Reachability and controllability of positive linear systems with state feedbacks

T. Kaczorek

Warsaw University of Technology, Faculty of Electrical Engineering
Institute of Control and Industrial Electronics
00-662 Warszawa, Koszykowa 75, Poland
e-mail: kaczorek @ nov.isep.pw.edu.pl

Abstract. It is shown that the reachability and controllability of positive linear systems is not invariant under the state-feedbacks.

Key Words. Controllability, invariance, positive linear system, reachability, state-feedback.

1. Introduction.

The reachability, controllability and observability of linear systems have been considered in many papers [1-4,14,15,17-19]. Necessary and sufficient conditions for the reachability and controllability of positive linear systems have been established in [5-7,16,17]. The reachability and controllability of weakly positive discrete-time and continuous-time linear systems have been studied in [9-13]. It is well-known [8] that the reachability and controllability of the standard linear systems is invariant under the state-feedbacks. It this paper it will be shown that the reachability and controllability of linear positive systems is not invariant under the state-feedbacks. In other words a positive linear system which is not n-step reachable (controllable) by suitable choice of the state-feedback gain matrix can be made n-step reachable (controllable).

2. Preliminaries.

Consider the linear discrete-time system

(1) $$x_{i+1} = Ax_i + Bu_i \quad , \; i \in Z_+ := \{0,1,...\}$$

where $x_i \in R^n$, $u_i \in R^m$ and A, B are real matrices of appropriate dimensions.

The system (1) is called positive if for every $x_0 \in R_+^n$ and any $u_i \in R_+^m$ we have $x \in R_+^n$, where R_+^n is the set of n-dimensional real vectors with nonnegative components. It is easy to show that [11] the system (1) is positive if and only if $A \in R_+^{n\times n}$ and $B \in R_+^{n\times m}$, where $R_+^{n\times m}$ denotes the set of $n \times m$ real matrices with nonnegative entries.

Definition 1. The positive system (1) is called h-step reachable if for every $x_f \in R_+^n$ (and $x_0 = 0$) there exists a input sequence $u_i \in R_+^m$, $i = 0,1,...,h-1$ such that $x_h = x_f$.

Definition 2. The positive system (1) is called reachable if for every $x_f \in R_+^n$ (and $x_0 = 0$) there exists $h \in Z_+$ and $u_i \in R_+^m$, $i = 0,1,...,h-1$ such that $x_h = x_f$.

Definition 3. The positive system (1) is called controllable if for every nonzero $x_f, x_0 \in R_+^n$ there exists $h \in Z_+$ and $u_i \in R_+^m$, $i = 0,1,...,h-1$ such that $x_h = x_f$.

Definition 4. The positive system (1) is called controllable to zero if for every $x_0 \in R_+^n$ there exists $h \in Z_+$ and $u_i \in R_+^m$, $i = 0,1,...,h-1$ such that $x_h = 0$.

Theorem 1. [7] The positive system (1) is n-step reachable if and only if:

a) rank $R_n = n$
b) there exists a non-singular matrix $\overline{R}_n$ consisting of n columns of R_n such that $R_n^{-1} \in R_+^{n\times n}$ or equivalently R_n has n linearly independent columns each containing only one positive entry

where

(2) $$R_n := [B, AB, ..., A^{n-1}B] \in R_+^{n\times nm}$$

If the positive system (1) is reachable then it is always n-step reachable [6,7,9-13].

Theorem 2. [7] The positive system (1) is controllable if and only if:

a) the matrix R_n (defined by (2)) has n linearly independent columns each containing only one positive entry
b) the spectral radius $r(A)$ of A is $r(A) < 1$ if the transfer from x_0 to x_f is allowed in an infinite number of steps and $r(A) = 0$ if the transfer from x_0 to x_f is required in a finite number of steps.

3. Reachability of positive systems.

3.1. Single-input systems

Let us assume that for $m = 1$ the matrices A and B of (1) have the canonical form

(3) $$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdotsfor{5} \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \in R_+^{n\times n}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R_+^{n\times 1}$$

It is easy to see that for (3)

(4) $$rank[B, AB, ..., A^{n-1}B] = n$$

but the condition ii) of theorem 1 is not satisfied if at least one $a_i \neq 0$ for $i = 1,...,n-1$. In this case the positive system (1) with (3) is not n-step reachable.

Consider the system (1) with state-feedback

(5) $$u_i = v_i + Kx_i$$

where $K \in R^{1\times n}$ and v_i is the new input.

Substitution of (5) into (1) yields

(6) $$x_{i+1} = A_c x_i + Bv_i, \quad i \in Z_+$$

where

(7) $$A_c = A + BK$$

For (3) and

(8) $$K = [a_0, a_1, ..., a_{n-1}]$$

the matrix (7) has the form

(9) $$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdotsfor{5} \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [a_0, a_1, ..., a_{n-1}] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdotsfor{5} \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Using (9) we obtain

$$[B, A_c B, ..., A_c^{n-1} B] = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \hdotsfor{5} \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Then the conditions of theorem 1 are satisfied and the closed-loop system is n-step reachable. Therefore, the following theorem has been proved.

Theorem 3. Let the positive system (1) with (3) is not n-step reachable. Then the closed-loop system (6) with (9) is n-step reachable if the state-feedback gain matrix K has the form (8).

Corollary 1. The n-step reachability of positive system (1) with (3) is not invariant under the state-feedback (5).

Remark 1. It is well-known [8] that if the pair (A, B) satisfies the condition (4) then it can be transformed by linear state transformation $\bar{x}_i = Px_i$, $\det P \neq 0$ to the canonical form (3)

$$\bar{A} = PAP^{-1}, \ \bar{B} = PB$$

and

$$[\bar{B}, \bar{A}\bar{B}, ..., \bar{A}^{n-1}\bar{B}] = P[B, AB, ..., A^{n-1}B]$$

Note that the conditions of theorem 1 are satisfied if and only if P is a monomial matrix (in each row and column has only one positive entry and the remaining entries are zero).

3.2. Multi-input systems.

Let the matrices A,B of (1) with $m > 1$ have the canonical form

(10a) $$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \hdotsfor{3} \\ A_{m1} & \cdots & A_{mm} \end{bmatrix}, \ B = diag[b_1, ..., b_m]$$

where

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdotsfor{5} \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0^{ii} & -a_1^{ii} & -a_2^{ii} & \cdots & -a_{d_i-1}^{ii} \end{bmatrix} \in R_+^{d_i \times d_i},$$

(10b)
$$A_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \hdashline 0 & 0 & \cdots & 0 \\ -a_0^{ij} & -a_1^{ij} & \cdots & -a_{d_j-1}^{ij} \end{bmatrix} \in R_+^{d_i \times d_j}, \; i, j = 1,...,m, \quad i \neq j$$

$$b_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R_+^{d_i \times 1}, \quad \sum_{i=1}^{m} d_i = n$$

It is easy to check that for (10) the condition (4) holds but the condition ii) of theorem 1 is not satisfied if at least m of the coefficients $a_k^{ii} \neq 0$ for $k = 1,...,d_{i-1}$ and $i = 1,...,m$. In this case the positive system (1) with (10) is not n-step reachable.
The closed-loop system matrix (7) with (10) and

(11)
$$K = \begin{bmatrix} a_0^{11}, & a_1^{11} & ,\cdots, & a_{d_1-1}^{11}, & a_0^{12}, & a_1^{12} & ,\cdots, & a_{d_2-1}^{12} & ,\cdots, & a_0^{1m}, & a_1^{1m} & ,\cdots, & a_{d_m-1}^{1m} \\ \hdashline a_0^{m1}, & a_1^{m1} & ,\cdots, & a_{d_1-1}^{m1}, & a_0^{m2}, & a_1^{m2} & ,\cdots, & a_{d_2-1}^{m2} & ,\cdots, & a_0^{mm}, & a_1^{mm} & ,\cdots, & a_{d_m-1}^{mm} \end{bmatrix} \in R^{m \times n}$$

has the form

(12)
$$A_c = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \hdashline 0 & 0 & \cdots & A_m \end{bmatrix}, \; A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in R_+^{d_i \times d_i}, \; i = 1,...,m$$

Then the conditions of theorem 1 are satisfied and the closed-loop system is n-step reachable. Therefore, the following theorem has been proved.

Theorem 4. Let the positive system (1) with (10) is not n-step reachable. Then the closed-loop system (6) with (12) is n-step reachable if the state-feedback gain matrix K has the form (11).

Corollary 2. The n-step reachability of positive system (1) with (10) is not invariant under the state-feedback (5).

Remark 2. It is well-known [8] that if the pair (A, B) satisfies the condition (4) then it can be transformed by linear transformation $\bar{x}_i = Px_i$, $\bar{u}_i = Qu_i$, $\det P \neq 0$, $\det Q \neq 0$, to the canonical form (10)

$$\bar{A} = PAP^{-1}, \; \bar{B} = PBQ^{-1}$$

and

$$\left[\bar{B}, \bar{A}\bar{B}, ..., \bar{A}^{n-1}\bar{B}\right] = P\left[B, AB, ..., A^{n-1}B\right] diag\left[Q^{-1}, ..., Q^{-1}\right]$$

Note that the conditions of theorem 1 are satisfied if and only if P and Q are monomial matrices.

4. Controllability of positive systems.

Consider the multi-inputs system (1) with matrices A,B in the canonical form (10). In a similar way as in the reachability case it can be shown that the condition i) of theorem 2 is not satisfied if at least m of the coefficients $a_k^{ii} \neq 0$ for $k = 1,...,d_{i-1}$ and $i = 1,...,m$. In this case the positive system (1) with (10) is not controllable.

The closed-loop system matrix (7) with (10) and state-feedback gain matrix (11) has the form (12). Note that the matrix (12) has all zero eigenvalues and its spectral radius $r(A_c) = 0$. Therefore, the following theorem has been proved.

Theorem 5. Let the positive system (1) with (10) is not controllable. Then the closed-loop system (7) with (12) is controllable in a finite number of steps if the state-feedback gain matrix K has the form (11).

5. Concluding remarks.

It has been shown that the positive discrete-time unreachable (uncontrollable) system (1) with matrices A and B in the canonical form (10) by suitable choice of the state-feedback gain matrix in the form (12) can be made reachable (controllable). This statement is also valid for positive continuous-time linear systems. An extension of this result for weakly positive linear systems [9,19] and for positive 2D linear systems [8] are open problems.

REFERENCES

[1] A. Banaszuk, M. Kociecki (1991), Observability with unknown input and dual properties for singular systems, *J.C.Baltzer AG, Scientific Publishing Co, IMACS*, pp. 125-129.

[2] S.L.Campbell, N.K. Nichols and W.J. Terrell (1991), Duality, observability and controllablity for linear time-varying descriptor systems, *Circuits Systems Signal Process*, vol. 10, No 4, pp. 455-470.

[3] K.-W. E. Chu (1987), Controllability of descriptor systems, *Int. J. Control*, vol. 46, No 5. pp. 1761-1770.

[4] D. Cobb (1984), Controllability, observability and duality in singular systems , *IEEE Trans. Automat.Contr.*, vol. AC-29, No 12, pp. 1076-1082.

[5] P.G. Coxson (1987), Positive input reachability and controllability of positive systems, *Linear Algebra and its Applications 94*, pp. 35-53.

[6] M.P. Fanti, B. Maione and B. Turchiano (1989), Controllability of linear single-input positive discrete-time systems, *Int. J. Control*, vol. 50, No 6, pp. 2523-2542.

[7] M.P. Fanti, B. Maione and B. Turchiano (1990), Controllability of multi-input positive discrete-time systems, *Int. J. Control*, vol. 51, No 6, pp. 1295-1308.

[8] T. Kaczorek (1993), Linear Control Systems, vol. 2, Research Studies Press and J. Wiley, New York.

[9] T. Kaczorek (1998), Reachability and controllability of weakly positive singular discrete linear systems, *IEEE Mediterranean Conference on Control and Systems*, Wieden, April 11-17, pp. 3-8.

[10] T. Kaczorek (1998), Computation of fundamental matrices and reachability of positive singular discrete linear systems, *Bull. Pol. Acad. Techn. Sci.*, vol. 47, No. 4, pp.501-511.

[11] T. Kaczorek (1997), Positive singular discrete linear systems, *Bull. Pol. Acad. Techn. Sci.*, vol. 46, No. 4, pp. 619-631.

[12] T. Kaczorek (1998), Reachability and controllability of weakly positive singular discrete linear systems, *6th IEEE Mediterranean Conference on Control and Systems*, Wieden, 11-17 kwiecien pp. 3-8.

[13] T. Kaczorek (1998), Weakly positive continuous-time linear systems, *Bull. Pol. Acad. Techn. Sci*., vol. 46, No. 2, pp. 233-245.

[14] J. Klamka (1991), Controllability of Dynamical Systems,*Kluwer Academic Publ,. Dordecht*.

[15] J. Klamka (1993), Controllability of dynamical systems A survey, *Arch.Contr. Sci*., vol. 2, No 3-4, pp. 281-307.

[16] D.N.P. Murthy (1986), Controllability of a linear positive dynamic system, *Int. J. Systems Sci.*, vol. 17, No 1, pp. 49-54.

[17] Y. Ohta, H. Madea and S. Kodama (1984), Reachability, observability and realizability of continuous-time positive systems, *SIAM J. Control and Optimization*, vol. 22, No 2, pp. 171-180.

[18] K. Ozcaldiran (1986), A geometric characterization of the reachable and the controllable subspaces of descriptor systems, *Circuits, Syst., Signal Processing*, vol. 5, no 1, pp. 37-48.

[19] E.L. Yip and E.F. Sinovec, Solvability, controllability and observability of continuous descriptor