ON THE EXISTENCE OF NASH EQUILIBRIUM SOLUTION FOR MIXED H_2/H_∞ CONTROL [1]

HELENICE O. FLORENTINO[2]
Dept. of Biostatistics
IB - UNESP-Botucatu SP, Brazil

ROBERTO M. SALES[3]
Dept. of Electronic. Eng
USP-Sao Paulo SP, Brazil

Abstract

The aim of this work is the application of the game theory in mixed H_2/H_∞ control problems, using convex optimization. We use the formulation of the mixed H_2/H_∞ control problem as a Nonzero-Sum NASH Game, where the two pay-off functions are associated with two players, which represent the H_2 and H_∞ criteria. We show that the necessary and sufficient conditions for the existence of a NASH equilibrium solution are related to the existence of a global optimal solution to a convex optimization problem. The plant is assumed linear and time-invariant and the resulting controller is a state-feedback law.
Keywords: Nash Game; H_2/H_∞ Control; Convex Optimization.

1 Introduction

The concepts of games and strategies were originally presented by Von Neumann in 1928. In 1954 R. Isaacs introduced the differential games theory. Ten years latter L.D. Berkowitz presented a rigorous approach to game theory using variational calculus. In 1965 Ho *et al.*, studied the optimal pursuit-evasion strategies problem. Since then, many real problems - missiles control, anti-missiles, aircraft traffic, etc. - were considered in the context.

Optimization theory, on the other hand, is undoubtedly one of the most relevant approach to modern control problems. Particularly, state feedback optimal control problems involving H_2 and H_∞ norms can be converted into equivalent convex problems, and solved through standard convex programming techniques. Yet, the convex approach allows the incorporation of additional constraints, as for example the minimization of disturbances external to the plant. In (Boyd *et al.*, 1994), a variety of control problems that can be formulated as a convex optimization problem involving matrix inequalities and solved numerically is presented.

This paper brings together some elements of both these theories. The H_2 /H_∞ mixed control problem is formulated as a non-zero sum Nash game, with two players and two pay-off functions, which represent H_2 and H_∞ criteria. Necessary and sufficient conditions for the existence of an equilibrium solution are then related to the existence of a global optimal solution to a convex optimization problem. The considered problem has been originally considered by (Limebeer *et al.*, 1994), where,

[1] **Grants by FUNDUNESP and FAPESP - N. 97/4668-1-Brazil**
[2] ***e-mail: helenice@ibb.unesp.br***
[3] ***e-mail: roberto@lac.usp.br***

in contrast to the results here presented, the existence of an equilibrium solution is associated to the existence of a solution to a pair of coupled Riccati equations.

In section 2 some preliminary results are presented. In section 3 the H_2 /H_∞ mixed control problem is formulated as a non-zero sum Nash game. In section 4 necessary and sufficient conditions for the existence of an equilibrium solution are obtained, and Section 5 presents the final conclusions.

2 Preliminaries

Let the linear, continuous, time invariant control system, described by the following equations:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\
u(t) &= Kx(t) \\
z_2(t) &= C_2 x(t) + D_2 u(t) \\
z_\infty(t) &= C_\infty x(t) + D_\infty u(t)
\end{aligned} \tag{2.1}$$

where $x \in R^n$ is the state vector,
$u \in R^m$ is the control vector,
$w \in R^\ell$ is the disturbance vector and
$z_2 \in R^{q_2}$ and $z_\infty \in R^{q_\infty}$ are the output vectors.

We assume that all the matrices are of known and appropriate dimensions and,

$$\begin{aligned}
&\text{i) } C_2' D_2 = 0, \\
&\text{ii) } D_2' D_2 > 0, \\
&\text{iii) } C_\infty' D_\infty = 0, \\
&\text{iv) } D_\infty' D_\infty > 0.
\end{aligned} \tag{2.2}$$

Associated with the system (2.1), we define the extended matrices,
$F \in R^{pxp}$, where $p = m + n$, $G \in R^{pxm}$, $R_2 \in R^{pxp}$ and $R_\infty \in R^{pxp}$,

$$F = \begin{bmatrix} A & -B_2 \\ 0 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 \\ I \end{bmatrix} \qquad Q = \begin{bmatrix} B_1 B_1' & 0 \\ 0 & 0 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} C_2' C_2 & 0 \\ 0 & D_2' D_2 \end{bmatrix}, \qquad R_\infty = \begin{bmatrix} C_\infty' C_\infty & 0 \\ 0 & D_\infty' D_\infty \end{bmatrix}, \tag{2.3}$$

the closed loop matrices,

$$A_{cl} = A + B_2 K, \tag{2.4}$$

$$C_{cl_2} = C_2 + D_2 K \quad \text{and} \quad C_{cl_\infty} = C_\infty + D_\infty K, \tag{2.5}$$

the set of admissible controllers,

$$K_{ad} = \{K \in R^{mxn} \,/\, A_{cl} \text{ is asymptotically stable}\} \tag{2.6}$$

and the closed-loop transfer functions from w to z_2 and de from w to z_∞ are, respectively,

$$T_{z_2 w} = C_{cl_2}[sI - A_{cl}]^{-1}B_1 \text{ and} \tag{2.7a}$$

$$T_{z_\infty w}(s) = C_{cl_\infty}[sI - A_{cl}]^{-1}B_1 . \tag{2.7b}$$

For all $K \in K_{ad}$ we have the H_2 norm associated with $T_{z_2 w}$, defined as:

$$\left\|T_{z_2 w}\right\|_2^2 = Tr(C_{cl_2} L_c C_{cl_2}^{'}) = Tr(B_1^{'} L_0 B_1) \tag{2.8}$$

where Tr(.) denotes the trace of (.) and L_c and L_o are, respectively, the controllability and the observability Gramians of (A_{cl}, B_1) and (C_{cl_2}, A_{cl}), and the H_∞ norm associated with $T_{z_\infty w}$, defined as:

$$\left\|T_{z_\infty w}\right\|_\infty = \sup_\omega \sigma_{max}[T_{z_\infty w}(j\omega)] = \sup_\omega \{\lambda_{max}^{1/2}[T_{z_\infty w}(-j\omega)' T_{z_\infty w}(j\omega)]\} \tag{2.9}$$

where σ_{max} [.] is the maximum singular value of [.], λ_{max} [.] is the maximum eigenvalue of [.] and $\omega \in R_+$.

From (2.3) we define the following functions:

$$\theta_\infty(.,.) : (R^{pxp}, R) \to R^{pxp}$$

$$\theta_\infty(W,\mu) = FW + WF' + WR_\infty W + \mu Q, \tag{2.10}$$

associated with the set

$$C_\infty = \{ W = W' \geq 0, \mu > 0 : v'\theta_\infty(W,\mu)v \leq 0, \forall\, v \in N\} \tag{2.11}$$

and

$$\theta_2(.) : R^{pxp} \to R^{pxp}$$

$$\theta_2(W) = FW + WF' + Q, \tag{2.12}$$

associated with the set

$$C_2 = \{ W = W' \geq 0 : v'\theta_2(W)v \leq 0, \forall\, v \text{ and } N \} \tag{2.13}$$

In both cases, N is the null space of G' and W is symmetric and partitioned in the form:

$$W = \begin{bmatrix} w_1 & w_2 \\ w_2{}' & w_3 \end{bmatrix} \tag{2.14}$$

where $w_1 \in R^{nxn}$, $w_1 > 0$.

It can be easily seen that

$v \in N$ iff $v' = [x' : 0]$, $x \in R^n$ (2.15)

In addition, (Colaneri et al., 1997)

C_2 and C_∞ are convex with respect to W. (2.16)

Theorem 2.1 Assume (A_{cl}, C_{cl_∞}) observable and let $\gamma > 0$ given. Then A_{cl} is asymptotically stable and $\|T_{z_\infty w}\|_\infty \leq \gamma$ if and only if the Riccati inequality

$$A_{cl}'P + PA_{cl} + \gamma^{-2}PB_1B_1'P + C_{cl_\infty}'C_{cl_\infty} \leq 0 \quad (2.17)$$

has a symmetric positive definite solution $P \in R^{nxn}$.

Proof: see (Sherer, 1989)

3 Statement of the Game

Let the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1w(t) + B_2u(t), \quad x(0) = x_0 \\ z(t) &= Cx(t) + Du(t) \end{aligned} \quad (3.1)$$

where $x \in R^n$ is the state vector , $u \in R^m$ is the control vector, $w \in R^\ell$ is the disturbance vector, $z \in R^q$ is the output vector and A, B_1, B_2, C and D are matrices of known and appropriate dimensions. This system corresponds to the particular case $z_2(t) = z_\infty(t) = z(t)$

problem: Find a control law u*(t,x) such that

$$\|z\|_2^2 \leq \gamma^2 \|w\|_2^2 \text{ for all } w(t) \neq 0 \in L_2, \text{ given } \gamma > 0, \quad (3.2)$$

where L_2 is a set of finite energy signals. Inequality (3.2) can be interpreted as a restriction on the H_∞ norm, i.e.,

$$\|T_{zw}\|_\infty \leq \gamma \quad (3.3)$$

where the operator T_{zw} represents the map from the disturbance signal w(t) to the output z(t) in the presence of the optimal control law u*(t,x).

It is desired that the control signal u*(t,x) produces regulation of the state x(t) and minimizes the output energy, in the presence of the worst disturbance signal $w^*(t,x)$, in the sense that $w^*(t,x)$ activates the maximum energy gain from the disturbance input to the output signal.

The above problem can be formulated as a non zero sum linear quadratic game, following the same lines presented in (Limebeer *et al.*, 1994). The two cost functions are given by:

$$J_1(u,w) = \int_0^\infty (\gamma^2 w(t)'w(t) - z(t)'z(t))dt \quad (3.4)$$

$$J_2(u,w) = \int_0^\infty (z(t)'z(t))dt \quad (3.5)$$

The cost function J_1 is related to the H_∞ criterion and J_2 optimizes the H_2 norm. The strategy sets for each player, denoted by Σ_1 and Σ_2, are defined as:

$$\Sigma_1 \overset{\Delta}{=} \{w(t)=K_2 x(t) : J_1(u,w)\geq 0 \text{ and } K_2 \text{ is real matrix}\} \tag{3.6}$$

$$\Sigma_2 \overset{\Delta}{=} \{u(t)=K_1 x(t) : K_1 \text{ is real matrix and } K_1 \in K_{ad}\} \tag{3.7}$$

We look for equilibrium strategies u* and w* in Σ_1 and Σ_2, respectively, which satisfy the Nash equilibrium conditions:

$$J_1(u^*,w^*) \leq J_1(u^*,w) \quad \text{for all } w\in\Sigma_1 \tag{3.8a}$$
$$J_2(u^*,w^*) \leq J_2(u,w^*) \quad \text{for all } u\in\Sigma_2 \tag{3.8b}$$

From (3.8a), if $J_1(u^*,w^*) \geq 0$, then $\|z\|_2^2 \leq \gamma^2\|w\|_2^2$ for all $w(t) \neq 0 \in L_2$, and $\|T_{zw}\|_\infty \leq \gamma$. The second inequality, (3.8b), says that u^* produces state regulation with minimum output energy in the presence of the worst disturbance input, w^*.

The game, denoted by Γ, is represented in the form

$$\Gamma \overset{\Delta}{=} \{ J_1 , J_2 , \Sigma_1 , \Sigma_2 \},$$

and the strategies set, denoted by Ω, is defined as

$$\Omega \overset{\Delta}{=} \{ (u(t,x),w(t,x)) : w(t,x) \in \Sigma_1 \text{ and } u(t,x) \in \Sigma_2 \} \tag{3.9}$$

4 The Necessary and Sufficient Conditions for the Existence of Nash Equilibrium Strategies

In this section we determine necessary and sufficient conditions for the existence of Nash equilibrium strategies for the game Γ formulated in Section 3. The main result is presented in Theorem 4.1.

Lemma 4.1 Let the system

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ &z(t) = Cx(t) + Du(t) \\ &u(t) = Kx(t), \quad K \in K_{ad} \end{aligned} \tag{4.1}$$

and define

$$J(u) \overset{\Delta}{=} \int_0^\infty [z'(t)z(t)]dt\,. \tag{4.2}$$

Assume that there exists $V(\xi) = \xi' P\xi$, such that

$$P > 0 \text{ and } \frac{d(V(x(t)))}{dt} = -z(t)'z(t) \tag{4.3}$$

for all x and z satisfying (4.1). Then,

$$J(u) = x_0' P x_0 \tag{4.4}$$

and

$$\min_u J(u) = \min_{\substack{P,K \\ P>0 \\ (A+BK)'P+P(A+BK)+(C+DK)'(C+DK)=0}} x_0' P x_0 \tag{4.5}$$

Proof. It follows by simple manipulations on (4.1)-(4.3).

Lemma 4.2 Let the system (3.1) with

$u = w_2' w_1^{-1} x$,

w_1, w_2 such that $w_1 > 0$ and

$$w_1^{-1}A + A' w_1^{-1} + w_1^{-1}B_2 w_2' w_1^{-1} + w_1^{-1} w_2 B_2' w_1^{-1} + C'C + w_1^{-1} w_2 w_2' w_1^{-1} + \gamma^{-2} w_1^{-1} B_1 B_1' w_1^{-1} = 0 \tag{4.6}$$

Then,

$$J_1(u,w) = \left\| w_2' w_1^{-1} x(t) + B_2' w_1^{-1} x(t) \right\|_2^2 - \left\| u(t) + B_2' w_1^{-1} x(t) \right\|_2^2 + \gamma^2 \left\| w(t) - \gamma^{-2} B_1 w_1^{-1} x(t) \right\|_2^2 - x_0' w_1^{-1} x_0$$

Proof. Computing $\dfrac{dx'(t) w_1^{-1} x(t)}{dt}$, and using (4.6), we get

$$\frac{dx'(t) w_1^{-1} x(t)}{dt} = \dot{x}'(t) w_1^{-1} x(t) + x'(t) w_1^{-1} \dot{x}(t) = (Ax(t) + B_1 w(t) + B_2 u(t))' w_1^{-1} x(t) + x'(t) w_1^{-1} (Ax(t) + B_1 w(t) + B_2 u(t)) =$$

$$= x'(t)(A' w_1^{-1} + w_1^{-1} A) x(t) + w'(t) B_1' w_1^{-1} x(t) + u'(t) B_2' w_1^{-1} x(t) + x'(t) w_1^{-1} B_1 w(t) + x'(t) w_1^{-1} B_2 u(t)) =$$

$$= x'(t)(-w_1^{-1} B_2 w_2' w_1^{-1} - w_1^{-1} w_2 B_2' w_1^{-1} - C'C - w_1^{-1} w_2 w_2' w_1^{-1} - \gamma^{-2} w_1^{-1} B_1 B_1' w_1^{-1}) x(t) + 2\langle w(t), B_1' w_1^{-1} x(t) \rangle +$$

$$+ 2\langle u(t), B_2' w_1^{-1} x(t) \rangle.$$

Using a standard completion of squares,

$$\frac{dx'(t) w_1^{-1} x(t)}{dt} = -\|z(t)\|^2 + \gamma^2 \|w(t)\|^2 + \left\| u(t) + B_2' w_1^{-1} x(t) \right\|^2 - \left\| w_2' w_1^{-1} x(t) + B_2' w_1^{-1} x(t) \right\|^2 - \gamma^2 \left\| w(t) - \gamma^{-2} B_1 w_1^{-1} x(t) \right\|^2.$$

But under (4.6), we have also that $x(\infty)=0$, (Colaneri et al., 1997), and since $w \in L_2$, integrating the above equation from 0 to ∞, we get

$$\gamma^2 \|w(t)\|_2^2 - \|z(t)\|_2^2 = \left\| w_2' w_1^{-1} x(t) + B_2' w_1^{-1} x(t) \right\|_2^2 - \left\| u(t) + B_2' w_1^{-1} x(t) \right\|_2^2 + \gamma^2 \left\| w(t) - \gamma^{-2} B_1 w_1^{-1} x(t) \right\|_2^2 - x_0' w_1^{-1} x_0$$

which concludes the proof.

Theorem 4.1. Let the game formulated in Section 3. Then, there exist Nash equilibrium strategies

$(u^*(t,x), w^*(t,x)) \in \Omega$

if and only if the optimization problem

$$\min_{(W,\gamma^2) \in C_\infty} \gamma^2 \mathrm{Tr}(RW) \tag{4.7}$$

where

$R = \begin{bmatrix} C'C & 0 \\ 0 & D'D \end{bmatrix}$ and $\gamma > 0$ given,

admits one global optimal solution, which is given by $W^* = \begin{bmatrix} w_1 & w_2 \\ w_2' & w_3 \end{bmatrix}$.

Furthermore, if such solution exists, then:

i) The Nash equilibrium strategy $(u^*(t,x), w^*(t,x))$ is given by

$u^*(t,x) = w_2' w_1^{-1} x(t)$

$w^*(t,x) = \gamma^{-2} B_1 w_1^{-1} x(t)$,

ii) In the case that $u(t) = u^*(t,x)$, with $x_0 = 0$, we have

$\|T_{zw}\|_\infty \le \gamma$ for all $w \in L_2$, $\gamma > 0$ given.

Proof. ($\Leftarrow$) Problem (4.7) consists of the minimization of a linear function on the closed convex set C_∞. Then, if it has an optimal solution, it is achieved at a boundary point of C_∞, that is, for all $v \in N$,

$$v\theta_\infty(W^*,\gamma^{-2})v = 0 \tag{4.8}$$

Thus, using the special form of v, (2.15),

$$(A+B_2w_2'w_1^{-1})w_1 + w_1(A+B_2w_2'w_1^{-1})' + w_1(C+Dw_2'w_1^{-1})'(C+Dw_2'w_1^{-1})w_1 + \lambda^{-2}B_1B_1' = 0 \tag{4.9}$$

and, K= $w_2 w_1^{-1}$ is a stabilizing state feedback gain and $\|T_{zw}\|_\infty \le \gamma$, (Colaneri et al., 1997). Equation (4.9) implies also that, for all (f ,Y) such that

$$f(A+B_2Yf^{-1})' + (A+B_2Yf^{-1})f + f(C+B_2Yf^{-1})'(C+B_2Yf^{-1})f + \gamma^{-2}B_1B_1' \le 0$$

we have

$$w_1 \ge f., \text{ (Ron } et\ al.\text{, 1988)} \tag{4.10}$$

In the following we determine u^* satisfying (3.8b). From lemma 4.1, we note that the problem

$$\min_{u \in \Sigma_2} J_2(u, w^*) \tag{4.11}$$

where

$$\dot{x}(t) = (A + \gamma^{-2}B_1B_1'w_1^{-1})x(t) + B_2u(t)$$

$$z(t) = Cx(t) + Du(t)$$

$$u(t) = Kx(t)$$

$$x(0) = x_0$$

is equivalent to

$$\begin{gathered}\min_{P,K,P>0} x_0'Px_0 \\ (A + \gamma^{-2}B_1B_1'w_1^{-1} + B_2K)'P + P(A + \gamma^{-2}B_1B_1'w_1^{-1} + B_2K) + (C+DK)'(C+DK) = 0\end{gathered} \tag{4.12}$$

for which, from (4.9), (P,K), $P = w_1^{-1}$ and $K = w_2' w_1^{-1}$, is a feasible solution. Moreover, f=w_1 and Y=w_2

$$\begin{gathered}\min_{f,Y,f>0} x_0' f^{-1} x_0 \\ f(A+B_2Yf^{-1})' + (A+B_2Yf^{-1})f + f(C+DYf^{-1})'(C+DYf^{-1})f + \gamma^{-2}B_1B_1' \le 0\end{gathered} \tag{4.13}$$

Now, suppose that there exists an optimal solution, (f*,Y*), to (4.13) .Assume by contradiction that $w_1 \ge f^*$, or equivalent that $w_1^{-1} \le f^{*-1}$. Then for all x $\in \mathbb{R}^n$,

$$x' f^{*-1} x \ \ge x' w_1^{-1} x$$

In particular, for x = x_0, we have

$$x_0' f^{*-1} x_0 \ge x_0' w_1^{-1} x_0$$

which is an absurd since (f*,Y*) is an optimal solution, and we conclude that

$$w_1 \le f^*. \tag{4.14}$$

On the other hand, since (f*, Y*) is an optimal solution to (4.13),

$$f^*(A+B_2Y^*f^{*-1})' + (A+B_2Y^*f^{*-1})f^* + f^*(C+DY^*f^{*-1})'(C+DY^*f^{*-1})f^* + \gamma^{-2}B_1B_1' \le 0, \tag{4.15}$$

and, from (4.10) and (4.15), we have

$$w_1 \ge f^* \tag{4.16}$$

and hence, from (4.14) and (4.16), it follows that

$w_1 = f^*.$

Thus, (w_1, w_2) is an optimal solution to (4.13), which implies that (P,K), $P = w_1^{-1}$ and $K = w_2' w_1^{-1}$, solves the problem (4.12), and consequently $u^*(t,x) = w_2' w_1^{-1} x(t)$ is a solution to (4.11) and satisfies

$J_2(u^*,w^*) \leq J_2(u,w^*)$, for all $u(t) \in \Sigma_2$.

In the next we determine $w^*(t,x)$ that satisfies the Nash equilibrium condition (3.8a). This is equivalent to solve the problem:

$$\min_{w \in \Sigma_1} J_1(u^*,w) \tag{4.17}$$

where

$$J_1(u^*,w) = \int_0^\infty (\gamma^2 w(t)' w(t) - z(t)' z(t)) dt$$

and

$$\dot{x}(t) = (A + w_2' w_1^{-1} B_2) x(t) + B_1 w(t)$$

$$z(t) = (C + D w_2' w_1^{-1}) x(t)$$

From the hypothesis, (see (4.9)),

$$w_1^{-1} A + A' w_1^{-1} + w_1^{-1} B_2 w_2' w_1^{-1} + w_1^{-1} w_2 B_2' w_1^{-1} + C'C + w_1^{-1} w_2 w_2' w_1^{-1} + \gamma^{-2} w_1^{-1} B_1 B_1' w_1^{-1} = 0$$

From lemma 4.2, we have

$$J_1(u,w) = \gamma^2 \; w(t) - \gamma^{-2} B_1 w_1^{-1} x(t) \; {}_2^{-2} - \; u(t) + + B_2 w_1^{-1} x(t) \; {}_2^{2} + \; w_2 w_1^{-1} x(t) + + B_2 w_1^{-1} x(t) \; {}_2^{2} - x_0 \, w_1^{-1} \, x_0 \tag{4.18}$$

which, for $u = u^* = w_2' w_1^{-1} x(t)$, gives

$$J_1(u^*,w) = \gamma^2 \; w(t) - \gamma^{-2} B_1 w_1^{-1} x(t) \; {}_2^{2} - x_0 \, w_1^{-1} \, x_0 \tag{4.19}$$

that assumes its minimum value for $w^*(t,x) = \gamma^{-2} B_1 w_1^{-1} x(t)$, and so

$J_1(u^*,w^*) \leq J_1(u^*,w)$ for all $w \in \Sigma_1$.

In addition, for $x_0 = 0$, we get

$J_1(u^*,w) = \gamma^2 \; w(t) - \gamma^{-2} B_1 w_1^{-1} x(t) \; {}_2 \geq 0$

and

$$\int_0^\infty (\gamma^2 w(t)' w(t) - z(t)' z(t)) dt \geq 0$$

or

$\|T_{zw}\|_\infty \leq \gamma$, for all $w(t) \neq 0 \in L_2$, given $\gamma > 0$.

($\Rightarrow$) The equilibrium strategy

$w^* = K_1 x(t)$

$u^* = K_2 x(t)$

applied to the system (3.1) produces

$$\begin{aligned} \dot{x}(t) &= (A + B_1 K_1 + B_2 K_2) x(t) \\ z(t) &= (C + D K_2) x(t) \end{aligned} \tag{4.20}$$

Defining $f(t,0)$ as the transition matrix associated with

$\dot{x}(t) = (A + B_1 K_1 + B_2 K_2)\, x(t)$, $\quad x(0) = x_0$; $\qquad (x(t) = f(t,0) x_0)$

we get

$$J_1(u^*,w^*) = x_0'[\int_0^\infty \phi'(t,0)\,(\gamma^2 K_1'K_1 - C'C - K_2'K_2)\,\phi(t,0)\,dt\ ]x_0 \tag{4.21}$$

From the hypothesis,

$J_1(u^*,w^*) \le J_1(u^*,w)$

which, for u = u* and $x_0 = 0$, gives,

$$0 \le \int_0^\infty (\gamma^2 w'(t)w(t)\ -\ z'(t)z(t))dt \tag{4.22}$$

and so

$||T_{zw}||_\infty \le \gamma \qquad \forall\ w(t) \ne 0 \in L_2$

It follows from Theorem 2.1 that the Riccati inequality

$(A+ B_2K)f+f(A+B_2 \quad f \qquad\qquad f +\gamma^{-2}B_1B_1\le 0$

has a solution > 0. Defining Y = K *f*, K= Y f^{-1}, we have

$(A+B_2\ Y\ f^{-1}\ )\ f + f\ (A+B_2\ Y\ f^{-1} \quad f(C+D\ Y\ f^{-1} \qquad f^{-1})\ f + \gamma^{-2}B_1B_1\le 0$

and

$f+ f \quad {}_2 \quad {}_2\ f \quad f \qquad \gamma^{-2}\ B_1B_1 \quad \le\ 0,\ x \in\ R^n$

Due to the special form of *v,* the above inequality implies that

$(W,\gamma^{-2})\in C_\infty$

where the matrix *W* is of the form

$$W=\begin{bmatrix} f & Y \\ Y' & L \end{bmatrix}\ \ge\ 0, \quad f > 0$$

and $\qquad f^{-1}Y \qquad\qquad (4.23)$

Then C_∞ is not empty and the convex optimization problem:

$$\min_{(W,\gamma^{-2})\in C_\infty} \gamma^2 \mathrm{Tr}(RW)$$

has an optimal solution of the form (4.23), which concludes the proof.

5 Conclusion

This work states a connection between the game theory and the mixed H_2/H_∞ control theory, following a convex approach. We determine the necessary and sufficient conditions for the existence of a Nash equilibrium solution for the Nonzero-Sum Nash Game, as proposed in (Limebeer *et al.*, 1994), where the control law and the output disturbance are game variables. We show that these necessary and sufficient conditions are related to the existence of a global optimal solution to a convex optimization problem, in contrast with the (Limebeer *et al.*, 1994), which associates these conditions to the existence of a solution to a pair of coupled Riccati equations.

References

Barmish, B.R. (1985). Necessary and Sufficient Conditions for Quadratic Stabilizability of an Uncertain System, *Journal of Optimization Theory and Applications*, **46**, pp. 399-408.

Basar, T. (1977). On the Uniqueness of NASH Solution in linear-Quadratic Differential Games, *International Journal of Game Theory*, **5**, pp. 65-90.

Bernstein, D. S., and M. W. Haddad (1989). LQG Control with an H_∞ Performance Bound: A Riccati Equation Approach, *International Journal of Control*, **34**, pp. 293-305.

Blaquere, A., F. Gerard, and G. Leitman (1969). Quantitative and Qualitative Games, *Academic Press, New York*.

Boyd, S., L.E. Ghaoui, E. Feron, and V. Balakrishnan (1994). Linear Matrix Inequalities in System and Control Theory. *SYAM*, *Philadelphia*.

Colaneri, P., A. Locatelli and C.G. Geromel (1997). Control Theory and Design in RH_2 and RH_∞, *Academic Press, New York*.

Doyle, J. C., K. Glover, P. P. Khargonekar and B. A. Francis (1989). State-Space Solutions to Standard H_2 and H_∞ Control Problems, *IEEE Transactions on Automatic Control*, **AC-34**, pp. 831-847.

Ho, Y.C., A .E. Bryson and S. Baron (1965). Differential Games and Optimal Pursuit-Evasion Strategies, *IEEE Transactions on Automatic Control*, **10**, pp. 385-389.

Isaacs, R. (1965). Differential Games, *John Wiley and Sons, New York*.

Khargonekar, P. P., and M. A. Rotea (1991). Mixed H_2 /H_∞ Control: A Convex Optimization Approach, *IEEE Transactions on Automatic Control*, **36**, pp. 824-837.

Limebeer, D. J. N., B. Anderson and B. Hendel (1994). A NASH Game Approach to Mixed H_2/H_∞ Control, *IEEE Transactions on Automatic Control*, **39**, pp. 69-82.

Petersen, I.R (1987). A Procedure for simultaneously Stabilizing a Collection of Single Input Linear Systems Using Non-linear State Feedback Control, *Automatica*, **23**, pp. 33-40.

Ron, A.C.M, and R. Vreugdenhil (1988). Existence and Comparison Theorems for Algebraic Riccati Equations for Continuous and Discrete time Systems, *Linear Algebra and its Applications*, **99**, pp. 63-83.

Scherer, C. (1989). H_∞ -Control by State Feedback and Characterization of High-Gain Occurrence, *Systems & Control Letters,* **12**, pp. 383-391.