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DYNAMICAL ADAPTIVE FIRST AND SECOND ORDER SLIDING MODE CONTROL OF NONLINEAR NON-TRIANGULAR UNCERTAIN SYSTEMS

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Abstract

In this paper combined algorithms for the control of non-triangular nonlinear systems with unmatched uncertainties will be presented. The controllers consist of a combination of Dynamical Adaptive Backstepping (DAB) and Sliding Mode Control (SMC) of first and second order. In order to solve a tracking problem, the DAB algorithm (a generalization of the backstepping technique) makes use of virtual functions as well as tuning functions to construct a transformed system for which a regulation problem has to be solved. The new state is extended by an $(n-\rho)$-th order subsystem in canonical form where n is the order of the original system and ρ is the relative degree. The role of the sliding mode control is to replace the last step of the design of the control law to obtain more robustness towards disturbances and unmodelled dynamics. The main advantages of the *second order* sliding mode algorithm are the prevention of chattering, higher accuracy and a significant simplification of the control law. A comparative study of these first and second order sliding controllers will be presented.

1 Introduction

The control of nonlinear systems with uncertainties is a challenging problem which has been the subject of research for many years. The various backstepping control design algorithms (Jiang and Praly, 1991; Kanellakopoulos *et al.*, 1991; Krstić *et al.*, 1992) provide a systematic framework for the design of tracking and regulation strategies suitable for large classes of nonlinear systems. The adaptive backstepping algorithm has enlarged the class of nonlinear systems controlled via a Lyapunov-based control law to uncertain systems transformable into the *parametric strict feedback* (PSF) form and the *parametric pure feedback* (PPF) form. In general, local stability is achieved for systems in the PPF form, whilst global stability is guaranteed for systems in the PSF form (Kanellakopoulos *et al.*, 1991). These two forms can be seen as special structural *triangular* forms of nonlinear systems which are adaptively input-output linearizable with the

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linearizing output $y = x_1$. A more general algorithm, the Dynamical Adaptive Backstepping-SMC (DAB-SMC) algorithm, has been developed (Rios-Bolívar *et al.*, 1995), which allows one to design a dynamical adaptive controller by following an input-output linearization procedure based upon the backstepping approach with tuning functions (Krstić *et al.*, 1992). It is applicable to both triangular and nontriangular uncertain observable minimum phase nonlinear systems. The role of the sliding mode control is to achieve more robustness towards disturbances and unmodelled dynamics. To control an even larger set of systems and to simplify the control law, the SMC part of the DAB design (Rios-Bolívar *et al.*, 1996), introduced to achieve robustness, can be effectively substituted with a second order sliding mode control (SOSMC) (Bartolini *et al.*, 1996). Using a suitable sliding function σ and a discontinuous law for the $(n-\rho)$-th derivative of the control (system order n, relative degree ρ), one can guarantee that the sliding mode condition $\sigma = 0$, $\dot{\sigma} = 0$ is reached in finite time. The main advantages of the *second order* sliding mode algorithm are the prevention of chattering, higher accuracy and a significant simplification of the control law.

We consider here a comparison between the DAB-SMC algorithm and the DAB-SOSMC algorithm, and present a comparative example. This paper is organized as follows: Section 2 outlines the combined backstepping algorithm. Section 3 outlines the extension to the second order sliding mode case (SOSMC). Section 4 presents a comparative example by application of the two algorithms to the adaptive regulation of a nonlinear continuous chemical process, namely the isothermal continuously stirred tank reactor. Conclusions are presented in Section 5.

2 Dynamical Adaptive Backstepping SMC

The Dynamical Adaptive Backstepping-SMC (DAB-SMC) algorithm is based upon a combination of dynamical input-output linearization and the adaptive backstepping algorithm with tuning functions. Its applicability to both triangular and nontriangular systems is guaranteed, but it requires that the controlled plant be *observable* and *minimum phase*. The observability condition is required to guarantee the existence of a local nonlinear mapping which transforms the plant into a suitable error system form, whilst the need for the minimum phase property is to guarantee stability of the closed-loop system. At the final step of this algorithm, a sliding surface is defined in terms of the error variables and both an update law and a dynamical discontinuous feedback law are synthesized (Rios-Bolívar *et al.*, 1997).

Consider a single-input single-output nonlinear system with linearly parameterized uncertainty

$$\begin{aligned} \dot{x} &= f_0(x) + \Phi(x)\theta + \Big(g_0(x) + \Psi(x)\theta\Big)u \qquad (1) \\ y &= h(x) \end{aligned}$$

where $x \in \Re^n$ is the state; $u, y \in \Re$ the input and output respectively; and $\theta = [\theta_1, \ldots, \theta_p]^T$ is a vector of unknown parameters. f_0, g_0 and the columns of the matrices $\Phi, \Psi \in \Re^{n\times p}$ are smooth vector fields in a neighbourhood R_0 of the origin $x = 0$ with $f_0(0) = 0$, $g_0(0) \neq 0$; and h is a smooth scalar function defined in R_0.

The steps leading to the design of the dynamical adaptive sliding mode compensator follow an input-output linearization procedure in which both a control dependent nonlinear mapping and a tuning function are constructed. In order to characterize the class of nonlinear systems for which this procedure is applicable, we set up a nonlinear mapping by considering the output

$y(t)$ and its first $n-1$ time derivatives as follows

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}\Big[f_0(x) + \Phi(x)\theta + (g_0(x) + \Psi(x)\theta)u\Big] \tag{2}$$

Due to the presence of the unknown parameter vector θ we rewrite (2) as

$$\begin{aligned}\dot{y} &= \mathcal{L}_h^1(x, \hat{\theta}, u) \\ &= \frac{\partial h}{\partial x}\Big[f_0(x) + \Phi(x)\hat{\theta} + (g_0(x) + \Psi(x)\hat{\theta})u\Big] + \omega_1(\theta - \hat{\theta})\end{aligned} \tag{3}$$

where $\hat{\theta}$ is an estimate of θ, and the vector ω_1 is defined as

$$\omega_1 = \frac{\partial h}{\partial x}\Big(\Phi(x) + u\Psi(x)\Big) \tag{4}$$

In other words, (3) may be rewritten as

$$\dot{y} = \mathcal{L}_h^1(x, \hat{\theta}, u) = \widehat{\mathcal{L}}_h^1(x, \hat{\theta}, u) + \omega_1(\theta - \hat{\theta}) \tag{5}$$

with

$$\widehat{\mathcal{L}}_h^1(x, \hat{\theta}, u) := \frac{\partial h}{\partial x}\Big[f_0(x) + \Phi(x)\hat{\theta} + (g_0(x) + \Psi(x)\hat{\theta})u\Big] \tag{6}$$

The second time derivative of the output is

$$\begin{aligned}\ddot{y} &= \frac{\partial\left(\mathcal{L}_h^1\right)}{\partial x}\dot{x} + \frac{\partial\left(\mathcal{L}_h^1\right)}{\partial \hat{\theta}}\dot{\hat{\theta}} + \frac{\partial\left(\mathcal{L}_h^1\right)}{\partial u}\dot{u} \\ &= \frac{\partial\left(\mathcal{L}_h^1\right)}{\partial x}\Big[f_0(x) + \Phi(x)\theta + (g_0(x) + \Psi(x)\theta)u\Big] \\ &\quad + \frac{\partial\left(\mathcal{L}_h^1\right)}{\partial \hat{\theta}}\dot{\hat{\theta}} + \frac{\partial\left(\mathcal{L}_h^1\right)}{\partial u}\dot{u}\end{aligned} \tag{7}$$

which can be rewritten as

$$\ddot{y} = \mathcal{L}_h^2(x, \hat{\theta}, u, \dot{u}) = \widehat{\mathcal{L}}_h^2(x, \hat{\theta}, u, \dot{u}) + \omega_2(\theta - \hat{\theta}) \tag{8}$$

with

$$\begin{aligned}\widehat{\mathcal{L}}_h^2 &:= \frac{\partial\left(\mathcal{L}_h^1\right)}{\partial x}\Big[f_0(x) + \Phi(x)\hat{\theta} + (g_0(x) + \Psi(x)\hat{\theta})u\Big] \\ &\quad + \frac{\partial\left(\mathcal{L}_h^1\right)}{\partial \hat{\theta}}\dot{\hat{\theta}} + \frac{\partial\left(\mathcal{L}_h^1\right)}{\partial u}\dot{u}\end{aligned} \tag{9}$$

and

$$\omega_2 = \frac{\partial\left(\mathcal{L}_h^1\right)}{\partial x}\Big(\Phi(x) + u\Psi(x)\Big) \tag{10}$$

By proceeding successively in this manner, we obtain the j-th time derivative of the output

$$\begin{aligned}y^{(j)} = \mathcal{L}_h^j(x, \hat{\theta}, u, \ldots, u^{(j-1)}) &= \widehat{\mathcal{L}}_h^j(x, \hat{\theta}, u, \ldots, u^{(j-1)}) \\ &\quad + \omega_j(\theta - \hat{\theta})\end{aligned} \tag{11}$$

with

$$\begin{aligned}\widehat{\mathcal{L}}_h^j \;:=\; & \frac{\partial\left(\mathcal{L}_h^{j-1}\right)}{\partial x}\Big[f_0(x)+\Phi(x)\hat{\theta}+(g_0(x)+\Psi(x)\hat{\theta})u\Big] \\ & +\frac{\partial\left(\mathcal{L}_h^{j-1}\right)}{\partial\hat{\theta}}\dot{\hat{\theta}}+\sum_{k=0}^{j-2}\frac{\partial\left(\mathcal{L}_h^{j-1}\right)}{\partial u^{(k)}}u^{(k+1)} \qquad (12)\end{aligned}$$

and

$$\omega_j=\frac{\partial\left(\mathcal{L}_h^{j-1}\right)}{\partial x}\Big(\Phi(x)+u\Psi(x)\Big) \qquad (13)$$

The expression (11) is valid if the relative degree is one. The general expression for systems with well-defined relative degree, i.e. $1\le\rho\le n$, has the form

$$\begin{aligned}y^{(j)}=\mathcal{L}_h^j(x,\hat{\theta},u,\ldots,u^{(j-\rho)}) \;=\; & \widehat{\mathcal{L}}_h^j(x,\hat{\theta},u,\ldots,u^{(j-\rho)}) \\ & +\omega_j(\theta-\hat{\theta}) \qquad (14)\end{aligned}$$

with

$$\begin{aligned}\widehat{\mathcal{L}}_h^j \;:=\; & \frac{\partial\left(\mathcal{L}_h^{j-1}\right)}{\partial x}\Big[f_0(x)+\Phi(x)\hat{\theta}+(g_0(x)+\Psi(x)\hat{\theta})u\Big] \\ & +\frac{\partial\left(\mathcal{L}_h^{j-1}\right)}{\partial\hat{\theta}}\dot{\hat{\theta}}+\sum_{k=0}^{j-\rho-1}\frac{\partial\left(\mathcal{L}_h^{j-1}\right)}{\partial u^{(k)}}u^{(k+1)} \qquad (15)\end{aligned}$$

In other words, the time derivatives of the output are obtained by the application of the following recursively defined operator

$$\begin{aligned}\mathcal{L}_h^0 \;=\; & h(x) \qquad (16) \\ \mathcal{L}_h^j \;:=\; & \frac{\partial\left(\mathcal{L}_h^{j-1}\right)}{\partial x}\Big[f_0(x)+\Phi(x)\theta+(g_0(x)+\Psi(x)\theta)u\Big] \\ & +\frac{\partial\left(\mathcal{L}_h^{j-1}\right)}{\partial\hat{\theta}}\dot{\hat{\theta}}+\sum_{k=0}^{j-\rho-1}\frac{\partial\left(\mathcal{L}_h^{j-1}\right)}{\partial u^{(k)}}u^{(k+1)}\; 1\le j\le n\end{aligned}$$

which also characterizes the control dependent nonlinear mapping

$$z=\Xi(x,\hat{\theta},u,\ldots,u^{(n-\rho-1)})=\begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)}\end{bmatrix}=\begin{bmatrix}\mathcal{L}_h^0 \\ \mathcal{L}_h^1 \\ \vdots \\ \mathcal{L}_h^{n-1}\end{bmatrix} \qquad (17)$$

Assumption 2.1 System (1) is locally *observable*, i.e. the mapping (17) satisfies the rank condition

$$\text{rank}\frac{\partial\Xi(\cdot)}{\partial x}=n \qquad (18)$$

in a subspace $R_1\subset R_0\subset\Re^n$.

Assumption 2.2 System (1) is *minimum phase* in $R_1\subset R_0\subset\Re^n$.

For observable minimum phase nonlinear systems of the form (1), the general problem of adaptively robustly tracking a bounded desired reference signal $y_r(t)$ with smooth and bounded derivatives can be solved through the DAB-SMC algorithm summarized as follows:

Coordinate transformation

$$\begin{aligned} z_1 &:= y - y_r(t) = h^{(0)}(x) - y_r(t) \\ z_k &:= \hat{h}^{(k-1)}(\cdot) - y_r^{(k-1)}(t) + \alpha_{k-1}(\cdot), \quad 2 \le k \le n \end{aligned} \tag{19}$$

with

$$\begin{aligned} \hat{h}^{(k)} &= \frac{\partial \hat{h}^{(k-1)}}{\partial \hat{\theta}} \tau_k + \frac{\partial \hat{h}^{(k-1)}}{\partial x} \Big[f_0 + \Phi\hat{\theta} + (g_0 + \Psi\hat{\theta}) v_1 \Big] \\ &\quad + \sum_{i=1}^{k-\rho-1} \frac{\partial \hat{h}^{(k-1)}}{\partial v_i} v_{i+1} + \frac{\partial \hat{h}^{(k-1)}}{\partial t} \end{aligned} \tag{20}$$

$$\omega_k = \left(\frac{\partial \hat{h}^{(k-1)}}{\partial x} + \frac{\partial \alpha_{k-1}}{\partial x} \right) \Big(\Phi(x) + u\Psi(x) \Big) \tag{21}$$

$$\begin{aligned} \alpha_k &= z_{k-1} + \left(\sum_{i=2}^{k-1} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{k-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) \Gamma \omega_k^T \\ &\quad + \sum_{i=1}^{k-\rho-1} \frac{\partial \alpha_{k-1}}{\partial v_i} v_{i+1} + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \tau_k + \frac{\partial \alpha_{k-1}}{\partial t} \\ &\quad + \frac{\partial \alpha_{k-1}}{\partial x} \Big[f_0 + \Phi\hat{\theta} + (g_0 + \Psi\hat{\theta}) v_1 \Big] + c_k z_k \end{aligned} \tag{22}$$

$$\tau_k = \Gamma \sum_{i=1}^{k} \omega_k^T z_k \qquad 1 \le k \le n-1 \tag{23}$$

Sliding surface
Define the sliding surface

$$\sigma = k_1 z_2 + k_2 z_2 + \ldots + k_{n-1} z_{n-1} + z_n = 0 \tag{24}$$

with the design parameters $k_i,\ i = 1, \ldots, n-1$, chosen such that the polynomial

$$p(s) = k_1 + k_2 s + \ldots + k_{n-1} s^{n-2} + s^{n-1} \tag{25}$$

in the complex variable s is Hurwitz.

Parameter update law

$$\dot{\hat{\theta}} = \tau_n = \tau_{n-1} + \Gamma\sigma \Big(\omega_n^T + \sum_{i=1}^{n-1} k_i \omega_i^T \Big) \tag{26}$$

Dynamical adaptive SMC law

$$\begin{aligned}
\dot v_1 &= v_2\\
\dot v_2 &= v_3\\
&\vdots\\
\dot v_{n-\rho} &= \frac{1}{\Delta}\Bigg[y_r^{(n)}(t) - \frac{\partial \hat h^{(n-1)}}{\partial t} - \frac{\partial \alpha_{n-1}}{\partial t}\\
&\qquad -\Big(\frac{\partial \hat h^{(n-1)}}{\partial \hat\theta} + \frac{\partial \alpha_{n-1}}{\partial \hat\theta}\Big)\tau_n\\
&\qquad -\sum_{i=1}^{n-\rho-1}\Big(\frac{\partial \hat h^{(n-1)}}{\partial v_i} + \frac{\partial \alpha_{n-1}}{\partial v_i}\Big)v_{i+1}\\
&\qquad -\Big(\frac{\partial \hat h^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x}\Big)\Big(f_0 + \Phi\hat\theta + (g_0 + \Psi\hat\theta)v_1\Big)\\
&\qquad -\sum_{i=2}^{n-1}\Big(\frac{\partial \hat h^{(n-1)}}{\partial \hat\theta} + \frac{\partial \alpha_{n-1}}{\partial \hat\theta}\Big)z_i\Gamma\Big(\omega_n^T + \sum_{i=1}^{n-1}k_i\omega_i^T\Big)\\
&\qquad +\sum_{i=1}^{n-1}k_i\Big(\frac{\partial \hat h^{(n-1)}}{\partial \hat\theta} + \frac{\partial \alpha_{n-1}}{\partial \hat\theta}\Big)(\tau_n - \tau_i)\\
&\qquad +\sum_{i=1}^{n-1}k_i\Big(\sum_{j=2}^{i-1}z_j\frac{\partial \hat h^{(j-1)}}{\partial \hat\theta} + \sum_{j=3}^{i-1}z_j\frac{\partial \alpha_{j-1}}{\partial \hat\theta}\Big)\\
&\qquad -\sum_{i=1}^{n-1}k_i(-z_{i-1} - c_iz_i + z_{i+1}) - \kappa\Big(\sigma + \beta\ \mathrm{sign}(\sigma)\Big)\Bigg]
\end{aligned} \tag{27}$$

with

$$v_1 = u; \quad \Delta = \left(\frac{\partial \hat h^{(n-1)}}{\partial v_{n-\rho}} + \frac{\partial \alpha_{n-1}}{\partial v_{n-\rho}}\right)$$

where the c_i's are constant design parameters and $\Gamma = \Gamma^T > 0$ is the adaptation gain matrix. The control u is obtained implicitly as the solution of the nonlinear time-varying differential equation (27). A complete description and proof of the DAB-SMC algorithm can be found in (Rios-Bolívar *et al.*, 1997; Rios-Bolívar, 1997) and a symbolic algebra design toolbox is available (Rios-Bolívar and Zinober, 1997).

The stability of the closed-loop system is proved by considering the Lyapunov function

$$V = \frac{1}{2}\sum_{i=1}^{n-1}z_i^2 + \frac{1}{2}\sigma^2 + \frac{1}{2}(\theta - \hat\theta)^T\Gamma^{-1}(\theta - \hat\theta) \tag{28}$$

whose time derivative is given by

$$\dot V = -z^TQz - \kappa\beta|\sigma| \tag{29}$$

where Q is a symmetric matrix with the following form

$$Q = \begin{bmatrix} c_1 + \kappa k_1^2 & \dots & \kappa k_1k_{n-1} & \kappa k_1\\ \kappa k_2k_1 & \dots & \kappa k_2k_{n-1} & \kappa k_2\\ \vdots & \ddots & \vdots & \vdots\\ \kappa k_{n-1}k_1 & \dots & c_{n-1} + \kappa k_{n-1}^2 & -\frac{1}{2} + \kappa k_{n-1}\\ \kappa k_1 & \dots & -\frac{1}{2} + \kappa k_{n-1} & \kappa \end{bmatrix}$$

Noting that the determinants of the principal minors of Q are all positive, a sufficient condition to guarantee that Q is positive definite is

$$\begin{aligned} |Q| &= \Big[-\frac{1}{4} + \kappa(c_{n-1} + k_{n-1})\Big] \prod_{i=1}^{n-2} c_i \\ &\quad -\frac{1}{4}\kappa \sum_{i=1}^{n-2} (c_1 \dots c_{i-1} k_i^2 c_{i+1} \dots c_{n-2}) > 0. \end{aligned} \tag{30}$$

Therefore, stability is guaranteed and asymptotic output tracking is achieved. Moreover, since the condition $\sigma\dot{\sigma} \leq 0$ holds, a sliding mode is generated on the sliding surface $\sigma = 0$. Asymptotic output regulation is achieved whenever the desired output y_r is constant.

3 Dynamical Adaptive Backstepping Second-Order SMC

The Dynamical Adaptive Backstepping Second-Order SMC (DAB-SOSMC) algorithm is simply an extension of the Dynamical Adaptive Backstepping (DAB) algorithm. The conditions on its applicability are therefore the same as those on the DAB algorithm. The main advantages of the DAB-SOSMC algorithm are the prevention of chattering, higher accuracy and a significant simplification of the control law.

The DAB algorithm is extended to incorporate second-order sliding in the following way: Suppose that we halt the DAB procedure at the calculation of the z_{n-1} error variable and use, instead of z_n,

$$y_1 = z_{n-1} + cz_{n-2} \tag{31}$$

With this substitution

$$\begin{aligned} \dot{V} &= -\sum_{k=1}^{n-3} c_k z_k^2 + f_1(x,v,\hat{\theta})(\dot{\hat{\theta}} - \tau_{n-3}) + (\theta - \hat{\theta})^T \Gamma^{-1}(-\dot{\hat{\theta}} + \tau_{n-3}) \\ &\quad +(1+c^2) z_{n-2}\dot{z}_{n-2} - cy_1\dot{z}_{n-2} - c\dot{y}_1 z_{n-2} \end{aligned} \tag{32}$$

Because z_{n-1} is selected such that

$$\dot{z}_{n-2} = f_2(x,v,\hat{\theta})(\theta - \hat{\theta}) + f_3(x,v,\hat{\theta})(\dot{\hat{\theta}} - \tau_{n-2}) - c_{n-2}z_{n-2} - z_{n-3} \tag{33}$$

(32) can be rewritten as

$$\begin{aligned} \dot{V} &= -\sum_{k=1}^{n-2} c_k z_k^2 + \tilde{f}_1(x,v,\hat{\theta})(\dot{\hat{\theta}} - \tau_{n-2}) + (\theta - \hat{\theta})^T \Gamma^{-1}(-\dot{\hat{\theta}} + \tau_{n-2} - c\Gamma z_{n-1}\omega_{n-2}^T) \\ &\quad -c^2 c_{n-2} z_{n-2}^2 - cy_1\dot{z}_{n-2} - c\dot{y}_1 z_{n-2} \end{aligned} \tag{34}$$

where

$$\begin{aligned} f_1(x,v,\hat{\theta}) &= \left(\sum_{i=2}^{n-3} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{n-3} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) \\ f_2(x,v,\hat{\theta}) &= \omega_{n-2} \\ f_3(x,v,\hat{\theta}) &= f_1(x,v,\hat{\theta}) \end{aligned}$$

$\tilde{f}_1(\cdot)$ is a suitable smooth function of its arguments, and IS-stability (Sontag, 1989) can be proved with respect to y_1, $\dot{y}_1$, provided that they are bounded signals and that $\dot{\hat{\theta}} = \tau_{n-2} - c\Gamma\omega_{n-2}^T z_{n-1}$.

If a way to steer y_1 to zero in finite time can be found, the overall closed-loop error system has the form

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{A}_z \tilde{z} + \tilde{W}(\theta - \hat{\theta}) && (35) \\ \dot{\hat{\theta}} &= \Gamma \tilde{W}^T \tilde{z} && (36) \end{aligned}$$

where $\tilde{z} = [z_1, \ldots, z_{n-1}]^T$, the matrix $\tilde{A}_z$ has the following skew-symmetric form

$$\tilde{A}_z = \begin{bmatrix} -c_1 & 1 & 0 & \ldots & 0 \\ -1 & -c_2 & 1 + \varrho_{2,3} & \ldots & \varrho_{2,n-1} \\ 0 & -1 - \varrho_{2,3} & -c_3 & \ldots & \varrho_{3,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\varrho_{2,n-1} & -\varrho_{3,n-1} & \ldots & 1 + \varrho_{n-1,n-1} \end{bmatrix}$$

with

$$\varrho_{i,j} = \left(\frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) \Gamma \omega_j^T, \tag{37}$$

and $\tilde{W}$ is a suitable vector of length $n-1$.

The skew-symmetric form of the matrix $\tilde{A}$ is important for the stability of the system (35)-(36), since the relation

$$\tilde{A}_z + \tilde{A}_z^T = -2 \begin{bmatrix} c_1 & 0 & \ldots & 0 \\ 0 & c_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & c_{n-1} \end{bmatrix} \tag{38}$$

yields

$$\dot{V} = -\sum_{i=1}^{n-1} c_i z_i^2 \tag{39}$$

with the quadratic Lyapunov function

$$V = \frac{1}{2} z^T z + \frac{1}{2} (\theta - \hat{\theta})^T \Gamma^{-1} (\theta - \hat{\theta}) \tag{40}$$

By a proof analogous to that used in (Rios-Bolívar *et al.*, 1995, 1997) for the DAB-system, it can be proved that the stability of the overall system is guaranteed and that asymptotic tracking is achieved.

Indeed, $y_1 = 0$ can be considered as a sliding surface, i.e., $\sigma(t) = y_1(t)$ and $\dot{s}igma(t) = y_2(t)$. So, the second order subsystem directly involved in the sliding process is

$$\begin{aligned} &\dot{y}_1(t) = y_2(t) \\ &\dot{y}_2(t) = k_1(x, \hat{\theta}, u, \ldots, u^{(n-\rho-1)}) - y_r^{(n)} - c y_r^{(n-1)} \\ &\qquad + k_2(x, \hat{\theta}, u, \ldots, u^{(n-\rho-1)})\theta + \theta^T k_3(x, \hat{\theta}, u, \ldots, u^{(n-\rho-1)})\theta \\ &\qquad + \left(\frac{\partial \hat{h}^{(n-2)}}{\partial u^{(n-\rho-2)}} + \frac{\partial \alpha^{(n-2)}}{\partial u^{(n-\rho-2)}} \right) u^{n-\rho} \end{aligned} \tag{41}$$

With implicit symbol definitions, and setting $\chi = (x, \hat{\theta}, u, \ldots, u^{(n-\rho-1)}, y_r, \ldots, y_r^{(n)})$, system (41) can be rewritten as

$$\begin{aligned} \dot{y}_1(t) &= y_2(t) \\ \dot{y}_2(t) &= H(\chi) + \beta_0(\chi) w(t) \end{aligned} \tag{42}$$

with $y_2(t)$ not determinable because of the presence of uncertainties.

The following assumptions relevant to $H(\chi)$ and $\beta_0 = \beta_0(\chi)$ are made for the sake of simplicity (actually more general cases can be dealt with via SOSMC, see (Bartolini *et al.*, 1997)):

$$|H(\chi)| < H_m \tag{43}$$

$$0 < B_1 \leq \beta_0 \leq B_2 \tag{44}$$

Thus, taking into account (42)–(44), the problem of steering to zero in a finite time both $y_1(t)$ and the unknown $y_2(t)$ (to attain the sliding regime on $\sigma(t) = \dot{\sigma}(t) = 0$) can be associated with the original control problem stated in Section 2.

In Bartolini *et al.* (1998) it has been proved that the control $w(t)$ can be chosen as bang–bang control (Kirk, 1970), switching between two values $-W_{Max}$, $+W_{Max}$. The classical switching logic for a double integrator ($H(\chi) = 0$, $B_1 = B_2 = 1$) is

$$w(t) = \begin{cases} -W_{Max} & \left\{y_1(t) > -\frac{1}{2}\frac{y_2(t)|y_2(t)|}{W_{Max}}\right\} \bigcup \\ & \left\{y_1(t) = -\frac{1}{2}\frac{y_2(t)|y_2(t)|}{W_{Max}} \bigcap y_1(t) < 0\right\} \\ & \\ +W_{Max} & \left\{y_1(t) < -\frac{1}{2}\frac{y_2(t)|y_2(t)|}{W_{Max}}\right\} \bigcup \\ & \left\{y_1(t) = -\frac{1}{2}\frac{y_2(t)|y_2(t)|}{W_{Max}} \bigcap y_1(t) > 0\right\} \end{cases} \tag{45}$$

This switching logic, instead of being based on the signs of $y_1(t) + (y_2(t)|y_2(t)|)/\ (2W_{Max})$ and $y_1(t)$, and therefore dependent upon both $y_1(t)$ and $y_2(t)$, can be expressed in terms of only $y_1(t)$ which, by assumption, is available for measurement. Indeed, it is easy to verify that the optimal trajectory is a sequence of two parabolic arcs. The second arc of the trajectory lies on the switching line $y_1(t) + (y_2(t)|y_2(t)|)/(2W_{Max}) = 0$. The modulus of the $y_1(t)$ component of the initial point of this second arc is equal to one half of the maximum modulus of the $y_1(t)$ component of the points of the previous part of the trajectory (Bartolini *et al.*, 1997).

Assume that the extremal value along each parabolic arc can be evaluated, and denote its abscissa with y_{Max}. Then, the previous considerations can be summarized by the algorithm presented in Bartolini *et al.* (1998), briefly recalled here for the reader's convenience.

Algorithm 1

i) Set $\alpha^* \in (0, 1] \cap \left(0, \dfrac{3B_1}{B_2}\right)$.

ii) Set $y_{max} = y_1(0)$.
Repeat, for any $t > 0$, the following steps:

(a) If $[y_1(t) - \frac{1}{2}y_{max}][y_{max} - y_1(t)] > 0$ then set $\alpha = \alpha^*$ else set $\alpha = 1$.

(b) If $y_1(t)$ is extremal value then set $y_{max} = y_1(t)$.

(c) Apply the control law

$$w(t) = -\alpha W_{Max} sign\{y_1(t) - \frac{1}{2}y_{max}\}$$

Until the end of the control time interval.

Algorithm 1 is equivalent to optimal control for the special case $H[\chi] = 0$, $B_1 = B_2 = 1$, $y_1(0)y_2(0) > 0$, $\alpha^* = 1$. Yet, it is valid also with $H[\chi] \neq 0$, $B_1 \neq B_2 \neq 1$, $y_1(0)y_2(0)$ not necessarily positive, in the sense that it allows the origin of the $y_1(t)$, $y_2(t)$ state space to be reached in a finite time. Indeed, in Bartolini *et al.* (1997) the following result has been proved.

Theorem 3.1 Given the state equation (42), with bounds as in (43)–(44), and $y_2(t)$ not available for measurement, then, for any $y_1(0)$, $y_2(0)$ the control strategy defined by Algorithm 1 with the additional constraint

$$W_{Max} > \max\left(\frac{H_m}{\alpha^* B_1}; \frac{4H_m}{3B_1 - \alpha^* B_2}\right) \tag{46}$$

causes the generation of a sequence of states with coordinates $(y_{max_i}, 0)$ featuring the contraction property $|y_{max_{i+1}}| < |y_{max_i}|$, $\quad i = 1, 2, \ldots$. Moreover, the convergence of the system trajectory to the origin of the state plane takes place in a finite time.

The convergence in a finite time of the sequence $\{y_{max_j}\}$ implies the convergence to zero of the phase trajectories, since in any time interval $[t_{max_j}, t_{max_{j+1}}]$ the maximum value of $|y_2(t)|$ is bounded by a function of $\sqrt{|y_{max_j}|}$ and this latter becomes zero in a finite time (Bartolini *et al.*, 1997).

In summary, the procedure proposed to design the control for uncertain nonlinear systems satisfying the mentioned assumptions can be expressed in algorithmic form as follows:

Algorithm 2

i) Apply the DAB algorithm until Step $n-1$, computing $z_1, \ldots, z_{n-1}$ (i.e. $\hat{h}^{(1)}, \ldots, \hat{h}^{(n-2)}, \alpha_1, \ldots, \alpha_{n-2}$);

ii) Design z_n as $y_1 = z_{n-1} + cz_{n-2}$;

iii) Apply Algorithm 1 to the system

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= H(z, v) + \beta_0(z, v)w(t) \end{aligned}$$

where $w(t) = u^{(n-\rho)}$, $z = [z_1, \ldots, z_{n-1}, y_1, y_2]$, $v = [u, \ldots, u^{(n-\rho-1)}]$, $H(\cdot)$, $\beta_0(\cdot)$ suitable functions to be upper bounded.

4 Comparative Example: Continuously Stirred Tank Reactor

Consider the following nonlinear third order dynamic model (see (Kravaris and Palanki, 1988)) of a Continuously Stirred Tank Reactor (CSTR) in which an isothermal liquid-phase, multicomponent chemical reaction takes place

$$\begin{aligned} \dot{x}_1 &= 1 - (1 + D_{a1})x_1 + D_{a2}x_2^2 \\ \dot{x}_2 &= D_{a1}x_1 - x_2 - (D_{a2} + D_{a3})x_2^2 + u \\ \dot{x}_3 &= D_{a3}x_2^2 - x_3 \\ y &= x_3 \end{aligned} \tag{47}$$

with

- x_1: normalized concentration C_A/C_{AF} of a species A
- x_2: normalized concentration C_B/C_{AF} of a species B

- x_2: normalized concentration C_C/C_{AF} of a species C
- C_{AF}: the feed concentration of the species A $(mol \cdot m^{-1})$
- u: the ratio of the per-unit volumetric molar feed rate of species B, denoted by N_{BF}, and the feed concentration C_{AF}, i.e. $u = N_{BF}/FC_{AF}$
- F: volumetric feed rate $(m^3 s^{-1})$
- $D_{a1} = k_1 V/F$ constant parameter
- $D_{a2} = k_2 V C_{AF}/F$ constant parameter
- $D_{a3} = k_3 V C_{AF}/F$ constant parameter
- V: the volume of the reactor (m^3)
- k_1, k_2, k_3: first order rate constants (s^{-1})

The system has a constant stable equilibrium point, for every constant volumetric feed rate value $u = U$, which is located in a minimum phase region of the system (Sira-Ramírez and Delgado, to be published)

$$\begin{aligned}
X_1 &= \frac{1 + D_{a2}X_2^2}{1 + D_{a1}} \\
X_2 &= (1 + D_{a1})\left[\frac{-1 + \left\{1 + 4\left(U + \dfrac{D_{a1}}{1 + D_{a1}}\right)\left(\dfrac{D_{a2} + D_{a3} + D_{a1}D_{a3}}{1 + D_{a1}}\right)\right\}^{\frac{1}{2}}}{2\left(D_{a2} + D_{a3} + D_{a1}D_{a3}\right)}\right] \\
X_3 &= D_{a3}X_2^2
\end{aligned} \tag{48}$$

The operating region of the system is, of course, the strict orthant in $\Re^3$, where all concentrations are positive. In other words,

$$\chi = \{x \in \Re^2, \quad \text{s.t.} \quad x_i > 0 \quad \text{for } i = 1, 2, 3\}$$

We assume that the constant parameters D_{a1}, D_{a2} and D_{a3} are all constant but unknown. Thus, system (47) can be rewritten as

$$\begin{aligned}
\dot{x}_1 &= 1 - x_1 + \varphi_1^T(x_1, x_2)\theta \\
\dot{x}_2 &= -x_2 + u + \varphi_2^T(x_1, x_2)\theta \\
\dot{x}_3 &= -x_3 + \varphi_3^T(x_2)\theta \\
y &= x_3
\end{aligned} \tag{49}$$

with $\theta = [\theta_1 \ \theta_2 \ \theta_3]^T = [D_{a1} \ D_{a_2} \ D_{a3}]^T$ the unknown parameter vector and

$$\varphi_1^T = [-x_1 \ \ x_2^2 \ \ 0] \quad ; \quad \varphi_2^T = [x_1 \ \ -x_2^2 \ \ -x_2^2] \quad ; \quad \varphi_3^T = [0 \ \ 0 \ \ x_2^2]$$

Both the DAB-SMC algorithm and the DAB-SOSMC algorithm can be applied to system (49) to synthesize a dynamical adaptive controller for its robust regulation.

4.1 DAB-SMC algorithm

Applying the DAB-SMC algorithm, we synthesize a dynamical adaptive SMC compensator for the regulation of system (49) using the Symbolic Algebra DAB-SMC MATLAB Toolbox (Rios-Bolívar and Zinober, 1997). This compensator is characterized by:

Coordinate transformation

$$\begin{aligned} z_1 &= y - X_3 = x_3 - X_3 \\ z_2 &= -x_3 + \varphi_3^T(x_2)\hat{\theta} + c_1 z_1 \\ z_3 &= \alpha(x,\hat{\theta}) + \frac{\partial \varphi_3^T}{\partial x_2}\hat{\theta}u \end{aligned} \tag{50}$$

Sliding surface

$$\sigma = k_1 z_1 + k_2 z_2 + z_3 = 0$$

Parameter update law

$$\begin{aligned} \dot{\hat{\theta}} = \tau_3 &= \tau_2 + \Gamma\sigma(k_1\varphi_3 + k_2\omega_2 + \omega_3) \\ &= \Gamma\Big[z_1\varphi_3 + z_2\omega_2 + \sigma(k_1\varphi_3 + k_2\omega_2 + \omega_3)\Big] \end{aligned} \tag{51}$$

with

$$\begin{aligned} \omega_2^T &= (c_1 - 1)\varphi_3^T(x_2) + \frac{\partial \varphi_3^T}{\partial x_2}\hat{\theta}\varphi_2^T(x_1, x_2) \\ \omega_3^T &= \frac{\partial \alpha}{\partial x_1}\varphi_1^T(x_1, x_2) + \Big(\frac{\partial \alpha}{\partial x_2} + \frac{\partial^2 \varphi_3^T}{\partial x_2^2}\hat{\theta}u\Big)\varphi_2^T(x_1, x_2) \\ &\quad + \frac{\partial \alpha}{\partial x_3}\varphi_3^T(x_2) \\ \alpha(x,\hat{\theta}) &= z_1 - (c_1 - 1)x_3 - \frac{\partial \varphi_3^T}{\partial x_2}\hat{\theta}x_2 + \omega_2^T\hat{\theta} \\ &\quad + \varphi_3^T\Gamma(z_1\varphi_3 + z_2\omega_2) + c_2 z_2 \end{aligned} \tag{52}$$

Dynamical adaptive SMC law

$$\begin{aligned} \dot{u} &= \frac{1}{\frac{\partial \varphi_3^T}{\partial x_2}\hat{\theta}}\Bigg[- (k_2 + z_2)\varphi_3^T(\tau_3 - \tau_2) - k_1(-c_1 z_1 + z_2) \\ &\quad -\omega_3^T\hat{\theta} - k_2(-z_1 - c_2 z_2 + z_3) - \frac{\partial \alpha}{\partial x_1}(1 - x_1) \\ &\quad -\Big(\frac{\partial \alpha}{\partial x_2} + \frac{\partial^2 \varphi_3^T}{\partial x_2^2}\hat{\theta}u\Big)(-x_2 + u) + \frac{\partial \alpha}{\partial x_3}x_3 \\ &\quad -\Big(\frac{\partial \alpha}{\partial \hat{\theta}} + u\frac{\partial \varphi_3^T}{\partial x_2}\Big)\tau_3 - \kappa(\sigma + \beta\mathrm{sign}(\sigma))\Bigg] \end{aligned} \tag{53}$$

where $\Gamma = \Gamma^T > 0$ is a diagonal matrix containing the adaptation parameter gains.

Finally, by satisfying the stability condition (30)

$$|Q| = [-\frac{1}{4} + \kappa(c_2 + k_2)]c_1 - \frac{1}{4}\kappa k_1^2 c_2 > 0$$

the output $y = x_3$ asymptotically converges to the desired value X_3.

4.2 DAB-SOSMC algorithm

The combined DAB-SOSMC algorithm can be applied to synthesize a dynamical adaptive discontinuous controller for the robust regulation of system (49).

Coordinate transformation

$$\begin{aligned} z_1 &= y - X_3 = x_3 - X_3 \\ z_2 &= -x_3 + \varphi_3^T(x_2)\hat{\theta} + c_1 z_1 \end{aligned} \tag{54}$$

Sliding surface

$$y_1 = \sigma = z_2 + cz_1 = 0$$

Auxiliary system

$$\begin{aligned} y_1 &= z_2 + cz_1 \\ y_2 &= (c - 1 + c_1)\left(x_3 - {\varphi_3}^T\theta\right) \\ &\quad + \frac{\partial \varphi_3}{\partial x_2}^T\left(-x_2 + u + {\varphi_2}^T\theta\right)\hat{\theta} + {\varphi_3}^T\dot{\hat{\theta}} \end{aligned} \tag{55}$$

Parameter update law

$$\begin{aligned} \dot{\hat{\theta}} = \tau_2 &= \tau_1 + \Gamma(-cz_2\varphi_3) \\ &= \Gamma\left[z_1\varphi_3 - cz_2\varphi_3\right] \end{aligned} \tag{56}$$

where $\Gamma = \Gamma^T > 0$ is a diagonal matrix containing the adaptation parameter gains.

Dynamical adaptive SOSMC law
Using Algorithm 1

$$\dot{w} = -\alpha W_{Max}\mathrm{sign}\left\{y_1 - \frac{1}{2}y_{1_{max}}\right\} \tag{57}$$

Algorithm 1 guarantees y_1 and y_2 to be bounded and dependent upon $y_1(0)$ and $y_2(0)$ (Bartolini *et al.*, 1998). It is always possible to choose c_1, c in such a way that $\dot{V}_2 < -c_1 z_1^2 - cz_2^2$ during the reaching phase, obtaining $\dot{V}_2 = -(c(c_1c+1) + c + c_1)z_1$ on the intersection of the manifolds described by the equations $y_1 = 0$, and $\dot{y}_1 = 0$. This guarantees that $z_1 \to 0$, and $z_2 \to 0$. Moreover, y_1 (which replaces the z_3 error variable of the pure DAB algorithm) tends to zero in finite time.

4.3 Simulations

Computer simulations were performed using both the DAB-SMC and DAB-SOSMC designed control laws for the robust regulation of a CSTR with the following “unknown” parameters

$$D_{a1} = 3.0 \quad ; \quad D_{a2} = 0.5 \quad ; \quad D_{a3} = 1.0$$

The desired equilibrium, corresponding to a constant value of u given by $U = 1$, is obtained as

$$X_1 = 0.3467 \quad ; \quad X_2 = 0.8796 \quad ; \quad X_3 = 0.7753$$

whilst the design parameters for the DAB-SMC law were selected to be

$$c_1 = 2 \quad , \quad c_2 = 1 \quad , \quad c_3 = 2 \quad , \quad \Gamma = 2I_3 \quad , \quad \kappa = 2 \quad , \quad \beta = 1 \quad , \quad k_1 = 1$$

and for the DAB-SOSMC law

$$c_1 = 2 \quad , \quad c = 1 \quad , \quad \Gamma = 2I_3 \quad , \quad W_{Max} = 500$$

Fig. 1 shows the DAB-SMC controlled CSTR output responses, whilst Fig. 2 depicts the DAB-SOSMC controlled output responses. It can be seen that the DAB-SMC controlled responses exhibit good transient performance to the equilibrium point, whilst achieving parameter convergence and very small control chatter. The DAB-SOSMC controlled responses exhibit good transient performance but only output tracking of the system can be guaranteed, i.e. $x_3 \rightarrow 0.7753$ as t increases, but the x_1 and x_2 states do not tend to the equilibrium values corresponding to $U = 1$. The overshoot transients of the states x_1, x_2 and x_3 are much larger for SOSMC. However, in comparison to the DAB-SMC algorithm, the DAB-SOSMC algorithm removes chattering completely from the control law and makes use of a simpler control, achieving a significant reduction in the number of computations.

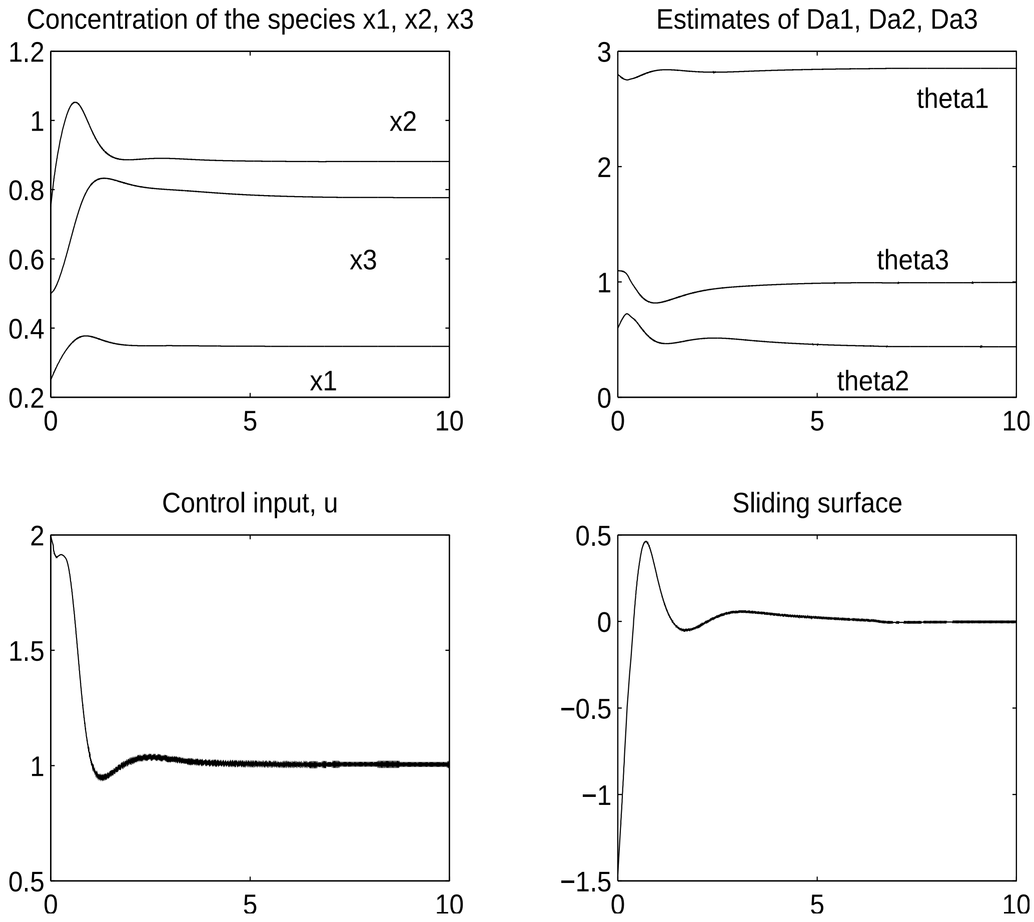

Figure 1: DAB-SMC Controlled responses of the Isothermal CSTR.

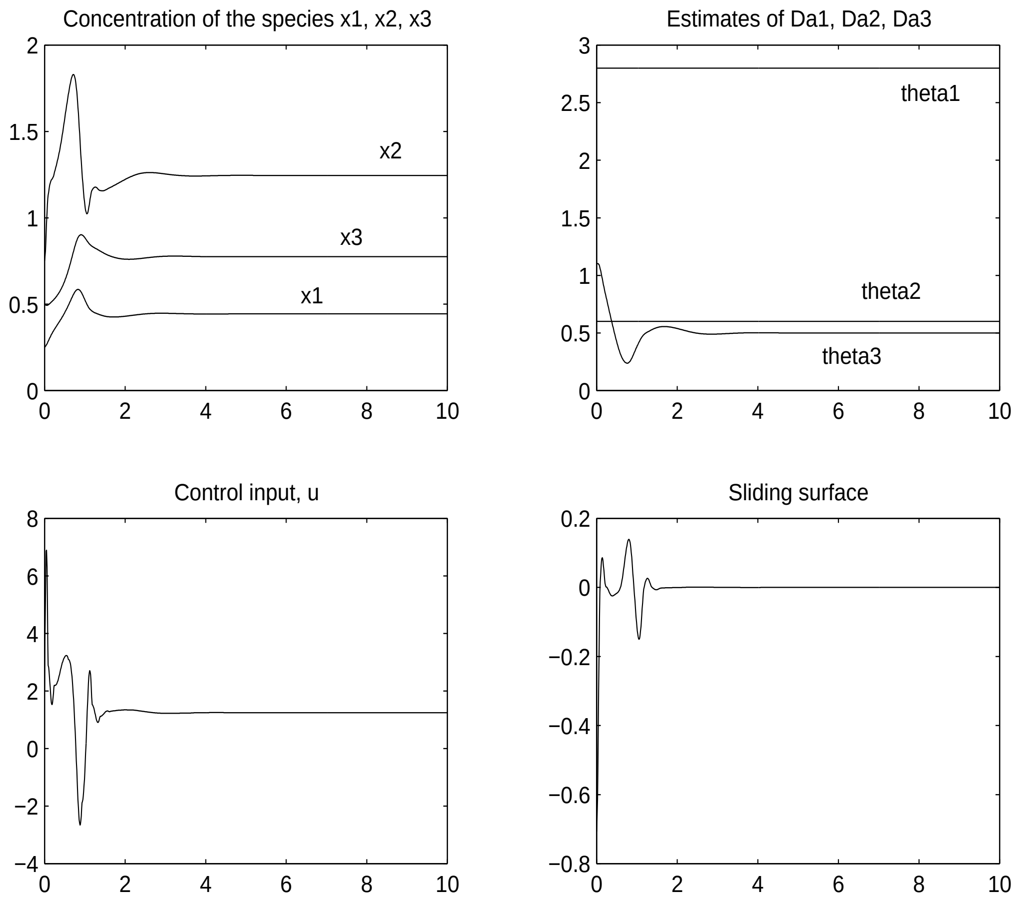

Figure 2: DAB-SOSMC Controlled responses of the Isothermal CSTR.

5 Conclusions

In this paper two different combined algorithms for the control of non-triangular nonlinear systems with unmatched uncertainties have been presented: the Dynamical Adaptive Backstepping-SMC (DAB-SMC) algorithm, based upon a combination of dynamical input-output linearization and the adaptive backstepping algorithm with tuning functions; and the Dynamical Adaptive-Second Order SMC (DAB-SOSMC) algorithm, based upon an extension of the DAB algorithm which combines the DAB algorithm with higher order sliding and bang-bang control. The application of both of these combined algorithms to a nonlinear continuous chemical process has also been presented. The mathematical model satisfies the minimum phase and observability conditions which are stringent conditions for the applicability of these control design approaches. Since the relative degree of the example is less than the corresponding system order, the designed compensators are dynamical, i.e. derivatives of the control input are involved. Both the DAB-SMC and SOSMC laws are synthesized in a systematic manner and the stability is proved in both cases by using a quadratic Lyapunov function. From the computer simulations, it can be seen that the adaptively controlled responses of these two approaches exhibit good transient performance. The DAB-SMC algorithm exhibits only small control chatter and guarantees parameter convergence. The DAB-SOSMC algorithm, in contrast, only guarantees output tracking but does remove chattering completely from the control law.

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