Optimal Adaptive Control of Uncertain Stochastic Discrete Linear Systems

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Abstract

The problem of optimal control of stochastic discrete linear time-invariant uncertain systems on finite time interval is formulated and partially solved. This optimal solution shows that previously published adaptive optimal control schemes and indirect adaptive control schemes do not need heuristics for their rationalization. It is shown that these schemes are suboptimal causal approximations of the optimal solution. The solution is achieved by the introduction of the State and Parameters Observability form - SPOF. This representation of linear time-invariant systems enables application of tools from the LQR-LQG theory of control and estimation of discrete linear time-varying systems. The optimal solution is exact and non causal. It is composed of a causal optimal estimator of the augmented state composed of the state of the system and the parameters and of a non-causal controller. The solution shows that certainty equivalence principle applies for the state and parameters, but the separation does not apply. A causal suboptimal controller, using certainty equivalence, is proposed as an ad-hoc solution. This controller needs only the knowledge of the order of the system. The scheme is bibo stable for sufficiently low noises. As an example, the proposed algorithm, is applied to an unstable nonminimum phase model of a dynamic vehicle.

1. Introduction

Optimal control is a well established theory, in general, and the optimal control of deterministic and stochastic linear known systems, the LQR-LQG theory [1], in particular. The treatment of control of uncertain systems is covered by the adaptive control theory. The main goal of the "first generation" adaptive controllers has been to maintain stability and performance in terms of the steady state tracking error. The transient, e.g., the performance on finite time interval, is not covered. Lately, the issue of optimizing the performance has emerged. The problem of optimal control of uncertain stochastic systems has been posed and solved for infinite time interval and for the case when the system's uncertainty set is finite [2]. The problem of adaptive optimal control on finite time interval has been posed and a so called "candidate adaptive controller" is presented [3]. A control of continuous deterministic uncertain system based on the state and parameters canonical form is dealt with in [4, 5]. Discussion of existing various adaptive algorithms is presented in [5] and comparison of the performance of state of the art adaptive algorithms in noisy environment on a common basis is presented in [12,21].

The objective of this paper is to formulate and partially solve the optimal control of uncertain discrete stochastic system on finite interval for the case when the system's parameters are treated as a random vector and the uncertainty set is specified by its mean and covariance. The solution is achieved by the introduction of the State and Parameters Observability Canonical form [6]. This new canonical representation of linear time-invariant systems enables application of tools from existing LQR-LQG theory of control and estimation of discrete linear time-varying systems. The solution is exact and noncausal. It is composed of the optimal estimator of the augmented state composed of the state of the system and the parameters, and of the optimal controller. The estimator is causal. The controller is

non-causal as it needs the future outputs and inputs to the plant. This shows the necessity of parameters identification for achieving the goal of optimal control of uncertain stochastic systems.
This optimal solution shows that previously published adaptive optimal control schemes[3] and indirect adaptive control schemes [7] do not need heuristics for their justification. It follows that these schemes are suboptimal causal approximations of the optimal solution. Moreover, a comparison between the presented algorithm and state of the art adaptive control algorithms on a common basis for stochastic continuous first order system as presented in [12] demonstrates the superiority of the presented algorithm over other adaptive control algorithms for the control of uncertain stochastic systems.
In this paper causal approximation of the optimal noncausal solution based on the SPOF form and the certainty equivalence principle is presented. The optimal adaptive control algorithm is presented. The asymptotic convergence properties of the algorithm are analyzed. Examples that demonstrate the performance are presented.

2. Statement of the problem

The following is the optimal control problem of uncertain stochastic discrete linear time-invariant systems. We consider the n^{th} order discrete stochastic linear time-invariant single-input single-output system

$$\begin{aligned} x(t+1) &= Ax(t) + bu(t) + w_1(t), \quad & x(t_o) = x_o, \\ y(t) &= c\,x(t) + w_2(t), \quad & t \geq t_o \end{aligned} \tag{2.1a}$$

where the input $u(t) \in L_2[t_o,t_f]$; $A \in R^{nxn}$; c^T, $b \in R^{nx1}$; $x(t) \in R^{nx1}$ is the state; and $y(t) \in R^1$ is the output; $w_1(t) \in R^{nx1}$ is the process driving noise; and $w_2(t) \in R^1$ is the output measurement noise. The noises are mutually uncorrelated, zero mean, white stochastic sequences, i.e.

$$E[w_1(t)]=0,\ E[w_2(t)]=0,\ E[w_1(t)w_1^T(\tau)]=V_1\delta_{t\tau},\ E[w_2(t)w_2(\tau)]=V_2\delta_{t\tau}$$

and $V_1 \in R^{nxn}$, $V_2 \in R^1$ are given. The initial state x_o is a random vector, with given mean $\bar{x}_o$ and covariance Q_o. We assume that the system is observable and without loss of generality that (A,b,c) is in the observer canonical form. That is

$$A = \begin{bmatrix} -a_1 & 1 & \dots & 0 & 0 \\ -a_2 & 0 & 1 & 0 & 0 \\ & & & \vdots & \vdots \\ -a_{n-1} & 0 & & 0 & 1 \\ -a_n & 0 & \cdots & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \\ b_{n-1} \end{bmatrix}, c = c_o = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{2.1b}$$

The unknown parameters are $\theta = [a_1\ a_2\ ...\ a_n\ b_1\ b_2\ ...\ b_n]^T \in R^{2n}$ and θ is ***a random vector*** with mean $\bar{\theta}$ and covariance Σ_θ. The initial conditions, the parameters and the stochastic processes are mutually uncorrelated.

The *optimal control of uncertain stochastic discrete linear time-invariant systems* is the problem of finding the functional

$$u(t) = f[\, y(\tau), t_o \le \tau \le t \,], t_o \le t \le t_f , \tag{2.2}$$

such that the criterion

$$J_c = \frac{1}{2} E\left[y(t_f)^T G y(t_f) + \sum_{t=t_o}^{t_f-1} [y(t)^T Q_c y(t) + u(t)^T R u(t)] \right] \tag{2.3}$$

is minimized subject to the difference equation constraint (2.1), for given terminal time, t_f, and such that $u(t) \in U$, where U is the admissible input set. The expectation is taken with respect to the stochastic sequences, initial conditions and the vector of parameters.

3. Simultaneous State and Parameters Observability

In this section we consider the problem of simultaneous state observability and parameter identification of single-input single-output stochastic linear time-invariant system.
In this paper simultaneous state and parameters estimation-identification is performed by the State and Parameters Observability form - SPOF. This form has been introduced for ARMA systems in [8, 9, 10] and in [5, 11, 12] for state space system model. Moreover, it is rederived here, that for stochastic systems where the parameters are random variables this estimator is the optimal state and parameters estimator. The SPOF is rederived here. The proof of optimality of this estimator is presented in [23].

3.1 The States and Parameters Observability Form

We consider the n^{th} order stochastic discrete linear time-invariant single-input single-output system in the observer canonical form (2.1). In this section we derive a time-varying canonical representation of the linear system (2.1). This canonical form is called the states and parameters observability form - SPOF. To derive it, let us choose (A_o, c_o) in the Brunovský form, namely we have

$$A_o = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & & 1 \\ 0 & 0 & & & 0 \end{bmatrix}, \quad c_o = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{3.1}$$

and (A_o, c_o) is observable. Then, there exist $h \in R^{nx1}$ such that $A = A_o + hc_o$ [13, 14]. We have $h=-[a_1\ a_2\ \dots\ a_{n-1}\ \ a_n]^T$. Further we can write

$$x(t+1) = (A_o + h_o c_o)x(t) + \delta h[\, y(t)-w_2(t)] + b_o\, u(t) + \delta b\, u(t) + w_1(t) = \tag{3.2}$$

$$= (A_o+h_oc_o)x(t)+\delta hy(t)+\delta bu(t)+b_ou(t)+w_1(t)-\delta hw_2(t)$$

where $E[h]=h_o$, $E[b]=b_o$, $\delta h=h-h_o$ and $\delta b=b-b_o$. Furthermore, we have time invariant system, so that formally we have

$$\begin{aligned} \delta h(t+1) &= \delta h(t), \quad \delta h(t_o) = h - h_o, \\ db(t+1) &= db(t), \quad \delta b(t_o) = b - b_o, \end{aligned} \tag{3.3}$$

However, in any practical system the parameters are not strictly constants. There are thermal drifts, drift of the setting points or the parameters are slowly time varying. Therefore, in this work we model this as

$$\begin{aligned} \delta h(t+1) &= \delta h(t) + w_h(t), \quad \delta h(t_o) = h - h_o, \\ db(t+1) &= db(t) + w_b(t), \quad \delta b(t_o) = b - b_o, \end{aligned} \tag{3.4}$$

where $w_h(t)$ and $w_b(t)$ are zero mean white stochastic sequences independent of x_o, $w_1(t)$, $w_2(t)$, and

$$E\left[\begin{bmatrix} w_h(t) \\ w_b(t) \end{bmatrix}\begin{bmatrix} w_h(t) \\ w_b(t) \end{bmatrix}^T\right] = \begin{bmatrix} V_{hh} & V_{hb} \\ V_{bh} & V_{bb} \end{bmatrix}\delta_{t\tau} = V_\theta \delta_{t\tau}. \tag{3.5}$$

The above means that the average system is time-invariant and the actual-true system is time-varying. In equation (3.2) we have exactly 2n parameters, in δh and δb. Equations (3.2, 3.3) are a different representation of equation (2.1). Now we can write (3.2, 3.3) in the augmented form, the *state and parameters observability form*, as

$$\begin{gathered} \begin{bmatrix} x(t+1) \\ \delta h(t+1) \\ \delta b(t+1) \end{bmatrix} = \begin{bmatrix} A_o + h_o c_o & Iy(t) & Iu(t) \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} x(t) \\ \delta h(t) \\ \delta b(t) \end{bmatrix} + \begin{bmatrix} b_o \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} w_1(t) - \delta h w_2(t) \\ w_h(t) \\ w_b(t) \end{bmatrix} \\ y(t) = \begin{bmatrix} c_o & 0 & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ \delta h(t) \\ \delta b(t) \end{bmatrix} + w_2(t), \quad t \geq t_o, \end{gathered} \tag{3.6}$$

which will be written as

$$\begin{aligned} &\mathbf{X}(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{b}u(t) + \mathbf{w}_1(t), \quad \mathbf{X}(t_o) = \mathbf{X_o}, \\ &y(t) = \mathbf{c}^T\mathbf{X}(t) + \mathbf{w}_2(t), \end{aligned} \tag{3.7a}$$

where

$$\mathbf{X}(t)=\begin{bmatrix} x(t+1) \\ \delta h(t+1) \\ \delta b(t+1) \end{bmatrix}, \mathbf{A}(t)=\begin{bmatrix} A_o+h_o c_o & Iy(t) & Iu(t) \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \mathbf{b}=\begin{bmatrix} b_o \\ 0 \\ 0 \end{bmatrix}, \mathbf{c}^T=\begin{bmatrix} c_o & 0 & 0 \end{bmatrix}. \quad (3.7b)$$

The initial state $\mathbf{X}(t_o)$ is random vector with mean $\overline{\mathbf{X}}_o$ and variance $\mathbf{Q}_O$,

$$\overline{\mathbf{X}}_o=\begin{bmatrix} x_o \\ 0 \\ 0 \end{bmatrix}, \mathbf{Q}_o=\begin{bmatrix} Q_o & 0 & 0 \\ 0 & \Sigma_{hh} & \Sigma_{hb} \\ 0 & \Sigma_{bh} & \Sigma_{bb} \end{bmatrix}, \quad (3.8a)$$

where

$$\Sigma_\theta=\begin{bmatrix} \Sigma_{hh} & \Sigma_{hb} \\ \Sigma_{bh} & \Sigma_{bb} \end{bmatrix}=\begin{bmatrix} E[(h-h_o)(h-h_o)^T] & E[(h-h_o)(b-b_o)^T] \\ E[(h-h_o)(b-b_o)^T] & E[(b-b_o)(b-b_o)^T] \end{bmatrix} \quad (3.8b)$$

The noises are

$$\mathbf{w}_1(t)+\begin{bmatrix} w_1(t)-\delta h w_2(t) \\ w_h(t) \\ w_b(t) \end{bmatrix}, \mathbf{w}_2(t)=w_2(t), \quad (3.9a)$$

$E[\mathbf{w}_1(t)]=0$, $E[\mathbf{w}_2(t)]=0$, $E[\mathbf{w}_1(t)\,\mathbf{w}_1^T(t)]=\mathbf{V}_1\,\delta_{t\tau}$,$E[\mathbf{w}_2(t)\mathbf{w}_2(\tau)]=V_2\delta_{t\tau}$,

where

$$\mathbf{V}_1=\begin{bmatrix} V_1+\Sigma_{hh}V_2 & 0 & 0 \\ 0 & V_{hh} & V_{hb} \\ 0 & V_{bh} & V_{bb} \end{bmatrix}, \quad (3.9b)$$

The stochastic processes remain uncorrelated as

$$\mathbf{w}(t)=\begin{bmatrix} \mathbf{w}_1(t) \\ \mathbf{w}_2(t) \end{bmatrix}=\begin{bmatrix} w_1(t)-\delta h w_2(t) \\ w_h(t) \\ w_b(t) \\ \mathbf{w}_2(t) \end{bmatrix}, \quad (3.10)$$

$$E\,[\mathbf{w}(t)\mathbf{w}^T(\tau)]=\mathbf{V}\,\delta_{t\tau}, \quad (3.11)$$

$$\mathbf{V}_1=\begin{bmatrix} V_1+\Sigma_{hh}V_2 & 0 & 0 & 0 \\ 0 & V_{hh} & V_{hb} & 0 \\ 0 & V_{bh} & V_{bb} & 0 \\ 0 & 0 & 0 & V_2 \end{bmatrix},, \quad (3.12)$$

Notice that, although the stochastic sequences $\mathbf{w}_1(t)$, $\mathbf{w}_2(t)$ are statistically dependent, they are uncorrelated. This is since

a) $E[(w_1(t) - \delta h(t)w_2(t))w_2(\tau)] = E[w_1(t)w_2(\tau) - \delta h(t)w_2(t)w_2(\tau)]$

$= -E[\delta h(t)w_2(t)w_2(\tau)] = -E[\delta h(t)]E[w_2(t)w_2(\tau)]=0$, since $E[\delta h] = 0$;

Similarly $\mathbf{w}_1(t)$ and the initial conditions $\mathbf{X}(t_o)$ are uncorrelated, namely

b) $E[(w_1(t) - \delta h(t)w_2(t))\delta h(t_o)^T] = E[(w_1(t)\delta h(t_o) - \delta h(t)w_2(t)\delta h(t_o)^T]$

$=E[(w_1(t)\delta h(t_o)^T]-E[\delta h(t)w_2(t)\delta h(t_o)^T]=-E[w_2(t)]E[\delta h(t)\delta h(t_o)^T]=0$, since $E[w_2(t)]=0$;

and similarly the other terms and by the assumptions is section 2 and 3.

Notice: When the uncertainty diminishes, i.e. $\delta h(t_o)=0$, $\delta h(t_o)=0$, $w_h(t)=0$, $w_b(t)=0$, the SPOF converges to the state space representation of certain stochastic system.

3.2 Simultaneous Optimal State Estimation and Parameters Identification

The SPOF enables the use of theory of observers-estimators for linear time-variant systems. The optimal estimator of the augmented state is [1 ch. 6, 15]

1) The state estimate extrapolation

$$\hat{\mathbf{X}}^{(-)}(t+1) = A(t)\hat{\mathbf{X}}^{(+)}(t) + \mathbf{b}u(t)$$

2) State estimate update

$$\hat{\mathbf{X}}^{(+)}(t+1) = A(t)\hat{\mathbf{X}}^{(-)}(t+1) + \mathbf{K}(t)[y(t+1) - \mathbf{c}\hat{\mathbf{X}}^{(-)}(t+1)], \quad \hat{\mathbf{X}}^{(+)}(t_o) = \overline{\mathbf{X}}_o,$$

where

3) The error covariance Extrapolation is

$$\mathbf{Q}^{(-)}(t) = \mathbf{A}(t-1)\mathbf{Q}^{(+)}(t-1)\mathbf{A}(t-1)^T+\mathbf{V}_1,\ \mathbf{Q}(t_o) = \mathbf{Q}_o$$

4) the Kalman Gain is

$$\mathbf{K}(t) = \mathbf{Q}^{(-)}(t)\mathbf{c}^T [\ \mathbf{c}\ \mathbf{Q}^{(-)}(t)\ \mathbf{c}^T + V_2]^{-1}.$$

5) The error Covariance update is

$$\mathbf{Q}^{(+)}(t) = [I- \mathbf{K}(t)\ \mathbf{c}]\mathbf{Q}^{(-)}(t) \qquad (3.13)$$

The proof is presented in [23].

This is an optimal estimator, by what we mean that given $\{y(\tau), u(\tau), t_o \le \tau \le t_f\}$ there is no other, input and output dependent, with a "linear" structure as described above, algorithm that derives smaller mean square error estimate of the augmented state. Notice that this is highly nonlinear algorithm. These equations are easily solved since up to time t, $\mathbf{A}(t)$, $\mathbf{b}$ and $\mathbf{c}$ are known and the iteration goes forward. The issue of how the selection of the input to the system, u(t), affects the quality of the estimation for continuous systems is dealt with in [16] and for discrete systems in [11]. For example the quality that we may look is the convergence of the estimation error, namely, asymptotic, exponential or in Lyapunov sense convergence, the magnitude of the estimation error covariance, etc. The following presents a result on the performance of the optimal estimator-identifier.

Theorem 3.1: If the input to the plant u(t) is such that ($\mathbf{A}$(t),$\mathbf{c}$) is uniformly completely observable and $\mathbf{V}_1$ is such that $(\mathbf{A}^{\mathbf{T}}(t), \mathbf{V}_1^{1/2})$ is uniformly completely controllable then the optimal observer (3.13) is exponentially stable, and the minimal mean square estimation error is $\mathbf{Q}$(t).
Proof: Direct outcome of [1, theorem 6.45].

Remark: theorem 3.1 states when the system

$$\xi(t+1) = [\mathbf{A}(t) - \mathbf{K}(t)\,\mathbf{c}\,]\,\xi(t)\,, \qquad \xi(t_o) = \xi_o, \tag{3.14}$$

is exponentially stable for any initial conditions.

Notice: When the uncertainty diminishes, i.e. $\delta h(t_o)=0$, $\delta h(t_o)=0$, $w_h(t)=0$, $w_b(t)=0$, then the SPOF based state estimation (the parameters are known) is algebraically identical to the Kalman filter of the certain stochastic system.

4. Formal solution of the optimal control of uncertain systems.

One possible approach to the solution of optimal control of uncertain systems has been presented by [2]. There the uncertainty set has been approximated by a finite set of models. Thus it was possible to derive finite dimensional causal approximation of the optimal solution. This means that the optimal solution for the case when the uncertainty set is not finite is infinite dimensional. In this section we will derive by the use of the SPOF a different solution. This solution is finite dimensional but not causal. This is the reason that we call this the formal solution, as it is not causal, i.e. it is not computable in real time and can not be applied to real time control.

4.1 Derivation of the Optimal Solution

In order to derive the optimal solution we use the SPOF (3.7). An important observation that is used in the derivation of the solution is that

$$E_{x_o w_1 w_2 \theta}[\quad] = E_\theta\{E_{x_o w_1 w_2}[\quad |\theta]\} = E_{\mathbf{x}_o \mathbf{w}_1 \mathbf{w}_2}[\quad] \tag{4.1}$$

Therefore (2.3) can be written as

$$E_\theta\left\{E_{x_o w_1 w_2}\left[x(t_f)^T c^T G c x(t_f) + \sum_{t=t_o}^{t_f-1}[x(t)^T c^T Q_c c x(t) + u(t)^T R u(t)]\right]\right\} \tag{4.2}$$

minimization of the inner expectation is the well known LQG problem and its solution is known [17] . We proceed as in [17] by introducing

$$\tilde{x}^{(+)}(t) = \hat{x}(t) - x(t) \tag{4.3}$$

then the inner expectation of (4.2) is given by as in [17]

$$E_{x_o w_1 w_2}\left[\hat{x}^{(+)}(t_f)^T c^T G c\,\hat{x}^{(+)}(t_f) + \sum_{t=t_o}^{t_f-1}[\hat{x}^{(+)}(t)^T c^T Q_c c\,\hat{x}^{(+)}(t) + u(t)^T R u(t)]\middle|\theta\right] +$$
$$+E_{x_o w_1 w_2}\left[\begin{array}{l}(x(t_f)-\hat{x}^{(+)}(t_f))^T c^T G c(x(t_f)-\hat{x}^{(+)}(t_f)) + \\ \qquad \sum_{t=t_o}^{t_f-1}[(x(t)-\hat{x}^{(+)}(t))^T c^T Q_c c(x(t)-\hat{x}^{(+)}(t)) + u(t)^T R u(t)]\middle|\theta\end{array}\right] \tag{4.4}$$

where the cross terms $E_{x_o w_1 w_2}[x^{(+)}(t)^T x^T Q_c c(x(t)-\hat{x}^{(+)}(t))|\theta] = 0$ cancel out due to the orthogonality principle. From the second term in (4.4), the estimation term, we have

$$E_\theta\left[E_{x_o w_1 w_2}\left[\begin{array}{l}(x(t_f)-\hat{x}^{(+)}(t_f))^T c^T G c(x(t_f)-\hat{x}^{(+)}(t_f)) + \\ \qquad \sum_{t=t_o}^{t_f-1}[(x(t)-\hat{x}^{(+)}(t))^T c^T Q_c c(x(t)-\hat{x}^{(+)}(t)) + u(t)^T R u(t)]\middle|\theta\end{array}\right]\right] = \tag{4.5}$$

$$E_{\mathbf{X}_o \mathbf{w}_1 \mathbf{w}_2}\left[\begin{array}{l}(\mathbf{X}(t_f)-\hat{\mathbf{X}}^{(+)}(t_f))^T \Gamma(\mathbf{X}(t_f)-\hat{\mathbf{X}}^{(+)}(t_f)) + \\ \qquad \sum_{t=t_o}^{t_f-1}[(\mathbf{X}(t)-\hat{\mathbf{X}}^{(+)}(t))^T \Theta(\mathbf{X}(t)-\hat{\mathbf{X}}^{(+)}(t)) + u(t)^T R u(t)]\end{array}\right] \tag{4.6}$$

where

$$\Gamma = \begin{bmatrix} c^T G c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Theta = \begin{bmatrix} c^T Q c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which means that this term is minimized by the Kalman Filter introduced in section 3.2. From the first term in (4.4), the control term, we have

$$E_\theta\left[E_{x_o w_1 w_2}\left[\hat{x}^{(+)}(t_f)^T c^T G c\,\hat{x}^{(+)}(t_f) + \sum_{t=t_o}^{t_f-1}[\hat{x}^{(+)}(t)^T c^T Q_c c\,\hat{x}^{(+)}(t) + u(t)^T R u(t)]\middle|\theta\right]\right] \tag{4.7}$$

the inner expectation is a LQR problem. Equation (4.7) can be rewritten as

$$E_{\mathbf{X}_o \mathbf{w}_1 \mathbf{w}_2}\left[\hat{\mathbf{X}}^{(+)}(t_f)^T \Gamma\,\hat{\mathbf{X}}^{(+)}(t_f) + \sum_{t=t_o}^{t_f-1}[\hat{\mathbf{X}}^{(+)}(t)^T \Theta\,\hat{\mathbf{X}}^{(+)}(t) + u(t)^T R u(t)]\right].. \tag{4.8}$$

Although it is possible to write an expression of the optimal controller, it is of no practical importance as it is not causal. The expression needs the future inputs and outputs $\{u(\tau), y(\tau), t \le \tau \le t_f\}$, where t is the current time. It means that it can not be realized in real time. This is the reason it is called a formal solution.

The original optimal problem of order n, in section 2, is incomputable as the parameters are unknown and we have no hint on the structure of the solution. Here we have increased the order of the problem to 3n and obtained the optimal solution. Although part of this solution is non causal it suggests a structure on the solution and points out that an observer of the augmented state is necessary to achieve the goal of optimal control of uncertain systems. We know the structure of the optimal observer of the augmented state and it is causal. Moreover, this solution shows that certainty equivalence principle holds and separation does not, the optimal controller (solution of (4.8)) depends on the outcome from the optimal observer (3.13) and the optimal observer (3.13) depends on the output of the controller (solution of (4.8)). This means that since the optimal controller can not be solved in real time, the observer that can be implemented in real time will not be optimal with respect to the control objective (2.3).

Notice: (a) for the estimation problem when the uncertainty diminishes the presented proof is identical to the proofs that result in the optimal LQG controller for certain systems.

(b) in the solution we required only the order of the observable subspace (along with the technical requirements that all stochastic processes and random variables are uncorrelated).

5. Certainty Equivalence Based Control

The purpose of this section is to apply the certainty equivalence principle as a causal, ad-hoc approximation of the optimal solution to the control of uncertain linear systems.

5.1 Certainty Equivalence Based Control on Finite Time Interval

In this section we present control algorithm based on the certainty equivalence principle as an approximation to the optimal non-causal solution of the optimal control of uncertain systems problem. This algorithm is called the ***optimal adaptive control of uncertain stochastic discrete linear systems***. In this section the algorithm is described. In the following subsection the asymptotic performance, conditions for exponential and bibo stability, are stated.

The observer-identifier is the presented in section 3.2. It derives the estimated state and parameters of the plant. We denote

$$\begin{aligned} \hat{h}(t) &= h_o + \delta\hat{h}(t), \\ \hat{b}(t) &= b_o + \delta\hat{b}(t), \quad , \\ \hat{A}(t) &= A_o + \hat{h}(t)c, \end{aligned} \tag{5.1}$$

The controller is the certainty equivalence version of the optimal controller for known plant, that is

$$\begin{aligned} u(t) &= -F_{cei}(t)\hat{x}(t) \\ F_{cei}(t) &= R^{-1}\hat{b}(t_i)^T P_{cei}(t+1)[I + \hat{b}(t_i)R^{-1}\hat{b}(t_i)^T P_{cei}(t+1)]^{-1}\hat{A}(t_i) \\ P_{cei}(t) &= \hat{A}(t_i)P_{cei}(t+1)[\hat{A}(t_i) - \hat{b}(t_i)F_{cei}(t)] + c^T Q_c c, \quad P_{cei}(t_f) = c^T G c, \end{aligned} \tag{5.2}$$

where the times t_i, i=0,1,2,3,..., are the update times at which the certainty equivalence optimal feedback, $P_{cei}(t)$, is recomputed. Equations (5.2) state that at the times t_i the gain to be used during the following period $[t_i,t_{i+1}]$, is computed with the available parameters estimates used as the true plant parameters - certainty equivalence. Formally, the update times can be the iteration times or larger. Their values do not influence the asymptotic performance, but will influence the transient-performance and the computational effort. When there is no uncertainty then (5.2) is the control for known system, where in this case there is only one iteration instant at t_o.

5.2 Asymptotic Properties of Certainty Equivalence Based Control for Deterministic Systems

When designing controls on finite time interval, stability is not necessarily an issue. However, due to testability, maintainability, safety of operation, or the system operates over long period, asymptotic stability is a necessary feature of a proposed control algorithm. In this section we analyze the asymptotic performance of certainty equivalence principle based control algorithm for deterministic systems. We show in this section, that the proposed algorithm, based on the certainty equivalence, possesses the same exponential stability properties under the same conditions, as the existing adaptive control algorithms [7], i.e., persistency of excitation of the input. In this section we analyze deterministic systems as explicit results on stability appear, in the literature, mainly, for deterministic systems. Thus we show that the proposed algorithm generalizes the existing adaptive control strategies.

I) The deterministic system is

$$\begin{aligned} x(t+1) &= Ax(t) + bu(t), \quad x(t_o) = x_o, \\ y(t) &= c\,x(t), \qquad t \geq t_o \end{aligned} \tag{5.3}$$

where A, b, c, u(t) and y(t) are as defined in section 2.

II) The states and parameters observability canonical form for deterministic system is from (3.7)

$$\begin{aligned} \mathbf{X}(t+1) &= \mathbf{A}(t)\,\mathbf{X}(t) + \mathbf{b}\,u(t), \quad \mathbf{X}(t_o) = \mathbf{X}_o, \\ y(t) &= \mathbf{c}\,\mathbf{X}(t), \qquad t \geq t_o \end{aligned} \tag{5.4}$$

where are **A**(t), **b**, **c** and **X**(t) are defined in section 3.

III) The observer-identifier is

$$\begin{aligned} \hat{\mathbf{X}}(t+1) &= \mathbf{A}(t)\,\hat{\mathbf{X}}(t) + \mathbf{b}\,u(t) + \mathbf{K}(t)[y(t+1) - \mathbf{c}\mathbf{A}(t)\,\hat{\mathbf{X}}(t) + \mathbf{c}\mathbf{b}u(t)], \quad \hat{\mathbf{X}}(t_o) = \overline{\mathbf{X}}_o, \\ \mathbf{Q}^{(-)}(t) &= \mathbf{A}(t-1)\mathbf{Q}^{(+)}(t-1)\mathbf{A}(t-1)^T + \mathbf{N}_1, \qquad \mathbf{Q}^{(+)}(t_o) = \mathbf{N}_o, \\ \mathbf{K}(t) &= \mathbf{Q}^{(-)}(t)\mathbf{c}^T[\mathbf{c}\mathbf{Q}^{(-)}(t)\mathbf{c}^T + N_2)^{-1}, \\ \mathbf{Q}^{(+)}(t) &= [I - \mathbf{K}(t)\mathbf{c}]\mathbf{Q}^{(-)}(t), \end{aligned} \tag{5.5}$$

As here we deal with the deterministic case the parameters associated to the noise and uncertainty $\mathbf{N}_o \geq 0$, $\mathbf{N}_1 \geq 0$ and $N_2 > 0$ serve as tuning parameters [6]. The estimation error is

$$\mathbf{E}(t) = \mathbf{X}(t) - \hat{\mathbf{X}}(t), \qquad \mathbf{E}(t) = \begin{bmatrix} x(t) - \hat{x}(t) \\ \delta h(t) - \delta\hat{h}(t) \\ \delta b(t) - \delta\hat{b}(t) \end{bmatrix} = \begin{bmatrix} \varepsilon(t) \\ \varepsilon h(t) \\ \varepsilon b(t) \end{bmatrix}, \tag{5.6}$$

and

$$\mathbf{E}(t) = [I - \mathbf{K}(t)\mathbf{c}]\mathbf{A}(t)\mathbf{E}(t), \qquad \mathbf{E}(t_o) = \mathbf{X}_o - \hat{\mathbf{X}}_o, \tag{5.7}$$

where $\hat{x}(t), \delta\hat{h}(t), \delta\hat{b}(t)$ are the estimates of x(t), δh(t) and δb(t) respectively, as derived by the observer (5.5).The following states a performance theorem of the deterministic observer-identifier.

Theorem 5.1: If the input to the plant u(t) is such that ($\mathbf{A}$(t),$\mathbf{c}$) is uniformly completely observable and the tuning matrix $\mathbf{N}_1$ is such that $(\mathbf{A}^\mathbf{T}(t),\mathbf{N}_1^{1/2})$ is uniformly completely controllable then the optimal observer (5.5) is exponentially stable.
Proof: Direct outcome of [1, theorem 6.45].

Remarks: 1) The pair $(\mathbf{A}^T(t),\mathbf{N}_1^{1/2})$ can be made uniformly controllable by setting $\mathbf{N}_1 = \alpha I$, $\alpha>0$. Any α will do, but its value will influence the convergence rate; 2) The condition of uniform complete observability of ($\mathbf{A}$(t),$\mathbf{c}$) is the condition that the input to the plant, u(t), is persistently exciting-PE. For further details see [6, 12].

IV) The certainty equivalence control law is given by

$$u(t) = -F(t)\hat{x}(t) + v(t) \tag{5.8}$$

where $v(t)\in R^1$ is an external input and $F(t)\in R^n$ is a certainty equivalence controller gain given by

$$F(t) = F(\hat{A},\hat{b},c) = F(A + (h_o + \delta\hat{h}(t))c_o, b_o + \delta\hat{b}(t), c)\,. \tag{5.9}$$

The last expression is valid as up to the current time the input and output are measured and therefore known functions. It follows that $\delta\hat{h}(t)$ and $\delta\hat{b}(t)$ are up to the current time known functions and therefore F(.) up to the current time is a known function of time. We do not specify the specific algorithm of computing the certainty equivalence controller gain except that it is such that

$$F(A,b,c) = F, \text{ and } A\text{-}bF \text{ is exponentially stable}, \tag{5.10}$$

and additional conditions as stated in theorem 5.2 are satisfied. Condition (5.10) states that the certainty equivalence controller F for the exact values of the parameters of the system is a stabilizing controller (exponentially stable). Before stating the main theorem, similarly to [18], we define several properties of function.

Definition 5.1: The function F: $R^s \rightarrow R^q$ is *bounded* if there exist a finite $F_{max} < \infty$, such that

$$\sup_{\xi} \| F(\xi) \| \le F_{max}, \quad \forall\, \xi\in R^s. \tag{5.11}$$

Definition 5.2: The function F: $R^s \rightarrow R^q$ has *finite incremental gain* (Lipschitz) if there exist a finite $k_{max} < \infty$, such that

$$\| F(\xi_1) - F(\xi_2) \| \le k_{max} \| \xi_1 - \xi_2 \|, \quad \forall\, \xi_1, \xi_2 \in R^s. \tag{5.12}$$

The following theorem states the main stability result.

Theorem 5.2: If

i) $F(\hat{A},\hat{b},c)$ is bounded and with finite incremental gain function of the variables $\hat{A},\hat{b},c$;

ii) A-bF(A,b,c) is exponentially stable (we denote F=F(A,b,c)) ;

iii) the external input v(t) is bounded and such that ($\mathbf{A}$(t),$\mathbf{c}$) is uniformly completely observable; and

iv) the tuning matrix $\mathbf{N}_1$ is such that $(\mathbf{A}^T(t), \mathbf{N}_1^{1/2})$ is uniformly completely controllable;

then the system (5.3) with the observer-estimator (5.5) and the certainty equivalence feedback (5.8) is bibo stable and the system transition matrix is exponentially stable .

Proof: See appendix A.

The most important outcome of theorems 5.1 and 5.2 is that the proposed certainty equivalence based control converges exponentially to the separated observer controller structure that is well known for uncertain systems.

Proposition 5.3: There almost always exists a bounded external input v(t) such that ($\mathbf{A}$(t),$\mathbf{c}$) is uniformly completely observable.

Proof: presented in [6].

Notice that in the derivation of the algorithm and the proofs we required the observability of the system and its order only. The rest of the parameters and requirements are in our control, "tuning parameters".

5.4 Asymptotic Properties for Stochastic Systems

In this section we analyze the asymptotic performance of certainty equivalence control algorithm for stochastic systems. For stochastic systems we are unable to guarantee global exponential stability. The reason is that if the noises are large then the estimation errors are large and stability can not be guaranteed. However, if the noises are sufficiently small, the signal-to-noise ratio is sufficiently large, then the performance in the presence of noises will be close to the performance without noises. This section will formalize these statements.

The stochastic system is (2.1). We use the SPOF (3.7) and the identifier is (3.13, 5.5). The performance of this identifier is stated in theorems 3.1 and 5.1.

The certainty equivalence controller is given by (5.8) and the certainty equivalence controller gain given by (5.9). The certainty equivalence controller gain is such that (5.10) is satisfied. The estimation error is (5.6) and

$$\mathbf{E}(t+1) = [I- \mathbf{K}(t)\mathbf{c}]\ \mathbf{A}(t)\mathbf{E}(t) + \mathbf{w}_1(t) - \mathbf{K}(t)\mathbf{w}_2(t), \quad \mathbf{E}(t_o) = \mathbf{E}(t) = \mathbf{X}_o - \overline{\mathbf{X}}_o.. \qquad (5.13)$$

In previous section we showed via Lyapunov analysis that the certainty equivalence based control system is globally bibo stable and the state transition matrix is globally exponentially stable. In the case of stochastic systems we have to specify what type of stability we deal with. The issue of stochastic stability of stochastic systems is dealt with in [19]. As the existence of stochastic Lyapunov function is not guaranteed we adopt a different approach. This approach will treat the first and second moments of the distribution of the variables. We will present conditions such that the mean is exponentially stable and the variance is bounded. Such results does not mean that the processes themselves are bounded but means that the power in the stochastic processes is finite.

The following theorems state the main asymptotic behavior of certainty equivalence principle based control of stochastic uncertain linear systems.

Theorem 5.4: If

i) $F(\hat{A}, \hat{b}, c)$ continuous and has bounded first derivative with respect to the variables $\hat{A}, \hat{b}, c$;

ii) A-bF(A,b,c) is exponentially stable (we denote F=F(A,b,c)) ;

iii) the external input v(t) is bounded and such that ($\mathbf{A}$(t),$\mathbf{c}$) is uniformly completely observable;

iv) the input disturbance noise $\mathbf{V}_1$ is such that $(\mathbf{A}^{\mathbf{T}}(t), \mathbf{V}_1^{1/2})$ is uniformly completely controllable; and

v) the noise levels of the input disturbance and measurement noise are sufficiently low;

then

(a) the transition matrix of the mean of the state of the system (2.1) with the estimator (3.13) and the certainty equivalence feedback (5.9, 5.10) is exponentially stable.

(b) the covariance of the state of the system (2.1) with the estimator (3.13) and the certainty equivalence feedback (5.9, 5.10) is bounded.

Proof: see appendix B.

Although theorem 5.4 states that the mean of the state is exponentially stable and the variance is bounded only if the noises are sufficiently small. The quantity that really matters is the "signal-to-noise ratio" at the input and output. This signal-to-noise ratio is directly governed by the amplitude of the external input v(t). Therefore, as the observability of the system (3.6) is not influenced by the amplitude of the input, only by its spectral support, then formally we can increase the amplitude of the external input and to achieve good signal-to-noise ratio.

6. Example

As an example we present the problem of control of uncertain system described in figure 1. This is a linear model of a launch vehicle [20, 3]. The second order transfer function $k/(s^2-\mu)$ represents the dynamics of the vehicle rigid body. In parallel with the rigid body dynamics are the first flexible mode dynamics. The output represents the vehicles attitude. None of the system parameters values are known precisely, but are known to vary during operation of the system within the following bounds: ζ=0.01, 0.01<μ<6, k=1, $2\pi 0.5<\omega<2\pi 4$.

We apply the certainty equivalence algorithm described in section 5.2, where the control law in (5.8) is computed to place the closed loop poles in s-domain at -1±j and -40±10j. The discrete control is applied with sampling interval of 100msec. The control scheme is presented in figure 2. The DC gain takes care that the overall transmission will have DC gain of 1. The pole placement algorithm does not satisfy the boundedness condition required by the theorems. The problem arises when the estimated parameters of the plant give an uncontrollable system. As the estimated plant approaches uncontrollability the gains tend to infinity. Therefore, the algorithm should bound the gains. In the presented simulations this bound is the maximal number of the computer.

Figure 3 presents the external input to the closed loop. The first 5sec have a representation of the launch transient, for the next 5sec the loop maintains constant attitude and at t=10sec a "step" with unit amplitude and time constant of 0.5sec is applied.

The true parameters of the plant (unknown to the algorithm) are μ=-0.01, ω=4¹(2Hz), k=1, and the average values are $\overline{\mu}$=0.05, $\overline{\omega}$=2π, k=1.

The example is presented for

$$w_1(t) = b\tilde{w}_1(t),$$

The specific values of the parameters are presented in table 10.1.

Table 10.1: The parameters used in the examples.

example no.	1	2	3
$\tilde{V}_1$	0	0.1	
V_2	0	0.1	
sign	+	+	-
$\mathbf{N}_1, \mathbf{V}_1$	diag(10^{-3},10^{-3},10^{-3},10^{-3},10^{-5},10^{-5},10^{-5},10^{-5},10^{-5},10^{-5},10^{-5},10^{-5})		
$\mathbf{N}_2, \mathbf{V}_2$	0.1		
$\mathbf{Q}_0$	diag(1,1,1,1,10^8,10^8,10^8,10^8,10^8,10^8,10^8,10^8)		

Example 1: This example is without noise. This is similar to the example presented for continuous plant in [3], however, this example is for discrete plant. Figure 4 presents the states of the plant. We can see the transient. This transient can be seen on the plot of the output and input on figures 5 and 6, respectively. Figure 7 presents the estimates of the parameters, and figure 8 presents the normalized parameters estimation norm.

$$\text{PEN = parameters estimation norm} = \sum_i \left[\frac{p - \hat{p}_i}{p_i}\right]^2$$

We can see the convergence rate and its quality, thus at t=10sec when the input is applied the algorithm is ready and the response of the closed loop is identical to the response of controller for perfectly known plant. Additional results on the performance without noise are presented in [21].

Example 2: This example is for the same parameters as example 1 but here noises are present at the input and the output. The results are presented in figures 9 to 13, respectively. We can see from the figures that the transient, when the noises are present , is much larger, respectively to the case without noises. Due to the noises the parameters estimation norm reduces to RMS value of about 1%.

Example 3:This example is for the same parameters as example 2 but here the sign of the input to the plant has been reversed, i.e. the average plant sign and the actual sign have unmatched signs. This example demonstrates that the presented algorithm is insensitive to the sign (neither low nor high frequency) of the plant. The results are presented in figures 14 to 18, respectively. We can see from the figures that the transient, for reversed sign, is larger respectively to the case when the average plant and the actual plant have matched signs.

We can see, in all examples, that within 2 sec the parameters converged closely to the true values and thus the controller is "ready" for the input at t=10sec. The input to the plant and the output have vigorous transient during the first seconds of operation. However, during this transient the values of the input and output are of the same order of magnitude as are during normal operation, i.e. after transient died out. Notice that the algorithm is applied at 10Hz rate while the flexible mode is at 2Hz, i.e. Nyquist rate of the plant is ~4Hz. This again demonstrates the performance of the algorithm.
A comparison between the presented algorithm and state of the art adaptive control algorithms on a common basis for stochastic continuous first order uncertain plant is presented in [12]. This comparison demonstrates the superiority of the presented algorithm over other adaptive control algorithms for the control of uncertain stochastic systems.

7. Conclusions

The problem of optimal control of stochastic discrete linear time-invariant uncertain systems on finite time interval has been formulated. By the use of the State and Parameters Canonical form the problem of optimal control of stochastic linear time-invariant uncertain systems on finite time interval is partially solved. The solution is explicit and noncausal. As causal approximation to the optimal noncausal solution, control schemes based on the certainty equivalence principle a called optimal adaptive control algorithms, are presented. The conditions for bibo stability of the algorithm had been presented and performance of the algorithm has been demonstrated by examples.

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Appendix A

Proof of theorem 5.2

The condition that $(\mathbf{A}(t),\mathbf{c})$ is uniformly completely observable and the tuning matrix $\mathbf{N}_1$ is such that $(\mathbf{A}^T(t),\mathbf{N}_1^{1/2})$ is uniformly completely controllable guarantees the exponential convergence of the estimation errors from theorem 5.1. That is, for every $\mathbf{E}(t_o)$ there exists some $\infty>M_1>0$ and $|\lambda_1|<1$ such that

$$\|E(t)\| \le M_1\lambda_1^{t-t_o}. \tag{A.1}$$

The system with the certainty equivalence feedback (5.8) is

$$x(t+1) = Ax(t) + bu(t) = (A-bF(t))\,x(t) - bF(t)\varepsilon(t) + bv(t) \tag{A.2}$$

The solution of (A.2) is given by

$$x(t) = \Phi(t,0)x(t_o) + \sum_{\tau=t_o}^{t-1} \Phi(t,\tau)[-bF(\tau)\varepsilon(\tau) + bv(\tau)] \qquad (A.3)$$

where the state transition matrix is

$$\Phi(t+1,t) = (A - b\,F(t))\,\Phi(t,t), \qquad \Phi(0,0)=I. \qquad (A.4)$$

Since $F(t) = F(\hat{A}, \hat{b}, c)$ is a bounded operator, $\varepsilon(t)$ is bounded, and $v(t)$ is bounded by assumption, the solution $x(t)$ is always bounded for every finite time $t <\infty$ (the solution can not "escape" in finite time).
The closed loop system (A-bF) is exponentially stable, therefore there exists $P>0$ and $Q_1 \geq 0$ such that

$$(A-bF)^T P(A-bF) - P = -Q_1. \qquad (A.5)$$

To investigate the stability of the solution of (A.4), we construct the Lyapunov function $V(t)=x(t)^T Px(t)$ for the autonomous part of (A.2)

$$x(t+1)= (A - b\,F(t))x(t), \qquad (A.6)$$

and we have

$$V(t+1) - V(t) = x(t)^T[(A-bF(t))^T P(A-bF(t)) - P\,]x(t). \qquad (A.7)$$

We denote $F(t) = F + F(t)$, and we have

$$V(t+1) - V(t) = x(t)^T[(A-bF)^T P(A-bF) - P -(A-bF)^T P\,F(t)- F(t)^T b^T P(A-bF) + F^T b^T Pb\,F)]\,x(t) \qquad (A.8)$$
$$= x(t)^T[-Q_1-(A-bF)^T bP\,F(t)- F(t)^T b^T P(A-bF)+ F^T b^T Pb\,F]\,x(t).$$

Now, we have,

$$F(t)= F(t) - F = F(A+\varepsilon h(t)c,b+\varepsilon b(t),c) - F(A,b,c). \qquad (A.6)$$

Since F(.) is bounded with finite incremental gain and due to (A.1), there exist $\infty>M_2,M_3>0$ and $|\lambda_2|<1$such that

$$\|\Delta F(t)\| = \|F(A+\varepsilon h(t)c, b+\varepsilon b(t), c)\| \leq M_2 \left\| \begin{matrix} \varepsilon h(t) \\ \varepsilon b(t) \end{matrix} \right\| \leq M_3 \lambda_2^{t-t_o}. \qquad (A.10)$$

and there exists $\infty>M_4$, $M_5>0$ such that

$$V(t+1) - V(t) \leq - x^T(t)\,Q_1\,x(t) + 2\,\|\,x(t)\,\|^2\,\|\,(A-bF)^T bP\,F(t)\,\| + \|\,x(t)\,\|^2\,\|\,F^T b^T Pb\,F\|$$
$$\leq - x^T(t)\,Q_1\,x(t) + \|\,x(t)\,\|^2 M_4\,\lambda_2^{t-t_o} + \|\,x(t)\,\|^2\,\lambda_2^{t-t_o}. \qquad (A.11)$$

Moreover, there exist a finite time $t_1 < \infty$, such that

$$V(t+1) - V(t) < 0, \text{ for } t > t_1. \tag{A.12}$$

This means that for $t > t_1$, $\Phi(.,.)$ is exponentially stable [5, Lemma 6.2.1], and

$$\begin{aligned}
\|x(t)\| &\le \|\Phi(t,0)\|\,\|x(t_o)\| + \left\|\sum_{\tau=t_o}^{t-1}\Phi(t,\tau)[-bF(\tau)\varepsilon(\tau)+bv(\tau)]\right\| \\
&\le \|\Phi(t,0)\|\,\|x(t_o)\| + F_{max}\left\|\sum_{\tau=t_o}^{t-1}\Phi(t,\tau)b\varepsilon(\tau)\right\| + v_{max}\left\|\sum_{\tau=t_o}^{t-1}\Phi(t,\tau)b\right\| \\
&\le \|\Phi(t,0)\|\,\|x(t_o)\| + F_{max}\left\|\sum_{\tau=t_o}^{t_1-1}\Phi(t,\tau)b\varepsilon(\tau) + \sum_{\tau=t_1}^{t-1}\Phi(t,\tau)b\varepsilon(\tau)\right\| + v_{max}\left\|\sum_{\tau=t_o}^{t_1-1}\Phi(t,\tau)b + \sum_{\tau=t_1}^{t-1}\Phi(t,\tau)b\right\|
\end{aligned} \tag{A.13}$$

$$\begin{aligned}
\|x(t)\| \le \|\Phi(t,0)\|\,\|x(t_o)\| &+ F_{max}\left\|\sum_{\tau=t_o}^{t_1-1}\Phi(t,\tau)b\varepsilon(\tau)\right\| + F_{max}\left\|\sum_{\tau=t_1}^{t-1}\Phi(t,\tau)b\varepsilon(\tau)\right\| \\
&+ v_{max}\left\|\sum_{\tau=t_o}^{t_1-1}\Phi(t,\tau)b\right\| + v_{max}\left\|\sum_{\tau=t_1}^{t-1}\Phi(t,\tau)b\right\|
\end{aligned}$$

all terms are bounded, therefore we proved the bibo stability. Q.E.D.

Appendix B

Proof of theorem 5.4: It is easy to get that

$$E[x(t+1)] = (A - bF)\, E[x(t)] - bE[\Delta F(t)\hat{x}(t)] + bv(t), \quad E[x(t_{oo})] = \bar{x}_{oo} \tag{B.1}$$

where

$$\Delta F(t) = F(t) - F \tag{B.2}$$

and where we redefined the time origin to the point, t_{oo}, where the initial conditions of the states and parameters observer-estimator faded out. Such point instant exists by assumption (iii) and lemma 5.4. From assumption (iv) the noise levels are sufficiently low and from (i) F(t) has first derivatives, therefore

$$\Delta F(t) = \varepsilon h(t)^T \frac{\partial F(t)}{\partial h} + \varepsilon b(t)^T \frac{\partial F(t)}{\partial b} + Z \tag{B.3}$$

where Z contains the high order terms so that

$$E[x(t+1)] = (A - bF)E[x(t)] - E\left[\varepsilon h(t)^T \frac{\partial F(t)}{\partial h}\hat{x}(t)\right] - E\left[\varepsilon b(t)^T \frac{\partial F(t)}{\partial b}\hat{x}(t)\right] - E\left[Z\hat{x}(t)\right] + bv(t) \tag{B.4}$$

From the orthogonality property of the Kalman filter estimates, we have

$$E\left[\varepsilon h(t)\hat{x}(t)^T\right] - E\left[\varepsilon b(t)^T \hat{x}(t)^T\right] = 0\ , \tag{B.5}$$

and for sufficiently small noises Z is small enough such that $E\left[Z\hat{x}(t)\right]$ will not influence the stability properties, so that we have

$$E[x(t+1)] = (A - bF)E[x(t)] + bv(t) - E\left[Z\hat{x}(t)\right], \quad E[x(t_{oo})] = \bar{x}_{oo}\,. \tag{B.6}$$

From here follows that the transition matrix of the mean of the plant state is exponentially stable. In other words, after the transient died out the average behavior, for sufficiently small noises, is as the behavior of the deterministic system. Q.E.D.

Proof of Theorem 5.5: As the noise levels decrease the estimation errors, **Q**(t) of (3.13) decreases accordingly. The covariance is

$$Q(t) \ = \ E[\ (x(t) - \bar{x}(t))(x(t) - \bar{x}(t))^T\ ] \tag{B.7}$$

where $\bar{x}(t) = E[x(t)]$ and

$$Q(t+1) = (A\text{-}bF)Q(t)(A\text{-}bF)^T + V_1 + W \tag{B.8}$$

where

$$\begin{aligned} W = {} & E[(Z\hat{x}(t) - E[Z\hat{x}(t)])(Z\hat{x}(t) - E[Z\hat{x}(t)])^T] + E[(Z\hat{x}(t) - E[Z\hat{x}(t)])(x(t) - \bar{x}(t))^T](A - bF)^T \\ & + (A - bF)E[(x(t) - \bar{x}(t))(Z\hat{x}(t) - E[Z\hat{x}(t)])^T] \end{aligned} \tag{B.9}$$

where Z is defined in (B.3) and W = 0 for known systems. The magnitude of W is governed by the "magnitude" of Z, which depends on the high order statistics of the stochastic sequences. For sufficiently small noises W will be small enough and will be neither influence the stability of (B.8) nor the boundedness of Q(t). Q.E.D.

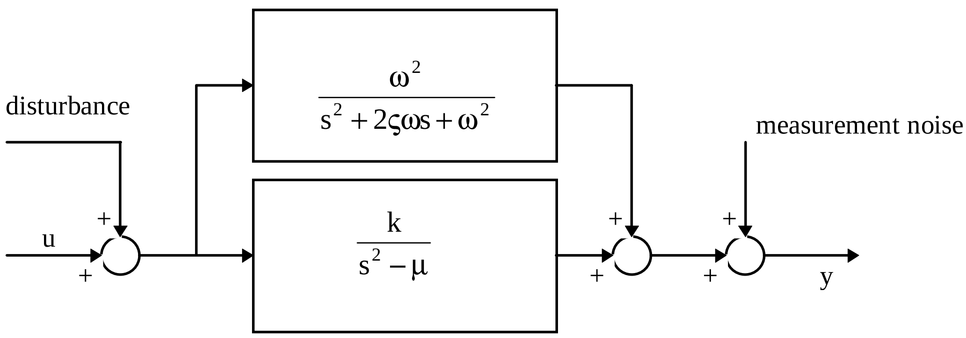

Figure 1: Block diagram of the dynamic vehicle model.

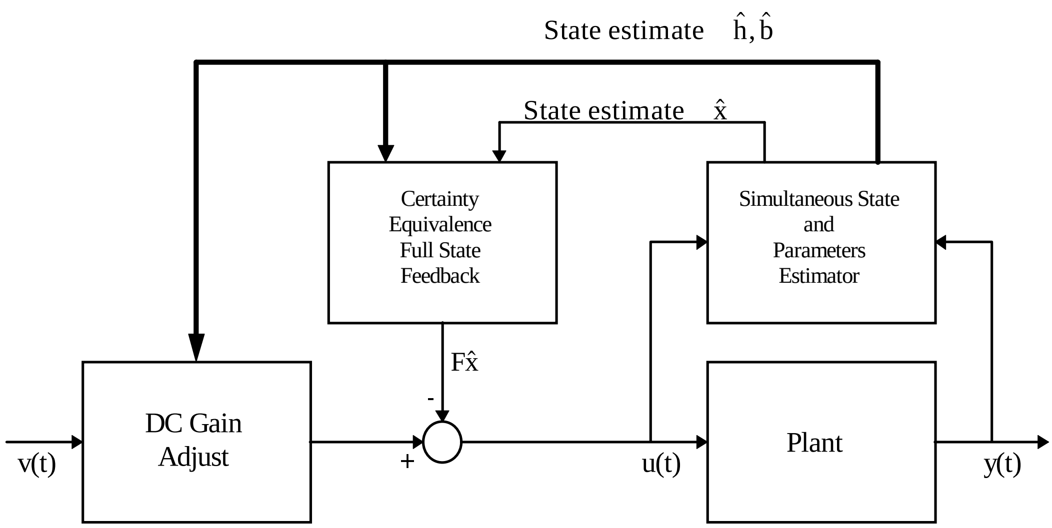

Figure 2: The bloc\k diagram of the controller for linear uncertain system.

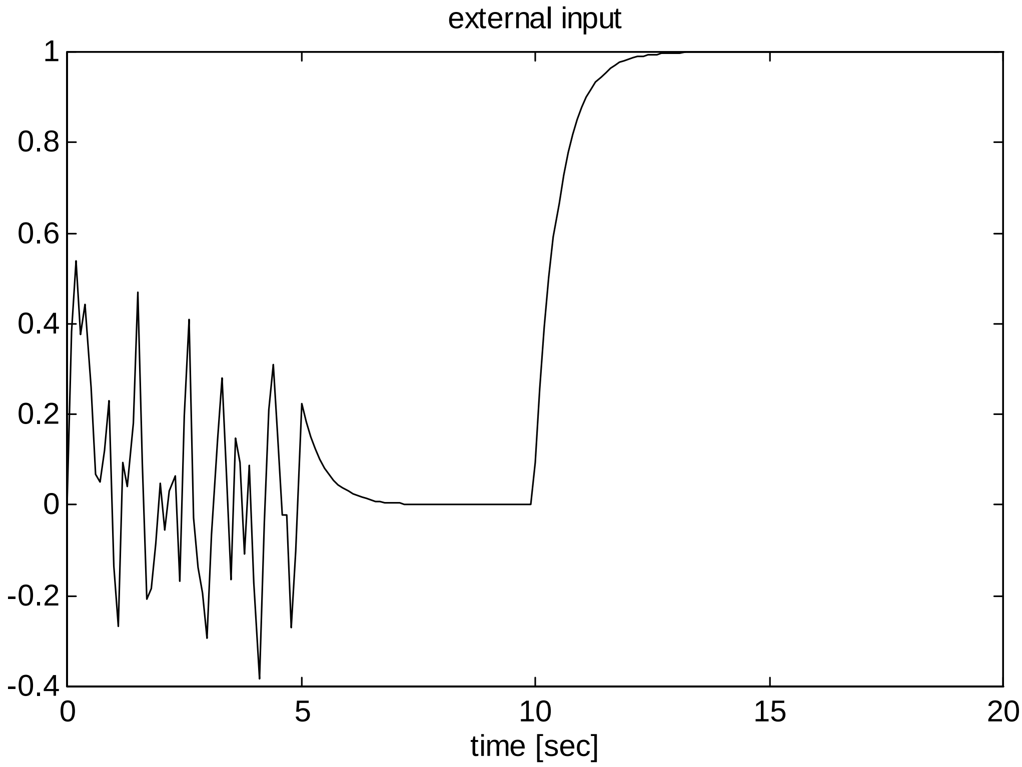

Figure 3: The external input from the moment of launch.

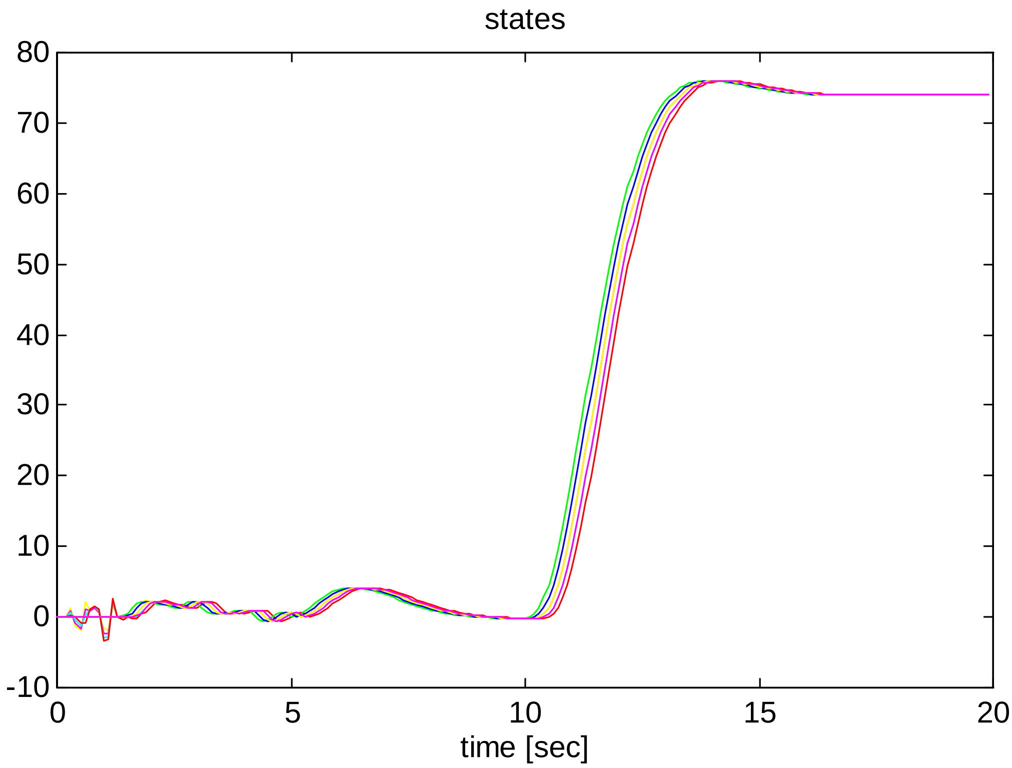

Figure 4: The states and the estimated states of the plant - example 1.

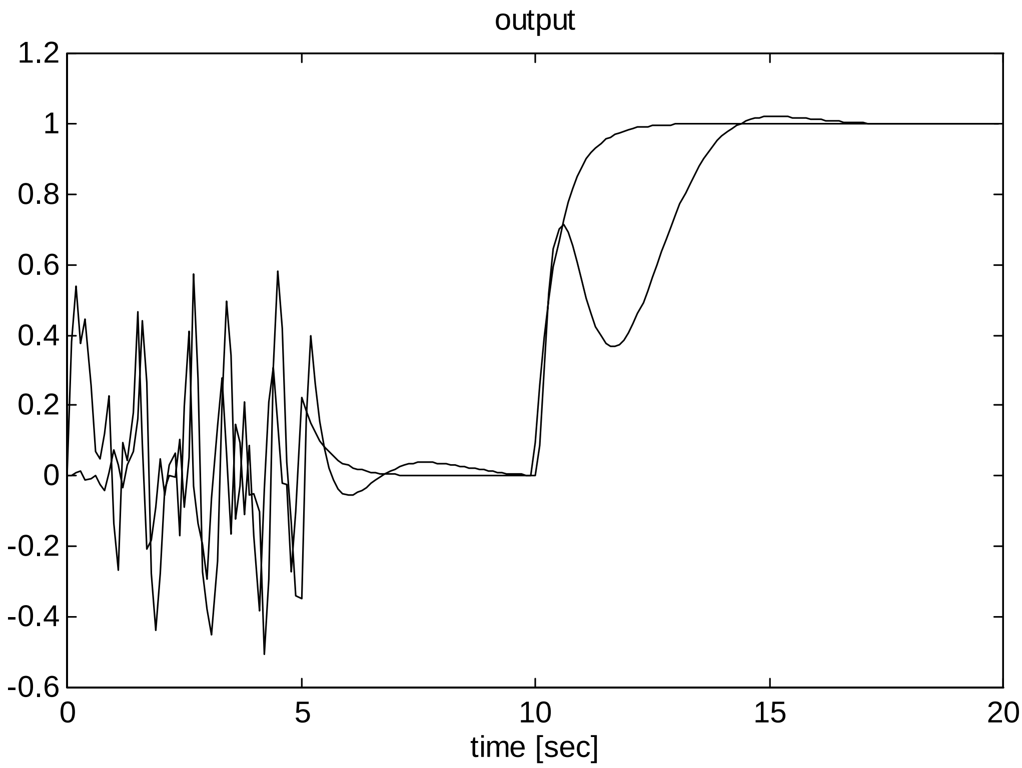

Figure 5: The external input and the output of the plant - example 1.

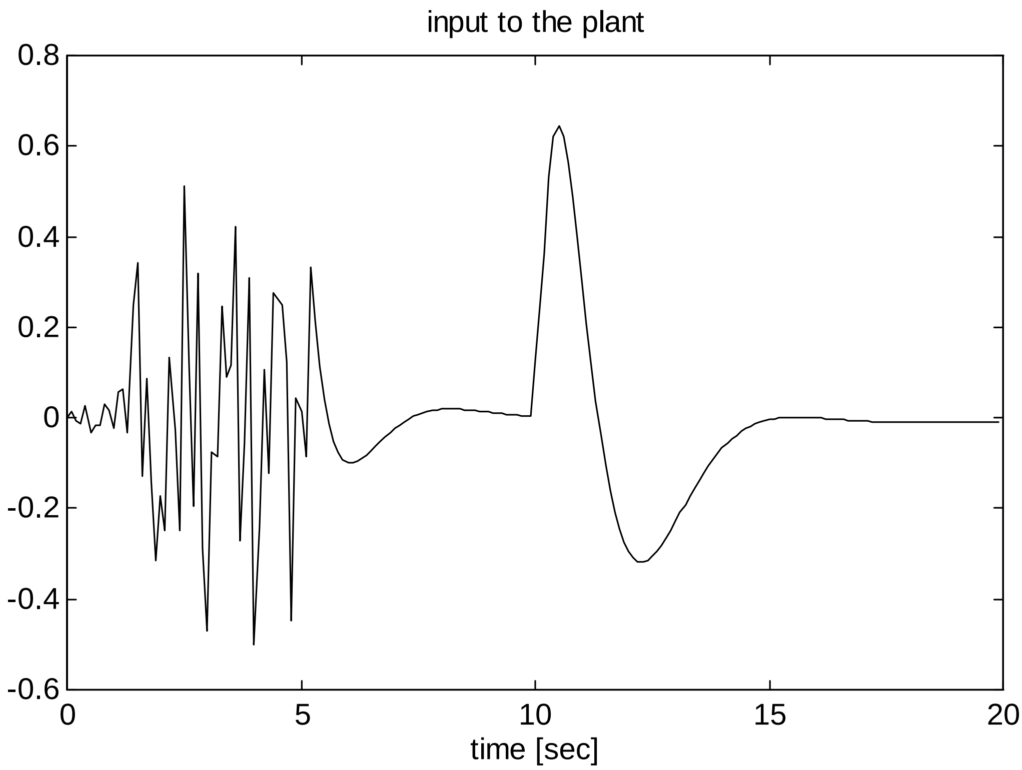

Figure 6: The input to the plant - example 1.

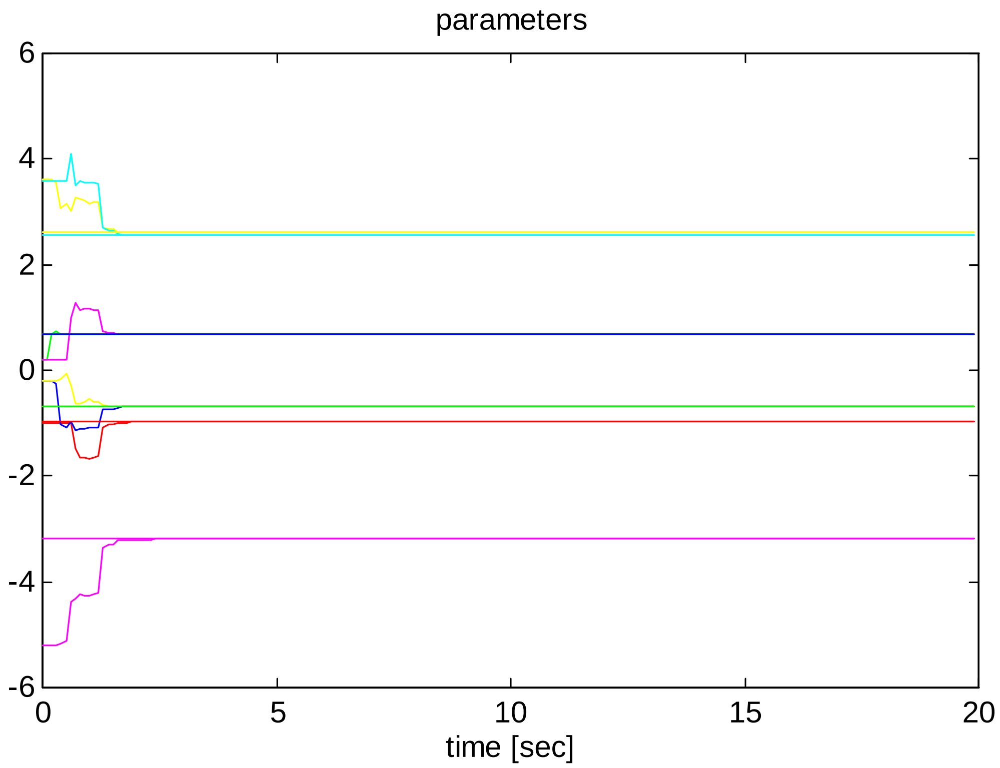

Figure 7:The parameters of the plant and their estimates - example 1.

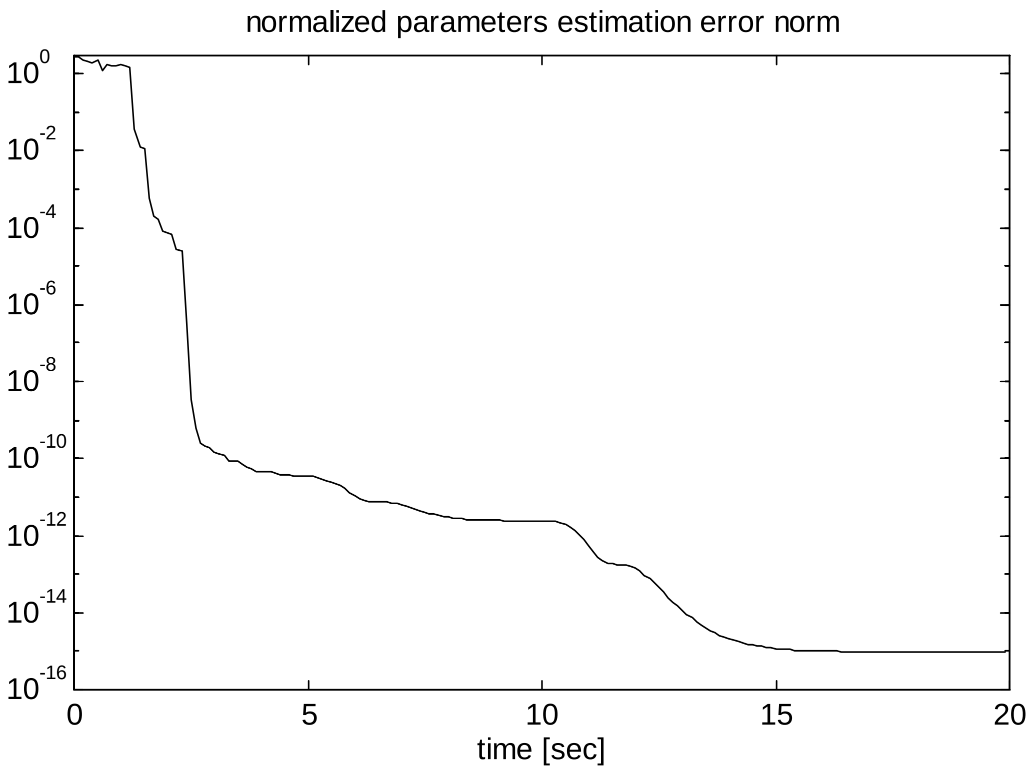

Figure 8: The normalized parameters estimation norm - example 1.

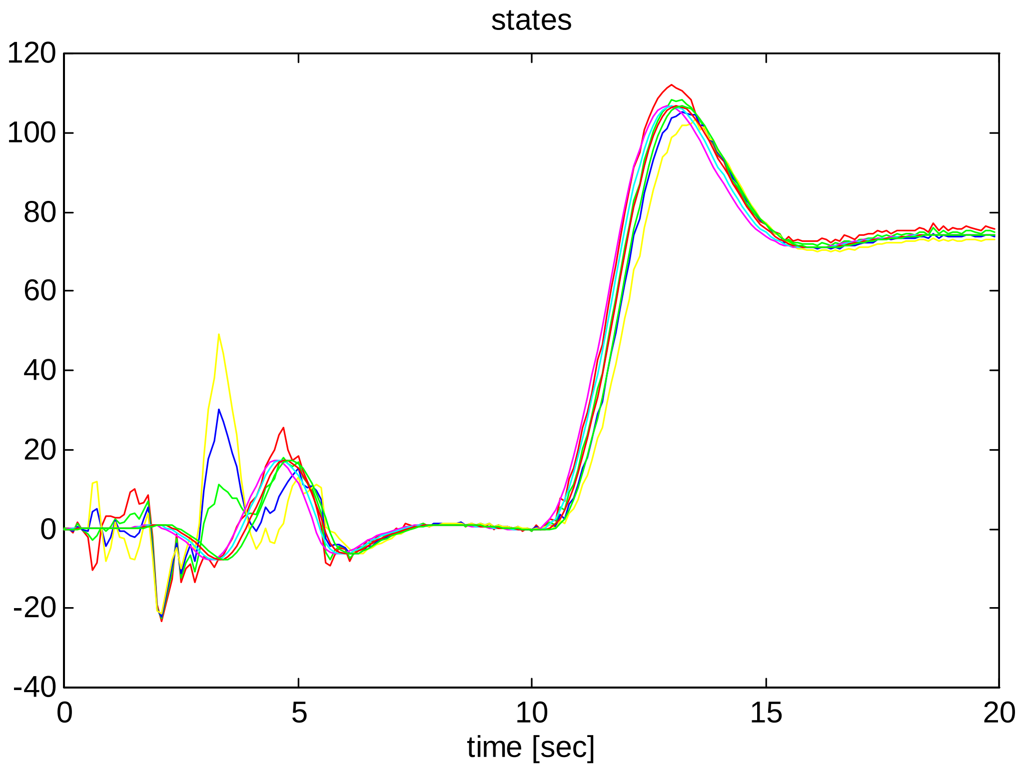

Figure 9: The states and the estimated states of the plant - example 2.

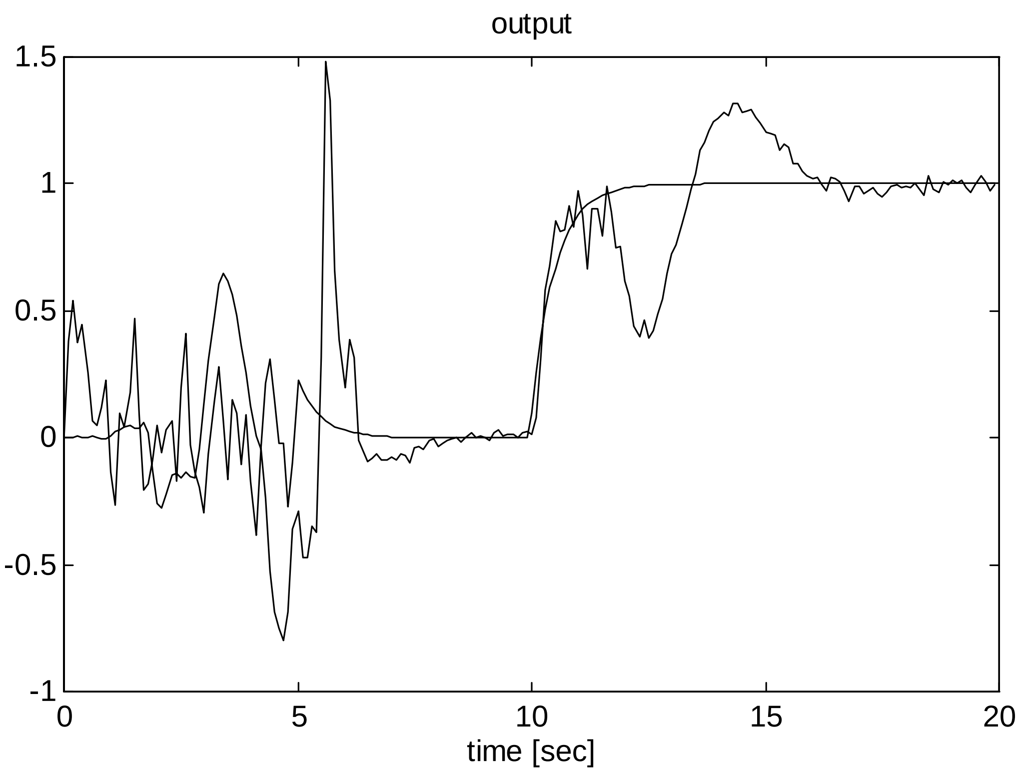

Figure 10: The external input and the output of the plant - example 2.

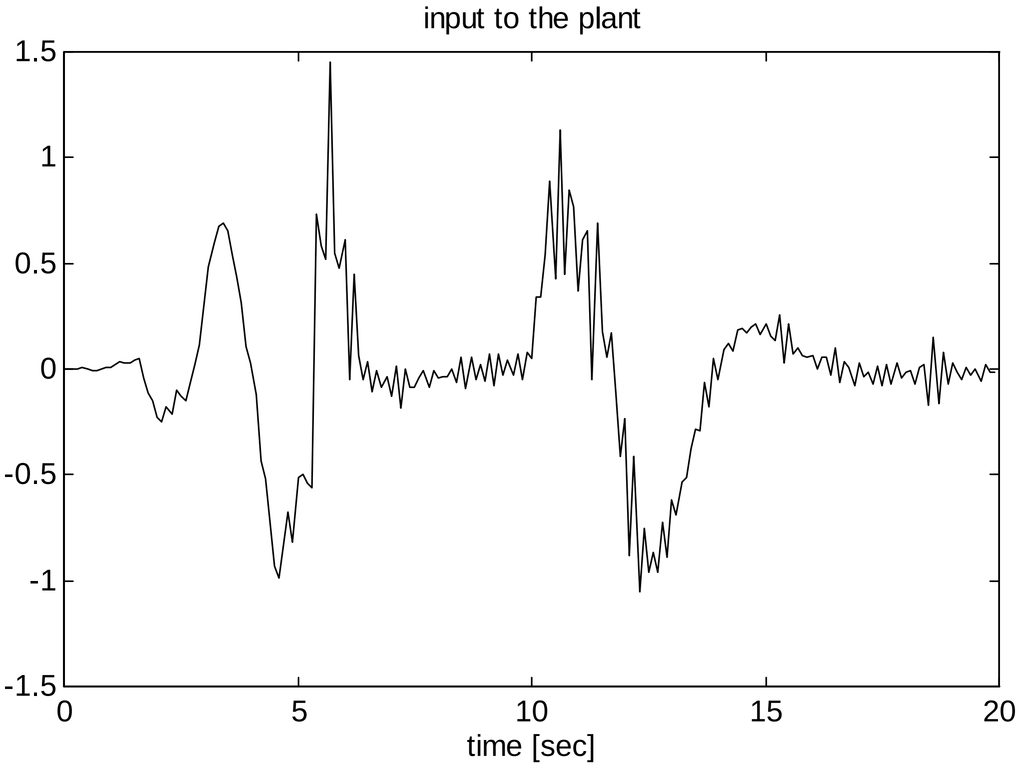

Figure 11: The input to the plant - example 2.

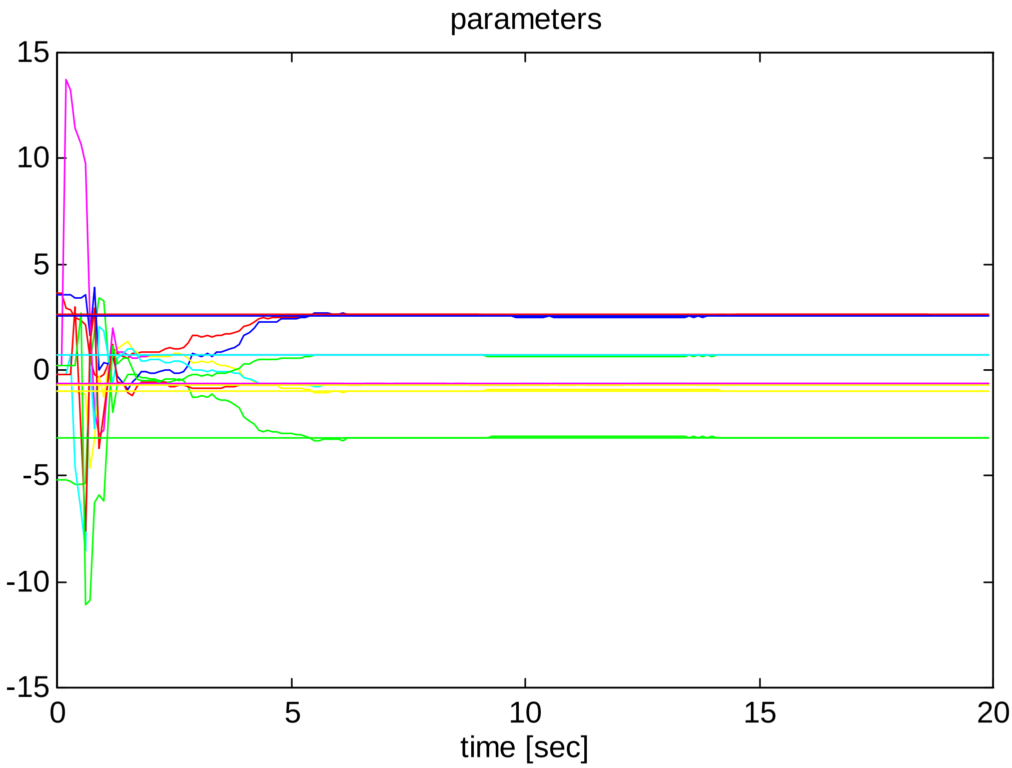

Figure 12:The parameters of the plant and their estimates - example 2.

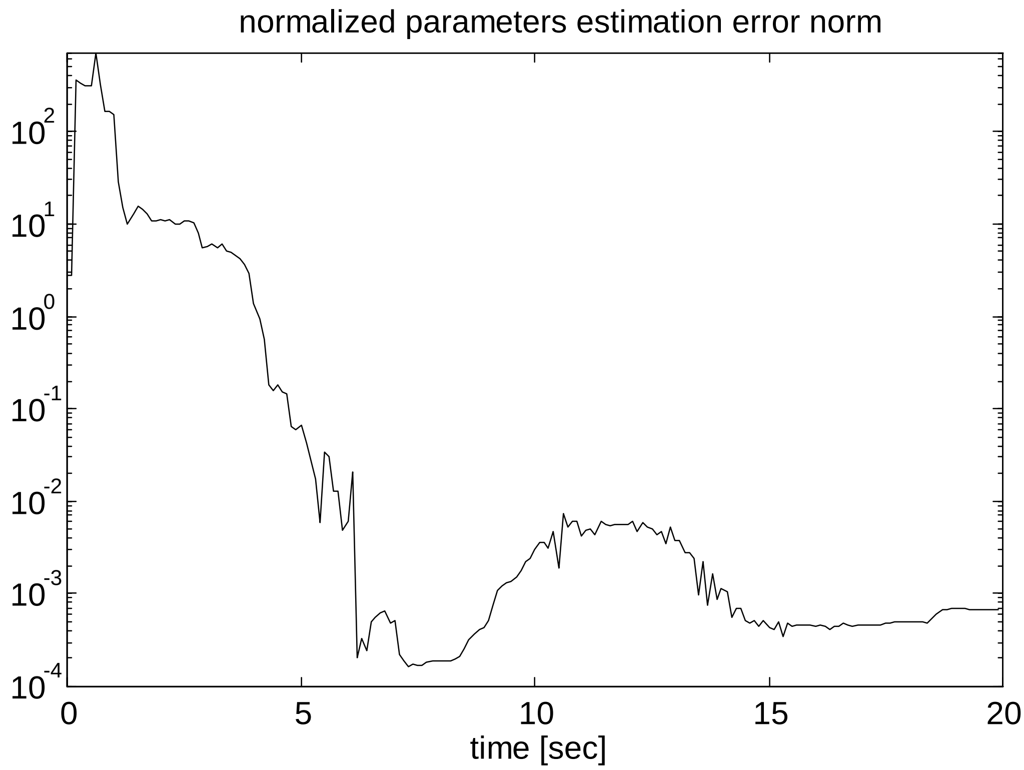

Figure 13: The normalized parameters estimation norm - example 2.

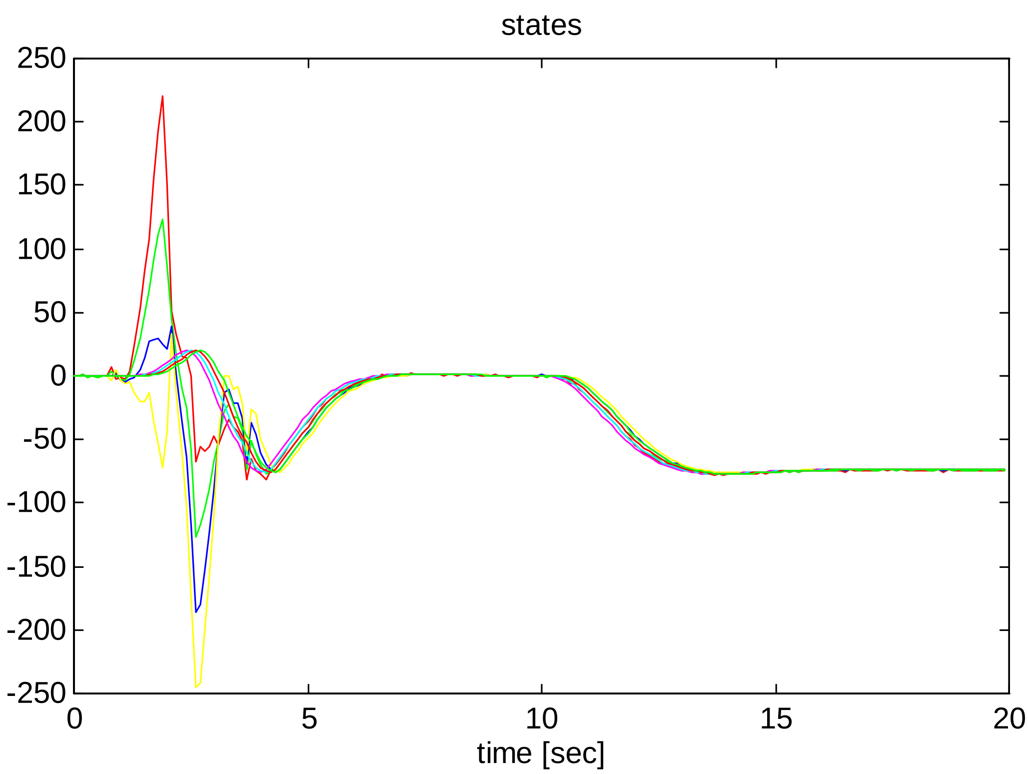

Figure 14: The states and the estimated states of the plant - example 3.

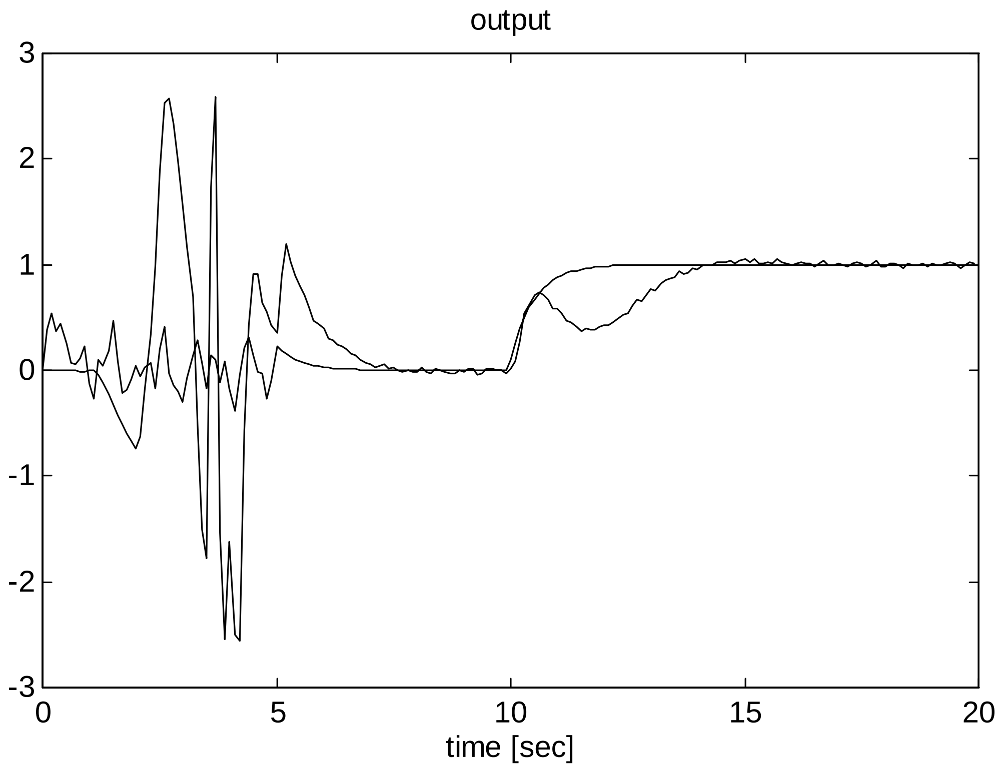

Figure 15: The external input and the output of the plant - example 3.

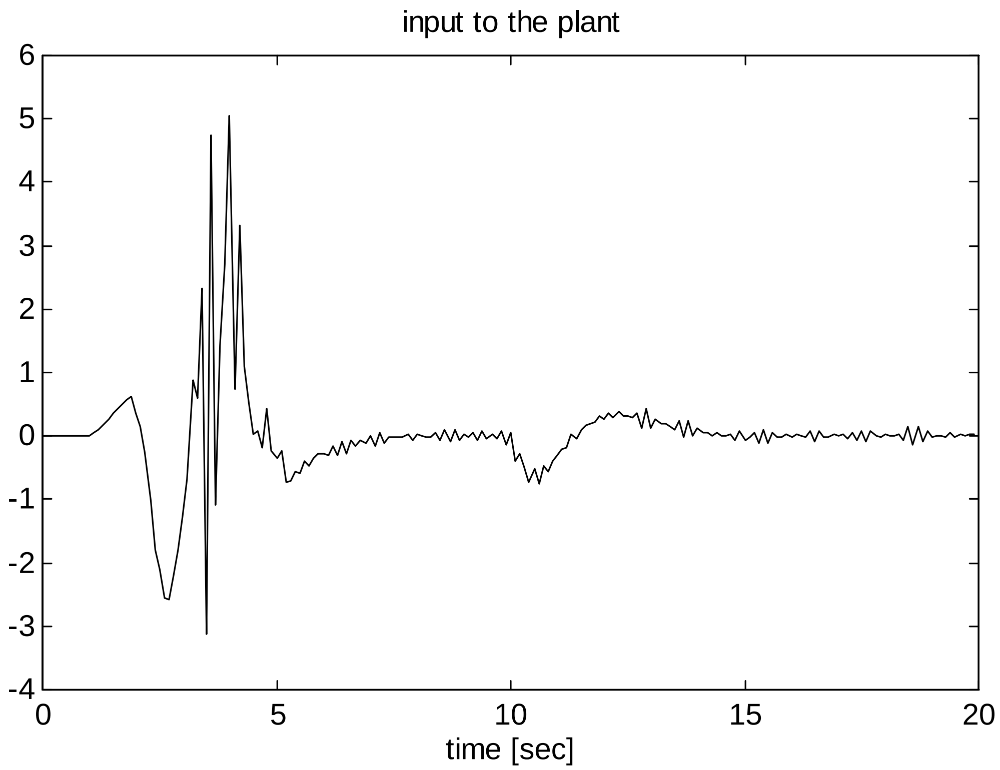

Figure 16: The input to the plant - example 3.

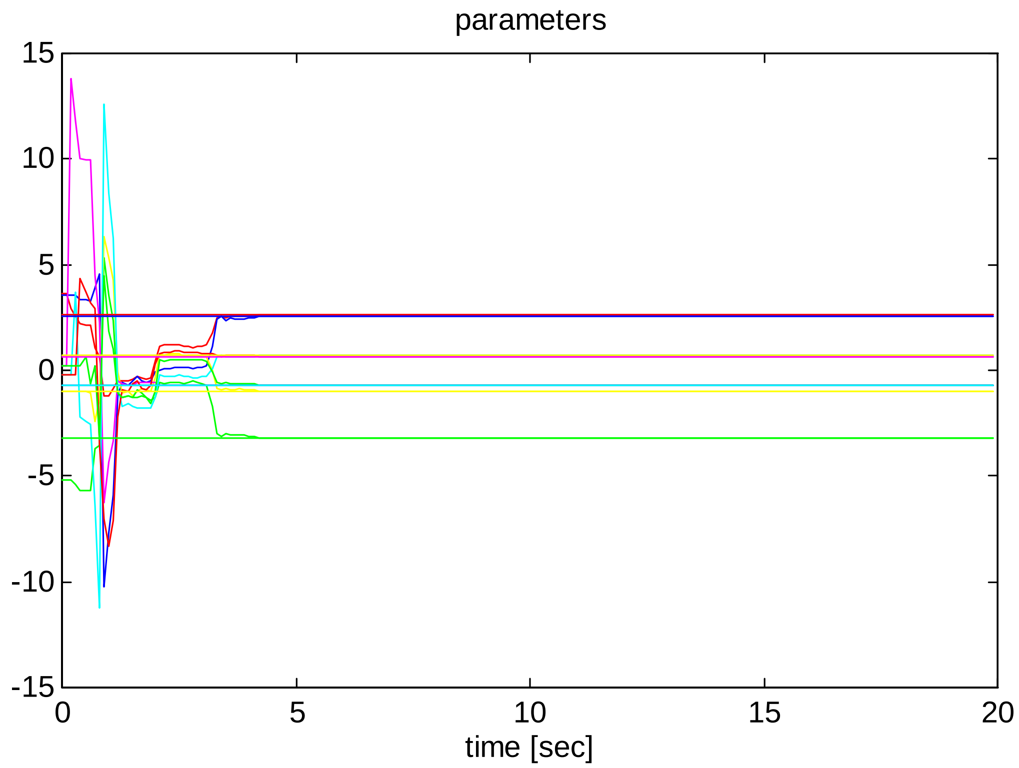

Figure 17:The parameters of the plant and their estimates - example 3.

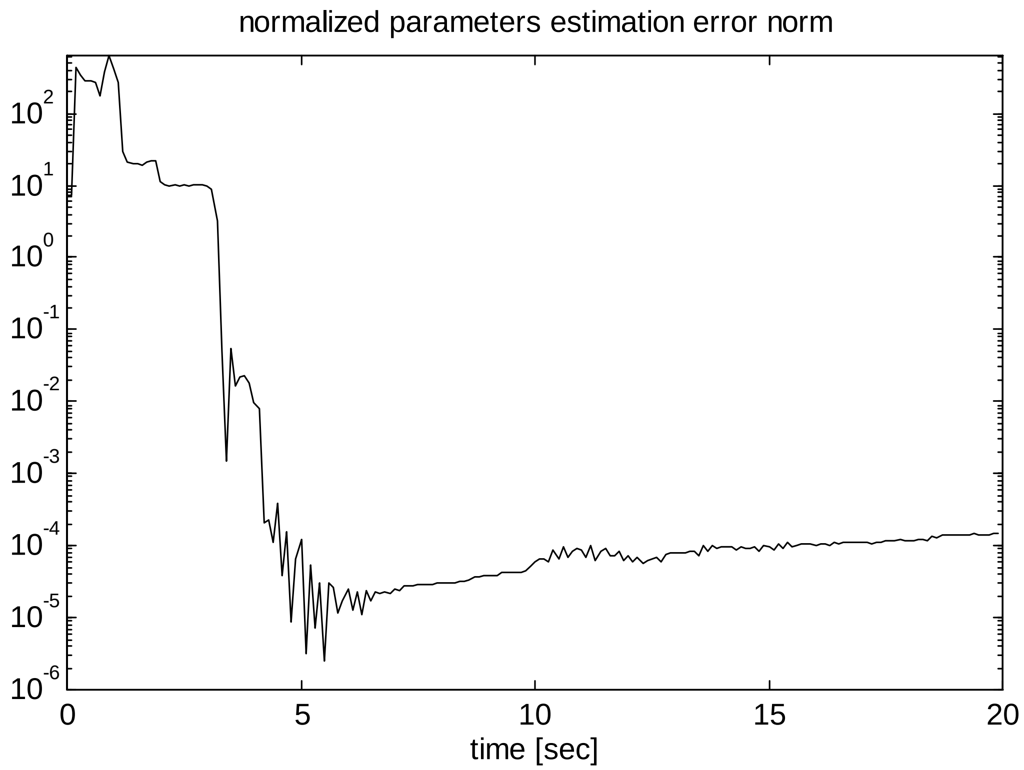

Figure 18: The normalized parameters estimation norm - example 3.