Approximation of High-Order Lumped Systems by using Non-Integer Order Transfer Functions

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Abstract

Non-integer order systems have been studied by several authors to model particular physical systems (electrical, biological etc.). In particular it can be shown that a non integer order system is equivalent to an infinite order LTI system. This feature can be useful considered for model order reduction purposes . The main aim of this paper is to show the mathematical background of this new approximation theory, the criteria for selecting the order of a non-integer order model which behaves as the original integer order ones and the quality indexes that can be considered for assessing the goodness of the approximated model. Some examples and simulations are reported.

1. Introduction

Model order reduction has recently become an interesting field of investigation, due to the increasing importance of synthesize controllers with more simple structure (reduced order controllers). The approximation of the plant is performed via different strategies, paying attention to avoid the deleting of most important dynamical features of the system.

In this paper a way for reducing the number of parameters describing an high order system is presented. It is based on selecting r zeros and r+1 poles in order to be well approximated by a one pole fractional order system, whose behavior has been studied in (Sun *et al.*, 1984, Charef *et al.*, 1992). Since this kind of system is identified by three parameters, globally is possible to spare 2(r-1) ones. It is quite evident that for r sufficiently large this fact can be a great advantage because an n-order system is so decomposed in an integer n-r-1 order system and a fractional m^{th} order (where $0<m<1$) system, allowing an easier approach in designing suitable regulation strategies.

After a brief discussion above fractional order systems and methods for approximating them, a detailed explanation of the strategy concerning the inversion of methods previously introduced is therefore presented, including a method that allows to quantify the effectiveness of the performed approximation. Several examples of applications compared with classical methods are also discussed in order to validate the performed approach.

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2. Fractional systems and approximation methods

Fractional Systems (or Non-Integer-Order Systems) can be considered as a generalization of integer order systems. Typical examples of natural fractional systems can be found in studying dielectric polarization, as shown in (Charef *et al.*,1992; Liu, 1985), transmission lines or noise spectrum in 1/f type processes like in (Keshner, 1982), while often fractional regulators are needed in control engineering and robotics as reported in (Oustaloup 1983; Oustaloup1991).
As a classical example of a non integer order linear system, the voltage-current relation of a semi-infinite lossy (RC) line is reported:

$$\frac{d^{0.5}v(t)}{dt^{0.5}} = \sqrt{\frac{R}{C}}i(t) \tag{1}$$

In (Oldham, 1974; Ross, 1975) it is assumed as m-order derivative, for 0<m<1, the following expression:

$$\frac{d^m v(t)}{dt^m} = \frac{d}{dt}\left[\frac{1}{\Gamma(1-m)}\int_0^t (t-y)^{-m} v(y)dy\right] \tag{2}$$

($\Gamma(*)$ being the factorial function cited in (Oldham, 1974)).
Referring to (1) and considering i(t) and v(t) as the input and the output of a fractional system respectively, the corresponding transfer function is represented by an half-integrator. Therefore the frequency response shows asymptotically a slope of -10 dB/dec and the phase delay is 45°.
A more general fractional system has the following transfer function:

$$F(s) = \frac{Y(s)}{U(s)} = \frac{1}{(s+p)^m} \tag{3}$$

that can be represented by the non-integer differential equations

$$\frac{d^m}{dt^m}\left(e^{pt}y(t)\right) = e^{pt}u(t)\,. \tag{4}$$

By using (4) and (2), it follows that the impulse response ($u(t)=\delta(t)$) of system (3) results:

$$y(t) = \frac{t^{m-1}e^{-pt}}{\Gamma(m)}\quad , \tag{5}$$

which has finite integral in $[0,+\infty[$. Therefore relation (5) can be used in classical convolution integral for deriving the response of the system to a general input signal u(t). As regards the frequency response of (3) it could be observed that such system shows a magnitude Bode's plot which resembles a first order system, except for the slope, which is proportional to the order *m* (*-20m* dB/dec).
An analogous dependence can be found for phase Φ:

$$\Phi[F(j\omega)] = -m\,\mathrm{arctg}\left(\frac{\omega}{p}\right) \tag{6}$$

Fractional systems like in (3) can be approximated by using integer order models with n zeros and $n+1$ poles. In (Sun *et al.*, 1984; Charef *et al.*, 1992), an alternative succession of poles and zeros is usually adopted to this purpose. The quality of the approximation depends on the desired bandwidth together with the maximum error in dB admissible. An example is given in fig. 1. The number of poles and zeros is clearly correlated with the desired bandwidth. In the example has been adopted an approximation with maximum error of 2 dB from $\omega=10^{-2}$ to $\omega=10^{2}$.

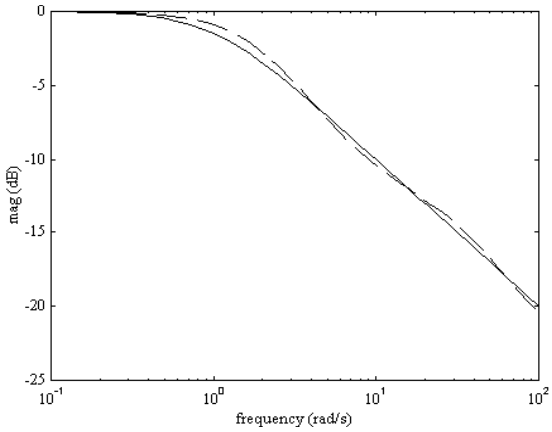


Figure 1: Bode's diagram of $1/(s+1)^{0.5}$ and (dashed) its 4° order approximation

The general form of the transfer function is reported below

$$H(s) = \frac{1}{\left(1+\frac{s}{p_T}\right)^m} \approx \frac{\prod_{i=0}^{N-1}\left(1+\frac{s}{z_i}\right)}{\prod_{i=0}^{N}\left(1+\frac{s}{p_i}\right)} \tag{7}$$

where $p_T=10^{-2}$ and p_i, z_i and N are chosen suitably, according to (Charef *et al.*, 1992), for each different order m. As it can be seen, there is a relation between non integer order systems and a particular kind of high order systems (relaxation systems). This fact can suggest that, inverting this assumption, a *suitably distributed* succession of poles and zeros can be represented by a one-pole fractional systems.
In next section this fact will be explained more in details.

3. From high order to fractional order

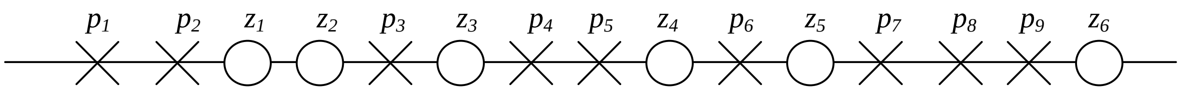
Let us consider a N order system, with all poles and zeros real, described by the following transfer function in Bode form:

$$G(s) = \frac{k\prod_{i=1}^{L}\left(1+\frac{s}{z_i}\right)}{\prod_{i=1}^{N}\left(1+\frac{s}{p_i}\right)} \tag{8}$$

Our aim is to approximate most of the poles and zeros of (8) with a fractional order transfer function in order to obtain the following approximation function:

$$\tilde{G}(s) = \frac{1}{\left(1+\frac{s}{p_T}\right)^m} \frac{k\prod_{i=1}^{l}\left(1+\frac{s}{\tilde{z}_i}\right)}{\prod_{i=1}^{n}\left(1+\frac{s}{\tilde{p}_i}\right)} \approx G(s) \tag{9}$$

where $\tilde{z}_i$ and $\tilde{p}_i$ are subsets of z_i and p_i respectively ($l<L$ and $n<N$). Expression (9) is more compact and simple than (8), especially if $n<<N$. The zeros and poles distribution of (8) can be whatever: an example of singularity succession could be the following (p_1 p_2 z_1 z_2 p_3 z_3 p_4 p_5 z_4 p_6 z_5 p_7 p_8 p_9 z_6) :



The first step is to "extract" the longest possible alternative succession of poles and zeros . In most of cases, different choices can be made. In order to select the best one, a test on every possible distribution must be applied. The longest succession contains five poles and four zeros and its general term is (P-Z-p_3-z_3-P-z_4-p_6-z_5-P). The multiplicity of each unspecified subset is 2, 2, 2 and 3 so the number of sequence to analyze is 2x2x2x3=24. For each sequence a test of "regularity" is applied: it consists in calculating the ratio between each singularity and the following in increasing order, for example

$$\begin{gathered} r_1 = \frac{p_1}{p_2}, r_2 = \frac{p_2}{p_3}, r_3 = \frac{p_3}{p_4}....... \\ f_1 = \frac{z_1}{z_2}, f_2 = \frac{z_2}{z_3}, f_3 = \frac{z_3}{z_4}....... \end{gathered} \tag{10}$$

In order to have a good approximation, the quantities r_i and f_i must be very close one to each other or approximately constant for every i.

For each pole-zero interval a partial slope or a partial fractional order can be obtained. It is given by the expression:

$$m_i = \frac{\log_{10}\frac{z_i}{p_i}}{\log_{10}\frac{p_{i+1}}{p_i}} \tag{11}$$

from which it derives

$$m = \overline{m} = \frac{\sum_{i=1}^{n} m_i}{n_z} \tag{12}$$

that is the optimal value to be substituted in expression (9). A quite good index of dispersion around a mean value is the *Standard Deviation*, which is defined as follows:

$$\sigma_m = \sqrt{\frac{\sum_{i=1}^{\nu}\left(m_i - \overline{m}\right)}{\nu - 1}} \tag{13}$$

where $\nu = n_z$ (being n_z the number of zeros selected) and $\overline{m}$ is the mean value of *m*. It can be fixed a threshold for σ_m over which the attempted approximation should be rejected. From experimental simulations, a reasonable choice for this value seems to be $\sigma_m = 0.08$. If does not exist any sequence with n_z zeros having the mentioned characteristic, the method is applied to all the sequences with $n_z - 1$ zeros and so on until a value under the threshold is obtained.
Analogous relations give the parameter p_T:

$$p_{Ti} = p_i 10^{-\frac{\log(p_{i+1}/z_i)}{2m_i}} \tag{14}$$

$$p_t = \overline{p}_{Ti} = \frac{\sum_{i=1}^{n} p_{Ti}}{n_z} \tag{15}$$

The presented algorithm is quite simple and shows an easy software implementation. However its application may be in some cases not convenient: in fact only if the selected succession of poles and zeros is sufficiently long , this technique provides a real advantage in terms of reduction of parameters.

4. Case Studies

4.1. Relaxation Systems

Let us have a "Relaxation" system represented by the following transfer function:

$$F(s) = \frac{10^4 (s+18)(s+44)(s+69)(s+96)}{(s+10)(s+35)(s+56)(s+83)(s+111)} \tag{16}$$

The peculiarity of this kind of systems is in having an alternative succession of poles and zeros. For this reason an approximation of the whole system may be attempted. By using expression (12), the following four values for can be found:
m_1=0.468, m_2=0.486, m_3=0.530, m_4=0.500
from which, taking into account relations (11), (12),(13), (14),(15) it derives

$$\begin{aligned} \sigma_m &= 0.026 \\ \overline{m} &= 0.496 \\ p_T &= 8.4 \end{aligned} \tag{17}$$

The value of σ_m is sufficiently small to provide a low error in the frequency range $0.01<\omega<111$. The desired fractional order transfer function is so given by

$$\tilde{F}(s) = \frac{290.5}{\left(\frac{s}{8.4}+1\right)^{0.496}} \tag{18}$$

A graphical example of the performed approximation is given by the magnitude bode diagram depicted in figure 2.

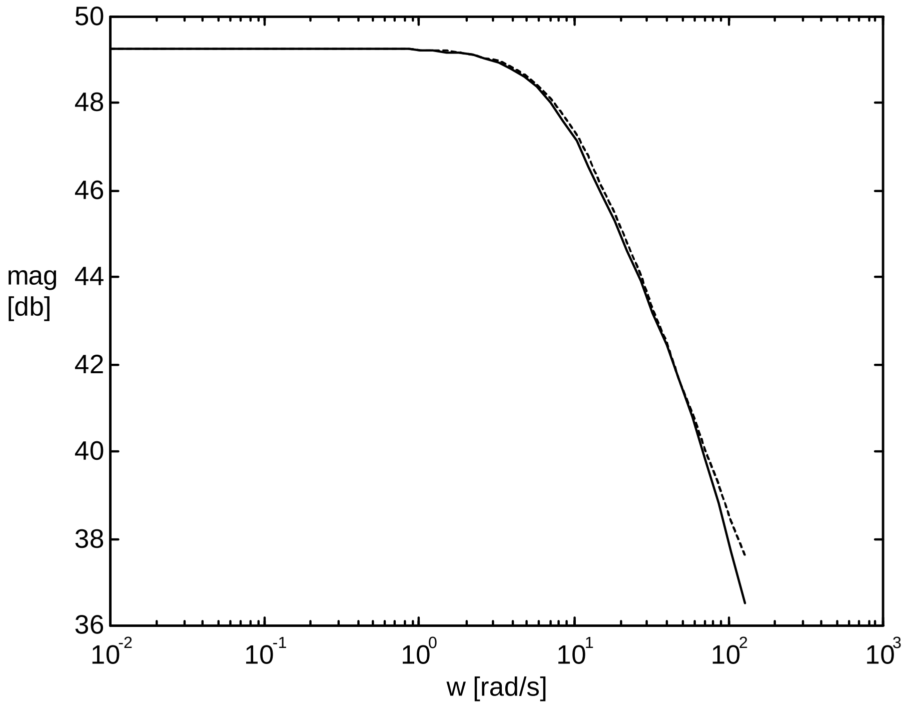


Figure 2: Bode's diagram of F(s) and (dashed) its fractional order approximation

4.2. General Systems

Let us consider a more general case. A SISO mechanical system (a flexible beam) is described by the following state-space representation:

$$\begin{cases} \dfrac{dx}{dt} = Ax + Bu \\ y = Cx \end{cases}$$

where

$$A = \begin{bmatrix} -25.0995 & -10.0248 & 0 & 0 \\ 10.0248 & 0 & 0 & 0 \\ -14.0995 & -9.0273 & -0.1095 & -0.0315 \\ 0 & 0 & 0.0315 & 0 \end{bmatrix};$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad ; \qquad C = \begin{bmatrix} 0 & 0 & 0 & 6340 \end{bmatrix}.$$

The relative transfer function is derived using the formula $F(s)=C(sI-A)^{-1}B$. Putting it in Bode form, it results:

$$F(s) = \frac{2 \cdot 10^4 (s+1)\left(\frac{s}{10}+1\right)}{\left(\frac{s}{0.01}+1\right)\left(\frac{s}{0.0995}+1\right)\left(\frac{s}{5}+1\right)\left(\frac{s}{20.0995}+1\right)} \tag{19}$$

Due to the position of zeros and poles, only two different sequences of maximum length (three poles and two zeros) are obtained. They are (P -1 -5 -10 -20.0995), where $P \in \{p_1,p_2\}=$ $=\{-0.01,-0.0995\}$. If $P=p_1$, by using relations (11), (12) and (13) it holds:

$$\overline{m} = 0.6196$$
$$\sigma_m = 0.1717$$

On the other hand, if $P=p_2$, it can be obtained

$$\overline{m} = 0.5436$$
$$\sigma_m = 0.0643$$

The lowest value of σ_m suggest us to choose this last sequence for the approximation. From this consideration, it derives that the simplified transfer function results to be the following:

$$\tilde{F}(s) = \frac{2 \cdot 10^4}{\left(\left(\frac{s}{0.01}+1\right)\left(\frac{s}{0.0376}+1\right)\right)^{0.5436}} \tag{20}$$

The value of the fractional pole is computed by using relation (14) and taking the mean value of p_{Ti}. As shown in figure 3, the performed approximation works very well for a wider range of frequencies, if compared with a second order model obtained using a singular perturbation approximation of an internally balanced realization like in (Muscato *et al.*,1997) . It must be noted that the number of parameters of a second order model is the same of that of the parameters present in (20).

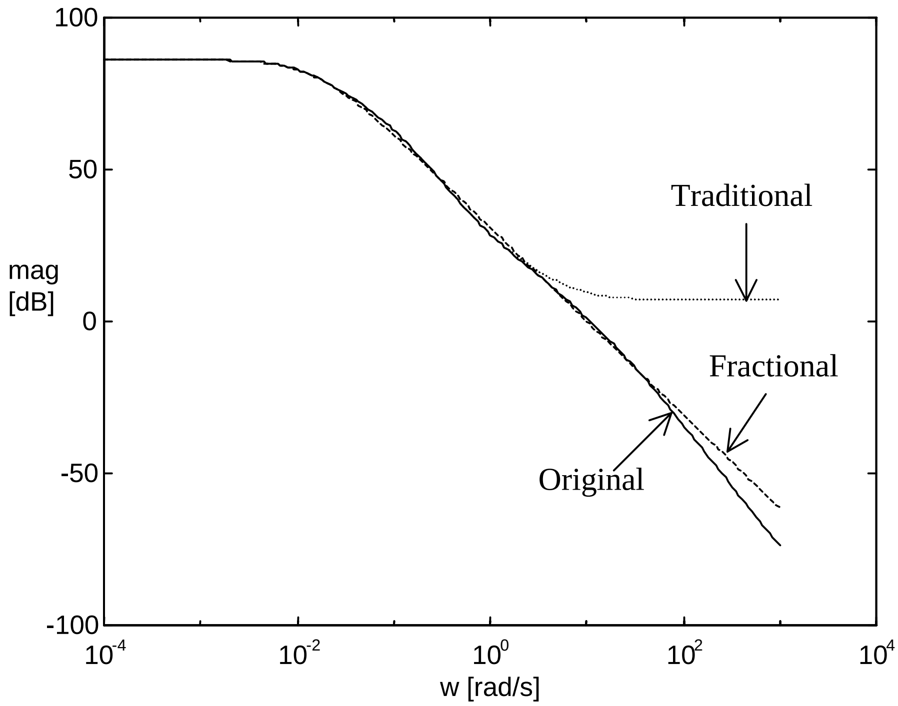


Figure 3: Bode's diagrams of F(s) and its fractional order approximation, which works better than the second order model obtained by using a traditional technique.

5. Conclusions

In this paper an innovative strategy for obtaining a simplified model with few parameters from an high order system has been proposed. The introduction of non integer order systems resulted to be an efficient way to compress frequency response information usually intrinsic in an high number of poles and zeros. It has been also demonstrated that σ_m is a quite good index in order to evaluate the quality of the performed approximation. Finally, a comparison with a traditional method of model order reduction proved that a reduced model with the same number of parameters is not able to get the same good performance in the frequency domain.

Future developments include further investigations on different systems in order to build a more complete theory on fractional order modelling with the aim of finding suitable solutions for the control problem.

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