Proceedings of the 7th Mediterranean Conference on Control and Automation (MED99) Haifa, Israel - June 28-30, 1999

Design of Non-Saturating Guidance Systems

Pini Gurfil* and Mario Jodorkovsky†
Rafael - Armament Development Authority
Dept. 39
P.O.B. 2250
Haifa 31021, Israel

Moshe Guelman‡
Department of Aerospace Engineering
Technion- Israel Institute of Technology
Haifa 32000, Israel

Abstract

Design of non-saturating guidance systems is considered. Assuming linearized kinematics, a proportional navigation guidance model is introduced. The missile guidance loop discussed contains nonlinearities such as limited missile maneuverability, limited acceleration command and constrained measured line-of-sight angular rate. A novel approach, based on input-output stability, renders design guidelines that assure operation in the non-saturating region, given the missile-target maneuver ratio. These guidelines yield a proportional navigation based guidance law that assures zero miss distance for any bounded target maneuver. It is shown that if the total dynamics of the guidance loop is designed to be positive real, and the effective proportional navigation constant is chosen to be a simple function of the maneuver ratio, no saturation shall occur. The illustrative examples validate the analysis, and show that the new guidance law is robust enough to guarantee a significant performance improvement even if the design guidelines are somewhat loosened.

Nomenclature

$\Vert\cdot\Vert_p$	= p-norm
$\Vert\cdot\Vert_\infty$	= ∞-norm
a	= lateral acceleration
L^1	= normed space
L^p	= normed space
L^∞	= normed space
LOS	= line of sight
N'	= effective proportional navigation constant
{PR}	= the class of positive real functions
R	= missile-target relative range
r	= relative order of a rational function
t_f	= flight time

* Email: pinig@rafael.co.il
† Email: marioj@rafael.co.il
‡ Email: aerglmn@aerodyne.technion.ac.il

V = velocity
V_C = closing velocity
y = missile-target relative vertical position
ZMD = zero miss distance
λ = line of sight angle
μ = bound on the required missile-target maneuver ratio
μ_0 = given missile-target maneuver ratio
μ_r = required missile-target maneuver ratio
ω_n = natural frequency
τ = time-to-go
τ_1 = time constant
ζ = damping coefficient

Subscripts:
$()_T$ = target
$()_M$ = missile
$()_C$ = commanded value
$()_f$ = final value
$()_0$ = initial value

Superscripts:
$(\dot{\ })$ = time differentiation

1. Introduction

The design of a guidance law for a given missile system is effected primarily from the a-priori knowledge of the missile acceleration capability. When the missile-target maneuver ratio is known, a suitable guidance method can be synthesized. Several approaches to confront this problem have been suggested in previous studies. Based on optimal control theory, explicit formulae of optimal guidance for acceleration constrained missile were derived both for the deterministic (Rusnak and Meir, 1991) and stochastic (Rusnak, 1993) cases. The differential games approach have also been utilized (Gutman, 1979). In both approaches, the guidance law is sought by solving a dynamic optimization problem.

However, in many actual applications, the guidance law is chosen without performing a constrained optimization. The designer chooses the guidance law based on considerations involving the available measurements, the minimization of miss distance and ease of implementation. The problem of limited acceleration is then dealt with by a proper tuning of the guidance law parameters. The basic question to be raised is whether after determining a guidance method, it can be improved such that missile acceleration (or any other state variable) saturation is *avoided*.

One of the most commonly used guidance laws is proportional navigation (PN). A vast amount of literature exists on the subject (Shneydor, 1984, Zarchan 1990). In a conventional PN guidance (PNG) system, it is known (Shinar, 1976, Shinar and Steinberg, 1977, Gurfil *et al.*, 1998), that an infinite missile acceleration is required near to interception. This means that saturation is always reached, and miss distance will be greater than predicted by linear analysis.

Our main goal is to characterize a set of PN based guidance systems, in which saturation is *avoided.* Thus, the basic guidance framework is set to be PNG, and an improvement to this guidance law is sought, such that an a-priori knowledge of the missile-target maneuver ratio will suffice to guarantee a non-saturating operating region. It will be shown, that this class of improved PN-based systems yields zero miss distance (ZMD) for any bounded target maneuver.

Although the general problem of PNG features nonlinear kinematics, in order to apply known techniques of analysis and design, the system equations are linearized, yielding an equivalent linear time-varying system. The linearization is valid when it is assumed that the missile and target approach the so-called collision course. It is known that the linearized model faithfully represents the guidance dynamics (Zarchan, 1979, Zarchan, 1990). Based on the assumption of linearized kinematics, we introduce nonlinearities associated with acceleration saturation either in the output or the command channels. An additional nonlinearly we consider is the measured line-of-sight (LOS) angular rate saturation.

The a-priori knowledge of missile-target maneuver ratio is used to characterize the PN-based guidance systems dynamics which render non-saturating behavior. This is performed by implementing functional analysis tools, mainly the concept of bounded-input bounded-output stability in appropriate functional spaces (Sandberg, 1965, Zames, 1966, Mossaheb, 1982). Valuable design guidelines are derived, which offer a simple improvement of a PNG system such that saturation is avoided. Based on previous works (Gurfil *et al.*, 1998 (2)), it is pointed out that when saturation is prevented, ZMD can be obtained for any bounded target maneuver. Furthermore, we show that the design guidelines could be somewhat relieved without a substantial modification of either the non-saturating or ZMD property. The novel approach offered in this paper will be illustrated in several examples.

The paper is organized as follows. Section 2 presents some mathematical background in the fields of functional analysis and positive-real functions. In Section 3, the problem of designing a non-saturating PNG-based guidance system is introduced. Section 4 gives a definition of input-output stability and discusses the appropriate theorem. In Section 5, this theorem is applied in a PNG loop. Section 6 suggests the design implications stem from the previous discussion. Section 7 expands the results to the prevention of saturation of state variables other then output maneuver acceleration. The Illustrative examples of Section 8 are used to validate the results. In section 9, some concluding remarks are outlined.

2. Mathematical Background

In the sequel, we extensively implement functional analysis. Therefore, some well known definitions of frequently used signal and system norms are hereby presented in brevity.

Let E be a linear space defined over the field of real numbers $\Re$. The following signal norms are defined on appropriate subsets of E for some causal signal $x(t)$ (Desoer and Vidyasagar 1975):

$$\|x\|_p \stackrel{\Delta}{=} \left(\int_0^{t_f} |x(t)|^p dt \right)^{1/p}, \quad 1 \le p < \infty \tag{1}$$

$$\|x\|_\infty \stackrel{\Delta}{=} \underset{t \in [0, t_f]}{ess\sup} |x(t)| \tag{2}$$

where as usual, $ess \sup\limits_{t\in[0,t_f)} |x(t)| \stackrel{\Delta}{=} \inf\{k \mid |x(t)| \le k \ \ almost\ everywhere\}$ *.

The corresponding normed spaces are denoted, respectively, $L^p[0,t_f]$ and $L^\infty[0,t_f]$. It will be said that $x(t) \in L^p[0,t_f]$ if *x(t)* is locally (Lebesgue) integrable and in addition

$$\|x\|_p < \infty,\ p \in [1,\infty) \tag{3}$$

Accordingly, $x(t) \in L^\infty[0,t_f]$ if:

$$\|x\|_\infty < \infty \tag{4}$$

We consider system norms as well. The systems are assumed linear, time-invariant, causal and finite-dimensional. In the time domain, input-output models for such systems have the form of a convolution equation,

$$y = h * u = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau \tag{5}$$

where, due to causality, *h(t)=0* for *t<0.*

Let *H(s)* denote the transfer function, The Laplace transform of *h(t)*. The following system norms are defined (Doyle *et al.*, 1992):

$$\|h\|_1 \stackrel{\Delta}{=} \int_0^\infty |h(t)| dt \tag{6}$$

$$\|h\|_2 \stackrel{\Delta}{=} \left(\int_{-\infty}^{\infty} |h(t)|^2 dt \right)^{1/2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\omega)|^2 dw \right)^{1/2} = \|H\|_2 \tag{7}$$

$$\|H\|_\infty \stackrel{\Delta}{=} \sup_{\omega\in\Re} |H(j\omega)| \tag{8}$$

The corresponding normed spaces are denoted, respectively, L^1, L^2 and L^∞. The notation $h \in L^1$, $h \in L^2$ and $H \in L^\infty$ means that the system norm in the appropriate normed space is finite.

For the implementation of the stability theorem, the following definitions are required:

* In the sequel, we use the notation *sup* instead of *ess sup*.

Definition 1: A function $h(t):[0,\infty]\rightarrow\Re$ is said to satisfy $h(t)\in A_1$ if and only if

$$(1+t)h(t)\in L^1\cap L^2 \tag{9}$$

Definition 2: A function $h(t):[0,\infty]\rightarrow\Re$, with $h(t)=h_1(t)+h_2(t)$ and $H_2(s)$ the Laplace transform of $h_2(t)$ is said to satisfy $h(t)\in A_2$ if and only if

$$h_1(t)\in A_1,\ H_2(s)\,\textit{is strictly proper} \tag{10}$$

The analysis hereafter shall utilize the so-called positive real (PR) functions, defined as follows:

Definition 3: It is said that some stable transfer function *H(s)* satisfies $H(s)\in\{PR\}$ if and only if

$$\operatorname{Re} H(j\omega)\geq 0\quad\forall\omega\in\Re \tag{11}$$

Notice that (11) is satisfied if and only if

$$-\pi/2\leq\angle H(j\omega)\leq\pi/2\quad\forall\omega\in\Re \tag{12}$$

where $\angle H(j\omega)$ denotes the phase of *H(s)*.

It is known (Slotine and Li, 1991) that for (11) to hold it is necessary, but not sufficient, that

$$r[H(s)]=1\ or\ 0\ or\ -1 \tag{13}$$

where *r* denotes the relative order of *H(s)* (i.e. the degree of the denominator minus the degree of the numerator).

3. Problem Formulation

The problem of designing a non-saturating proportional navigation (PN) based guidance system will be addressed using a familiar formulation of a simplified PN guidance (PNG) model. Although the three dimensional PN interception problem is rather complicated, a considerable simplification is obtained when assuming that the lateral and longitudinal maneuver planes are de-coupled by means of roll-control. Thus, we shall further assume that the geometry is two-dimensional.

The above assumption permits the formulation of a general planar intercept missile-target geometry as depicted in Fig.1. The figure describes a missile employing PN to intercept a maneuvering target.

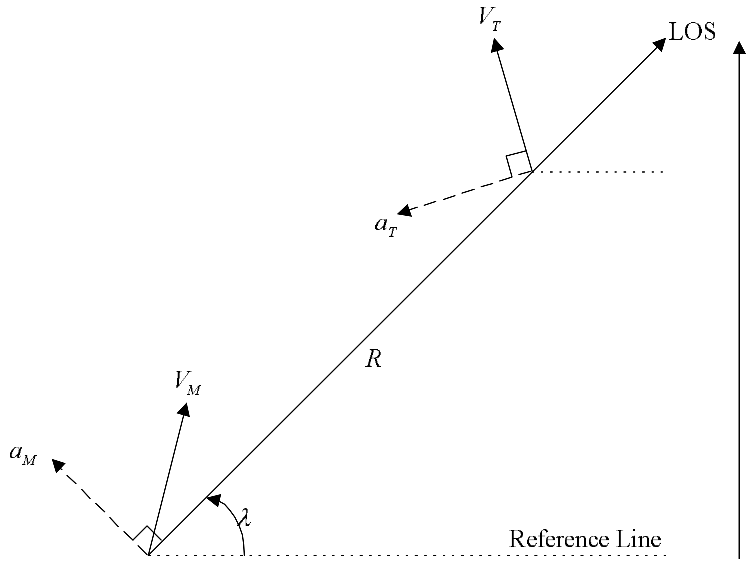

Figure 1: Interception Geometry

Based on Fig. 1, a linearized model of the guidance dynamics can be developed. Such a model is widely used in the analysis of PNG (Shinar, 1977, Zarchan, 1979, Shneydor, 1984, Zarchan, 1990). A block diagram describing it is given in Fig. 2. In this linear time-varying system, missile acceleration a_M is subtracted form target acceleration a_T to form a relative acceleration $\ddot{y}$. A double integration yields the relative vertical position *y* (see Fig. 1), which at the end of the engagement is the miss distance $y(t_f)$. By assuming that the closing velocity V_C is constant, the relative range is given by:

$$R = V_C \cdot \tau \tag{14}$$

where τ is the time to go, defined as

$$\tau \overset{\Delta}{=} t_f - t \tag{15}$$

Dividing the relative vertical position *y* by the range given in (14), yields the geometric line-of-sight (LOS) angle λ. It is assumed that λ is a small angle. The missile seeker is represented in Fig. 2 as an ideal differentiator with an additional transfer function $G_1(s)$, representing the LOS measurement and noise filtering dynamics. The seeker generates a LOS rate command $\dot{\lambda}_C$, which is multiplied by the PN gain $N' \cdot V_C$ to form a commanded missile maneuver acceleration a_C, with N' being the effective PN constant. The flight control system, whose dynamics are represented by the transfer function $G_2(s)$, attempts to adequately maneuver the missile to follow the desired acceleration command.

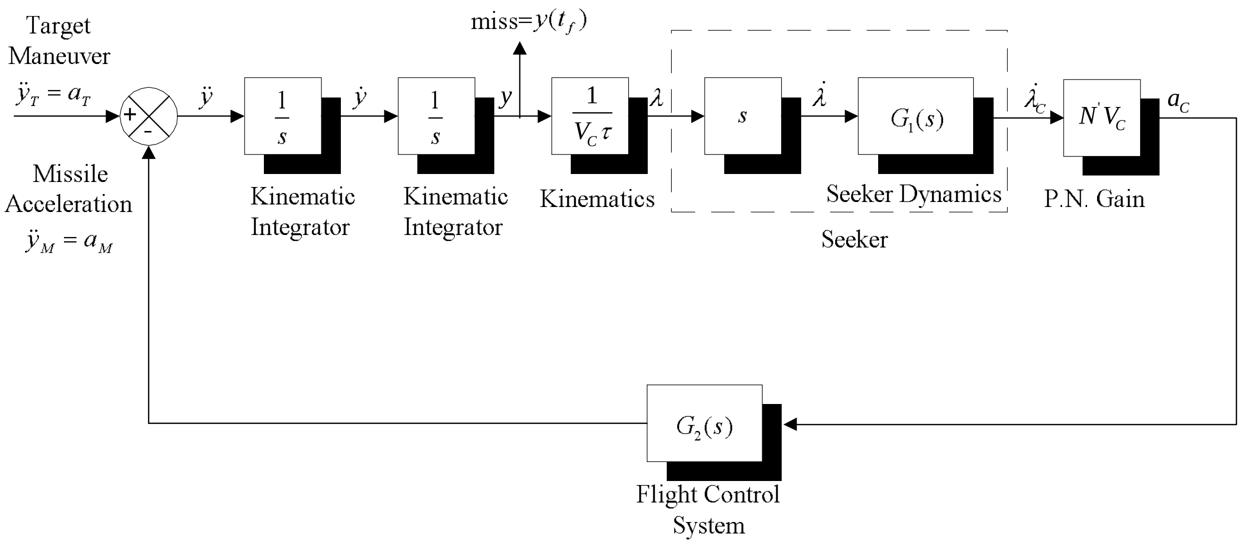

Figure 2: Linear PNG model block diagram

The linear model presented above is based implicitly on the assumption of non-limited missile maneuverability. Actually, every real missile system is subject to maneuverability saturation due to aerodynamic or structural constraints. To complete the guidance model description in this case, a new variable, the required missile maneuver acceleration a_R , is introduced. The nonlinear relationship between the actual and required missile maneuver is defined by

$$a_M = a_{M_{\max}} \operatorname{sat}\left\{ \frac{a_R}{a_{M_{\max}}} \right\} \tag{16}$$

Where

$$\operatorname{sat}(x) \overset{\Delta}{=} \begin{cases} x & if\, x \le 1 \\ 1 & if\, |x| > 1 \end{cases} \tag{17}$$

Equation (16) implicitly assume that the limit on missile maneuverability is of the aerodynamic type, due to mechanical limits of control fin deflection or to hinge moment saturation. This limit is at the output of the guidance channel, as shown in Fig. 3.

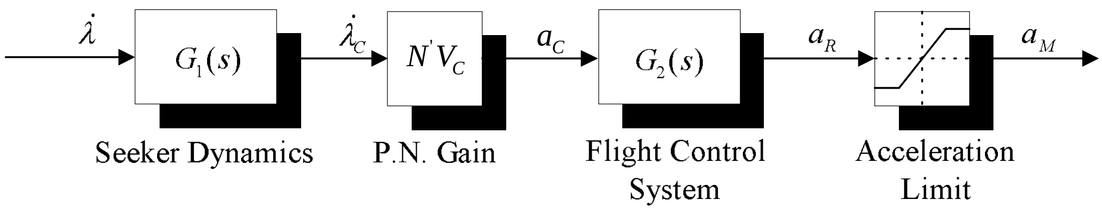

Figure 3: Limited output acceleration

However, in many missiles the limit is imposed on the acceleration command. This limit is usually designed to be sufficiently conservative, such that an aerodynamic saturation, mentioned earlier, is not reached. In this case, the limited acceleration command is denoted a_L, where:

$$a_L = a_{C_{\max}} \operatorname{sat}\left\{ \frac{a_C}{a_{C_{\max}}} \right\} \tag{18}$$

This limit is depicted in Fig. 4.

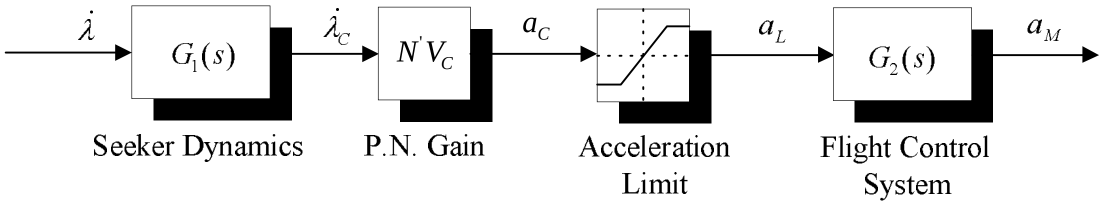

Figure 4: Limited acceleration command

An additional state variable of the guidance loop is usually limited: The measured LOS angular rate. This limit stems from either full-scale measurement capability of the seeker rate gyro or a maximal tracking rate. In this case, described in Fig. 5, the limited LOS angular rate $\dot{\lambda}_L$ satisfies

$$\dot{\lambda}_L = \dot{\lambda}_{\max} \operatorname{sat}\left\{ \frac{\dot{\lambda}}{\dot{\lambda}_{\max}} \right\} \tag{19}$$

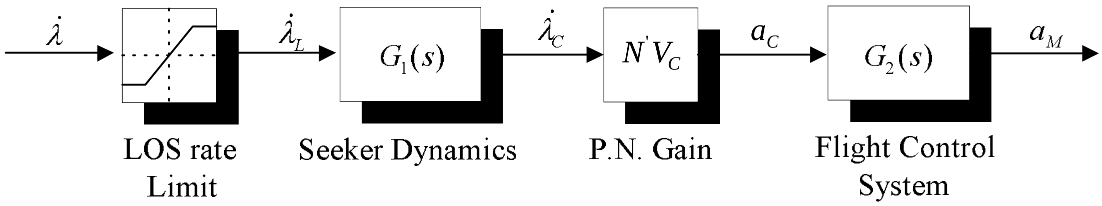

Figure 5: Limited LOS angular rate

In a conventional PNG system, it is known (Shinar, 1976, Shinar, 1977, Gurfil *et al.*, 1998), that an infinite missile acceleration is required near to intercept $(t \to t_f)$. This means that saturation is always reached, and miss distance will be greater than predicted by linear analysis. Our goal is to characterize a set of PN based guidance systems, in which saturation is *avoided.* It will be shown, that this class of systems yields zero miss distance (ZMD) for any type of bounded target maneuver. The treatment hereafter will consider the case of a limited output acceleration. Nevertheless, the other types of limits mentioned will be discussed later in the study.

Consequently, it is necessary to find some bound μ on the required missile-target maneuver ratio, μ_r, defined as

$$\mu_r \overset{\Delta}{=} \frac{\sup_{t\in[0,t_f]} |a_M(t)|}{\sup_{t\in[0,t_f]} |a_T(t)|} \tag{20}$$

If μ is found to be smaller than the *a-priori* known missile-target maneuver ratio μ_0, no saturation will occur.

In the case of PNG, due to the divergence of the state variables at the vicinity of the interception, we have $\lim_{t\to t_f} |a_M(t)| \to \infty \;\; \forall a_T(t) \neq 0$. We shall prove, that there is a way to modify the PNG law such that $\sup_{t\in[0,tf]} |a_M(t)| < \infty$. Moreover, we shall find a constant μ, such that

$$\sup_{t\in[0,tf]} |a_M(t)| = \|a_M(t)\|_\infty \leq \mu \|a_T(t)\|_\infty = \mu \sup_{t\in[0,tf]} |a_T(t)| \tag{21}$$

A proper assessment of μ, together with a proper design modification of the PNG, might yield the aspired non-saturating guidance system.

Remark 1: Clearly, μ is not uniquely defined by the above inequality. It is clear that we are interested in the smallest μ such that (21) still holds.

Inequality (21) describes a bounded-input bounded-output stability problem in the functional space $L^\infty[0,t_f]$. An application of this problem to a PNG loop is the main issue dwelt upon in the forthcoming sections.

4. Input-Output Stability Definition and Sufficiency Theorem

Consider the nonlinear time-varying feedback system depicted in Fig. 6. The input to the system is $u(t) \in \Re$ and the output is $z(t) \in \Re$. These signals are defined for $t \in [0, t_f]$. $K_1(s),\ K_2(s),\ K_3(s)$ are linear time-invariant (LTI), whereas $\psi(x,t)$ is a nonlinear time-varying operator satisfying $\psi : \Re \times \Re_+ \to \Re$. It is assumed that $\psi(x,t)$ is a continuous sector-bounded nonlinearity, i.e.

$$\alpha x(t) \leq \psi(x,t) \leq \beta x(t),\ \forall x(t) \in \Re,\ \ \alpha, \beta \in \Re \tag{22}$$

If $\psi(x,t) = \psi(t)x(t)$, the sector condition (22) becomes

$$\alpha \leq \psi(t) \leq \beta,\ \ \alpha, \beta \in \Re \tag{23}$$

Also, let

$$H(s) \overset{\Delta}{=} K_1(s)K_2(s)K_3(s) \tag{24}$$

Assumption 1: H(s) is stable, and its poles on the imaginary axis are distinct.

Referring to the system of Fig.6, the following definition is used (Desoer and Vidyasagar 1975):

Definition 4: The system of Fig.6 is said to be L^∞ / L^∞ stable if and only if $u(t) \in L^\infty[0, t_f]$ implies $z(t) \in L^\infty[0, t_f]$ and, moreover,

$$\|z(t)\|_\infty \le \mu \|u(t)\|_\infty \quad \forall u(t) \in L^\infty[0, t_f], \quad \mu \ne \mu(u) \tag{25}$$

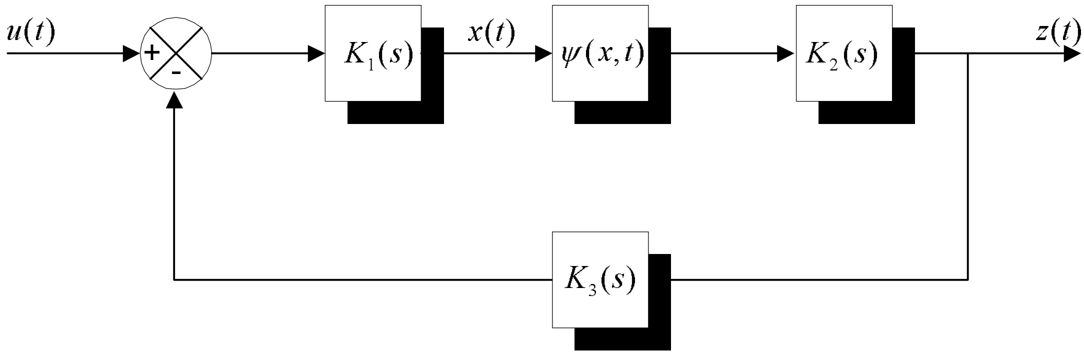

Figure 6: Nonlinear time-varying feedback system

Several theorems providing sufficient conditions for L^∞ / L^∞ stability can be found in the literature (Sandberg, 1965, Zames, 1966, Mossaheb, 1982, Sandberg and Johnson, 1990). The theorem we shall apply is based upon the well-known small gain theorem (Desoer and Vidyasagar, 1975, Mossaheb, 1982). The small gain approach states a sufficient stability condition of L^P stability of a closed-loop system, based on the L^P induced norms of the forward and feedback paths. Thus, the non-linearity must be confined in a sector (22), (23). To obtain a useful result concerning the L^∞ / L^∞ stability of the PNG loop, the following formulation of the small-gain theorem is used (Mossaheb 1982):

Theorem 1: Consider the system depicted in Fig. 6. Under Assumptions 1, if

$$\gamma = \frac{\beta}{2} \left\| \frac{H(j\omega)}{1 + \beta/2 \cdot H(j\omega)} \right\|_\infty \le 1 \tag{26}$$

$$h(t) \in A_2 \tag{27}$$

Then the system is L^∞ / L^∞ stable and $\|z\|_\infty \le \mu(\gamma) \|u\|_\infty$, where the constant $\mu(\gamma)$ is, at most, a function of γ only.

Proof: See (Mossaheb 1982). ❑

Remark 2: A celebrated L^2 / L^2 stability theorem for the system of Fig. 6, known as the circle criterion, was obtained in (Sandberg, 1964) based upon (26). It can be shown that the circle criterion is an application of the small-gain theorem (Desoer and Vidyasagar, 1975). However, to extend the result to L^∞ / L^∞ stability, the additional condition

$e^{\varepsilon t}h(t) \in L^1 \cap L^2, \varepsilon > 0$ together with the shifted Nyquist plot of $H(j\omega)$ were used (Zames 1966). In a more recent work (Mossaheb 1982), it was shown that (27) could be used as an additional condition needed for L^∞ / L^∞ stability. This condition is less conservative then previous results. Furthermore, the shifted Nyquist plot need not be used.

To clarify the relationship between (26) and the circle criterion, we consider the following lemma.

Lemma 1: (26) is equivalent to the condition $\mathrm{Re}\,H(j\omega) \geq -\frac{1}{\beta}\ \forall \omega \in \Re$.

Proof: Notice that (26) can be re-written as:

$$\frac{\beta}{2}\left|\mathrm{H}(j\omega)\right| \leq \left|1 + \frac{\beta}{2}\mathrm{H}(j\omega)\right| \quad \forall \omega \in \Re \tag{28}$$

Thus,

$$\beta/2 \cdot \sqrt{\mathrm{Re}^2 H(j\omega) + \mathrm{Im}^2 H(j\omega)} \leq \sqrt{[1 + \beta/2 \cdot \mathrm{Re}H(j\omega)]^2 + [\beta/2 \cdot \mathrm{Im}H(j\omega)]^2} \quad \forall \omega \in \Re \tag{29}$$

Simplifying both parts of the inequality yields $\mathrm{Re}\,H(j\omega) \geq -\frac{1}{\beta}\ \forall \omega \in \Re$ ❑

In the next section, an application of Theorem 1 and Lemma 1 in a PNG loop will be discussed.

5. Application of the Stability Theorem in a PNG Loop

One can observe that there is a complete equivalence between the problem formulated in Section 3, especially (21), and the L^∞ / L^∞ stability definition given in regard to the system delineated in Fig. 6. Furthermore, it can be shown in a straightforward manner that a PNG loop is a particular case of the general setting.

Consider the simplified PNG loop described in Fig.7. The resemblance to the system of Fig. 6 is obvious:

$$u(t) = a_T(t),\ z(t) = a_M(t) \tag{30}$$

$$K_3(s) = 1\ ,K_2(s) = N' \cdot s \cdot G(s)\ ,K_1(s) = \frac{1}{s^2} \tag{31}$$

$$\psi(x,t) = \psi(t) = \frac{1}{\tau} = \frac{1}{t_f - t} \in [0,\infty) \quad \forall t \in [0,\ t_f) \tag{32}$$

$$H(s) = K_1(s)K_2(s)K_3(s) = \frac{N'G(s)}{s} \tag{33}$$

Eq. (33) renders the total dynamics of the PNG loop. Restricting conditions on the transfer function *H(s)* will be found, that characterize the non-saturating class of PN-based guidance systems.

Assumption 2: G(0)=1 , i.e. N' is the total gain of the LTI portion of the PNG loop.

Assumption 3: G(s) is asymptotically stable.

It should be noted that the Assumption 3 does not confine the generality of the treatment. This is because *G(s)* represents the missile closed-loop auto-pilot and seeker measurement dynamics, which are designed such that asymptotic stability is guaranteed.

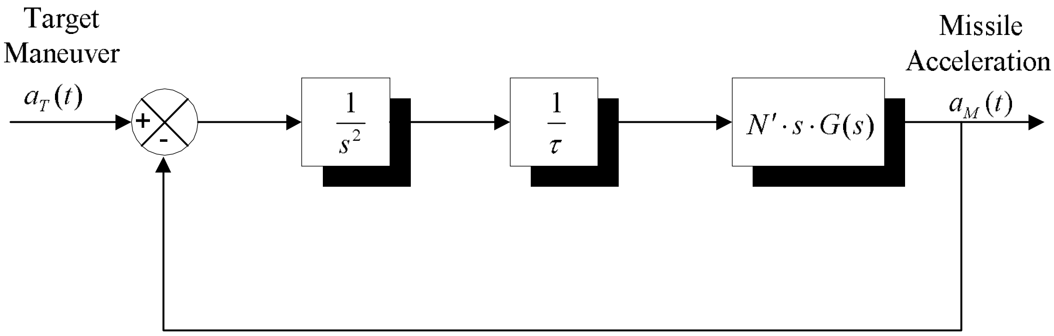

Figure 7: Simplified PNG loop

Next, consider (26). In the case of PNG, the nonlinear time-varying element $\psi(x,t)$ reduces to a linear time-varying function (32). Note that due to (32), this function is confined in the sector $\alpha = 0, \beta \to \infty$. Therefore, according to Lemma 1, (26) is satisfied if

$$\operatorname{Re} H(j\omega) = \operatorname{Re} \frac{N'G(j\omega)}{j\omega} \geq 0 \quad \forall \omega \in \Re \tag{34}$$

or equivalently,

$$H(j\omega) \in \{PR\} \tag{35}$$

Lemma 2: If (34) holds, then $\gamma = 1$.

Proof: According to Lemma 1, (34) is equivalent to (26). Thus, define

$$f(\omega) \stackrel{\Delta}{=} \frac{\beta}{2} \left| \frac{H(j\omega)}{1 + \beta/2 \cdot H(j\omega)} \right| \leq 1 \tag{36}$$

Substituting $H(j\omega) = \dfrac{N'G(j\omega)}{j\omega}$ into (36) yields

$$f(\omega) = \frac{\beta}{2} \left| \frac{N'G(j\omega)}{j\omega + \beta/2 \cdot N'G(j\omega)} \right| \leq 1 \tag{37}$$

Recall Assumption 2. It is simple to note that assigning $\omega = 0$ into (37) yields

$$f(0) = \frac{\beta}{2}\left|\frac{N'}{\beta/2 \cdot N'}\right| = 1 \tag{38}$$

But according to (36) $f(\omega) \le 1$, so we have $\gamma = \frac{\beta}{2}\sup_{\omega}\left|\frac{H(j\omega)}{1+\beta/2 \cdot H(j\omega)}\right| = f(0) = 1$ ❑

In the sequel, Lemma 2 is exploited in the derivation of the main result of the study. We proceed with the application of the additional condition of Theorem 1.

Lemma 3: Under Assumption 3 and *H(s)* as in (33), $h(t) \in A_2$.

Proof: Assumption 3 assures that the residue of the pole *s=0* is 1, so *H(s)* can be written in the following partial fraction description:

$$H(s) = H_1(s) + \frac{1}{s} \tag{39}$$

with $H_1(s)$ strictly proper. Assumption 3 also guarantees that $H_1(s)$ consists of a sum of asymptotically stable transfer functions. Therefore, $\|H_1\|_2 < \infty$ (Doyle *et al.*, 1992) which implies $h_1(t) \in L^2$. From the same reasons, it stems that $\|H_1\|_1 < \infty$ (Doyle *et al.*, 1992). Consequently,

$$h_1(t) \in L^1 \cap L^2 \tag{40}$$

Now, it is required to show that $th_1(t) \in L^1 \cap L^2$. This will be done by applying the following characteristic of the Laplace transform:

$$\mathcal{L}\,[th_1(t)] = -\frac{dH_1(s)}{ds} \tag{41}$$

$H_1(s)$ is a rational function, i.e. $H_1(s) = \frac{N(s)}{D(s)}$. Let *deg[N(s)]=q* and *deg[D(s)]=p*. Due to the fact that $H_1(s)$ is strictly proper, its relative order satisfies:

$$r \overset{\Delta}{=} p - q > 0 \tag{42}$$

Note that

$$r\left[\frac{dH_1(s)}{ds}\right] = r\left[\frac{dN(s)/ds \cdot D(s) - dD(s)/ds \cdot N(s)}{D^2(s)}\right] = 2p-(q-1+p) = (p-q)+1 > 0 \tag{43}$$

Where the last inequality in (43) stems from (42).

(43) shows that $-\frac{dH_1(s)}{ds}$ is strictly proper. Since $H_1(s)$ is asymptotically stable, $-\frac{dH_1(s)}{ds}$ is asymptotically stable as well, because the differentiation does not alter the denominator polynomial. Thus, we have

$$\frac{-dH_1(s)}{ds} \in L^1 \cap L^2 \Rightarrow th_1(t) \in L^1 \cap L^2 \tag{44}$$

(40) and (44) yield the result:

$$h_1(t) \in A_1 \tag{45}$$

Now, observe (39). The term *1/s* is strictly proper. Together with (45), we obtain $h(t) \in A_2$ (see Definition 2). ❑

In the following section, the results obtained are applied to the establishment of design guidelines which prevent the guidance system saturation.

6. Design Implications

The results obtained thus far suggested that a PNG system is L^∞ / L^∞ stable provided that $H(s) \in \{PR\}$. In this case, for any $a_T \in L^\infty[0, t_f]$ there exists $\mu(\gamma)$ such that $\|a_M\|_\infty \leq \mu(\gamma)\|a_T\|_\infty$. It was also proved that if $H(s) \in \{PR\}$, $\gamma = 1$. In order to get a quantitative information regarding the required missile-target maneuver ratio, $\mu(\gamma)$ should be found. Thusly, the following theorem is contemplated.

Theorem 2: If $H(s) \in \{PR\}$, then $\|a_M\|_\infty \leq \frac{N'}{N'-2}\|a_T\|_\infty \quad \forall a_T \in L^\infty[0, t_f]$.

Proof: The required maneuver acceleration of a PN guided missile with ideal dynamics, i.e. *G(s)=1* and *H(s)=1/s* , against a constantly maneuvering target is (Zarchan 1990):

$$\frac{a_M(t)}{a_T} = \frac{N'}{N'-2}\left[1 - \left(1 - \frac{t}{t_f}\right)^{N'-2}\right] \tag{46}$$

Note that in this case

$$\mu(\gamma) = \frac{\sup\limits_{t\in[0,tf]}|a_M|}{\sup\limits_{t\in[0,tf]}|a_T|} = \frac{\sup\limits_{t\in[0,tf]}|a_M|}{a_T} = \frac{N'}{N'-2} \quad \forall N' > 2, \quad a_T = const \tag{47}$$

However, the case *H(s)=1/s* is a particular case of $H(s) \in \{PR\}$. According to Theorem 1, $\mu(\gamma)$ is a function of γ *only* $\forall a_T \in L^\infty[0, t_f]$. Lemma 2 states that *for any*

$H(s) \in \{PR\}$, i.e., $\mathrm{Re}\, H(j\omega) \geq 0 \quad \forall \omega \in \Re$, we have $\gamma = 1$. Thus, $\mu(\gamma)$ has the value given in (47) for *any* $H(s) \in \{PR\}$ and $\forall a_T \in L^\infty[0, t_f]$. ❑

The consequence of Theorem 2 should be interpreted as follows. If the PNG system is designed such that $H(s) \in \{PR\}$, and $N'/(N'-2)$ is chosen to be higher then the a-priori known missile-target maneuver ratio, acceleration saturation will be avoided. This is provided that the target maneuver satisfies $a_T \in L^\infty[0, t_f]$. Nevertheless, most physical target maneuvers, such as a constant maneuver, a sinusoidal maneuver or a "bang-bang" maneuver, are L^∞-bounded. Theorem 2 expands the results thus known in the literature, since it shows that the required missile-target maneuver ratio should be $N'/(N'-2)$ not only for an ideal missile and a constant target maneuver, but also for any missile dynamics satisfying $\mathrm{Re}\, H(j\omega) \geq 0 \quad \forall \omega \in \Re$ and any target maneuver with bounded maximal value.

It is desirable to formulate The condition $H(s) \in \{PR\}$ in terms of the transfer function *G(s)*. For this reason, the next Lemma is introduced:

Lemma 4: $H(s) \in \{PR\}$ if and only if *G(s)* is a phase lead network with maximal phase lead not exceeding 180°.

Proof: Recall (12). Since $\angle H(j\omega) = \angle G(j\omega) - \pi/2$, $H(s) \in \{PR\}$ if and only if $0 \leq \angle G(j\omega) \leq \pi$. ❑

In view of the above discussion, we may summarize the design guidelines sufficient to avoid output acceleration saturation:

Design Guideline 1: Given flight control dynamics $G_1(s)$ and seeker dynamics $G_2(s)$, design a controller *K(s)* such that $G(s)=K(s)G_1(s)G_2(s)$ is a phase lead network with a maximal phase lead not exceeding 180°.

Design Guideline 2: Given the missile-target maneuver ratio μ_0, choose N' such that $N'/(N'-2) \geq \mu_0$, or equivalently $N' \geq \dfrac{2\mu_0}{\mu_0 - 1}$.

From Design Guideline 2 it is obvious that the smaller μ_0 is, the larger N' should be chosen.

When following the design guidelines, it is assured that saturation is prevented. Yet, the most obvious question asked is how this design influences the miss distance. It is subsequently shown that L^∞ / L^∞ stability is strongly associated with miss distance.

It was proved in a previous study (Gurfil *et al.*, 1998 (2)), that if the guidance system is linear, and in addition

$$H(s) \in \mathbf{Z} \overset{\Delta}{=} \{H(s) \mid r[H(s)] = 1\} \tag{48}$$

zero miss distance (ZMD) will be obtained for any bounded target maneuver.

In our case, the nonlinearities embedded in the guidance system are not active, and it is assured that the operating region of the state variables is linear. Hence, we can examine the miss distance utilizing the assumption of linearity, as was done in (Gurfil *et al.*, 1998 (2)). Also, due to (13), it is clear that

$$\{PR\} \subseteq \mathbf{Z} \tag{49}$$

Which guarantees that if Design Principles 1, 2 are followed, not only saturation is prevented, but ZMD shall be rendered. Thus, the PNG-based systems satisfying Design Principles 1,2, will be called hereafter ZMD-PNG systems.

7. Preventing Saturation of Additional State Variables

The treatment thus far can be repeated in the context of preventing saturation in state variables such as a_C and $\dot{\lambda}$. It is known, that for an ideal missile dynamics (Zarchan, 1990):

$$\frac{a_C(t)}{a_T} = \frac{N'}{N'-2}\left[1-\left(1-\frac{t}{t_f}\right)^{N'-2}\right] \tag{50}$$

Consequently, as was the case for a_M, $\mu(\gamma) = \dfrac{N'}{N'-2}$. Thus, if Design Principles 1 & 2 are followed, commanded acceleration saturation will be avoided as well.

It is also known (Zarchan 1990) that for an ideal missile dynamics,

$$\frac{\dot{\lambda}(t)}{a_T} = \frac{1}{V_C(N'-2)}\left[1-\left(1-\frac{t}{t_f}\right)^{N'-2}\right] \tag{51}$$

In this case we have $\mu(\gamma) = \dfrac{1}{V_C(N'-2)}$. Without a loss of generality, we can normalize a_T to obtain $\|a_T\|_\infty = 1$. Given the maximal allowed LOS angular rate $\dot{\lambda}_{\max}$, Design Guideline 1 remains unchanged. Equivalently to Design Guideline 2, N' should be chosen as follows:

$$N' \geq 2 + \frac{1}{V_C \dot{\lambda}_{\max}} \tag{52}$$

8. Illustrative Examples

The design process proposed will be illustrated using a third order model of *G(s)*. The flight control dynamics are assumed a second order transfer function with damping ζ and natural frequency ω_n , and the seeker LOS measurement dynamics are modeled by a single time lag. As a result, we get

$$G_1(s)G_2(s) = \frac{1}{(\tau_1 s + 1)\cdot(s^2/\omega_n^2 + 2\zeta s/\omega_n + 1)} \tag{53}$$

Let

$$\tau_1 = 0.3\,sec, \zeta = 0.5, \omega_n = 10\,rad/sec \tag{54}$$

With missile-target maneuver ratio

$$\mu_0 = 2 \tag{55}$$

According to Design Principle 1, we should design a controller *K(s)* such that $G(s)=K(s)G_1(s)G_2(s)$ satisfies $0 \le \angle G(s) \le 180^\circ$. Consider the following controller:

$$K(s) = \prod_{i=1}^{3}(\tau_{Z_i} + 1) \tag{56}$$

Let, for simplicity, $\tau_{Z_i} = \tau_Z$. To minimize noise amplification, the smallest τ_Z should be chosen. A straightforward examination shows that $\tau_Z = 0.23$ satisfies the phase lead requirement. Fig. 8 depicts the phase plot of *G(s)* before and after the adding of the controller. It is seen that the controller shifts the phase plot such that $0 \le \angle G(s) \le 58^\circ < 180^\circ$. Thus, Design Principle 1 is satisfied.

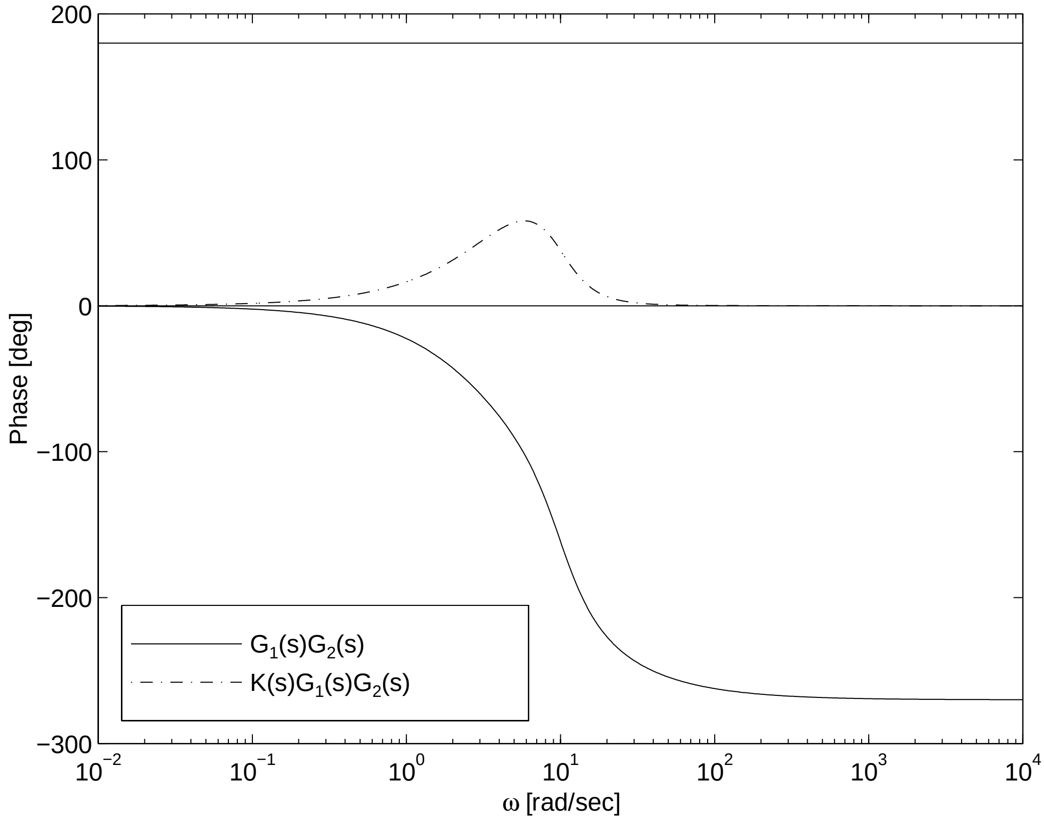

Figure 8: The phase plot before and after adding a controller

Controller (56) shall be called an "Ideal" controller, and the resulting PN-based guidance law will be called "Ideal" ZMD-PNG.

Since *K(s)* is a PD controller, noise amplification problems might arise. To partially overcome the problem, the following controller is suggested:

$$K(s) = \prod_{i=1}^{3} \frac{(\tau_{Z_i} + 1)}{(\tau_{P_i} + 1)} \tag{57}$$

It is apparent that with this type of a controller, Design Principle 1 can not be accomplished, because (13) is not satisfied and hence $H(s) \notin \{PR\}$. Nonetheless, if the additional lag is not too large, we can still prevent acceleration saturation. Let $\tau_{P_i} = \tau_P$ and $\tau_{Z_i} = \tau_Z$. if τ_P is small enough, the phase lag will occur at high frequencies, higher than the typical missile operating frequencies. Thus, no substantial degradation of performance is expected although Design Principle 1 is not fulfilled. We chose the value $\tau_p = 0.05$.

The controller (57) shall be called an "Actual" controller, and the resulting PN-based guidance law will be called "Actual" ZMD-PNG.

After designing the appropriate controller, according to Design Principle 2 it is required that $N' \geq \frac{2 \cdot 2}{2-1} = 4$.

The performance of the guidance loop is evaluated in 3 cases: PNG, "Ideal" ZMD-PNG and "Actual" ZMD-PNG. Also, we consider two types of target maneuvers: A constant maneuver and a sinusoidal ("weave") maneuver.

8.1 Constant target maneuver

Let a_T=1g, and t_f=5 sec. Fig. 9 depicts a comparison of the missile maneuver acceleration for the 3 cases. Notice that when PNG is employed, the acceleration reaches its 2g limit 2 seconds before flight end, resulting in a miss of 1m. However, in the case of ZMD-PNG, no saturation occurs since, $\sup_t |a_M| \leq \frac{N'}{N'-2} = 2$. The miss distance is negligible. It is also apparent that the ZMD-PNG performance remains almost unchanged when the ideal case is corrupted by using an additional small lag.

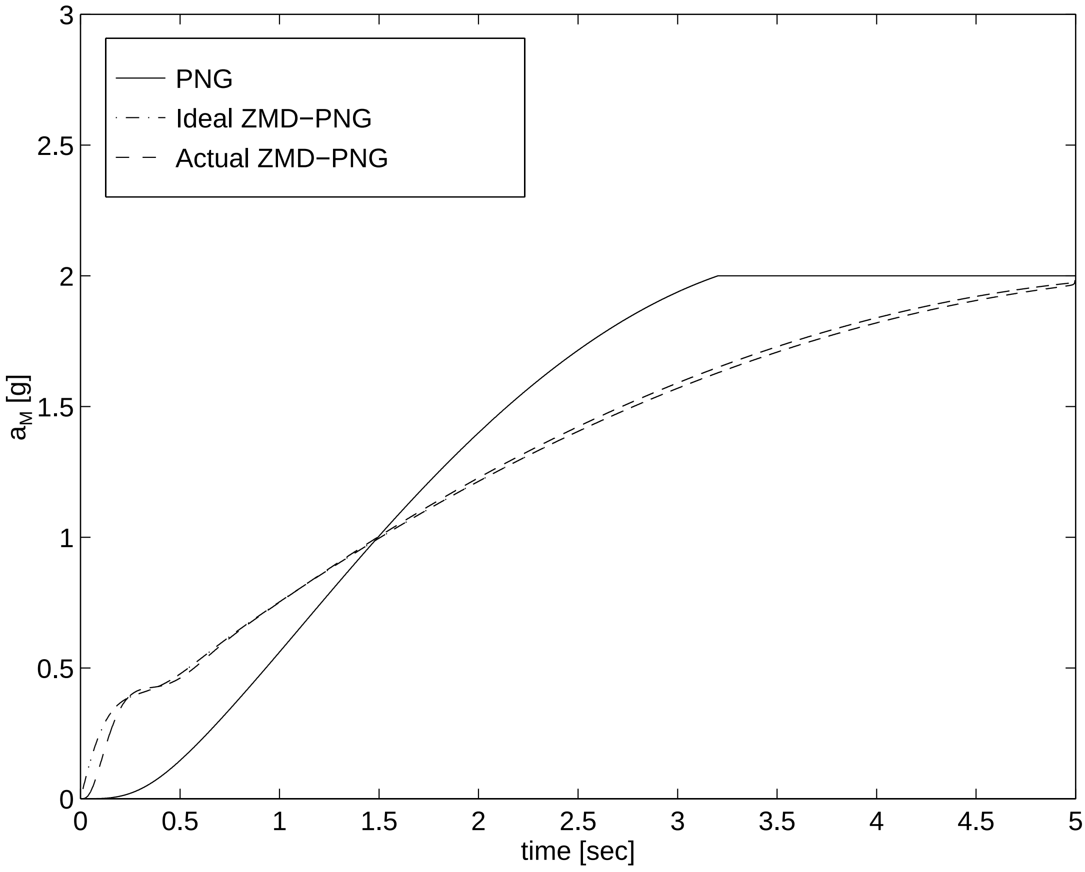

Figure 9: Maneuver acceleration comparison for a constant target maneuver

8.2 Sinusoidal target maneuver

The sinusoidal target maneuver satisfies:

$$a_T = a_{T_0} \sin(\omega_T t) \tag{58}$$

Let $a_{T_0} = 1g$, $\omega_T = 2.5\ rad/\sec$, t_f=5sec. Figure 10 depicts the absolute value of missile acceleration for the 3 cases considered. Notice that in the case of PNG, the acceleration saturates, causing a miss distance of 0.5*m*. However, ZMD-PNG does not saturate, both in the "Ideal" and "Actual" cases, and the miss distance is negligible.

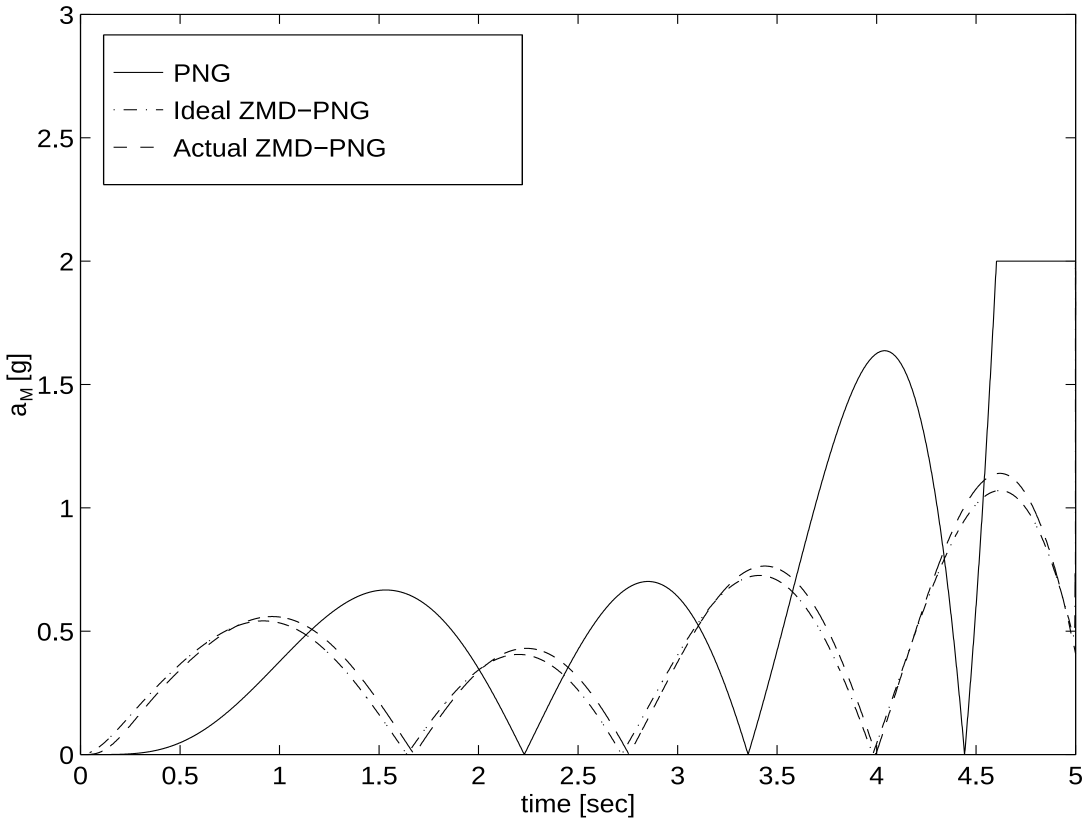

Figure 10: Maneuver acceleration comparison for a sinusoidal target maneuver

9. Concluding Remarks

This paper presented a simple improvement to the well-known PNG law, which assured that saturation of maneuver acceleration or other limited state variables was prevented. Hence, zero miss distance could be obtained. The improved guidance, called ZMD-PNG, was based upon the assumption that the target maneuver was bounded. ZMD-PNG offered 2 design guidelines to follow: First, the total dynamics of the guidance system should be designed positive real; Second, the effective PN constant should be chosen according to a simple function of the given missile-target maneuver ratio. The new guidance law exhibited a significant improvement comparing to PNG. The main disadvantage of the proposed law is that it might increase noise sensitivity; However, this obstacle could be overcome by introducing some lag into the system. Some examples illustrated that the new guidance law is robust to small deviations from the design guidelines.

References

Desoer, C. A., Vidyasagar, M. (1975). *Feedback Systems: Input-Output Properties*, Academic Press, New York.

Doyle, J. C., Francis, B. A. and Tannenbaum, A. R. (1992). *Feedback Control Theory,* Macmillan Publishing, New York.

Gurfil P., Jodorkovsky, M. and Guelman, M., (1998). Stability Approach to Proportional Navigation Systems Analysis *Journal of Guidance, Control and Dynamics,* Vol. 21, No.6, pp. 853-861.

Gurfil, P., Jodorkovsky, M., Guelman, M. (1998). "Simple Guidance Law against Highly Maneuvering Targets", in *Proceedings of the 38th AIAA Guidance, Navigation and Control Conference,* Boston, MA, pp. 570-580

Gutman, S. (1979). "On Optimal Guidance for Homing Missiles", *Journal of Guidance, Control and Dynamics*, Vol. 2, No. 4, pp. 296-300.

Mossaheb, S. (1982). "The Circle Criterion and the L^p Stability of Feedback Systems", *SIAM Journal on Control and Optimization*, Vol. 20, No.1, January 1982, pp. 144-151

Sandberg, I. W. (1964). "A Frequency Domain Condition for the Stability of Feedback Systems Containing a Single Time-Varying Nonlinear Element", *Bell Systems Technical Journal*, Vol.43, pp. 1601-1608.

Sandberg, I. W. (1965). "Some Results on the Theory of Physical Systems Governed by Nonlinear Functional Equations", *Bell Systems Technology Journal*, Vol. 44, No. 5, pp. 871-898.

Sandberg, I. W., Johnson, K. K. (1990). *IEEE Transactions on Automatic Control*, Vol. 35, No.1, pp. 530-534.

Shinar, J. (1976). "Divergence Range of Homing Missiles", *Israel Journal of Technology*, Vol.14, pp.47-55.

Shinar, J., Steinberg, D., (1977). Linearized Two-Dimensional Kinematic Model *Journal of Aircraft*, Vol. 14, No. 8, pp. 795-802.

Shneydor, N. (1984). "Proportional Navigation Guidance-A Survey", RAFAEL Report no. MT/P/013/30/84.

Slotine, J. J. E., Li, W. (1991). *Applied Nonlinear Control*, Prentice Hall.

Rusnak, I., Meir., L. (1991). "Optimal Guidance for High Order and Acceleration Constrained Missile", *Journal of Guidance, Control and Dynamics,* Vol. 14, No. 3, pp. 589-596.

Rusnak, I. (1993). "Advanced Guidance Laws for Acceleration Constrained Missile, Randomly Maneuvering Target and Noisy Measurement", *Transactions of the IEEE Regional Conference on Aerospace Control Systems*, Westlake, California.

Zames, G. (1966). "On input-Output Stability of Time-varying Nonlinear Feedback Systems-Part II: Conditions Involving Circles in the Frequency Plane and Sector Nonlinearities", *IEEE Transactions on Automatic Control*, Vol. AC-11, pp. 465-476.

Zarchan, P., (1979),
Systems-SLAM *Journal of Guidance and Control*, Vol.2, pp. 71-78.

Zarchan, P. (1990). *Tactical and Strategic Missile Guidance*, Progress in Astronautics and Aeronautics, Vol.124, AIAA, Washington, DC.