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A Composite Semigroup for the Infinite-Dimensional Differential Sylvester Equation*

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Abstract

This paper presents a certain approach to the study of the operator differential Sylvester equation which arises in various control problems on finite time horizon. A crucial role in this approach is played by the so-called composite semigroup. It is a strong-operator continuous semigroup defined on a Banach space of linear bounded operators obtained as a composition of two 'classical' strongly continuous semigroups defined on a Hilbert space. We investigate basic properties of the solution to this equation in the case when the operators occuring in in the equation are unbounded.

1 Introduction

This paper introduces to the concept of a *composite semigroup* and its application to the analysis of the operator differential Sylvester equation. This equation arises in various control problems on finite time horizon $[0,\tau]$, $\tau \in (0,\infty)$, for linear infinite-dimensional systems with unbounded input or output operators. In order to formulate the problems precisely we have to introduce the following notation and assumptions.

$\mathbf{A}_1$ and $\mathbf{A}_2$ are linear unbounded operators on a real Hilbert space H (identified with its dual) with the domains $D(\mathbf{A}_1)$ and $D(\mathbf{A}_2)$. We assume that both $\mathbf{A}_1$ and $\mathbf{A}_2$ generate strongly continuous semigroups $\mathbf{T}_1(t) \in \mathcal{L}(H)$ and $\mathbf{T}_2(t) \in \mathcal{L}(H)$, $t \in [0,\infty)$. $H_1^1 = D(\mathbf{A}_1)$ is a Hilbert space with appropriately defined scalar product and H_{-1}^1 is a completion of H with respect to the norm $\|\cdot\|_{H_{-1}^1} = \|(\lambda\mathbf{I}-\mathbf{A}_1)^{-1}(\cdot)\|_H$. Notice that H_{-1}^1 can be equivalently defined as $D(\mathbf{A}_1^*)^*$, i.e. the dual to the domain $D(\mathbf{A}_1^*)$ of the unbounded adjoint operator $\mathbf{A}_1^*$ on H. Analogously, we define spaces H_1^2 and H_{-1}^2 for the operator $\mathbf{A}_2$.

The main objective of our examination is the following operator differential equation (for the time being written formally)

$$\dot{\mathbf{X}}(t) = \mathbf{A}_1\mathbf{X}(t) + \mathbf{X}(t)\mathbf{A}_2 + \mathbf{B}_1\mathbf{C}_2, \qquad \mathbf{X}(0) = \mathbf{X}_0, \tag{1}$$

where $t \in [0,\tau]$, $\mathbf{A}_1$ are $\mathbf{A}_2$ are unbounded operators defined above, $\mathbf{C}_2 \in \mathcal{L}(H_1^2, Z)$ is an admissible observation operator for $\mathbf{T}_2(t)$ (see Weiss (1989a)) and $\mathbf{B}_1 \in \mathcal{L}(Z, H_{-1}^1)$ is an admissible

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control operator for $\mathbf{T}_1(t)$ (see Weiss (1989b)), where Z is a real Hilbert space. Usually, equation (1) is referred to as the *differential Sylvester equation*, e.g. Gajic and Qureshi (1995), and it is almost impossible to overestimate the importance of this equation in systems and control theory.

Probably, the most important special cases of equation (1), frequently appearing in analysis of numerous control problems on finite time horizon $[0,\tau]$, see e.g. Gajic and Qureshi (1995) or Emirsajlow and Townley (1995), are the following operator differential Lyapunov equations

$$\dot{\mathbf{X}}(t)=\mathbf{A}\mathbf{X}(t)+\mathbf{X}(t)\mathbf{A}^*+\mathbf{B}\mathbf{B}^*, \qquad \mathbf{X}(0)=\mathbf{X}_0, \tag{2}$$

or

$$\dot{\mathbf{X}}(t)=\mathbf{A}^*\mathbf{X}(t)+\mathbf{X}(t)\mathbf{A}+\mathbf{C}^*\mathbf{C}, \qquad \mathbf{X}(0)=\mathbf{X}_0, \tag{3}$$

where $t\in[0,\tau]$.

The main **goal** of this paper is to develop a certain mathematical framework whithin which the basic properties of a solution to the differential equation (1), can be analyzed.

The **main idea** we explore is an introduction of the so-called *composite semigroup* $\mathbb{T}(t)\in\mathcal{L}(\mathcal{L}(H))$ (i.e. $\mathbb{T}(t):\mathcal{L}(H)\rightarrow\mathcal{L}(H)$), $t\in[0,\infty)$, defined as follows

$$\mathbb{T}(t)\mathbf{X}=\mathbf{T}_1(t)\mathbf{X}\mathbf{T}_2(t), \quad \mathbf{X}\in\mathcal{L}(H), \quad t\in[0,\infty), \tag{4}$$

where $\mathbf{T}_1(t),\mathbf{T}_2(t)\in\mathcal{L}(H)$, $t\in[0,\infty)$, are generated by $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively.

2 Properties of the composite semigroup

We start with examination of basic properties of the operator $\mathbb{T}(t)\in\mathcal{L}(\mathcal{L}(H))$.

Lemma 2.1 *(a) If* $\|\mathbf{T}_1(t)\|_{\mathcal{L}(H)}\leq M_1e^{\omega_1t}$ *and* $\|\mathbf{T}_2(t)\|_{\mathcal{L}(H)}\leq M_2e^{\omega_2t}$*, then*

$$\|\mathbb{T}(t)\mathbf{X}\|_{\mathcal{L}(H)}\leq M_1M_2e^{(\omega_1+\omega_2)t}\|\mathbf{X}\|_{\mathcal{L}(H)}, \quad t\in[0,\infty). \tag{5}$$

(b) The family $\mathbf{T}(t)\in\mathcal{L}(\mathcal{L}(H))$*,* $t\in[0,\infty)$*, is a semigroup, i.e.*

$$\begin{aligned}\mathbb{T}(0)\mathbf{X} &= \mathbf{X}, \quad \mathbf{X}\in\mathcal{L}(H),\\ \mathbb{T}(t+s)\mathbf{X} &= \mathbb{T}(t)(\mathbb{T}(s)\mathbf{X})=\mathbb{T}(s)(\mathbb{T}(t)\mathbf{X}), \quad \mathbf{X}\in\mathcal{L}(H), \quad t,s\in[0,\infty).\end{aligned}$$

(c) $\mathbf{T}(\cdot)\in\mathcal{L}(\mathcal{L}(H))$ *is strong-operator continuous at origin, i.e.*

$$\lim_{t\rightarrow0^+}\|(\mathbb{T}(t)\mathbf{X})h-(\mathbb{T}(0)\mathbf{X})h\|_H=0, \quad h\in H, \quad \mathbf{X}\in\mathcal{L}(H).$$

This lemma can be proven rather easily using the definition (4). It can be also shown that Part (c) implies that $\mathbf{T}(\cdot)\in\mathcal{L}(\mathcal{L}(H))$ is strong-operator continuous at every $t\in[0,\infty)$ and, in general cannot be strongly continuous unless the semigroups $\mathbf{T}_1(\cdot)$ and $\mathbf{T}_2(\cdot)$ are uniformly continuous.

The *infinitesimal generator* $\mathbb{A}$ of $\mathbb{T}(t)$, understood as an operator on $\mathcal{L}(H)$, is defined as the limit

$$(\mathbb{A}\mathbf{X})h=\lim_{t\rightarrow0^+}\frac{(\mathbb{T}(t)\mathbf{X})h-\mathbf{X}h}{t}\}, \quad \mathbf{X}\in D(\mathbb{A}), \quad h\in H, \tag{6}$$

where $D(\mathbb{A})\subset\mathcal{L}(H)$ is its *domain* defined as follows

$$D(\mathbb{A})=\{\mathbf{X}\in\mathcal{L}(H): \quad \lim_{t\rightarrow0^+}\frac{(\mathbb{T}(t)\mathbf{X})h-\mathbf{X}h}{t}\}, \tag{7}$$

with the limit existing for every $h\in H$.

Lemma 2.2 *The operator* $\mathbb{A}: \quad D(\mathbb{A}) \mapsto \mathcal{L}(H)$ *enjoys the following properties.*

(a) $D(\mathbb{A})$ *is strong-operator dense in* $\mathcal{L}(H)$.

(b) $\mathbb{A}$ *is uniform-operator closed on* $\mathcal{L}(H)$.

(c) For $\mathbf{X} \in \mathcal{L}(H)$

$$\int_0^t (\mathbb{T}(r)\mathbf{X})dr \in D(\mathbb{A}) \quad \textit{and} \quad \mathbb{A}(\int_0^t (\mathbb{T}(r)\mathbf{X})dr) = \mathbb{T}(t)\mathbf{X} - \mathbf{X}.$$

(d) For $\mathbf{X} \in D(\mathbb{A})$

$$\mathbb{T}(t)\mathbf{X} \in D(\mathbb{A}) \quad \textit{and} \quad \frac{d}{dt}(\mathbb{T}(t)\mathbf{X}) = \mathbb{A}(\mathbb{T}(t)\mathbf{X}) = \mathbb{T}(t)(\mathbb{A}\mathbf{X}).$$

(e) For $\mathbf{X} \in D(\mathbb{A})$ *and* $h \in D(\mathbf{A}_2)$

$$(\mathbb{A}\mathbf{X})h = \mathbf{A}_1\mathbf{X}h + \mathbf{X}\mathbf{A}_2h. \tag{8}$$

Proof : We omit the proofs of Parts (a)-(d) since they are essentially the same as for the strongly continuous case (see Pazy (1983)). Part (c) can proven as follows. Let $h \in D(\mathbf{A}_2) \subset H$, $g \in D(\mathbf{A}_1^*) \subset H$, $\mathbf{X} \in D(\mathbb{A}) \subset \mathcal{L}(H)$ and then

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{1}{t}\langle(\mathbb{T}(t)\mathbf{X})h - \mathbf{X}h, g\rangle_H \\
&\qquad = \lim_{t\to 0^+} \frac{1}{t}\langle \mathbf{X}(\mathbf{T}_2(t) - \mathbf{I})h, (\mathbf{T}_1^*(t) - \mathbf{I})g\rangle_H \\
&\qquad\quad + \lim_{t\to 0^+} \frac{1}{t}\langle \mathbf{X}(\mathbf{T}_2(t) - \mathbf{I})h, g\rangle_H + \lim_{t\to 0^+} \frac{1}{t}\langle \mathbf{X}h, (\mathbf{T}_1^*(t) - \mathbf{I})g\rangle_H \\
&\qquad = \langle \mathbf{X}\mathbf{A}_2h, g\rangle_H + \langle \mathbf{X}h, \mathbf{A}_1^*g\rangle_H \\
&\qquad = \langle(\mathbf{A}_1\mathbf{X} + \mathbf{X}\mathbf{A}_2)h, g\rangle_{D(\mathbf{A}_1^*)^*, D(\mathbf{A}_1^*)}.
\end{aligned}$$

Hence

$$\langle(\mathbb{A}\mathbf{X})h, g\rangle_H = \langle(\mathbf{A}_1\mathbf{X} + \mathbf{X}\mathbf{A}_2)h, g\rangle_H$$

and since this extends to all $g \in H$ we have

$$(\mathbf{A}_1\mathbf{X} + \mathbf{X}\mathbf{A}_2)h \in H, \quad \mathbf{A}_1\mathbf{X}h \in H.$$

$\square$

Using the above concept of a composite semigroup we can look at the following operator differential equation

$$\dot{\mathbf{X}}(t) = \mathbb{A}\mathbf{X}(t) + \mathbf{Y}, \quad \mathbf{X}(0) = \mathbf{X}_0, \quad t \in [0, \infty), \tag{9}$$

where $\mathbf{Y} \in \mathcal{L}(H)$. It can be easily shown that for $\mathbf{X}_0 \in D(\mathbb{A})$ this equation has a unique solution $\mathbf{X}(t) \in D(\mathbb{A})$ which is strong-operator differentiable in $\mathcal{L}(H)$. It is also clear that this solution is given by

$$\mathbf{X}(t) = \mathbb{T}(t)\mathbf{X}_0 + \int_0^t (\mathbb{T}(t-r)\mathbf{Y})dr. \tag{10}$$

3 Some generalization

If we take into considerations properties of the semigroups $\mathbf{T}_1(t)$ and $\mathbf{T}_2(t)$ then it can be seen that the second term in the formula (10) makes also sense for $\mathbf{Y} \in \mathcal{L}(D(\mathbf{A}_2), D(\mathbf{A}_1^*)^*)$, as an integral on $\mathcal{L}(D(\mathbf{A}_2), D(\mathbf{A}_1^*)^*)$ with the strong-operator topology. Now we need the following definition.

Definition 3.1 An operator $\mathbf{Y} \in \mathcal{L}(D(\mathbf{A}_2), D(\mathbf{A}_1^*)^*)$ is said to be *admissible* for $\mathbb{T}(t) \in \mathcal{L}(\mathcal{L}(H))$ if for every $t > 0$ the following inequality holds

$$|\langle \int_0^t (\mathbb{T}(t-r)\mathbf{Y})hdr, g\rangle_{D(\mathbf{A}_1^*)^*, D(\mathbf{A}_1^*)}| \leq m(t)\|h\|_H \|g\|_H, \quad h \in D(\mathbf{A}_2), \quad g \in D(\mathbf{A}_1^*). \tag{11}$$

It is clear that in this case (11) extends to all $h, g \in H$ which can be equivalently written as

$$\int_0^t (\mathbb{T}(t-r)\mathbf{Y})dr \in \mathcal{L}(D(\mathbf{A}_2), D(\mathbf{A}_1^*)^*) \cap \mathcal{L}(H).$$

Lemma 3.2 *If $\mathbf{C}_2 \in \mathcal{L}(D(\mathbf{A}_2), Z)$ is an admissible observation operator for $\mathbf{T}_2(t)$ and $\mathbf{B}_1 \in \mathcal{L}(Z, D(\mathbf{A}_1^*)^*)$ is an admissible control operator for $\mathbf{T}_1(t)$, then $\mathbf{B}_1\mathbf{C}_2 \in \mathcal{L}(D(\mathbf{A}_2), D(\mathbf{A}_1^*)^*)$ is admissible for $\mathbb{T}(t) \in \mathcal{L}(\mathcal{L}(H))$.*

The proof follows rather easily from the Cauchy-Schwarz inequality, so we omit details. Let us now consider the following differential equation

$$\dot{\mathbf{X}}(t) = \mathbb{A}\mathbf{X}(t) + \mathbf{B}_1\mathbf{C}_2, \quad \mathbf{X}(0) = \mathbf{X}_0 \tag{12}$$

and notice that for $\mathbf{X} \in \mathcal{L}(H)$ the expression $\mathbb{A}\mathbf{X}$ makes sense as an operator in $\mathcal{L}(D(\mathbf{A}_2), D(\mathbf{A}_1^*)^*)$. Our final result reads as follows.

Theorem 3.3 *If $\mathbf{C}_2 \in \mathcal{L}(D(\mathbf{A}_2), Z)$ is an admissible observation operator for $\mathbf{T}_2(t)$ and $\mathbf{B}_1 \in \mathcal{L}(Z, D(\mathbf{A}_1^*)^*)$ is an admissible control operator for $\mathbf{T}_1(t)$, then for every $\mathbf{X}_0 \in \mathcal{L}(H)$ equation (12) has a unique solution $\mathbf{X}(t) \in \mathcal{L}(H)$ such that $\dot{\mathbf{X}}(t) \in \mathcal{L}(D(\mathbf{A}_2), D(\mathbf{A}_1^*)^*)$, both with weak-operator topologies. This solution is explicitly given by the formula*

$$\mathbf{X}(t) = \mathbb{T}(t)\mathbf{X}_0 + \int_0^t (\mathbb{T}(t-r)\mathbf{B}_1\mathbf{C}_2)dr. \tag{13}$$

References

Emirsajlow Z., Townley S. (1995). Uncertain systems and minimum energy control, *Applied Mathematics and Computer Science*, Vol. 5, No. 3.

Gajic Z., Qureshi M. (1995). *Lyapunov Matrix Equation in Systems Stability and Control*. Academic Press, London.

Pazy A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations* , Springer-Verlag, New York.

Weiss G. (1989a). Admissible observation operators for linear semigroups, *Israel Journal of Mathematics*, Vol. 65, No. 1, pp. 17–43.

Weiss G. (1989b). Admissibility of unbounded control operators, *SIAM Journal on Control and Optimization*, Vol. 27, No. 3, pp. 527–545.