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Stabilization of quasilinear systems and hybrid feedback controls

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1 Introduction

Consider the following controlled system

$$\begin{aligned} \dot{x} &= F(x) + Bu \\ y &= Cx, \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$ is the physical state of the system, $y \in \mathbf{R}^m$ is the output, $u \in U \subset \mathbf{R}^\ell$ is the control, $F : \mathbf{R}^n \to \mathbf{R}^n$ is a differentiable at 0 vector field, B, C are given real matrices of the sizes $n \times \ell$, $m \times n$, respectively. Denote by A the Frechet derivative (the Jacobi matrix) $\left[\frac{\partial F_i}{\partial x_j}(0)\right]_{i,j=1}^n$ of the vector field F evaluated at 0, and rewrite the system (1) as follows

$$\begin{aligned} \dot{x} &= Ax + Bu + f(x) \\ y &= Cx, \end{aligned} \tag{2}$$

where $f(x) = o(|x|)$ (here and below $|\cdot|$ stays for Euclidean norm in a finite-dimensional space).

According to the general theory of ordinary differential equations (see, for example (Arnold, 1992; Hubbard and West, 1991)), if the eigenvalues of the matrix A have negative real parts, then under absence of any control (i.e. if $u \equiv 0$), the corresponding solution of the system (2) starting at $x(0)$ satisfies the following estimate

$$|x(t)| \le M \exp(-\lambda t)\, |x(0)|, \qquad t \ge 0, \tag{3}$$

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where the constants $\lambda > 0$, $M > 0$ do not depend on $|x(0)|$, and $|x(0)|$ is sufficiently small. This fact is sometimes called the first Lyapunov method in stability theory. This method provides, as we see, the (local) exponential stability of the zero solution with respect to small perturbations of initial data.

For the controlled system (2) with an arbitrary matrix A, it is always possible (see for example (Sontag, 1989; Wonham, 1979; Litsyn *et al.*, 1998)) to make the zero solution exponentially stable using a linear feedback control $u = Gx$, provided that the pair (A, B) is controllable and rank $C = n$ (which describes the case of complete observability of the solution). This justifies an analog of the first Lyapunov method for controlled system like (2).

However, it is known that in practice the case of complete observability is rather an exception than the rule. The most typical and interesting situation for applications is therefore the case when rank $C < n$.

Let us consider an illustrating example where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = (1,\, 0)$ This is nothing, but the classical harmonic oscillator where we can only observe the physical state and control the velocity. In this case, the matrix A is perturbed by a matrix of rank 1 representing any linear control $u = Gy$ chosen. It can be shown (Artstein, 1996) that no ordinary linear controls of the form $u = Gy$ asymptotically stabilize the solutions of the linearized system (2), so that stability properties of (2) essentially depend on the nonlinearity $f(x)$, or more precisely, on the higher order terms, rather than the linear ones, in the Taylor expansion of the function $F(x)$ at 0 (as e. g. in the Lyapunov test for weak source in the Hopf bifurcation; see (Arnold, 1992), or (Hubbard and West, 1991, p. 292) for details). As it is pointed out by Sontag (1989, Example 6.2.1) and Artstein (1996, Example 3.1), no stabilization can be achieved by means of nonlinear autonomous controls of the type $u(x) = \mathrm{col}\,\{0,\, u_2(y)\}$ as well.

However, one will be able to stabilize systems like the above harmonic oscillator if one uses special feedback controls called *hybrid feedback controls* (abbr. HFC) (Nerode and Kohn, 1993; Artstein, 1996). Such controls can often stabilize systems where the ordinary feedback paradigm does not work (see e.g. (Litsyn *et al.*, 1998)). On one side, HFC generalizes the classical autonomous control $u(t) = u(y(t))$, on the other side, the discrete nature of this control makes its realization easier in practice.

In (Litsyn *et al.*, 1998) the question whether it is possible to stabilize an arbitrary linear system of the form (2) with $f \equiv 0$ was formulated and positively answered.

There is no doubt that also the problem of how to stabilize quasi-linear controlled systems like (2) with small non-linear perturbations f, where the linearized system is supposed to be asymptotically stable, is of interest. Recently, some breakthrough attempts to extend the Lyapunov stability theory to investigate the asymptotic stability of systems with HFCs have been undertaken (see e.g. (Branicky, 1994, 1995), and references therein). However, as far as we know, no complete solution of the discussed problem has been achieved by now (even for 2×2 systems).

In this paper, we will try to contribute to this problem in a way described below.

Namely, we shall justify the first Lyapunov method for controlled 2×2 systems of the type (2), where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = (1,\, 0)$, and $u(\cdot)$ is an *elementary* hybrid control (see e.g. Artstein (1996, Example 5.2)). The technique to be used will be based on direct calculations rather than on the method of Lyapunov functions. Referring the reader to Section 6 for further explanations, let us only remark here that the dynamics of such systems does not fit in with the traditional framework to study stability properties of ordinary differential equations and, probably, autonomous switched systems investigated in (Branicky, 1994, 1995). For example, trajectories of systems with HFCs may have intersections (which is impossible in "usual" autonomous systems), and it may even happen that two trajectories stick together resulting in a new single trajectory.

2 Basic notions and definitions

Consider a general nonlinear system

$$\begin{aligned} \dot{x} &= F(x,u) \\ y = c(x),\ &F(0,u) \equiv 0, \end{aligned} \tag{4}$$

where $x \in \mathbf{R}^n$ describes the physical state of the process ("plant") $y \in \mathbf{R}^m$ is the output, that is, the quantity which can be measured, and the control u is a element of a fixed subset U of the space $\mathbf{R}^\ell$.

The question is how to choose a control strategy $u(\cdot)$, so that a prescribed state, say $x = 0$, becomes stable. Only the quantity that can be measured, namely the output y, can be used by the control. In the classical output feedback paradigm, one tries to find a feedback $u(y) = u(c(x))$ making the zero solution asymptotically stable. If the feedback $u(y)$ is continuous, solutions to the resulting differential equation are well defined (for the formal definition of a stabilizer under the continuity assumption see Definition 2.2 in (Artstein, 1996)).

However, the output feedback may fail to stabilize even quite simple systems. This justifies a need to modify the "classical" output feedback approach.

We will follow our previous paper (Litsyn *et al.*, 1998) to specify our setting. Let us start with some exact definitions.

Definition 2.1. *(see e.g. (Artstein, 1996)) The system (4) is stabilizable by a control* $u(\cdot)$ *($u-$ stabilizable), if two following conditions hold:*

(a) *For any $\varepsilon > 0$ there is $\delta > 0$ such that every solution $x(t)$ of the system (4) with the property $|x(0)| < \delta$ satisfies the inequality $\sup\limits_{t\geq 0} |x(t)| < \varepsilon$.*

(b) *$x(t) \to 0$ as $t \to \infty$, the convergence being uniform w.r.t. the starting point $x(0)$ taken from an arbitrary compact set $K \subset \mathbf{R}^n$.*

Definition 2.2. *Denoting by Z the set of all admissible controls, we say that the system (4) is $Z-$stabilizable if there exists a control $u \in Z$ under which (4) is $u-$stabilizable.*

A typical automaton is given by a triple $\mathcal{A} = (Q, I, M)$, where

(i) Q is a set of all possible automaton states (locations), $\operatorname{card} Q \leq \aleph_0$;

(ii) the set I contains the input alphabet, $\operatorname{card} I \leq \aleph_0$;

(iii) the transition map $M : Q \times I \to Q$ indicates the location after a transition time, based on the previous location q and input i at the time of transition.

The automaton is supposed to follow solutions of the system (4). This fact is described by another given triplet $\mathcal{B} = (T, i, q_0)$ where

(iv) $T : Q \to (0, \infty)$ is a mapping which sets a period $T(q)$ between transitions times,

(v) $i : \mathbf{R}^m \to I$ is a function providing the element $i(y)$ of the alphabet I for any output y of the system (4),

(vi) and $q_0 = q(\tau_0)$ is a state of the automaton at the initial time τ_0.

The case $\tau_0 = -\infty$ is excluded from our considerations, so that later on we can assume, without loss of generality, that $\tau_0 = 0$.

Definition 2.3 ((Litsyn *et al.*, 1998)). *By an automaton, we mean a 6-tuple $\Delta = (Q, I, M, T, i, q_0)$.*

For arbitrary sets X, Y (topological spaces X, Y) we denote by $\mathbf{P}(X,Y)$ (resp. $\mathbf{C}(X,Y)$) the set of all functions (resp. continuous functions) from X to Y.

Now, for any automaton Δ satisfying (i)-(vi) we define by induction a Volterra operator $F_\Delta : \mathbf{C}([0,\infty), \mathbf{R}^m) \to \mathbf{P}([0,\infty), Q)$. For each $y \in \mathbf{C}([0,\infty), \mathbf{R}^n)$ F_Δ is given by:

1. $(F_\Delta y)(0) = q(0); \quad \tau_1 = T(q(0)); \quad (F_\Delta y)(t) \equiv q(0), \quad t \in [0, \tau_1);$

2. $(F_\Delta y)(\tau_1) = M(q(0), i(y(\tau_1))) := q(\tau_1); \quad \tau_2 = \tau_1 + T(q(\tau_1));$
$(F_\Delta y)(t) = q(\tau_1), \quad t \in [\tau_1, \tau_2);$

3. If $\tau_0, \tau_1, \ldots, \tau_k$ and the values $(F_\Delta y)(t)$ for $t \in [0, \tau_k)$ are already known, then τ_{k+1} and $(F_\Delta y)(t)$ for $t \in [\tau_k, \tau_{k+1})$ are defined by the equalities

$$(F_\Delta y)(\tau_k) = M(q(\tau_{k-1}), i(y(\tau_k))) := q(\tau_k); \quad \tau_{k+1} = \tau_k + T(q(\tau_k));$$
$$(F_\Delta y)(t) \equiv q(\tau_k), \quad t \in [\tau_k, \tau_{k+1}).$$

Definition 2.4. *A control $u(\cdot)$ of the type*

$$u(t) = \Phi(y(t), (F_\Delta y)(t)), \tag{5}$$

where $\Phi : \mathbf{R}^m \times Q \to \mathbf{R}^\ell$ is a certain function, will be addressed as a **hybrid feedback control** *(abbr. HFC).*

Further on we shall denote by $\mathcal{H}_\sigma$ the class of all HFCs described in Definition 2.4. Thus, any control $u \in \mathcal{H}_\sigma$ is uniquely defined by the pair (Δ, Φ), consisting of an automaton $\Delta = (Q, I, M, T, i, q_0)$ and a function $\Phi : \mathbf{R}^m \times Q \to \mathbf{R}^\ell$. Below we use the following notation: $u = (\Delta, \Phi) \in \mathcal{H}_\sigma$.

We will need some special subclasses of the class $\mathcal{H}_\sigma$:

$\mathcal{H}_0 := \{(\Delta, \Phi) \in \mathcal{H}_\sigma \mid \Phi(y, q)$ does not depend on $q\}$, it is the class of ordinary (*not* hybrid) controls.

$\mathcal{H}_n := \{(\Delta, \Phi) \in \mathcal{H}_\sigma \mid \operatorname{card} Q \le n, \ \operatorname{card} I \le n\}, \qquad n = 1, 2, \ldots$

$\mathcal{H}_e := \bigcup_{n=1}^{\infty} \mathcal{H}_n$ is the class of elementary HFCs (Artstein, 1996).

$\mathcal{LH}_\sigma := \{(\Delta, \Phi) \in \mathcal{H}_\sigma \mid \Phi(y, q)$ linearly depends on $y\}$ is the class of linear HFCs.

$\mathcal{LH}_e := \mathcal{LH}_\sigma \cap \mathcal{LH}_e, \quad \mathcal{LH}_n := \mathcal{LH}_\sigma \cap \mathcal{LH}_n, \quad n = 0, 1, 2, \ldots$

It is natural to call HFCs belonging the class $\mathcal{LH}_e$ *elementary*, while the subclass $\mathcal{LH}_n$ can be addressed as the class of HFCs with n locations.

It is clear that

$$\begin{array}{ccccccccccccccc}
\mathcal{H}_0 & = & \mathcal{H}_1 & \subset & \mathcal{H}_2 & \subset & \ldots & \subset \mathcal{H}_n & \subset & \mathcal{H}_{n+1} & \subset & \mathcal{H}_e & \subset & \mathcal{H}_\sigma \\
\cup & & \cup & & \cup & & & \cup & & \cup & & \cup & & \cup \\
\mathcal{LH}_0 & = & \mathcal{LH}_1 & \subset & \mathcal{LH}_2 & \subset & \ldots & \subset \mathcal{LH}_n & \subset & \mathcal{LH}_{n+1} & \subset & \mathcal{LH}_e & \subset & \mathcal{LH}_\sigma
\end{array}.$$

However, these inclusions need some comments.

1. The class $\mathcal{H}_0$ consists of ordinary (nonlinear) feedback controls which are of the type $u = \Phi(y)$, where $\Phi : \mathbf{R}^m \to \mathbf{R}^\ell$. It is also evident that $\mathcal{LH}_0$ is nothing, but the class of linear feedback controls of the form $\Phi x = Gx$ for some $\ell \times n$-matrix G.

2. Evidently, $\mathcal{H}_1 = \mathcal{H}_0$, i.e. in case Q degenerates into a singleton, any hybrid feedback control is given by a feedback control of the type $u = f(y)$ with a given function $f : \mathbf{R}^m \to \mathbf{R}^\ell$.

3. An elementary hybrid system is that with a finite number of locations $Q = \{q_1, \ldots, q_n\}$ (see (Artstein, 1996)). An elementary hybrid system gives rise to an **elementary HFC**. In our notation, an elementary HFC is nothing but n-HFC (or HFC of the class $\mathcal{H}_n$) for some natural number n. A typical elementary (or, more general, discrete) hybrid system's dynamic is continuous, and the solution satisfies (4) on the intervals $(\tau_i, \tau_{i+1}]$ if $u = \Phi(y, q_i)$. In (Artstein, 1996) more specific examples of HFCs belonging to the classes $\mathcal{H}_2$, $\mathcal{LH}_2$, $\mathcal{H}_3$, $\mathcal{LH}_3$ are given.

4. In (Litsyn *et al.*, 1998) the following result is proved: under assumptions of controllability of the pair (A, B) and observability of the pair (A, C) any linear system (2) of an arbitrary order with $f \equiv 0$ is $\mathcal{H}_\sigma$-stabilizable. Let us observe that, in fact, the stabilizing control constructed in (Litsyn *et al.*, 1998) belongs to the set $\mathcal{H}_\sigma \setminus (\mathcal{LH}_\sigma \cup \mathcal{H}_e)$.

By the present paper we intend to start investigating stability properties of nonlinear systems like (4). Our specific objective in this paper is a detailed study of a perturbed harmonic oscillator, i.e. a specific nonlinear 2×2 system. This system cannot be stabilized by ordinary $\mathcal{H}_0$-controls, but, as we will show, it admits stabilization by more general $\mathcal{LH}_3$-controls introduced in (Artstein, 1996, Example 5.2).

3 $\mathcal{LH}_3$-controls for a perturbed harmonic oscillator and formulation of the main result

Consider the following controlled system

$$\begin{aligned} \dot{\xi} &= \eta + f_1(\xi, \eta), \\ \dot{\eta} &= -\xi + f_2(\xi, \eta) + u, \\ y &= \xi \end{aligned} \tag{6}$$

(known as a perturbed harmonic oscillator).

If we assume ξ, η to be the coordinates of a point $x \in \mathbf{R}^2$, then we get the following matrix representation of the system (6):

$$\begin{aligned} \dot{x} &= Ax + Bu + f(x) \\ y &= Cx, \end{aligned}$$

where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = (1, 0)$.

We are going to investigate an asymptotic behaviour of the trajectories of the controlled system (6), or its equivalent matrix form.

We refer the reader to the paper (Artstein, 1996) where the author introduces a specific HFC u belonging to the class $\mathcal{LH}_3$ which stabilizes the corresponding *linear* harmonic oscillator ($f \equiv 0$). This HFC (which is also described below) will be denoted in this paper by $\mathcal{A}(\delta)$.

We shall use Definitions 2.3, 2.4 to define the HFC $\mathcal{A}(\delta)$. To do it we put

$$(\Delta, \Phi) = ((Q, I, M, T, i, q_0), \Phi),$$

where $Q = \{q_+, q_-, q_d\}$, $I = \{I_+, I_-\}$, $q_0 \in Q$ are arbitrary abstract objects, while the functions $M : Q \times I \to Q$, $T : Q \to (0, \infty)$, $i : \mathbf{R}^1 \times I \to Q$, $\Phi : \mathbf{R}^1 \times Q \to \mathbf{R}^1$ are given by

$$M(q_+, I_+) = M(q_d, I_+) = q_+, \quad M(q_-, I_-) = M(q_d, I_-) = q_-; \quad M(q_+, I_-) = M(q_-, I_+) = q_d,$$

$$T(q_+) = T(q_-) = \delta, \qquad T(q_d) = \frac{\pi}{4} - \delta;$$

$$i(\xi) = \begin{cases} I_1, & \text{if} \quad \xi \geq 0 \\ I_2, & \text{if} \quad \xi < 0 \end{cases};$$

$$\Phi(\xi, q_+) = \Phi(\xi, q_-) = 0, \qquad \Phi(\xi, q_d) = -3\xi.$$

Let us remark that we slightly modified the definition of $\mathcal{A}(\delta)$ as compared with the paper (Artstein, 1996, Example 5.2) (in (Artstein, 1996) it is assumed that $T(q_d) = \pi/4 - 2\delta$). This modification was done just for the sake of a technical convenience and does not really matter.

We observe now that $\mathcal{A}(\delta) \in \mathcal{LH}_3$.

It was shown in (Artstein, 1996) that the linear system

$$\begin{aligned} \dot{\xi} &= \eta, \\ \dot{\eta} &= -\xi. \end{aligned} \tag{7}$$

with the control $u \in \mathcal{A}(\delta)$ is asymptotically stable if $\delta > 0$ is small enough. We again remind the reader that it is impossible to stabilize this system via "ordinary" feedback controls.

The main result of the present paper is a generalization of the above linear theorem proved by Artstein to the case of controlled systems (6) with small nonlinear perturbations.

Theorem 3.1. *There exist positive numbers δ, M, λ such that for any Lebesgue-measurable function $f : \mathbf{R}^2 \to \mathbf{R}^2$, satisfying $|f(x)| = o(|x|)$, $|x| \to 0$, one can point out $\varepsilon > 0$ for which any solution $x(t)$ of the system (6) controlled by the HFC $u = \mathcal{A}(\delta)$, satisfies the estimate*

$$|x(t)| \leq M\, exp(-\lambda\, t)|x(0)|, \qquad t \geq 0. \tag{8}$$

for an arbitrary initial value $x(0)$, $|x(0)| \leq \varepsilon$. The theorem implies the following

Colollary 3.2. *Let $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = (1,\, 0)$. Then there exist positive numbers δ, M, λ such that for any differentiable function $F : \mathbf{R}^2 \to \mathbf{R}^2$, $F(0) = 0$ with the Jacobi matrix (evaluated at 0) $J := \left[\frac{\partial F_i}{\partial x_j}(0)\right]_{i,j=1}^{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, one can point out $\varepsilon > 0$ for which any solution $x(t)$ of the non-linear system*

$$\begin{aligned} \dot{x} &= F(x) + Bu \\ y &= Cx, \end{aligned}$$

controlled by the HFC $u = \mathcal{A}(\delta)$, satisfies the estimate (8) for an arbitrary initial value $x(0)$, $|x(0)| \leq \varepsilon$.

As an example, let us consider a 2×2 Hamiltonian system with one degree of freedom. The system is described by the Newton equation

$$\ddot{\xi} = F(\xi), \tag{9}$$

where $F : \mathbf{R} \to \mathbf{R}$ is a differentiable function (Arnold, 1992). The equation (9) can also be rewritten as follows:

$$\begin{aligned} \dot{\xi} &= \eta, \\ \dot{\eta} &= F(\xi), \end{aligned} \tag{10}$$

where ξ is the state coordinate of the oscillator, η is the velocity, $\dot{\eta}$ is the acceleration, while F describes the corresponding force field.

In applications, it may be possible to control the acceleration $\dot{\eta}$ as the function depending on the coordinate $\xi(t)$ evaluated at any time t. We consider therefore the following control system

$$\begin{aligned} \dot{\xi} &= \eta, \\ \dot{\eta} &= F(\xi) + u. \end{aligned} \tag{11}$$

If $F(0) = 0$, $F'(0) = -1$, the (linearized) system will be of the form (7), so that no "ordinary" feedback control $u(\xi)$, $(u \in \mathcal{H}_0)$ can get the spectrum of the system's matrix out of the imaginary axis. Thus, the linearized system cannot be asymptotically stable. As it was mentioned in the introduction, in such a case we cannot say much about asymptotic properties of the non-linearized system, unless we have an additional information about the higher derivatives of the function $F(x)$.

On the other hand, as it follows from (Artstein, 1996) the linearized system (11) is stabilizable by the HFC $u = \mathcal{A}(\delta)$ (Artstein, 1996). Using our Theorem 3.1 and Corollary 3.2 above we immediately get

Corollary 3.3. *There exist $\delta, M, \lambda > 0$, such that for any differentiable function $F : \mathbf{R} \to \mathbf{R}$, $F(0) = 0$, $F'(0) = -1$ one can point out $\varepsilon > 0$, for which any solution $x(t)$ of (11) controlled by the HFC $u = \mathcal{A}(\delta)$, admits the asymptotic estimate (8) if the initial value $x(0)$ satisfies the condition $|x(0)| \leq \varepsilon$.*

For example, Corollary 3.3 can be applied to the non-linear pendulum which is a well-known "test" object in mechanics (see, for example, (Arnold, 1992)):

$$\ddot{\xi} = -\sin \xi.$$

There exist more specific examples of (6) which are not stabilizable by any "ordinary" feedback control, but which (according to Theorem 3.1) admit stabilization via hybrid feedback controls. We just mention here two nonlinear systems which are used in (Hubbard and West, 1991, Chapter 9) to illustrate two kinds of an asymptotic behaviour near the equilibrium:

$$\begin{cases} \dot{\xi} = \eta + \xi^2 \\ \dot{\eta} = -\xi \end{cases} \quad \text{(center)}; \qquad \begin{cases} \dot{\xi} = \eta + \xi\eta^2 \\ \dot{\eta} = -\xi + \xi\eta - \eta^2 \end{cases} \quad \text{(weak source)}.$$

Assume we are able to influence the velocity $\dot{\eta}$ by making use of values of the state coordinate ξ, only. Our Corollary 3.3 guarantees then that the HFC $u = \mathcal{A}(\delta)$ asymptotically stabilizes both systems in the vicinity of 0.

Remark 3.4. *In fact, Theorem 3.1 and all the corollaries contain even more information about stability properties of the system in question. Namely, the zero solution of the quasi-linear equation (6) is not only asymptotically stable, but also exponentially stable, and this stability is uniform with respect to the perturbations $f := col\{f_1, f_2\}$.*

The next two sections include a proof of Theorem 3.1. We start with two (rather technical) auxiliary lemmas in Section 4 and use them in Section 5 to give a proof of our main result (Theorem 3.1).

4 Some auxiliary results

Let $T, a, b, c, \tau_1, \tau_2, \alpha_1, \alpha_2 > 0$ be arbitrary. By $\mathcal{E}(T, a, b, c, \tau_1, \tau_2, \alpha_1, \alpha_2)$ we denote a set of functions $r : [0, T] \to (0, \infty)$, satisfying the following conditions:

1) For any $t \in [0, \min\{a, T\}]$

$$r(t) \leq b \cdot r(0). \tag{12}$$

2) If $T \geq a$, then there exist a finite increasing sequence $t_0, t_1, \ldots t_{2m} \in [0, \infty)$ such that

$$t_0 \in [0, a], \qquad t_m \leq T < t_m + \tau_1 + \tau_2,$$

$$r(t) \leq c \cdot r(t_{2n}), \qquad t \in [t_{2n}, \min\{T, t_{2n+2}\}], \qquad n = 0, \ldots, m. \tag{13}$$

Moreover, for $m > 0$

$$t_{2n+i} - t_{2n+i-1} \leq \tau_i, \qquad i = 1, 2, \quad n = 0, \ldots, m-1; \tag{14}$$

$$\frac{r(t_{2n+i})}{r(t_{2n+i-1})} \leq \alpha_i, \qquad i = 1, 2, \quad n = 0, \dots, m-1. \tag{15}$$

Notice that if $T < a$, then r should satisfy the estimate (12), only; if $T \geq a$, then m cannot be zero; if $T > a + \tau_1 + \tau_2$ then necessarily $m > 0$. Lemma 4.1 below is of technical nature and should be regarded as a rather convenient tool to study the dynamics $|x(t)|$ of a system with a HFC. The lemma will be essentially used in the proof of Theorem 3.1 in Section 5.

Lemma 4.1. *We are supposed given some constants $T, a, b, c, \tau_1, \tau_2, \alpha_1, \alpha_2 > 0$, where $\alpha_1 \cdot \alpha_2 < 1$. Then for any $r \in \mathcal{E}(T, a, b, c, \tau_1, \tau_2, \alpha_1, \alpha_2)$ we have*

$$r(t) \leq M\, e^{-\lambda t} r(0), \qquad t \in [0, T], \tag{16}$$

where $\lambda > 0$, $M > 0$ are defined by

$$\lambda = -\frac{\ln \alpha_1 \alpha_2}{\tau_1 + \tau_2}, \qquad M = b\, c\, e^{\lambda(a + \tau_1 + \tau_2)}. \tag{17}$$

Proof. We fix an arbitrary function $r \in \mathcal{E}(T, a, b, c, \tau_1, \tau_2, \alpha_1, \alpha_2)$.

1) The inequality (12) implies that

$$r(t) \leq (b\, e^{\lambda t})(e^{-\lambda t}) r(0) \leq b\, e^{\lambda a} e^{-\lambda t} r(0) \leq M\, e^{-\lambda t} r(0), \qquad t \in [0, \min\{a, T\}] \tag{18}$$

where the constants λ, M are defined by (17).

2) Let $t \leq a$ and let the numbers $t_0, t_1, \dots\, t_{2m}$ be chosen according to the definition of $\mathcal{E}(\cdot)$. Then the following two cases can occur:

a) $m = 0$. According to (13) and (12) we have $\forall t \in [t_0, T]$

$$r(t) \leq c\, r(t_0) \leq b\, c\, e^{\lambda a} e^{-\lambda t} r(0) \leq M\, e^{-\lambda t} r(0). \tag{19}$$

Due to (18), the inequality $r(t) \leq M\, e^{-\lambda t}$ is also valid $\forall t \in [0, T]$. Thus, (16) is proved.

b) $m > 0$. For any $t \in [t_0, T]$ we let $n(t) = \max\{n \in \mathbf{Z} \,|\, t \geq 2n\}$. Then the formulae (14), (13) imply that

$$r(t) \leq c\, r(t_{2n(t)}), \qquad t \in [t_0, T], \tag{20}$$

and from (15) it follows that

$$r(t_{2n+2}) \leq \alpha_1 \alpha_2 f(t_{2n}), \qquad n = 0, \dots, m-1. \tag{21}$$

According to (14) we get

$$t \leq t_0 + (\tau_1 + \tau_2)(n(t) + 1),$$

which in turn implies the estimate

$$n(t) \geq \frac{t - t_0}{\tau_1 + \tau_2} - 1, \qquad t \in [t_0, T]. \tag{22}$$

Finally, from (14), (20)–(22) and $\alpha_1 \alpha_2 < 1$, we obtain

$$r(t) \leq c\, e^{(\ln \alpha_1 \alpha_2) \cdot n(t)} r(t_0) \leq c \cdot \exp\left\{ (\ln \alpha_1 \alpha_2) \left(\frac{t - t_0}{\tau_1 + \tau_2} - 1 \right) \right\} r(t_0) \leq M\, e^{-\lambda t} r(0)$$

which holds $\forall t \in [t_0, T]$. A similar inequality for $t \in [0, t_0]$ vytekaet is (18). The lemma is thereby proved. □

Lemma 4.2. *Let a function $\zeta : [0, \infty)^2 \to [0, \infty)$ satisfy the following conditions:*
1) $\zeta(\cdot, \mu)$ is continuous for all $\mu \in [0, \infty)$;
2) $\zeta(0, \mu) = \mu$ for all $\mu \in [0, \infty)$;
3) there exist constants $M > 0$, $\varepsilon > 0$, $\lambda > 0$, such that for all $t \geq 0$ the following implication is true

$$\left[\sup_{s \in [0,t]} \zeta(s, \mu) \leq \varepsilon \right] \Longrightarrow \left[\forall s \in [0, t] \quad \zeta(s, \mu) \leq M\, e^{-\lambda s} \cdot \mu \right].$$

Then $\forall \mu \in \left(0, \dfrac{\varepsilon}{M+1}\right)$

$$\zeta(t, \mu) \leq M\, e^{-\lambda t} \cdot \mu, \qquad t \geq 0.$$

Proof. Suppose, on the contrary, that this is not true. Then by virtue of the condition 3), for some $\mu \in \left(0, \dfrac{\varepsilon}{M+1}\right)$ one can find $t^* > 0$ such that $\zeta(t^*, \mu) > \varepsilon$.

Since $\zeta(0, \mu) = \mu < \varepsilon$ and $\zeta(\cdot, \mu)$ is continuous, we have

$$t_1 := \min\{\, t \in [0, \infty) \,|\, \zeta(t, \mu) = \varepsilon \,\} \in (0, t^*). \tag{23}$$

Then $\forall t \in (t_1/2, t_1)$ we obtain the estimate $\sup\limits_{s \in [0,t]} \zeta(s, \mu) \leq \varepsilon$, and according to the condition 3) of the lemma we get

$$\zeta(t, \mu) \leq M\mu\, e^{-\lambda t} \leq \frac{M\varepsilon}{M+1}\, e^{-\lambda t} \leq \varepsilon\, e^{-\lambda t_1/2}, \qquad t \in (t_1/2, t_1).$$

Because of the continuity of the function $\zeta(\cdot, \mu)$ at the point t_1 we have

$$\zeta(t_1, \mu) \leq \varepsilon\, e^{-\lambda t_1/2} < \varepsilon,$$

which contradicts (23). $\square$

5 Proof of the Theorem 3.1

1^0. *A definition of some constants.*

Consider (6) with the HFC $u = \mathcal{A}(\delta)$ where $\delta > 0$ is an arbitrary fixed number satisfying the following property

$$\max\left\{2\, \tan 2\delta_1, \sqrt{6}\, \sin^2 \delta_1\right\} < 1, \qquad \forall \delta_1 \in [0, \delta]. \tag{24}$$

We define three functions $\omega_i : [0, 1) \to \mathbf{R}$ as follows:

$$\omega_1(\gamma) = \left(\frac{\pi}{4} - \delta\right) \sqrt{(1+\gamma)(4+\gamma)} - \arctan\left(\sqrt{\frac{1+\gamma}{4+\gamma}} \cdot \cot\left(\delta(1+\gamma)\right)\right),$$

$$\omega_2(\gamma) = \sqrt{\frac{4-\gamma}{1-\gamma}} \cdot \tan\left(\frac{\pi}{2} + \frac{\pi}{4}\,(4\delta - 1)\, \sqrt{(1-\gamma)(4-\gamma)}\right) - 1,$$

$$\omega_3(\gamma) = \sqrt{\frac{4 + 6\sin^2 \delta}{5}} \cdot e^{5\pi\gamma/4} - 1.$$

Taking into account (24), we have

$$\lim_{\gamma \to 0} \omega_i(\gamma) = \omega_i(0) < 0, \qquad i = 1, 2, 3.$$

This implies existence of a number $\gamma \in (0,1)$ for which

$$\omega_i(\gamma) < 0, \qquad i = 1, 2, 3. \tag{25}$$

Finally, fix an arbitrary $\gamma \in (0,1)$ satisfying (25) and put

$$\lambda = -\frac{4\,\ln(\omega_3(\gamma)+1)}{5\pi}, \qquad M = \exp\left\{\pi\gamma\left(\frac{2}{1-\gamma}+\frac{1}{4}\right)+\lambda\left(\frac{2\pi}{1-\gamma}+\frac{5\pi}{4}\right)\right\}. \tag{26}$$

By definition, $\lambda > 0$, $M > 0$.

2^0. *Some inequalities related to the dynamics of (6) with $u = \mathcal{A}(\delta)$.*

Let us fix an arbitrary Lebesgue measurable function $f{:=}\mathrm{col}\,\{f_1, f_2\} : [0,\infty) \to \mathbf{R}^2$, satisfying $|f(x)| = o(|x|)$ for $x \to 0$, as well as an arbitrary initial state $q_0 \in \{q_+, q_-, q_d\}$ of the automaton.

Consider a trajectory $x(t)$ of (6) which starts at $x(0)$. If $x(0) = 0$, then, evidently, $x \equiv 0$ which proves (8).

Assume now that $x(0) \neq 0$.

Let Γ be the plane transformation given by $\xi = r\cos\varphi$, $\eta = r\sin\varphi$. The Jacobi matrix of the transformation will be called J. Then any solution $x(t) = (\xi(t), \eta(t))$ of (6) is uniquely determined by a pair of functions $r : [0,\infty) \to [0,\infty)$, $\varphi : [0,\infty) \to \mathbf{R}/(2\pi\mathbf{Z})$ where $\xi(t) = r(t)\cos\varphi(t)$, $\eta(t) = r(t)\sin\varphi(t)$.
In what follows we assume that the function φ takes values from the interval $(-\pi, \pi]$. We let also $h = \mathrm{col}\,\{h_1, h_2\} := J^{-1} f\Gamma$.

Within any interval $S = (s_1, s_2) \subset [0,\infty)$ where the automaton does not change its location, the solution $x(t)$ satisfies one of the following systems of differential equations:

$$(\ q(t) = q_+,\ t \in S\ )\ \vee\ (\ q(t) = q_-,\ t \in S\ ) \implies \begin{cases} \frac{\dot r}{r} = h_1(r,\varphi) \\ \dot\varphi = -1 + h_2(r,\varphi), \end{cases}$$

$$(\ q(t) = q_d,\ t \in S\ ) \implies \begin{cases} \frac{\dot r}{r} = -\frac{3}{2}\sin 2\varphi + h_1(r,\varphi) \\ \dot\varphi = -1 - 3\cos^2\varphi + h_2(r,\varphi). \end{cases}$$

Clearly, $|h(r,\varphi)| = o(1)$, and $r \to 0$ uniformly with respect to φ. For the number γ, defined in the item $\mathbf{1}^0$ of the proof, one can find $\varepsilon > 0$ such that

$$|h_i(r,\varphi)| \le \gamma, \qquad \forall r \in [0,\varepsilon],\ \ \forall\varphi,\ \ i = 1, 2.$$

We fix now an interval $(s_1, s_2) \subset [0,\infty)$. The estimates above imply the following statements:
1) If $\forall t \in (s_1, s_2)$, $r(t) \le \varepsilon$, and either $q(\cdot) \equiv q_+$, or $q(\cdot) \equiv q_+$ on (s_1, s_2), then

$$-\gamma \le \frac{\dot r(t)}{r(t)} \le \gamma, \qquad -1-\gamma \le \dot\varphi(t) \le -1+\gamma, \qquad t \in (s_1, s_2). \tag{27}$$

2) If $\forall t \in (s_1, s_2)$, $r(t) \le \varepsilon$ and $q(\cdot) \equiv q_d$ on (s_1, s_2), then

$$\begin{gathered} -\gamma - \frac{3}{2}\sin 2\varphi(t) \le \frac{\dot r(t)}{r(t)} \le \gamma - \frac{3}{2}\sin 2\varphi(t), \\ -1-\gamma-3\cos^2\varphi(t) \le \dot\varphi(t) \le -1+\gamma-3\cos^2\varphi(t), \quad t \in (s_1, s_2). \end{gathered} \tag{28}$$

Integrating these inequalities, we obtain the following:
1) If $\forall t \in (s_1, s_2)$, $r(t) \le \varepsilon$, and either $q(\cdot) \equiv q_+$, or $q(\cdot) \equiv q_+$ on (s_1, s_2), then

$$(1-\gamma)(s_2-s_1) \le \varphi(s_1)\varphi(s_2) \le (1+\gamma)(s_2-s_1), \tag{29}$$

$$\frac{r(s_2)}{r(s_1)} \le e^{\gamma(s_2-s_1)}. \tag{30}$$

2) If $\forall t \in (s_1, s_2),\ \ r(t) \le \varepsilon$ and $q(\cdot) \equiv q_d$ on (s_1, s_2), then

$$\psi(1+\gamma,\varphi)\left|_{\varphi=\varphi(t_2)}^{\varphi=\varphi(s_1)}\right. \le s_2-s_1 \le \psi(1-\gamma,\varphi)\left|_{\varphi=\varphi(s_2)}^{\varphi=\varphi(s_1)}\right., \tag{31}$$

where ψ is given by

$$\psi(\alpha,\varphi) = \frac{1}{\sqrt{\alpha(\alpha+3)}} \arctan\left(\sqrt{\frac{\alpha}{\alpha+3}}\,\tan\varphi\right). \tag{32}$$

If, in addition, $\varphi([s_1,s_2]) \subset [0,\pi/2] \pmod{\pi}$, then

$$\frac{r(s_2)}{r(s_1)} \le \sqrt{\frac{1+\gamma+3\cos^2\varphi(s_1)}{1+\gamma+3\cos^2\varphi(s_2)}}\, e^{\gamma(s_2-s_1)}. \tag{33}$$

In the items $\mathbf{3^0}$–$\mathbf{6^0}$ below we fix an arbitrary $T \in (0,\infty)$ and assume that the estimate

$$r(t) \le \varepsilon, \qquad t \in [0,T] \tag{34}$$

holds true.

We shall investigate the dynamics of the system (6) for various T. We notice first that (27) and (28) imply $\dot\varphi(t) < 0,\ \ \dot r(t) \le \gamma\ \ (\forall t \in [0,T])$. Therefore,

$$r(t) \le e^{\gamma t} r(0) \le e^{\frac{2\pi\gamma}{1-\gamma}} r(0), \qquad t \in \left[0, \min\left\{\frac{2\pi\gamma}{1-\gamma}, T\right\}\right]. \tag{35}$$

We shall also assume in the three forthcoming items of the proof that $T \ge \frac{2\pi}{1-\gamma}$.

$\mathbf{3^0}$**.** *A definition of a switching sequence for the automaton defined on* $[0,T]$*.*

First of all we note that if the locations q_+ or q_- are switched on within a sub-interval $[t_1,t_2] \subset \left[0, \frac{2\pi}{1-\gamma}\right]$, then (35) holds, while in case the location q_d is switched on within $[t_1,t_2]$, we will, by virtue of (31), have that

$$\varphi(t_1) - \varphi(t_2) \le 2\cdot\left(\frac{\pi}{4} - \delta\right) \le \frac{\pi}{2}. \tag{36}$$

A simple straightforward analysis of the trajectories' behavior with different $x(0)$ and q_0 as well as the observation that (29), (36) and (35) hold for $t \in \left[0, \frac{2\pi}{1-\gamma}\right]$ imply that the automaton's location q_d will be switched on at some time point $t_0 \in \left[0, \frac{2\pi}{1-\gamma}\right]$ for which

$$\varphi(t_0) \in \left[\frac{\pi}{2} - \delta(1+\gamma), \frac{\pi}{2}\right]. \tag{37}$$

In other words, the time $t = t_0$ is nothing, but a transition time of the automaton when the latter comes to the location q_d from another location (the number of previous switches to the location q_d does not matter here).

Continuing our proof we call by $t_0, t_1, \ldots, t_m$, respectively, all the transition times of the automaton from a location q_i to a different location $q_j \ne q_i$ until the moment $t = T$ (we ignore all "switches" from q_+ to q_+ and from q_- to q_-). Assume without loss of generality that $\{t_i\}$ is an increasing sequence. By now, we have just verified the fact that the time t_0 does exist and coincides with the time when the automaton gets to the location q_d (see (37)). Note that the number m depends on T and that we do not assume that $m > 0$. The case $m = 0$ is therefore not excluded.

4^0. *The dynamics of $q(t)$ and $\varphi(t)$ on $[0, T]$.*
Let $T_d = \pi/4 - \delta$, $t^* = \min\{T, t_0 + T_d\}$. By virtue of (31) we have

$$T_d \geq t - t_0 \geq \psi(1+\gamma, \varphi) \Big|_{\varphi(t)}^{\varphi(t_0)} \geq \psi(1+\gamma, \varphi) \Big|_{\varphi(t)}^{\pi/2-2\delta},$$

where ψ is defined by (32). This and the inequality $\omega_1(\gamma) < 0$ (see (25)) imply that $\varphi(t) \geq 0$.

Thus, in the case $T \geq t_0 + T_d$ we obtain that $\varphi(t_0 + T_d) \geq 0$. For such T the estimates (31) imply that

$$T_d \leq \psi(1-\gamma, \varphi) \Big|_{\varphi(t_0+T_d)}^{\varphi(t_0)} \leq \psi(1+\gamma, \varphi) \Big|_{\varphi(t_0+T_d)}^{\pi/2}.$$

¿From this and from the inequality $\omega_2(\gamma) < 0$ it immediately follows that $\varphi(t_0 + T_d) \leq \pi/4$.

Assuming $T \geq t_0 + T_d$, we can therefore define the time $t_1 = t_0 + T_d$ when switching from q_d to q_+ occurs and where we also have $0 \leq \varphi(t_1) \leq \pi/4$.

According to (29), if the automaton's location q_+ is switched on, the function $\varphi(t)$ will decrease for $t \geq t_1$. It is also clear that no transition will occur as long as $\varphi(t) > -\pi/2$. If $m \geq 2$, then t_2 becomes a time when the automaton switches over to the location q_d, and due to (29) we get

$$t_2 \in [t_1, t_1 + \pi], \qquad \varphi(t_2) \in \left[-\frac{\pi}{2} - \delta(1+\gamma), -\frac{\pi}{2}\right].$$

This implies, in particular, that $m \geq 2$ for $T \geq t_1 + \pi$.

Repeating this argument inductively between the transition times t_{2n} and t_{2n+2}, we arrive at the following conclusion (referred in the sequel as *Property A*) describing the dynamics $q(t)$ and the sequence of the transition times $\{t_i\}$.

Property A. *Let $T \geq \frac{2\pi}{1-\gamma}$, $\tau_1 = T_d$, $\tau_2 = \pi$. Then the sequence of transition times $t_0, t_1, \ldots, t_{2m}$ (from q_i to q_j, where $q_i \neq q_j$) possesses the following properties:*

1) $t_0 \in \left[0, \frac{2\pi}{1-\gamma}\right]$, $\varphi(t_0) \in \left[\frac{\pi}{2} - \delta(1+\gamma), \frac{\pi}{2}\right] \pmod{\pi}$, $q(t_0) = q_d$;
2) $t_{2m} \leq T \leq t_{2m} + \tau_1 + \tau_2$;
3) If $m \geq 1$, then for $n = 0, 1, \ldots m-1$ we have

$$0 < t_{2n+i} - t_{2n+i-1} \leq \tau_i, \qquad i = 1, 2;$$

$$q(t_{2n+2}) = q_d, \qquad \varphi(t_{2n+2}) \in \left[\frac{\pi}{2} - \delta(1+\gamma), \frac{\pi}{2}\right] \pmod{\pi};$$

$$q(t_{2n+1}) \in \{q_+, q_-\}, \quad \varphi(t_{2n+1}) \in \left[0, \frac{\pi}{4}\right] \pmod{\pi}.$$

5^0. *The dynamics of $r(t)$ on $[0, T]$.*
Let a finite sequence $t_0, t_1, \ldots, t_{2m}$ be chosen according to Property A. The estimates in Property A imply that for $\varphi(t_i)$ the following holds:

$$-\frac{3}{2}\sin 2\varphi(t) < 0, \qquad t \in [t_{2n}, \min\{T, t_{2n+2}\}], \qquad n = 0, 1, \ldots, m-1.$$

According to the two first inequalities in (27), (28) we get

$$\frac{\dot{r}(t)}{r(t)} \leq \gamma, \qquad t \in [t_{2n}, \min\{T, t_{2n+2}\}], \qquad n = 0, 1, \ldots, m-1.$$

Thus, we obtain the estimate

$$r(t) \leq r(t_{2n})\, e^{\gamma(t-t_{2n})} \leq e^{5\pi\gamma/4} r(t_{2n}), \qquad t \in [t_{2n}, \min\{T, t_{2n+2}\}], \quad n = 0, 1, \ldots, m-1. \quad (38)$$

Now, (30) and Property A imply that

$$\frac{r(t_{2n+2})}{r(t_{2n+1})} \leq e^{\gamma(t_{2n}-t_{2n-1})} \leq e^{\pi\gamma} =: \alpha_2. \qquad n = 0, 1, \ldots, m-1. \quad (39)$$

Similarly, for $n = 0, 1, \ldots, m-1$ the inequality (33) and Property A yield

$$\frac{r(t_{2n+1})}{r(t_{2n})} \leq \sqrt{\frac{1+\gamma+3\cos^2\varphi(t_{2n})}{1+\gamma+3\cos^2\varphi(t_{2n+1})}}\, e^{\gamma(t_{2n+1}-t_{2n})} \leq \sqrt{\frac{4+6\sin^2\delta}{5}}\, e^{\pi\gamma/4} =: \alpha_1. \quad (40)$$

6^0. According to the definition of the constants α_1,α_2 v (39) i (40) and the inequality $\omega_3(\gamma) < 0$ (see (25)) we have $\alpha_1\alpha_2 < 1$. This, the estimates (35), Property A and the inequalities (38)–(40) imply that under the assumption (34) the function $r(t)$ belongs to the class $\mathcal{E}(T, a, b, c, \tau_1, \tau_2, \alpha_1, \alpha_2)$, introduced in Lemma 4.1, where $\tau_1 = T_d$, $\tau_2 = \pi$, a postoyannye a, b, c opredelyayutsya ravenstvami $a = \frac{2\pi}{1-\gamma}$, $b = \exp\left\{\frac{2\pi\gamma}{1-\gamma}\right\}$, $c = e^{\pi\gamma/4}$.

Applying Lemma 4.1, we conclude that $\forall\, T \in (0, \infty)$

$$[\, r(t) \leq \varepsilon,\ t \in [0, T]\,] \implies \left[\, r(t) \leq M\, e^{-\lambda t} r(0)\,\right], \quad (41)$$

where the constants $\lambda > 0$, $M > 0$ are defined by (26).

7^0. Finally, we define a function $\zeta : [0, \infty)^2 \to [0, \infty)$ as follows: $\zeta(t, \mu) = |x(t)|$, where $x(t)$ is a trajectory of (6) controlled by $u = \mathcal{A}(\delta)$, and $|x(0)| = \mu$. From (41) it follows that ζ satisfies all the conditions of Lemma 4.2.

According to this lemma the inequality (8) holds true for all sufficiently small $|x(0)|$ the constants $\lambda > 0$, $M > 0$ being defined by (26). It is straightforward that the constants do not depend on the choice of a function $f : \mathbf{R}^2 \to \mathbf{R}^2$ satisfying $|f(x)| = o(|x|)$. This completes the proof of the theorem. □

6 Final remarks

To prove Theorem 3.1 we employed an approach based on certain direct calculations applied to the hybrid dynamics, rather than the method of Lyapunov functions widely used to study asymptotic stability of various dynamical systems. This can be explained by a number reasons.

1. The dynamics described by the equation (6) equipped with the hybrid control $u = \mathcal{A}(\delta)$ is normally quite different from the standard dynamics arising from ordinary differential equations. The dynamics of a hybrid system like (6) rather comes from delay equations and inherits some of properties typical for them (see e.g., (Azbelev *et al.*, 1996; Berezansky and Braverman, 1995; Chukwu, 1992; Kolmanovski and Nosov, 1986)).

2. Of course, one may try to use generalizations of the Lyapunov method to the case of functional differential equations by introducing Lyapunov functionals (see (Chukwu, 1992; Kolmanovski and Nosov, 1986)). We suspect that this, in principle, is possible, but one should always remember that such a technique may lead to serious difficulties, both technical and ideological, related to the problem how to construct such functionals and how to use them properly. One of the difficulties is that the operators one gets from (even linear) systems with HFCs are usually nonlinear. Moreover, one should be prepared to deal with differential equations with time lags depending on the solution itself (known as systems with auto-control). A typical operator related to such a system is, in general, discontinuous. It is not clear for us for the time being how to study such operators and their (dis)continuity properties in functional spaces, and it is not clear for us either how to deal with the corresponding Lyapunov functionals which cannot properly be defined outside the trajectories of solutions. By these reasons, any usage of Lyapunov functionals was beyond the scope of this particular paper.

3. We failed as well to use the machinery of the so-called *multiple Lyapunov functions* developed in (Branicky, 1994, 1995). In particular, we encountered a problem of how to pass from the system (6) to a switching system in the sense of Branicky (1995). However, it does not mean of course that it is impossible at all.

Moreover, we do believe that any further considerable progress in developing the Lyapunov method focused on systems with HFCs could only occur in the form of either the method of Lyapunov functionals, or the method of multiple Lyapunov functions for switching systems.

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