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Adaptive Control of a Time-Varying Parabolic System: Averaging Analysis†

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Abstract

Related to the error dynamics of an adaptive system, averaging theorems are developed for coupled differential equations which consist of ordinary differential equations and a parabolic partial differential equation. The results are then applied to the convergence analysis of the parameter estimate errors to zero in the model reference adaptive control of a nonautonomous parabolic partial differential equation with slowly time varying parameters.

Keywords: Adaptive control, averaging method, convergence analysis, parabolic partial differential equation, slow varying system

1 Introduction

In recent control literature the adaptive control/identification of distributed parameter systems are getting more attention (Balas, 1983; Baumeister and Scondo, 1987; Kobayashi, 1988; Banks and Kunisch, 1989; Miyasato, 1990; Hong and Bentsman, 1992; Bentsman et al., 1992; Hong and Bentsman, 1994a,b; Demetriou and Rosen, 1994; Hong, 1997; Baumeister et al., 1997). There have been increasing efforts for the last several years in explicit incorporation of time-varying parameters into adaptive control analysis (Tsakalis and Ioannou, 1993). Also the averaging method has been emerged as a powerful tool for the analysis of adaptive algorithms. The aim of this note is to bring these two streams together with the hope that the averaging method can yield extra insights on the ada-

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ptive control process in the presence of time-varying parameters. The adaptive control design follows that in (Hong and Bentsman, 1994a), which dealt with an autonomous parabolic plant, but the presence of time-varying parameters considered here calls for new averaging theorems. It is shown that a similar design methodology can be extended to the systems with time-varying coefficients.

In Section II, the model reference adaptive control (MRAC) of a time-varying parabolic system is firstly introduced with the Lyapunov redesign method. In Section III, averaging theorems are developed for coupled ordinary and partial differential equations, which are motivated from the error dynamics of the adaptive control. Next the convergence of parameter errors to zero is investigated through averaging. Even both the treatment of time-varying parameters of the parabolic system and the averaging analysis for coupled ODE/PDE are new, the main focus of this note is to show the convergence of parameter errors to zero through averaging. For detailed construction of adaptive controller or related issues refer to (Astrom and Wittenmark, 1995; Sastry and Bodson, 1989; Bank and Kunisch, 1989). In this manuscript, B, C and L denote generic constants.

2 Problem setup

Consider a nonautonomous parabolic partial differential equation as

$$\begin{aligned} &x_t = a(et)x_{xx} + b(et)x + u\,, \quad 0 \le x \le 1\,, \quad t \ge 0\,, \\ &x(0,t) = g_a(t)\,, \quad x(1,t) = g_b(t)\,, \\ &x(x,0) = x_0(x)\,, \end{aligned} \tag{2.1}$$

where $u = u(x,t)$ is the control input, $x = x(x,t)$ is the distributed state, $x_t = dx/dt$, and $x_{xx} = \partial^2 x/\partial x^2$. The coefficients $a(et)$ and $b(et)$ are bounded slowly varying parameters, where e indicates the slowly varying nature of the system. Specifically in heat transfer they are referred to as heat conductivity and radiation factor, respectively. Assume $0 < e << 1$, so that the variation is slow, i.e. $da(et)/dt = O(e)$ and $db(et)/dt = O(e)$. However, the amplitude variations of the parameters are large, i.e. $a(et) = O(1)$, $b(et) = O(1)$. For instance, $a(et) = 2 - \sin(et)$, $b(et) = -2 + \cos(et)$ would be plausible examples.

Along with system (2.1) consider a reference model with the same boundary conditions as

$$\begin{aligned} &x_{mt} = a_m x_{mxx} + b_m x_m + r\,, \quad 0 \le x \le 1\,, \quad t \ge 0 \\ &x_m(0,t) = g_a(t)\,, \quad x_m(1,t) = g_b(t)\,, \\ &x_m(x,0) = x_{m0}(x)\,, \end{aligned} \tag{2.2}$$

where the subscript m stands for model, and $r = r(x,t)$ is the reference signal. Note that $g_a(t)$ and $g_b(t)$ could also be thought as the boundary reference signals. Constant coefficients $a_m > 0$ and $b_m < 0$ are assumed.

Now adopting the procedure in (Hong and Bentsman, 1994a), consider a control law of the form

$$u = (a_m - \hat{a})x_{xx} + (b_m - \hat{b})x + r\,, \tag{2.3}$$

where $\hat{a}$ and $\hat{b}$ are adaptive estimates to be specified. Substituting (2.3) into (2.1) yields the closed loop plant equation as

$$x_t = (a_m - \tilde{a})x_{xx} + (b_m - \tilde{b})x + r\,, \tag{2.4}$$

where $\tilde{a} \overset{D}{=} \hat{a} - a$, $\tilde{b} \overset{D}{=} \hat{b} - b$ are the parameter estimation errors. Note that if $\tilde{a} = \tilde{b} = 0$, then (2.4) is exactly the same as (2.2). Introducing the state error $e = x - x_m$, the following state error equation is derived.

$$\begin{aligned} &e_t = (a_m - \tilde{a})e_{xx} + (b_m - \tilde{b})e - (\tilde{a}x_{mxx} + \tilde{b}x_m), \qquad (2.5)\\ &e(0,t) = e(1,t) = 0,\\ &e(x,0) = x_0(x) - x_{m0}(x). \end{aligned}$$

Now, consider a functional $V : L_2(0,1) \times R^2 \to R^+$ such that

$$V(t) = \frac{1}{2}\langle e, e\rangle + \frac{1}{2e}(\tilde{a}^2 + \tilde{b}^2), \qquad (2.6)$$

where $\langle \cdot, \cdot \rangle$ is the inner product in space $L_2(0,1)$ defined as $\langle h, g\rangle = \int_0^1 h(x,t)g(x,t)dx$, and with the induced norm $\| \cdot \|$. Differentiating (2.6) with respect to t along (2.5) yields

$$\begin{aligned} \dot{V}(t) &= \langle e, \dot{e}\rangle + \frac{1}{e}(\tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}})\\ &= a_m\langle e_x, e_{xx}\rangle + b_m\langle e, e\rangle + \tilde{a}\left(-\langle e, e_{xx}\rangle - \langle e, x_{mxx}\rangle + \frac{1}{e}\dot{\tilde{a}}\right) \qquad (2.7)\\ &\quad + \tilde{b}\left(-\langle e, e\rangle - \langle e, x_m\rangle + \frac{1}{e}\dot{\tilde{b}}\right). \end{aligned}$$

Choose

$$\dot{\tilde{a}} = e\left(\langle e, e_{xx}\rangle + \langle e, x_{mxx}\rangle\right) - s\hat{a} - \dot{a}, \qquad (2.8a)$$

$$\dot{\tilde{b}} = e\left(\langle e, e\rangle + \langle e, x_m\rangle\right) - s\hat{b} - \dot{b}. \qquad (2.8b)$$

where s is a positive constant (s-modification, see Tsakalis and Ioannou, 1993). Substituting (2.8a,b) into (2.7) yields.

$$\begin{aligned} \dot{V} &= -a_m\langle e_x, e_x\rangle + b_m\langle e, e\rangle\\ &\quad - \frac{s}{2e}\left\{\tilde{a}^2 + \tilde{b}^2 + \left(\hat{a} + \frac{\dot{a}}{s}\right)^2 + \left(\hat{b} + \frac{\dot{b}}{s}\right)^2 - \left(a + \frac{\dot{a}}{s}\right)^2 - \left(b + \frac{\dot{b}}{s}\right)^2\right\} \qquad (2.9)\\ &\le -a_1\|e\|^2 + a_2(\tilde{a}, \tilde{b}). \end{aligned}$$

where a_1 and a_2 are positive constants. The existence and uniqueness of the solutions of (2.5) and (2.8a,b) are addressed in Appendix A. Now the above development is summarized as follows

Theorem 1: Consider equations (2.5) and (2.8a,b). Assume that a and b are bounded with bounded derivatives. Then, all signals in the closed loop are uniformly ultimately bounded.

Proof: The conclusion directly follows by extending the work of (Corless and Leitmann, 1981).

Remark 1: If a and b are constant, i.e. $\dot{a} = \dot{b} = 0$, then a_2 in (2.9) can be set to zero. This implies that (2.6) is a Lyapunov function and therefore the stability of an equilibrium point $(e, \tilde{a}, \tilde{b}) = (0,0,0)$ is guaranteed. Furthermore, the convergence of state error e to zero is also guaranteed by Barbalat's Lemma (Narendra and Annaswamy, 1989) or by the uniqueness and semigroup properties of the solution (Hong, 1997). In our case, however, the time-varying behavior of the system does not allow this situation in general.

Remark 2: It will be shown in Section III, however, that if the system is slowly varying, $\tilde{a}$ and $\tilde{b}$ converge to zero and therefore a_2 converges to zero. This will eventually achieve the model following control problem for the time-varying plant. In averaging analysis, the adaptation law is also assumed slow, but it is relatively faster than the time-varying behavior of the plant.

Remark 3: The differential equations for the controller parameters in (2.3) are written as

$$\begin{aligned} \dot{\hat{a}} &= e\left(\langle e, e_{xx}\rangle + \langle e, x_{mxx}\rangle\right) - s\hat{a}, \\ \dot{\hat{b}} &= e\left(\langle e, e\rangle + \langle e, x_m\rangle\right) - s\hat{b}. \end{aligned} \tag{2.10}$$

Two things are noted for (2.10): The tuning laws are implementable and the positive constant s has been intentionally introduced to improve the robustness of the adaptive system. It is remarked that the same conclusions of averaging analysis can be deduced with or without this s -modification.

Now we turn to the analysis of parameter error convergence to zero through averaging. Define $q = \left[\tilde{a}\ \tilde{b}\right]^T$, then (2.8a,b) can be written as follows

$$\dot{q} = ef\left(t, q, e, et\right) \tag{2.11}$$

Note that $f(\cdot)$ is a functional of e rather than a function of e. The explicit appearance of time t as an argument in f comes from the exogenous signal x. The first e in front of f denotes the adaptation gain and the second e, as an argument of f, denotes the existence of slow-varying parameters. If the two e 's are different, the bigger one can be chosen as the representative one.

In (Bentsman et al., 1992; Hong and Bentsman, 1994b), averaged systems corresponding to (2.8a,b) have been explicitly computed. Associated with (2.5) a frozen state $e_{t,\tilde{q}}(\cdot)$ is defined through

$$e_{t|t,\tilde{q}} = (a_m - \tilde{a})e_{xx|t,\tilde{q}} + (b_m - \tilde{b})e_{t,\tilde{q}} - \left[x_{mxx} \quad x_m\right]\tilde{q} \tag{2.12}$$

where parameters $\tilde{a}$ and $\tilde{b}$ are assumed to be frozen, and

$$\begin{aligned} e_{t|t,\tilde{q}} &= \partial e_{t,\tilde{q}}(x,t) \big/ \partial t, \\ e_{xx|t,\tilde{q}} &= \partial^2 e_{t,\tilde{q}}(x,t) \big/ \partial x^2 . \end{aligned}$$

3 Averaging analysis

Consider a coupled system as

$$\dot{q}^e = ef\left(t, q^e, e^e, et\right), \tag{3.1}$$

$$e_t^e = \left(a_m - \tilde{a}^e\right)e_{xx}^e + \left(b_m - \tilde{b}^e\right)e^e - \left[\tilde{a}^e x_{mxx} + \tilde{b}^e x_m\right]. \tag{3.2}$$

(3.1) and (3.2) correspond to (2.11) and (2.5), respectively. The superscript e is affixed to denote the variables in fast time t prior to a time scaling. An averaged system associated with (3.1) is introduced as

$$\dot{q}_{av}(t) = ef_{av}\left(q_{av}(t), et\right) \tag{3.3}$$

where the averaged function is defined by

$$f_{av}\left(q_{av},et\right)=\lim_{T\to\infty}\frac{1}{T}\int_{t}^{t+T}f\left(s,q_{av},e^{e}(s),et\right)ds \tag{3.4}$$

if the limit exists uniformly in t. It is assumed that $\left|f(s,0,0,0)\right|\le B$, and $\left|f_{av}(0,0)\right|\le B$. With a new time scale (slow time) $t=et$, (3.3) can be rewritten as

$$\dot{q}_a(t)=f_{av}(q_a(t),t) \tag{3.5}$$

The closeness of the two solutions of primary system (3.1) and averaged system (3.3) is now investigated. With a sufficiently small e, (3.1) is integrated in t and is re-scaled to the slow time $t=et$ as follows

$$q_e(t)=q^{e}(te^{-1})=q^{e}(0)+e\int_{0}^{te^{-1}}f_s\,ds\ , \tag{3.6}$$

where $f_s=f\left(s,q^{e}(s),e^{e}(s),es\right)$. The subscript e denotes the re-scaled variable in the slow time. Similarly, for averaged system (3.3) and re-scaled system (3.5) the following is obtained

$$q_a(t)=q_{av}(te^{-1})=q_a(0)+e\int_{0}^{te^{-1}}f_{av}\left(q_a(es),es\right)ds\ . \tag{3.7}$$

Now subtract (3.7) from (3.6), then we have

$$D_e(t)=q_e(t)-q_a(t)=D_e(0)+I_e(t), \tag{3.8}$$

where

$$D_e(0)=q^{e}(0)-q_a(0), \tag{3.9}$$

$$I_e(t)=e\int_{0}^{te^{-1}}\left[f_s-f_{av}(q_a(es),es)\right]ds\ . \tag{3.10}$$

By introducing a frozen state as an intermediate state between f_s and f_{av}, the evaluation of (3.10) can be carried out in two stages as follows

$$I_e(t)=I_{eA}(t)+I_{eB}(t),$$

where $I_{eA}(t)=e\int_{0}^{te^{-1}}\left[f_s-f\left(s,q_a(es),e_{es,q_a(es)}(s),es\right)\right]ds\ ,$ (3.12)

$$I_{eB}(t)=e\int_{0}^{te^{-1}}\left[f\left(s,q_a(es),e_{es,q_a(es)}(s),es\right)-f_{av}\left(q_a(es),es\right)\right]ds\ . \tag{3.13}$$

Note that the first integral $I_{eA}(t)$ deals with the approximation of the fast system f_s by the frozen system $f_{t,q_a}(\cdot)$, whereas the second integral $I_{eB}(t)$ deals with the approximation of the frozen system $f_{t,q_a}(\cdot)$ by the averaged system $f_{av}(\cdot)$.

Now, specific bounds for both integrals (3.12), (3.13) are obtained. For proper implication, a bound for the second integral $I_{eB}(t)$ is firstly derived as follows.

$$\left|I_{eB}(t)\right|\le C_T(e),\quad 0\le t\le T, \tag{3.14}$$

where $C_T(e)\to 0$ as $e\to 0$. Detailed derivations of (3.14) are gathered in Appendix B. In obtaining (3.14), a general Lipschitz condition on f has been assumed

$$\left|f(s,q,e,t)-f(s,q',e',t')\right|\le L_1\left|q-q'\right|+L_1\left\|e-e'\right\|+L_1\left|t-t'\right|,\quad L_1=\text{constant}>0. \tag{3.15}$$

The satisfaction of (3.15) for our case is obvious. Now for the first integral $I_{eA}(t)$, a bound as following is obtained

$$
\begin{aligned}
\left|I_{eA}(t)\right| &= e\int_0^{te^{-1}} \left| f\left(s, q^e(s), e^e(s), es\right) - f\left(s, q_a(es), e_{es, q_a(es)}(s), es\right)\right| ds \\
&\le eL_1 \int_0^{te^{-1}} \left(\left|D_e(es)\right| + \left\|e^e(s) - e_{es, q_a(es)}(s)\right\| \right) ds .
\end{aligned}
\tag{3.16}
$$

It is shown in Appendix C that (3.15) again ensures

$$\int_0^{te^{-1}} \left\|e^e(s) - e_{es, q_a(es)}(s)\right\| ds \le B . \tag{3.17}$$

Finally, the combination of (3.8), (3.14) and (3.16)-(3.17) yields

$$\left|D_e(t)\right| \le \left|D_e(0)\right| + L_1 \int_0^t \left|D_e(s)\right| ds + C_T(e) + eL_1 B . \tag{3.18}$$

The Bellman-Gronwall inequality then gives

$$\left|D_e(t)\right| \le e^{L_1 t}\left(D_e(0) + C_T(e) + eL_1 B\right), \quad 0 \le t \le T \tag{3.19}$$

which again implies

$$\sup_{0\le t\le T} \left|D_e(t)\right| = \sup_{0\le t\le T} \left|q_e(t) - q_a(t)\right| \le B_T(e) \tag{3.20}$$

where $B_T(e) \to 0$ as $e \to 0$.

All above development is now summarized as follows.

Theorem 2: Consider (3.1)-(3.2) and (3.3) with appropriate regularity conditions. Then for fixed T and sufficiently small e

$$\sup_{0\le t\le T/e} \left|q^e(t) - q_{av}(t)\right| \le B_T(e) \tag{3.21}$$

where $B_T(e) \to 0$ as $e \to 0$.

Remark 4: Theorem 2 asserts the closeness of the two solutions of (3.1) and (3.3) for sufficiently small e. It does not yet connect the stability properties between (3.1)-(3.2) and (3.3). However with further assumptions on f_{av} the following theorem can be stated.

Theorem 3: Consider (3.1)-(3.2) and (3.3) with appropriate regularity conditions. Assume further that averaged system (3.3) is exponentially stable, then the trivial solutions $q^e(t) = 0$, $e^e(t) = 0$ are exponentially stable for sufficiently small e.

Proof: The proof follows exactly that of Theorem 4.3 in (Hong and Bentsman, 1994a).

4 Application and Simulations

In this section the averaging theorems are applied to the convergence analysis of the controller parameter errors to zero.

1) Linear Analysis: To see an explicit expression for the averaged system, (2.5) and (2.8a,b) are linearized, following the work of Anderson et al. (1986), around zero. Then, we have

$$\dot{\tilde{a}} = e < e, x_{mxx} > - \dot{a} \tag{3.22a}$$

$\dot{\tilde{b}} = e < e, x_m > - \dot{b}$ (3.22b)

$$e_t = a_m e_{xx} + b_m e - (\tilde{a} x_{mxx} + \tilde{b} x_m),\ e(0,t) = e(1,t) = 0,\ e(x,0) = 0 \tag{3.23}$$

$$x_{mt} = a_m x_{mxx} + b_m x_m + r,\ x_m(0,t) = x_m(1,t) = 0,\ x_m(x,0) = 0 \tag{3.24}$$

Note that all initial conditions are set to zero since they do not affect the final form of averaged system. The solution of (3.24) with $r(x,t) = f(x)$ is of the form

$$x_m(x,t) = \sum_{n=1}^{\infty} \frac{f_n}{k_n}(1 - e^{-k_n t}) j_n(x) \tag{3.25}$$

where $k_n = a_m (np)^2 - b_m$, $j_n(x) = \sin(npx)$, and $f_n = 2 < f(x), j_n(x) >$. Similarly, the solution of (3.23) is of the form

$$e(x,t) = \sum_{n=1}^{\infty} \left[\int_0^1 e^{-k_n(t-s)} F_n(s)\, ds \right] j_n(x) \tag{3.26}$$

where $F_n(t) = -2 < \tilde{a} x_{mxx} + \tilde{b} x_m, j_n(x) > = \dfrac{f_n (np)^2}{k_n}(1 - e^{-k_n t})\tilde{a} - \dfrac{f_n}{k_n}(1 - e^{-k_n t})\tilde{b}$.

The substitution of (3.25) and (3.26) into (3.22a,b) and the application of (3.4) to the right hand side of (3.22a,b) yield

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{a}} \\ \dot{\tilde{b}} \end{bmatrix} &= -e \begin{bmatrix} \displaystyle\sum_{n=1}^{\infty} \frac{f_n^2 (np)^4}{2k_n^3} & \displaystyle -\sum_{n=1}^{\infty} \frac{f_n^2 (np)^2}{2k_n^3} \\ \displaystyle -\sum_{n=1}^{\infty} \frac{f_n^2 (np)^2}{2k_n^3} & \displaystyle\sum_{n=1}^{\infty} \frac{f_n^2}{2k_n^3} \end{bmatrix} \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} - \lim_{T \to \infty} \int_{t_0}^{t_0+T} \begin{bmatrix} \dot{a}(et) \\ \dot{b}(et) \end{bmatrix} dt \\ &= eAq + \text{2nd Term} \end{aligned} \tag{3.27}$$

where $tr\, A < 0$ and $\det A > 0$ (see Hong and Bentsman (1994b)). Note that if $\dot{a}$ and $\dot{b}$ are almost periodic functions, the second term in (3.27) becomes zero. Therefore, the trivial solution of (3.27) is exponentially stable if there exists at least one $f_n \neq 0$, which is one of the Fourier coefficients of $f(x)$, which is the case that $f(x) \neq 0$ on at least one interval of nonzero measure. Combined with the results of Theorem 3, this implies that the zero equilibrium of (2.5) and (2.8a,b) is uniformly asymptotically stable, and that there is a neighborhood of zero equilibrium where both $\tilde{a}$ and $\tilde{b}$ have exponential convergence to zero.

2) Frozen State Analysis: The error dynamics with a frozen state error equation are defined by (2.8a,b) and (2.12), where $\tilde{a}$ and $\tilde{b}$ in (2.12) are assumed to be frozen. The solution of (2.12) is in the same form of (3.26) except $k_n = (a_m - \tilde{a})(np)^2 - (b_m - \tilde{b})$. The averaged system corresponding to (2.5) and (2.8a,b) will be nonlinear. Since the stability of the zero equilibrium of a nonlinear system can be determined by the linearized system at that equilibirum point, the result of case 1) already implies the stability of the zero solution of case 2).

3) Simulations: Let the plant be given with homogeneous boundary conditions as

$$x_t = a(et) x_{xx} + b(et) x + u,\quad 0 \le x \le 1,\quad t \ge 0$$

$$x(0,t) = x(1,t) = 0,\quad x(x,0) = 0.2\sin(px)$$

where $a(et)$ and $b(et)$ are unknown time-varying coefficients. In simulations, however, they are assumed to be $2.5 - \sin(et)$ and 0, respectively, with $e = 0.1$. Let the reference model be

$$x_{mt} = 0.5x_{mxx} + 5.0\,,\quad 0 \le x \le 1\,,\quad t \ge 0\,.$$

$$x_m(0,t) = x_m(1,t) = 0\,,\quad x_m(x,0) = -\sin(px)$$

The adaptive gain in (2.10) is chosen as 0.4. Fig. 1 and Fig. 2 show the behaviors of the reference model and the plant, respectively. Fig. 3 shows the exponential convergence of state error $e(x,t)$ to zero. Fig. 4 shows the exponential convergence of the estimated parameter $\hat{a}(t)$ to the plant parameter $a(et)$.

5 Conclusions

In this note averaging theorems are developed for coupled ordinary and partial differential equations and applied to the asymptotic convergence analysis of parameter estimate errors to zero in the model reference adaptive control of a time-varying parabolic partial differential equation.

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Appendix A. Existence and Uniqueness

Rewrite nonlinear error equations (2.5) and (2.8a,b), replaced by $\tilde{a} = \hat{a} - a$ and $\tilde{b} = \hat{b} - b$, in the following form

$$\dot{z} = A(t)z + F(t,z)\,, \quad z(0) = z_0 \tag{A.1}$$

where $z = \left(e, \hat{a}, \hat{b}\right)^T$, and

$$A(t)=\begin{bmatrix} (a_m+a-\hat{a})\dfrac{\partial^2}{\partial x^2}+\left(b_m+b-\hat{b}\right) & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & -s \end{bmatrix}=\begin{bmatrix} A_0(t) & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & -s \end{bmatrix},$$

$$F(t,z)=\begin{bmatrix} (a-\hat{a})x_{mxx}+(b-\hat{b})x_m \\ e\langle e,e_{xx}\rangle+e\langle e,x_{mxx}\rangle \\ e\langle e,e\rangle+e\langle e,x_m\rangle \end{bmatrix}.$$

where $A_0(t)\overset{D}{=}(a_m+a-\hat{a})\dfrac{\partial^2}{\partial x^2}+\left(b_m+b-\hat{b}\right)$. Define a state space as $H\overset{D}{=}L_2(0,1)\times R^2$, and

$$D(A)=\left\{\left(e,\hat{a},\hat{b}\right)\in H: e\in H^2(0,1)\cap H_0^1(0,1) \text{ with } e(0)=0=e(1), \text{and } \hat{a},\hat{b}\in R\right\}. \tag{A.2}$$

Note that the boundary conditions of (2.5) have been incorporated in the space $H^2(0,1)\cap H_0^1(0,1)$, which is the domain of the differential operator A_0. $D(A)$ is dense, and A is a closed operator (Walker, 1980).

For $z\in D(A)$

$$\begin{aligned}\langle z,Az\rangle_H &=\int_0^1 e(x)\left[(a_m+a-\hat{a})\frac{\partial^2 e(x)}{\partial x^2}+\left(b_m+b-\hat{b}\right)e(x)\right]dx-s\hat{a}^2-s\hat{b}^2\\ &\le\left[-(a_m+a-\hat{a})p^2+\left(b_m+b-\hat{b}\right)\right]\int_0^1 e^2(x)dx-s\hat{a}^2-s\hat{b}^2\\ &\le -C_1\langle z,z\rangle_H ,\end{aligned} \tag{A.3}$$

where $C_1=\min\left\{(a_m+a-\hat{a})p^2-\left(b_m+b-\hat{b}\right),s\right\}>0$, and $(a_m+a-\hat{a})\ge Da\ge 0$ is assumed. Now by the linearity of A, we see that $wI-A$ is monotone (accretive) for every $w\le C_1$. Hence $A:D(A)\subset H\to H$ is the infinitesimal generator of a linear process $\{S(t)\}_{t\ge 0}=\left\{\left(E(t,0),\hat{A}(t),\hat{B}(t)\right)\right\}_{t\ge 0}$ on H (See (Walker, 1980), Theorem 3.2, p. 92). Note that the first component $E(t,0)$ is generated by A_0. Note also that $E(t,0)e_0$ is the strong solution of the evolution equation $\dot{e}(t)=A_0 e(t)$ for every $e_0\in D(A_0)$.

Now set $z=\left(e,\hat{a},\hat{b}\right)$ and $z'=\left(e',\hat{a}',\hat{b}'\right)$. Then

$$\begin{aligned}\left\|F(t,z)-F(t,z')\right\|_H^2 &=\left\|(a-\hat{a})x_{mxx}+(b-\hat{b})x_m-(a-\hat{a}')x_{mxx}-(b-\hat{b}')x_m\right\|^2\\ &\quad+e^2\left|\langle e,e_{xx}\rangle+\langle e,x_{mxx}\rangle-\langle e',e_{xx}\rangle+\langle e',e_{xx}\rangle-\langle e',e'_{xx}\rangle-\langle e',x_{mxx}\rangle\right|^2\\ &\quad+e^2\left|\langle e,e\rangle+\langle e,x_m\rangle-\langle e',e\rangle+\langle e',e\rangle-\langle e',e'\rangle-\langle e',x_m\rangle\right|^2\\ &\le \|x_{mxx}\|^2|\hat{a}-\hat{a}'|^2+\|x_m\|^2\left|\hat{b}-\hat{b}'\right|^2+e^2\left(\|e_{xx}\|^2+\|x_{mxx}\|^2+\|e'_{xx}\|^2\right)\|e-e'\|^2\\ &\quad+\mathring{a}^2\left(\|e\|^2+\|\imath_m\|^2+\|e'\|^2\right)\|e-e'\|^2.\end{aligned}$$

Hence $\qquad \left\|F(t,z)-F(t,z')\right\|_H\le C_2\|z-z'\|_H,$ (A.4) where C_2 is a constant. Therefore $F : H \to H$ is locally Lipschitz continuous in H. Thus a unique solution exists. Finally the solution of (2.5) can be written in the following variation of constant formula (Henry, 1981; Pazy, 1983)

$$e(t) = E(t,0)e(0) + \int_0^t E(t,t)\left(-\tilde{a}(t)x_{mxx}(t) - \tilde{b}(t)x_m(t)\right)dt \ , \tag{A.5}$$

where $E(t,s)$ is the evolution operator associated with A_0 in the space $L_2(0,1)$.

Appendix B. Bound (3.14)

Following the work of (Solo, 1996), the integral (3.13) is divided into small pieces as follows. Introduce a sequence of increasing integers $N_e \to \infty$ as $e \to 0$ such that $\{n_e\}$ is an increasing sequence:

$$n_e = \frac{1}{eN_e} = \frac{d_e}{e} \to \infty \text{ as } e \to 0 .$$

Then, (3.13) can be written as

$$I_{eB}(s) = d_e \sum_{l=1}^{N_e} j_e\left(s(l-1)d_e\right), \tag{B.1}$$

where

$$j_e(t) = n_e^{-1} \int_{te^{-1}}^{te^{-1}+n_e s} \left[f\left(s, q_a(es), e_{es,q_a(es)}(s), es\right) - f_{av}\left(q_a(es), es\right)\right] ds \ ,$$

and $0 \le s \le T$. In the sequel it will be shown that

$$j_e(t) \to 0 , \tag{B.2}$$

as $e \to 0$ uniformly in $0 \le t \le T$. Therefore, the following result holds

$$\left|I_{eB}(s)\right| \le C_T(e) \overset{D}{=} T \sup_{0 \le t \le T} \left|j_e(t)\right| \to 0 , \tag{B.3}$$

as $e \to 0$.

To prove (B.2) $j_e(t)$ is considered into three pieces as follows

$$j_e(t) = j_{ea}(t) + j_{eb}(t) + j_{ec}(t) ,$$

where

$$j_{ea}(t) = n_e^{-1} \int_{te^{-1}}^{te^{-1}+n_e s} \left[f\left(s, q_a(es), e_{es,q_a(es)}(s), es\right) - f\left(s, q_a(t), e_{t,q_a(t)}(s), t\right)\right] ds \ , \tag{B.4a}$$

$$j_{eb}(t) = n_e^{-1} \int_{te^{-1}}^{te^{-1}+n_e s} \left[f\left(s, q_a(t), e_{t,q_a(t)}(s), t\right) - f_{av}\left(q_a(t), t\right)\right] ds \ , \tag{B.4b}$$

$$j_{ec}(t) = n_e^{-1} \int_{te^{-1}}^{te^{-1}+n_e s} \left[f_{av}\left(q_a(t), t\right) - f_{av}\left(q_a(es), es\right)\right] ds \ . \tag{B.4c}$$

From the Lipschitz condition (3.15), the first term (B.4a) is bounded as

$$\begin{aligned} \left|j_{ea}(t)\right| \le\ & L_1 \sup_{te^{-1} \le s \le te^{-1}+n_e s} \left|q_a(es) - q_a(t)\right| \\ & + L_1 \sup_{te^{-1} \le s \le te^{-1}+n_e s} \left|e_{es,q_a(es)}(s) - e_{t,q_a(t)}(s)\right| + L_1 d_e . \end{aligned} \tag{B.5}$$

To bound the first term in (B.5) we use the following

$$q_a(es) - q_a(t) = \int_t^{es} f_{av}\left(q_a(t'), t'\right) dt' .$$

Since $f_{av}(\cdot)$ also obeys the Lipschitz condition, it follows that

$$\left| q_a(es) - q_a(t) \right| \le L_1\left(h + \left| es - t \right|\right)\left| es - t \right| ,$$

which again implies that

$$\sup_{te^{-1} \le s \le te^{-1} + n_e s} \left| q_a(es) - q_a(t) \right| \le d_e s L_1 (h + d_e s) \le d_e T L_1 (h + d_e T) ,$$

for $0 \le t \le T$.

To bound the second term in (B.5), differentiate (2.12) with respect to q .

$$e^q_{t|t,q} = (a_m - \tilde{a}) e^q_{xx|t,q} + \left(b_m - \tilde{b}\right) e^q_{t,q} - \begin{bmatrix} e_{xx|t,q} \\ e_{t,q} \end{bmatrix} - \begin{bmatrix} x_{mxx} \\ x_m \end{bmatrix}$$

where $e^q_{t,q} = \dfrac{\partial e_{t,q}(s)}{\partial q}$ and the other terms with subscripts are defined similarly. By differentiating (2.12) with respect to t , a similar equation for $e^t_{t,q}$ is derived. The following results now hold from the parabolic boundedness lemma of Appendix D.

$$\left\| \partial e_{t,q}(s) / \partial t \right\| \le B , \text{ and } \left\| \partial e_{t,q}(s) / \partial q \right\| \le B . \qquad \text{(B.6)}$$

Then the second term in (B.5) is bounded by

$$L_1 B \sup_{te^{-1} \le s \le te^{-1} + n_e s} \left(\left| es - t \right| + \left| q_a(es) - q_a(t) \right| \right) \le d_e C$$

by similar arguments given above. Finally, we have

$$\left| j_{ea}(t) \right| \le d_e B ,$$

for $0 \le t \le T$. Therefore, (B.2) holds for $j_{ea}(t)$. Through similar analysis it can be shown that (B.2) holds for $j_{ec}(t)$ and $j_{eb}(t)$, respectively. Q.E.D.

Appendix C. Fast state dynamics

In this appendix a bound for the fast state e is obtained. Rewrite (2.5) as

$$e_t = (a_m - \tilde{a}) e_{xx} + \left(b_m - \tilde{b}\right) e - h\left(t, \tilde{q}\right),$$

where $h\left(t, \tilde{q}\right) = \left[x_{mxx} \quad x_m\right] \tilde{q}$. Denoting $\hat{e}(t) = e_{et, q_a(et)}(t)$, the following approximate system is introduced using (2.12)

$$\begin{aligned} \frac{d\hat{e}}{dt} &= \frac{\partial \hat{e}}{\partial t} + \left[\frac{\partial \hat{e}}{\partial \tilde{q}} \right]^T \frac{\partial \tilde{q}}{\partial t} \\ &= (a_m - \tilde{a}) \hat{e}_{xx} + \left(b_m - \tilde{b}\right) \hat{e} - h\left(t, \tilde{q}\right) + e f^T \left[\frac{\partial \hat{e}}{\partial \tilde{q}} \right], \end{aligned}$$

where $\hat{e}_{xx} = e_{xx|et, q_a(et)}(t)$. Thus the error dynamics between $e(t)$ and $\hat{e}(t)$ is obtained as

$$\begin{aligned}\tilde{e}_t &= e_t - \hat{e}_t \\ &= (a_m - \tilde{a})\tilde{e}_{xx} + \left(b_m - \tilde{b}\right)\tilde{e} - e f_t^T \frac{\partial \hat{e}}{\partial \tilde{q}}.\end{aligned}$$

Now apply the parabolic boundedness lemma of Appendix D to deduce

$$\|\tilde{e}\| \le eB^2$$

provided $|f| \le B$, $\left\|\frac{\partial \hat{e}}{\partial \tilde{q}}\right\| \le B$. The first inequality follows from (3.15). The second one follows from Appendix C, since $\|q_a\| \le h$.

Appendix D. Parabolic boundedness lemma

Lemma D: Consider the following time-varying parabolic system:

$$u_t = [a_m - a(t)]u_{xx} + [b_m - b(t)]u + h(x,t), \tag{D.1}$$

$$u(0,t) = u(1,t) = 0, \quad u(x,0) = 0.$$

Assume that $a_m - a(t) \ge D_a =$ constant, $b_m - b(t) \ge 0$, and $\|h\| \le B$ for all t where B is a constant. Then, $\|u\| \le BD_a^{-1}\left(\sum_{n=1}^{\infty} l_n^{-4}\right)^{1/2}$.

Proof. The parabolic system (D.1) has a solution

$$u(x,t) = \sum_{n=1}^{\infty} j_n(x) u_n(t), \tag{D.2}$$

where $u(t)$ satisfies

$$\dot{u}_n(t) = -\tilde{k}_n(t)u_n(t) + \langle j_n, h\rangle, \tag{D.3}$$

$$\tilde{k}_n(t) = (a_m - a(t))(np)^2 + (b_m - b(t)).$$

Also the solution to (D.3) is given by

$$u_n(t) = \int_0^t e^{-\int_s^t \tilde{k}_n(u)du} \langle h, j_n\rangle ds.$$

Since $\tilde{k}_n(t) \ge D_a l_n^2$ and $\|h\| \le B$, we have

$$|u_n| \le \int_0^t e^{-D_a l_n^2 (t-s)} ds\, B \le BD_a^{-1} \big/ l_n^2.$$

Which finally implies

$$\|u(x,t)\| = \left(\sum_{n=1}^{\infty} u_n^2(t)\right)^{1/2} \le BD_a^{-1}\left(\sum_{n=1}^{\infty} l_n^{-4}\right)^{1/2}$$

as required.

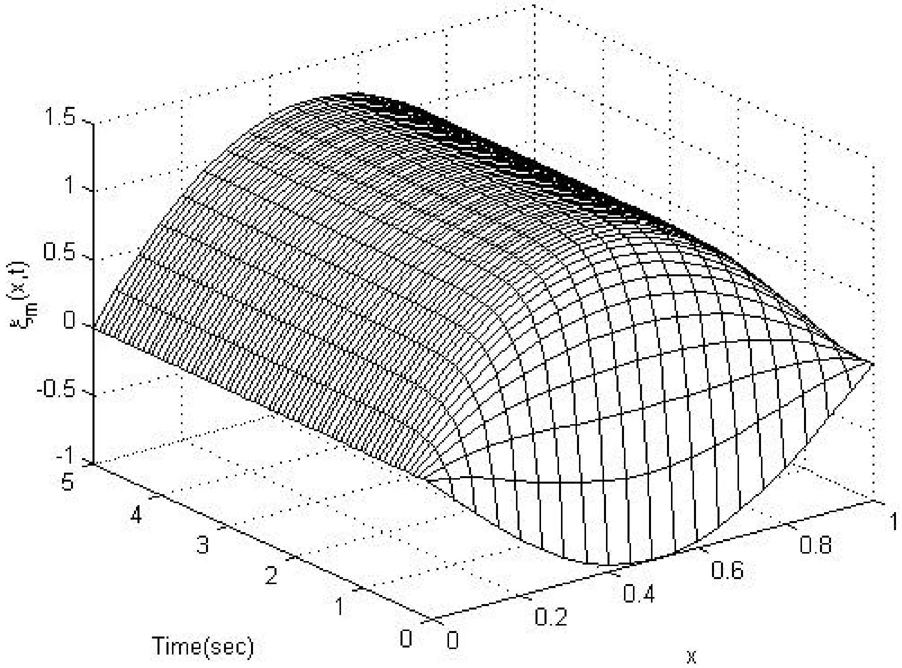

Fig. 1. Solution of reference model (2.2): $x_m(x,t)$.

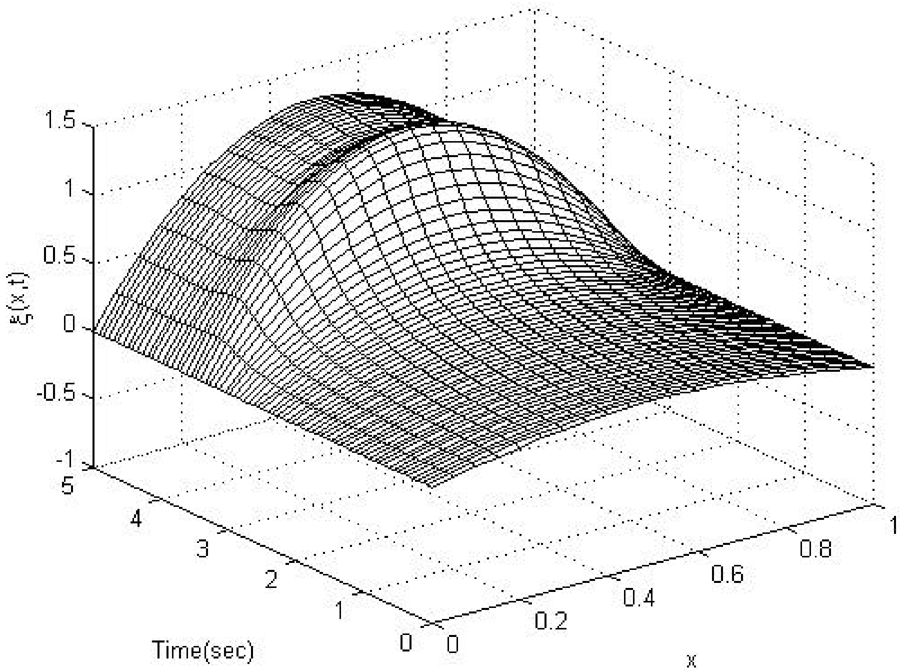

Fig. 2. Solution of plant (2.1) which follows (2.2): $x(x,t)$.

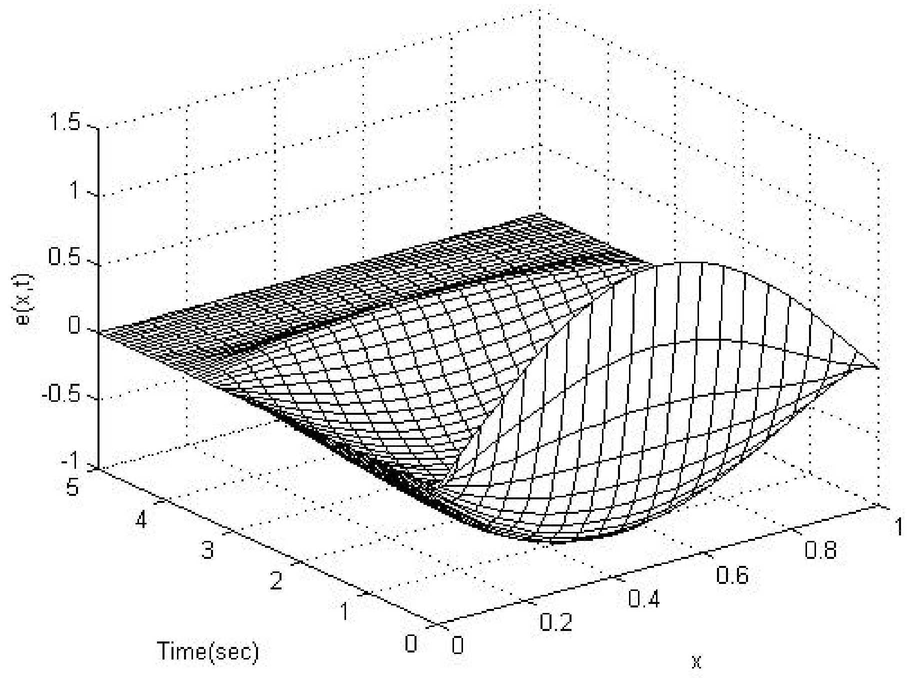

Fig. 3. Exponential convergence of state error $e(x,t)$ to zero.

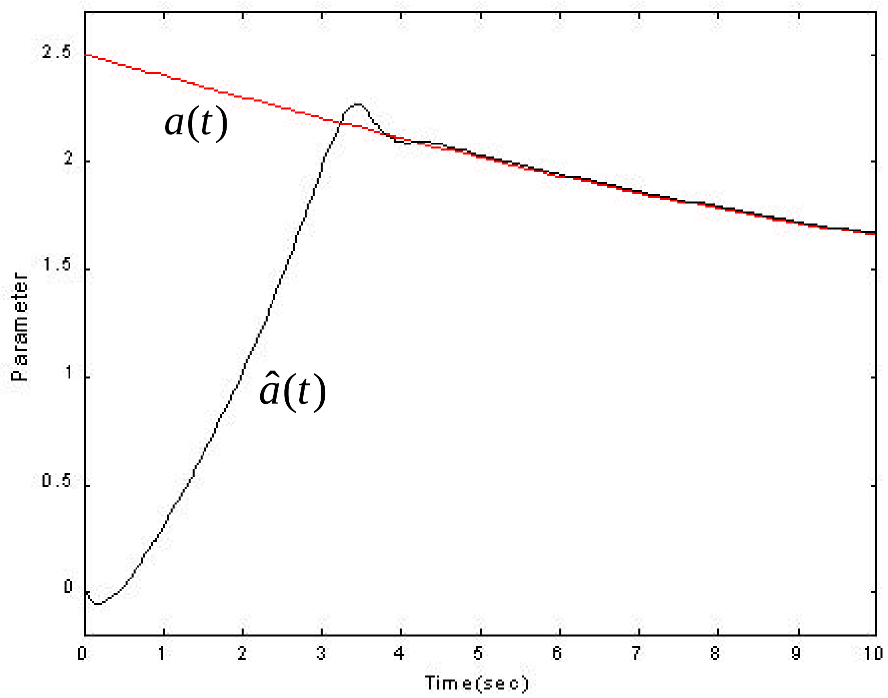

Fig. 4. Exponential convergence of estimated parameter $\hat{a}(t)$ to plant parameter $a(t)$.