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Analytic Conditions for Stabilizability*

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Abstract

The Nyquist criterion gives a graphical condition for closed loop stability stated in terms of the Nyquist plot of the open loop system transfer function $G(s)$. In this paper we develop an equivalent, but new, *analytical* criterion for closed loop stability, based on analysis of the behaviour of a real polynomial function $X(u)$ constructed from $G(s)$. It is shown that the real negative zeros u_i of $X(u)$ and the signs of $\dot{X}(u)|_{u=u_i}$ determine the range of stabilizing gains K completely, and in closed form. Besides providing a nongraphical and computationally simpler alternative to the Nyquist criterion and root locus techniques, this solution is a first step towards investigating stabilizability by higher order controllers.Some illustrative examples are given.

1 Introduction

Consider the single-input single-output feedback control system in Figure 1.

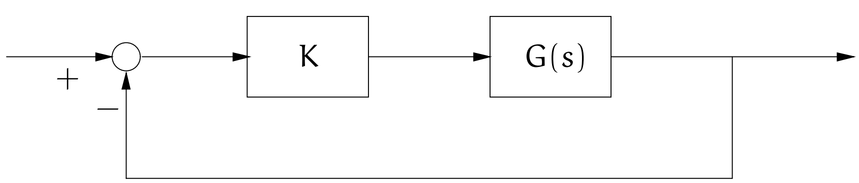

Figure 1: A unity feedback system

The problem of interest is to determine the range of stabilizing gains K, if any, from $G(s) = \frac{N(s)}{D(s)}$. There are two classical methods of solution for this problem, namely, the Nyquist criterion (Nyquist, 1932) and the Root Locus (Evans, 1950) technique. Both methods are graphical and computationally intensive.

In this paper we develop an alternative analytic solution of this problem. This new solution is developed by combining information from the root locus method and the Nyquist criterion.

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The result is that the complete set of stabilizing gains can be obtained in closed form from some simple computations performed on a real polynomial $X(u)$ constructed from $G(s)$. The only computations involved are 1) finding the real negative zeros of $X(u)$, and 2) finding the signs of $\dot{X}(u)$ at these zeros. Although this computational simplicity is itself of some interest our real motivation for developing this new solution is that its simplicity may help in shedding some light of stabilizability by controllers of fixed order and structure.

2 Notation and preliminary considerations

The characteristic equation of the closed loop system can be written in the polynomial form

$$D(s) + KN(s) = 0 \tag{1}$$

or the rational form

$$1 + K\frac{N(s)}{D(s)} := 1 + KG(s) = 0. \tag{2}$$

Under the assumption that $N(s)$ and $D(s)$ are coprime (see the standing assumptions below) the zeros of equations (1) and (2) are identical and are the *closed loop characteristic (clc) roots*. We consider the two ranges $K \in [0, +\infty) := \mathbb{R}^+$ and $K \in (-\infty, 0] := \mathbb{R}^-$. For solvability of the constant gain stabilization problem it is required that there exist values of $K \in \mathbb{R}^+ \cup \mathbb{R}^-$ for which all clc roots are in the open left half plane (LHP), or equivalently no clc roots are in the closed right half plane (RHP).

To proceed, decompose $D(s)$ and $N(s)$ into its odd and even parts as follows:

$$D(s) = D_e\left(s^2\right) + sD_o\left(s^2\right)$$
$$N(s) = N_e\left(s^2\right) + sN_o\left(s^2\right)$$

where D_e, N_e, D_o, N_o are polynomials in s^2. Define the real polynomial

$$X(u) = D_e(u)N_o(u) - N_e(u)D_o(u). \tag{3}$$

We make the following standing assumptions:

Assumptions

1. $D_e(u)N_o(u)$ and $N_e(u)D_o(u)$ are coprime.
2. The zeros of $X(u)$ are distinct.
3. $X(0) \neq 0$,
4. $G(s)$ has no poles on the imaginary axis.
5. $G(s)$ is not improper.

Note that assumptions 1, 2, 3 and 4 will hold generically and in case they fail a slight perturbation of the coefficients of $D(s)$ and $N(s)$ will satisfy them, without changing the conclusions. Assumption 5) always holds in practical cases.

We introduce the set of real negative zeros of $X(u)$ along with 0 and $-\infty$. Let $\mathcal{U}$ denote the elements of this set, *ordered* in the specific way defined below. Write

$$\mathcal{U} = \{u_1, u_2, \cdots u_p, \cdots, u_q, \cdots, u_m\}. \tag{4}$$

where

$$u_p = 0, \quad u_q = -\infty \tag{5}$$

and the remaining $u_i, i \neq p, q$ are the negative real zeros of $X(u)$. Define the *crossover gains*

$$K_i := -\frac{D_e(u_i)}{N_e(u_i)}, \quad i = 1, \cdots, m; \quad i \neq p, q \tag{6}$$

$$K_p := -\frac{D_e(0)}{N_e(0)} \tag{7}$$

and

$$K_q := -\frac{1}{G(\infty)}. \tag{8}$$

Note that as a result of the assumptions all the K_i, except K_q are well defined and finite and K_q has infinite magnitude when $G(s)$ is strictly proper.

To order the set of crossover gains and thereby the elements of $\mathcal{U}$, we adopt the convention that by *increasing* K we shall mean that K starts from 0^+ and traverses the set

$$[0^+ \ \rightarrow \ \infty \vdots -\infty \ \rightarrow \ 0^-]$$

from left to right. In other words the ordered set of gains

$$\mathcal{K} := \{K_1, K_2, \cdots, K_t \cdots, K_m\}$$

which may include positive, negative and infinite values, satisfy

$$0^+ := K_0 < K_1 < K_2 \cdots K_t \leq +\infty, \quad -\infty \leq K_{t+1}, \cdots, K_m < K_{m+1} := 0^- \tag{9}$$

and this ordering is what *induces* the ordering of the u_i in the set $\mathcal{U}$.

Using the standard sign function

$$\mathrm{Sign}[x] := \begin{cases} +1, & \text{when } x > 0 \\ -1, & \text{when } x < 0 \\ \ \ 0, & \text{when } x = 0 \end{cases}$$

introduce the real numbers

$$\begin{aligned} l_i &= 2\mathrm{Sign}\left[\frac{dX(u)}{du}\right]_{u=u_i}, \quad i = 1, \cdots, m; \quad i \neq p, q \\ l_p &= -\mathrm{Sign}[X(0)] \\ l_q &= +\mathrm{Sign}[X(-\infty)] \end{aligned}$$

and set $l_0 = 0$.

With these preliminaries we are ready to state the main result.

3 Main result

With the notation and definitions of the previous section we can state the main result of the paper. Let the number of closed right half plane (RHP) roots of $D(s) = 0$ be denoted by r.

Theorem 1 *The number of clc roots in the RHP for* $K \in (K_i, K_{i+1})$ *is given by*

$$r_i = r - \sum_{j=0}^{j=i} l_j, \quad i = 0, 1, \cdots, m \tag{10}$$

The proof of this theorem is based on a combination of results from Nyquist and Root Locus theories stated below as Lemma 1. As K increases from 0^+, the clc roots move from the n roots of

$$D(s) = 0.$$

The trace of these clc roots, which consists of n directed arcs in the complex plane, make up the *root loci*, with the direction corresponding to increasing K. We are interested in determining the *numbers* and *directions* of crossings of the stability boundary, namely the imaginary axis, by the root loci, as K increases from 0. The answer to this is given in the Lemma below.

Lemma 1 *The root loci cross the imaginary axis as follows:*

1. *At* $s = 0$*, one root crosses the imaginary axis when*

$$K = -\frac{1}{G(0)} \tag{11}$$

the direction of crossing, for increasing K*, being*

$$\text{RHP to LHP if } X(0) < 0$$

and

$$\text{LHP to RHP if } X(0) > 0.$$

2. *At* $s = \infty$ *one root crosses the imaginary axis when*

$$K = -\frac{1}{G(\infty)} \tag{12}$$

the direction of crossing, for increasing K*, being*

$$\text{RHP to LHP if } X(-\infty) > 0$$

and

$$\text{LHP to RHP if } X(-\infty) < 0.$$

3. *At* $s = \pm j\omega, 0 < \omega < \infty$ *two roots cross the imaginary axis (one at* $s = +j\omega$ *and another at* $s = -j\omega$*) if and only if with* $u = -\omega^2$*,*

$$X(u) = D_e(u)N_o(u) - D_o(u)N_e(u) = 0 \tag{13}$$

is satisfied and the gain

$$K = -\frac{1}{G(j\omega)}. \tag{14}$$

The direction of crossing for both roots, for increasing K*, is:*

$$\text{RHP to LHP if } \dot{X}(u) > 0$$

and

$$\text{LHP to RHP if } \dot{X}(u) < 0.$$

Proof. For a root locus branch to cross the imaginary axis at $s = 0$ we must have

$$1 + KG(0) = 0. \tag{15}$$

Similarly for a crossing at $s = \infty$ it is necessary and sufficient that

$$1 + KG(\infty) = 0 \tag{16}$$

gives a finite value of K. For root locus crossings to occur at $s \pm j\omega$ for $0 < \omega < \infty$, it is necessary and sufficient that

$$D(j\omega) + KN(j\omega) = 0. \tag{17}$$

has a real solution K. The above equation separates into the two real equations

$$D_e(-\omega^2) + KN_e(-\omega^2) = 0 \tag{18}$$
$$D_o(-\omega^2) + KN_o(-\omega^2) = 0. \tag{19}$$

Therefore there exists a real solution K to these equations if and only if

$$D_e(-\omega^2)N_o(\omega^2) - D_o(-\omega^2)N_e(\omega^2) = 0 \tag{20}$$

which is just the equation

$$X(u) = 0. \tag{21}$$

This proves that root loci crossings of the imaginary axis occur at the points and gain values stated in the Lemma.

To determine the number and direction of crossing of the roots we use the Nyquist criterion. Consider the Nyquist plot of $G(s)$ as s traverses the Nyquist contour clockwise. The Nyquist contour consists, as usual, of the imaginary axis and a semicircle of infinite radius enclosing the right half plane. In a clockwise traversal of the imaginary axis we have frequency increasing along the Nyquist plot. Thus at $s = j0$ the frequency increases from $0^- \rightarrow 0^+$, and at $s = j\infty$ the frequency moves from $+\infty \rightarrow -\infty$ along the Nyquist contour.

Now consider the point $-\frac{1}{K}$ located on the real axis of the s plane. As K increases, this point moves to the right along the negative real axis. As this point moves past a real axis intercept of the Nyquist plot, say at $G(j\omega_i)$, a branch of the root locus crosses the imaginary axis at $s^* = j\omega_i$ and at the corresponding gain value of $K_i = -\frac{1}{G(j\omega_i)}$. This is illustrated in Figures 2 - 7.

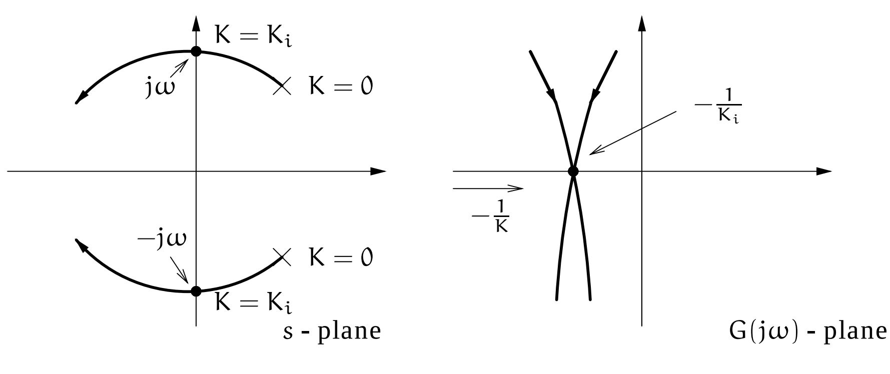

Figure 2: Root locus boundary crossings and Nyquist plot real axis crossings: 2 RHP to LHP crossings

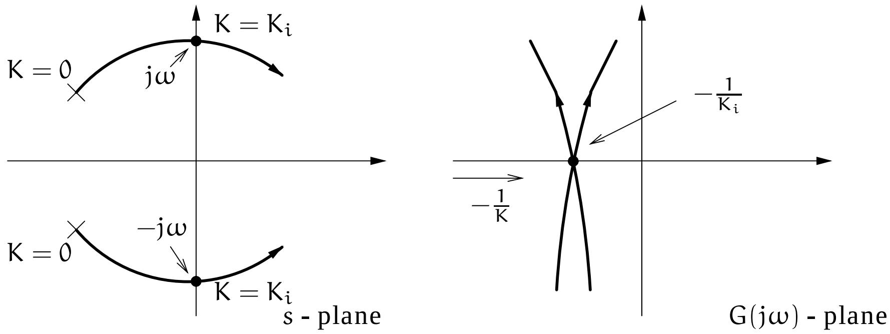

Figure 3: Root locus boundary crossings and Nyquist plot real axis crossings: 2 LHP to RHP crossings

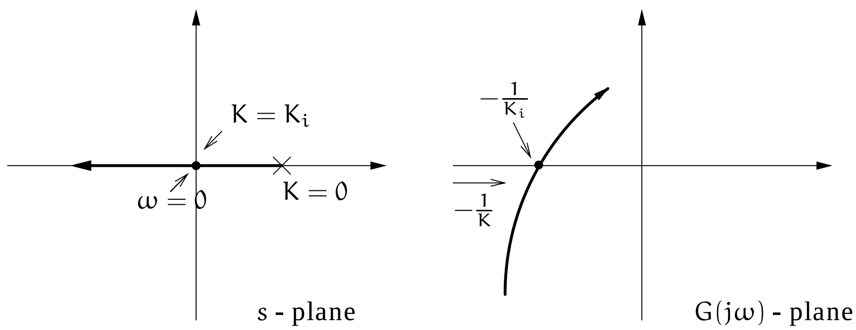

Figure 4: Root locus boundary crossings and Nyquist plot real axis crossings: 1 RHP to LHP through the origin

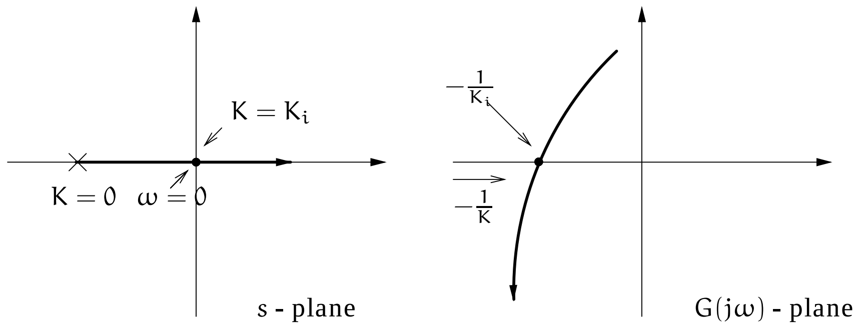

Figure 5: Root locus boundary crossings and Nyquist plot real axis crossings: 1 LHP to RHP through the origin

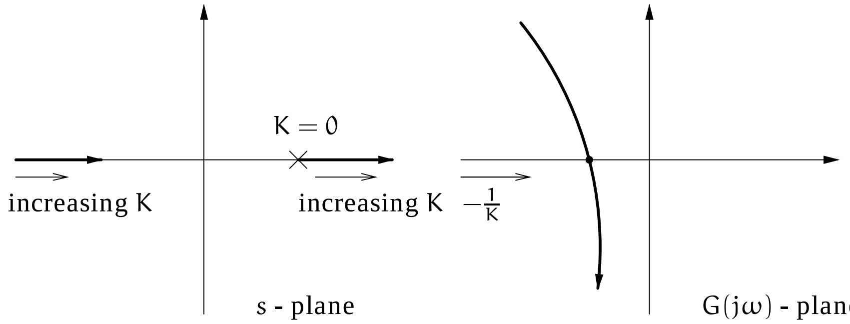

Figure 6: Root locus boundary crossings and Nyquist plot real axis crossings: 1 RHP to LHP through $s = \infty$

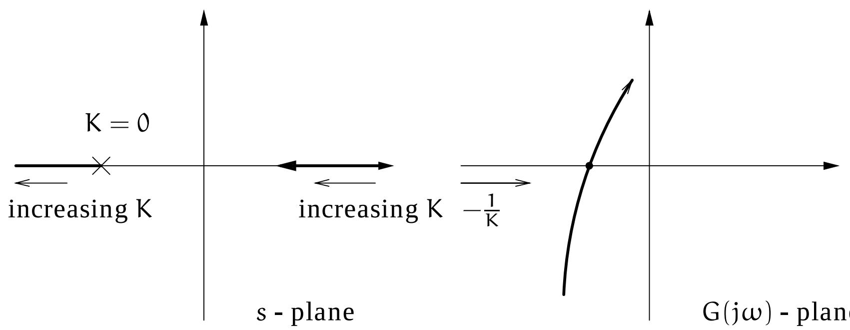

Figure 7: Root locus boundary crossings and Nyquist plot real axis crossings: 1 LHP to RHP through $s = \infty$

If the direction of movement of the Nyquist plot at this point is downward it follows that as K passes increasingly through this value $K = K_i$ a net increase of the counterclockwise encirclements of the point $-\frac{1}{K}$ will occur. From the Nyquist criterion it follows that this corresponds to roots crossing from the RHP to the LHP. Of course if the direction of movement of the Nyquist plot at this point is upward exactly the opposite is true, namely that the root locus crosses from the LHP to the RHP. These cases are illustrated in Figures 2 - 7. The proof of the Lemma requires two additional facts. First if $s^* = j\omega_i$ is a boundary crossing point and $\omega_i \neq 0$ then $-j\omega_i$ is also a boundary crossing point. Thus at the real axis point $K_i = -\frac{1}{G(j\omega_i)}$ *two* branches of the Nyquist plot will cross the real axis and a pair of root loci branches cross the imaginary axis at $s = \pm j\omega_i$ in the same direction. If a root locus branch crosses the boundary at the origin then the corresponding point $K_0 = -\frac{1}{G(0)}$ will correspond to a real axis intercept of the Nyquist plot. However in this case only one branch of the Nyquist plot can pass through this point. Again a downward movement with increasing frequency, of the Nyquist plot will correspond to a RHP to LHP transition of the root locus and an upward movement will correspond to an LHP to RHP transition. Exactly the same considerations apply at $s = \infty$ where one root crossing occurs if and only if $K = -\frac{1}{G(\infty)}$. The direction of the root crossing, for increasing K, is again RHP to LHP if the Nyquist plot is moving downward, and LHP to RHP if the Nyquist plot is moving upward as s makes the transition from $+\infty$ to $-j\infty$.

We show below that these facts can be converted to computations involving only the function $X(u)$. Write

$$
\begin{aligned}
G(s) &= \frac{N(s)}{D(s)} \\
&= \frac{N(s)D^*(s)}{D(s)D^*(s)} \\
&= \frac{\left[N_e\left(s^2\right) + sN_o\left(s^2\right)\right]\left[D_e\left(s^2\right) - sD_o\left(s^2\right)\right]}{D_e\left(s^2\right)D_e\left(s^2\right) - s^2D_o\left(s^2\right)D_o\left(s^2\right)} \\
&= \frac{N_e\left(s^2\right)D_e\left(s^2\right) - s^2N_o\left(s^2\right)D_o\left(s^2\right)}{D_e\left(s^2\right)D_e\left(s^2\right) - s^2D_o\left(s^2\right)D_o\left(s^2\right)} + \frac{s\left[N_o\left(s^2\right)D_e\left(s^2\right) - N_e\left(s^2\right)D_o\left(s^2\right)\right]}{D_e\left(s^2\right)D_e\left(s^2\right) - s^2D_o\left(s^2\right)D_o\left(s^2\right)}.
\end{aligned}
\tag{22}
$$

Therefore

$$
\begin{aligned}
G(j\omega) &= \underbrace{\frac{N_e\left(-\omega^2\right)D_e\left(-\omega^2\right) + \omega^2N_o\left(-\omega^2\right)D_o\left(-\omega^2\right)}{D_e^2\left(-\omega^2\right) + \omega^2D_o^2\left(-\omega^2\right)}}_{R(\omega)} \\
&\qquad + j\underbrace{\frac{\omega\left[N_o\left(-\omega^2\right)D_e\left(-\omega^2\right) - N_e\left(-\omega^2\right)D_o\left(-\omega^2\right)\right]}{D_e^2\left(-\omega^2\right) + \omega^2D_o^2\left(-\omega^2\right)}}_{I(\omega)} \\
&= R(\omega) + jI(\omega).
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
I(\omega) &= \frac{\omega\left[N_o\left(-\omega^2\right)D_e\left(-\omega^2\right) - N_e\left(-\omega^2\right)D_o\left(-\omega^2\right)\right]}{D_e^2\left(-\omega^2\right) + \omega^2D_o^2\left(-\omega^2\right)} \\
&:= \frac{\omega X(-\omega^2)}{Z(-\omega^2)}
\end{aligned}
\tag{24}
$$

and

$$
\begin{aligned}
R(\omega) &= \frac{N_e\left(-\omega^2\right)D_e\left(-\omega^2\right) + \omega^2N_o\left(-\omega^2\right)D_o\left(-\omega^2\right)}{D_e^2\left(-\omega^2\right) + \omega^2D_o^2\left(-\omega^2\right)} \\
&:= \frac{Y(-\omega^2)}{Z(-\omega^2)}.
\end{aligned}
\tag{25}
$$

If we set

$$u := -\omega^2 \tag{26}$$

we have $u \le 0$ for all real values of ω and

$$X\left(-\omega^2\right) := X(u) := N_o(u)D_e(u) - N_e(u)D_o(u) \tag{27}$$

$$Y\left(-\omega^2\right) := Y(u) := N_e(u)D_e(u) - uN_o(u)D_o(u) \tag{28}$$

and

$$Z(-\omega^2) := Z(u) := D_e^2(u) - uD_o^2(u) \tag{29}$$

and therefore

$$G(j\omega) = \frac{Y(u)}{Z(u)} + \frac{\omega X(u)}{Z(u)}. \tag{30}$$

At a real axis crossing of the Nyquist plot we must have

$$I(\omega) = 0 \tag{31}$$

and therefore the real axis crossings occur at $\omega = 0$, $\omega = \infty$ and ω_i corresponding to the real negative zeros $u_i = \omega_i^2$ of

$$X(u) = 0. \tag{32}$$

A downward movement of the Nyquist plot at each of these points corresponds to

$$\frac{dI(\omega)}{d\omega} < 0 \tag{33}$$

and an upward movement corresponds to

$$\frac{dI(\omega)}{d\omega} > 0 \tag{34}$$

evaluated at each of these points. Using the notation

$$\dot{f}(x) := \frac{df(x)}{dx}$$

it is easy to show that

$$\dot{I}(\omega) = \frac{Y(u)X(u) - 2\omega^2 Y(u)\dot{X}(u) + 2\omega^2 X(u)\dot{Y}(u)}{Y^2(u)}.$$

Now consider a point ω_i with $0 < \omega_i < \infty$ where the Nyquist plot cuts the real axis. At such a point $X(u_i) = 0$ and therefore

$$\dot{I}(\omega)|_{\omega=\omega_i} = \frac{-2\omega_i^2 \dot{X}(u_i)}{Y(u_i)}.$$

Since $Y(u_i) > 0$ we have

$$\dot{I}(\omega)|_{\omega=\omega_i} > 0 \quad \Longleftrightarrow \quad \dot{X}(u)\Big|_{u=u_i} < 0$$

and

$$\dot{I}(\omega)|_{\omega=\omega_i} < 0 \quad \Longleftrightarrow \quad \dot{X}(u)\Big|_{u=u_i} > 0.$$

Now let $\epsilon > 0$ be a small number and consider the transition $s = 0 - j\epsilon \to 0 + j\epsilon$ on the Nyquist contour. At such a point the Nyquist plot moves downward if the transition

$$\frac{-\epsilon X(0)}{Y(0)} \quad \Longrightarrow \quad \frac{+\epsilon X(0)}{Y(0)} \tag{35}$$

is $+ \to -$ and upward otherwise. Therefore, since $Y(0) > 0$ we have

$$\dot{I}(\omega)\Big|_{\omega=0} > 0 \quad \Longleftrightarrow \quad X(0) > 0$$

and

$$\dot{I}(\omega)\Big|_{\omega=0} < 0 \quad \Longleftrightarrow \quad X(0) < 0.$$

Now consider the transition point $s = +j\infty - j\epsilon \rightarrow s = -j\infty + j\epsilon$ on the Nyquist contour where $\epsilon > 0$ is a small number. Clearly in this case we have

$$\dot{I}(\omega)\Big|_{\omega=\infty} > 0 \quad \Longleftrightarrow \quad X(-\infty) < 0$$

and

$$\dot{I}(\omega)\Big|_{\omega=\infty} < 0 \quad \Longleftrightarrow \quad X(-\infty) > 0.$$

To summarize the above conclusions we have shown that:

1. One branch of the root locus crosses from the RHP to the LHP at $s = 0$ if and only if $X(0) < 0$
2. One branch of root locus crosses from the LHP the RHP at $s = 0$ if and only if $X(0) > 0$
3. Two branches of the root locus cross from the RHP to the LHP at $s = \pm j\omega_i$ if and only if with $-\omega_i^2 = u_i$, $X(u_i) = 0$ and $\dot{X}(u)|_{u=u_i} > 0$
4. Two branches of the root locus cross from the LHP to the RHP at $s = \pm j\omega_i$ if and only if with $-\omega_i^2 = u_i$, $X(u_i) = 0$ and $\dot{X}(u)|_{u=u_i} < 0$
5. One branch of the root locus crosses from the RHP to the LHP at $s = j\infty$ if and only if $X(-\infty) > 0$
6. One branch of root locus crosses from the LHP the RHP at $s = j\infty$ if and only if $X(-\infty) < 0$

This completes the proof of the Lemma. $\bigtriangledown\bigtriangledown\bigtriangledown$

Note that because of the assumptions Sign$[X(0)]$, Sign$[X(-\infty)]$, and Sign $\left[\dot{X}(u)\right]_{u=u_i}$ are never zero. Therefore, based on the calculations shown in the Lemma 1, and counting an RHP to LHP transition as a positive one, and vice versa, we define:

$$\text{for } \omega = 0, \qquad l_j = \begin{cases} -1 & \text{if } X(0) > 0 \\ \;\;1 & \text{if } X(0) < 0 \end{cases} \tag{36}$$

$$\text{for } \omega = \infty, \qquad l_j = \begin{cases} -1 & \text{if } X(-\infty) < 0 \\ \;\;1 & \text{if } X(-\infty) > 0 \end{cases} \tag{37}$$

$$\text{for } \omega_i \ \left(u_j = -\omega_j^2\right), \qquad l_j = \begin{cases} -2 & \text{if } \dot{X}(u)\Big|_{u=u_j} < 0 \\ \;\;2 & \text{if } \dot{X}(u)\Big|_{u=u_j} > 0 \end{cases} \tag{38}$$

Proof of Theorem 1

The proof of the theorem now follows from Lemma 1 in a straightforward fashion. We increase the gain K from $0+$ as described before and count the number of roots l_i crossing the imaginary axis at the gain $K = K_i$ for increasing gain. For a crossing at $s = 0$ or $s = \infty$ the value of l_i is $1, -1$ or 0 according as a root crosses from the RHP to the LHP, from the LHP to the RHP or does not cross the imaginary axis. For a crossing at $s = \pm j\omega_j$ to occur $u_j = -\omega_j^2$ must be a zero of $X(u)$ and then a pair of roots crossover at $K = K_j$ from the RHP to LHP ($l_j = 2$) or LHP to RHP ($l_j = -2$) or does not cross the imaginary axis. It is clear that in the process of increasing

K from 0^+ to the interval $K \in (K_i, K_{i+1})$ the algebraic sum of the number of clc roots that have *crossed* the imaginary axis from the RHP to the LHP is $\sum_{j=0}^{j=i} l_j$. Thus the number r_i of roots *remaining* in the RHP, for this range of gains is given by $r - \sum_{j=0}^{j=i} l_j$. This completes the proof of the Theorem 1. $\bigtriangledown\bigtriangledown\bigtriangledown$

4 Examples

The results of the last section lead to the following algorithm for calculating the stabilizing gains:

Algorithm

Input	$G(s)$
Output	Ranges of stabilizing gain K
Step 1	Get $G(s)$
Step 2	Identify $N^e\left(s^2\right), N^o\left(s^2\right), D^e\left(s^2\right), D^o\left(s^2\right)$
Step 3	Construct $X(u)$ where $u = -\omega^2$
Step 4	Find the negative real roots of $X(u)$
Step 5	Evaluate $K = -\dfrac{D^e(u)}{N^e(u)}$ at the negative real roots of $X(u)$
Step 6	Evaluate $K = -\dfrac{D^e(u)}{N^e(u)}\Big\vert_{u=0}$
Step 7	**If** $G(s)$ is proper, **then** evaluate $K = -\dfrac{1}{G(\infty)}$
	Else set $K = +\infty$ (leading coefficient of D(s) positive)
	or $-\infty$ (leading coefficient of D(s) negative)
Step 8	Determine the sign values of all Ks found in Steps 5, 6, and 7
	based on the Theorem 1
Step 9	Rearrange according to K values with their signs Ks should be arranged so that
	$[0^+ \rightarrow \infty \vdots -\infty \rightarrow 0^-]$ and label them as $K_1, \cdots, K_m$.
Step 10	Evaluate partial sums and determine the ranges of K

Example 1 Consider the open loop transfer function:

$$G(s) = \frac{s^7 + 27s^6 + 289s^5 + 1589s^4 + 4833s^3 + 8121s^2 + 7020s + 2430}{s^7 + 8s^6 + 23s^5 + 35s^4 + 16s^3 - 23s^2 - 42s - 18}$$

It has one RHP pole. Using the same notations, we have

$$N^e(u) = 27u^3 + 1589u^2 + 8121u + 2430$$
$$N^o(u) = u^3 + 289u^2 + 4833u + 7020$$

$$D^e(u) = 8u^3 + 35u^2 - 23u - 18$$
$$D^o(u) = u^3 + 23u^2 + 16u - 42$$

and

$$X(u) = -19u^6 + 137u^5 + 3656u^4 + 5147u^3 + 10251u^2 + 53748u - 24300.$$

The negative real roots of $X(u)$ are -9.8227 and -2.9019. Thus, the elements of the set $\mathcal{U}$ are -9.8227, -2.9019, 0, and ∞. We now evaluate

$$-\frac{D^e(u)}{N^e(u)}\bigg|_{u=-9.8227} = 0.0793$$
$$-\frac{D^e(u)}{N^e(u)}\bigg|_{u=-2.9019} = 0.0176$$
$$-\frac{D^e(u)}{N^e(u)}\bigg|_{u=0} = 0.0074$$
$$-\frac{1}{G(\infty)} = -1.$$

Thus, K_is are ordered as follows.

$$\{K_1,\ K_2,\ K_3,\ K_4\} = \{0.0074,\ 0.0176,\ 0.0793,\ -1\}$$

In this example, we see that $K_p = K_1, = K_q = K_4$, and $K_t = K_3$. This induces the ordering of the elements of $\mathcal{U}$, which is the ordered set.

$$\mathcal{U} = \{0,\ -2.9019,\ -9.8227,\ -\infty\}.$$

Then we determine the respective signs as following.

$$X(u)|_{u=u_1=0} < 0 \quad \Rightarrow \quad l_1 = +1$$
$$\frac{dX(u)}{du}\bigg|_{u=u_2=-9.8227} > 0, \quad \Rightarrow \quad l_2 = +2$$
$$\frac{dX(u)}{du}\bigg|_{u=u_3=-2.9019} < 0, \quad \Rightarrow \quad l_3 = -2$$
$$X(u)|_{u=u_4=-\infty} < 0, \quad \Rightarrow \quad l_4 = -1$$

Arranging these in tabular form, we have

i :	1	2	3	4
K_i :	0.0074	0.0176	0.0793	-1
u_i :	0	-2.9019	-9.8227	$-\infty$
l_i :	$+1$	-2	$+2$	-1

From the Theorem, the numbers of RHP poles of the closed loop system are as follows:

K :	$(0, K_1)$	(K_1, K_2)	(K_2, K_3)	(K_3, ∞)	$(-\infty, K_4)$	$(K_4, 0)$
	$(0, 0.0074)$	$(0.0074, 0.0176)$	$(0.0176, 0.0793)$	$(0.0793, \infty)$	$(-\infty, -1)$	$(-1, 0)$
r_i :	1	0	2	0	0	1

We can see from the above table that the ranges of stabilizing K are

$$(0.0074, 0.0176),\quad (0.0793, +\infty),\quad (-\infty, -1).$$

Example 2 Consider the open loop transfer function:

$$G(s) = \frac{4.3333s^4 + 17.667s^3 + 24.333s^2 + 17.667s + 4}{s^5 - 2s^4 - 10s^3 + 8s^2 + 33s + 18}$$

It has 2 RHP poles. Using the same notations, we have

$$\begin{aligned} N^e(u) &= 4.333u^2 + 24.3333u + 4 \\ N^o(u) &= 17.6667u + 17.6667 \\ D^e(u) &= -2u^2 + 8u + 18 \\ D^o(u) &= u^2 - 10u + 33 \end{aligned}$$

and

$$X(u) = -4.333u^4 - 16.3367u^3 + 202.3442u^2 - 303.66478u + 186.0006.$$

The negative real root of $X(u)$ is -9.5074. Thus the elements of the set $\mathcal{U}$ are -9.5074, 0, and ∞. We now evaluate

$$\begin{aligned} -\frac{D^e(u)}{N^e(u)}\bigg|_{u=-9.5074} &= 1.4535 \\ -\frac{D^e(u)}{N^e(u)}\bigg|_{u=0} &= -4.5 \\ -\frac{1}{G(\infty)} &= -\infty \end{aligned}$$

Thus, K_is are order as follows.

$$\{K_1, \ \ K_2, \ \ K_3\} = \{1.4535, \ \ -\infty, \ \ -4.5\}$$

In this example, $K_p = K_3$, $K_q = K_2$, and $K_t = K_1$. This induces the ordering of the elements of $\mathcal{U}$, which is the ordered set.

$$\mathcal{U} = \{-9.5074, \ \ -\infty \ \ , 0\}.$$

Then we determine the respective signs as following.

$$\begin{aligned} \frac{dX(u)}{du}\bigg|_{u=-9.5074} > 0, &\quad \Rightarrow \quad l_1 = +2 \\ X(u)|_{u=-\infty} < 0 &\quad \Rightarrow \quad l_2 = -1 \\ X(u)|_{u=0} > 0 &\quad \Rightarrow \quad l_3 = -1 \end{aligned}$$

Arranging these in tabular form, we have

i :	1	2	3
K_i :	1.4535	$-\infty$	-4.5
u_i :	-9.5074	$-\infty$	0
l_i :	$+2$	-1	-1

From the theorem, the numbers of RHP poles of the closed loop system are as follows.

K :	$(0, K_1)$	(K_1, ∞)	$(-\infty = K_2, K_3)$	$(K_3, 0)$,
	$(0, 1.4535)$	$(1.4535, \infty)$	$(-\infty, -4.5)$	$(-4.5, 0)$
r_i :	2	0	1	2

From the above table, the range of stabilizing K is $(1.4535, +\infty)$.

Example 3 Consider the open loop transfer function:

$$G(s) = \frac{14.5s^6 - 27s^5 + 328s^4 - 274s^3 + 926.5s^2 - 236s + 240}{s^7 - 17s^6 + 119s^5 - 447s^4 + 980s^3 - 1276s^2 + 940s - 300}$$

The plant has all its poles at RHP and the total number RHP poles of is 7. Using the same notations, we have

$$\begin{aligned} N^e(u) &= 14.5u^3 + 328u^2 + 926.5u + 240 \\ N^o(u) &= -27u^2 - 274u - 236 \\ D^e(u) &= -17u^3 - 447u^2 - 1276u - 300 \\ D^o(u) &= u^3 + 119u^2 + 980u + 940 \end{aligned}$$

and

$$X(u) = \Big(-0.0001u^6 - 0.0159u^5 - 0.3744u^4 - 2.8462u^3 - 7.8163u^2 - 7.2277u - 1.5480\Big) \times 10^5.$$

The negative real roots of $X(u)$ are

$$80.968, \quad -17.8091, \quad -6.6815, \quad -2.9737, \quad -1.2304, \quad -0.3029.$$

Thus, the elements of the set $\mathcal{U}$ are

$$80.968, \quad -17.8091, \quad -6.6815, \quad -2.9737, \quad -1.2304, \quad -0.3029, \ \ 0, \ \ \infty.$$

We now evaluate

$$\begin{aligned} -\frac{D^e(u)}{N^e(u)}\bigg|_{u=-80.968} &= 1.1023 \\ -\frac{D^e(u)}{N^e(u)}\bigg|_{u=-17.8091} &= 3.9751 \\ -\frac{D^e(u)}{N^e(u)}\bigg|_{u=-6.6815} &= 1.5247 \\ -\frac{D^e(u)}{N^e(u)}\bigg|_{u=-2.9737} &= 2.7886 \\ -\frac{D^e(u)}{N^e(u)}\bigg|_{u=-1.2304} &= 1.4520 \\ -\frac{D^e(u)}{N^e(u)}\bigg|_{u=-0.3029} &= 4.2065 \\ -\frac{D^e(u)}{N^e(u)}\bigg|_{u=0} &= 1.25 \\ -\frac{1}{G(\infty)} &= -\infty. \end{aligned}$$

Thus, K_is are ordered as follows.

$$\{K_1,\ K_2,\ K_3,\ K_4,\ K_5,\ K_6,\ K_7,\ K_8\} = \{1.1023,\ 1.25,\ 1.4520,\ 1.5247,\ 2.7886,\ 3.9751,\ 4.2065,\ -\infty\}.$$

In this example, $K_p = K_2$, $K_q = K_8$, and $K_t = K_7$. This induces the ordering of the elements of $\mathcal{U}$, which is the ordered set.

$$\mathcal{U} = \{-80.968,\ 0,\ -1.2304,\ -6.6815,\ -2.9737,\ -17.8091,\ -0.3029,\ -\infty\}.$$

Then we determine the respective signs as following.

$$\left.\frac{dX(u)}{du}\right|_{u=-80.968} > 0, \quad \Rightarrow \quad l_1 = +2$$
$$X(u)|_{u=0} < 0 \quad \Rightarrow \quad l_2 = +1$$
$$\left.\frac{dX(u)}{du}\right|_{u=-1.2304} > 0, \quad \Rightarrow \quad l_3 = +2$$
$$\left.\frac{dX(u)}{du}\right|_{u=-6.6815} > 0, \quad \Rightarrow \quad l_4 = +2$$
$$\left.\frac{dX(u)}{du}\right|_{u=-2.9737} < 0, \quad \Rightarrow \quad l_5 = -2$$
$$\left.\frac{dX(u)}{du}\right|_{u=-17.8091} < 0, \quad \Rightarrow \quad l_6 = -2$$
$$\left.\frac{dX(u)}{du}\right|_{u=-0.3029} < 0, \quad \Rightarrow \quad l_7 = -2$$
$$X(u)|_{u=-\infty} < 0 \quad \Rightarrow \quad l_8 = -1$$

Arranging these in tabular form, we have

i	:	1	2	3	4	5	6	7	8
K_i	:	1.1023	1.25	1.452	1.5247	2.7886	3.9751	4.2065	$-\infty$
u_i	:	-80.968	0	-1.2304	-6.6815	-2.9737	-17.8091	-0.3029	$-\infty$
l_i	:	$+2$	$+1$	$+2$	$+2$	-2	-2	-2	-1

From the Theorem, the numbers of RHP poles of the closed loop system are as follows.

K	:	$(0, K_1)$	(K_1, K_2)	(K_2, K_3)	(K_3, K_4)	(K_4, K_5)
		$(0, 1.1023)$	$(1.1023, 1.25)$	$(1.25, 1.452)$	$(1.452, 1.5247)$	$(1.5247, 2.7886)$
r_i	:	7	5	4	2	0

K	:	(K_5, K_6)	(K_6, K_7)	(K_7, ∞)	$(K_8 = -\infty, 0)$
		$(2.7886, 3.9751)$	$(3.9751, 4.2065)$	$(4.2065, \infty)$	$(-\infty, 0)$
r_i	:	2	4	6	7

From the above table, the range of stabilizing K is $(1.5247, 2.7886)$.

Example 4 Consider the open loop transfer function:

$$G(s) = \frac{2s^6 - 7s^5 - 15s^4 + 55s^3 - 15s^2 + 105s + 7}{s^6 + 4s^5 + 3s^4 - 66s^3 + 34s^2 - 456s + 44}$$

It has 4 RHP poles. Using the same notations, we have

$$N^e(u) = 2u^3 - 15u^2 - 15u + 7$$
$$N^o(u) = -7u^2 + 55u + 105$$
$$D^e(u) = u^3 + 3u^2 + 34u + 44$$
$$D^o(u) = 4u^2 - 66u - 456$$

and

$$X(u) = -15u^5 + 226u^4 + 14u^3 - 5981u^2 - 388u + 7812.$$

The negative real roots of $X(u)$ are -4.3610 and -1.2119. Thus, the elements of the set $\mathcal{U}$ are -4.3610, -1.2119, 0, and ∞. We now evaluate

$$\begin{aligned}
-\frac{D^e(u)}{N^e(u)}\bigg|_{u=-4.3610} &= -0.3437 \\
-\frac{D^e(u)}{N^e(u)}\bigg|_{u=-1.2119} &= 13.1881 \\
-\frac{D^e(u)}{N^e(u)}\bigg|_{u=0} &= -6.2857 \\
-\frac{1}{G(\infty)} &= -\frac{1}{2}.
\end{aligned}$$

Then, K_is are ordered as follows.

$$\{K_1,\ K_2,\ K_3,\ K_4\} = \{13.1881,\ -6.2857,\ -0.5,\ -0.3437\}$$

In this example, $K_p = K_2$, $K_q = K_3$, and $K_t = K_1$. This induces the ordering of the elements of $\mathcal{U}$, which is the ordered set.

$$\mathcal{U} = \{-1.2119,\ 0,\ -\infty,\ -4.3610\}.$$

Then we determine the respective signs as following.

$$\begin{aligned}
\frac{dX(u)}{du}\bigg|_{u=-1.2119} &> 0, &\Rightarrow\quad & l_1 = +2 \\
X(u)|_{u=0} &> 0 &\Rightarrow\quad & l_2 = +1 \\
X(u)|_{u=-\infty} &> 0 &\Rightarrow\quad & l_3 = +1 \\
\frac{dX(u)}{du}\bigg|_{u=-4.3610} &< 0, &\Rightarrow\quad & l_4 = -2
\end{aligned}$$

Arranging these in tabular form, we have

i	:	1	2	3	4
K_i	:	13.1881	-6.2857	-0.5	-0.3437
u_i	:	-1.2119	0	$-\infty$	-4.3610
l_i	:	$+2$	-1	1	-2

From the Theorem, the numbers of RHP poles of the closed loop system are as follows.

K	:	$(0, K_1)$	(K_1, ∞)	$(-\infty, K_2)$	(K_2, K_3)
		$(0, 13.1881)$	$(13.1881, \infty)$	$(-\infty, -6.2857)$	$(-6.2857, -0.5)$
r_i	:	4	2	2	3

K	:	(K_3, K_4)	$(K_4, 0)$
		$(-0.5, -0.3437)$	$(-0.3437, 0)$
r_i	:	2	4

From the above table, there is no stabilizing K exists.

5 Concluding remarks

In this paper we have shown that the determination of stabilizing gains can be completely reduced to finding the negative real roots of the fixed polynomial $X(u)$ and the signs of the derivatives $\dot{X}(u)$ at these real roots. We note that the idea of obtaining stability information only from the real axis cuts of the Nyquist plot was presented in (Vidyasaga *et al.*, 1988). The present paper can be considered as an extension and completion of this idea. The results of this paper give in essence a new criterion for constant gain stabilizability of a linear time invariant system, in terms of the characterizing function $X(u)$. Since stabilization by an arbitrary controller of prescribed order or structure such as PID, can always be regarded as a two step problem where the first step will consist of reduction to a problem solvable by constant gain, we expect these results to have application to the problem of fixed order or structure stabilization problem.

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