# Optimal Guidance Laws with Uncertain Time-of-Flight

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## Abstract

The existing optimal guidance laws assume that the time-to-go is known exactly. The time-to-go is usually estimated and thus is a random variable. This paper deals with the issue of optimal guidance with uncertain time-to-go. A problem of control of linear discrete systems with unknown time-to-go is formulated and solved. The solution is applied to derive guidance laws. The solution depends on the probability density function of the time-of-flight. This guidance law has the structure of a rendezvous guidance law where the guidance gains are time-dependent and depend on the distribution of the time-to-go. Examples that demonstrate these dependencies are presented.

## 1. Introduction

The Proportional Navigation guidance law is a well know guidance law that can be derived from the optimal control theory [1]. It is optimal under certain assumption such as constant closing velocity, the intercept is close to collision course, and more. More advanced guidance laws depend on more detailed models of the target and missile and the scenario [5,6], such as their time constant, maneuver capability, etc.. One of the assumptions is that the terminal time is known. In practice the terminal time is not known exactly and is estimated [7]. That is, the terminal time is a random variable. It has been shown [2] that optimal guidance laws are sensitive to the value of the terminal time. In [3,4] an approach to desensitize the performance with respect to the terminal time is outlined by formulating the guidance problem as an optimal rendezvous problem.

The problem of optimal control of linear continuous systems with unknown final time, treated as a random variable, is formulated and solved in [8].

In this paper an approach to design of guidance laws in the case of time-of-flight uncertainty is presented. This replaces heuristic and ad-hoc approaches. The presented formulation inherently assumes that the time-to-go (the terminal time) is a random variable with known probability density function. First, a problem of optimal control of linear discrete systems with unknown terminal time is posed and solved. Then the solution is applied to derive guidance laws.

As an example, a scenario, that for known time-to-go gives the Proportional Navigation guidance law, is used to derive a guidance law for uncertain time-to-go. This guidance law has the structure of a rendezvous guidance law where the guidance gains are time dependent and depend on the distribution of the time-to-go. Explicit examples that demonstrate the dependency of the guidance gains on time and the distribution final time distribution are presented.

## 2. Statement of The Problem

The following is the problem of terminal guidance with uncertain terminal time. We consider the n-th order linear discrete time-invariant system

$$x(i+1) = A\,x(i) + b\,u(i), \quad x(0) = x_o, \tag{1}$$

where $x(i) \in R^n$ is the state, $u(i) \in R^1$ is the input, $A \in R^{nxn}$ and $b \in R^n$.
Optimal guidance laws are derived by posing the following linear quadratic regulator problem on finite time: given the final time N find the function u(i), i=0,...,N, such that the cost function

$$C_N = \tfrac{1}{2}[x(N)^T G(N)x(N) + \sum_{i=0}^{N-1} u(i)^T u(i)] \tag{2}$$

where $G \in R^{nxn}$ is a non-negative define matrix, is minimized subject to (1).
The solution of the above problem is well known. Here we consider the case then the final time N is a random variable with known distribution. The probability density function is given by

$$p_N(i),\ i = 0,1,2,...,\infty, p_N(0),\quad \sum_{i=0}^{\infty} p_N(i) = 1.. \tag{3}$$

and we have

$$\text{Probability}\{N \le i\} = \sum_{i=0}^{N} p_N(i). \tag{4}$$

The problem being posed here is to find the function u(i), i=0,1,2,...∞, such that the cost criterion

$$C = E_N[C_N\} = \tfrac{1}{2}E_N[\ x(N)^T G(N)x(N) + \sum_{i=0}^{N-1} u(i)^T u(i)] \tag{5}$$

is minimized subject to the dynamic constraint (1), where $E_N[\ ]$ denotes the expectation operator with respect to the random variable N.

## 3. The General Solution

Equation (5) can be written

$$\begin{aligned} C &= \tfrac{1}{2}\sum_{n=1}^{\infty}[\ x(n)^T G(n)x(n) + \sum_{i=0}^{n-1} u(i)^T u(i)]\ p_N(n) \\ &= \tfrac{1}{2}\sum_{n=1}^{\infty} x(n)^T G(n)x(n)\ p_N(n) + \tfrac{1}{2}\sum_{n=1}^{\infty}[\sum_{i=0}^{n-1} u(i)^T u(i)]\ p_N(n) \end{aligned} \tag{6}$$

by the change of summation

$$\sum_{n=1}^{\infty}\ \sum_{i=0}^{n-1} = \sum_{i=0}^{\infty}\ \sum_{n=i}^{n-1} \tag{7}$$

we get

$$C = \tfrac{1}{2}\sum_{n=1}^{\infty} x(n)^T [G(n)\, p_N(n)]\, x(n)) \;\; + \tfrac{1}{2}\sum_{i=0}^{\infty} u(i)^T [\sum_{n=i}^{n-1} p_N(n)]\, u(i)] \tag{8}$$

and we have

$$C = \tfrac{1}{2}\sum_{i=1}^{\infty} [\, x(i)^T Q(i)\, x(i) \;\; + \sum_{i=0}^{\infty} u(i)^T R(i) u(i)] \tag{9}$$

where

$$C = Q(i) p_N(i), \;\; R(i) = \sum_{n=i}^{\infty} p_N(n)^T = \text{Probability}\{i \le N\}. \tag{10}$$

Thus the stochastic criterion (5) has been transformed into the deterministic criterion (10). However, the weights depend on time. The minimization of (10) subject to (1) is a standard Deterministic Linear Quadratic Regulator problem. The solution is as follows

$$\begin{aligned} &u(i) = K(i)\, x(i) \\ &K(i) = R(i)^{-1} b^T P(i+1)\, [I+bR(i)^{-1} b^T P(i+1)]^{-1} A \\ &P(i) = Q(i) + A^T P(i+1)\, [I+bR(i)^{-1} b^T P(i+1)]^{-1} A, \; P(\infty) = 0. \end{aligned} \tag{11}$$

## 4. Derivation of Guidance Law

To derive a guidance law we assume the following continuous system:

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{12}$$

$$G(t_f) = \begin{bmatrix} g(t_f) & 0 \\ 0 & 0 \end{bmatrix} \tag{13}$$

It is the same model that is used to derive the Proportional Navigation law in [1] (see there for figure of the geometry). In derivation of the guidance law we discretize (12) and have

$$A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} \tfrac{1}{2} T^2 \\ T \end{bmatrix}, \tag{14}$$

where T is the sampling period.
In order to arrive at a guidance law of a form that is similar to the Proportional Navigation we write the solution as (for simplicity it is derived in the continuous time domain)

$$\begin{aligned} u(t) &= k_x x + k_{\dot{x}} \dot{x} = k'_x \frac{x}{(t_f - t)^2} + k'_{\dot{x}} \frac{\dot{x}}{t_f - t} \\ &= k'_x [\frac{x}{(t_f - t)^2} + \frac{\dot{x}}{t_f - t}] + (k'_x - k'_{\dot{x}}) \frac{x}{(t_f - t)^2} \end{aligned} \tag{15}$$

where $t_f$ is defined as

$$N_f = \arg\max \{x \mid p_N(x)>0\},\ t_f = N_f\, T. \tag{16}$$

We recall that the inertial line-of-sight angle and the inertial line-of-sight angular rate are

$$\sigma = \frac{x}{V_c(t_f - t)}, \quad \dot{\sigma} = \frac{x + \dot{x}(t_f - t)}{V_c(t_f - t)^2} \tag{17}$$

where $V_C$ is the closing velocity, thus

$$u(t) = k'_x V_c \dot{\sigma} + (k'_x - k'_{\dot{x}}) V_c \frac{\sigma}{t_f - t} = N' V_c [\dot{\sigma} + k_\sigma \frac{\sigma}{t_f - t}] \tag{18}$$

where N' is the effective Navigation ratio. The effective navigation ratio, N', and the gain $k_\sigma$, are time dependent. We see that we arrived at an augmented form of the Proportional Navigation guidance law. The augmentation term is directing the missile so that the intercept will be along the line-of-sight. Notice that the guidance law has the structure of optimal rendezvous guidance law that is [1]:

$$u(t) = 4V_c[\dot{\sigma} + \frac{1}{2}\frac{\sigma}{t_f - t}] \tag{19}$$

This explains the results in [3, 4] that "rendezvous" guidance laws are less sensitive to time-of-flight.

## 5. Examples

Here we present some examples of the time dependency of the guidance gains on the cost matrix and the time-of-flight distribution. In the following examples we assume T=0.1sec,

(1) for no uncertainty, i.e. $p_N(N_f)=1$, $g(N_f)=1e5$, we arrive at the well known Proportional Navigation Law, that is N' = 3 ( asymptotically) and $k_\sigma=0$. This can be seen on figure 1;

(2) time-of-flight uncertainty, $p_N(N_f)=0.5$, $p_N(N_f-20)=0.5$, $g(N_f)=1e3$, $g(N_f-20)=1e3$. Results are presented in figure 2.

(3) time-of-flight uncertainty, $p_N(N_f)=0.5$,$p_N(N_f-20)=0.5$ $g(N_f)=1e3$, $g(N_f-20)=1e1$. Results are presented in figure 3.

(4) time-of-flight uncertainty, $p_N(i)$-uniform on the last second, g=1e3. Results are presented in figure 4.

(5) time-of-flight uncertainty, $p_N(i)$-uniform on the last second, $g(N_f)=1e5$, $g(N_f-9)=1e3$. Results are presented in figure 5.

## 6. Conclusions

Optimal guidance laws with uncertain time-to-go are derived. The solution depends on the probability density function of the time-of-flight. This guidance law has the structure of a rendezvous guidance law where the guidance gains are time-dependent and depend on the distribution of the time-to-go. This explains previous the results that optimal "rendezvous" guidance laws are less sensitive to time-of-flight.

## References

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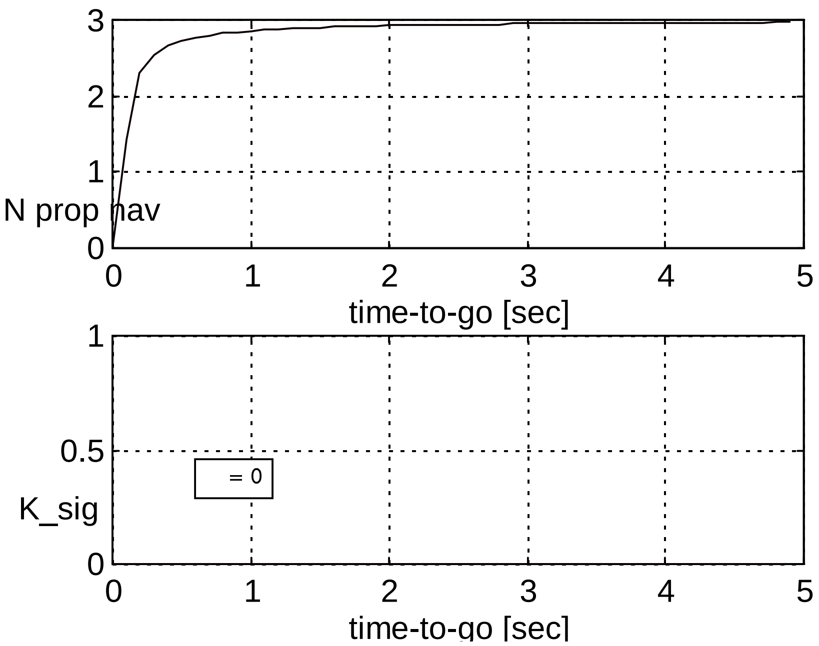


Figure 1: The effective Proportional Navigation gain, for no uncertainty in time-of-flight, $p_N(N_f)=1$, $g(N_f)=1e5$ (notice N' = 3 ( asymptotically) and $k_\sigma=0$).

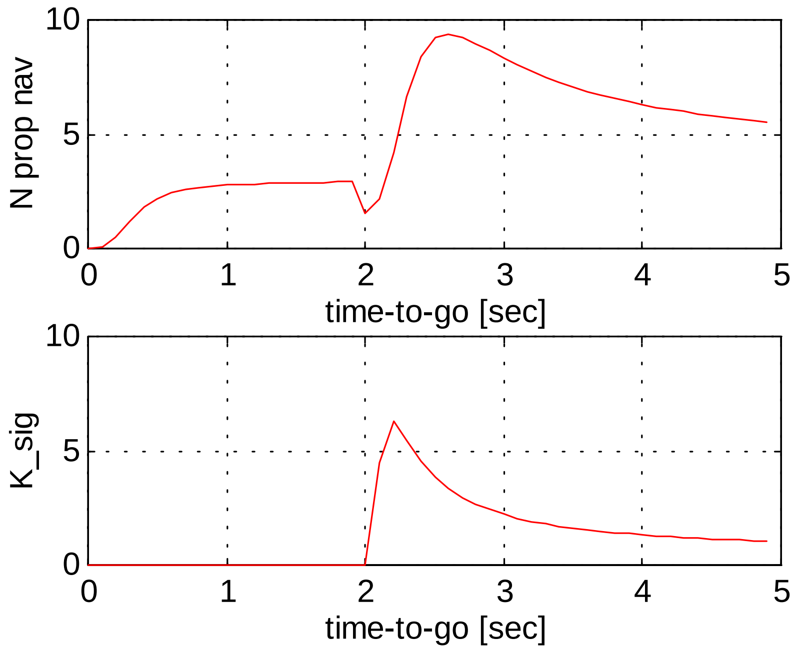


Figure 2: The effective Proportional Navigation gain, for uncertainty in time-of-flight, $p_N(N_f)=0.5, p_N(N_f-20)=0.5$ $g(N_f)=1e3$, $g(N_f-20)=1e3$.

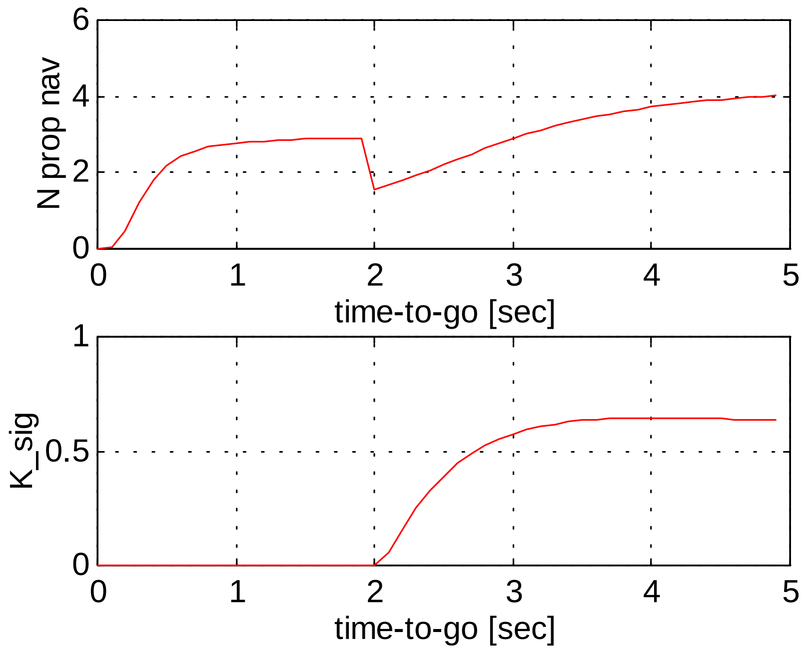


Figure 3: The effective Proportional Navigation gain, for uncertainty in time-of-flight, $p_N(N_f)=0.5, p_N(N_f-20)=0.5$ $g(N_f)=1e3$, $g(N_f-20)=1e1$.

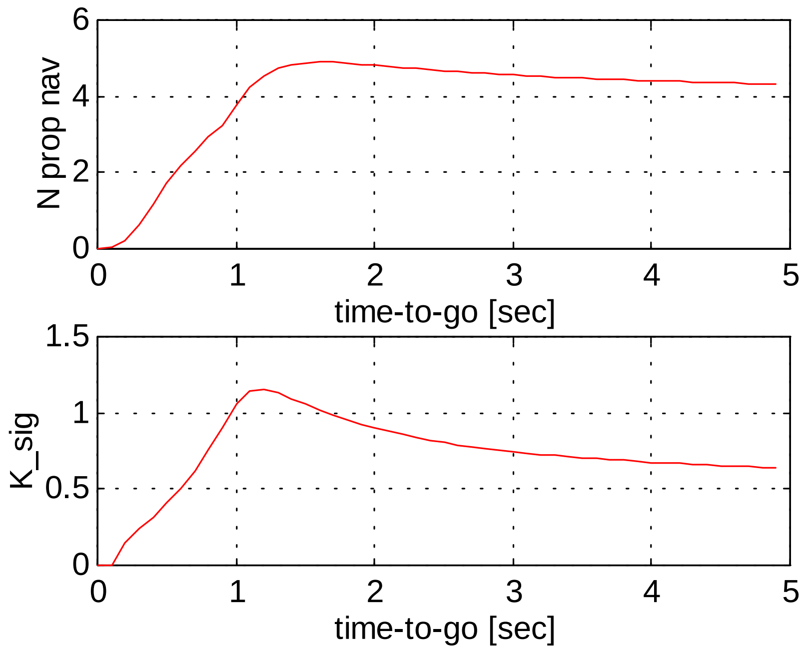


Figure 4: The effective Proportional Navigation gain, for uncertainty in time-of-flight, $p_N(i)$-uniform on the last second, g=1e3.

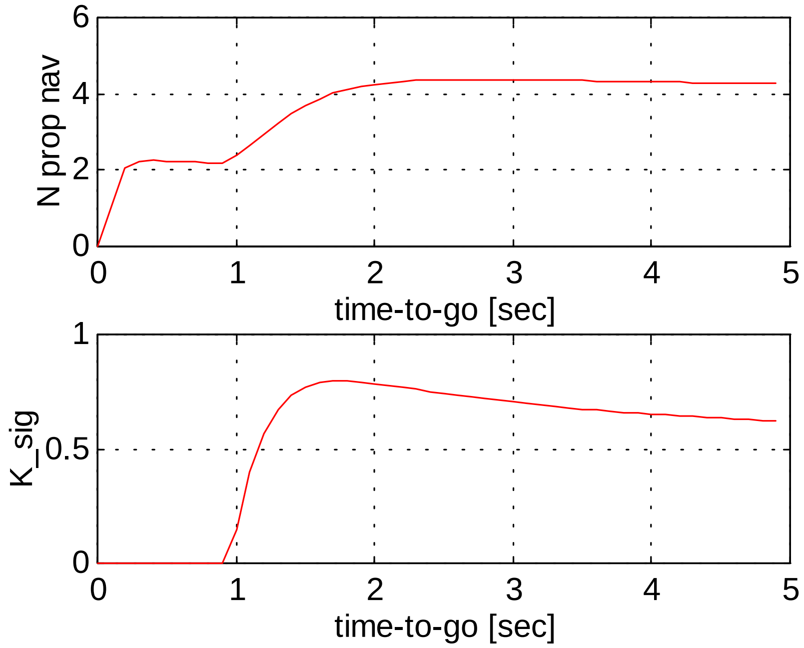


Figure 5: The effective Proportional Navigation gain, for uncertainty in time-of-flight, $p_N(i)$-uniform on the last second, $g(N_f)=1e5$, $g(N_f-9)=1e3$,