Proceedings of the 7th Mediterranean Conference on Control and Automation (MED99) Haifa, Israel - June 28-30, 1999

A Linear Matrix Inequality Approach towards H_∞ Control of Descriptor Systems

A. Rehm† and F. Allgöwer‡

† Institut für Systemdynamik und Regelungstechnik, Universität Stuttgart,
Pfaffenwaldring 9, D-70550 Stuttgart, Germany.
Email: `ansgar@isr.uni-stuttgart.de`

‡ Institut für Automatik, ETH Zürich, CH-8092 Zürich, Switzerland.
Email: `allgower@aut.ee.ethz.ch`

Abstract

In this paper H_∞ control of high index and non-regular linear descriptor systems is addressed. Based on a generalization of the bounded real lemma (BRL) to index one systems, all controllers solving the H_∞ control problem can be characterized via biaffine matrix inequalities (BMIs). These inequalities imply a certain structure of candidate matrix solutions. Making use of this structure, standard linear algebra tools can be used in order to show the equivalence of the BMI synthesis conditions to a numerically appealing characterization of the solution of the H_∞ control problem via linear matrix inequalities (LMIs). We also address the computation of full- and reduced order controllers.

1 Introduction

Descriptor systems (sometimes also referred to as Differential-algebraic-equation (DAE), singular or semistate systems) describe a broad class of systems which are not only of theoretical interest but also have great practical significance. Models of chemical processes for example typically consist of differential equations describing the dynamic balances of mass and energy while additional algebraic equations account for thermodynamic equilibrium relations, steady-state assumptions, empirical correlations, etc. (Pantelides *et al.*, 1988; Kumar and Daoutidis, 1997). In mechanics descriptor system descriptions, that are typically of index less or equal than three, result from holonomic and non-holonomic constraints (Schüpphaus, 1995). Also in electronics and even in economics descriptor descriptions are frequently encountered (Luenberger, 1979).

Descriptor systems are able to describe a system behavior that cannot be captured by “non-descriptor” systems (i.e. ssystems governed only by differential equations) (Verghese *et al.*, 1981). Therefore index reduction techniques (i.e. reduction of a descriptor description to an ODEs (Pantelides *et al.*, 1988)) necessarily are connected to a loss of information. Due to this fact in recent years much work has been focused on analysis and design techniques for descriptor systems (see (Campbell, 1980; Dai, 1989)). For linear systems many of the standard design techniques for non-descriptor systems have been extended to descriptor systems. Based on a generalization of J-spectral factorization (Green *et al.*, 1990) also H_∞ controller design for descriptor systems was established recently (Takaba *et al.*, 1994). However the approach in (Takaba *et al.*, 1994) is restricted to the so called DGKF assumptions (Doyle *et al.*, 1989). These assumptions, that are rather restrictive for practical applications, were overcome in (Masubuchi *et al.*, 1997) by means of a Riccati inequality approach.

In our paper we present an elementary linear algebra approach to the synthesis problem, which is almost completely based on the equivalence (Skelton *et al.*, 1998)

$$\Pi + PXQ + (PXQ)^{\mathrm{T}} < 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{rcl} P^{\perp}\Pi P^{\perp\mathrm{T}} & < & 0 \\ Q^{\mathrm{T}\perp}\Pi Q^{\mathrm{T}\perp\mathrm{T}} & < & 0 \end{array} \right. \tag{1}$$

for matrices $\Pi = \Pi^{\mathrm{T}} \in I\!R^{n\times n}$, $P \in I\!R^{n\times m}$, $Q \in I\!R^{k\times n}$, $X \in I\!R^{m\times k}$. Here $P^{\perp}$ denotes a matrix of maximal full row rank such that $P^{\perp}P = 0$, i.e. the rows of $P^{\perp}$ represent a basis of the left null space of P. This approach reveals the similarities and differences between H_∞ control of descriptor systems and the "classical" LMI approach towards H_∞ control of non-descriptor systems (Gahinet and Apkarian, 1994; Iwasaki and Skelton, 1994). Especially this approach provides the possibility to discuss the existence of reduced order controllers for descriptor systems.

The paper is structured as follows: Firstly the necessary background on descriptor systems is provided and a characterization of the aim of H_∞ control design for descriptor systems, i.e. an analysis result, is given. Then, in the main part of the paper, the LMI conditions for the existence of a sub-optimal output feedback controller in descriptor form are derived. These conditions are constructive in the sense, that their solution transforms the (nonlinear) analysis conditions for the closed loop system into numerically tractable LMI conditions for the controller matrices.

2 Linear descriptor systems and a generalized version of the bounded real lemma

We consider the descriptor system

$$E\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{w}(t), \quad \boldsymbol{y}(t) = C\boldsymbol{x}(t) \tag{2}$$

with descriptor variable $\boldsymbol{x}(t) \in I\!R^{n_x}$, input variable $\boldsymbol{w}(t) \in I\!R^{n_w}$, output variable $\boldsymbol{y} \in I\!R^{n_y}$, constant quadratic matrices A, E, and constant matrices B, C of compatible dimension.

In contrast to standard linear systems with $E = I$, system (2) with $\mathrm{rank}(E) < n_x$ may have no solution, one solution, or even multiple solutions. In general the solutions exhibit impulsive behavior (i.e. are *generalized* solutions (Doetsch, 1971)) even if the input $\boldsymbol{w}(\cdot)$ is continuous (Dai, 1989). A necessary and sufficient condition for the existence and uniqueness of a solution is, that the pencil $sE - A$ is *regular*, i.e. $\det(sE - A) \not\equiv 0$ (Dai, 1989). Regular descriptor systems are termed *stable* if $\{s|s \in \mathbb{C}, \det(sE - A) = 0\} \subset \mathbb{C}^-$(Dai, 1989). If the pencil $sE - A$ is *singular*, i.e. $\det(sE - A) \not\equiv 0$ it can be shown (Gantmacher, 1959) that the unforced ($\boldsymbol{w}(\cdot) = \mathbf{0}$) descriptor system admits non-trivial solutions to the homogeneous initial value problem. Therefore the following reformulation of the term "internal stability" seems natural:

Definition 2.1 *A descriptor system is said to be* internally stable *if it is regular, stable, and has no impulsive solutions* [1].

Internal stable descriptor systems are system equivalent to asymptotically stable non-descriptor systems (Gantmacher, 1959). Therefore the H_∞ norm of internal stable systems (2) can be defined as the H_∞ norm of the equivalent non-descriptor system. However, analysis of (2) by a two step procedure, namely firstly to establish internal stability and then to compute the H_∞ norm of the associated non-descriptor system, is complicated and furthermore such an procedure cannot be applied to synthesis of descriptor systems since the open loop description may be unstable or even singular. For this reason we derive a simple characterization of internal stable descriptor systems (2) with an H_∞ norm bound:

Proposition 2.1 *The system (2) is internally stable and the transfer matrix T_E with $T_E(s) := C(sE - A)^{-1}B$ is H_∞ norm bounded, i.e. $\|T_E\|_\infty < \gamma$ for a given $\gamma > 0$, if and only if there exists a matrix X such that the matrix inequalities*

$$E^{\mathrm{T}}X = X^{\mathrm{T}}E \geq 0, \quad \mathcal{B}_{[A,B,C]}(\gamma, X) := \begin{bmatrix} A^{\mathrm{T}}X + X^{\mathrm{T}}A & X^{\mathrm{T}}B & C^{\mathrm{T}} \\ B^{\mathrm{T}}X & -\gamma I & 0 \\ C & 0 & -\gamma I \end{bmatrix} < 0 \tag{3}$$

hold true.

Remark. For a given system (2) the inequalities (3) constitute analysis LMIs in X. Due to the (1,1) element in $\mathcal{B}_{[A,B,C]}(\gamma, X)$ the matrix X is always non-singular. With $E = I$ and $E^{\mathrm{T}}X = X^{\mathrm{T}}E \geq 0$ we get $X = X^{\mathrm{T}} > 0$. Therefore Proposition 2.1 contains the bounded real lemma for non-descriptor systems (Scherer, 1990) as special case.

Proof. We only show sufficiency here. The lengthy proof of necessity can be found in (Rehm and Allgöwer, 1998a).

Assume (3) holds true for some matrix X. By means of a Schur complement argument (Boyd *et al.*, 1994) $\mathcal{B}_{[A,B,C]}(\gamma, X) < 0$ is equivalent to

$$\begin{bmatrix} A^{\mathrm{T}}X + X^{\mathrm{T}}A & X^{\mathrm{T}}B \\ B^{\mathrm{T}}X & 0 \end{bmatrix} + \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \frac{1}{\gamma} & 0 \\ 0 & -\gamma I \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} < 0 \tag{4}$$

[1] In view of the Weierstrass canonical form (Gantmacher, 1959) of a descriptor system we frequently use the term "index one system" instead of "regular system without impulsive solution".

The (1,1)- entry in (4) implies $A^{\mathrm{T}}X + X^{\mathrm{T}}A + \frac{1}{\gamma}C^{\mathrm{T}}C < 0$ and due to (3) we get $A^{\mathrm{T}}X + X^{\mathrm{T}}A < 0$, $E^{\mathrm{T}}X = X^{\mathrm{T}}E \geq 0$. These inequalities imply internal stability of (2) (Masubuchi *et al.*, 1997, Lemma 2). Now define $\mathcal{V}(\boldsymbol{\xi}) := \boldsymbol{\xi}^{\mathrm{T}}E^{\mathrm{T}}X\boldsymbol{\xi}$. Differentiation along trajectories of (2) renders

$$\frac{d}{dt}\mathcal{V}(\boldsymbol{\xi}(t)) = \dot{\boldsymbol{\xi}}^{\mathrm{T}}(t)E^{\mathrm{T}}X\boldsymbol{\xi}(t) + \boldsymbol{\xi}^{\mathrm{T}}(t)X^{\mathrm{T}}E\dot{\boldsymbol{\xi}}(t) = \begin{bmatrix}\boldsymbol{\xi}(t)\\ \boldsymbol{w}(t)\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}A^{\mathrm{T}}X + X^{\mathrm{T}}A & X^{\mathrm{T}}B\\ B^{\mathrm{T}}X & 0\end{bmatrix} \begin{bmatrix}\boldsymbol{\xi}(t)\\ \boldsymbol{w}(t)\end{bmatrix}.$$

Together with (3) (pre- and post-multiplied by $[\boldsymbol{\xi}^{\mathrm{T}}(t), \boldsymbol{w}^{\mathrm{T}}(t)]$ and $\begin{bmatrix}\boldsymbol{\xi}(t)\\ \boldsymbol{w}(t)\end{bmatrix}$ respectively) and (2) we derive

$$\frac{d}{dt}\mathcal{V}(\boldsymbol{\xi}(t)) + \begin{bmatrix}\boldsymbol{z}(t)\\ \boldsymbol{w}(t)\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\frac{1}{\gamma} & 0\\ 0 & -\gamma I\end{bmatrix} \begin{bmatrix}\boldsymbol{z}_i(t)\\ \boldsymbol{w}_i(t)\end{bmatrix} \leq 0, \qquad \begin{array}{l}\text{with equality for}\\ \boldsymbol{w}(t) = \mathbf{0},\ \boldsymbol{z}(t) = \mathbf{0}.\end{array}$$

Integration from $t = 0$ to $t = T$ with $T > 0$, $\boldsymbol{\xi}(0) := \mathbf{0}$ together with $\boldsymbol{\xi}^{\mathrm{T}}(T)E^{\mathrm{T}}X\boldsymbol{\xi}(T) \geq 0$ renders $\int_0^T \|\boldsymbol{z}\|^2 - \gamma^2\|\boldsymbol{w}\|^2\,dt \leq 0$, i.e. the time domain condition for an H_∞ norm bound γ. $\square$

In the following section we will use this analysis tool in order to derive necessary and sufficient conditions for the existence of a γ suboptimal controller.

3 The H_∞ control problem for descriptor systems

We consider a generalized plant description Γ

$$\Gamma: \quad \begin{aligned} E\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B_1\boldsymbol{w}(t) + B_2\boldsymbol{u}(t) && (5')\\ \boldsymbol{z}(t) &= C_1\boldsymbol{x}(t) + D_{11}\boldsymbol{w}(t) + D_{12}\boldsymbol{u}(t) &&\\ \boldsymbol{y}(t) &= C_2\boldsymbol{x}(t) + D_{21}\boldsymbol{w}(t) && (5)\end{aligned}$$

with $\boldsymbol{x}(t) \in \mathbb{R}^{n_x}$, $\boldsymbol{w}(t) \in \mathbb{R}^{n_w}$, $\boldsymbol{u} \in \mathbb{R}^{n_u}$, $\boldsymbol{z}(t) \in \mathbb{R}^{n_z}$, and $\boldsymbol{y} \in \mathbb{R}^{n_y}$ denoting the descriptor variables, the external input variables, the control input variables, the external output variables, and the measurement variables, respectively. E and A are square constant matrices where E explicitly is allowed to be singular, i.e. $\operatorname{rank}(E) =: r_p \leq n_x$. The remaining matrices are constant matrices of appropiate dimension. $\operatorname{rank} E = r < n$. By means of the auxiliaray variables $\boldsymbol{x}_1(t)$, $\boldsymbol{x}_2(t)$, $\boldsymbol{x}_3(t)$

$$\begin{aligned} \boldsymbol{x}_1(t) &:= \boldsymbol{x}(t)\\ \boldsymbol{x}_2(t) &:= D_{11}\boldsymbol{w}(t) + D_{12}\boldsymbol{u}(t)\\ \boldsymbol{x}_3(t) &:= D_{21}\boldsymbol{w}(t)\end{aligned} \tag{6}$$

it is always possible to reformulate (5) as

$$\begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \\ \dot{\boldsymbol{x}}_3 \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{bmatrix} + \begin{bmatrix} B_1 \\ D_{11} \\ D_{21} \end{bmatrix} \boldsymbol{w} + \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix} \boldsymbol{u}$$

$$\boldsymbol{z} = \begin{bmatrix} C_1 & I & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} C_2 & 0 & I \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{bmatrix}, \tag{7}$$

i.e. as a system (5) with new descriptor variable $\boldsymbol{x} := [\boldsymbol{x}_1^{\mathrm{T}}, \boldsymbol{x}_2^{\mathrm{T}}, \boldsymbol{x}_3^{\mathrm{T}}]^{\mathrm{T}}$ and $D_{11} = 0$, $D_{12} = 0$, $D_{21} = 0$. In the following we therefore assume a plant description (5) with $D_{ij} = 0$. Also without loss of generality (Rehm and Allgöwer, 1998b) it is possible to assume that E is given as

$$E = \begin{bmatrix} I_{r_p} & 0 \\ 0 & 0 \end{bmatrix}. \tag{8}$$

The control problem is to find a dynamic output feedback controller K

$$K: \quad \begin{array}{rcrcl} E_K\dot{\boldsymbol{\zeta}}(t) & = & A_K\boldsymbol{\zeta}(t) & + & B_K\boldsymbol{y}(t), \quad \boldsymbol{\zeta}(t) \in \mathbb{R}^{n_\zeta}, \\ \boldsymbol{u}(t) & = & C_K\boldsymbol{\zeta}(t) & + & D_K\boldsymbol{y}(t) \end{array} \quad E_K = \begin{bmatrix} I_{r_{\mathrm{K}}} & 0 \\ 0 & 0 \end{bmatrix}, r_{\mathrm{K}} < n_\zeta \tag{9}$$

with $E_K, A_K \in \mathbb{R}^{n_\zeta \times n_\zeta}$ (i.e. the controller is in descriptor form), $B_K \in \mathbb{R}^{n_\zeta \times n_y}$, $C_K \in \mathbb{R}^{n_u \times n_\zeta}$, and $D_K \in \mathbb{R}^{n_u \times n_y}$ such that the closed loop system

$$\begin{aligned} E_{cl}\dot{\boldsymbol{\xi}}(t) &= A_{cl}\boldsymbol{\xi}(t) + B_{cl}\boldsymbol{w}(t), \quad \boldsymbol{\xi}(t) \in \mathbb{R}^{(n_x+n_\zeta)} \\ \boldsymbol{z}(t) &= C_{cl}\boldsymbol{\xi}(t) \end{aligned} \tag{10}$$

with

$$E_{cl} = \begin{bmatrix} E & 0 \\ 0 & E_K \end{bmatrix}, \; A_{cl} = \begin{bmatrix} A + B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K \end{bmatrix}, \; B_{cl} = \begin{bmatrix} B_1 \\ 0_{n_\zeta \times n_w} \end{bmatrix}, \; C_{cl} = \begin{bmatrix} C_1 & 0_{n_z \times n_\zeta} \end{bmatrix} \tag{11}$$

is internally stable and such that the H_∞ norm of the closed loop is bounded by a given number $\gamma > 0$, i.e. $\|G_{cl}\|_\infty < \gamma$ with $G_{cl}(s) := C_{cl}(sE_{cl} - A_{cl})^{-1})B_{cl}$.
In view of the bounded real lemma for descriptor systems (Proposition 2.1) the problem can be reformulated as the problem to find matrices A_K, B_K, C_K, D_K, and X such that the inequalities

$$E_{cl}^{\mathrm{T}}X = X^{\mathrm{T}}E_{cl} \geq 0, \tag{12}$$

$$\mathcal{B}_{[A_{cl},B_{cl},C_{cl}]}(\gamma, X) < 0 \tag{13}$$

hold true. In a sequence of propositions we will now establish, that the *nonlinear* inequalities (12), (13) are equivalent to certain LMI conditions. For simplicity of notation we assume henceforth, that the number of descriptor variables and the number of dynamic modes of plant and controller are equal (i.e. $n_x = n_\zeta$, $r_p = r_{\mathrm{K}}$). The necessary modifications for the examination of reduced order controllers are discussed in connection with the LMI synthesis conditions.

Proposition 3.1 *Consider a plant (5) with $D_{ij} = 0$, a matrix E as in (8) and a controller as in (9). Define matrices*

$$A_0 := \begin{bmatrix} A & 0 \\ 0 & 0_{n_\zeta} \end{bmatrix},\ \mathcal{B} := \begin{bmatrix} 0 & B_2 \\ I_{n_\zeta} & 0 \end{bmatrix},\ \mathcal{C} := \begin{bmatrix} 0 & I_{n_\zeta} \\ C_2 & 0 \end{bmatrix},\quad \begin{array}{l} P := \begin{bmatrix} \mathcal{B}^{\mathrm{T}} & 0 & 0 \end{bmatrix}, \\ Q := \begin{bmatrix} \mathcal{C} & 0 & 0 \end{bmatrix}. \end{array} \tag{14}$$

Then the inequalities (12), (13) equivalently can be written as

$$E_{cl}^{\mathrm{T}} X = X^{\mathrm{T}} E_{cl} \geq 0, \quad P^{\perp} \Phi P^{\perp \mathrm{T}} < 0, \quad Q^{\mathrm{T}\perp} \Psi Q^{\mathrm{T}\perp\mathrm{T}} < 0, \text{ with} \tag{15}$$

$$\Phi := \begin{bmatrix} A_0 X^{-1} + X^{-T} A_0^{\mathrm{T}} & B_{cl} & X^{-T} C_{cl}^{\mathrm{T}} \\ B_{cl}^{\mathrm{T}} & -\gamma I & 0 \\ C_{cl} X^{-1} & 0 & -\gamma I \end{bmatrix}, \quad \Psi := \begin{bmatrix} A_0^{\mathrm{T}} X + X^{\mathrm{T}} A_0 & X^{\mathrm{T}} B_{cl} & C_{cl}^{\mathrm{T}} \\ B_{cl}^{\mathrm{T}} X & -\gamma I & 0 \\ C_{cl} & 0 & -\gamma I \end{bmatrix}. \tag{16}$$

Proof. We make use of the fact that the controller data occurs in (13) in an affine way, i.e. (13) can be written as

$$\Psi + P_X \theta Q + (P_X \theta Q)^{\mathrm{T}} < 0, \quad \text{with } \theta := \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix},\ P_X := \begin{bmatrix} X^{\mathrm{T}} \mathcal{B} \\ 0 \\ 0 \end{bmatrix}. \tag{17}$$

With the explicit expression

$$P_X^{\perp} = P^{\perp} X_I^{-T}, \qquad X_I := \begin{bmatrix} X & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

of $P_X^{\perp}$ and the application of (1) to (17) the claim $P^{\perp} \Phi P^{\perp \mathrm{T}} < 0$, $Q^{\mathrm{T}\perp} \Psi Q^{\mathrm{T}\perp\mathrm{T}} < 0$ follows immediately from $\Phi = X_I^{-T} \Psi X_I^{-1}$. □

Although we removed the controller matrices in this characterization, it is also not computationally attractive since the inequalities (15) contain the matrix X as well as the inverse X^{-1}. This problem can be overcome by an explicit parameterization of X and X^{-1}. A possible solution X of (15) is necessarily non-singular and $E_{cl}^{\mathrm{T}} X = X^{\mathrm{T}} E_{cl}$ implies that X and X^{-1} can be written as

$$X = \left[\begin{array}{cc|cc} S_1 & 0 & N_1 & 0 \\ S_3 & S_4 & N_3 & N_4 \\ \hline N_1^{\mathrm{T}} & 0 & L_1 & 0 \\ N_7 & N_8 & L_3 & L_4 \end{array}\right], \quad X^{-1} = \left[\begin{array}{cc|cc} R_1 & 0 & M_1 & 0 \\ R_3 & R_4 & M_3 & M_4 \\ \hline M_1^{\mathrm{T}} & 0 & K_1 & 0 \\ M_7 & M_8 & K_3 & K_4 \end{array}\right], \quad \begin{array}{ll} S_1 = S_1^{\mathrm{T}} & L_1 = L_1^{\mathrm{T}} \\ R_1 = R_1^{\mathrm{T}} & K_1 = K_1^{\mathrm{T}} \end{array} \tag{18}$$

and $S_1, R_1 \in \mathbb{R}^{r_p \times r_p}$, $S_4, R_4 \in \mathbb{R}^{(n_x - r_p) \times (n_x - r_p)}$, $L_1, K_1 \in \mathbb{R}^{r_K \times r_K}$, $L_4, K_4 \in \mathbb{R}^{(n_\zeta - r_K) \times (n_\zeta - r_K)}$, and the other sub-matrices of appropriate dimension. Due to this partition of X and X^{-1} a refinement of (15) is possible:

Proposition 3.2 *Assume the existence of matrices X, X^{-1} as in (18) such that (15) holds true. Define*

$$A =: \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B_1 =: \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \quad C_1 =: \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \quad \begin{array}{l} A_{11} \in \mathbb{R}^{r_p \times r_p}, \\ B_{11} \in \mathbb{R}^{r_p \times n_w}, \\ C_{11} \in \mathbb{R}^{n_z \times r_p}. \end{array} \tag{19}$$

Then $P^{\perp}\Phi P^{\perp\mathrm{T}} < 0$, $Q^{\mathrm{T}\perp}\Psi Q^{\mathrm{T}\perp\mathrm{T}} < 0$ *equivalently can be written as*

$$\begin{bmatrix} A_{12} & B_2 \\ A_{22} & \\ C_{12} & 0 \\ 0 & 0 \end{bmatrix}^{\perp} \underbrace{\begin{bmatrix} AR_0 + R_0^{\mathrm{T}}A^{\mathrm{T}} & R_0^{\mathrm{T}}C_1 & B_1 \\ C_1R_0 & -\gamma I & 0 \\ B_1^{\mathrm{T}} & 0 & -\gamma I \end{bmatrix}}_{=:\,\Phi_0} \begin{bmatrix} A_{12} & B_2 \\ A_{22} & \\ C_{12} & 0 \\ 0 & 0 \end{bmatrix}^{\perp T} < 0, \quad R_0 := \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} A_{21}^{\mathrm{T}} & C_2^{\mathrm{T}} \\ A_{22}^{\mathrm{T}} & \\ B_{12}^{\mathrm{T}} & 0 \\ 0 & 0 \end{bmatrix}^{\perp} \underbrace{\begin{bmatrix} A^{\mathrm{T}}S_0 + S_0^{\mathrm{T}}A & S_0^{\mathrm{T}}B_1 & C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}}S_0 & -\gamma I & 0 \\ C_1 & 0 & -\gamma I \end{bmatrix}}_{=:\,\Psi_0} \begin{bmatrix} A_{21}^{\mathrm{T}} & C_2^{\mathrm{T}} \\ A_{22}^{\mathrm{T}} & \\ B_{12}^{\mathrm{T}} & 0 \\ 0 & 0 \end{bmatrix}^{\perp T} < 0, \quad S_0 := \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{21}$$

Proof. We introduce the shorthand notation

$$R_l := \begin{bmatrix} R_3 & R_4 \end{bmatrix}, \; S_l := \begin{bmatrix} S_3 & S_4 \end{bmatrix}, \text{ and } X = \begin{bmatrix} S & N_u \\ N_l & L \end{bmatrix}, \; X^{-1} = \begin{bmatrix} R & M_u \\ M_l & K \end{bmatrix} \tag{22}$$

for the indicated block partition in (18). The matrices Φ, Ψ in (16) then become

$$\Phi = \begin{bmatrix} AR + R^{\mathrm{T}}A^{\mathrm{T}} & AM_u & B_1 & R^{\mathrm{T}}C_1^{\mathrm{T}} \\ M_u^{\mathrm{T}}A^{\mathrm{T}} & 0 & 0 & M_u^{\mathrm{T}}C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}} & 0 & -\gamma I & 0 \\ C_1R & C_1M_u & 0 & -\gamma I \end{bmatrix}, \Psi = \begin{bmatrix} A^{\mathrm{T}}S + S^{\mathrm{T}}A & A^{\mathrm{T}}N_u & S^{\mathrm{T}}B_1 & C_1^{\mathrm{T}} \\ N_u^{\mathrm{T}}A & 0 & N_u^{\mathrm{T}}B_1 & 0 \\ B_1^{\mathrm{T}}S & B_1^{\mathrm{T}}N_u & -\gamma I & 0 \\ C_1 & 0 & 0 & -\gamma I \end{bmatrix}. \tag{23}$$

$P^{\perp}$ and $Q^{\mathrm{T}\perp}$ can be expressed as

$$P^{\perp} = \begin{bmatrix} B_2^{\perp} & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}, \qquad Q^{\mathrm{T}\perp} = \begin{bmatrix} C_2^{\mathrm{T}\perp} & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{24}$$

Due to the zero column in (24) and (23) the inequalities in (15) are equivalent to

$$\begin{bmatrix} B_2^{\perp} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \underbrace{\begin{bmatrix} AR + R^{\mathrm{T}}A^{\mathrm{T}} & R^{\mathrm{T}}C_1^{\mathrm{T}} & B_1 \\ C_1R & -\gamma I & 0 \\ B_1^{\mathrm{T}} & 0 & -\gamma I \end{bmatrix}}_{=:\,\Phi'} \begin{bmatrix} B_2^{\mathrm{T}\perp} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0 \tag{25}$$

$$\begin{bmatrix} C_2^{\mathrm{T}\perp} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{\mathrm{T}} \underbrace{\begin{bmatrix} A^{\mathrm{T}}S + S^{\mathrm{T}}A & S^{\mathrm{T}}B_1 & C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}}S & -\gamma I & 0 \\ C_1 & 0 & -\gamma I \end{bmatrix}}_{=:\,\Psi'} \begin{bmatrix} C_2^{\mathrm{T}\perp\mathrm{T}} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0. \tag{26}$$

With $P_\Phi := [B_2^{\mathrm{T}}, 0, 0]^{\mathrm{T}}$ and $P_\Psi := [C_2, 0, 0]^{\mathrm{T}}$ the inequalities (25), (26) can be written as

$$P_\Phi^\perp \Phi' P_\Phi^{\perp\mathrm{T}} < 0, \quad P_\Psi^\perp \Psi' P_\Psi^{\perp\mathrm{T}} < 0. \tag{27}$$

If we additionally introduce $Q_\Phi := [I_{n_x}, 0, 0]$ and $Q_\Psi := [I_{n_x}, 0, 0]$ the inequalities

$$Q_\Phi^{\mathrm{T}\perp} \Phi' Q_\Phi^{\mathrm{T}\perp\mathrm{T}} < 0, \quad Q_\Psi^{\mathrm{T}\perp} \Psi' Q_\Psi^{\mathrm{T}\perp\mathrm{T}} < 0 \tag{28}$$

are trivially fulfilled. Together with (1) the inequalities (27), (28) then become

$$\exists \beta, \delta : \quad \Phi' + P_\Phi \beta Q_\Phi + (P_\Phi \beta Q_\Phi)^{\mathrm{T}} < 0, \quad \Psi' + P_\Psi \delta Q_\Psi + (P_\Psi \delta Q_\Psi)^{\mathrm{T}} < 0 \tag{29}$$

with matrices β, δ of suitable dimension. Now we can split Φ_0 from Φ' (and analogous for Ψ_0):

$$\Phi' = \Phi_0 + [A_{12}^{\mathrm{T}}, A_{22}^{\mathrm{T}}, C_{12}^{\mathrm{T}}, 0]^{\mathrm{T}} R_l [I, 0, 0] + [I, 0, 0]^{\mathrm{T}} R_l^{\mathrm{T}} [A_{12}^{\mathrm{T}}, A_{22}^{\mathrm{T}}, C_{12}^{\mathrm{T}}, 0] \tag{30}$$

In conjunction with the corresponding inequality in (29) we end up with

$$\Phi_0 + \begin{bmatrix} A_{12}^{\mathrm{T}} & A_{22}^{\mathrm{T}} & C_{12}^{\mathrm{T}} & 0 \\ & B_2 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R_l \\ \beta \end{bmatrix} [I, 0, 0] + [I, 0, 0]^{\mathrm{T}} \begin{bmatrix} R_l \\ \beta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A_{12}^{\mathrm{T}} & A_{22}^{\mathrm{T}} & C_{12}^{\mathrm{T}} & 0 \\ & B_2 & 0 & 0 \end{bmatrix} < 0. \tag{31}$$

A final application of (1) renders the proposition. □

The inequalities (20), (21) are linear inequalities in R_1, R_2. However, these inequalities are based on the assumption that matrices X, X^{-1} as in (18) actually exists. This problem partly is addressed in the following proposition.

Proposition 3.3 *A parameterization of X, X^{-1} as in (18) with $E_{cl}^{\mathrm{T}} X = X^{\mathrm{T}} E_{cl} \geq 0$ is possible if and only if*

$$\begin{bmatrix} S_1 & I \\ I & R_1 \end{bmatrix} \geq 0, \;\; S_1 > 0, \;\; R_1 > 0 \qquad \textit{hold true.} \tag{32}$$

In order to proof the proposition we need a basic matrix dilation result:

Proposition 3.4 *(Gahinet and Apkarian, 1994) Suppose that $X_{11} = X_{11}^{\mathrm{T}}, Y_{11} = Y_{11}^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ with $X_{11} > 0, Y_{11} > 0$ are given. Let r be a non-negative integer. Then there exists matrices $X_{12} \in= \mathbb{R}^{n \times r}$, $X_{22} = X_{22}^{\mathrm{T}} \in \mathbb{R}^{r \times r}$, and*

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \end{bmatrix} > 0, \quad \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \end{bmatrix}^{-1} = \begin{bmatrix} Y_{11} & ? \\ ?^{\mathrm{T}} & ? \end{bmatrix} \tag{33}$$

$$\textit{if and only if} \quad \begin{bmatrix} X_{11} & I \\ I & Y_{11} \end{bmatrix} \geq 0, \quad \textit{and} \quad \operatorname{rank} \begin{bmatrix} X_{11} & I \\ I & Y_{11} \end{bmatrix} \leq n + r. \tag{34}$$

Proof.(of Proposition 3.3) ¿From $E_{cl}^{\mathrm{T}}X = X^{\mathrm{T}}E_{cl}$ we get the parameterization (18). Due to $\mathrm{rank}(E_{cl}^{\mathrm{T}}X) = r_p + r_{\mathrm{K}}$, $E_{cl}^{\mathrm{T}}X \geq 0$ is equivalent to $\begin{bmatrix} S_1 & N_1 \\ N_1^{\mathrm{T}} & L_1 \end{bmatrix} > 0$. The parameterization (18) furthermore implies $\begin{bmatrix} S_1 & N_1 \\ N_1^{\mathrm{T}} & L_1 \end{bmatrix}\begin{bmatrix} R_1 & M_1 \\ M_1^{\mathrm{T}} & K_1 \end{bmatrix} = I$, i.e. $R_1 > 0$. Application of Proposition 3.4 then renders the inequalities (32). The rank condition in (34) is always fulfilled since we have $n := n_x = r_{\mathrm{K}} =: r$. $\square$

Theorem 3.1 *Consider a plant (5) with $D_{ij} = 0$, a matrix E as in (8) and a controller as in (9). The H_∞ control problem to render the closed loop system (10) internally stable with H_∞ norm $\|G_{cl}\| < \gamma$, $\gamma > 0$ has a solution if and only if the linear matrix inequalities (20), (21), (32) have a solution R_1, S_1.*

Proof. The theorem is a straightforward consequence of Proposition 3.1, 3.2, and 3.3 except one technical detail: in Proposition 3.1 the decoupled LMIs (20), (21) are derived under the nonlinear coupling condition due to (18). The coupling between S_1, R_1 is captured by the LMIs from Proposition 3.3 but for the remaining submatrices in (18) the point is open. An analysis of the proof of Proposition 3.1 shows, that the original inequality conditions due to the generalized bounded real lemma also affects the submatrices R_l, S_l (due to (30) and the corresponding inequality for S_l). However, the reformulation

$$X\Pi_1 = \Pi_2, \quad \Pi_1 := \left[\begin{array}{cc|cc} R_1 & 0 & I_{r_p} & 0 \\ R_3 & R_4 & 0 & I_{n_x - r_p} \\ \hline M_1^{\mathrm{T}} & 0 & 0 & 0 \\ M_7 & M_8 & 0 & 0 \end{array}\right], \quad \Pi_2 := \left[\begin{array}{cc|cc} I_{r_p} & 0 & S_1 & 0 \\ 0 & I_{n_x - r_p} & S_3 & S_4 \\ \hline 0 & 0 & N_1^{\mathrm{T}} & 0 \\ 0 & 0 & N_7 & N_8 \end{array}\right] \tag{35}$$

of (18) shows, that any restriction of R_l, S_l does not affects the existence of a matrix X such that (18) or (35) holds true: If R_i, S_i, $i = 1(1)3$ are given, we always can choose the matrices M_i, N_i, $i \in \{1, 7, 8\}$ such that Π_1, Π_2 and therefore X are non-singular, i.e. such that (18) holds true. $\square$

4 Controller Computation

Theorem 3.1 is an existence result which do not address the computation of the controller itself. This issue is now discussed in some detail. Full order controller design consists of the following steps:

a) Solution of the LMIs (20), (21), (32) for R_1, S_1.

b) Parameterization of the LMIs (25), (26) with R_1, S_1 from a) and solution for R_l, S_l.

c) The matrices N_i, M_i in Π_1, Π_2 in (35) must be chosen such that Π_1, Π_2 are non-singular. The matrix X then can be computed as $X = \Pi_2\Pi_1^{-1}$.

d) With a *known* matrix X the generalized bounded real lemma inequality $\mathcal{B}_{[A_{cl},B_{cl},C_{cl}]}(\gamma, X) < 0$ for the closed loop system is a linear inequality with respect to the controller variables. This inequality can be solved by efficient numerical methods or, due to its special structure, by explicit formulas (Skelton *et al.*, 1998).

If we want to consider reduced order controllers (i.e. $r_{\mathrm{K}} < r_p$) some minor modifications in the presented proof are necessary: Additionally to the LMI conditions in Theorem 3.1 we then have to consider the (non-convex) rank condition (34) from Proposition 3.4, i.e.

$$\operatorname{rank}\begin{bmatrix} S_1 & I \\ I & R_1 \end{bmatrix} \leq r_p + r_{\mathrm{K}}.$$

With respect to controller computation the matrices N_i, M_i then must be chosen such that Π_1, Π_2 have full row rank.

5 Conclusions

We considered the H_∞ control problem for descriptor systems, that are allowed to be of high index and even can be non-regular. Based on a generalization of the bounded real lemma we provided a novel algebraic approach to the synthesis problem. The resulting LMI existence conditions parallels the ones given in (Gahinet and Apkarian, 1994) for the non-descriptor case. Based on the synthesis conditions a numerically reliable computation of the controller matrices is possible. Also the reduced order controller case is treated within the presented framework. The controller renders the closed loop internally stable and imposes an H_∞ norm constraint on the input/output behavior of the closed loop system. In future work the similarities to the non-descriptor case may be used in order to translate the presented approach to discrete time descriptor systems.

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