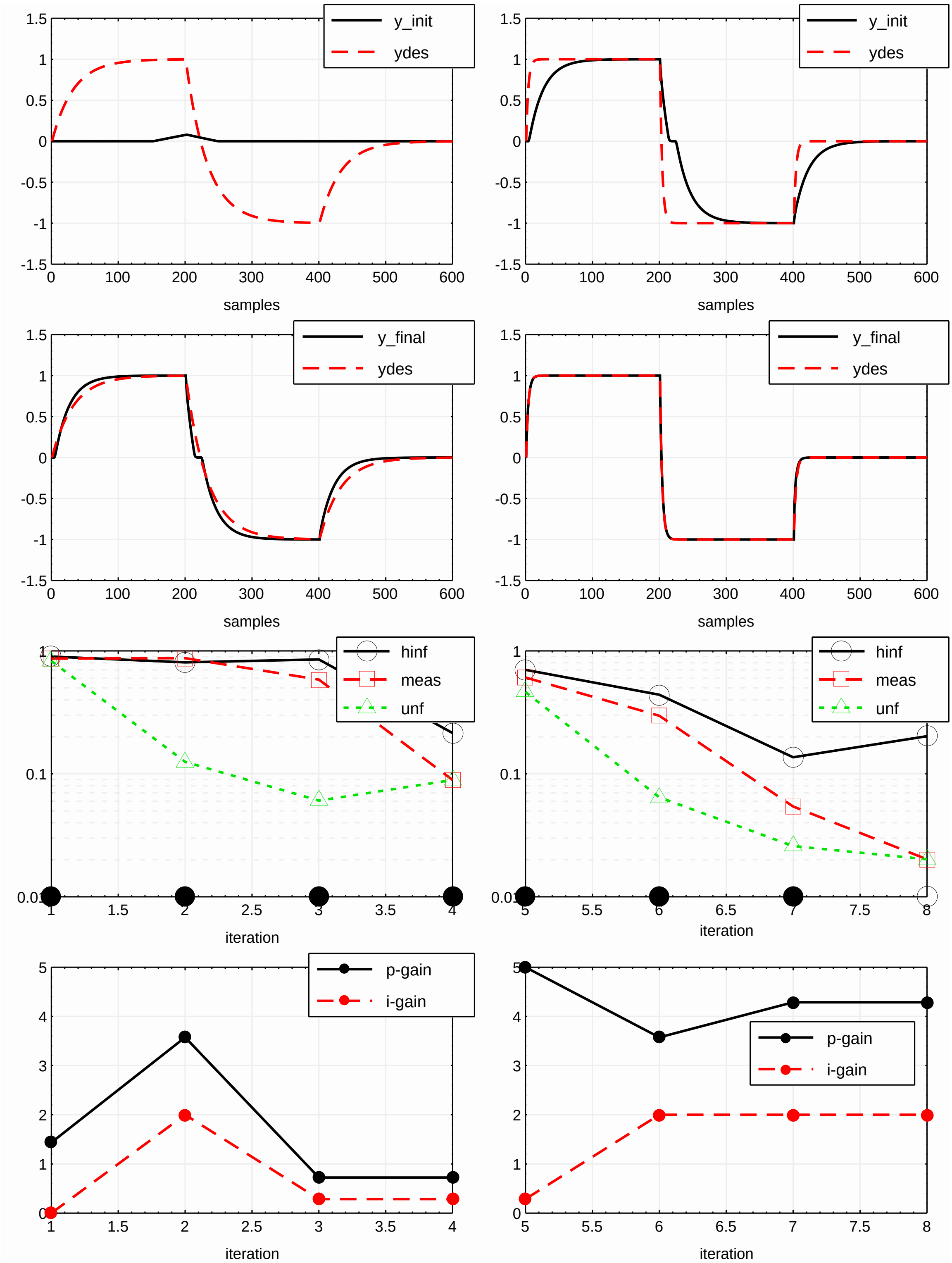

Figure 4: Deadband width $d = 1.5$; RMS-disturbance $\sigma = 0$; reference bandwidth $f_{\text{ref}} = .005$ hz (col 1), $f_{\text{ref}} = .05$ hz (col 2).

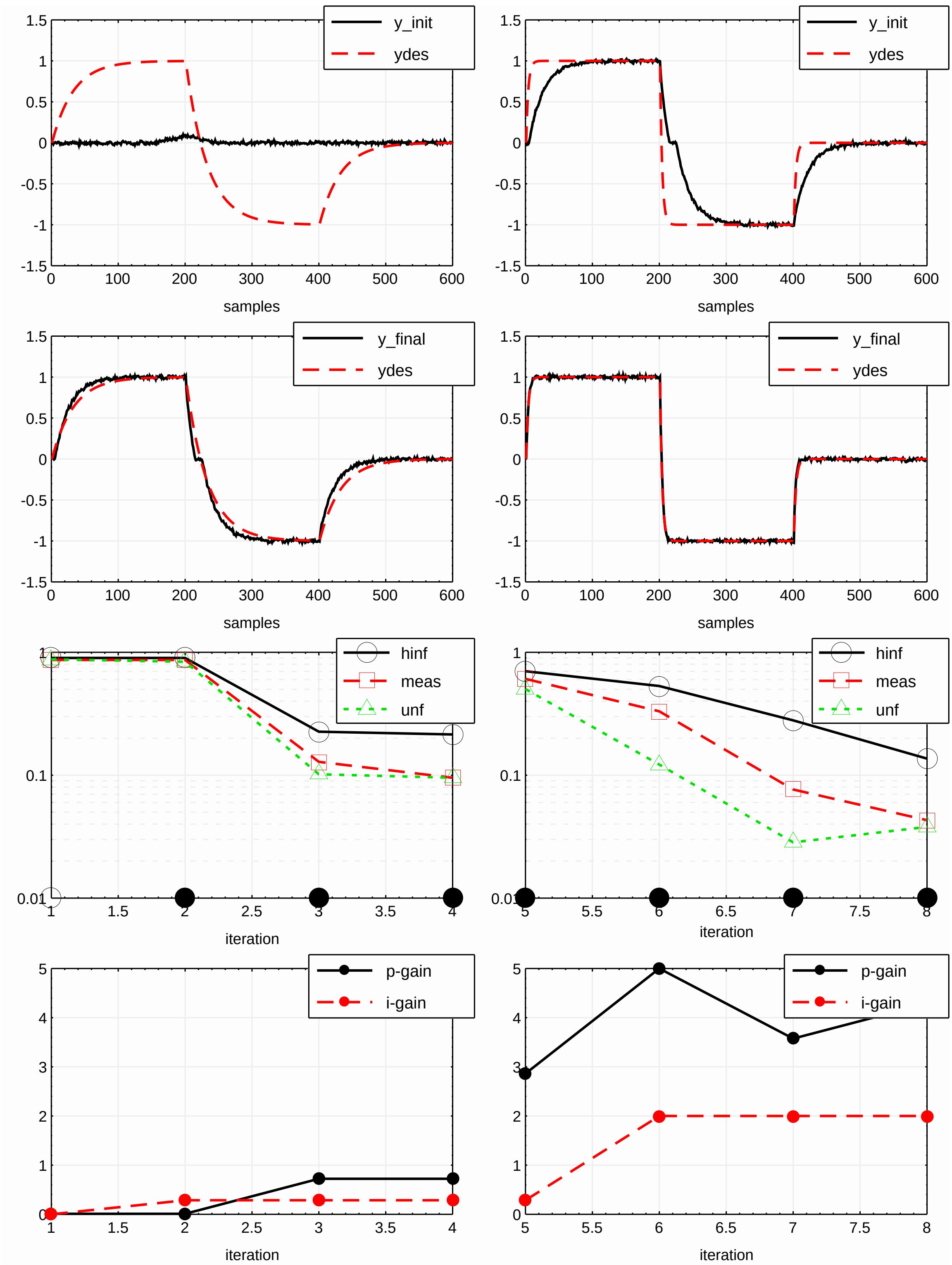

Figure 5: Deadband width $d = 1.5$; RMS-disturbance $\sigma = .1$; reference bandwidth $f_{\mathrm{ref}} = .005$ hz (col 1), $f_{\mathrm{ref}} = .05$ hz (col 2).