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Generalized PID Controllers

Ilan Rusnak
RAFAEL(88), P.O.Box 2250, Haifa, Israel, 31021

Abstract

The PI, PD and PID controllers are widely used and successfully applied controllers to many applications. Their successful application, good performance, easiness of tuning are sufficient rational for their use, although their structure is justified by heuristics.
In this paper by the use of optimal control theory we formulate a tracking problem and show those cases when their solution gives the PI, PD and PID controllers, thus avoiding heuristics and giving a systematic approach to explanation for their excellent performance. It is shown that the PI controller is optimal for a first order system, the PID controller is optimal for a 2nd order systems with no zero. The reference trajectory is generated by a system identical to the plant.
Then, the same approach, that led to the PI and PID controller, is applied to a general linear, strictly proper system and a generalized PID controller is derived. Such controller is called here PID^{n-1} controller. As an example, a generalized PID controller for a DC motor with one flexible mode is presented.

1. Introduction

The PD, PI and PID controllers are successfully applied controllers to many applications, almost from the beginning of controls applications [1,2].
The facts of their successful application, good performance, easiness of tuning are speaking for themselves and are sufficient rational for their use, although their structure is justified by heuristics: "These ... controls - called proportional-integral-derivative (PID) control - constitute the heuristic approach to controller design that has found wide acceptance in the process industries." [2, pp. 168].
In [3, pp. 114] it is shown that Internal Model Control - "IMC leads to PID controllers for virtually all models common in industrial practice."
In [4] the linear quadratic regulator (LQR) theory has been used to formulate tracking problems and show those cases when their solution gives the PID controllers. Namely, a problem has been state whose solution leads to the PD, PI and PID controllers. This enables avoiding heuristics and is giving a systematic approach to explanation for the good performance of the PID controllers. The main contribution of the results in [4] is that it shows for what problems the PID controllers are the optimal controllers and for which they are not. The importance of this is:

1) From theoretical point of view it is important to know that a widely used control architecture can be derived from an optimal control problem.

2) The solution shows for what kind of systems the PID controllers are optimal and will show for which systems it is not, thus enabling to show why a PID controller does not perform well for all systems. This will enable to forecast what control designs not to apply a PID controller.

3) For those systems that the PID is not the optimal controller architecture the optimal control approach shows what is the optimal controller architecture, thus achieving generalization.

It is shown in [4] that the PI controller is optimal for a first order system, the PID controller is optimal for a 2nd order systems with no zero. The reference trajectory is generated by a system identical to the plant. The differences are the initial conditions and the input to the reference

trajectory generator. The tracking error is the position error, and zero steady state is imposed by integral action on the tracking error. This is the reason that the PID controllers are so well performing in servo applications and chemical processes, as these are of this type.
In this paper we apply the approach that has been developed in [4] to more general cases. First, for the coherence of presentation, the main result from [4] are rederived. Then the approach is applied to a general linear strictly proper system and a generalized PID controller is derived, the PID^{n-1} controller. As an example, a generalized PID controller for a DC motor with one flexible mode is presented.

2. Optimal Tracking Problem

We assume the n-th order multi-input multi-output system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in R^n$ is the state; $u \in R^m$ is the input and $y \in R^p$ is the measured output; $A \in R^{n \times n}$, $B \in R^{n \times m}$ and $C \in R^{p \times n}$. The reference trajectory generator is

$$\begin{aligned} \dot{x}_r &= A_r x_r + B_r u_r \\ y_r &= C_r x_r \end{aligned} \tag{2}$$

where $x_r \in R^\nu$ is the state; $u_r \in R^\mu$ is the input and $y \in R^p$ is the reference output; $A \in R^{\nu \times \nu}$, $B \in R^{\nu \times \mu}$ and $C \in R^{p \times \nu}$.
The integral action is introduced into the control in order to "force" zero tracking errors for polynomial inputs, and to attenuate disturbances. This is done by introducing the auxiliary variable (the integral of the tracking error)

$$\dot{\eta} = e = y - y_r \tag{3}$$

The control objective is

$$\begin{aligned} J = \tfrac{1}{2}\{ &(y(t_f) - y_r(t_f))^T G_1 (y(t_f) - y_r(t_f)) + \eta(t_f)^T G_2 \eta(t_f) \\ &+ \int_{t_o}^{t_1} [y(t) - y_r(t))^T Q_1 (y(t) - y_r(t)) + \eta(t)^T Q_2 \eta(t) + u(t)^T R u(t)]dt\} \end{aligned} \tag{4}$$

The optimal tracking problem [6] is to find an admissible input u(t) such that the tracking objective (4) is minimized subject to the dynamic constraints (1, 2, 4). All vectors and matrices are of the proper dimensions.

3. Solution of the Tracking Problem

In order to solve the Optimal Tracking Problem we augment the state variables to the form [6]

$$\bar{x} = \begin{bmatrix} x \\ \eta \\ x_r \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & 0 & 0 \\ C & 0 & -Cr \\ 0 & 0 & A_r \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & 0 & -C_r \end{bmatrix}, \tag{5}$$

then the problem is minimization of (4) subject to (1, 2) is the problem of minimization of

$$J = \tfrac{1}{2}\{x(t_f)^T Gx(t_f) + \int_{t_o}^{t_1} [x(t)^T Qx(t) + u(t)^T Ru(t)]dt\} \tag{6}$$

subject to

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \tag{7}$$

where

$$Q = \bar{C}^T Q_1 \bar{C} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} Q_2 \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$G = \bar{C}^T G_1 \bar{C} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} G_2 \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \tag{8}$$

Notice that the solution is not affected by u_r.
The solution is [5,6]

$$\begin{aligned} u &= -R^{-1}\bar{B}P\bar{x} \\ -\dot{P} &= P\bar{A} + \bar{A}^T P + Q - P\bar{B}R^{-1}\bar{B}P, \quad P(t_f) = G, \end{aligned} \tag{9}$$

if we write P={[P_{ij}]; i,j=1,2,3}, then

$$u = -R^{-1} \; [\, B^T P_{11} \;\; B^T P_{12} \;\; B^T P_{13} \,], \tag{10}$$

where in steady state we have

$$0 = P_{11}A - P_{12}C + A^T P_{11} - C^T P_{12} + C^T Q_1 C - P_{11}BR^{-1}B^T P_{11} \tag{11.11}$$
$$0 = A^T P_{12} - C^T P_{22} - P_{11}BR^{-1}B^T P_{12} \tag{11.12}$$
$$0 = P_{12}C_r + P_{13}A_r + A^T P_{13} - C^T P_{23} - C^T Q_1 C_r - P_{11}BR^{-1}B^T P_{13} \tag{11.13}$$
$$0 = Q_2 - P_{12}BR^{-1}B^T P_{12} \tag{11.22}$$
$$0 = P_{22}C_r + P_{23}A_r - P_{12}BR^{-1}B^T P_{13} \tag{11.23}$$
$$0 = P_{23}C_r + P_{33}A_r + C_r^T P_{23} + A_r^T P_{33} + C_r^T Q_1 C_r - P_{13}BR^{-1}B^T P_{13} \tag{11.33}$$

4. First Order system -The PI controller

If $A = A_r = 0$, $C = C_r = 1$ and $B = B_r = 1$, i.e. one integrator, then from (11)

$$0 = -P_{12} - P_{12} + Q_1 - P_{11}R^{-1}P_{11} \tag{12.11}$$
$$0 = -P_{22} - P_{11}R^{-1}P_{12} \tag{12.12}$$
$$0 = P_{12} - P_{23} - Q_1 - P_{11}R^{-1}P_{13} \tag{12.13}$$
$$0 = Q_2 - P_{12}R^{-1}P_{12} \tag{12.22}$$
$$0 = P_{22} - P_{12}R^{-1}P_{13} \tag{12.23}$$
$$0 = P_{23} + P_{23} + Q_1 - P_{13}R^{-1}P_{13} \tag{12.33}$$

from (12.22)

$$P_{12} = -\sqrt{Q_2 R} \quad \text{(we select the negative root)} \tag{13.1}$$

from (12.11) and (13.1)

$$P_{11} = -\sqrt{R(Q_1 - 2P_{12})} = R\sqrt{\frac{Q_1}{R} + 2\sqrt{\frac{Q_2}{R}}} \tag{13.2}$$

from (12.12)

$$P_{22} = -\frac{P_{12}P_{11}}{R} \tag{13.3}$$

from (12.23) and (13.3)

$$P_{13} = \frac{P_{22}R}{P_{12}} = -P_{11}\,. \tag{13.4}$$

This means, by the use of (3), that

$$\begin{aligned} u &= -\frac{1}{R}\begin{bmatrix} P_{11} & P_{12} & -P_{11} \end{bmatrix}\begin{bmatrix} x \\ \eta \\ x_r \end{bmatrix} = -\begin{bmatrix} k_1 & k_2 & -k_1 \end{bmatrix}\begin{bmatrix} x \\ \eta \\ x_r \end{bmatrix} \\ &= k_1(x - x_r) + k_2\eta = k_1 e + k_2\int e\,dt \end{aligned} \tag{14}$$

This is the PI controller.

5. General solution - The General PID Controller

Now, for the general case $A = A_r$, $C = C_r = I$ and $B = B_r$, i.e. the reference trajectory generator is identical to the plant, it can be shown that $P_{13} = -P_{11}$. Then

$$u = -R^{-1}\begin{bmatrix} B^T P_{11} & B^T P_{12} & B^T P_{13} \end{bmatrix}\begin{bmatrix} x \\ \eta \\ x_r \end{bmatrix} = K_1 e + K_2 \int edt \tag{15}$$

where

$$K_1 = -R^{-1} B^T P_{11}, \qquad K_2 = R^{-1} B^T P_{12} . \tag{16}$$

6. Second Order System with no Zero

Here we assume that the plant (for example, DC motor with low electrical time constant) and the trajectory generator are

$$A = A_r = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ b_2 \end{bmatrix},\ C = C_r = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$\text{i.e. } H(s) = H_r(s) = \frac{b_2}{s^2 + a_1 s + a_2}. \tag{17}$$

The state of the plant and trajectory generator are denoted $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix}$, respectively.

6.1 We want to force zero steady state tracking error on the output. Here $y = x_1, y_r = x_{1r}$ and $\dot{y} = x_2, \dot{y}_r = x_{2r}$. Then, since the tracking error is $e = y - y_r$, we have

$$\dot{\eta} = e = y - y_r \tag{18}$$

$$u = \begin{bmatrix} k_1 & k_2 & k_3 & -k_1 & -k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \eta \\ x_{1r} \\ x_{2r} \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & k_3 & -k_1 & -k_2 \end{bmatrix}\begin{bmatrix} y \\ \dot{y} \\ \eta \\ y_r \\ \dot{y}_r \end{bmatrix} \tag{19}$$

$$= k_1 e + k_2 \dot{e} + k_3 \int edt$$

That is, we get PID controller.

6.2 Here we want to force zero steady state tracking error on the rate of the output as well, $e = y - y_r$,

$$\eta = \begin{bmatrix} y - y_r \\ \dot{y} - \dot{y}_r \end{bmatrix} = \begin{bmatrix} e \\ \dot{e} \end{bmatrix}, \quad \eta = \begin{bmatrix} \int edt \\ e \end{bmatrix}, \tag{20}$$

$$u = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & -k_1 & -k_2 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \eta \\ \dot{\eta} \\ y_r \\ \dot{y}_r \end{bmatrix} = (k_1 + k_4)e + k_2\dot{e} + k_3\int edt \tag{21}$$

These are PID controllers. For other second order plants, or when the trajectory generator is not identical to the plant we will not get generally such structure.
Notice that the solution is independent of the reference trajectory generator input, u_r. This means that the optimality criterion induced some smoothness conditions on the trajectory and its derivative.

6.3 The PD controlled is derived if no integral action is required, i.e. $k_3 = 0$, then we get

$u = = (k_1+k_4)\ e + k_2.$

7. Second Order System with Zero

Here we assume that the plant and the trajectory generator are

$$A = A_r = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix},\ B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},\ C = C_r = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$\text{i.e. } H(s) = H_r(s) = \frac{b_1 s + (a_1 b_1 + b_2)}{s^2 + a_1 s + a_2},\quad \frac{x_2}{x_1} = \frac{b_2 s - a_2 b_1}{b_1 s + (a_1 b_1 + b_2)}. \tag{22}$$

7.1
We want to force zero steady state tracking error on the output. Here $y=x_1$, $y_r=x_{1r}$ but _x_2, r_x_{2r} .
Then, since the tracking error is

$$\dot{\eta} = e = y - y_r = x_1 - x_{1r} \tag{23}$$

we have

$$u = \begin{bmatrix} k_1 & k_2 & k_3 & -k_1 & -k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \eta \\ x_{1r} \\ x_{2r} \end{bmatrix} \tag{24}$$

$$= k_1(x_1 - x_{1r}) + k_2(x_2 - x_{2r}) + k_3\eta = k_1 e + k_2(x_2 - x_{2r}) + k_3\int edt$$

To proceed we write the controller in Laplace domain

$$
\begin{aligned}
u &= k_1 e + k_2 (x_2 - x_{2r}) + \frac{k_3}{s} e = k_1 e + k_2 \frac{x_2}{x_1}(x_1 - x_{1r}) + \frac{k_3}{s} e \\
&= [k_1 + k_2 \frac{b_2 s - a_2 b_1}{b_1 s + (a_1 b_1 + b_2)} + \frac{k_3}{s}] e
\end{aligned}
\tag{25}
$$

$$
\frac{u}{e} = \frac{(k_1 b_1 + k_2 b_2) s^2 + [(a_1 b_1 + b_2) k_1 + b_1 k_3 - a_2 b_1 k_2] s + k_3 (a_1 b_1 + b_2)}{s(b_1 s + (a_1 b_1 + b_2))} \tag{26}
$$

We used the assumption that $\frac{x_2}{x_1} = \frac{x_{2r}}{x_{1r}}$ and ignored the response to initial conditions. This is a proper PID controller, i.e. no direct derivative is required. To see this we write

$$
\frac{u}{e} = k_p + k_d \frac{s}{sT_d + 1} + \frac{k_I}{s} \tag{27}
$$

$$
\frac{u}{e} = \frac{(k_d + k_p T_d) s^2 + (k_p + k_I T_d) s + k_I}{s(sT_d + 1)} \tag{28}
$$

That is, for 2nd order plant with one zero, the optimal controller is a proper PID controller.

7.2 Here we want to force zero steady state tracking error on the second state, as well, i.e.

$$
\dot{\eta} = \begin{bmatrix} x_1 - x_{1r} \\ x_2 - x_{2r} \end{bmatrix}, \quad \eta = \begin{bmatrix} \int e dt \\ \int (x_2 - x_{2r}) dt \end{bmatrix}, \tag{29}
$$

$$
\begin{aligned}
u &= \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & -k_1 & -k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \int e dt \\ \int (x_2 - x_{2r}) dt \\ x_{1r} \\ x_{2r} \end{bmatrix}. \\
&= k_1 e + k_2 (x_2 - x_{2r}) + k_3 \int e dt + k_4 \int (x_2 - x_{2r}) dt
\end{aligned}
\tag{30}
$$

To proceed we write the controller in Laplace domain

$$
\begin{aligned}
u &= k_1 e + k_2 (x_2 - x_{2r}) + \frac{k_3}{s} e + \frac{k_3}{s}(x_2 - x_{2r}) = (k_1 + \frac{k_3}{s}) e + (k_2 + \frac{k_3}{s}) \frac{x_2}{x_1}(x_1 - x_{1r}) \\
\frac{u}{e} &= k_1 + \frac{k_3}{s} + (k_2 + \frac{k_3}{s}) \frac{b_2 s - a_2 b_1}{b_1 s + (a_1 b_1 + b_2)} = \\
&\frac{(k_1 b_1 + k_2 b_2) s^2 + [(a_1 b_1 + b_2) k_1 + b_1 k_3 + b_2 k_4 - a_2 b_1 k_2] s + k_3 (a_1 b_1 + b_2) - a_2 b_1 k_4}{s(b_1 s + (a_1 b_1 + b_2))}
\end{aligned}
\tag{31}
$$

Notice that the PI controller on each of the states created a PID controller on the error. For a plant with zero, in the PID structure controller a pole that cancels out this zero has been introduced. This means that as long as the zero is in LHP (minimum phase) the optimal PID controller can be build without measuring all states. However, if this zero is in RHP (non minimum phase) it introduces an unstable pole that cancels the plants zero, and this can not be realized in the PID structure. Therefore, for a plant with unstable zero the optimal PID can not be realized, and measurement of the two states, or an observer are required if one wishes to build the optimal controller.

7.3 If no integral action is required, i.e. $k_3 = 0$, $k_4 = 0$, then we get

$$\frac{u}{e} = k_1 + k_2 \frac{b_2 s - a_2 b_1}{b_1 s + (a_1 b_1 + b_2)} = \frac{(k_1 b_1 + k_2 b_2)s + [(a_1 b_1 + b_2)k_1 - a_2 b_1 k_2]}{b_1 s + (a_1 b_1 + b_2)} \tag{32}$$

This is the lead-lag/lag-lead controller.

7.4 We arrived at the family of PID controllers. For other second order plants, or when the trajectory generator is not identical to the plant we will not get generally such structure.
Notice that the solution is independent of the reference trajectory generator input, u_r. This means that the optimality criterion induced some smoothness conditions on the trajectory and its derivative.

8. Third Order System

8.1 As an example of a third order system we consider a DC motor. The differential equation that are describing a linear DC motor are

$$\begin{aligned} m\ddot{x} &= k_F I - D\dot{x} \\ v - k_E\dot{x} &= L\frac{dI}{dt} + RI \end{aligned} \tag{33}$$

where

m	- is the mass
x	- is the position
k_F	- is the force constant
I	- is the current of the motor
D	- is the friction coefficient
v	- is the applied voltage
k_E	- is the back emf coefficient
L	- is the inductance of the motor
R	- is the resistance of the motor.

an we have

$$\begin{bmatrix} ms^2 + Ds & -k_F \\ k_E s & Ls + R \end{bmatrix} \begin{bmatrix} x \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ v \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} x \\ I \end{bmatrix} = \frac{1}{(ms^2 + Ds)(Ls + R) + k_F k_E s} \begin{bmatrix} Ls + R & k_F \\ -k_E & ms^2 + Ds \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix} \tag{35}$$

$$\frac{x}{v} = \frac{k_F}{(ms^2 + Ds)(Ls + R) + k_F k_E s}$$

$$\frac{I}{v} = \frac{ms^2 + Ds}{(ms^2 + Ds)(Ls + R) + k_F k_E s} \qquad (36)$$
$$\frac{x}{I} = \frac{k_F}{ms^2 + Ds}$$

From equation (31), one can see that our method applies a PI controller on each state. This is denoted as

$$C = k + \frac{k_I}{s} \qquad (37)$$

The full state feedback controller (parallel structure) is

$$u = C_1(x - x_r) + C_2(\dot{x} - \dot{x}_r) + C_3(I - I_r) \qquad (38)$$

This structure is presented in figure 1. Usually the control engineers prefer the architecture, that is presented in figure 2, the cascade architecture. Then the controller is

$$u = C_3\left\{(I - I_r) + \frac{C_2}{C_3}\left[(\dot{x} - \dot{x}_r) + \frac{C_1}{C_2}(x - x_r)\right]\right\} \qquad (39)$$

and we have

$$C_I = C_3 = k_3 + \frac{k_{I3}}{s}\text{, the controller of the current loop;}$$
$$C_v = \frac{C_2}{C_3} = \frac{k_2 s + k_{I2}}{k_3 s + k_{I3}}\text{, the controller of the velocity loop;} \qquad (40)$$
$$Cp = \frac{C_1}{C_2} = \frac{k_1 s + k_{I1}}{k_2 s + k_{I2}}\text{, the controller of the position loop;}$$

i.e. PI controller in the current loop and lead-lag/lag-lead controllers in the velocity and position loops. Notice that there are only six free parameters and not ten as one might guess.

8.2 In the previous examples we assumed measurement of all state variables. If we assume that only the error is measured then the generalized PID controller is

$$
\begin{aligned}
u &= [C_1 + C_2 s + C_3 \frac{I}{x}]e \\
\frac{u}{e} &= k_1 + \frac{k_{I1}}{s} + [k_2 + \frac{k_{I2}}{s}]s + [k_3 + \frac{k_{I3}}{s}]\frac{ms^2 + Ds}{k_F} \\
&= \frac{\frac{m}{k_F}k_3 s^3 + (k_2 + k_{I3}\frac{m}{k_F} + \frac{D}{k_F}k_3)s^2 + (k_1 + k_{I2} + \frac{D}{k_F}k_{I3})s + k_{I1}}{s} \\
&= \frac{C_a s^3 + C_v s^2 + C_p s + C_I}{s}
\end{aligned}
\qquad (41)
$$

This is a generalized PID controller denoted here PID^2 controller. This controller is not proper and it might create problems in realization.

8.3 Here we assume that the plant and the trajectory generator are a general third order system

$$A = A_r = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix},\ B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},\ C = C_r = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

$$\text{i.e. } \frac{y}{u} = H(s) = H_r(s) = \frac{b_1 s^2 + (a_1 b_1 + b_2)s + a_1 b_2 + b_1 a_2 + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}, \tag{42}$$

$$\begin{aligned} \frac{x_2}{x_1} &= \frac{b_2 s^2 + (a_1 b_2 + b_3)s + a_3 b_1}{b_1 s^2 + (a_1 b_1 + b_2)s + a_1 b_2 + b_1 a_2 + b_3} \\ \frac{x_3}{x_1} &= \frac{b_3 s^2 - (a_3 b_1 + a_2 b_2)s - a_3 b_2}{b_1 s^2 + (a_1 b_1 + b_2)s + a_1 b_2 + b_1 a_2 + b_3} \end{aligned} \tag{43}$$

We want to force zero steady state tracking error on the output. Here $y = x_1, y_r = x_{1r}$ but $\dot{y} \neq x_2, \dot{y}_r \neq x_{2r}$. Then, since the tracking error is

$$\dot{\eta} = e = y - y_r = x_1 - x_{1r} \tag{44}$$

we have

$$\begin{aligned} u &= \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & -k_1 & -k_2 & -k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \eta \\ x_{1r} \\ x_{2r} \\ x_{3r} \end{bmatrix} \\ &= k_1(x_1 - x_{1r}) + k_2(x_2 - x_{2r}) + k_3(x_3 - x_{3r}) + k_4 \eta \\ &= k_1 e + k_2(x_2 - x_{2r}) + k_3(x_3 - x_{3r}) + k_4 \int e dt \end{aligned} \tag{45}$$

to proceed we write the controller in Laplace domain

$$\begin{aligned} u &= k_1 e + k_2(x_2 - x_{2r}) + k_3(x_3 - x_{3r}) + \frac{k_4}{s} e \\ &= k_1 e + k_2 \frac{x_2}{x_1}(x_1 - x_{1r}) + k_3 \frac{x_2}{x_1}(x_1 - x_{1r}) + \frac{k_4}{s} e \\ &= [k_1 + k_2 \frac{b_2 s^2 + (a_1 b_2 + b_3)s + a_3 b_1}{b_1 s^2 + (a_1 b_1 + b_2)s + a_1 b_2 + b_3} \\ &\qquad + k_3 \frac{b_3 s^2 - (a_3 b_1 + a_2 b_2)s - a_3 b_2}{b_1 s^2 + (a_1 b_1 + b_2)s + a_1 b_2 + b_3} + \frac{k_4}{s}]e \end{aligned} \tag{46}$$

$$\frac{u}{e} = \frac{(\quad)s^3 + (\quad)s^2 + (\quad)s + (\quad)}{s(b_1 s^2 + (a_1 b_1 + b_2)s + a_1 b_2 + b_3)} \tag{47}$$

This is a proper PID^2 controller, i.e. no direct derivative is required. To see this we write

$$\frac{u}{e} = k_p + k_d \frac{s(sT_d + 1)}{s^2 2\zeta\omega_d s + \omega_d^2} + \frac{k_I}{s} \tag{48}$$

$$\frac{u}{e} = \frac{(\quad)s^3 + (\quad)s^2 + (\quad)s + (\quad)}{s(s^2 + 2\zeta\omega_d s + \omega_d^2)} \tag{49}$$

That is, for 3rd order plant with zeros, the optimal controller is a proper PID^2 controller. Notice that the controller cancels out the zeros of the plant.

9. Fifth Order System

This section deals with a fifth order system. It is, for emphasis of its relevance, a DC motor with one flexible mode. The differential equation that are describing the system are

$$\begin{aligned} m_1 \ddot{x}_1 &= -K(x_1 - x_2) - D(\dot{x}_1 - \dot{x}_2) + k_F I \\ m_2 \ddot{x}_2 &= K(x_1 - x_2) + D(\dot{x}_1 - \dot{x}_2) \\ v - k_E \dot{x}_1 &= L\frac{dI}{dt} + RI \end{aligned} \tag{50}$$

and we have

$$\begin{bmatrix} Ls + R & k_E s & 0 \\ -k_F & m_1 s^2 + Ds + K & -(Ds + K) \\ 0 & -(Ds + K) & m_2 s^2 + Ds + K \end{bmatrix} \begin{bmatrix} I \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} \tag{51}$$

$$\frac{x_1}{v}=\frac{1}{s}\frac{k_F}{m_1}\frac{s^2+\frac{D}{m_2}s+\frac{K}{m_2}}{s(Ls+R)(s^2+(\frac{1}{m_1}+\frac{1}{m_2})Ds+(\frac{1}{m_1}+\frac{1}{m_2})K)+k_E\frac{k_F}{m_1}(s^2+\frac{D}{m_2}s+\frac{K}{m_2})}$$

$$\frac{x_2}{v}=\frac{1}{s}\frac{k_F}{m_1}\frac{\frac{D}{m_2}s+\frac{K}{m_2}}{s(Ls+R)(s^2+(\frac{1}{m_1}+\frac{1}{m_2})Ds+(\frac{1}{m_1}+\frac{1}{m_2})K)+k_E\frac{k_F}{m_1}(s^2+\frac{D}{m_2}s+\frac{K}{m_2})}$$

$$\frac{I}{v}=\frac{s}{L}\frac{s^2+(\frac{1}{m_1}+\frac{1}{m_2})Ds+(\frac{1}{m_1}+\frac{1}{m_2})K}{s(Ls+R)(s^2+(\frac{1}{m_1}+\frac{1}{m_2})Ds+(\frac{1}{m_1}+\frac{1}{m_2})K)+k_E\frac{k_F}{m_1}(s^2+\frac{D}{m_2}s+\frac{K}{m_2})}$$

$$\frac{x_2}{x_1}=\frac{1}{s}\frac{k_F}{m_1}\frac{\frac{D}{m_2}s+\frac{K}{m_2}}{s^2+\frac{D}{m_2}s+\frac{K}{m_2}}$$

$$\frac{x_1}{I}=\frac{1}{s^2}\frac{k_F}{m_1}\frac{s^2+\frac{D}{m_2}s+\frac{K}{m_2}}{s^2+(\frac{1}{m_1}+\frac{1}{m_2})Ds+(\frac{1}{m_1}+\frac{1}{m_2})K} \tag{52}$$

The full state feedback controller (parallel structure) is

$$u=C_1(x_1-x_{1r})+C_2(\dot{x}_1-\dot{x}_{1r})+C_3(x_2-x_{2r})+C_4(\dot{x}_2-\dot{x}_{2r})+C_5(I-I_r) \tag{53}$$

9.1 There are many possibilities of measurements. Here we assume that the position, x_1 , velocity, $_1$, and current, I, of the motor are measured. Then we can write

$$u=(C_1+C_3\frac{x_2}{x_1})(x_1-x_{1r})+(C_2+C_4\frac{x_2}{x_1})(\dot{x}_1-\dot{x}_{1r})+C_5(I-I_r) \tag{54}$$

$$u=C_1'(x_1-x_{1r})+C_2'(\dot{x}_1-\dot{x}_{1r})+C_5(I-I_r)$$

The cascade architecture is

$$u=C_5\left\{(I-I_r)+\frac{C_2'}{C_5}\left[(\dot{x}-\dot{x}_r)+\frac{C_1'}{C_2'}(x-x_r)\right]\right\} \tag{55}$$

and we have

$$C_I=C_5=k_5+\frac{k_{I5}}{s},\ \text{the controller of the current loop;} \tag{56}$$

$$C_v = \frac{C_2'}{C_5} = \frac{C_2 + C_4 \frac{x_2}{x_1}}{C_5}; \qquad \text{the controller of the velocity loop}$$

$$= \frac{(k_2 s + k_{I2})(s^2 + \frac{D}{m_2}s + \frac{K}{m_2}) + (k_4 s + k_{I4})(\frac{D}{m_2}s + \frac{K}{m_2})}{(k_5 s + k_{I5})(s^2 + \frac{D}{m_2}s + \frac{K}{m_2})} = \quad , \tag{57}$$

$$C_p = \frac{C_1'}{C_2} = \frac{C_1 + C_3 \frac{x_2}{x_1}}{C_2 + C_4 \frac{x_2}{x_1}}; \qquad \text{the controller of the position loop;}$$

$$= \frac{(k_1 s + k_{I1})(s^2 + \frac{D}{m_2}s + \frac{K}{m_2}) + (k_3 s + k_{I3})(\frac{D}{m_2}s + \frac{K}{m_2})}{(k_2 s + k_{I2})(s^2 + \frac{D}{m_2}s + \frac{K}{m_2}) + (k_4 s + k_{I4})(\frac{D}{m_2}s + \frac{K}{m_2})} = \tag{58}$$

i.e. PI controller in the current loop and third order proper controllers in the velocity and position loops. Notice that there are only ten free parameters and not fourteen as one might guess. The total controller order is 7 relative to 5 with full state feedback.

9.2 Here we assume that the current of the motor, I, the position, x_2, and velocity, 2, are measured. Then we can write

$$u = (C_3 + C_1 \frac{x_1}{x_2})(x_2 - x_{2r}) + (C_4 + C_2 \frac{x_1}{x_2})(\dot{x}_2 - \dot{x}_{2r}) + C_5(I - I_r) \tag{59}$$

This architecture can not be implemented, as the controllers are not proper. Even the cascade architecture is not proper. To arrive at a proper controller we proceed as follows: we can derive that

$$x_1 = \frac{\frac{k_F}{m_1}}{s^2 + \frac{D}{m_1}s + \frac{K}{m_1}} I + \frac{\frac{D}{m_1}s + \frac{K}{m_1}}{s^2 + \frac{D}{m_1}s + \frac{K}{m_1}} x_2 = (\left.\frac{x_1}{I}\right|_{x_2=0})I + (\left.\frac{x_1}{x_2}\right|_{I=0})x_2 \tag{61}$$

Then

$$u = (C_1 + C_3 \left.\frac{x_2}{x_1}\right|_{I=0})(x - x_{1r}) + (C_2 + C_4 \left.\frac{x_2}{x_1}\right|_{I=0})(\dot{x}_1 - \dot{x}_{1r}) + [C_5 + (C_1 + C_2 s)\left.\frac{x_1}{I}\right|_{x_2=0}](I - I_r) \tag{62}$$

$$u = C_3'(x_2 - x_{2r}) + C_4'(\dot{x}_2 - \dot{x}_{2r}) + C_5'(I - I_r) \tag{63}$$

The cascade architecture is

$$u = C_5'\left\{(I - I_r) + \frac{C_4'}{C_5'}\left[(\dot{x} - \dot{x}_r) + \frac{C_3'}{C_4'}(x - x_r)\right]\right\} \tag{64}$$

and we have the controller of the current loop

$$\begin{aligned} C_I = C_5' &= [C_5 + (C_1 + C_2 s)\frac{x_1}{I}\bigg|_{x_2=0}] \\ &= \frac{(k_5 s + k_{I5})(s^2 + \frac{D}{m_1}s + \frac{K}{m_1}) + \frac{k_F}{m}(k_2 s^2 + (k_1 + k_{I2})s + k_{I1})}{s(s^2 + \frac{D}{m_1}s + \frac{K}{m_1})} \end{aligned} \tag{65}$$

the controller of the velocity loop

$$\begin{aligned} C_v = \frac{C_4'}{C_5'} &= \frac{C_4 + C_2 \frac{x_1}{x_2}\bigg|_{I=0}}{C_4 + (C_1 + C_2 s)\frac{x_1}{I}\bigg|_{x_2=0=0}} \\ &= \frac{(k_4 s + k_{I4})(s^2 + \frac{D}{m_1}s + \frac{k}{m_1}) + (k_2 s + k_{I2})(\frac{D}{m_1}s + \frac{k}{m_1})}{(k_5 s + k_{I5})(s^2 + \frac{D}{m_1}s + \frac{k}{m_1}) + \frac{k_F}{m_1}(k_2 s^2 + (k_1 + k_{I2})s + k_{I1})} \end{aligned} \tag{66}$$

the controller of the position loop

$$\begin{aligned} C_p = \frac{C_3'}{C_4'} &= \frac{C_3 + C_1 \frac{x_1}{x_2}\bigg|_{I=0}}{C_4 + C_2 \frac{x_1}{x_2}\bigg|_{I=0}} \\ &= \frac{(k_3 s + k_{I3})(s^2 + \frac{D}{m_1}s + \frac{k}{m_1}) + (k_1 s + k_{I1})(\frac{D}{m_1}s + \frac{k}{m_1})}{(k_4 s + k_{I4})(s^2 + \frac{D}{m_1}s + \frac{k}{m_1}) + (k_2 s + k_{I2})(\frac{D}{m_1}s + \frac{k}{m_1})} \end{aligned} \tag{67}$$

these are third order proper controllers in the current, velocity and position loops. Notice that there are only ten free parameters and not eighteen as one might guess. The total controller is of order 9 relative to 5 with full state feedback (parallel architecture).

9.3 Here we assume that only the error, $e = x_1 - x_{1r}$, is measured then the generalized PID controller is

$$u = [C_1 + C_2 s + (C_3 + C_4 s)\frac{x_2}{x_1} + C_5 \frac{I}{x_1}]e \tag{68}$$

$$\frac{u}{e} = [k_1 + \frac{k_{I1}}{s} + (k_2 + \frac{kI2}{s})s] + [k_3 + \frac{k_{I3}}{s} + (k_4 + \frac{k_{I4}}{s})s]\frac{\frac{D}{m_2}s + \frac{K}{m_2}}{s^2 + \frac{D}{m_2}s + \frac{K}{m_2}}$$

$$[k_5 + \frac{k_{I5}}{s}]\frac{m_1}{k_F}\frac{s^2(s^2 + (\frac{1}{m_2} + \frac{1}{m_1})Ds + (\frac{1}{m_2} + \frac{1}{m_1})K)}{s^2 + \frac{D}{m_2}s + \frac{K}{m_2}} \tag{69}$$

$$= \frac{C_\zeta s^5 + C_j s^4 + C_a s^3 + C_v s^2 + C_p s + C_I}{s(s^2 + 2\zeta\omega_d s + \omega_d^2)}$$

This is a generalized PID controller denoted here PID^4 controller. This controller is not proper and it might create problems in realization.

10. Discussion

Similarly, for higher order systems generalized PID controller can be derived. These controllers have the structure of proper and not proper PID^{n-1}, where n is the order of the plant. The controller's poles cancel out the plant's zeros.
Although from input-output transfer function, there is no difference between the full state feedback PI controller and the generalized PID^{n-1} controller, there is difference with respect to the response to initial conditions, effects of saturation etc..

11. Conclusions

By the use of Linear Quadratic Tracking theory we formulated a control-tracking problem and showed those cases when their solution gives the PID family of controllers. This way we avoided heuristics and gave a systematic approach to explanation for the good performance of the PID controllers. The PID controller architecture is optimal for Linear Quadratic Tracking problem of a 2nd order systems with no zero. The reference trajectory is generated by a system identical to the plant.

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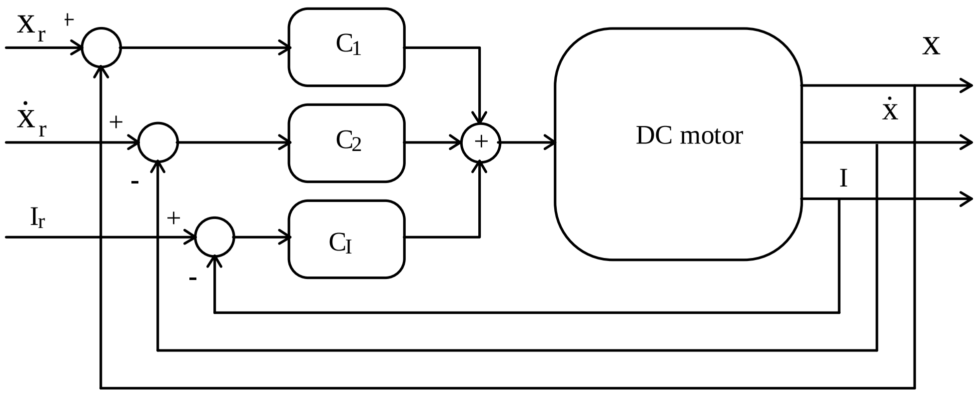

Figure 1: Parallel PI controller structure of a DC motor.

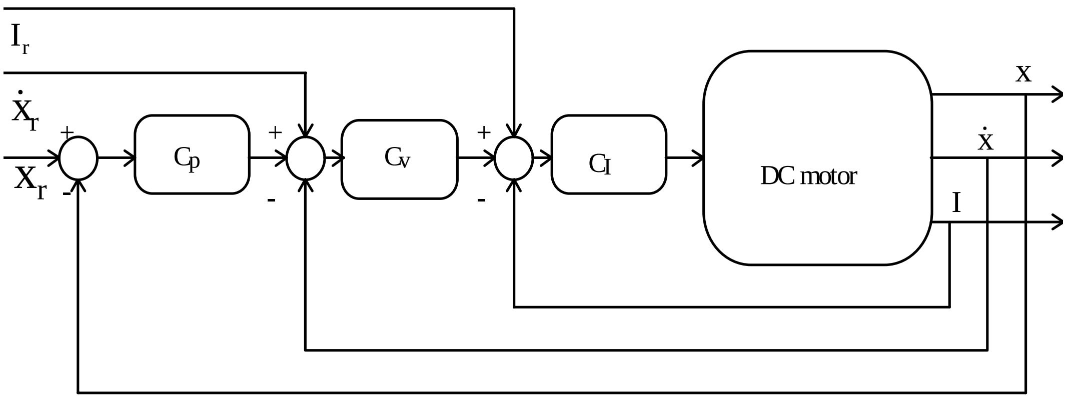

Figure 2: Cascade PI controller structure of a DC motor.