BALANCED REALIZATION OF FLEXIBLE STRUCTURES WITH GENERAL DAMPING: A POWER SERIES APPROACH

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Abstract

A method of approximating the balanced realization for lightly damped flexible structures is presented. The damped system is treated as a perturbation from the undamped system, and the controllability and observability gramians, as well as the balancing transformation are given as a power series in the perturbation scaling factor. The approximation utilizes the special structure of the system i.e. the positive definitness of the inertia, damping and stiffness matrices, and the fact that the damping is small., to obtain closed form expressions for the series coefficient matrices. These expressions lead to interesting structural properties, which are discussed and related to physical properties of vibrating systems. The results can be obtained at any level of accuracy by appropriate truncation of the series.

1 Introduction

The problem of approximating a high-order, linear, time invariant dynamic system by a lower order model is one of the fundamental problems of system theory and has received renewed interest in the last two decades. The method that represents the beginning of the new era in model order reduction is the truncated balanced realization method (Moore, 1982 Pernebo and Silverman, 1983). In this method a state transformation is used to obtain a realization with controllability and observability gramians which are diagonal and equal. This identifies the strong modes of the system which are retained while other modes are truncated. The method is heuristic and formally does not incorporate any explicit criterion. However, it is closely related to L_2 minimization, (Kabamba, 1985, Hyland and Bernstein, 1985, Halevi, 1992) and in many cases results in reduced models which are near optimal in that sense. Moreover, reduced models obtained by this method have guarantees bound on the H_∞ error (Enns,1984). There is a considerable volume of works dealing with the properties and applications of the truncated balanced realization method and its usefulness seems to be evident.

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The computational aspects have also been considered and efficient and reliable numerical algorithms for a general system were presented (Laub et al, 1987, Safonov and Chiang, 1989). Nevertheless the main problem in the application of the method to structures seems to be the computational burden. The steps that are involved in this method are

a) Calculating the controllability and observability gramians.

b) Calculating the balancing transformation and the balanced realization.

c) Truncation of the balanced realization.

The orders of models of structures can be as large as hundreds of thousands and steps a) and b) in such cases may require unacceptably long computation time. Another source of difficulty is the fact that due to the small damping, the system matrices contain terms which may be several orders of magnitude apart.

Several works deal with the application of the truncated balanced realization to structures (Mottershead and Friswell. 1993, Williams, 1990 and 1994, Gawronski and Juang 1990, and Gawronski 1996 and 1997) and the algorithms there exploit some properties particular to those systems. There are two main differences between those methods and the method that is proposed in this paper. First, most of the methods consider systems with *modal damping* while we consider the case of a *general damping matrix*. Secondly, they consider the *exact* solution of the balancing problem of while we look for an *approximate* one.

In undamped or lightly damped structures the most common method of order reduction is modal truncation, which is a special case of the method of partial fraction expansion. In general, i.e. for systems that may be overdamped or with large damping factor, this method does not yield good approximations. However, for undamped systems with disjoint natural frequencies it gives the optimal approximation, and, by continuity, good approximations for systems with proportional light damping. where the accuracy depends on the level of damping and the distance between the natural frequencies. When the damping matrix has a general structure the modal methods are no longer accurate and cannot be applied

The main idea in the suggested method is to express the lightly damped system as a small perturbation from the undamped and to write the gramians as a power series in the perturbation scalar factor. The analysis that follows gives closed form formulas for the coefficients of the series as well as for the balancing transformation. For light damping only a few terms are required to calculate the gramians for the desired accuracy and thus a substantial reduction in the computation is achieved.

The material is organized as follows. Section 2 contains the statement of the problem and some preliminary results. In section 3 the power series approximation of the controllability gramian is introduced and calculated. Section 4 discusses these results from a deterministic time domain interpretation of the controllability gramian. In section 5 the dual results for the observability gramian are derived. The approximated balancing transformation is given in section 6. The results of the paper are summarized in section 7.

2 Problem statement and preliminaries

The dynamics of a flexible structure is given by

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = Fu(t) \tag{2.1}$$

where $q \in R^N$ is a vector of generalized coordinates and $u \in R^m$ is a vector of generalized forces. The inertia matrix M and the stiffness matrix K are symmetric and positive definite. That means that the system does not have rigid body degrees of freedom, possibly because of a definition as the deviation from the rigid body motion. The damping matrix C is nonnegative definite and may be singular. However it is assumed that the system is asymptotically stable, i.e. all the solutions of

$$\det(Ms^2 + Cs + K) = 0 \tag{2.2}$$

have strictly negative real part. A necessary and sufficient condition for that is that

$$C\phi_i \neq 0 \quad \forall i \tag{2.3}$$

where ϕ_i is a modeshape of the undamped system (M, K). Any $C > 0$, and generically all $C \geq 0$ satisfy this mild condition. Since systems with light damping are considered, C is small (one way to express that 'smallness' is $\|C\| << (\|M\| \, \|K\|)^{1/2}$) and we use the parametrization

$$C = \alpha C_o \tag{2.4}$$

$1 >> \alpha > 0$ is a scalar and $\|C_o\|$ is (roughly) of the order of magnitude $(\|M\| \, \|K\|)^{1/2}$. Thus C_o contains information regarding the existence of damping elements, their connectivity and relative values. Since the stability condition (2.3) is geometric it is a property of C_o only. Furthermore, it will be shown later that the all the results of this paper are independent of the magnitude distribution between α and C_o.

The output, $y \in R^r$ consists of linear combinations of the generalized velocities.

$$y(t) = H_v \dot{q}(t) \tag{2.5}$$

(The case of displacements output is analogous and will not be discussed in this paper). Defining the state vector $z = \left[q^T \ \dot{q}^T\right]^T$, the state space realization of the system which has an order $n = 2N$ is given as

$$\dot{z}(t) = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix} u(t) \tag{2.6a}$$

$$y(t) = [0 \quad H_V] \; z(t) \tag{2.6b}$$

The first step in the derivation is the calculation of the modal form of the undamped system (M, K). To simplify the analysis we assume that the natural frequencies of the system are distinct. Let Ω be the diagonal matrix of the natural frequencies and Φ the mass normalized modal matrix. Then we have

$$\Phi^T M \Phi = I_N \tag{2.7}$$

$$\Phi^T K \Phi = \Omega^2 \tag{2.8}$$

The state transformation z=Tx, where

$$T = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \tag{2.9}$$

and

$$T^{-1} = \begin{bmatrix} \Phi^T M & 0 \\ 0 & \Phi^T M \end{bmatrix} \tag{2.10}$$

Results in the following realization, known as modal realization.

$$\dot{x}(t) = \begin{bmatrix} 0 & I \\ -\Omega^2 & -\Phi^T C \Phi \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \Phi^T F \end{bmatrix} u(t) \tag{2.11a}$$

$$y(t) = [\,0 \quad H_v \Phi\,]\, x(t) \tag{2.11b}$$

The frequently made assumption, e.g. (Williams, 1990 and 1994, Gawronski 1996) of modal damping implies that $\Phi^T C \Phi$ is diagonal as well and that simplifies the derivation considerably. However this assumption is not made here and $\Phi^T C \Phi$ is considered to be a full symmetric matrix. An observation, which is important for the subsequent derivation, is given as a lemma.

Lemma 2.1: If the system (2.11) is asymptotically stable the diagonal entries of $\Phi^T C \Phi$ are strictly positive even if C is singular.

Proof : For nonnegative matrices the stability condition (2.3) implies that

$$\left(\Phi^T C \Phi\right)_{ii} = \phi_i^T C \phi_i > 0 \,\forall i \tag{2.12}$$

3 Series approximation of the controllability gramian

We start with the realization (2.11) which is written as

$$\dot{x}(t) = (A_0 + \alpha A_1) x(t) + B u(t) \tag{3.1a}$$

$$y = Hx \tag{3.1b}$$

where

$$A_0 = \begin{bmatrix} 0 & I \\ -\Omega^2 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\Phi^T C_0 \Phi \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \Phi^T F \end{bmatrix}, \quad H = \begin{bmatrix} 0 & H_v \Phi \end{bmatrix} \tag{3.2a-d}$$

The Lyapunov equation which determines the controllability gramian Q is

$$(A_0 + \alpha A_1) Q + Q (A_0 + \alpha A_1)^T + BB^T = 0\ . \tag{3.3}$$

It is well known that eq. (3.3) has a solution if and only if $(A_0 + \alpha A_1)$ does not have a pair of poles which are symmetric about the origin (Kwakernnak and Sivan, 1972), which in our case means purely imaginary poles. Hence no solution exists for α=0 and a unique solution exists for $\alpha > 0$. In (Halevi, 1999) it has been shown that the power series

$$Q^K = \alpha^{-1}Q_{-1} + Q_0 + \alpha Q_1 + \ldots + \alpha^K Q_K \tag{3.4}$$

is an adequate approximation since as $\alpha \to 0$, Q goes to infinity at a rate proportional to α^{-1}.

To simplify the notation in the forthcoming derivation we define

$$C^0 = \Phi^T C_0 \Phi \tag{3.5}$$

$$\widetilde{F} = \Phi^T F F^T \Phi \tag{3.6}$$

The first step in the calculation of the gramians is substituting the series (3.4) into eq. (3.3) and equating like powers of α. The following equations are obtained for α^{-1}, α^{o} and α^{k} $(k \geq 1)$ respectively.

$$A_0 Q_{-1} + Q_{-1} A_0{}^T = 0 \tag{3.7}$$

$$A_0 Q_0 + A_1 Q_{-1} + Q_0 A_0{}^T + Q_{-1} A_1{}^T + BB^T = 0 \tag{3.8}$$

$$A_0 Q_k + A_1 Q_{k-1} + Q_k A_0{}^T + Q_{k-1} A_1{}^T = 0 \tag{3.9}$$

We partition each Q_k to four square $n \times n$ sub-blocks as

$$Q_k = \begin{bmatrix} Q_{k,a} & Q_{k,ab} \\ Q^T_{k,ab} & Q_{k,b} \end{bmatrix} \tag{3.10}$$

Then eq. (3.5) can be written as

$$\begin{bmatrix} Q_{-1,ab} + Q^T_{-1,ab} & Q_{-1,b} - Q_{-1,a}\Omega^2 \\ \text{sym} & -\Omega^2 Q_{-1,ab} - Q^T_{-1,ab}\Omega^2 \end{bmatrix} = 0 \tag{3.11}$$

Since the natural frequencies are distinct it follows immediately that $Q_{-1,ab} = 0$ and that $Q_{-1,a}$ and $Q_{-1,b}$ are diagonal. Eq. (3.8) can now be written as

$$\begin{bmatrix} Q_{0,ab} + Q^T_{0,ab} & Q_{0,b} - Q_{0,a}\Omega^2 \\ \text{sym} & -\Omega^2 Q_{0,ab} - Q^T_{0,ab}\Omega^2 - C^0 Q_{-1,b} - Q_{-1,b} C^0 + \widetilde{F} \end{bmatrix} = 0 \tag{3.12}$$

The upper left sub-block implies that $Q_{0,ab}$ is skew symmetric, which in turn implies that the diagonal elements of $\Omega^2 Q_{0,ab}$ are zero. By inspecting the diagonal entries of the lower right sub-block we find that

$$(Q_{-1,b})_{ii} = \frac{\widetilde{F}_{ii}}{2C^0_{ii}} \tag{3.13}$$

$Q_{-1,a}$, which is diagonal as well, is given by

$$(Q_{-1,a})_{ii} = \frac{(Q_{-1,b})_{ii}}{\omega_i^2} = \frac{\tilde{F}_{ii}}{2C_{ii}^0\omega_i^2} \tag{3.14}$$

Substituting $Q_{0,ab}^T = -Q_{0,ab}$ into the lower right sub-block we obtain

$$Q_{0,ab}\Omega^2 - \Omega^2 Q_{0,ab} = C^0 Q_{-1,b} + Q_{-1,b}C^0 - \tilde{F} \tag{3.15}$$

which leads, for $i \neq j$, to

$$(Q_{0,ab})_{ij} = \frac{\left(C^0 Q_{-1,b} + Q_{-1,b}C^0 - \tilde{F}\right)_{ij}}{\omega_j^2 - \omega_i^2} \tag{3.16}$$

A more explicit formula is

$$(Q_{0,ab})_{ij} = \frac{C_{ij}^0\left((Q_{-1,b})_{ii} + (Q_{-1,b})_{jj}\right) - \tilde{F}_{ij}}{\omega_j^2 - \omega_i^2} \tag{3.17}$$

Since $Q_{0,ab}$ is skew symmetric only the upper triangular part (j>i) needs to be calculated. To find the diagonal entries of $Q_{0,a}$ and $Q_{0,b}$ we consider now eq. (3.9) with k=1.

$$\begin{bmatrix} Q_{1,ab} + Q_{1,ab}^T & Q_{1,b} - Q_{1,a}\Omega^2 - Q_{0,ab}C^0 \\ \text{sym} & -\Omega^2 Q_{1,ab} - Q_{1,ab}^T\Omega^2 - C^0 Q_{0,b} - Q_{0,b}C^0 \end{bmatrix} = 0 \tag{3.18}$$

As in the discussion that follows eq. (3.12), the upper left sub-block implies that $Q_{1,ab}$ is skew symmetric, hence the diagonal elements of $-\Omega^2 Q_{1,ab} - Q_{1,ab}^T\Omega^2$ are zero. The diagonal entries of the lower right sub-block are therefore

$$2(C^0)_{ii}(Q_{0,b})_{ii} = 0, \tag{3.19}$$

but since $(C^0)_{ii} \neq 0$ it follows that $(Q_{0,b})_{ii} = 0$. Hence

$$Q_{0,b} = Q_{0,a} = 0 \tag{3.20}$$

Using this, it is concluded that

$$Q_{1,ab} = 0 \tag{3.21}$$

Writing the (i,j) and (j,i) entries of the upper-right sub-block, recalling that $Q_{1,a}$ and $Q_{1,b}$ are symmetric we have

$$(Q_{1,b})_{ij} - (Q_{1,a})_{ij}\omega_j^2 = (Q_{0,ab}C^0)_{ij} \tag{3.22}$$

$$(Q_{1,b})_{ij} - (Q_{1,a})_{ij}\omega_i^2 = (Q_{0,ab}C^0)_{ji} \tag{3.23}$$

These equations yield, for $i \neq j$

$$(Q_{1,a})_{ij} = \frac{\left(Q_{0,ab}C^0\right)_{ji} - \left(Q_{0,ab}C^0\right)_{ij}}{\omega_j^2 - \omega_i^2} \tag{3.24}$$

$$(Q_{1,b})_{ij} = \frac{\omega_j^2\left(Q_{0,ab}C^0\right)_{ji} - \omega_i^2\left(Q_{0,ab}C^0\right)_{ij}}{\omega_j^2 - \omega_i^2} \tag{3.25}$$

Continuing to k=2 we obtain

$$\begin{bmatrix} Q_{2,ab} + Q_{2,ab}^T & Q_{2,b} - Q_{2,a}\Omega^2 \\ \text{sym} & -\Omega^2 Q_{2,ab} - Q_{2,ab}^T\Omega^2 - C^0 Q_{1,b} - Q_{1,b}C^0 \end{bmatrix} = 0 \tag{3.26}$$

As in eq. (3.18), $Q_{2,ab}$ is skew-symmetric and as such does not affect the diagonal elements of the lower-right sub-block. Comparing these elements to zero we obtain

$$(Q_{1,b})_{ii} = -\frac{1}{(C^0)_{ii}} \sum_{\substack{j=1 \\ j\neq i}}^{n} (Q_{1,b})_{ij}(C^0)_{ji} \tag{3.27}$$

and from the upper-right sub-block of (3.18)

$$(Q_{1,a})_{ii} = \frac{\left(Q_{1,b}\right)_{ii} - \left(Q_{0,ab}C^0\right)_{ii}}{\omega_i^2} \tag{3.28}$$

After $Q_{1,b}$ is known, the lower-right sub-block of (3.26) completely determines (the skew symmetric) $Q_{2,ab}$ which is given by

$$(Q_{2,ab})_{ij} = \frac{\left(C^0 Q_{1,b}\right)_{ji} + \left(C^0 Q_{1,b}\right)_{ij}}{\omega_j^2 - \omega_i^2} \tag{3.29}$$

Examining k=3 reveals, similar to what follows (3.18) that the diagonal matrices $Q_{2,a}$ and $Q_{2,b}$ are zero. The derivation then repeats itself for odd and even terms. It is summarized in the following Theorem.

Theorem 4.1: The matrices Q_k of the series (3.4) are given by

$$Q_{-1} = \begin{bmatrix} \operatorname{diag}\left\{\frac{\tilde{F}_{ii}}{2C_{ii}^0\omega_i^2}\right\} & 0 \\ 0 & \operatorname{diag}\left\{\frac{\tilde{F}_{ii}}{2C_{ii}^0}\right\} \end{bmatrix} \tag{3.30}$$

$$Q_0 = \begin{bmatrix} 0 & Q_{0,ab} \\ Q_{0,ab}^T & 0 \end{bmatrix} \tag{3.31}$$

where

$$(Q_{0,ab})_{ij} = \begin{cases} \dfrac{C^0_{ij}((Q_{-1,b})_{ii} + (Q_{-1,b})_{jj}) - \widetilde{F}_{ij}}{\omega_j{}^2 - \omega_i{}^2} & i \neq j \\ 0 & i = j \end{cases} \tag{3.32}$$

<u>k=2m-1</u>

$$Q_k = \begin{bmatrix} Q_{k,a} & 0 \\ 0 & Q_{k,b} \end{bmatrix} \tag{3.33}$$

where

$$(Q_{k,b})_{ij} = \begin{cases} \dfrac{\omega_j^2 \left(Q_{k-1,ab} C^0\right)_{ji} - \omega_i^2 \left(Q_{k-1,ab} C^0\right)_{ij}}{\omega_j^2 - \omega_i^2} & i \neq j \\ \dfrac{1}{(C^0)_{ii}} \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} (Q_{k,b})_{ij} (C^0)_{ji} & i = j \end{cases} \tag{3.34}$$

$$(Q_{k,a})_{ij} = \begin{cases} \dfrac{\left(Q_{k-1,ab} C^0\right)_{ji} - \left(Q_{k-1,ab} C^0\right)_{ij}}{\omega_j^2 - \omega_i^2} & i \neq j \\ \dfrac{\left(Q_{k,b}\right)_{ii} - \left(Q_{k-1,ab} C^0\right)_{ii}}{\omega_i^2} & i = j \end{cases} \tag{3.35}$$

<u>k=2m</u>

$$Q_k = \begin{bmatrix} 0 & Q_{k,ab} \\ Q^T_{k,ab} & 0 \end{bmatrix} \tag{3.36}$$

where

$$(Q_{k,ab})_{ij} = \begin{cases} \dfrac{\left(C^0 Q_{k-1,b}\right)_{ji} + \left(C^0 Q_{k-1,b}\right)_{ij}}{\omega_j^2 - \omega_i^2} & i \neq j \\ 0 & i = j \end{cases} \tag{3.37}$$

4 Properties of the controllability gramian series

We start by looking at the physical interpretation of the results obtained in section 3. There are several point of view regarding the controllability gramian, but the deterministic time domain seems to provide the best insight for our analysis. Assume for simplicity single input and let **u**(t) be a unit impulse and **g**(t) the state response. Then

$$Q_{ij} = \int_0^\infty g_i(t) g_j(t) dt \tag{4.1}$$

We start with the fact that being a sum of zero (for odd k) and skew symmetric matrices (for even k) Q_{ab} is a skew symmetric matrix. Notice that $(Q_{ab})_{ij} = Q_{i,N+j}$ and that $x_{N+j}(t) = \dot{x}_j(t)$. Hence

$$(Q_{ab})_{ij} = \int_0^\infty g_i(t) \dot{g}_j(t) dt \tag{4.2}$$

and

$$\begin{aligned} (Q_{ab})_{ij} + (Q_{ab})_{ji} &= \int_0^\infty [g_i(t)\dot{g}_j(t) + g_j(t)\dot{g}_i(t)] dt \\ &= \int_0^\infty d[g_i(t) g_j(t)] \\ &= g_i(\infty) g_j(\infty) - g_i(0) g_j(0) = 0 \end{aligned} \tag{4.3}$$

At t=0 both displacements are zero because due to the impulse force there is a jump in the velocity but not in the displacement. At $t \to \infty$ they go to zero as a result of the stability of the system. This skew symmetry is independent of the special properties of the system such as symmetry, definiteness or small damping. It is common to all stable A matrices of the form

$$A = \begin{bmatrix} 0 & I \\ * & * \end{bmatrix}$$

The second topic is the orders of α in the various sub-matrices. For small α it can be shown that

$$g_i(t) \cong X_{ii} e^{-\alpha\beta_i t} \sin \omega_i t + \alpha \sum_{i \neq j} X_{ij} e^{-\alpha\beta_j t} \sin \omega_j t \tag{4.4}$$

and consequently

$$\dot{g}_i(t) \cong \omega_i X_{ii} e^{-\alpha\beta_i t} \cos \omega_i t + \alpha \left(-\alpha\beta_i X_{ii} e^{-\alpha\beta_i t} \sin \omega_i t + \sum_{i \neq j} \omega_j X_{ij} e^{-\alpha\beta_j t} \cos \omega_j t \right) \tag{4.5}$$

This expressions, together with the following formulas

$$\int_0^\infty (e^{-\alpha\beta_i t}\sin\omega_i t)(e^{-\alpha\beta_j t}\sin\omega_j t)dt = \begin{cases} \frac{1}{4\beta_i}\alpha^{-1} + o(\alpha) & i = j \\ \gamma_{ij}\alpha + o(\alpha^3) & i \neq j \end{cases} \tag{4.6}$$

$$\int_0^\infty (e^{-\alpha\beta_i t}\sin\omega_i t)(e^{-\alpha\beta_j t}\cos\omega_j t)dt = \mu_{ij}\alpha^0 + o(\alpha^2) \tag{4.7}$$

lead to the same orders of magnitudes that were found in the previous section.

Another interesting question is the accuracy and the convergence properties of the series (3.4). First we have the following result.

Lemma 4.1: Let $C=\alpha_1 C_{01}$ and $C=\alpha_2 C_{02}$ be two factorizations of C. Then

$$Q_k^2 = \left(\frac{\alpha_2}{\alpha_1}\right)^k Q_k^1 \tag{4.8}$$

Proof: Q_{-1} is inversely proportional to C and then the coefficients of the recursion (4.34)-(4.39) are linear in C.

An immediate consequence of Lemma 4.1 is that $Q_k^2 {\alpha_2}^k = Q_k^1 {\alpha_1}^k$. Hence the terms in the power series (3.4) are independent of the factorization of C. The accuracy of the approximation, for a given number of terms, and also the convergence of the series to the true gramian, are properties of the damping matrix C. The only role of α is to let C_0 have a convenient magnitude from a numerical point of view. Choosing α=1 leads to a different interpretation of the series (3.4). It can be looked at as a recursive algorithm for the solution of the Lyapunov equation.

$$Q = \sum_{k=-1}^{\infty} Q_k \tag{4.9}$$

$$Q_{k+1} = f(Q_k) \tag{4.10}$$

where the summation stops when a certain convergence criterion is met.

5 Calculation of the observability gramian

The observability gramian is given by

$$P(A_0 + \alpha A_1) + (A_0 + \alpha A_1)^T P + C^T C = 0\,. \tag{5.1}$$

The same reasoning as for the controllability gramian applies also to this equation and therefore the same type of power series is used for P.

$$P^K = \alpha^{-1}P_{-1} + P_0 + \alpha P_1 + \ldots + \alpha^K P_K \tag{5.2}$$

The counterpart results for the observability gramian are analogous to the results in section 4 and are given by the following theorem.

Theorem 5.1: The matrices P_k of the series (5.2) are given by

$$P_{-1} = \begin{bmatrix} \mathrm{diag}\left\{\frac{\widetilde{H}_{ii}\omega_i^2}{2C_{ii}^0}\right\} & 0 \\ 0 & \mathrm{diag}\left\{\frac{\widetilde{H}_{ii}}{2C_{ii}^0}\right\} \end{bmatrix} \tag{5.3}$$

$$P_0 = \begin{bmatrix} 0 & P_{0,ab} \\ P_{0,ab}^T & 0 \end{bmatrix} \tag{5.4}$$

where

$$(P_{0,ab})_{ij} = \begin{cases} \dfrac{\left[C_{ij}^0((P_{-1,b})_{ii} + (P_{-1,b})_{jj}) - \widetilde{H}_{ij}\right]\omega_i^2}{\omega_j^2 - \omega_i^2} & i \neq j \\ 0 & i = j \end{cases} \tag{5.5}$$

<u>k=2m-1</u>:

$$P_k = \begin{bmatrix} P_{k,a} & 0 \\ 0 & P_{k,b} \end{bmatrix} \tag{5.6}$$

where

$$(P_{k,b})_{ij} = \begin{cases} \dfrac{\omega_j^2\left(P_{k-1,ab}C^0\right)_{ji} - \omega_i^2\left(P_{k-1,ab}C^0\right)_{ij}}{\omega_j^2 - \omega_i^2} & i \neq j \\ \dfrac{1}{(C^0)_{ii}}\sum\limits_{\substack{j=1 \\ j\neq i}}^{n}(P_{k,b})_{ij}(C^0)_{ji} & i = j \end{cases} \tag{5.7}$$

$$(P_{k,a})_{ij} = \begin{cases} \dfrac{\left[\left(P_{k-1,ab}C^0\right)_{ji} - \left(P_{k-1,ab}C^0\right)_{ij}\right]\omega_j^2\omega_i^2}{\omega_j^2 - \omega_i^2} & i \neq j \\ \left[\left(P_{k,b}\right)_{ii} - \left(P_{k-1,ab}C^0\right)_{ii}\right]\omega_i^2 & i = j \end{cases} \tag{5.8}$$

<u>k=2m</u>

$$P_k = \begin{bmatrix} 0 & P_{k,ab} \\ P_{k,ab}^T & 0 \end{bmatrix} \tag{5.9}$$

where

$$(P_{k,ab})_{ij} = \begin{cases} \dfrac{\left[\left(C^0 P_{k-1,b}\right)_{ji} + \left(C^0 P_{k-1,b}\right)_{ij}\right]\omega_i^2}{\omega_j^2 - \omega_i^2} & i \neq j \\ 0 & i = j \end{cases} \tag{5.10}$$

Proof: While the derivation can be done directly along the same lines as for **Q**, it is easier to use the state transformation

$$T = \begin{bmatrix} -\Omega^{-2} & 0 \\ 0 & I \end{bmatrix} \tag{5.11}$$

Then it follows that

$$A' = T^{-1}AT = \begin{bmatrix} 0 & -\Omega^2 \\ I & -\alpha C^0 \end{bmatrix} = A^T \tag{5.12}$$

and

$$H' = HT = \begin{bmatrix} 0 & \Phi^T H_v \end{bmatrix} = H \tag{5.13}$$

Hence the matrices $P_k' = T^T P_k T$ are given by the formulas for Q_k with

$$\tilde{H} = \Phi^T H_v H_v{}^T \Phi \tag{5.14}$$

replacing $\tilde{F}$. Due to the structure of the transformation T the inverse transformation from P_k' to P_k is as follows. $P_{k,b}'$ remains unchanged, $P_{k,ab}'$ is pre-multiplied by Ω^2 and $P_{k,a}'$ is pre and post-multiplied by Ω^2. This leads to (5.3)-(5.10).

6 Balancing

In sections 3 and 5 the power series for the controllability and observability gramians were derived. At this point one can calculate these matrices to the desired accuracy and then balance exactly, using any standard method, to get the balanced realization of the system. However, the motivation for the approximations was to reduce the amount of computational and to see the effect of the damping on the results. We therefore continue and present now an approximation of the balancing transformation.

First recall that a state transformation T changes the gramians in the following way

$$Q \to Q' = T^{-1} Q T^{-T} \qquad P \to P' = T^T P T$$

A transformation is called balancing if in the new realization Q' and P' are diagonal and equal. As a first step towards the approximated balanced realization we employ the state transformation

$$T_1 = \begin{bmatrix} \Omega^{-1}T_{b1} & 0 \\ 0 & T_{b1} \end{bmatrix} \tag{6.1}$$

where

$$T_{b1} = \mathrm{diag}\left\{ \left(\frac{\tilde{F}_{ii}}{\tilde{H}_{ii}} \right)^{1/4} \right\} \tag{6.2}$$

This transformation is diagonal and is merely scaling of the state variables. Then

$$Q' = \begin{bmatrix} \alpha^{-1}\Sigma + \alpha\overline{Q}_{1,a} + \alpha^3\overline{Q}_{3,a} + \ldots & \overline{Q}_{0,ab} + \alpha^2\overline{Q}_{2,ab} + \ldots \\ \overline{Q}_{0,ab}{}^T + \alpha^2\overline{Q}_{2,ab}{}^T + \ldots & \alpha^{-1}\Sigma + \alpha\overline{Q}_{1,b} + \alpha^3\overline{Q}_{3,b} + \ldots \end{bmatrix} \tag{6.3}$$

$$P' = \begin{bmatrix} \alpha^{-1}\Sigma + \alpha\overline{P}_{1,a} + \alpha^3\overline{P}_{3,a} + \ldots & \overline{P}_{0,ab} + \alpha^2\overline{P}_{2,ab} + \ldots \\ \overline{P}_{0,ab}{}^T + \alpha^2\overline{P}_{2,ab}{}^T & \alpha^{-1}\Sigma + \alpha\overline{P}_{1,b} + \alpha^3\overline{P}_{3,b} + \ldots \end{bmatrix} \tag{6.4}$$

where

$$\Sigma = \mathrm{diag}\left\{ \frac{\left(\tilde{F}_{ii}\tilde{H}_{ii}\right)^{1/2}}{C_{ii}^0} \right\} \tag{6.5}$$

and

$$\overline{Q}_{k,a} = T_{b1}{}^{-1}\Omega Q_{k,a}\Omega T_{b1}{}^{-1}, \quad \overline{Q}_{k,b} = T_{b1}{}^{-1}Q_{k,b}T_{b1}{}^{-1}, \quad \overline{Q}_{k,ab} = T_{b1}{}^{-1}\Omega Q_{k,ab}T_{b1}{}^{-1}$$
$$\overline{P}_{k,a} = \Omega^{-1}T_{b1}P_{k,a}T_{b1}\Omega^{-1}, \quad \overline{P}_{k,b} = T_{b1}P_{k,b}T_{b1}, \quad \overline{P}_{k,ab} = \Omega^{-1}T_{b1}P_{k,ab}T_{b1}$$

Considering only the leading elements of order α^{-1} the system is already balanced. This is the "almost balanced realization" in (Gawronski, 1997). For further refinement we apply the second transformation

$$T_2 = \begin{bmatrix} I + \alpha^2 D_1 & \alpha L_1 \\ \alpha L_2 & I + \alpha^2 D_2 \end{bmatrix} \tag{6.6}$$

This structure of T_2 is a result of a formal development as in sections 3 and 5 but for the sake of brevity only the correct form is given. Since α is small

$$T_2{}^{-1} \cong \begin{bmatrix} I + \alpha^2(L_1L_2 - D_1) & -\alpha L_1 \\ -\alpha L_2 & I + \alpha^2(L_2L_1 - D_2) \end{bmatrix} \tag{6.7}$$

The purpose of this second transformation is to cancel the α^0 terms. They disappear if

$$(L_1)_{ij} = \frac{\Sigma_i(\overline{P}_{0,ab})_{ij} + \Sigma_j(\overline{Q}_{0,ab})_{ij}}{\Sigma_j{}^2 - \Sigma_i{}^2} \tag{6.8}$$

$$(L_2)_{ij} = \frac{\Sigma_i (\overline{P}_{0,ab})_{ji} + \Sigma_j (\overline{Q}_{0,ab})_{ji}}{\Sigma_j^2 - \Sigma_i^2} \tag{6.9}$$

$$(L_1)_{ii} = -(L_2)_{ii} \tag{6.10}$$

The exact values of the diagonal of L_1, L_2, as well as the matrices D_1, D_2 affect only higher order terms whose analysis is beyond the scope of this paper. The gramians of the final realization are given by

$$Q'' = \begin{bmatrix} \alpha^{-1}\Sigma + o(\alpha) & o(\alpha^2) \\ o(\alpha^2) & \alpha^{-1}\Sigma + o(\alpha) \end{bmatrix} \tag{6.11}$$

$$P'' = \begin{bmatrix} \alpha^{-1}\Sigma + o(\alpha) & o(\alpha^2) \\ o(\alpha^2) & \alpha^{-1}\Sigma + o(\alpha) \end{bmatrix} \tag{6.12}$$

7 Summary

A method for approximating the controllability gramian, the observability gramian and the balancing transformation for a lightly damped structure was presented. The actual structure was presented as a small perturbation from an undamped system where the small parameter α multiplied a 'non-dimensional' matrix C_0. Then in the analysis that was carried out, the gramians were expressed by a power series in α, where closed form formulas for the coefficient matrices were given. That enables the simple calculation of the gramians to a desired level of accuracy and an approximated balancing transformation. The approximation was also investigated from a deterministic time domain interpretation of the controllability gramian.

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