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ASYMPTOTIC BEHAVIOR OF INFINITE PRODUCTS OF ORDER-PRESERVING MAPPINGS IN BANACH SPACES

Simeon Reich and Alexander J. Zaslavski

Department of Mathematics
The Technion-Israel Institute of Technology
32000 Haifa, Israel

Abstract

In this paper we present several results concerning the asymptotic behavior of (random) infinite products of generic sequences of order-preserving mappings on intervals and cones in an ordered Banach space. Such operators find application in many areas of mathematics and, in particular, in dynamical models of economics and biology. In addition to weak ergodic theorems we also obtain convergence to a unique common fixed point (for self-mappings of an interval) and to an operator of the form $f(\cdot)\eta$, where f is a functional and η is a common fixed point. More precisely, we show that in appropriate complete metric spaces of sequences of operators there exists a subset which is a countable intersection of open everywhere dense sets such that for each sequence belonging to this subset the corresponding infinite products converge. Thus, instead of considering a certain convergence property for a single sequence of operators, we investigate it for a space of all such sequences equipped with some natural metric, and show that this property holds for most of these sequences. This allows us to establish convergence without restrictive assumptions on the space and on the operators themselves.

1 Introduction

Order-preserving mappings and infinite products of operators find application in many areas of mathematics. See, for example, (Aliprantis and Burkinshaw, 1985; Amann, 1976; Bauschke and Borwein, 1996; Bauschke *et al.,* 1997; Censor and Reich, 1996; Dye *et al.,* 1996; Dye and Reich, 1992; Fujimoto and Krause, 1988; Lin, 1995; Nussbaum, 1990) and the references mentioned there. Our goal in this paper is to study the asymptotic behavior of (random) infinite products of generic sequences of order-preserving mappings on intervals and cones in an ordered Banach space. More precisely, we show that in appropriate complete metric spaces of sequences of operators there exists a subset which is a countable intersection of open everywhere dense sets such that for each sequence belonging to this subset the corresponding infinite products converge. Results of this kind for powers of a single nonexpansive operator were already established by De Blasi and Myjak (1976) while such results for infinite products have recently been obtained in (Reich and Zaslavski, 1999a). The approach used in these papers and in the present paper is common in global analysis and the theory of dynamical systems (De Blasi and Myjak, 1983; Myjak, 1983). Recently it has also been used in the study of the structure of extremals of variational and optimal control problems (Zaslavski, 1995; Zaslavski, 1996). Thus, instead of considering a certain convergence property for a single sequence of operators, we investigate it for a space of all such sequences equipped with some natural metric, and show that this property

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holds for most of these sequences. This allows us to establish convergence without restrictive assumptions on the space and on the operators themselves.

2 Order-preserving mappings on intervals

Let $(X, ||\cdot||)$ be a Banach space ordered by a closed convex cone X_+ such that $||x|| \le ||y||$ for each $x, y \in X_+$ satisfying $x \le y$. For each $u, v \in X$ such that $u \le v$ denote

$$< u, v >= \{x \in X : u \le x \le v\}.$$

For each $x, y \in X_+$ we define

$$\lambda(x, y) = \sup\{r \ge 0 : \; rx \le y\}.$$

Let $b \in X_+ \setminus \{0\}$. We consider the space $< 0, b > \subset X$ with the topology induced by the norm $||\cdot||$. Denote by $\mathfrak{A}$ the set of all continuous operators $A :< 0, b > \to < 0, b >$ such that

$$Ax \le Ay \text{ for each } x, y \in < 0, b > \text{ satisfying } x \le y$$

and

$$A(\alpha z) \ge \alpha Az \text{ for each } z \; \in < 0, b > \text{ and each } \alpha \in [0, 1].$$

For the space $\mathfrak{A}$ we define a metric $\rho : \mathfrak{A} \times \mathfrak{A} \to [0, \infty)$ by

$$\rho(A, B) = \sup\{||Ax - Bx|| : x \in < 0, b >\}, \; A, B \in \mathfrak{A}.$$

It is easy to see that the metric space $\mathfrak{A}$ is complete.

For more information on ordered Banach spaces, order-preserving mappings and their applications see, for example, (Aliprantis and Burkinshaw, 1985; Amann, 1976).

In (Reich and Zaslavski, 1999b) we established the following result.

Theorem 2.1. *There exists a set $\mathfrak{F} \subset \mathfrak{A}$ which is a countable intersection of open everywhere dense sets in $\mathfrak{A}$ such that for each $B \in \mathfrak{F}$ the following assertions hold:*

1. There exists $x_B \in < 0, b >$ such that $Bx_B = x_B$ and

$$B^T x \to x_B \text{ as } T \to \infty \text{ uniformly on } < 0, b > .$$

2. For each $\epsilon > 0$ there exists a neighborhood U of B in $\mathfrak{A}$ and an integer $N \ge 1$ such that for each $C \in U$, $z \in < 0, b >$ and each integer $T \ge N$,

$$||C^T z - x_B|| \le \epsilon.$$

Assume now that b is an interior point of the cone X_+. Define

$$||x||_b = \inf\{r \in [0, \infty) : -rb \le x \le rb\}, \quad x \in X.$$

Clearly $||\cdot||_b$ is a norm on X which is equivalent to the norm $||\cdot||$.

Denote by $\mathfrak{M}$ the set of all sequences $\{A_t\}_{t=1}^{\infty}$, where each $A_t \in \mathfrak{A}$, $t = 1, 2, \ldots$ Such a sequence will occasionally be denoted by a boldface $\mathbf{A}$. For the set $\mathfrak{M}$ we consider the metric $\rho_s : \mathfrak{M} \times \mathfrak{M} \to [0, \infty)$ defined by

$$\rho_s(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) = \sup\{||A_t x - B_t x||_b : \quad x \in < 0, b >,$$

$$t = 1, 2, \dots \}, \quad \{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty} \in \mathfrak{M}.$$

It is easy to see that this metric space $(\mathfrak{M}, \rho_s)$ is complete. The topology generated in $\mathfrak{M}$ by the metric ρ_s will be called the strong topology.

In addition to this topology on $\mathfrak{M}$ we will also consider the uniformity which is determined by the base

$$E(N, \epsilon) = \{(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) \in \mathfrak{M} \times \mathfrak{M} :$$

$$||A_t x - B_t x||_b \leq \epsilon, \ t = 1, \dots N, \ x \in <0, b>\},$$

where N is a natural number and $\epsilon > 0$. It is easy to see that the space $\mathfrak{M}$ with this uniformity is metrizable (by a metric $\rho_w : \mathfrak{M} \times \mathfrak{M} \to [0, \infty)$) and complete. The topology generated by ρ_w will be called the weak topology. The following two results established in (Reich and Zaslavski, 1999b) describe the asymptotic behavior of (random) infinite products of a generic sequence $\{C_t\}_{t=1}^{\infty}$ in the space $\mathfrak{M}$. Such results are usually called weak ergodic theorems in the population biology literature (Cohen, 1979; Fujimoto and Krause, 1988; Nussbaum, 1990).

Theorem 2.2. *There exists a set $\mathfrak{F} \subset \mathfrak{M}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) sets in $\mathfrak{M}$ and such that for each $\{B_t\}_{t=1}^{\infty} \in \mathfrak{F}$ the following assertions hold:*

For each $\epsilon \in (0, 1)$ there exist a neighborhood U of $\{B_t\}_{t=1}^{\infty}$ in $\mathfrak{M}$ with the weak topology and an integer $N \geq 1$ such that for each $\{C_t\}_{t=1}^{\infty} \in U$, each integer $T \geq N$ and each $x \in <0, b>$,

$$\lambda(C_T \cdot \ldots \cdot C_1 x, C_T \cdot \ldots C_1(0)) \geq 1 - \epsilon.$$

Theorem 2.3. *There exists a set $\mathfrak{F} \subset \mathfrak{M}$ which is a countable intersection of open everywhere dense sets in $\mathfrak{M}$ with the strong topology and such that for each $\{B_t\}_{t=1}^{\infty} \in \mathfrak{F}$ the following assertion holds:*

For each $\epsilon \in (0, 1)$ there exists a neighborhood U of $\{B_t\}_{t=1}^{\infty}$ in $\mathfrak{M}$ with the strong topology and an integer $N \geq 1$ such that for each $\{C_t\}_{t=1}^{\infty} \in U$, each $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$, each integer $T \geq N$ and each $x \in <0, b>$,

$$\lambda(C_{r(T)} \cdot \ldots \cdot C_{r(1)} x, C_{r(T)} \cdot \ldots \cdot C_{r(1)}(0)) \geq 1 - \epsilon.$$

Let $a \in <0, b>$ be an interior point of X_+. Denote by $\mathfrak{M}_a$ the set of all sequences $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}$ such that

$$A_t a = a, \ t = 1, 2, \dots.$$

Clearly $\mathfrak{M}_a$ is a closed subset of $\mathfrak{M}$ with the weak topology. We consider the topological subspace $\mathfrak{M}_a \subset \mathfrak{M}$ with the relative weak and strong topologies.

Now we present the theorems proved in (Reich and Zaslavski, 1999b) which establish generic convergence to a unique fixed point in the space $\mathfrak{M}_a$.

Theorem 2.4. *There exists a set $\mathfrak{F} \subset \mathfrak{M}_a$ which is a countable intersection of open (in the relative weak topology) everywhere dense (in the relative strong topology) sets in $\mathfrak{M}_a$ such that the following assertion holds:*

For each $\{B_t\}_{t=1}^{\infty} \in \mathfrak{F}$ and each $\epsilon > 0$ there exist a neighborhood U of $\{B_t\}_{t=1}^{\infty}$ in $\mathfrak{M}_a$ with the relative weak topology and a natural number N such that for each $\{C_t\}_{t=1}^{\infty} \in U$, each integer $T \geq N$ and each $z \in <0, b>$,

$$||C_T \cdot \ldots \cdot C_1 z - a||_b \leq \epsilon.$$

Theorem 2.5. *There exists a set $\mathfrak{F} \subset \mathfrak{M}_a$ which is a countable intersection of open everywhere dense sets in $\mathfrak{M}_a$ with the relative strong topology such that the following assertion holds:*

For each $\{B_t\}_{t=1}^{\infty} \in \mathfrak{F}$ and each $\epsilon > 0$ there exist a neighborhood U of $\{B_t\}_{t=1}^{\infty}$ in $\mathfrak{M}_a$ with the relative strong topology and a natural number N such that for each $\{C_t\}_{t=1}^{\infty} \in U$, each $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$, each integer $T \geq N$ and each $z \in < 0, b >$,

$$||C_{r(T)} \cdot \ldots \cdot C_{r(1)} z - a||_b \leq \epsilon.$$

Finally, denote by $\mathfrak{M}_*$ the set of all sequences $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}$ such that

$$A_t a = a, \quad t = 1, 2, \dots$$

for some $a \in < 0, b >$ such that a is an interior point of X_+. Denote by $\bar{\mathfrak{M}}_*$ the closure of $\mathfrak{M}_*$ in the space $\mathfrak{M}$ with the strong topology and consider the topological subspace $\bar{\mathfrak{M}}_* \subset \mathfrak{M}$ with the relative strong topology.

The last theorem in this section establishes generic uniform convergence of random infinite products to a unique common fixed point in the space $\bar{\mathfrak{M}}_*$. The proof of this theorem is given in (Reich and Zaslavski, 1999b).

Theorem 2.6. *There exists a set $\mathfrak{F} \subset \bar{\mathfrak{M}}_*$ which is a countable intersection of open everywhere dense sets in $\bar{\mathfrak{M}}_*$ such that for each $\{B_t\}_{t=1}^{\infty} \in \mathfrak{F}$ the following assertion holds:*

1. There exists an interior point $x(\mathbf{B}) \in < 0, b >$ of the cone X_+ which satisfies

$$B_t x(\mathbf{B}) = x(\mathbf{B}), \quad t = 1, 2, \dots$$

2. For each $\epsilon > 0$ there exist a neighborhood U of $\{B_t\}_{t=1}^{\infty}$ in $\bar{\mathfrak{M}}_*$ and a natural number N such that for each $\{C_t\}_{t=1}^{\infty} \in U$, each $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$, each integer $T \geq N$ and each $z \in < 0, b >$,

$$||C_{r(T)} \cdot \ldots \cdot C_{r(1)} z - x(\mathbf{B})||_b \leq \epsilon.$$

3 Homogeneous order-preserving mappings

In this section we let $(X, ||\cdot||)$ be a Banach space ordered by a closed cone X_+ with a nonempty interior such that $||x|| \leq ||y||$ for each $x, y \in X_+$ satisfying $x \leq y$. When $u, v \in X$ and $u \leq v$ we again set

$$< u, v > = \{x \in X : \ u \leq x \leq v\}.$$

For each $x, y \in X_+$ we define

$$\lambda(x, y) = \sup\{r \in [0, \infty) : \ rx \leq y\}, \quad r(x, y) = \inf\{\lambda \in [0, \infty) : \ y \leq \lambda x\}.$$

(We assume that the infimum of the empty set is ∞.)

For an interior point η of the cone X_+ we define

$$||x||_\eta = \inf\{r \in [0, \infty) : \ -r\eta \leq x \leq r\eta\}.$$

It is known that $||\cdot||_\eta$ is a norm on X which is equivalent to the norm $||\cdot||$.

Denote by $\mathcal{A}$ the set of all mappings $A : \ X_+ \to X_+$ such that

$$Ax \leq Ay \text{ for each } x \in X_+ \text{ and each } y \geq x,$$

$$A(\alpha z) = \alpha Az \text{ for each } \alpha \in [0, \infty) \text{ and each } x \in X_+.$$

Fix an interior point η of the cone X_+.

For the space $\mathcal{A}$ we define a metric $\rho : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ by

$$\rho(A, B) = \sup\{||Ax - Bx||_\eta : \ x \in <0, \eta>\}, \quad A, B \in \mathcal{A}.$$

It is easy to see that the metric space $(\mathcal{A}, \rho)$ is complete.

Denote by $\mathcal{M}$ the set of all sequences $\{A_t\}_{t=1}^\infty \subset \mathcal{A}$. A member of $\mathcal{M}$ will occasionally be denoted by a boldface $\mathbf{A}$. For the set $\mathcal{M}$ we consider the uniformity which is determined by the following base:

$$E(N, \epsilon) = \{(\{A_t\}_{t=1}^\infty, \{B_t\}_{t=1}^\infty) \in \mathcal{M} \times \mathcal{M} : \ \rho(A_t, B_t) \le \epsilon, \ t = 1, \dots N\},$$

where N is a natural number and $\epsilon > 0$. It is easy to see that the uniform space $\mathcal{M}$ is metrizable (by a metric $\rho_w : \mathcal{M} \times \mathcal{M} \to [0, \infty)$) and complete. This uniformity generates a topology which we call the weak topology in $\mathcal{M}$.

For the set $\mathcal{M}$ we also consider the uniformity which is determined by the following base:

$$E(N, \epsilon) = \{(\{A_t\}_{t=1}^\infty, \{B_t\}_{t=1}^\infty) \in \mathcal{M} \times \mathcal{M} : \ \rho(A_t, B_t) \le \epsilon, \ t = 1, 2, \dots\},$$

where $\epsilon > 0$. It is easy to see that the space $\mathcal{M}$ with this uniformity is metrizable (by a metric $\rho_s : \mathcal{M} \times \mathcal{M} \to [0, \infty)$) and complete. This uniformity generates a topology which we call the strong topology in $\mathcal{M}$.

Denote by $\mathcal{M}_{reg}$ the set of all sequences $\{A_t\}_{t=1}^\infty \in \mathcal{M}$ for which there exist positive constants $c_1 < c_2$ such that for each integer $T \ge 1$,

$$c_2\eta \ge A_T \cdot \ldots \cdot A_1\eta \ge c_1\eta.$$

Denote by $\bar{\mathcal{M}}_{reg}$ the closure of $\mathcal{M}_{reg}$ in $\mathcal{M}$ with the weak topology. We consider the topological subspace $\bar{\mathcal{M}}_{reg} \subset \mathcal{M}$ with the relative weak and strong topologies.

We now list the results established in (Reich and Zaslavski, 1999c).

Theorem 3.1. *There exists a set $\mathcal{F} \subset \bar{\mathcal{M}}_{reg}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) sets in $\bar{\mathcal{M}}_{reg}$ such that for each $\{B_t\}_{t=1}^\infty \in \mathcal{F}$ the following assertions hold:*

1. $B_T \cdot \ldots \cdot B_1\eta$ is an interior point of X_+ for each integer $T \ge 1$.

2. For each $\Delta > 1$ and each $\epsilon \in (0, 1)$ there exist an integer $N \ge 1$ and an open neighborhood U of $\{B_t\}_{t=1}^\infty$ in $\bar{\mathcal{M}}_{reg}$ with the weak topology such that for each $\{C_t\}_{t=1}^\infty \in U$,

$$C_T \cdot \ldots \cdot C_1\eta$$

is an interior point of X_+ for all $T \in \{1, \dots N\}$ and

$$r(C_N \cdot \ldots \cdot C_1\eta, C_N \cdot \ldots \cdot C_1x) - \lambda(C_N \cdot \ldots \cdot C_1\eta, C_N \cdot \ldots \cdot C_1x) \le \epsilon$$

for all $x \in <0, \Delta\eta>$.

(As mentioned in the previous section such results are usually called weak ergodic theorems in the population biology literature (Cohen, 1979; Fujimoto and Krause, 1988; Nussbaum, 1990).

Let $0 < c_1 < c_2$. Denote by $\mathcal{M}(c_1, c_2)$ the set of all sequences $\{A_t\}_{t=1}^\infty \in \mathcal{M}$ such that

$$A_T \cdot \ldots A_1\eta \ \in <c_1\eta, c_2\eta> \text{ for all integers } T \ge 1.$$

It is easy to verify that $\mathcal{M}(c_1, c_2)$ is a closed subset of $\mathcal{M}$ with the weak topology. We first consider the topological subspace $\mathcal{M}(c_1, c_2) \subset \mathcal{M}$ with the relative weak and strong topologies.

Theorem 3.2. *There exists a set $\mathcal{F}_0 \subset \mathcal{M}(c_1, c_2)$ which is a countable intersection of open (in the weak topology) and everywhere dense (in the strong topology) sets in $\mathcal{M}(c_1, c_2)$ such that for each $\{B_t\}_{t=1}^{\infty} \in \mathcal{F}_0$ assertion 2 of Theorem 3.1 is valid.*

Denote by $\mathrm{int}(X_+)$ the set of interior points of the cone X_+. Let $\mathcal{M}_*$ be the set of all $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ for which there exists a point $\xi \in \mathrm{int}(X_+)$ such that

$$A_t \xi = \xi, \quad t = 1, 2, \dots$$

Denote by $\bar{\mathcal{M}}_*$ the closure of $\mathcal{M}_*$ in the strong topology. Next we consider the topological subspace $\bar{\mathcal{M}}_* \subset \mathcal{M}$ with the relative strong topology.

Theorem 3.3. *There exists a set $\mathcal{F} \subset \bar{\mathcal{M}}_*$ which is a countable intersection of open everywhere dense sets in $\bar{\mathcal{M}}_*$ such that for each $\{B_t\}_{t=1}^{\infty} \in \mathcal{F}$ there exists an interior point ξ_B of X_+ satisfying*

$$B_t \xi_B = \xi_B, \ t = 1, 2, \dots, \quad ||\xi_B||_\eta = 1,$$

and the following assertions hold:

1. For each $s : \{1, 2, \dots\} \to \{1, 2, \dots\}$ there exists a function $g_s : X_+ \to [0, \infty)$ such that

$$\lim_{T \to \infty} B_{s(T)} \cdot \ldots \cdot B_{s(1)} x = g_s(x) \xi_B, \quad x \in X_+.$$

2. For each $\epsilon > 0$ there exist a neighborhood U of $\{B_t\}_{t=1}^{\infty}$ in $\bar{\mathcal{M}}_*$ and an integer $N \geq 1$ such that for each $\{C_t\}_{t=1}^{\infty} \in U \cap \mathcal{M}_*$, each integer $T \geq N$, each $s : \{1, 2, \dots\} \to \{1, 2, \dots\}$ and each $x \in < 0, \eta >$,

$$||C_{s(T)} \cdot \ldots \cdot C_{s(1)} x - g_s(x) \xi_B||_\eta \leq \epsilon.$$

Denote by $\mathcal{M}_\eta$ the set of all sequences $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ such that $A_t \eta = \eta$, $t = 1, 2, \dots$ Clearly $\mathcal{M}_\eta$ is a closed subset of $\mathcal{M}$ with the weak topology. We now consider the topological subspace $\mathcal{M}_\eta \subset \mathcal{M}$ with the relative weak and strong topologies.

Theorem 3.4. *There exists a set $\mathcal{F} \subset \mathcal{M}_\eta$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) sets in $\mathcal{M}_\eta$ such that for each $\{B_t\}_{t=1}^{\infty} \in \mathcal{F}$ the following assertions holds:*

1. There exists $f : X_+ \to R^1$ such that

$$\lim_{T \to \infty} B_T \cdot \ldots \cdot B_1 x = f(x) \eta, \quad x \in X_+.$$

2. For each $\epsilon > 0$ there exist a neighborhood U of $\{B_t\}_{t=1}^{\infty}$ in $\mathcal{M}_\eta$ with the weak topology and an integer $N \geq 1$ such that for each $\{C_t\}_{t=1}^{\infty} \in U$, each integer $T \geq N$ and each $x \in < 0, \eta >$,

$$||C_T \cdot \ldots \cdot C_1 x - f(x) \eta||_\eta \leq \epsilon.$$

Theorem 3.5. *There exists a set $\mathcal{F} \subset \mathcal{M}_\eta$ which is a countable intersection of open everywhere dense sets in $\mathcal{M}_\eta$ with the strong topology such that for each $\{B_t\}_{t=1}^{\infty} \in \mathcal{F}$ the following assertions hold:*

1. For each $s : \{1, 2, \dots\} \to \{1, 2, \dots\}$ there exists a function $g_s : X_+ \to R^1$ such that

$$\lim_{T \to \infty} B_{s(T)} \cdot \ldots \cdot B_{s(1)} x = g_s(x) \eta, \quad x \in X_+.$$

2. For each $\epsilon > 0$ there exist a neighborhood U of $\{B_t\}_{t=1}^{\infty}$ in $\mathcal{M}_\eta$ with the strong topology and an integer $N \geq 1$ such that for each $\{C_t\}_{t=1}^{\infty} \in U$, each integer $T \geq N$, each $s : \{1, 2, \dots\} \to \{1, 2, \dots\}$ and each $x \in < 0, \eta >$,

$$||C_{s(T)} \cdot \ldots \cdot C_{s(1)} x - g_s(x) \eta||_\eta \leq \epsilon.$$

Denote by $\mathcal{A}_*$ the set of all $A \in \mathcal{A}$ such that there is an interior point ξ_A of X_+ satisfying $A \xi_A = \xi_A$. Let $\bar{\mathcal{A}}_*$ be the closure of $\mathcal{A}_*$ in $\mathcal{A}$. We equip the topological subspace $\bar{\mathcal{A}}_* \subset \mathcal{A}$ with the relative topology.

Theorem 3.6. *There exists a set $\mathcal{F} \subset \bar{\mathcal{A}}_*$ which is a countable intersection of open everywhere dense sets in $\bar{\mathcal{A}}_*$ such that for each $B \in \mathcal{F}$ there exists an interior point ξ_B of X_+ satisfying*

$$B\xi_B = \xi_B, \quad ||\xi_B||_\eta = 1,$$

and the following assertions hold:

1. There exists a function $g_B : X_+ \to R^1$ such that

$$\lim_{T\to\infty} B^T x = g_B(x)\xi_B, \quad x \in X_+.$$

2. For each $\epsilon > 0$ there exist a neighborhood U of B in $\bar{\mathcal{A}}_*$ and an integer $N \geq 1$ such that for each $C \in U \cap \mathcal{A}_*$, each integer $T \geq N$ and each $x \in <0, \eta>$,

$$||C^T x - g_B(x)\xi_B||_\eta \leq \epsilon.$$

Finally, denote by $\mathcal{A}_\eta$ the set of all $A \in \mathcal{A}$ satisfying $A\eta = \eta$. Clearly $\mathcal{A}_\eta$ is a closed subset of $\mathcal{A}$. We endow the topological subspace $\mathcal{A}_\eta \subset \mathcal{A}$ with the relative topology.

Theorem 3.7. *There exists a set $\mathcal{F} \subset \mathcal{A}_\eta$ which is a countable intersection of open everywhere dense sets in $\mathcal{A}_\eta$ such that for each $B \in \mathcal{F}$ the following assertions hold:*

1. There exists a functional $g_B : X_+ \to R^1$ such that

$$\lim_{T\to\infty} B^T x = g_B(x)\eta, \quad x \in X_+.$$

2. For each $\epsilon > 0$ there exist a neighborhood U of B in $\mathcal{A}_\eta$ and an integer $N \geq 1$ such that for each $C \in U$, each integer $T \geq N$ and each $x \in <0, \eta>$,

$$||C^T x - g_B(x)\eta||_\eta \leq \epsilon.$$

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